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Higher T-duality of super M-branes

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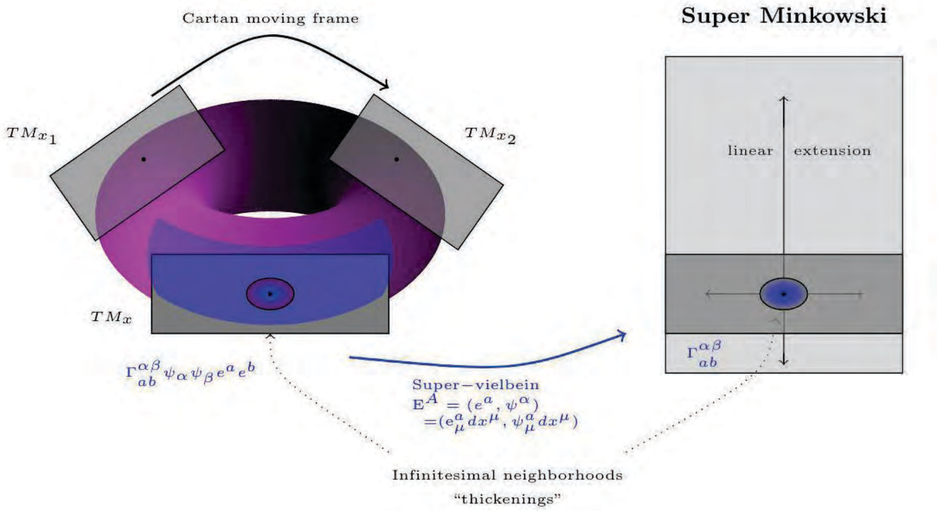
We establish a higher generalization of super L_∞-algebraic T-duality of super WZW-terms for super p-branes. In particular, we demonstrate spherical T-duality of super M5-branes propagating on exceptional-geometric 11d super spacetime.

1. Introduction and summary

The torsion constraints of supergravity and of super p-brane models have remarkably far-reaching consequences. This is well-known and goes back to [GWZ79], but since it is key to our study of M-brane phenomena, we briefly recall it, putting it into a geometric perspective that will be useful for presenting our results. The following picture illustrates how super-Cartan geometry models a super-manifold by moving local super-Minkowski frames around it, and that super-torsion freedom of the super-vielbein $(E^a) := (e^a, \psi^\alpha)$ means that, moreover, each first-order infinitesimal neighborhood is supersymmetrically identified with the local model. This is explained in [Lot90, EE12], following the seminal result of [Gui65].



Explicitly, vanishing of the super-torsion in eleven-dimensional supergravity says, in particular, that the bifermionic component of the torsion tensor is constrained to be in each super-tangent space given by (see [BST87, (14)])

$$T^a_{\alpha\beta} = \Gamma^a_{\alpha\beta} \,.$$

In terms of Cartan calculus on super-Minkowski spacetime, this means that (see Section 2.1 below)

$$de^a = \overline{\psi}\Gamma^a\psi \,, \tag{1}$$

which makes it manifest that this identifies each super-tangent space of a supergravity background with super-Minkowski spacetime not just as a super-vector space, but as a super Lie algebra: the translational part of the supersymmetry algebra. This is the basis for the powerful super-Cartan-geometric perspective on supergravity advocated in [D'AF82, CDF91]; and in disguise the innocent-looking differential (1) governs much of the structure of supergravity (also known as *τ-cohomology* [BBLPT90]). Remarkably, in 11d the constraint of vanishing super-torsion alone is already equivalent to the full supergravity equations of motion [CL94, Ho97, FOS17], a first indication that the implications of these super tangent space-wise constraints are far-reaching.

Moreover, the bifermionic component $H_{1,2} := H_{a\alpha\beta} e^a \wedge \psi^\alpha \wedge \psi^\beta$ of the background field strength H, to which the type I superstring [GS84] in dimensions 3, 4, 6 and 10 couples, is constrained in each super tangent-space to be of the form [BST86, (2.11), (2.15)] (see expression (39) below))

$$H_{a\alpha\beta} = \Gamma_{a\alpha\beta} \,. \tag{2}$$

Similarly, the bifermionic component $G_{2,2} := G_{ab\alpha\beta} e^a \wedge e^b \wedge \psi^\alpha \wedge \psi^\beta$ of the background field strength, to which the supermembrane in dimension 4, 5, 7 and 11 couples, is constrained in each super-tangent space to be given by [BST87, (15)] (see (42) below)

$$H_{ab\alpha\beta} = \Gamma_{ab\alpha\beta} \,. \tag{3}$$

From the point of view of the kappa-symmetric super-brane sigma-model, this is due to the fact that precisely in these dimensions these forms are super Lie algebra cocycles for the translational supersymmetry algebra [AETW87, AzTo89, BH10, BH11], a fact known as the "old brane scan". Strikingly, in 11d the same forms are implied by the super torsion-free super Cartan geometry that also implies the 11d supergravity equations of motion [CL94, Ho97, FOS17].

Directly analogous statements, with more complicated local expressions, hold for all the super D-branes [CGNSW97, (3.9)] and for the M5-brane [LT96, (5)] [BLNPST97, (6)] (recalled below in Example 4.3). A key difference here is that understanding these as super-cocycles requires passage from ordinary to extended super Minkowski spacetime [CAIB00, Sak00]. In [FSS15c] we pointed out, following [SSS09, p. 54] as expanded on in [Hu12, Hu14], that this means to pass to *homotopy* super Lie algebras, called super L_∞*-algebras* (see Section 2) and we showed that this perspective completes the "old brane scan" to a "brane bouquet" of iterative higher central extensions of super L_∞-algebras:

$$\begin{tikzcd}
 & & & \mathfrak{m5brane} \arrow[d] & & & & \text{[FSS15d]} \\
 & & & \mathfrak{m2brane} \arrow[dd] & & & & \\
\mathfrak{d5brane} \arrow[dddr] & \mathfrak{d3brane} \arrow[ddd] & \mathfrak{d1brane} \arrow[dddl] & & \mathfrak{d0brane} \arrow[dl, dotted] \arrow[dddr] & \mathfrak{d2brane} \arrow[ddd] & \mathfrak{d4brane} \arrow[dddl] & \\
\mathfrak{d7brane} \arrow[ddr] & & & \mathbb{R}^{10,1|\mathbf{32}} \arrow[dddr] & & & \mathfrak{d6brane} \arrow[ddl] & \text{[FSS15c, FSS16a]} \\
 & & & & \scriptstyle (pb) & & & \\
\mathfrak{d9brane} \arrow[r] & \mathfrak{string}_{\mathrm{IIB}} \arrow[dr] & & \mathfrak{string}_{\mathrm{het}} \arrow[d] & & \mathfrak{string}_{\mathrm{IIA}} \arrow[dl] & \mathfrak{d8brane} \arrow[l] & \\
 & & \mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & & & \\
 & & & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} & & & \\
 & & \mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & & & & \text{[HSc17]} \\
 & & \mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & & & & \\
 & & \mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & & & &
\end{tikzcd}$$

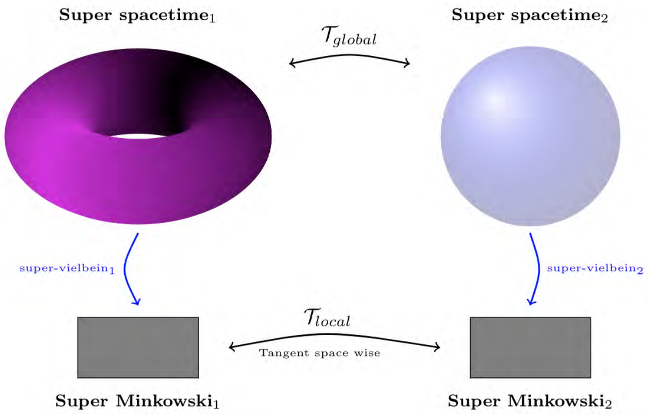
$$\begin{tikzcd}
\boxed{\text{Type IIB}} \arrow[rr, leftrightarrow, bend right, "\text{T-Duality}"'] & \boxed{\text{Type I}} & \boxed{\text{Type IIA}} & \text{[FSS16b]}
\end{tikzcd}$$

The full implication of these super tangent space-wise constraints for super WZW-terms for super p-branes has perhaps not been fully appreciated yet. Notice that any duality in string/M-theory, when fully taking into account all the fermionic degrees of freedom, will have to respect all these constraints from super tangent space to super tangent space. This is a strong condition on any duality.

Indeed, in [FSS16b] we had shown that the super tangent space-wise torsion constraints/super-WZW terms of the super F1/Dp branes in type II super-spacetime already completely reveal the structure of T-duality on brane charges; we recall this below in Section 4.1. This structure had previously been proposed under the name "topological T-duality" ([BEM04, BHM04, BS05]) and had been conjectured to underlie the actual T-duality of string theory (see also Remark 3.11 below).

The following picture illustrates how a global duality, such as topological T-duality $\mathcal{T}_{global}$, restricts to a duality on super-tangent spaces, such as the super L_∞-algebraic T-duality $\mathcal{T}_{\mathrm{loc}}$ of [FSS16b] (Section 4.1 below):

Turning this around, it means that analysis of the super-tangent space-wise super-WZW terms of the super p-branes may be used to systematically discover and analyze previously unknown facts about super M-brane



Note that the double-headed arrows here indicate not direct maps, but rather "spans" of maps, called *correspondences*; see Def. 3.8 below.

physics, and hence about the elusive theory of which they are a part. This is what we consider here. In this paper we establish further extensions of the above brane bouquet which may be organized into the following diagrammatic table of contents. In the remainder of this introduction we will explain informally what (some of) the boxed items in this diagram mean.

Spherical T-duality (Prop. 4.17)

528-toroidal T-duality (Prop. 4.11 (ii))

$$\begin{tikzcd}
\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s} \arrow[rr, "\mathrm{comp}"] \arrow[d, "\mathrm{hofib}(\mu_{\mathrm{exc},s})"'] & & \mathfrak{m2brane} \arrow[dd, "\mathrm{hofib}(\mu_{M2})"] & \\
\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc}} \arrow[drr, "\mathrm{hofib}(\mu_{\mathrm{exc}})"] \arrow[dddddd, "\mathrm{hofib}(\psi\wedge\overline{\psi})"'] & & & \\
& & \mathbb{R}^{10,1|\mathbf{32}} \arrow[dr] & \\
& \mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \\
& \mathbb{R}^{5,1|\mathbf{8}+\mathbf{8}} & \mathbb{R}^{5,1|\mathbf{8}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} \\
& \mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
& \mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{R}^{0|\mathbf{32}} & \mathbb{R}^{0|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] &
\end{tikzcd}$$

Spherical T-duality (Prop. 4.4)

517-toroidal T-duality (Prop. 4.11 (iii))

Cyclic T-duality (Sect. 4.1)

Cyclic T-duality (Prop. 4.2)

Exceptional | Type IIB | Type I | Type IIA

It is noteworthy that this is a derivation of M-theoretic structure from first principles, not involving any extrapolation from perturbative string theory nor any conjectures or informal analogies from other sources. Instead, this derivation is the systematic rigorous analysis of the progression of higher extensions in super homotopy theory that is emanating from the superpoint (see [Sc17b] for exposition of our perspective).

We now explain this conceptually. We first observe that the WZW-term of the M5-brane sigma-model [BLNPST97, FSS15d] exhibits 3-spherical topological T-duality, which is interestingly a self-duality; this is Prop. 4.4 below. It may be understood by observing that the relation between the joint M2/M5-cocycle (recalled as Example 4.3 below)

$$d\mu_{{}_{M5}} + \tfrac{1}{2}\mu_{{}_{M2}} \wedge \mu_{{}_{M2}} = 0\,,$$

which is the super L_∞-algebraic avatar of the equations of motion on the 11d supergravity flux forms

$$dG_7 + \tfrac{1}{2}G_4 \wedge G_4 = 0,$$

is a higher analog of the relation

$$d\mu_{{}_{F1}}|_{8+1} + c_2^{\mathrm{IIA}} \wedge c_2^{\mathrm{IIA}} = 0$$

that encodes super-topological T-duality for type II superstrings (recalled in Section 4.1 below). Equivalently (by Prop. 3.13 below) this is the fact that the genuine M5-brane supercocycle (see (43) below)

$$\tilde{\mu}_{{}_{M5}} := 2\mu_{{}_{M5}} + c_3 \wedge \mu_{{}_{M2}}\,, \tag{4}$$

which is the super L_∞-algebraic avatar of the higher WZW term of the M5-brane sigma-model

$$d\mathbf{L}_{M5,\mathrm{WZW}} = G_7 + \tfrac{1}{2}C_3 \wedge G_4,$$

has an algebraic structure which is a higher degree analog of that of a "T-dualizable H-flux" whose super L_∞-algebraic avatar is (see (39) below)

$$\mu_{{}_{F1}}^{\mathrm{IIA/B}} = \mu_{{}_{F1}}|_{8+1} + e^9_{{}_{A/B}} \wedge c_2^{\mathrm{IIB/A}}\,. \tag{5}$$

In comparing expression (4) with expression (5), one sees that the role of the fiberwise Maurer-Cartan 1-form e^9 in cyclic type II string T-duality is now taken by the C-field c_3. This shows that when passing from T-duality for type II string theory (recalled in Section 4.1 below) to spherical T-duality of M5-branes (Section 4.3 below), the analog of the role of the 10d super-spacetime fibered over the 9d super spacetime is now the 2-gerbe (or, rationally, the 3-sphere fibration) that is classified by the M2-brane charge, which is fibered *over 11d spacetime*. This phenomenon is also indicated in the table at the beginning of Section 3.2 below.

$$\begin{tikzcd}
\boxed{\text{M2-extended super-spacetime}} \arrow[rr, leftrightarrow, "\textit{Spherical T-self-duality}"] \arrow[dr, "\textit{Fibration classified by}"'] & & \boxed{\text{M2-extended super-spacetime}} \arrow[dl, "\textit{M2-brane WZW-term}"] \\
& \boxed{\text{11d super-spacetime}} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{m2brane} \arrow[rr, leftrightarrow, bend left, "\textit{Homotopy fiber-product}"] \arrow[dr, "\mathrm{hofib}(\mu_{M2})"'] & & \mathfrak{m2brane} \arrow[dl, "\mathrm{hofib}(\mu_{M2})"] \\
& \mathbb{R}^{10,1|\mathbf{32}} &
\end{tikzcd}$$

This indeed makes sense, as highlighted before in [FSS15c, Remark 3.11, Sec. 4.4]: the component of a plain sigma-model field in this 2-gerbe fiber is equivalently the higher gauge field on the M5-brane's worldvolume.

$$\begin{tikzcd}
& & \mathfrak{m2brane} \arrow[d, "\mathrm{hofib}(\mu_{M2})"] & \boxed{\text{Super 2-gerbe over super-spacetime}} \\
\boxed{\text{M5-brane worldvolume}} & \Sigma_{M5} \arrow[ur, dotted, "\textit{Sigma-model- \& gauge-field}"] \arrow[r, bend right, "\textit{Plain sigma-model field}"'] & \mathbb{R}^{10,1|\mathbf{32}} & \boxed{\text{Target Super-spacetime}}
\end{tikzcd}$$

This way higher geometry provides a unification of sigma-model fields with worldvolume gauge fields, and our Prop. 4.4 says that this unified perspective reveals spherical T-duality in M-brane theory.

In order to shed more light on this subtle point, we next observe that the decomposed C-field on the exceptional super-tangent space of 11d super-spacetime serves as a transgression element for the M2-brane WZW terms. This is Prop. 4.14 below, which provides a re-interpretation of the "hidden" D'Auria-Fré algebra from [D'AF82, BAIPV04] (see also [BdAPV05, ADR16, ADR17]) over which the supergravity C-field decomposes super-equivariantly. We expand on that in Section 4.6 below, where we explain how the D'Auria-Fré algebra may be regarded as providing the supersymmetric refinement of the exceptional generalized geometry for the C-field proposed in [H07b, PW08][1] and how spherical T-duality acts by duality

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[1] Finding a suitable supergeometric refinement of exceptional/generalized geometry is stated as an open problem in [CEK13, p. 18][Ced14, p. 4,7]. In [Ban17,

transformation on the resulting super-moduli spaces.

$$\begin{tikzcd}
\boxed{\text{Def. 4.13}} & \mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s} \arrow[d, "\mathrm{hofib}(\mu_{\mathrm{exc},s})"] & \simeq_{\mathbb{R}} & \underbrace{\mathbb{R}^{10,1} \oplus \wedge^2\big(\mathbb{R}^{10,1|\mathbf{32}}\big)^* \oplus \wedge^5\big(\mathbb{R}^{10,1|\mathbf{32}}\big)^* \oplus \mathbf{32}_{\mathrm{odd}} \oplus \mathbf{32}_{\mathrm{odd}}}_{\text{Exceptional generalized super-geometry}} \\
\boxed{\text{Def. 4.5, Prop. 4.6}} & \mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc}} \arrow[d, "\mathrm{hofib}(\mu_{\mathrm{exc}})"] & \simeq_{\mathbb{R}} & \underbrace{\mathbb{R}^{10,1} \oplus \wedge^2\big(\mathbb{R}^{10,1|\mathbf{32}}\big)^* \oplus \wedge^5\big(\mathbb{R}^{10,1|\mathbf{32}}\big)^*}_{\text{Exceptional generalized geometry}} \oplus \mathbf{32}_{\mathrm{odd}} \\
\boxed{\text{Ex. 2.1}} & \mathbb{R}^{10,1|\mathbf{32}} & \simeq_{\mathbb{R}} & \underbrace{\underbrace{\mathbb{R}^{10,1}}_{\text{Spacetime}} \oplus \mathbf{32}_{\mathrm{odd}}}_{\text{Super spacetime}}
\end{tikzcd}$$

In particular, using the relation to the Hořava-Witten boundary of the decomposed C-field as in [ES03], this explains the nature of the extra fermion generator in the D'Auria-Fré algebra, which was a concern in [ADR16, ADR17] (see Example 4.23 below). [2]

This means that sigma-model fields with values in the super 2-gerbe, hence the pairs of ordinary sigma-model fields and worldvolume higher gauge fields, may be obtained from plain sigma-model fields into the exceptional tangent space of 11d super-spacetime. Here, again, the sigma-model field components into the fibers over spacetime transmute into gauge fields on the brane's worldvolume:

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{Exceptional tangent space}\\ \text{over super-spacetime}\end{array}} & \mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s} \arrow[r, "\textit{Decomposition}\\ \textit{of C-field}"] & \mathfrak{m2brane} & \boxed{\begin{array}{c}\text{Super 2-gerbe}\\ \text{over super-spacetime}\end{array}} \\
\Sigma_{M5} \arrow[ur, dotted, "\textit{U-duality equivariant}\\ \textit{sigma-model- \& gauge-field}"] \arrow[urr, dashed, bend right, "\textit{Sigma-model-}\\ \textit{\& gauge-field}"'] & & &
\end{tikzcd}$$

That the M5-brane indeed has a formulation as a plain sigma-model on the exceptional tangent space over 11d-spacetime this way was recently observed in [SU16]. Related observations are due to [FLST10].

It may be noteworthy that passing from $\mathfrak{m2brane}$ to $\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s}$ this way, means to trade a (rational) 3-sphere fibration over 11d superspacetime for

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p.10,11] it was proposed that $\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc}}$ (in our notation) is the answer. But for our argument in Section 4.6 the further fermionic extension to the DF-algebra $\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s}$ is crucial.

[2] Note that decomposability of the corresponding field strength G_4 also arises naturally in the classification of backgrounds for 11-dimensional supergravity [FOP02].

a high-dimensional (rational) torus fibration. This is reminiscent of a recent interest in relating toroidal and spherical backgrounds [CCGS13, CS18].

Putting all the above together we arrive at the following global picture:

$$\begin{tikzcd}
 & & \boxed{\textit{Exceptional tangent space over super-spacetime}} & \boxed{\textit{Super 2-gerbe over super-spacetime}} & \\
 & & \mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s} \arrow[r, "\textit{Decomposition of C-field}", "\mathrm{comp}_s"'] & \mathfrak{m2brane} \arrow[d, "\mathrm{hofib}(\mu_{M2})"'] \arrow[dr, leftrightarrow, bend left, "\textit{Spherical T-self-duality}"] & \\
 & & & & \mathfrak{m2brane} \arrow[dl, "\mathrm{hofib}(\mu_{M2})"] \\
\boxed{\textit{M5-brane worldvolume}} & \Sigma_{M5} \arrow[uur, dotted, "\textit{U-duality equivariant sigma-model- \& gauge-field}"] \arrow[uurr, dashed, "\textit{Sigma-model- \& gauge-field}"] \arrow[rr, bend right, "\textit{Plain Sigma-model-field}"'] & & \mathbb{R}^{10,1|\mathbf{32}} & \\
 & & & \boxed{\textit{Target super-spacetime}} &
\end{tikzcd}$$

Finally we show that the spherical T-duality on $\mathfrak{m2brane}$ passes along the decomposition map to spherical T-duality on the exceptional super spacetime $\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s}$; this is Prop. 4.17 below. The key step in establishing this (via Theorem 3.23 below) is to show that the decomposed C-field on the exceptional super-spacetime $\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s}$ still allows to distinguish the 3-spherical wrapping modes of the M5-brane that get exchanged with the non-wrapping modes under spherical T-duality. Prop. 4.15 below shows that this is the case, except possibly for summands in the M5-brane charge twisted supercocycles which are multiples in the gravitino field ψ of the 528-volume form on the exceptional superspacetime; see Remark 4.19 below.

Note that a higher spherical version of T-duality had been established in vast mathematical generality in [LSW16], and in a special case in [BEM15]. In [BEM15] the spherical fiber bundle was suggested to be identified with actual spacetime itself, not a fibration over spacetime; however, the direct relation to string theory or M-theory had remained unclear. Notice that the spherical T-duality which we discover takes place entirely in M-theory; notably it is distinct from the strong coupling lift of ordinary T-duality to M-theory (see [Ru97][Sch96]) which is the non-perturbative version of 10-dimensional T-duality, with one side inevitably involving a string theory. The super L_∞-algebraic formulation of this strong coupling T-duality, interpreted as F-theory, we had discussed already in [FSS16b, Section 8].

In **summary**, in this article we offer the following insights:

1) A new duality in M-theory: spherical T-duality for M5-branes (Section 4.3).
2) Toroidal T-duality on exceptional super-spacetime (Section 4.4);
3) Clarification of the supersymmetric exceptional generalized geometry of M-theory (Section 4.6);
4) Identification of spherical T-duality as a duality on the exceptional-generalized super-geometry (Section 4.5);
5) Parity symmetry as an isomorphism in M5-brane charge twisted cohomology akin to spherical T-duality (Section 4.8);

We provide the above physical insights within the proper mathematical setting, developed in Section 3. We highlight the power of cohomological techniques in the supergeometric setting, including C-cohomology to study T-duality for decomposed C-fields in Section 3.4. Similar techniques in other contexts have allowed, for instance, for eleven-dimensional supergravity to be recovered from the Spencer cohomology of the Poincaré superalgebra [FOS17].

The **outline** of the article is as follows: Section 2 is mainly to fix concepts and notations; the reader roughly familiar with super L_∞-algebras and their cohomology may want to skip this section and start with Section 3, coming back to Section 2 when the need arises. Section 3 contains the main mathematical results. They are presented in a general form, with all examples from M-brane theory postponed to Section 4. The reader principally interested into these examples is invited to start directly from Section 4, going back to Section 3 for the proofs of the general statements used in Section 4.

2. Super L_∞-homotopy theory

We work in the homotopy theory of super L_∞-algebras as in [FSS15c, FSS15d, FSS16a, FSS16b, FSS17, Sc17b] (see Remark 2.2 below on differing terminology). Here we set up the basics that we need in the following sections.

The bosonic sector of super L_∞-homotopy theory is a model for *rational homotopy theory* (for review see e.g. [Hes06] or [FSS16a, Section A]), where topological spaces as well as spectra parameterized over them [Bra18] are studied in the coarse-grained perspective that regards two of them as essentially the same as soon as there is a map between them that induces an isomorphism on all rationalized homotopy groups. This amounts to disregarding (for the time being) all information contained in torsion-subgroups of homotopy group and to retaining only the information that may be represented by differential form data. Notably the Chevalley-Eilenberg algebras of nilpotent L_∞-algebras are Sullivan's models for rational homotopy types [Sul77].

For example, in rational homotopy theory the spheres of odd dimension $2n+1$ are equivalent to Eilenberg-MacLane spaces concentrated in this odd degree:

$$\begin{tikzcd}
K(\mathbb{Z}, 2n+1) \arrow[r, "\text{rationalize}"] & K(\mathbb{Q}, 2n+1) & S^{2n+1} \arrow[l, "\text{generator}"'] \arrow[ll, bend left, "\text{rational equivalence}"]
\end{tikzcd}$$

and both are algebraically represented by the simplest possible Sullivan model, namely by the differential-graded commutative algebra (dgc-algebra) that has a single generator c in degree $2+1$, and whose differential vanishes: $dc = 0$.

The observation that rational homotopy theory sits inside the homotopy theory of L_∞-algebras is implicit already in [Qui69], but was made fully explicit only in [Hin01], on which the modern model [Pri10] is based. A review of rational homotopy theory from the perspective of L_∞-algebras is given in [BFM11, Section 2]. Therefore, the homotopy theory of super L_∞-algebras may be regarded as a model for rational supergeometric homotopy theory.

In the supergravity literature which goes back to [D'AF82, CDF91], the Chevalley-Eilenberg algebras of super L_∞-algebras are known as "FDA"s, following [vN82]. In supergravity these serve to neatly unify supersymmetry super Lie algebras (such as super Poincaré algebras) with the higher degree form fields that are crucial ingredients of higher dimensional supergravity theories.

From the point of view of super homotopy theory this phenomenon is interpreted [FSS15c] as saying that super-Minkowki super Lie algebras carry a finite number of exceptional $\mathrm{Spin}(p,1)$-invariant cohomology classes (Section 2.1 and Section 2.2 below) that iteratively classify a Whitehead-like tower of higher central extensions (Section 2.3 below), analogous to the case of connected covers of Lie groups studied in [SSS09][SSS12][Sa14], which interestingly admit extensions to the Lorentzian case [SS15] and to the rational case [SW] as we consider here. The tower we construct has higher equivariant connections which are precisely the higher WZW terms of the super-p-branes appearing in string/M-theory; see the diagram on page 624.

In [FSS16b] we had shown that not only the brane content and brane intersection laws may be "read off" from super homotopy this way, but the "super-topological" T-duality between type IIA and type IIB super F1/Dp-branes may be discovered (in fact together with the very axioms of topological T-duality themselves) establishing a kind of reflection symmetry in the above *brane bouquet* diagram. It is curious that all vertical-going arrows in this diagram are given from first principles: they are the maximal R-symmetry invariant higher extension in each case [HSc17]. We discuss this phenomenon further below in Section 4.4.

Basic concepts and notation used.

- $\mathfrak{g}$: A (super-) L_∞-algebra, i.e. an algebraic structure akin to a super Lie algebra but with brackets and higher Jacobi relations of any higher degree. A (super-) L_∞-algebra encodes the structure of infinitesimal (super-)symmetries and of ever higher order infinitesimal (super-)symmetries of infinitesimal (super-)symmetries. A key example for us is $\mathfrak{g}$ = extended Minkowski super spacetime. For background on (super-)L_∞-algebras in the context that we use see [SSS09, BH11, FSS16b].
- $b^n\mathfrak{u}_1 \simeq b^n\mathbb{R}$: The super L_∞-algebras which are the higher versions ("deloopings") of the abelian Lie algebras $\mathbb{R} \simeq \mathfrak{u}_1$. The underlying chain complex has a copy of $\mathbb{R}$ in degree n, and all brackets vanish. These are the rational models for Eilenberg-MacLane spaces $K(\mathbb{Z}, n+1)$. Just as the latter classify ordinary integral cohomology, these L_∞-algebra serve as classifying objects for super L_∞-cohomology; see Def. 2.3 below.
- $\mathrm{CE}(\mathfrak{g})$: The Chevalley-Eilenberg algebra of the super L_∞-algebra $\mathfrak{g}$, i.e., a differential graded-commutative (DGC) superalgebra of elements dual to $\mathfrak{g}$ whose differential encodes the brackets and higher brackets on $\mathfrak{g}$. The CE-algebra may be presented as

  $$\mathrm{CE}(\mathfrak{g}) = [\text{generators}]/(\text{differentials of generators}).$$

  These are also known as "FDA"s in some of the supergravity literature; see Remark 2.2 below.
- $\mu \in \mathrm{CE}(\mathfrak{g})$ a super L_∞-cocycle, hence a closed elements in the Chevalley-Eilenberg algebra.

- $\hat{\mathfrak{g}}$: A higher central extension of super-L_∞-algebra $\mathfrak{g}$. A key example is $\widehat{\mathfrak{g}} = \mathfrak{m2brane}$, which is the extension of 11d super Minkowski spacetime by the M2-brane supercocycle $\mu_{{}_{M2}}$.
- hofib(μ_n): Homotopy fiber of a super L_∞-cocycle μ of degree n. The latter may be viewed as a map to the classifying algebra $b^n\mathbb{R}$.

$$\begin{tikzcd}
b^{n-1}\mathfrak{u}_1 \arrow[r] \arrow[d] & \hat{\mathfrak{g}} \arrow[r] \arrow[d, "\mathrm{hofib}(\mu)"] & * \arrow[d] \\
* \arrow[r] & \mathfrak{g} \arrow[r, "\mu"] & b^n\mathfrak{u}_1
\end{tikzcd}$$

See Prop. 2.10 below. This is the super L_∞-algebra counterpart of principal bundles obtained as pullback of universal bundles

$$\begin{tikzcd}
G \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & EG \arrow[d] \\
* \arrow[r] & X \arrow[r] & BG
\end{tikzcd}$$

for G a (higher) abelian topological group [NSS12].

- $\mu_{{}_M}$: Cocycle corresponding to an M-brane; of degree 4 for the M2-brane and degree 7 for the M5-brane; see Example 4.3 below. Jointly these are valued in the rational 4-sphere [Sa13][FSS15d][FSS16a].
- dd$_n$: Cocycle regarded as the rational version of a higher Dixmier-Douady class (see [Sch09] for a very readable account) generalized from to higher degrees, as described in [FSS13, FSS15a, FSS15b]. The original class dd$_3$ is in turn a generalization of the Chern class of a line bundle.
- *Higher torus*: A product of shifted circles, i.e. of $b^n\mathfrak{u}_1$'s ; see Def. 3.1 below.
- $\overrightarrow{\mathrm{dd}}$: A tuple of cocycles, in our case a k-vector of $(2n+1)$-cocycles classifying an extension by a higher k-torus of degree $2n+1$; see Def. 3.1 below.
- $H^{i+\mu}$: Cohomology in degree i twisted by the cocycle μ, see Def. 2.9 below. Other notations for twisted cohomology include $H^i_\mu(-)$ and $H^i(-;\mu)$ but the first might be confused with equivariant cohomology (which we use) and the second would lead to cumbersome notation when we introduce coefficients to our cohomology groups.

- $s\mathrm{LieAlg}_{\mathbb{R}}^{\mathrm{fin}}$: The category of finite dimensional super Lie algebras, i.e., the collection of all super Lie algebras with appropriate homomorphisms between then.
- $sL_\infty\mathrm{Alg}_{\mathbb{R}}^{\mathrm{fin}}$: The category of super L_∞-algebras of finite type, i.e., the collection of all super L_∞-algebras with appropriate homomorphisms between them.
- $\mathrm{sdgcAlg}_{\mathbb{R}}^{\mathrm{op}}$: The category of super differential graded-commutative (sCDG) algebras, i.e., the collection of all super CDG-algebras with appropriate homomorphisms between them.

2.1. Super L_∞-cohomology

The cocycles that we encounter are built out of bosonic and fermionic fields and are closed under an appropriate differential. We now provide the proper setting for describing such fields or cocycles, namely *super L_∞-algebras.* For details we refer the reader to [FSS16b, Section 2] and references therein; see also Remark 2.2 below on differing terminology.

To every finite-dimensional super Lie algebra $(\mathfrak{g}, [-,-])$ one associates its *Chevalley-Eilenberg algebra* $\mathrm{CE}(\mathfrak{g})$, which is the free $(\mathbb{Z}, \mathbb{Z}/2)$-bigraded-commutative algebra $\wedge^\bullet(\mathfrak{g})^*$ equipped with the differential $d_{\mathfrak{g}} := [-,-]^*$, which on generators is the linear dual of the super Lie bracket, and from there uniquely extended as graded derivation of bidegree $(1, \mathrm{even})$.

Example 2.1 (Translational supersymmetry super Lie algebra). A key class of examples is the Lorentzian supersymmetry super Lie algebras which are specified by a spacetime dimension $d = p+1$ and a choice of real representation $\mathbf{N}$ (of real dimension $N \in \mathbb{N}$) of the corresponding Spin group $\mathrm{Spin}(p,1)$. Their translational part may be thought of as the corresponding $(p+1)$-dimensional and "N-supersymmetric" super-Minkowski spacetime $\mathbb{R}^{p,1|\mathbf{N}}$ equipped with its super-translation super Lie action on itself. From this point of view the corresponding Chevalley-Eilenberg algebra is generated from the standard super-left invariant super-vielbein

$$(e^a = dx^a + \overline{\theta}\Gamma^a d\theta,\ \psi^\alpha = d\theta^\alpha)$$

and the CE-differential is given on generators by the torsion constraint equation (1):

$$\mathrm{CE}\big(\mathbb{R}^{p,1|\mathbf{N}}\big) = \mathbb{R}\Big[\underbrace{(e^a)}_{\mathrm{deg}=(1,\mathrm{even})}{}_{a\in\{0,\ldots,p\}} \;,\; \underbrace{(\psi^\alpha)}_{\mathrm{deg}=(1,\mathrm{odd})}{}_{\alpha\in\{1,\ldots,N\}} \Big] \Big/ \left(\begin{array}{l} de^a = \overline{\psi}\Gamma^a\psi \\ d\psi^\alpha = 0 \end{array} \right). \tag{6}$$

Here

$$\overline{(-)}\Gamma(-) \;:\; \mathbf{N}\otimes\mathbf{N} \longrightarrow \mathbb{R}^{p,1}$$

denotes the bilinear spinor-to-vector pairing that is canonically associated with ever real spin representation.

The operation that takes a finite-dimensional super Lie algebra $\mathfrak{g}$ to its Chevalley-Eilenberg algebra $\mathrm{CE}(\mathfrak{g})$ turns out to be a fully-faithful embedding into the opposite of differential $(\mathbb{Z}, \mathbb{Z}/2)$-bigraded commutative algebras, hence super DGC-algebras. [3] This means that a homomorphism of super Lie algebras is equivalently a super DGC (differential graded commutative) algebra homomorphism of their CE-algebras in the other direction

$$\frac{\mathfrak{g}_1 \xrightarrow{\qquad\quad} \mathfrak{g}_2}{\mathrm{CE}(\mathfrak{g}_1) \xleftarrow{\qquad\quad} \mathrm{CE}(\mathfrak{g}_2)}$$

This makes it evident that there is a generalization of finite dimensional super Lie algebras to *super L_∞-algebras* $\mathfrak{g}$ of finite type which may be *defined* to be the formal duals of super dgc-algebras $\mathrm{CE}(\mathfrak{g})$ whose underlying graded-commutative algebra is *free*, i.e., is a super-graded Grassmann-algebra [SSS09, Def. 13]. [4]

Remark 2.2 (Differing terminology for super L_∞-algebras). The history of the concept of (super-)L_∞-algebras is a bit interwined, which tends to hide the great unity of the subject behind the different terminology of disjoint schools. Traditionally the concept of L_∞-algebras is attributed to Stasheff (see [LS93]), who had introduced A_∞-algebras much earlier. But, in fact, Stasheff indicates that he got the concept from Zwiebach (see [Sta16, slide 17]), who had discovered infinite-dimensional bosonic L_∞-algebra in closed string field theory in 1989. However, the *evident* linear dualization [SSS09, Def. 13] allows us to interpret it as arising a decade

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[3] This may be indicated as $\mathrm{CE} \;:\; s\mathrm{LieAlg}_{\mathbb{R}}^{\mathrm{fin}} \hookrightarrow \mathrm{sdgcAlg}_{\mathbb{R}}^{\mathrm{op}}$.

[4] We write this as $\mathrm{CE} \;:\; sL_\infty\mathrm{Alg}_{\mathbb{R}}^{\mathrm{fin}} \hookrightarrow \mathrm{sdgcAlg}_{\mathbb{R}}^{\mathrm{op}}$.

earlier in the supergravity literature with [vN82, D'AF82], where the CE-algebras of finite-type super L_∞-algebras are referred to as "FDA"s. This somewhat non-standard terminology may be one cause that the ubiquity of super L_∞-algebra theory in supergravity and superstring theory remains under-appreciated, even with the recent renewed interest in L_∞-algebras, for instance in [HZ17].

A first curious fact about (super-)L_∞-algebras is that even if one starts out being interested just in (super-)Lie algebras, the concept of (super-)L_∞-algebras serves to provide classifying "spaces" for (super)Lie algebra cohomology. This simple but powerful change of perspective is paramount for much of our discussion. In the following the notation dd_{n+1} is meant to indicate the generalization of the Dixmier-Douady class from degree 3 to degree $n+1$.

Definition 2.3 (Super L_∞-cocycles). For $n \in \mathbb{N}$, write $b^n\mathfrak{u}_1 \in sL_\infty\mathrm{Alg}_{\mathbb{R}}^{\mathrm{fin}}$ for the super L_∞-algebra dually given by

$$\mathrm{CE}(b^n\mathfrak{u}_1) := \mathbb{R}[\ \underbrace{\mathrm{dd}_{n+1}}_{\mathrm{deg}=(n+1,\mathrm{even})}\ ]/(d(\mathrm{dd}_{n+1}) = 0)\,.$$

Hence for $\mathfrak{g} \in sL_\infty\mathrm{Alg}_{\mathbb{R}}^{\mathrm{fin}}$ any super L_∞-algebra, we have that morphisms from $\mathfrak{g}$ to $b^n\mathfrak{u}_1$ are in natural bijection to closed elements of degree $n+1$ in the Chevalley-Eilenberg algebra of $\mathfrak{g}$

$$\left\{\mathfrak{g} \xrightarrow{\mu} b^n\mathfrak{u}_1\right\} \simeq \left\{\mu \in \mathrm{CE}^{n+1}(\mathfrak{g}) \,|\, d_{\mathfrak{g}}\mu = 0\right\}.$$

Such a μ is a *super L_∞-cocycle* of degree $n+1$ on $\mathfrak{g}$.

We would like to account for the occurrence of the fields in particular degrees with a specific spacing. We interpret this as having fields in a certain rational *periodic* cohomology theory, that we collectively call $K(t)$, where t is the periodicity parameter. This includes and generalizes the set-up of rational K-theory in [FSS17], where t is the usual Bott periodicity parameter. One can account for degrees by taking the suspension, i.e. $K^n = \Sigma^n K$.

Definition 2.4 (Periodic super L_∞-cohomology). For $t, n \in \mathbb{N}$ with $t \geq 1$, let $\mathfrak{l}(\Sigma^n K(t))$ be the super L_∞-algebra defined by

$$\mathrm{CE}(\mathfrak{l}(\Sigma^n K(t))) := \mathbb{R}\big[\big(\ \underbrace{\omega_{2kt+n}}_{\mathrm{deg}=(2kt+n,\mathrm{even})}\ , k \in \mathbb{Z}\big)\big] / \left(d\omega_{(2kt+n)} = 0\right).$$

Hence for $\mathfrak{g} \in \mathrm{sL}_\infty\mathrm{Alg}$ any super L_∞-algebra, a morphism

$$\mathfrak{g} \xrightarrow{\omega_{2t\bullet+n}} \mathfrak{l}(\Sigma^n K(t))$$

is equivalently a sequence of super L_∞-cocycles on $\mathfrak{g}$, according to Def. 2.3 of degrees n mod $2t$. We write

$$H^{\bullet \bmod 2t}(\mathfrak{g}/K) \tag{7}$$

for the corresponding periodic cohomology groups.

2.2. Equivariant super L_∞-cohomology

The setting we have will involve an action of a group. We now describe the proper way to account for that in our framework.

Definition 2.5 (Quotient of super L_∞-algebra by group action). For $\mathfrak{g}$ a super L_∞-algebra, an *action* of a group K on $\mathfrak{g}$ is a linear group action on the underlying graded vector space which preserves the bi-grading

$$\rho \;:\; K \times \mathfrak{g}_\bullet \longrightarrow \mathfrak{g}_\bullet \,,$$

such that its induced dual action

$$(\rho(-))^* \;:\; K \times \wedge^\bullet(\mathfrak{g}^*) \longrightarrow \wedge^\bullet(\mathfrak{g}^*)$$

is compatible with the Chevalley-Eilenberg differential d_{CE}, in that for all $k \in K$ we have

$$d_{\mathrm{CE}} \circ \rho(k)^* = \rho(k)^* \circ d_{\mathrm{CE}} \,.$$

This means that the subspace of K-invariant elements in the CE-algebra is a sub-dgc-algebra, to be denoted

$$\mathrm{CE}(\mathfrak{g})^K \hookrightarrow \mathrm{CE}(\mathfrak{g}). \tag{8}$$

We may think of this as the CE-algebra of the quotient $\mathfrak{g}/K$, which is thereby defined. Accordingly, given two super L_∞-algebras $\mathfrak{g}_1$, $\mathfrak{g}_2$ equipped with actions by groups K_1 and K_2, respectively, then we say that a homomorphism

$$\mathfrak{g}_1/K_1 \xrightarrow{\phi} \mathfrak{g}_2/K_2$$

between them is equivalently a dgc-algebra homomorphism the other way around, between their invariant CE-algebras (8):

$$\mathrm{CE}\left(\mathfrak{g}_1\right)^{K_1} \xleftarrow{\phi^*} \mathrm{CE}\left(\mathfrak{g}_2\right)^{K_2} .$$

Example 2.6 (Invariant super L_∞-cocycle). If $K = 1$ is the trivial group, then every super L_∞-algebra $\mathfrak{g}$ carries a unique action by that group and is canonically identified with the quotient $\mathfrak{g}/1$. Under this identification, if $\mathfrak{g}$ now is equipped with a general group action, then a homomorphism of the form

$$\mu/K \ : \ \mathfrak{g}/K \longrightarrow b^n\mathfrak{u}_1$$

is equivalently a super L_∞-cocycle according to Def. 2.3 which in addition is K-invariant, namely a plain homomorphism μ such that

$$\rho(k)^*(\mu) = \mu \qquad \begin{tikzcd} \mathfrak{g} \arrow[rr, "\rho(k)"] \arrow[dr, "\mu"'] & & \mathfrak{g} \arrow[dl, "\mu"] \\ & b^n\mathfrak{u}_1 & \end{tikzcd}$$

for all $k \in K$. Notice that, more generally, one may consider super L_∞-cocycles which are not necessarily K-invariant, but which are *K-equivariant.* This means first of all that the cocycle is K invariant only up to specified homotopies

$$\eta_k \ : \ \rho(k)^*(\mu) \Rightarrow \mu \qquad \begin{tikzcd} \mathfrak{g} \arrow[rr, "\rho(k)"] \arrow[dr, "\mu"'] & {} \arrow[d, Rightarrow, shorten=2mm, "\eta_k"] & \mathfrak{g} \arrow[dl, "\mu"] \\ & b^n\mathfrak{u}_1 & \end{tikzcd}$$

such that, moreover, these homotopies are compatible up to specified higher homotopies

$$\eta_{k_1} \cdot \rho(k_1)^*\eta_{k_2} \Longrightarrow \eta_{k_1 k_2}$$

and so on. In a broader context of higher supergeometry one may sum this up by saying all this data is equivalently a homomorphism out of the *homotopy quotient* of $\mathfrak{g}$ by K, denoted

$$(\mu, \eta, \dots) \ : \ \mathfrak{g}//K \longrightarrow b^n\mathfrak{u}_1 \, . \tag{9}$$

Definition 2.7 (K-invariant super L_∞-cohomology). Given a super L_∞-algebra $\mathfrak{g}$ equipped with an action by a group K (Def. 2.5), then its *K-invariant super L_∞-cohomology*

$$H^\bullet(\mathfrak{g}/K) := H^\bullet\left(\mathrm{CE}(\mathfrak{g})^K\right)$$

is the $(\mathbb{Z} \times (\mathbb{Z}/2))$-bigraded cochain cohomology groups of the K-invariant subcomplex (8) of its Chevalley-Eilenberg algebra.

Remark 2.8 (Different notions of equivariant cohomology). (i) One may also consider the group $H^\bullet(\mathfrak{g}//K)$ of equivalence classes of equivariant cocycles (9) as well as the subgroup $(H^\bullet(\mathfrak{g}))^K \hookrightarrow H^\bullet(\mathfrak{g}) := H^\bullet(\mathrm{CE}(\mathfrak{g}))$ of those cohomology classes in the full Chevalley-Eilenberg complex which are invariant under K. There are canonical comparison maps to these from the group of Def. 2.7

$$H^\bullet(\mathfrak{g}/K) \longrightarrow H^\bullet(\mathfrak{g}//K) \longrightarrow H^\bullet(\mathfrak{g})^K \,, \tag{10}$$

where the first one regards an invariant cocycle as an equivariant cocycle with trivial equivariance data, and the second forgets the choice of equivariance data.

(ii) For (super-)Lie algebras (i.e., (super-)Lie 1-algebras) the study of the composite comparison map in (10) is the topic of the Hochschild-Serre spectral sequence, which may be used to extract sufficient conditions for the total comparison map to be an isomorphism. One such sufficient condition is that K is a compact topological group (which is however not the case for the Lorentzian spin groups $K = \mathrm{Spin}(p,1)$ of interest below.)

(iii) But notice that from the point of view of equivariant homotopy theory the group $H^\bullet(\mathfrak{g})^K$ has no intrinsic meaning in itself, since its elements are just "in-coherently equivariant" cocycles.

(iv) In contrast, the group $H^\bullet(\mathfrak{g}/K)$ does have intrinsic meaning in equivariant homotopy theory, despite superficial appearance, namely in the context of what is called *Bredon equivariant homotopy theory* [Rez14, Section 5.1]. This is the group we will be considering here.

2.3. Twisted super L_∞-cohomology

Our setting will also involve twists, so twisted versions of the above constructions are needed. The main statements below in Theorem 3.17, Theorem 3.23

and Cor. 3.18 (with various examples in Section 4) establish isomorphisms in *twisted* invariant super L_∞-cohomology. On Chevalley-Eilenberg algebras this concept of twisted cohomology is straightforward, made explicit by Def. 2.9 below.

A key example of twisted super L_∞-cocycles are the super-WZW-terms for the F1/D*p*-branes on type II super-Minkowski spacetime [FSS16a], [FSS16b], recalled below in Section 4.1. These may be extracted, via Prop. 2.12 below, from untwisted cocycles on higher central extensions (Prop. 2.10 below) of super-Minkowski spacetimes, as found originally in [CAIB00, Sak00]. This transformation of Prop. 2.12 is an example of a general equivalence [NSS12, Theorem 4.39] between twisted cohomology and non-twisted but higher equivariant cohomology on the extension classified by the twist, we make this homotopy-theoretic perspective explicit in Prop. 2.18 below.

Definition 2.9 (Twisted super L_∞-algebra cohomology). Let $\mathfrak{g}$ be a super L_∞-algebra equipped with the action of a group K (Def. 2.5), and let $\mathfrak{g}/K \xrightarrow{\mu} b^{2t}\mathfrak{u}_1$, i.e., $\mu \in \mathrm{CE}(\mathfrak{g})^K$ be a K-invariant cocycle (Example 2.6). Then the Chevalley-Eilenberg differential $d_{\mathfrak{g}}$ plus the wedge product with μ defines a differential of degree 1 mod $2t$

$$d_{\mathfrak{g}} + \mu\wedge \,:\, \bigoplus_{k\in\mathbb{Z}} \mathrm{CE}^{2kt+\bullet}(\mathfrak{g})^K \longrightarrow \bigoplus_{k\in\mathbb{Z}} \mathrm{CE}^{2kt+\bullet+1}(\mathfrak{g})^K\,.$$

The cochain cohomology of this differential is the *K-invariant μ-twisted super L_∞-cohomology* of $\mathfrak{g}$

$$H^{n+(\mu)}(\mathfrak{g}/K) := \frac{\ker\Big(\bigoplus\limits_{k\in\mathbb{Z}} \mathrm{CE}^{2kt+n}(\mathfrak{g})^K \xrightarrow{d_{\mathfrak{g}}+\mu\wedge} \bigoplus\limits_{k\in\mathbb{Z}} \mathrm{CE}^{2kt+n+1}(\mathfrak{g})^K\Big)}{\mathrm{im}\Big(\bigoplus\limits_{k\in\mathbb{Z}} \mathrm{CE}^{2kt+n-1}(\mathfrak{g})^K \xrightarrow{d_{\mathfrak{g}}+\mu\wedge} \bigoplus\limits_{k\in\mathbb{Z}} \mathrm{CE}^{2kt+n}(\mathfrak{g})^K\Big)}\,.$$

The concept of twisted super L_∞-cohomology (Def. 2.9) is closely related to the non-twisted cohomology of higher central extensions:

Proposition 2.10 (Homotopy fiber functor [FSS15c, Theorem 3.8], based on [FRS13, Theorem 3.1.13]). *Let $\mathfrak{g} \in sL_\infty\mathrm{Alg}$ be a super L_∞-algebra and let $\mu \,:\, \mathfrak{g} \longrightarrow b^n\mathfrak{u}_1$ be an $n+1$-cocycle on it. Then*

(i) *A model for its* homotopy fiber

(11)
$$\begin{tikzcd} \hat{\mathfrak{g}} \arrow[d, "\mathrm{hofib}(\mu)"] \\ \mathfrak{g} \end{tikzcd}$$

is the super L_∞-algebra dually given by adjoining to the CE-algebra of $\mathfrak{g}$ a generator b of degree n which trivializes the cocycle:

$$\mathrm{CE}(\hat{\mathfrak{g}}) := \mathrm{CE}(\mathfrak{g})[b]/(db = \mu)\,.$$

(ii) *This construction clearly extends to a functor*

$$\mathrm{hofib}\ :\ sL_\infty\mathrm{Alg}/b^n\mathfrak{u}_1 \longrightarrow sL_\infty\mathrm{Alg}$$

from super L_∞-algebras over $b^n\mathfrak{u}_1$ to plain super L_∞-algebras.

(iii) *If $\mathfrak{g}$ is equipped with an action by a group K (Def. 8) and if the cocycle is K-invariant, $\mu \in \mathrm{CE}(\mathfrak{g})^K$ (8), then $\widehat{\mathfrak{g}}$ inherits a K-action, such that the projection (11) respects the K-actions.*

We also say that $\hat{\mathfrak{g}}$ in Def. 2.10 is the higher central extension of $\mathfrak{g}$ classified by μ.

Remark 2.11. The consideration of higher central extensions of supersymmetry super Lie algebras may be identified [FSS15c], under the translation provided by Prop. 2.10, as the core tool for supergravity and superstring theory that was established in [D'AF82, CDF91], there referred to as the "Free Differential Algebra" or "FDA" approach.

The following proposition compares the concepts of twisted super L_∞-cohomology (Prop. 2.9) with the non-twisted cohomology of higher central extensions of super L_∞-algebras (Prop. 2.10) at the purely algebraic level.

Proposition 2.12 (Twisted cohomology maps into the periodic cohomology of the higher central extension). *Let $\mathfrak{g}$ be a super L_∞-algebra equipped with an action by a group K (Def. 2.5) and let $\mu \in \mathrm{CE}(\mathfrak{g})^K$, $d_{\mathfrak{g}}\mu = 0$ be a K-invariant cocycle of degree $2t+1$ (Example 2.6) for $t \geq 1$. Then there is an injection*

$$H^{\bullet+\mu}(\mathfrak{g}/K) \longrightarrow H^\bullet(\widehat{\mathfrak{g}}/K)$$

from the K-invariant μ-twisted cohomology of $\mathfrak{g}$ (Def. 2.9) to the non-twisted periodic K-invariant cohomology (Def. 2.4) of the higher central extension $\widehat{\mathfrak{g}}$ classified by μ according to Prop. 2.10.

Proof. For any degree $n \in \mathbb{Z}$ consider the following linear map on cochains:

(12)
$$\begin{tikzcd}
\bigoplus\limits_{k\in\mathbb{Z}} \mathrm{CE}^{2kt+n}(\mathfrak{g}) \arrow[r, "\phi"] & \bigoplus\limits_{k\in\mathbb{Z}} \mathrm{CE}^{2kt+n}(\widehat{\mathfrak{g}}) \\
\{\omega_{2kt+n}\}_{k\in\mathbb{Z}} \arrow[r, mapsto, "\phi"] \arrow[dd, "d_{\mathfrak{g}+\mu\wedge}"'] & \left\{\left(e^b \wedge \sum\limits_{j\in\mathbb{Z}} \omega_{2jt+n}\right)_{2kt+n}\right\}_{k\in\mathbb{Z}} \arrow[d, "d_{\widehat{\mathfrak{g}}}"] \\
 & \left\{\left(\underbrace{d_{\widehat{\mathfrak{g}}}(e^b)}_{\mu\wedge e^b} \wedge \sum\limits_{j\in\mathbb{Z}} \omega_{2jt+n} + e^b \wedge \sum\limits_{j\in\mathbb{Z}} d_{\mathfrak{g}}\omega_{2jt+n}\right)_{2kt+n}\right\}_{k\in\mathbb{Z}} \arrow[d, equal] \\
\{d_{\mathfrak{g}}\omega_{2kt+n} + \mu\wedge\omega_{2kt+n-1}\}_{k\in\mathbb{Z}} \arrow[r, mapsto, "\phi"'] & \left\{\left(e^b \wedge \sum\limits_{j\in\mathbb{Z}} \left(d_{\mathfrak{g}}\omega_{2jt+n} + \mu\wedge\omega_{2jt+n-1}\right)\right)_{2kt+n}\right\}_{k\in\mathbb{Z}}
\end{tikzcd}$$

which intertwines the twisted CE-differential $d_{\mathfrak{g}} + \mu\wedge$ of $\mathfrak{g}$ with the plain CE-differential $d_{\widehat{\mathfrak{g}}}$ of $\widehat{\mathfrak{g}}$, as shown. It is clear that this is an injective linear map, hence an injective chain map from the $(d_{\mathfrak{g}} = +\mu\wedge)$-complex to the $d_{\widehat{\mathfrak{g}}}$-complex, whose image is closed unde the pre-image of $d_{\mathfrak{g}}$. This implies the claim. ☐

The following example show that the map of Prop. 2.12 is in general not surjective, hence that there are in general cohomology classes not in its image.

Example 2.13 (Non-twisted periodic cohomology of higher central extension is strictly larger than twisted cohomology). Under the assumptions of Prop. 2.12, consider two elements $\omega_{2t+n}, \omega_{4t+n} \in \mathrm{CE}(\mathfrak{g})$ and consider the following three equations:

$$d_{\mathfrak{g}}\omega_{4t} = -\mu\wedge\omega_{2t}, \qquad d_{\mathfrak{g}}\omega_{2t} = 0, \qquad \mu\wedge\omega_{4t} = 0.$$

The combination of the first two of these conditions is equivalent to the statement that the element

$$\omega_{4t+n} + b\wedge\omega_{2t+n} \in \mathrm{CE}(\widehat{\mathfrak{g}})$$

is a $d_{\widehat{\mathfrak{g}}}$-cocycle. On the other hand, the combination of all three conditions is equivalent to the stronger statement that also the element

$$b\wedge\omega_{4t+n} + \tfrac{1}{2} b\wedge b\wedge\omega_{2t+n} \in \mathrm{CE}(\widehat{\mathfrak{g}})$$

is a $d_{\widehat{\mathfrak{g}}}$-cocycle, which in turn is equivalent to the statement that the tuple

$$(\omega_{2t+n}, \omega_{4t+n})$$

is a $(d_{\mathfrak{g}} + \mu\wedge)$-cocycle.

We now give the homotopy-theoretic explanation of the phenomenon seen in Prop. 2.12, exhibiting it as a special case of a general statement [NSS12, Theorem 4.39] about twisted cohomology.

Definition 2.14 (Coefficients for twisted L_∞-cocycles). For $n \geq 0$ and $t \geq 1$, write $\mathfrak{l}(\Sigma^n K(t)/b^t\mathfrak{u}_1) \in sL_\infty\mathrm{Alg}_{\mathbb{R}}^{\mathrm{fin}}$ for

$$\begin{aligned}
&\mathrm{CE}\big(\mathfrak{l}(\Sigma^n K(t)/b^{2t-1}\mathfrak{u}_1)\big)\\
&:= \mathbb{R}\Big[\ \underbrace{h_{2t+1}}_{\mathrm{deg}=(2t+1,\mathrm{even})}\ ,\ \big(\underbrace{\omega_{2kt+n}}_{(2kt+n,\mathrm{even})}, k \in \mathbb{Z}\big)\Big]\Big/\left(\begin{array}{l} dh_{2t+1} = 0, \\ d\omega_{2(k+1)t+n} = -h_{2t+1} \wedge \omega_{2kt+n} \end{array}\right)
\end{aligned}$$

and write

$$\begin{tikzcd}
\mathfrak{l}\Sigma^n K(t) \arrow[r, "\rho"] & b^{2t}\mathfrak{u}_1 \\
h_{2t+1} & \mathrm{dd}_{2t+1} \arrow[l, mapsto]
\end{tikzcd} \tag{13}$$

for the canonical morphism. Then for $\mathfrak{g} \in SL_\infty\mathrm{Alg}_{\mathbb{R}}^{\mathrm{fin}}$ any super L_∞-algebra, and for

$$\mu \ :\ \mathfrak{g} \longrightarrow b^2t\mathfrak{u}_1$$

a super L_∞-cocycle on $\mathfrak{g}$ according to Def. 2.3, a *μ-twisted cocycle* $\omega_\bullet$ in degree n mod $2t$ is a homomorphism over $b^{2t}\mathfrak{u}_1$ (see [NSS12, Def. 4.21]):

$$\begin{aligned}
&\left\{\begin{tikzcd}
\mathfrak{g} \arrow[rr, "\omega_\bullet"] \arrow[dr, "\mu"'] & & \mathfrak{l}(\Sigma^p K(n)/b^{n-1}\mathfrak{u}_1) \arrow[dl, "\rho"] \\
& b^{2t}\mathfrak{u}_1 &
\end{tikzcd}\right\} \\
&\simeq \big\{\omega_{2kt+n} \in \mathrm{CE}(\mathfrak{g}) \mid d_{\mathfrak{g}}\omega_{2(k+1)t+n} + h_{2t+1} \wedge \omega_{2kt+n} = 0,\ \ k \in \mathbb{Z}\big\}\,.
\end{aligned} \tag{14}$$

The homotopy fiber of the canonical morphism (13) is the coefficient $\mathfrak{l}\Sigma^n K(t)$ for untwisted periodic cohomology from Def. 2.4

$$\begin{tikzcd}
\mathfrak{l}\Sigma^n K(t) \arrow[r, "\mathrm{hofib}(\rho)"] & \mathfrak{l}\Sigma^n K(t)/b^{2t-1} \arrow[d, "\rho"] \\
 & b^{2t}\mathfrak{u}_1 .
\end{tikzcd}$$

Definition 2.15 (CE-algebra of degree n, rational, $2t$-periodic, $(2t+1)$-twisted (generalized) cohomology). For $n \geq 0$ and $t \geq 1$, the super L_∞-algebra

$$\mathfrak{l}(\Sigma^n K(t))_{\mathrm{res}} \in sL_\infty \mathrm{Alg}^{\mathrm{fib}}_{\mathbb{R}}$$

is defined dually by

$$\mathrm{CE}\left(\mathfrak{l}(\Sigma^n K(t))_{\mathrm{res}}\right) := \mathbb{R}\Big[\underbrace{b}_{\mathrm{deg}=(2t,\mathrm{even})} , \underbrace{h_{2t+1}}_{(2t+1,\mathrm{even})} , \big(\underbrace{\omega_{2kt+n}}_{(2kt+n,\mathrm{even})}, k \in \mathbb{Z} \big) \Big] \Big/ \left(\begin{array}{l} db_{2t} = h_{2t+1}, \quad dh_{2t+1} = 0, \\ d\omega_{2(k+1)t+n} = -h_{2t+1} \wedge \omega_{2kt+n} \end{array} \right).$$

For the proof of Prop. 2.18 below we need the following fibration resolution of this homotopy fiber:

Lemma 2.16 (Fibration resolution of coefficients for untwisted periodic cohomology). *The algebra* $\mathrm{CE}\left(\mathfrak{l}(\Sigma^n K(t))_{\mathrm{res}}\right)$ *of Def. 2.15 provides a fibration resolution of the homotopy fiber inclusion*

$$\begin{tikzcd}
\mathfrak{l}(\Sigma^n K(t)) \arrow[r, hook, "\simeq"] \arrow[rr, bend left, "\mathrm{hofib}(\rho)"] & \mathfrak{l}(\Sigma^n K(t))_{\mathrm{res}} \arrow[r, two heads, "\mathrm{hofib}(\rho)_{\mathrm{res}}"] & \mathfrak{l}(\Sigma^n K(t)/b^{2t-1}\mathfrak{u}_1) \\
\omega_{2kt+n} & \omega_{2kt+n} \arrow[l, mapsto] & \omega_{2kt+n} \arrow[l, mapsto] \\
0 & h \arrow[l, mapsto] & h \arrow[l, mapsto] \\
0 & b \arrow[l, mapsto] &
\end{tikzcd} \tag{15}$$

$$\begin{tikzcd}
\mathfrak{l}(\Sigma^n K(n)) & \mathfrak{l}(\Sigma^n K(t))_{\mathrm{res}} \arrow[l, "\phi", "\simeq"'] \\
\omega_{2kt+n} \arrow[r, mapsto] & \left(e^b \wedge \sum\limits_{j \in \mathbb{Z}} \omega_{2jt+n} \right)_{2kt+n} .
\end{tikzcd}$$

Proof. That $\mathfrak{l}(\Sigma^n K(t))_{\mathrm{res}}$ provides a fibration resolution as claimed follows just as the proof of Prop. 2.10 from [FRS13, Theorem 3.1.13]. From this the other statements follow by inspection. ☐

Remark 2.17. At the bottom of (15) we indicated a homotopy inverse ϕ of the resolution, which will be of use below. In terms of this resolution, the long homotopy fiber sequence of ρ starts out as the following ordinary fibration sequence:

$$\begin{tikzcd}
b^{2t-1}\mathfrak{u}_1 \arrow[r, "b"] & \mathfrak{l}(\Sigma^n K(t))_{\mathrm{res}} \arrow[r, "\mathrm{hofib}(\rho)_{\mathrm{res}}"] & \mathfrak{l}(\Sigma^n K(t)/b^{2t-1}\mathfrak{u}_1) \arrow[r, "\rho"] & b^{2t}\mathfrak{u}_1 \\
0 & h_{2t+1} \arrow[l, mapsto] & h_{2t+1} \arrow[l, mapsto] & \mathrm{dd}_{2t+1} \arrow[l, mapsto] \\
0 & \omega_{2kt+n} \arrow[l, mapsto] & \omega_{2kt+n} \arrow[l, mapsto] & \\
b & b \arrow[l, mapsto] & &
\end{tikzcd}$$

Proposition 2.18 (Map of twisted cohomology of $\mathfrak{g}$ into non-twisted periodic cohomology of $\widehat{\mathfrak{g}}$ is a homotopy pullback). *The inclusion from Prop. 2.12 of the twisted super L_∞-cohomology on $\mathfrak{g}$ into the non-twisted periodic cohomology of $\widehat{\mathfrak{g}}$ is equivalently the image on cohomology classes of forming the homotopy pullback along the homotopy fiber inclusion of the projection morphism ρ (see the mapping* (13)*):*

$$[\mathrm{hofib}(\rho)^*] \;:\; H^{\bullet+\mu}(\mathfrak{g}/K) \longrightarrow H^{\bullet \bmod 2t}(\widehat{\mathfrak{g}}/K)\,.$$

Proof. Consider a twisted cocycle in degree n mod t

$$\omega_\bullet := \{\omega_{2kt+n}\}_{k\in\mathbb{Z}}\;.$$

Via the homomorphism (14) we may regard this equivalently as a super L_∞-homomorphism of the form

$$\begin{tikzcd}
\mathfrak{g} \arrow[rr, "\omega_\bullet"] \arrow[dr, "\mu"'] & & \mathfrak{l}\left(\Sigma^n K(t)/b^{2t-1}\mathfrak{u}_1\right). \arrow[dl, "\rho"] \\
& b^{2t}\mathfrak{u}_1 &
\end{tikzcd}$$

By forming homotopy pullbacks and using the pasting law for homotopy pullbacks this induces a homotopy-commutative diagram of the form

$$\begin{tikzcd}
b^{2t-1}\mathfrak{u}_1 \arrow[r] \arrow[d] \arrow[rr, bend left, "b"] & \widehat{\mathfrak{g}} \arrow[r, "(\mathrm{hofib}(\rho))^*(\omega_\bullet)"] \arrow[d] \arrow[dl, Rightarrow] & \mathfrak{l}(\Sigma^n K(t)) \arrow[r] \arrow[d, "\mathrm{hofib}(\rho)"] \arrow[dl, Rightarrow] & * \arrow[d] \arrow[dl, Rightarrow] \\
* \arrow[r] & \mathfrak{g} \arrow[r, "\omega_\bullet"] \arrow[rr, bend right, "\mu"'] & \mathfrak{l}(\Sigma^n K(t)/b^{2t-1}\mathfrak{u}_1) \arrow[r, "\rho"] & b^{2t}\mathfrak{u}_1
\end{tikzcd}$$

Hence this induces, in particular, a super L_∞-homomorphism

$$(\mathrm{hofib}(\rho))^*(\omega_\bullet) \ : \ \widehat{\mathfrak{g}} \longrightarrow \mathfrak{l}(\Sigma^n K(t))$$

which, via Def. 2.4, represents a periodic cohomology class on the higher central extension $\widehat{\mathfrak{g}}$:

$$[(\mathrm{hofib}(\rho))^*(\omega_\bullet)] \in H^{\bullet \bmod t}(\widehat{\mathfrak{g}})\,.$$

So it is now sufficient to show that this cohomology class is represented by $\phi(\omega_\bullet)$ according to formula (12). In order to obtain such an explicit formula for $(\mathrm{hofib}(\rho))^*(\omega_\bullet)$ we may apply the resolution of Lemma 2.16 to represent, up to weak equivalence, the middle homotopy pullback in the above diagram by an ordinary pullback, as shown in the middle of the following diagram:

$$\begin{tikzcd}
 & & \mathfrak{l}(\Sigma^n K(t)) \arrow[d, hook, "\simeq"'] & \\
b^{n-1}\mathfrak{u}_1 \arrow[r] \arrow[d] & \widehat{\mathfrak{g}} \arrow[r, "(\mathrm{hofib}(\rho)_{\mathrm{res}})^*(\omega_\bullet)"] \arrow[d] \arrow[ur, bend left, "\left(e^b \wedge \sum\limits_{j\in\mathbb{Z}} \omega_{2jt+n}\right)_{2t\bullet+n}"] & \mathfrak{l}(\Sigma^n K(t))_{\mathrm{res}} \arrow[r] \arrow[d, "\mathrm{hofib}(\rho)_{\mathrm{res}}"'] \arrow[u, bend left, "\phi"] & * \arrow[d] \\
* \arrow[r] & \mathfrak{g} \arrow[r, "\omega_\bullet"] \arrow[rr, bend right, "\mu"'] & \mathfrak{l}(\Sigma^n K(t)/b^{t-1}\mathfrak{u}_1) \arrow[r, "\rho"] & b^{2t}\mathfrak{u}_1.
\end{tikzcd}$$

Inspection reveals that the resolved homotopy pullback of $\omega_\bullet$ that is obtained thereby is simply given by

$$\begin{tikzcd}
\widehat{\mathfrak{g}} \arrow[r, "(\mathrm{hofib}(\rho)_{\mathrm{res}})^*(\omega_\bullet)"] & \mathfrak{l}(\Sigma^n K(t))_{\mathrm{res}} \\
\omega_{2kt+n} & \omega_{2kt+n} \arrow[l, mapsto] \\
\mu & h \arrow[l, mapsto] \\
b & b \arrow[l, mapsto]
\end{tikzcd}$$

In order to manifestly identify this with a periodic cocycle we may postcompose with the weak equivalence ϕ from Lemma 2.16, as shown in the above diagram. By the formula (15) for ϕ this implies the claim:

$$\begin{aligned}[\mathrm{hofib}(\rho)^*(\omega_\bullet)] &= [\phi \circ (\mathrm{hofib}(\rho)_{\mathrm{res}})^*(\omega_\bullet)] \\ &= \left[e^b \wedge \Big(\sum_{j \in \mathbb{Z}} \omega_{2jt+n} \Big)_{2t\bullet+n} \right] . \end{aligned}$$

□

2.4. Transgression elements

The following simple but crucial structure in super L_∞-homotopy theory will play a key role in Section 4.5 and Section 4.6 below:

Definition 2.19 (Transgression of super L_∞-cocycles, [SSS09, Def. 21] [FSS12, Def. 4.1.20]). Consider the homotopy fiber sequence

$$\begin{tikzcd} \mathfrak{f} \arrow[r, hook, "\iota"] & \mathfrak{g} \arrow[d, "\pi"] \\ & \mathfrak{b} \end{tikzcd}$$

for π a fibration and let $\mu_{\mathfrak{b}} \in \mathrm{CE}(\mathfrak{b})$ be a cocycle. Then a *transgression* of $\mu_{\mathfrak{b}}$ to a cocycle in the fiber $\mu_{\mathfrak{f}} \in \mathrm{CE}(\mathfrak{f})$ is an element $\mathrm{cs} \in \mathrm{CE}(\mathfrak{g})$ such that

(16)
$$\begin{tikzcd} \mu_{\mathfrak{f}} & & & \mathrm{CE}(\mathfrak{f}) \\ \mathrm{cs} \arrow[u, mapsto, "\iota^*"] \arrow[r, mapsto, "d"] & d\mathrm{cs} & & \mathrm{CE}(\mathfrak{g}) \arrow[u, "\iota^*"] \\ & \mu_{\mathfrak{b}} \arrow[u, mapsto, "\pi^*"] & & \mathrm{CE}(\mathfrak{b}). \arrow[u, "\pi^*"] \arrow[uu, bend right, "0"'] \end{tikzcd}$$

We will be interested in elements which arise in the following manner.

Example 2.20 (Transgression elements in higher central extensions). For $n \in \mathbb{N}$, $n \geq 1$, let

$$\begin{tikzcd} b^{n-1}\mathfrak{u}_1 \arrow[r, hook] & \widehat{\mathfrak{b}} \arrow[d, "\mathrm{hofib}(\mu)"'] & \\ & \mathfrak{b} \arrow[r, "\mu"] & b^n\mathfrak{u}_1 \end{tikzcd}$$

be a homotopy fiber sequence. Then the element $b \in \mathrm{CE}(\widehat{\mathfrak{g}})$ from Def. 2.10 is a transgression element in the sense of Def. 2.19.

$$\begin{tikzcd}
b & & & \mathrm{CE}(b^{n-1}\mathfrak{u}_1) \\
b \arrow[u, mapsto, "\iota^*"'] \arrow[r, mapsto, "d"] & db & & \mathrm{CE}(\widehat{\mathfrak{b}}) \arrow[u, "\iota^*"'] \\
& \mu \arrow[u, mapsto, "\pi^*"'] & & \mathrm{CE}(\mathfrak{b}) \arrow[u, "\pi^*"']
\end{tikzcd}$$

In fact this is the *universal* transgression element for μ: For every other transgression element cs of μ on any other fibration $\mathfrak{g} \twoheadrightarrow \mathfrak{b}$ as in diagram (16), there is a unique morphisms of fibrations

$$\begin{tikzcd}
\mathfrak{g} \arrow[rr, two heads, "\phi"] \arrow[dr, two heads, "\pi"'] & & \widehat{\mathfrak{b}} \arrow[dl, two heads, "\mathrm{hofib}(\mu)"] \\
& \mathfrak{b} &
\end{tikzcd} \tag{17}$$

such that $\mathrm{cs} = \phi^*(b)$.

Of particular interest are the following transgression elements for higher cocycles existing on ordinary (i.e., not higher) super Lie algebras.

Proposition 2.21 (Left-invariant differential forms on a super Lie group represent CE-elements). *Let G be a super Lie group with super Lie algebra $\mathfrak{g}$. Then restriction to the super tangent space at the neutral element constitutes an isomorphism of differential $(\mathbb{Z} \times \mathbb{Z}/2)$-graded-commutative algebras*

$$\Omega^\bullet(G)_{\mathrm{LI}} \xrightarrow{\simeq} \mathrm{CE}(\mathfrak{g})$$

between the sub-DGCA-algebra of left-invariant differential forms inside the super de Rham algebra of the super Lie group and the Chevalley-Eilenberg algebra of the super Lie algebra.

We will leave this isomorphism notationally implicit. The application of this isomorphism in the following simple setup is what drives the exceptional generalized geometry that we discover below in Section 4.5.

Example 2.22 (Primitives from sections via transgression elements). Consider a transgression of super Lie algebra cocycles (Def. 2.19)

$$\begin{tikzcd}
\mathfrak{g} \arrow[rr, "\phi"] \arrow[dr, "\pi"', two heads] & & \widehat{\mathfrak{b}} \arrow[dl, "\mathrm{hofib}(\mu)", two heads] \\
& \mathfrak{b} &
\end{tikzcd}$$

for $\mathfrak{g}$ and $\mathfrak{b}$ both super Lie 1-algebras. We are interested in studying the corresponding fibration at the level of the Lie groups. Consider a Lie integration to left-invariant super-differential forms on corresponding super Lie groups according to Prop. 2.21:

$$\begin{tikzcd}
F \arrow[d, hook, "\iota"'] & d\iota^* b = 0 \\
G \arrow[d, "\pi"'] & b \in \Omega^\bullet(G)_{\mathrm{LI}}, \qquad db = \pi^*(\mu) \\
B & \mu \in \Omega^\bullet(B)_{\mathrm{LI}}.
\end{tikzcd}$$

Then every smooth section σ of π, regarded as a fibration of super-manifolds $G \underset{\pi}{\overset{\sigma}{\leftrightarrows}} B$ induces a primitive $\sigma^*(b) \in \Omega^\bullet(B)$ of μ:

$$d\sigma^*(b) = \sigma^*(db) = \underbrace{\sigma^*\pi^*}_{\mathrm{id}}\mu = \mu\,.$$

Of course $\sigma^*(b)$ here is necessarily *not* left-invariant if $[\mu] \neq 0 \in H^\bullet(\mathfrak{g})$.

3. Higher T-duality

In this section we establish the higher and super-geometric generalization (Theorem 3.17 and Cor. 3.18 below) of toroidal “topological T-duality” as originally proposed in [BHM04] and as derived from analysis of the super-WZW terms of type II F1/Dp-branes in [FSS16b].

After considering some basics of rational higher torus fibrations in Section 3.1, we introduce the concept of *higher T-duality correspondences* between twisting cocycles on such higher torus fibrations in Section 3.2 and we prove, in Section 3.3, that these induce isomorphisms in the corresponding twisted cohomology groups. These are the higher T-duality isomorphisms as such, generalizing Hori’s formula for the T-duality isomorphisms on Ramond-Ramond (RR) charges twisted by F1-brane charges (recalled in Section 4.1 below) to higher twisting degree, notably by M5-brane charges (Section 4.3 below).

Finally in Section 3.4 we consider a phenomenon in higher T-duality that has no analog in ordinary T-duality, namely its passage to "decomposed" higher form fields (Theorem 3.23 below); the realization of this effect in M-brane theory is discussed in Section 4.5.

3.1. Higher torus fibrations

Where a torus is, topologically, a Cartesian product of circles $S^1 \simeq U(1)$, by a "higher torus" we shall mean here a Cartesian product of shifted circles $B^n U(1)$, i.e. an Eilenberg-MacLane space $K(\mathbb{Z}, n+1)$. Rationally, we think of this via the corresponding L_∞-algebras $b^n \mathfrak{u}_1$. In every odd degree these are indistinguishable, up to rational weak equivalence, from the corresponding odd-dimensional spheres (see the discussion at the beginning of Section 2). As such the rational *spherical* T-duality for M-branes discussed below in Sections 4.3, 4.7 and 4.5 involves higher torus fibrations in the following sense.

Beyond its plain homotopy type, the Riemannian structure on a flat torus is equivalently encoded in the corresponding universal cup product on the tuples of universal first Chern classes that classify torus-principal bundles. This cohomological incarnation of the Riemannian structure on a torus immediately generalizes to higher tori, in this sense, in terms of cup products on tuples of universal higher Dixmier-Douady classes. This is what the following definitions formalizes.

Definition 3.1 (Higher tori and cup products). For $n, k \in \mathbb{N}$ two natural numbers, consider the L_∞-algebra

$$b^n(\mathfrak{u}_1)^k \in sL_\infty$$

whose Chevalley-Eilenberg algebra has k generators in degree $n+1$ and vanishing differential:

$$\mathrm{CE}(b^n(\mathfrak{u}_1)^k) = \Big(\mathbb{R}[\underbrace{\mathrm{dd}_n^{(1)}}_{\mathrm{deg}=n+1}, \ldots, \underbrace{\mathrm{dd}_n^{(k)}}_{\mathrm{deg}=n+1}],\, d = 0\Big).$$

For

$$\langle -, - \rangle \;:\; \mathbb{R}^k \otimes \mathbb{R}^k \longrightarrow \mathbb{R}$$

a non-degenerate bilinear pairing, symmetric if $n+1$ is even, skew-symmetric otherwise, on the vector space spanned by these generators, we say that the corresponding *universal cup product* is the morphism

$$\begin{tikzcd} b^n(\mathfrak{u}_1)^k \times b^n(\mathfrak{u}_1)^k \arrow[r, "\langle(-)\cup(-)\rangle"] & b^{2n+1}\mathfrak{u}_1 \\ \left\langle \mathrm{pr}_1^*(\overrightarrow{\mathrm{dd}}_n) \wedge \mathrm{pr}_2^*(\overrightarrow{\mathrm{dd}}_n) \right\rangle & \mathrm{dd}_{2n+1} \arrow[l, maps to] \end{tikzcd} \tag{18}$$

Definition 3.2 (Higher torus fibrations). Consider the higher cup product pairing (18) (Def. 3.1). Then for $\mathfrak{g} \in sL_\infty\mathrm{Alg}$ a super L_∞-algebra and for

$$\vec{\mu} \;:\; \mathfrak{g} \longrightarrow b^n(\mathfrak{u}_1)^k$$

a k-tuple of $(n+1)$-cocycles, the corresponding *higher torus fibration* is the homotopy fiber of μ

$$\begin{tikzcd} \hat{\mathfrak{g}} \arrow[d, "\pi:=\mathrm{hofib}(\vec{\mu})"] \\ \mathfrak{g} \end{tikzcd} \tag{19}$$

with Chevalley-Eilenberg algebra given via Prop. 2.10 as

$$\mathrm{CE}(\hat{\mathfrak{g}}) = \mathrm{CE}(\mathfrak{g})[\vec{b}](d\vec{b} = \vec{\mu}).$$

This factors of course as a sequence of plain higher central extensions. In particular, we have the following.

Definition 3.3 (Fiber integration). Let

$$\begin{tikzcd} \hat{\mathfrak{g}} \arrow[d, "\pi:=\mathrm{hofib}(\mu)"] \\ \mathfrak{g} \end{tikzcd}$$

be a higher central extension according to Prop. 2.10, classified by a cocycle of even degree $\mu\colon \mathfrak{g} \to b^{2n-1}\mathbb{R}$. Let

$$\int : \mathrm{CE}(b^{2n-2}\mathbb{R}) = (\mathbb{R}[b], db = 0) \longrightarrow \mathbb{R}[\deg b]$$

be the morphism of cochain complexes defined by $\int b = 1$, where b has degree $2n-1$. Then the morphism "$\int$" extends to a morphism of graded vector spaces, called the *fiber integration morphism*,

$$\pi_* \,:\, \mathrm{CE}(\hat{\mathfrak{g}}) \longrightarrow \mathrm{CE}(\mathfrak{g})[\deg b]$$

as the composition

$$\mathrm{CE}(\hat{\mathfrak{g}}) \cong \mathrm{CE}(\mathfrak{g}) \otimes \mathrm{CE}(b^{2n-2}\mathbb{R}) \xrightarrow{\mathrm{id}\otimes\int} \mathrm{CE}(\mathfrak{g})[\deg b].$$

Remark 3.4 (Wrapping and non-wrapping modes). As $\mathrm{CE}(\hat{\mathfrak{g}}) = \mathrm{CE}(\mathfrak{g})[b]/(db = \mu)$, any cochain ω on $\hat{\mathfrak{g}}$ can be uniquely written as

$$\omega = \omega_{\mathrm{nw}} + b \wedge \omega_{\mathrm{w}} \tag{20}$$

for suitable "non-wrapping" and "wrapping" coefficients $\omega_{\mathrm{nw}}, \omega_{\mathrm{w}} \in \mathrm{CE}(\mathfrak{g})$. Under fiber integration

$$\pi_* \,:\, \omega \longmapsto (-1)^{\deg(b)}\omega_{\mathrm{w}}\,, \tag{21}$$

so that π_* picks the coefficient of the "wrapping component", up to a sign.

Remark 3.5. The fiber integration morphism does indeed respect the differentials and so it is a $\deg(b)$-graded morphism of chain complexes $\mathrm{CE}(\hat{\mathfrak{g}}) \to \mathrm{CE}(\mathfrak{g})$. Namely, one has

$$\begin{aligned} \pi_*(d\omega) &= \pi_*\big(d\omega_{\mathrm{nw}} + \underbrace{db}_{\mu} \wedge \omega_{\mathrm{nw}} + (-1)^{\deg(b)} b \wedge d\omega_{\mathrm{nw}}\big) \\ &= (-1)^{\deg(b)} d\omega_{\mathrm{nw}} \\ &= (-1)^{\deg(b)} d\pi_*(\omega). \end{aligned} \tag{22}$$

We may think of an ordinary k-torus T^k as being an S^1-fibration over the $(k-1)$-torus in k different ways

$$\begin{tikzcd} S^1 \arrow[r, hook] & T^k \arrow[d, "\pi_j"] \\ & T^{k-1} \end{tikzcd}$$

and of course the same remains true for higher tori, according to Def. 3.1. Accordingly from every higher k-torus there are k *partial fiber integration* maps that fiber integrate, in the sense of Def. 3.6, over one of the factors, Def. 3.6 below. This is needed in the key condition (29) on higher T-duality correspondences below in Def. 3.10.

Definition 3.6 (Partial fiber integration). For any choice $j \in \{1, \ldots, k\}$, π factors as

$$\begin{tikzcd}
\hat{\mathfrak{g}} \arrow[d, "\pi^{(j)} := \mathrm{hofib}(\mu^{(j)})"] \\
{} \arrow[d, "\pi^{(1,\ldots,j-1,j+1,\ldots,k)} := \mathrm{hofib}(\mu^{(1)},\ldots,\mu^{(j-1)},\mu^{(j+1)},\ldots,\mu^{(k)})"] \\
\mathfrak{g}
\end{tikzcd}$$

so that there are k different intermediate fiber integration maps (Def. 3.6), defined by (see Remark 3.4)

$$\pi_*^{(j)}\big(\omega_{\mathrm{nw}} + \big\langle \vec{b} \wedge \vec{\omega}_{\mathrm{w}} \big\rangle + \cdots\big) := \omega_{\mathrm{w}}^{(j)} \tag{23}$$

or

$$\vec{\pi}_*(\omega) := \vec{\omega}_{\mathrm{w}} \,,$$

for short. This is well-defined by the assumption that the pairing $\langle -, - \rangle$ is non-degenerate.

3.2. Higher T-duality correspondences

The higher topological T-duality itself, below in Section 3.3, is a non-trivial isomorphism between two different twisted cohomology groups. But, first of all, the twists on the two sides of this isomorphisms need to be T-dual themselves. This is encoded in the concept of a *higher T-duality correspondence* which we discuss now (Def. 3.10 below) being a special case of a correspondence of twisting cocycles (Def. 3.8 below.)

In application to super p-brane physics in Section 4 below, the twisting cocycles correspond to the charges of a given brane species that may end of some other branes (e.g. the fundamental string in type II or the M5-brane in 11d) and the twisted cohomology classes twisted thereby correspond to the charges of the branes it may end on (e.g. the D-branes or (possibly) the M9-brane, respectively); see [FSS15c, Section 3] for details on this homotopy-theoretic incarnation of the brane intersection laws.

Definition 3.7 (Higher Poincaré form). Let $n, k \in \mathbb{N}$, and let $b^n(\mathfrak{u}_1)^k \times b^n(\mathfrak{u}_1)^k \xrightarrow{\langle(-)\cup(-)\rangle} b^{2n+1}\mathfrak{u}_1$ be a choice of cup product (18) as in Def. 3.1.

Concepts	**Higher T-duality correspondence**	**Pull-tensor-push through correspondence**	**Higher T-duality**
	Def. 3.10	Def. 3.16	Theorem 3.17, Cor. 3.18
Examples			
String theory	*Type IIA/B string* (38, 39) $\mu_{F1}^{\mathrm{IIA/B}} = \underbrace{\mu_{F1}\vert_{8+1}}_{\text{basic}} + e^9_{A/B} \wedge \underbrace{c_2^{\mathrm{IIB/A}}}_{\text{dual extension class}}$	D-branes	Hori's formula for Buscher rules for RR-fields (40) from [FSS16b, Prop. 6.4]
M-theory	*M5-brane* (44, 43) $\widetilde{\mu}_{M5} = \underbrace{2\mu_{M5}}_{\text{basic}} + c_3 \wedge \underbrace{\mu_{M2}}_{\text{dual extension class}}$	Remark 4.18	Sections 4.3, 4.5

Moreover, let $\mathfrak{g}$ be a super-L_∞-algebra and let

$$\overrightarrow{\mathrm{dd}}^A, \overrightarrow{\mathrm{dd}}^B \ : \ \mathfrak{g} \longrightarrow b^{2n+1}(\mathfrak{u}_1)^k$$

be two cocycles, classifying two higher extensions $\hat{\mathfrak{g}}^A$ and $\hat{\mathfrak{g}}^B$, respectively, via Prop. 2.10: [5]

$$\mathrm{CE}\big(\widehat{\mathfrak{g}}^{A/B}\big) = \mathrm{CE}(\mathfrak{g})\big[\vec{b}^{A/B}\big]/\big(d\big(\vec{b}^{A/B}\big) = \overrightarrow{\mathrm{dd}}^{A/B}\big)\,. \tag{24}$$

Then on the fiber product

$$\begin{tikzcd}
& \hat{\mathfrak{g}}^A \times_{\mathfrak{g}} \hat{\mathfrak{g}}^B \arrow[dl, "p_A"'] \arrow[dr, "p_B"] & \\
\hat{\mathfrak{g}}^A \arrow[dr, "\pi^A"'] & & \hat{\mathfrak{g}}^B \arrow[dl, "\pi^B"] \\
& \mathfrak{g} &
\end{tikzcd} \tag{25}$$

whose CE-algebra is

$$\mathrm{CE}\left(\hat{\mathfrak{g}}^A \times_{\mathfrak{g}} \hat{\mathfrak{g}}^B\right) = \mathrm{CE}(\mathfrak{g})\big[\vec{b}^A, \vec{b}^B\big]/\big(d(\vec{b}^A) = \overrightarrow{\mathrm{dd}}^A, d(\vec{b}^B) = \overrightarrow{\mathrm{dd}}^B\big)$$

there is the cochain

$$\mathcal{P} := \big\langle \vec{b}^A \wedge \vec{b}^B \big\rangle \in \mathrm{CE}\big(\hat{\mathfrak{g}}^A \times_{\mathfrak{g}} \hat{\mathfrak{g}}^B\big)\,, \tag{26}$$

which we call the *higher Poincaré form* on this fiber product.

---

[5] We will use the notation $(-)^{A/B}$ to indicate the two cases A and B respectively.

Definition 3.8 (Correspondence of twisting cocycles). In the context of Def. 3.7, if

$$h^{A/B} \;:\; \hat{\mathfrak{g}}^{A/B} \longrightarrow b^{4n+2}\mathfrak{u}_1$$

are two $(4k+3)$-cocycles on the two extensions, respectively, we say that they are *in correspondence*, or that the diagram

$$\begin{tikzcd}
 & & \hat{\mathfrak{g}}^A \times_{\mathfrak{g}} \hat{\mathfrak{g}}^B \arrow[dl, "p_A"'] \arrow[dr, "p_B"] & & \\
b^{4n+2}\mathfrak{u}_1 & \hat{\mathfrak{g}}^A \arrow[l, "h^A"'] & & \hat{\mathfrak{g}}^B \arrow[r, "h^B"] & b^{4k+2}\mathfrak{u}_1
\end{tikzcd} \tag{27}$$

is a *correspondence between* the cocycles, if the Poincaré form (26) trivializes the differences of their pullbacks to the fiber product:

$$d\mathcal{P} = (p_B)^*(h^B) - (p_A)^*(h^A)\,, \tag{28}$$

Remark 3.9. The above definition, Def. 3.8, may naturally be stated more homotopy-theoretically: The algebraic condition (28) witnesses a 2-dimensional diagram in the $(\infty,1)$-category of super L_∞-algebras of the following form:

$$\begin{tikzcd}
 & \hat{\mathfrak{g}}^A \times_{\mathfrak{g}} \hat{\mathfrak{g}}^B \arrow[dl, "p_A"'] \arrow[dr, "p_B"] \arrow[dd, phantom, "\overset{\mathcal{P}}{\Longleftarrow}"] & \\
\hat{\mathfrak{g}}^A \arrow[dr, "h^A"'] & & \hat{\mathfrak{g}}^B, \arrow[dl, "h^B"] \\
 & b^{4n+2}\mathfrak{u}_1 &
\end{tikzcd}$$

where $\mathcal{P}$ is viewed as a homotopy interpolating between the two compositions.

Definition 3.10 (Higher T-duality correspondence). We call a correspondence of cocycles according to Def. 3.8

$$\begin{tikzcd}
 & & \hat{\mathfrak{g}}^A \times_{\mathfrak{g}} \hat{\mathfrak{g}}^B \arrow[dl, "p_A"'] \arrow[dr, "p_B"] & & \\
b^{4n+2}\mathfrak{u}_1 & \hat{\mathfrak{g}}^A \arrow[l, "h^A"'] \arrow[dr, "\pi^A := \mathrm{hofib}(\overrightarrow{\mathrm{dd}}^A)"'] & & \hat{\mathfrak{g}}^B \arrow[r, "h^B"] \arrow[dl, "\pi^B := \mathrm{hofib}(\overrightarrow{\mathrm{dd}}^B)"] & b^{4n+2}\mathfrak{u}_1 \\
 & & \mathfrak{g} \arrow[dl, "\overrightarrow{\mathrm{dd}}^A"] \arrow[dr, "\overrightarrow{\mathrm{dd}}^B"'] & & \\
 & b^{2n+1}(\mathfrak{u}_1)^k & & b^{2n+1}(\mathfrak{u}_1)^k &
\end{tikzcd}$$

a *higher topological T-duality correspondence* if the partial integration (Def. 3.6) of the A-cocycle along the A-fibration coincides with minus the B-cocycle on the base, and conversely:

$$\vec{\pi}_*^A(h^A) = -\overrightarrow{\mathrm{dd}}^B \qquad \text{and} \qquad \vec{\pi}_*^B(h^B) = -\overrightarrow{\mathrm{dd}}^A\,. \tag{29}$$

We denote this situation by either of the following two notations:

$$(\overrightarrow{\mathrm{dd}}^A, h^A) \xleftrightarrow{\;\mathcal{T}\;} (\overrightarrow{\mathrm{dd}}^B, h^B) \qquad \text{or} \qquad (\widehat{\mathfrak{g}}^A, h^A) \xleftrightarrow{\;\mathcal{T}\;} (\widehat{\mathfrak{g}}^B, h^B)\;.$$

Remark 3.11 (The T-duality axiom). **(i)** Notice that in relations (29), by usual abuse of notation, we are notationally suppressing the pullback of $\vec{\mathrm{dd}}^{B/A}$ to the codomain of $\vec{\pi}_*^{A/B}$ (Def. 3.6).

(ii) This condition (29) is the evident generalization of the condition for toroidal topological T-duality that has been considered before: The latter was conjecturally proposed in [BHM04, (2.1)] and argued for from actual T-duality in [BHM07, around (2.5)]. Its evident lift to integral cohomology was considered in [BRS06, (2.3)]. We had *derived* this condition from analysis of type II superstring super-WZW terms in [FSS16b, (4) and (5)].

(iii) In the language of "dg-manifolds", this condition for ordinary T-duality was considered in [LRU14, Def. 3.2]. Notice that dg-manifolds are just L_∞-algebras that vary over a base manifold, in a suitable sense: namely dg-manifolds are L_∞*-algebroids* [SSS12, Section A.1]. This way the setup in [LRU14] is closely related to our super L_∞-algebraic discussion. The key difference is the higher generalization of T-duality which we consider and, crucially, the generalization to super-dg-geometry: It is the *higher* and *fermionic* cocycles on super-L_∞-algebras which induce all the nontrivial structure in higher T-duality of super p-branes, both for ordinary T-duality of type II superstrings from [FSS16b] (recalled as Section 4.1 below), as well as in the spherical T-duality of M-branes discussed below in Section 4.

It will be convenient to have notation for the set of all possible T-duality correspondences over a given base:

Definition 3.12 (Set of higher T-duality correspondences). Given a super L_∞-algebra $\mathfrak{g}$ we write

$$\mathfrak{g} \longmapsto \mathrm{TCorr}_n(\mathfrak{g}) \tag{30}$$

for the set of all higher T-duality correspondences over $\mathfrak{g}$; by Prop. 2.10 this is a contravariant functor in $\mathfrak{g}$.

We offer the following more succinct way to *induce* the data of T-duality correspondences purely from cocycle data on $\mathfrak{g}$.

Proposition 3.13 (Classifying data for higher topological T-duality correspondences). *Given a super L_∞-algebra $\mathfrak{g}$, then the set* $\mathrm{TCorr}_n(\mathfrak{g})$ *(30) of higher T-duality correspondences above it is in natural bijection with the set of choices of cocycles* $\overrightarrow{\mathrm{dd}}^{A/B}$ *on* $\mathfrak{g}$ *equipped with a trivialization of (minus) their cup product (Def. 3.1):*

$$\mathrm{TCorr}_n(\mathfrak{g}) \simeq \Big\{ \underbrace{\overrightarrow{\mathrm{dd}}{}^{A/B}_{\mathfrak{g}}}_{\mathrm{deg}=2n},\ \underbrace{h_{\mathfrak{g}}}_{4n-1} \in \mathrm{CE}(\mathfrak{g}) \mid d\big(\overrightarrow{\mathrm{dd}}{}^{A/B}\big) = 0,\ dh_{\mathfrak{g}} = -\big\langle \overrightarrow{\mathrm{dd}}{}^{A}_{\mathfrak{g}} \wedge \overrightarrow{\mathrm{dd}}{}^{B}_{\mathfrak{g}} \big\rangle \Big\} .$$

Proof. First we observe that given the data $(\overrightarrow{\mathrm{dd}}{}^{A/B}, h^{A/B})$ underlying a T-duality correspondence, then the two conditions saying that this does indeed constitute a T-duality correspondence (Def. 3.10),

1) $d\mathcal{P} = (p_B)^*(h^B) - (p_A)^*(h^A)$;
2) $(\vec{\pi}^{A/B})_*(h^{A/B}) = -\overrightarrow{\mathrm{dd}}{}^{A/B}$,

are equivalent to the statement that $h^{A/B}$ has the following form:

$$h^{A/B} = (\pi^{A/B})^*(h_{\mathfrak{g}}) + \Big\langle \vec{b}^{A/B} \wedge (\pi^{A/B})^*\big(\overrightarrow{\mathrm{dd}}{}^{B/A}\big) \Big\rangle \quad \text{for some } h_{\mathfrak{g}} \in \mathrm{CE}(\mathfrak{g}). \tag{31}$$

To see this, observe by the definition of fiber integration, Def. 3.6, and noticing that the prefactor in (21) is -1 in the present case, that the second condition above is equivalent to

$$h^{A/B} = (\pi^{A/B})^*(q^{A/B}) + \Big\langle \vec{b}^{A/B} \wedge (\pi^{A/B})^*\big(\overrightarrow{\mathrm{dd}}{}^{B/A}\big) \Big\rangle \quad \text{for some } q^{A/B} \in \mathrm{CE}(\mathfrak{g}).$$

With this the first condition is equivalent to

$$\begin{aligned} 0 &= (p^B)^*(h^B) - (p^A)^*(h^A) - d\mathcal{P} \\ &= (p^B)^*(\pi^B)^*(q^B) - (p^A)^*(\pi^A)^*(q^A) \\ &\quad + \underbrace{\Big((p^B)^* \Big\langle \vec{b}^B \wedge (\pi^B)^*(\overrightarrow{\mathrm{dd}}{}^A) \Big\rangle - (p^A)^* \Big\langle \vec{b}^A \wedge (\pi^A)^*(\overrightarrow{\mathrm{dd}}{}^B) \Big\rangle \Big) - d\mathcal{P}}_{=0}, \end{aligned}$$

where the term over the brace vanishes by the definition of $\mathcal{P}$ (expression (26)). But since $(p^A)^*(\pi^A)^* = (p^B)^*(\pi^B)^*$, by the fiber product diagram

(25), and since the pullback operation along a fibration is injective, this is equivalent to $q^A = q^B$. Hence $h_{\mathfrak{g}} := q^{A/B}$. This proves the claim.

Now from this claim it is immediate that Def. 3.10 implies

$$\begin{aligned} 0 &= (p^{A/B})^*(dh^{A/B})) \\ &= (p^{A/B})^* \Big(d\Big((\pi^{A/B})^*(h_{\mathfrak{g}}) + \Big\langle \vec{b}^{A/B} \wedge (\pi^{A/B})^* \big(\overrightarrow{\mathrm{dd}}^{B/A}\big) \Big\rangle \Big) \Big) \\ &= (p^{A/B})^*(\pi^{A/B})^* \Big(dh_{\mathfrak{g}} + \langle \overrightarrow{\mathrm{dd}}^A \wedge \overrightarrow{\mathrm{dd}}^B \rangle \Big) . \end{aligned}$$

Consequently, we have the equation $dh_{\mathfrak{g}} = -\mathrm{dd}^A \wedge \mathrm{dd}^B$, because the pullback operation along higher central extensions is injective, by Prop. 2.10. Conversely, given this equation, then the same computation shows that setting

$$h^{A/B} := (\pi^{A/B})^*(h_{\mathfrak{g}}) + b^{A/B} \wedge (\pi^{A/B})^*(\mathrm{dd}^{B/A})$$

defines cocycles. Then, as in the proof of the claim above, it follows that these satisfy the two conditions above. □

This immediately implies that there is a universal L_∞-algebra that serves as the classifying space for higher topological T-duality correspondences, provided we give it the right context first.

Definition 3.14 (Super L_∞ T-fold algebra). For $n, k \in \mathbb{N}$, $n \geq 1$ write

$$\mathfrak{btfold}_n \in sL_\infty\mathrm{Alg}$$

for the super L_∞-algebra dually defined as having the following Chevalley-Eilenberg algebra:

$$\begin{aligned} \mathrm{CE}(\mathfrak{btfold}_n) := \mathbb{R}\Big[&\underbrace{\overrightarrow{\mathrm{dd}}^A}_{\mathrm{deg}=2n}, \underbrace{\overrightarrow{\mathrm{dd}}^B}_{2n}, \underbrace{h}_{4n-1} \Big] \\ &\Big/ \Big(d\big(\overrightarrow{\mathrm{dd}}^{A/B}\big) = 0,\ dh = -\big\langle \overrightarrow{\mathrm{dd}}^A \wedge \overrightarrow{\mathrm{dd}}^B \big\rangle \Big). \end{aligned} \tag{32}$$

Proposition 3.15 (Higher T-duality L_∞-algebra). *For $\mathfrak{g} \in sL_\infty\mathrm{Alg}$ any super L_∞-algebra there is a natural bijection*

$$\mathrm{Hom}(\mathfrak{g}, \mathfrak{btfold}_n) \simeq \mathrm{TCorr}_n(\mathfrak{g})$$

between the set of super L_∞-homomorphisms of the form $\mathfrak{g} \to \mathfrak{btfold}_n$ and the set of higher T-duality correspondences over $\mathfrak{g}$ (Def. 3.10).

Proof. This is the composite of natural bijections

$$\begin{aligned}\mathrm{Hom}(\mathfrak{g}, b\mathfrak{tfold}_n) \simeq \Big\{ & \underbrace{h_{\mathfrak{g}}}_{\mathrm{deg}=4n-1}, \underbrace{\overrightarrow{\mathrm{dd}}{}_{\mathfrak{g}}^{A/B}}_{2n} \in \mathrm{CE}(\mathfrak{g}) \mid \\ & d\big(\overrightarrow{\mathrm{dd}}{}^{A/B}\big) = 0, dh_{\mathfrak{g}} = -\Big\langle \overrightarrow{\mathrm{dd}}{}_{\mathfrak{g}}^{A} \wedge \overrightarrow{\mathrm{dd}}{}_{\mathfrak{g}}^{B} \Big\rangle \Big\} \\ \simeq {} & \mathrm{TCorr}_n(\mathfrak{g}) \,,\end{aligned}$$

where the first one is given by expression (32) and the second is given by Prop. 3.13. □

3.3. Higher T-duality transformations

Finally we discuss how every correspondence of cocycles (Def. 3.8) induces a pull-push transformation on cochains (Def. 3.16 below), which is an isomorphism on the corresponding higher twisted cohomology groups (Theorem 3.17 below). Applied to the case of higher T-duality correspondences (Def. 3.10) this yields the genuine higher topological T-duality (Cor. 3.18 below).

Definition 3.16 (Pull-push through correspondences). Consider a correspondence (Def. 3.8)

$$\begin{tikzcd} & & \hat{\mathfrak{g}}^A \times_{\mathfrak{g}} \hat{\mathfrak{g}}^b \arrow[dl, "p_A"'] \arrow[dr, "p_B"] & & \\ b^{4n+2}\mathbb{R} & \hat{\mathfrak{g}}^A \arrow[l, "h^A"'] & & \hat{\mathfrak{g}}^B \arrow[r, "h^B"] & b^{4n+2}\mathbb{R}. \end{tikzcd}$$

Then we say that the *pull-push transform through the correspondence* is the linear map $(p_A)_* \circ e^{\mathcal{P}} \circ (p_B)^*$ from the cochains on $\hat{\mathfrak{g}}^B$ to those of $\hat{\mathfrak{g}}^A$ which is the composite of

1) pullback $(p_B)^*$ to the fiber product,
2) multiplication with the exponential $e^{\mathcal{P}} := 1 + \mathcal{P} + \frac{1}{2}\mathcal{P} \wedge \mathcal{P} + \cdots$ of the Poincaré form (26), and
3) fiber integration $(p_A)_*$ (Def. 3.6).

Theorem 3.17 (Pull-push through correspondences is isomorphism on twisted cohomology). (i) *The pull-push through correspondences of*

Def. 3.16 induces an isomorphism between the corresponding twisted cohomology groups (Def. 2.9):

$$H^{(\bullet+2n+1)+h^A}(\hat{\mathfrak{g}}^A) \xleftarrow[\simeq]{\mathcal{T}:=(p_A)_* e^{\mathcal{P}}(p_B)^*} H^{\bullet+h^B}(\hat{\mathfrak{g}}^B)\ .$$

(ii) *Moreover, if the base super L_∞-algebra $\mathfrak{g}$ is equipped with an action by a group K (Def. 2.5), and if the cocycles $\vec{\mathrm{dd}}^{A/B} \in \mathrm{CE}(\mathfrak{g})^K$ are K-invariant (8), so that also the extensions $\widehat{\mathfrak{g}}^{A/B}$ are canonically equipped with a K-actions (by Prop. 2.10) and if finally the twisting cocycles $h^{A/B} \in \mathrm{CE}(\widehat{\mathfrak{g}}^{A/B})^K$ are K-invariant, then this isomorphism restricts to an isomorphism of K-invariant twisted cohomology groups*

$$H^{(\bullet+2n+1)+h^A}(\hat{\mathfrak{g}}^A/K) \xleftarrow[\simeq]{\mathcal{T}:=(p_A)_* e^{\mathcal{P}}(p_B)^*} H^{\bullet+h^B}(\hat{\mathfrak{g}}^B/K)\ .$$

Proof. First consider the case that $k = 1$ in Def. 3.7. This means that there is a single generator b^B and a single generator b^A. Hence in this case the Poincaré form $\mathcal{P}$ given in expression (26) is a linear multiple of $b^A \wedge b^B$. Without restriction of generality we may take the linear multiple to be one. Then

$$e^{\mathcal{P}} = 1 + b^A \wedge b^B\,.$$

Now, let (see Remark 3.4)

$$\omega = \omega_{\mathrm{nw}} + b^B \wedge \omega_{\mathrm{w}} \in \mathrm{CE}(\hat{\mathfrak{g}}^B) \tag{33}$$

be any, possibly inhomogeneous, cochain on $\hat{\mathfrak{g}}^B$, where on the right we are displaying its unique coefficients $\omega_{\mathrm{nw}}, \omega_{\mathrm{w}} \in \mathrm{CE}(\mathfrak{g})$ with respect to the fibration π^B as in expression (20).

We directly compute the action of the transformation on this cochain in terms of generators:

$$\begin{aligned}(p_A)_* e^{\mathcal{P}}(p_B)^*(\omega) &= (p_A)_* e^{\mathcal{P}}(p_B)^*\big(\omega_{\mathrm{nw}} + b^B \wedge \omega_{\mathrm{w}}\big)\\ &= (\pi^B)_*\big(\underbrace{(1 + b^A \wedge b^B)}_{e^{\mathcal{P}}} \wedge (\omega_{\mathrm{nw}} + b^B \wedge \omega_{\mathrm{w}})\big)\\ &= -\omega_{\mathrm{w}} + b^A \wedge \omega_{\mathrm{nw}} \in \mathrm{CE}(\hat{\mathfrak{g}}^A).\end{aligned} \tag{34}$$

This says that the transform just swaps the wrapping/non-wrapping components of cochains, up to a sign. Hence it is manifestly a linear isomorphism on cochains.

For general k the argument is directly analogous: Generally, each cochain is expanded in coefficients of monomials $b^B_{j_1} \wedge \cdots \wedge b^B_{j_r}$, and the operation $(\pi^B)_* e^{\mathcal{P}}$ amounts to re-interpreting these as coefficients of the corresponding Hodge dual powers of the b^A's. Since every monomial in the b^B's has a unique Hodge dual monomial in the b^A's, this is still a linear isomorphism on cochains.

Therefore, to conclude it is sufficient to see that the transform operation intertwines the twisted differentials

$$(d + h^A) \circ (\pi_A)_* e^{\mathcal{P}} (\pi_B)^* = -(\pi_A)_* e^{\mathcal{P}} (\pi_B)^* \circ (d + h^B)\,.$$

We may check this as follows:

$$\begin{aligned} (d + h^A)\ (p_A)_* e^{\mathcal{P}} (p_B)^*(\omega) &= -(p_A)_* (d + h^A) e^{\mathcal{P}} (p_B)^*(\omega) \\ &= -(p_A)_* e^{\mathcal{P}} (d + \underbrace{h^A + d\mathcal{P}}_{h^B}) (p_B)^*(\omega) \\ &= -(p_A)_* e^{\mathcal{P}} (p_B)^*\ (d + h^B)(\omega). \end{aligned} \tag{35}$$

Here in the first step we used that the plain differential graded-commutes with fiber integration by (22), as does multiplication by h^A, trivially. Then under the brace we applied the defining condition (28) for a correspondence.

□

Corollary 3.18 (Higher topological T-duality in twisted cohomology). *Since every higher T-duality correspondence (Def. 3.10) is in particular a correspondence of cocycles in the sense of Def. 3.8, its induced pull-push transform (Def. 3.16) is an isomorphism in higher twisted cohomology, by Theorem 3.17. We call this the actual* higher topological T-duality *induced by the T-duality correspondence:*

$$\begin{tikzcd} H^{(\bullet+2n+1)+h^A}(\hat{\mathfrak{g}}^A, \mathbb{R}) & H^{\bullet+h^B}(\hat{\mathfrak{g}}^B, \mathbb{R}) \arrow[l, "\mathcal{T} := (p_A)_* e^{\mathcal{P}} (p_B)^*"', "\simeq"] \\ {[-\omega_{\mathrm{w}} + b^A \wedge \omega_{\mathrm{nw}})]} & {[\omega_{\mathrm{nw}} + b^B \wedge \omega_{\mathrm{w}}].} \arrow[l, mapsto] \end{tikzcd}$$

Remark 3.19 (component-wise analysis of higher T-duality). While the slick computation in (35) implies that this map on cochains indeed respects the twisted differentials, it is instructive to check this alternatively in components, in terms of the characteristic condition (29) on a higher T-duality correspondence. This will pave the way for the generalization of higher T-duality to higher T-duality of decomposed form fields in Theorem 3.23 below:

Condition (29) implies that the cocycles $h^{A/B}$ decompose as in (31). Using this and collecting coefficients of b^A and b^B, one obtains the respect for the twisted differentials under $\mathcal{T}$ as follows:

$$
\begin{aligned}
(d + h^A)\mathcal{T}(\omega) &= (d + h^A)(-\omega_{\mathrm{w}} + b^A \wedge \omega_{\mathrm{nw}}) \\
&= -\Big(- \underbrace{\big(- d\omega_{\mathrm{w}} - h_{\mathfrak{g}} \wedge \omega_{\mathrm{w}} + \mathrm{d}\mathrm{d}^A \wedge \omega_{\mathrm{nw}}\big)}_{=((d+h^B)\omega)_{\mathrm{w}}} \\
&\qquad + b^A \wedge \underbrace{\big(d\omega_{\mathrm{nw}} + h_{\mathfrak{g}} \wedge \omega_{\mathrm{nw}} + \mathrm{d}\mathrm{d}^B \wedge \omega_{\mathrm{w}}\big)}_{=((d+h^B)\omega)_{\mathrm{nw}}} \Big) \\
&= -\mathcal{T}\big((d + h^B)(\omega)\big).
\end{aligned}
\tag{36}
$$

3.4. Higher T-duality for decomposed form fields

Corollary 3.18 shows that every higher topological T-duality correspondence (Def. 3.10) induces an isomorphism in higher twisted super L_∞-cohomology (Def. 2.9), obtained there as an example of a general class of pull-push transforms through correspondences of cocycles (Theorem 3.17). But the computation in (36) shows that this T-duality isomorphism may alternatively be understood without reference to either the correspondence space or the Poincaré form on it (Def. 3.7). Instead, a brief reflection on (36) reveals that this alternative proof relies, apart from the T-duality condition (29) itself, only on the fact that in the Chevalley-Eilenberg algebras of $\widehat{\mathfrak{g}}^{A/B}$ (see (24)) every cochain has a *unique* decomposition of the form $\omega = \omega_{\mathrm{nw}} + b^{A/B} \wedge \omega_{\mathrm{w}}$ with *unique* coefficients $\omega_{\mathrm{nw}}, \omega_{\mathrm{w}}$; see expression (33).

While such a unique decomposition is of course immediate for generators $b^{A/B}$ in a free graded-commutative algebra (re-amplified as Example 3.22 below) it is not restricted to this situation. Indeed, the same happens equivalently (Prop. 3.21 below) in algebras on which the *C-cohomology* (Def. 3.20 below) of the given element vanishes. Hence in this situation higher topological T-duality generalizes; this is Theorem 3.23 below. Below in Section 4.5 we discover a curious example of this theorem in M-brane physics.

Definition 3.20 (C-cohomology). Let $\mathcal{A}$ be a graded-commutative algebra in characteristic zero, and let $C \in \mathcal{A}$ be an element in odd degree, hence multiplicatively nilpotent:

$$C^2 := C \cdot C = 0 \in \mathcal{A}\,.$$

Then the cochain cohomology of the resulting complex with differential $C \cdot (-)$ we call the *C-cohomology* $H(\mathcal{A}, C)$ of C:

$$H(\mathcal{A}, C) := \frac{\ker(C \cdot (-))}{\operatorname{im}(C \cdot (-))} .$$

For closed elements of degree-3 in a de Rham algebra of differential forms, this cohomology is called 'H-cohomology" in [Cav05, p. 19], but of course the concept as such is elementary and appears elsewhere under different names or under no special name, e.g. [Šev05, p. 1]. Since the letter "H" is alluding to the NS-NS field strength for the string, which is closed, for emphasis we use "C" to allude instead to the supergravity C-field, which is not, in general, closed. So our formulation is more general than previous ones. With the decomposition into wrapping and non-wrapping modes (Remark 3.4), we have the following.

Proposition 3.21 (Vanishing C-cohomology equivalent to unique expansions in C). *Let $\mathcal{A}$ be a graded-commutative algebra in characteristic zero, and let $C \in \mathcal{A}$ be an element in odd degree. Then the following are equivalent:*

(i) *There exists a linear subspace $\mathcal{A}_0 \hookrightarrow \mathcal{A}$ such that every element $\omega \in \mathcal{A}$ has an expansion*

$$\omega = \omega_{\mathrm{nw}} + C \cdot \omega_{\mathrm{w}}$$

for unique $\omega_{\mathrm{w}}, \omega_{\mathrm{nw}} \in \mathcal{A}_0$.

(ii) *The C-cohomology of C (Def. 3.20) vanishes.*

Proof. Having a unique expansion of the form $\omega_{\mathrm{nw}} + C \cdot \omega_{\mathrm{w}}$ with $\omega_{\mathrm{w}}, \omega_{\mathrm{nw}} \in \mathcal{A}_0$ for every ω in $\mathcal{A}$ is equivalent to saying that the linear map

$$\begin{aligned} \phi_C \colon \mathcal{A}_0 \oplus \mathcal{A}_0 &\longrightarrow \mathcal{A} \\ (\omega_{\mathrm{nw}}, \omega_{\mathrm{w}}) &\longmapsto \omega_{\mathrm{nw}} + C \cdot \omega_{\mathrm{w}} \end{aligned}$$

is an isomorphism. In one direction, assume that ϕ_C is an isomorphism, and consider an element $\omega \in \mathcal{A}$ which is a C-cocycle, i.e., such that $C \cdot \omega = 0$. By the nilpotency of C this implies $C \cdot \omega_{\mathrm{nw}} = 0$, and so $\phi_C(0, \omega_{\mathrm{nw}}) = \phi_C(0, 0)$. As we are assuming ϕ_C is an isomorphism, we get $\omega_{\mathrm{nw}} = 0$. This in turn means that $\omega = C \cdot \omega_{\mathrm{w}}$, hence that ω is a C-coboundary. This shows that the C-cohomology vanishes.

Conversely, assume that the C-cohomology vanishes, i.e., $\ker(C \cdot (-)) = \operatorname{im}(C \cdot (-))$, and let $\mathcal{A}_0 \subseteq \mathcal{A}$ be a linear complement of $\operatorname{im}(C \cdot (-))$, so that we have a linear direct sum decomposition $\mathcal{A} = \mathcal{A}_0 \oplus \mathrm{im}(C \cdot (-))$. As $C \cdot C = 0$, this implies that $C \cdot \mathcal{A} = C \cdot \mathcal{A}_0$, and so the map $\phi_C \colon \mathcal{A}_0 \oplus \mathcal{A}_0 \to \mathcal{A}$ is surjiective. If $(\omega_{\mathrm{nw}}, \omega_{\mathrm{w}}) \in \ker \phi_C$, then $\omega_{\mathrm{nw}} + C \cdot \omega_{\mathrm{w}} = 0$ with $\omega_{\mathrm{nw}} \in \mathcal{A}_0$ and $C \cdot \omega_{\mathrm{w}} \in \mathrm{im}(C \cdot (-))$. Since $\mathcal{A}_0$ is a linear complement of $\mathrm{im}(C \cdot (-))$ in $\mathcal{A}$, this gives $\omega_{\mathrm{nw}} = 0$ and $C \cdot \omega_{\mathrm{w}} = 0$. The second equation gives $\omega_{\mathrm{w}} \in \mathcal{A}_0 \cap \ker(C \cdot (-)) = \mathcal{A}_0 \cap \mathrm{im}(C \cdot (-)) = 0$. So ϕ_C is injective and therefore an isomorphism. This finishes the proof.

Alternatively, we may argue for the converse direction more abstractly as follows. That the C-cohomology vanishes means that we have a long exact sequence of vector spaces

$$\cdots \longrightarrow \mathcal{A} \xrightarrow{C\cdot} \mathcal{A} \xrightarrow{C\cdot} \mathcal{A} \xrightarrow{C\cdot} \mathcal{A} \longrightarrow \cdots .$$

Since exact sequences of vector spaces split, this fits into the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{A} \arrow[rr, "C\cdot"] \arrow[dr, "C\cdot"] & & \mathcal{A} \arrow[rr, "C\cdot"] \arrow[dr, "C\cdot"] & & \mathcal{A} \arrow[r] & \cdots \\
& & (C) \arrow[ur, hook, "\iota"] \arrow[dl, hook', "{(0,(C\cdot)^{-1})}"'] \arrow[dr, hook, "{(\mathrm{id},0)}"] & & (C) \arrow[ur, hook, "\iota"] \arrow[dl, hook', "{(0,(C\cdot)^{-1})}"'] \arrow[dr, hook, "{(\mathrm{id},0)}"] & & \\
\cdots \arrow[r] & (C) \oplus \mathcal{A}/(C) \arrow[uu, "\iota+\sigma", "\simeq"'] \arrow[rr, "{\begin{pmatrix} 0 & 0 \\ C\cdot & 0 \end{pmatrix}}"'] & & (C) \oplus \mathcal{A}/(C) \arrow[uu, "\iota+\sigma", "\simeq"'] \arrow[rr, "{\begin{pmatrix} 0 & 0 \\ C\cdot & 0 \end{pmatrix}}"'] & & (C) \oplus \mathcal{A}/(C) \arrow[uu, "\iota+\sigma", "\simeq"'] \arrow[r] & \cdots
\end{tikzcd}$$

where $C = \mathrm{im}(C\cdot)$ is the ideal of $\mathcal{A}$ generated by the element C, the morphism $\iota \colon (C) \hookrightarrow \mathcal{A}$ is the inclusion, and $\sigma \colon \mathcal{A}/(C) \hookrightarrow \mathcal{A}$ is a linear section of the natural projection $\mathcal{A} \to \mathcal{A}/(C)$. The isomorphism $(C\cdot)^{-1} \colon (C) \xrightarrow{\sim} \mathcal{A}/(C)$ is the inverse of the natural isomorphism

$$\mathcal{A}/(C) = \mathcal{A}/\mathrm{im}(C\cdot) = \mathcal{A}/\ker(C\cdot) \xrightarrow[\simeq]{C\cdot} \mathrm{im}(C\cdot) = (C)$$

induced by the vanishing of C-cohomology. Under this identification between (C) and $\mathcal{A}/(C)$, the isomorphisms in the above diagram become

$$\mathcal{A}/(C) \oplus \mathcal{A}/(C) \xrightarrow[\simeq]{(C\cdot)+\sigma} \mathcal{A} \ . \tag{37}$$

As a result, setting $\mathcal{A}_0 = \sigma(\mathcal{A}/(C))$, one sees that there is a unique decomposition of elements of $\mathcal{A}$ as claimed. □

The following example highlights the case in which C-cohomology of an element C vanishes for the trivial reason that the element C is a free generator. The purpose of the concept of C-cohomology here is to allow for generalization away from this example.

Example 3.22 (Trivial C-cohomology of fiber generator in higher central extension). Let

$$\begin{tikzcd} b^{2n}\mathfrak{u}_1 \arrow[r, hook] & \widehat{\mathfrak{g}} \arrow[d, "\mathrm{hofib(dd)}"'] & \\ & \mathfrak{g} \arrow[r, "\mathrm{dd}"] & b^{2n+1}\mathfrak{u}_1 \end{tikzcd}$$

be a higher central extension, classified by a cocycle dd in even degree. Then the element $b \in \mathrm{CE}(\widehat{\mathfrak{g}})$ of degree $2n+1$, from Prop. 2.10, has vanishing C-cohomology (Def. 3.20). This is because b is a free generator in the underlying graded-commutative algebra of $\mathrm{CE}(\widehat{\mathfrak{g}}) = \mathrm{CE}(\mathfrak{g})[b]/(db = \mathrm{dd})$, which by definition means that every element $\omega \in \mathrm{CE}(\widehat{\mathfrak{g}})$ has a unique expansion

$$\omega = \omega_{\mathrm{nw}} + b \wedge \omega_{\mathrm{w}}$$

for unique

$$\omega_{\mathrm{nw}},\ \omega_{\mathrm{w}} \in \mathrm{CE}(\mathfrak{g}) \hookrightarrow \mathrm{CE}(\mathfrak{g})[b] \ ,$$

as required by Prop. 3.21. This unique expansion is used notably in the definition of fiber integration along higher central extensions in Def. 3.6.

With appeal to vanishing C-cohomology, we may thus generalize the concept of higher T-duality:

Theorem 3.23 (Higher T-duality for decomposed fields). *Consider a higher self-T-duality correspondence according to Def. 3.10*

$$\begin{tikzcd} b^{4n+2}\mathfrak{u}_1 & \hat{\mathfrak{g}} \arrow[l, "h"'] \arrow[dr, "\pi:=\mathrm{hofib(dd)}"'] & & \hat{\mathfrak{g}} \arrow[r, "h"] \arrow[dl, "\pi:=\mathrm{hofib(dd)}"] & b^{4n+2}\mathfrak{u}_1 \\ & & \mathfrak{g} \arrow[dl, "\mathrm{dd}"'] \arrow[dr, "\mathrm{dd}"] & & \\ & b^{2n+1}\mathfrak{u}_1 & & b^{2n+1}\mathfrak{u}_1 & \end{tikzcd}$$

possibly equipped with the action of a group K (Def. 2.5) such that all cocycles are K-invariant (Example 2.6) Let $\mathfrak{e} \longrightarrow \mathfrak{g}$ be a fibration over the *base, equipped with a K-invariant transgression element (Def. 2.19)*

$$\mathrm{cs} \in \mathrm{CE}(\mathfrak{e})^K$$

for dd, *where such that*

(a) *the C-cohomology (Def. 3.20) of the transgression element* cs *vanishes in* $\mathrm{CE}(\mathfrak{e})^K$*;*

(b) *the inclusion ι from Prop. 3.21 may be found such as to contain the CE-algebra of the base*

$$\begin{tikzcd}
\mathrm{CE}(\mathfrak{e})^G/(\mathrm{cs}) \arrow[rr, hook, "\iota"] & & \mathrm{CE}(\mathfrak{e})^K . \\
& \mathrm{CE}(\mathfrak{g})^K \arrow[ul, hook] \arrow[ur, hook] &
\end{tikzcd}$$

Then there is an isomorphism of ($\phi^*(h)$-twisted cohomology groups (Def. 2.9) which covers the higher T-duality isomorphism from Example 3.18, in that we have a diagram:

$$\begin{tikzcd}
H^{(\bullet+2n+1)+h^A}(\widehat{\mathfrak{g}}^A)^K \arrow[d, "(\phi^A)^*"'] & H^{\bullet+h^B}(\widehat{\mathfrak{g}})^K \arrow[l, "\mathcal{T}"', "\simeq"] \arrow[d, "(\phi^B)^*"] \\
H^{(\bullet+2n+1)+h}(\mathfrak{e})^K & H^{\bullet+h^B}(\mathfrak{e}^B)^K \arrow[l, "\mathcal{T}_{\mathrm{comp}}"', "\simeq"] \\
{[-\omega_{\mathrm{w}} + (\phi)^*(b) \wedge \omega_{\mathrm{nw}})]} & {[\omega_{\mathrm{nw}} + (\phi)^*(b) \wedge \omega_{\mathrm{w}}].} \arrow[l, mapsto]
\end{tikzcd}$$

Here the vertical morphisms come from pullback along the classifying morphisms (17)

$$\mathfrak{e} \xrightarrow{\phi} \widehat{\mathfrak{g}}$$

according to Example 2.20

Proof. By Prop. 3.21, the vanishing of the C-cohomology implies that the decompositions $\omega_{\mathrm{nw}} + \mathrm{cs}^{A/B} \wedge \omega_{\mathrm{w}}$ are unique. Consequently, the linear map on cochains

$$-\omega_{\mathrm{w}} + \mathrm{cs}^A \wedge \omega_{\mathrm{nw}} \longleftarrow\!\!\shortmid\; \omega_{\mathrm{nw}} + \mathrm{cs}^B \wedge \omega_{\mathrm{w}}$$

is a well-defined linear isomorphism. With this it is now sufficient to see that this linear isomorphism on cochains intertwines the twisted differentials. But

$$d\mathrm{cs} = \mathrm{dd}\,,$$

by the defining condition on transgression elements, this follows verbatim by the same computation as in expression (36). □

4. Higher T-duality of M-branes

We now present and discuss examples of the super L_∞-algebraic higher T-duality that we introduced in Section 3.

First we observe in Section 4.1 that the super-topological T-duality of F1/Dp-branes on 10d type II super-Minkowski spacetimes established in [FSS16b] is an example. This serves to put the generalization to higher T-dualities in the following examples into perspective.

To illustrate that there are further examples even of ordinary (i.e. non-higher) super-topological T-duality we observe in Section 4.2 that there is super-topological T-self-duality for superstrings on 6d super spacetime.

After this warmup, we pass attention to the higher T-duality of genuine interest here:

(i) First we observe in Section 4.3 that the fact that the joint supercocycle for the M2/M5 brane takes values in the rational 4-sphere [FSS15d] implies, by Prop. 3.13, spherical self-T-duality of the M5-brane on the M2-brane extended superspacetime.

(ii) In order to gain a deeper understanding of what this means, we turn attention in Section 4.4 to 11d exceptional superspacetime and show that it exhibits 528-toroidal T-duality over the superpoint, and 517-toroidal T-duality over ordinary 11d superspacetime.

(iii) Then in Section 4.5 we first recall the decomposition of the C-field over the 11d exceptional tangent superspacetime due to [D'AF82, BAIPV04]. Then we compute the C-cohomology of the decomposed C-field in Prop. 4.15 and thus establish that the spherical T-duality of M5-branes passes to the exceptional tangent superspacetime (Prop. 4.17 below).

(iv) To conclude the role of exceptional super-spacetime in spherical T-duality, we explain in Section 4.6 how the decomposed C-field on the exceptional super tangent spacetime realizes the proposal of [H07b, Wes03] (see Remark 4.7 below) that M-theoretic field configurations should have a moduli space in exceptional generalized geometry. We observe that spherical

T-duality implements a duality relation on these moduli spaces which renders duality-equivalent the decomposition of the C-field at different values of the parameter s.

(v) In Section 4.7 we observe that the mechanism of spherical T-duality immediately passes to various Kaluza-Klein (KK) compactifications of 11d superspacetime, notably it passes to minimal 7d superspacetime with its decomposition of the C-field due to [ADR16].

(vi). Finally we prove in Section 4.8 that, different from but akin to spherical T-duality, also the parity symmetry of the 11d supergravity action functional lifts to isomorphism on M5-brane-charge twisted cohomology on the exceptional super tangent spacetime. Since they thus act on the same spaces of brane charges, we may think of spherical T-duality and parity symmetry to jointly constitute a new system of M-theoretic duality relations.

4.1. Ordinary T-duality of super F1/D*p*-branes on 10d type II superspacetime

Ordinary *T-duality* is a fundamental symmetry in string theory (see [GPR94], [AAL95] for standard review), important both for the inner structure of the theory as well as for its phenomenology. Previously the formalization of "topological T-duality" — which is meant to be the restriction of T-duality to the brane charges, disregarding metric information — had been inspired by, but not directly derived from string theory: for circle bundles in [BEM04, BS05] and, more generally for torus bundles in [BHM04, BRS06].

- For $n = 0$ our definition of higher T-duality correspondences in Def. 3.10 structurally reduces to this formulation of ordinary topological T-duality (see Remark 3.11) and our Corollary 3.18 reduces to the corresponding isomorphism in degree-3 twisted cohomology, rationally.
- A key difference is that even for $n = 0$ our formulation captures also the super-geometry, where the super-WZW terms of the super p-branes take values. This had allowed us in [FSS16b] to show that the super WZW-terms of the type II F1/Dp branes constitute the archetypical example of "super-topological T-duality" ([FSS16b, Theorem 5.3, Remark 5.4]):

First, the T-duality correspondence from Def. 3.10 in this case is that of [FSS16b, Prop. 6.2]:
(38)

$$\begin{tikzcd}
 & & \mathbb{R}^{8+(1,1),1|32} \arrow[dl, "p_A"'] \arrow[dr, "p_B"] & & \\
b^2\mathfrak{u}_1 & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \arrow[l, "\mu^{\mathrm{IIA}}_{F1}"'] \arrow[dr, "\mathrm{hofib}(c_2^{\mathrm{IIA}})"'] & & \mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} \arrow[r, "\mu^{\mathrm{IIB}}_{F1}"] \arrow[dl, "\mathrm{hofib}(c_2^{\mathrm{IIB}})"] & b\mathfrak{u}_1 \\
 & & \mathbb{R}^{8,1|\mathbf{16}+\mathbf{16}} \arrow[dl, "c_2^{\mathrm{IIB}}"'] \arrow[dr, "c_2^{\mathrm{IIA}}"] & & \\
 & b\mathfrak{u}_1 & & b\mathfrak{u}_1 &
\end{tikzcd}$$

Here $\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$ and $\mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}}$ denote the 10d Type IIA/B super-Minkowski spacetimes (Example 2.1) carrying the 3-cocycles

$$\begin{aligned}
\mu_{F1}^{\mathrm{IIA/B}} &= i\overline{\psi}\Gamma_a^{\mathrm{IIA/B}}\Gamma_{10}\psi \wedge e^a_{A/B} \\
&= \mu_{F1}|_{8+1} + e^9_{A/B} \wedge c_2^{\mathrm{IIB/A}}
\end{aligned} \tag{39}$$

corresponding to the type IIA/B superstring super-WZW terms, respectively, i.e. the type II version of (2). Both the super-spacetime are super Lie algebra extensions (via Prop. 2.10, hence rational circle fibrations) over the 9d type II super-Minkowski spacetime $\mathbb{R}^{8,1|\mathbf{16}+\mathbf{15}}$ ([FSS16b, Prop. 2.14]), and their (homotopy-)fiber product as such is the "doubled" superspacetime $\mathbb{R}^{8+(1,1),1|32}$ ([FSS16b, Section 6]).

Moreover, the induced isomorphism on twisted cohomology from Corollary 3.18 via Theorem 3.17, in this case is that of [FSS16b, Prop. 6.4]
(40)

$$\begin{tikzcd}
H^{0+\mu^{\mathrm{IIA}}_{F1}}\big(\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}/\mathrm{Spin}(9,1)\big) & H^{1+\mu^{\mathrm{IIB}}_{F1}}\big(\mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}}/\mathrm{Spin}(9,1)\big) \arrow[l, "(p_A)_*\circ e^{\mathcal{P}}\circ p_B^*"', "\simeq"] \\
\left(\begin{array}{c} \mu_{D0},\ \mu_{D2} \\ \mu_{D4},\ \mu_{D6} \\ \mu_{D8},\ \mu_{D10} \end{array}\right) & \left(\begin{array}{c} \mu_{D1},\ \mu_{D3} \\ \mu_{D5},\ \mu_{D7} \\ \mu_{D9} \end{array}\right) \arrow[l, mapsto]
\end{tikzcd}$$

which takes the cocycles in even/odd-degree Spin(9,1)-invariant $\mu_{F1}^{IIA/B}$-twisted cohomology (Def. 2.9) corresponding to the super-WZW terms for the super Dp-branes ([FSS16b, Section 4]) into each other, as indicated. Structurally, by pull-tensor-push through the doubled super-spacetime, this is *Hori's formula for the Buscher rules for RR-fields* [Ho99, (1.1)] refined to the superspace components of the RR-fields. Both generalize and globalize the original rules [Bu87][Bu88].

Since in following we will be lifting this ordinary T-duality to exceptional spaces and involving higher gerbes, We comment on related literature, which in a certain sense this generalizes: Global topological structures in T-duality had also been considered in [AABL94, H07a]. A geometric description of T-duality may be given by identifying the cotangent bundles of the original and the dual manifold, exhibiting the duality as a symplectomorphism of the string phase spaces [KS95][Al95]. This has been extended to cotangent bundles of the corresponding loop spaces [BHM07]. T-duality is also described in generalized geometry [Per07][LRRvUZ07][CG10], in non-geometry [GMPW09], in doubled geometry [H07c]. Topological T-duality is extended to include automorphisms determined by the twists, which can be viewed as a topological approximation to a gerby gauge transformation of spacetime [Pa14]. The effect of the gauging on the B-field terms in the sigma model lead to restrictions on the corresponding curvature [HS89, HS91]. Treating the 2-form B as a gerbe connection captures the gauging obstructions and the global constraints on the T-duality [BHM07].

4.2. Self T-duality on 6d super-spacetime

As an other example of an ordinary (i.e. not higher) super-topological T-duality we observe that there is an example of a cyclic topological T-duality for superstrings on a D0-brane extension of 5d super-Minkowski spacetime (Prop. 4.2 below). To put this in perspective, we first recall some background.

$D = 5$ simple supergravity can be obtained directly as a Calabi-Yau compactification of D = 11 supergravity [CCDF96][FKM96][FMS96] on a threefold Y with Hodge number $h_{1,1} = 1$, together with the truncation of scalar multiplets. Therefore, the two objects in $D = 5$, namely the string and the dual D0-brane, have 11-dimensional origins. The first may be viewed as an M5-brane wrapped around the unique 4-cycle of Y, while the latter is an M2-brane wrapped around the unique 2-cycle dual to the 4-cycle. The fact that we are getting T-duality for the lower cocycles is perhaps then not surprising and can be naturally explained by the above direct relation between the branes and by uniqueness of the cycles on which they wrap.

Note that this theory resembles $D = 11$ supergravity in many respects. This is illustrated by using extended symmetries in [MO98] and higher gauge symmetries in [Sa10]. Hence this simpler model might give an insight into the unsolved interesting problems of M-theory.

In 5d there is a direct analog of what in 11d is the S^4-valued supercocycle of M2/M5-branes (42) from example 4.3:

Proposition 4.1 (Two-sphere valued supercocycle in 5d). *In* $\mathrm{CE}\left(\mathbb{R}^{4,1|\mathbf{8}+\mathbf{8}}\right)$ *consider the cochains*

$$\mu^{5d}_{D0} := \overline{\psi}_A \psi_A \qquad \textit{and} \qquad \mu^{5d}_{\mathrm{string}} := \overline{\psi}_A \Gamma_a \psi_A \wedge e^a \, .$$

(i) *Then we have the relations*

$$d\mu^{5d}_{D0} = 0 \qquad \textit{and} \qquad d\mu^{5d}_{\mathrm{string}} = \mu^{5d}_{D0} \wedge \mu^{5d}_{D0} \, .$$

(ii) *Hence jointly these constitute a rational 2-sphere valued coycle.*

$$\mathbb{R}^{4,1|\mathbf{8}+\mathbf{8}} \xrightarrow{\left(\mu^{5d}_{D0},\ \mu^{5d}_{\mathrm{string}}\right)} \mathfrak{l}(S^2)$$

Proof. Due to the relation $de^a = \overline{\psi}_A \Gamma^a \psi_A$, the condition to be shown is equivalently

$$\overline{\psi}_A \Gamma_a \psi_A \wedge \overline{\psi}_B \Gamma^a \psi_B = \overline{\psi}_A \psi_A \wedge \overline{\psi}_B \psi_B \, .$$

But this is precisely the Fierz identity that holds from [CDF91, (III.5.50.a)]. □

Consequently, we can discuss the appropriate T-duality in this setting.

Proposition 4.2 (Cyclic T-duality on D0-brane extension of 5d superspacetime). *There is a T-duality self-correspondence (Def. 3.10) on the D0-brane extension of 5d super-Minkowski spacetime of the form*

$$\begin{tikzcd}
b^2\mathfrak{u}_1 & \widehat{\mathbb{R}}^{4,1|\mathbf{8}+\mathbf{8}} \arrow[l, "h_3"'] \arrow[dr, "\mathrm{hofib}(\overline{\psi}_A\psi_A)"'] & & \widehat{\mathbb{R}}^{4,1|\mathbf{8}+\mathbf{8}} \arrow[r, "h_3"] \arrow[dl, "\mathrm{hofib}(\overline{\psi}_A\psi_A)"] & b^2\mathfrak{u}_1 \\
& & \mathbb{R}^{4,1|\mathbf{8}+\mathbf{8}} \arrow[dl, "\overline{\psi}_A\psi_A"] \arrow[dr, "\overline{\psi}_A\psi_A"'] & & \\
& b\mathfrak{u}_1 & & b\mathfrak{u}_1 . &
\end{tikzcd}$$

Proof. By Prop. 4.1 used in Prop. 3.13 the result is established. □

4.3. Spherical T-duality of M5-branes on M2-extended M-theory spacetime

We discuss spherical topological T-duality in the sense of Section 3 realized for M5-branes on the M2-brane extended 11d super-spacetime. We first recall that the M2/M5-brane geometry is governed by the following data.

Example 4.3 (M2/M5-Brane cocycle). Consider the the super-Minkowski spacetime $\mathbb{R}^{10,1|\mathbf{32}}$ (Example 2.1) underlying 11-dimensional supergravity. By [D'AF82, (3.27a)] [FSS16b, Prop. 4.3] the elements

$$\begin{aligned}\mu_{M2} &:= \tfrac{i}{2}\overline{\psi}\Gamma_{a_1 a_2}\psi \wedge e^{a_1} \wedge e^{a_2} \\ \mu_{M5} &:= \tfrac{1}{5!}\overline{\psi}\Gamma_{a_1\cdots a_5}\psi \wedge e^{a_1} \wedge \cdots \wedge e^{a_5}\end{aligned} \tag{41}$$

in $\mathrm{CE}(\mathbb{R}^{10,1|\mathbf{32}})$ satisfy the relations

$$d\mu_{M2} = 0 \qquad \text{and} \qquad d\mu_{M5} = -\tfrac{1}{2}\mu_{M2} \wedge \mu_{M2}\,. \tag{42}$$

By [FSS15c, FSS15d] μ_{M2} defines the WZW-term for the Green-Schwarz sigma-model of the M2-brane, and the following combination defines the WZW term for the GS sigma-model of the M5-brane:

$$\widetilde{\mu}_{M5} := 2\mu_{M5} + c_3 \wedge \mu_{M2} \in \mathrm{CE}(\mathfrak{m2brane})\,, \tag{43}$$

where

$$\mathfrak{m2brane} := \mathrm{hofib}(\mu_{M2})$$

is the M2-brane extension of 11d super-Minkowski spacetime ([FSS15c, Section 4.1]), which by Prop. 2.10 is given by

$$\mathrm{CE}(\mathfrak{m2brane}) = \mathrm{CE}(\mathbb{R}^{10,1|\mathbf{32}})[c_3]/\left(dc_3 = \mu_{M2}\right)\,.$$

We now indeed uncover that T-self-duality associated with the M-brane cocycle.

Proposition 4.4 (M5-brane cocycle is spherical T-dual to itself). *The M5-brane cocycle (43) on the M2-brane extension of 11-dimensional super-Minkowski spacetime (Example 4.3) is spherical T-dual (Def. 3.10) to*

itself, as exhibited in the following diagram
(44)

$$\begin{tikzcd}
 & & \mathfrak{m2brane} \times_{\mathbb{R}^{10,1|\mathbf{32}}} \mathfrak{m2brane} \arrow[dl, "p_A"'] \arrow[dr, "p_B"] & & \\
b^6\mathfrak{u}_1 & \mathfrak{m2brane} \arrow[l, "\widetilde{\mu}_{M5}"'] \arrow[dr, "\pi^A:=\mathrm{hofib}(\mu_{M2})"'] & & \mathfrak{m2brane} \arrow[r, "\widetilde{\mu}_{M5}"] \arrow[dl, "\pi^B:=\mathrm{hofib}(\mu_{M2})"] & b^6\mathfrak{u}_1 \\
 & & \mathbb{R}^{10,1|\mathbf{32}} \arrow[dl, "\mu_{M2}"] \arrow[dr, "\mu_{M2}"'] & & \\
 & b^3\mathfrak{u}_1 & & b^3\mathfrak{u}_1 &
\end{tikzcd}$$

$$\begin{tikzcd}
(\mathfrak{m2brane}, \widetilde{\mu}_{M5}) \arrow[rr, leftrightarrow, "\mathcal{T}"] & & (\mathfrak{m2brane}, \widetilde{\mu}_{M5}).
\end{tikzcd}$$

Proof. Establishing the diagram amounts to checking that the cocycles are compatible. With the equivalent reformulation from Prop. 3.13 we need to show that

$$d\mu_{M2} = 0$$

and

$$d(2\mu_{M5}) = -\mu_{M2} \wedge \mu_{M2} \,.$$

But this is precisely the M2/M5-cocycle presented in equation (42). ☐

4.4. Toroidal T-duality on exceptional M-theory spacetime

Definition 4.5 (Maximal central extension of the $N = 32$ superpoint). Consider the $N = 32$ superpoint, hence the super L_∞-algebra

$$\mathbb{R}^{0|32} \in sL_\infty \mathrm{Alg}_{\mathbb{R}}^{\mathrm{fin}}$$

defined dually by (cf. expression (6))

$$\mathrm{CE}(\mathbb{R}^{0|32}) = \mathbb{R}[\underbrace{(\psi^\alpha)}_{\mathrm{deg}=(1,\mathrm{odd})}{}_{\alpha \in \{1,2,\ldots,32\}}]/(d\psi^\alpha = 0)\,.$$

We write

$$\begin{tikzcd}
\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc}} \arrow[d, "\star"'] \\
\mathbb{R}^{0|\mathbf{32}}
\end{tikzcd}$$

for its maximal invariant central extension according to [HSc17, Def. 6]. We will call this the *exceptional tangent superspacetime.*

Proposition 4.6 (Spin-action on exceptional tangent superspacetime). (i) *The exceptional tangent superspacetime super Lie algebra* $\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc}}$ *from Def. 4.5 is, up to isomorphism, dually given by the following Chevalley-Eilenberg algebra:*

$$\mathrm{CE}(\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc}}) = \mathbb{R}\big[\ \underbrace{(e^a)}_{\deg=(1,\mathrm{even})}\ ,\ \underbrace{(B_{a_1a_2})}_{(1,\mathrm{even})},\ \underbrace{(B_{a_1\cdots a_5})}_{(1,\mathrm{even})},\ \underbrace{(\psi^\alpha)}_{(1,\mathrm{odd})}\ \big] \Big/ \left(\begin{array}{ll} d\psi^\alpha = 0, & dB_{a_1a_2} = \frac{i}{2}\overline{\psi}\Gamma_{a_1a_2}\psi, \\ de^a = \overline{\psi}\Gamma^a\psi, & dB_{a_1\cdots a_5} = \frac{1}{5!}\overline{\psi}\Gamma_{a_1\cdots a_5}\psi \end{array} \right), \tag{45}$$

where Γ *denotes a Clifford algebra representation on the 32-dimensional vector space spanned by the* ψ^α*, corresponding to the irreducible real representation* **32** *of* $\mathrm{Spin}(10,1)$.

(ii) *Hence the bosonic body of the exceptional super tangent spacetime is*

$$\left(\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc}}\right)_{\mathrm{bos}} \simeq \mathbb{R}^{10,1} \oplus \wedge^2\left(\mathbb{R}^{10,1}\right)^* \oplus \wedge^5\left(\mathbb{R}^{10,1}\right)^* . \tag{46}$$

(iii) *This induces on* $\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc}}$ *a* $\mathrm{Spin}(10,1)$*-action in the sense of Def. 2.5, and the projection to the superpoint factors* $\mathrm{Spin}(10,1)$*-equivariantly over the 11d super-Minkowski spacetime (Example 2.1) as*

$$\begin{tikzcd}
\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc}} \arrow[dd, "\mathrm{hofib}(\psi\wedge\overline{\psi})"'] \arrow[dr, "\mathrm{hofib}(\mu_{\mathrm{exc}})"] & \\
& \mathbb{R}^{10,1|\mathbf{32}} \arrow[dl, "\mathrm{hofib}(\overline{\psi}\Gamma_a\psi)"] . \\
\mathbb{R}^{0|\mathbf{32}} &
\end{tikzcd}$$

Proof. The maximal invariant central extension, in the sense of [HSc17], of $\mathbb{R}^{0|\mathbf{32}}$ is classified by the maximal tuple of linearly independent 2-cocycles. Since the ψ^α commute with each other, there is one 2-cocycle for every symmetric 32×32-matrix. These may be identified with the elements Γ_a, $\Gamma_{a_1a_2}$, $\Gamma_{a_1\cdots a_5}$ of a Clifford algebra representation of the **32** of $\mathrm{Spin}(10,1)$.

Under this identification and with notation from Def. 3.1 the cocycle classifying the maximal central extension is, up to isomorphism, given by

$$\Big((\overline{\psi}\Gamma_a\psi),\ \underbrace{\big((\tfrac{i}{2}\overline{\psi}\Gamma_{a_1a_2}\psi), (\tfrac{1}{5!}\overline{\psi}\Gamma_{a_1\cdots a_5}\psi) \big)}_{\mu_{\mathrm{exc}}} \Big) \;:\; \mathbb{R}^{0|\mathbf{32}} \longrightarrow b(\mathfrak{u}_1)^{528}. \tag{47}$$

This implies the claim by Prop. 2.10. ☐

Remark 4.7 (The 11d exceptional super tangent spacetime). That "exceptional generalized tangent spaces" of the form $\mathbb{R}^{d-1,1} \oplus \bigwedge^2(\mathbb{R}^{d-1,1})^* \oplus \bigwedge^5(\mathbb{R}^{d-1,1})^*$ should play a role in M-theory compactified on a d-dimensional fiber was proposed in [H07b, PW08]. The full compactification with $d = 11$

$$\mathbb{R}^{10,1} \oplus \bigwedge^2(\mathbb{R}^{10,1})^* \oplus \bigwedge^5(\mathbb{R}^{10,1})^*$$

that appears in (46) has been the basis of various conjectures in [Wes03], see also [Bar12, Sec. 2.2]. That this 11d exceptional tangent bundle (46) happens to be the bosonic body of the D'Auria-Fré (DF) algebra $\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s}$ (Def. 4.13, already due to [D'AF82]) has been highlighted in [Vau07]. However, it seems that no relation between the DF-algebra $\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s}$ and the actual idea of M-theoretic generalized geometry promoted in [H07b, PW08, Bar12] has been observed before. This is indeed what we discuss below in Section 4.6.

Definition 4.8 (528-Bein on exceptional super tangent spacetime). It is useful to abbreviate the CE-generators in expression (45) as

$$\begin{aligned} \vec{\mathcal{E}} &:= (\mathcal{E}_A) \\ &:= \big((e_a), (B_{a_1a_2}), (B_{a_1\cdots a_5}) \big). \end{aligned}$$

This $(\mathcal{E}_A)$ may be thought of as the 528-bein on the exceptional tangent superspacetime.

The following basic example will play a key role in the proof of Prop. 4.15 below:

Example 4.9. The *exceptional volume form* on the exceptional tangent superspacetime is

$$\begin{aligned} \mathrm{vol}_{528} &:= \bigwedge_{A=1}^{528} \mathcal{E}_A \\ &:= \Big(\bigwedge_{0\leq a\leq 10} e^a \Big) \wedge \Big(\bigwedge_{0\leq a_1,<a_2\leq 10} B^{a_1a_2} \Big) \wedge \Big(\bigwedge_{0\leq a_1,<\cdots<a_5\leq 10} B^{a_1\cdots a_5} \Big). \end{aligned} \tag{48}$$

Definition 4.10 (Inner product on exceptional tangent superspacetime). On $b(\mathfrak{u}_1)^{528}$ consider, for $s \in \mathbb{R} \setminus \{0\}$, the following universal cup product (Def. 3.1)

$$\begin{aligned}\big\langle \mathrm{pr}_1^*(\mathrm{d}\vec{\mathrm{d}}),\ \mathrm{pr}_2^*(\mathrm{d}\vec{\mathrm{d}}) \big\rangle_{\mathrm{exc},s} = {} & (s+1)\mathrm{pr}_1^*(\mathrm{dd}^a) \wedge \mathrm{pr}_2^*(\mathrm{dd}_a) \\ & - \mathrm{pr}_1^*(\mathrm{dd}^{a_1 a_2}) \wedge \mathrm{pr}_2^*(\mathrm{dd}_{a_1 a_2}) \\ & + (1 + \tfrac{s}{6})\mathrm{pr}_1^*(\mathrm{dd}^{a_1 \cdots a_5}) \wedge \mathrm{pr}_2^*(\mathrm{dd}_{a_1 \cdots a_5}).\end{aligned}$$

Proposition 4.11 (528-Toroidal T-duality correspondence on exceptional tangent super-spacetime). **(i)** *On the exceptional tangent superspacetime* $\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc}}$ *(Def. 4.5) the following CE-element is a* $\mathrm{Spin}(10,1)$*-invariant 3-cocycle (Example 2.6) for each* $s \in \mathbb{R} \setminus \{0\}$*:*

$$\begin{aligned}\big\langle \vec{\mathcal{E}} \wedge \overline{\psi}\vec{\Gamma}\psi \big\rangle_{\mathrm{exc},s} = {} & (s+1) e_a \wedge \overline{\psi}\Gamma^a\psi - B_{a_1 a_2} \wedge \overline{\psi}\Gamma^{a_1 a_2}\psi \\ & + (1 + \tfrac{s}{6}) B_{a_1 \cdots a_5} \wedge \overline{\psi}\Gamma^{a_1 \cdots a_5}\psi \\ & \in \mathrm{CE}\big(\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc}}\big)^{\mathrm{Spin}(10,1)},\end{aligned} \tag{49}$$

where on the left we used the 528-bein from Def. 4.8 and the inner product from Def. 4.10.

(ii) *The cocycle (49) is in 528-toroidal T-duality correspondence with itself (Def. 3.10) with respect to the exceptional tangent superspacetime regarded as a 528-torus fibration over the superpoint via Def. 4.10:*

$$\begin{tikzcd}
b^2\mathfrak{u}_1 & \mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc}} \arrow[l, "\langle \vec{\mathcal{E}} \wedge \overline{\psi}\vec{\Gamma}\psi \rangle_{\mathrm{exc},s}"'] \arrow[dr, "\mathrm{hofib}(\psi\wedge\overline{\psi})"'] & & \mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc}} \arrow[r, "\langle \vec{\mathcal{E}} \wedge \overline{\psi}\vec{\Gamma}\psi \rangle_{\mathrm{exc},s}"] \arrow[dl, "\mathrm{hofib}(\psi\wedge\overline{\psi})"] & b^2\mathfrak{u}_1 \\
& & \mathbb{R}^{0|32} & &
\end{tikzcd}$$

(iii) *The cocycle 49 is also in 517-toroidal T-duality with itself (Def. 3.10) factoring via Prop.4.6 over the 11d super-Minkowski spacetime* $\mathbb{R}^{10,1|\mathbf{32}}$

$$\begin{tikzcd}
b^2\mathfrak{u}_1 & \mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc}} \arrow[l, "\langle \vec{\mathcal{E}} \wedge \overline{\psi}\vec{\Gamma}\psi \rangle_{\mathrm{exc},s}"'] \arrow[dr] & & \mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc}} \arrow[r, "\langle \vec{\mathcal{E}} \wedge \overline{\psi}\vec{\Gamma}\psi \rangle_{\mathrm{exc},s}"] \arrow[dl] & b^2\mathfrak{u}_1 \\
& & \mathbb{R}^{10,1|\mathbf{32}} & &
\end{tikzcd}$$

$$\big(\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc}},\ \langle \vec{\mathcal{E}} \wedge \overline{\psi}\vec{\Gamma}\psi \rangle_{\mathrm{exc},s}\big) \xleftrightarrow{\ \mathcal{T}_{\mathrm{self}}\ } \big(\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc}},\ \langle \vec{\mathcal{E}} \wedge \overline{\psi}\vec{\Gamma}\psi \rangle_{\mathrm{exc},s}\big).$$

Proof. The cocycle property to be checked requires that

$$
\begin{aligned}
(50)\qquad d\big\langle \vec{\mathcal{E}} \wedge \overline{\psi}\vec{\Gamma}\psi \big\rangle_{\mathrm{exc},s} &= \big\langle d\vec{\mathcal{E}} \wedge \overline{\psi}\vec{\Gamma}\psi \big\rangle_{\mathrm{exc},s} \\
&= (s+1)\left(\overline{\psi}\Gamma_a\psi\right) \wedge \left(\overline{\psi}\Gamma^a\psi\right) \\
&\quad - \tfrac{i}{2}\left(\overline{\psi}\Gamma_{a_1a_2}\psi\right) \wedge \left(\overline{\psi}\Gamma^{a_1a_2}\psi\right) \\
&\quad + (1+\tfrac{s}{6})\tfrac{1}{5!}\left(\overline{\psi}\Gamma_{a_1\cdots a_5}\psi\right) \wedge \left(\overline{\psi}\Gamma^{a_1\cdots a_5}\psi\right) \\
&= 0.
\end{aligned}
$$

This does indeed vanish by a trilinear Fierz identity for **32** that was originally observed in [D'AF82, (6.4)]; our parameterization by s follows [BAIPV04, (23)]:

$$
\begin{aligned}
(51)\quad (s+1)\left(\overline{\psi}\Gamma_a\psi\right) \wedge \overline{\psi}\Gamma^a - \tfrac{i}{2}\left(\overline{\psi}\Gamma_{a_1a_2}\psi\right) \wedge \overline{\psi}\Gamma^{a_1a_2}& \\
+ (1+\tfrac{s}{6})\tfrac{1}{5!}\left(\overline{\psi}\Gamma_{a_1\cdots a_5}\psi\right) \wedge \overline{\psi}\Gamma^{a_1\cdots a_5} &= 0\,.
\end{aligned}
$$

With this it is straightforward to verify that we have a T-duality self-correspondence according to Def. 3.10. □

Remark 4.12 (Fermionic 2-cocycles on exceptional tangent super-spacetime). The Fierz identity (51) is stronger than the cocycle condition (50) that it implies, since it says that already the cubic fermion term inside the quartic term vanishes by itself. This means that the single bosonic 3-cocycle (49) is in fact a linear combination of the following 32 fermionic 2-cocycles:

$$
(52)\qquad \big\langle \vec{\mathcal{E}} \wedge \overline{\psi}\vec{\Gamma} \big\rangle \;:\; \mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s} \longrightarrow b(\mathbb{R}^{0|1})^{32}.
$$

4.5. Spherical T-duality for M5-branes on exceptional M-theory spacetime

We have seen two different extensions of M-theory super-spacetime above: In Section 4.3 we considered the M2-brane extension by the higher degree M2-cocycle and found spherical T-duality for M5-branes on it, while in Section 4.4 we discussed the exceptional tangent space extension by 517 cocycles of degree 2 on which we found toroidal topological T-duality. Here we discuss how the spherical T-duality on the M2-brane extended super-spacetime passes to a spherical T-duality on the exceptionally extended super-spacetime.

Definition 4.13 (exceptional generalised super spacetime). For $s \in \mathbb{R} \setminus \{0\}$, write

$$\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s} \in sL_\infty\mathrm{Alg}^{\mathrm{fin}}_{\mathbb{R}}$$

for the fermionic central extension of the exceptional tangent superspacetime $\mathbb{R}^{10,1|\mathbf{32}}$ (Def. 4.5) which is classified by the 32 fermionic 2-cocycles

$$\mu_{\mathrm{exc},s} := \left\langle \vec{\mathcal{E}} \wedge \overline{\psi}\vec{\Gamma} \right\rangle$$

(52) from Remark 4.12:

$$\begin{tikzcd}
\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s} \arrow[d, "\mathrm{hofib}(\mu_{\mathrm{exc},s})"'] & \\
\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc}} \arrow[r, "\mu_{\mathrm{exc},s}"] & b(\mathbb{R}^{0|1})^{32}.
\end{tikzcd}$$

By Prop. 2.10 this means that we may take

$$\mathrm{CE}\big(\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s}\big) = \mathrm{CE}\big(\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc}}\big)[(\eta^\alpha)]/\big(d\overline{\eta} = \langle \vec{\mathcal{E}} \wedge \overline{\psi}\vec{\Gamma} \rangle\big). \tag{53}$$

Proposition 4.14 (Transgression of M2-cocycle via decomposed C–field). *For $s \in \mathbb{R} \setminus \{0\}$, the fermionic extension of exceptional tangent superspacetime $\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s}$ (Def. 4.13) regarded a fibered over 11d super-Minkowski spacetime $\mathbb{R}^{10,1|\mathbf{32}}$*

$$\begin{tikzcd}
\wedge^2(\mathbb{R}^{10,1})^* \oplus \wedge^5(\mathbb{R}^{10,1})^* \oplus \mathbf{32} \arrow[r, hook, "\iota"] & \mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s} \arrow[d, two heads, "\pi_{\mathrm{exc},s}"] \\
 & \mathbb{R}^{10,1|\mathbf{32}},
\end{tikzcd}$$

carries a transgression element (Def. 2.19) $c_{\mathrm{exc},s} \in \mathrm{CE}(\mathbb{R}^{10,1|\mathbf{32}})$ *for the M2-brane 4-cocycle (41):*

(54)
$$\begin{tikzcd}
c_{\mathrm{fib},s} & & & \mathrm{CE}\big(\wedge^2(\mathbb{R}^{10,1})^* \oplus \wedge^5(\mathbb{R}^{10,1})^* \oplus \mathbf{32}\big)^{\mathrm{Spin}(10,1)} \\
c_{\mathrm{exc},s} \arrow[u, mapsto, "\iota^*"'] \arrow[r, mapsto, "d"] & dc_{\mathrm{exc},s} & & \mathrm{CE}\big(\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s}\big)^{\mathrm{Spin}(10,1)} \arrow[u, "\iota^*"'] \\
& \mu_{{}_{M2}} \arrow[u, mapsto, "\pi^*_{\mathrm{exc},s}"'] & & \mathrm{CE}\big(\mathbb{R}^{10,1|\mathbf{32}}\big)^{\mathrm{Spin}(10,1)} \arrow[u, "\pi^*"']
\end{tikzcd}$$

Proof. The condition

$$dc_{\mathrm{exc},s} = \pi^*_{\mathrm{exc},s}(\mu_{{}_{M2}})$$

had been solved for two values of s in [D'AF82, Section 6], for the other values of s in [BAIPV04, Section 3]. Explicitly, in terms of the CE-generators from (45) and (53), the transgression element reads

$$c_{\mathrm{exc},s} = (c_{\mathrm{exc},s})_{\mathrm{bos}} + (c_{\mathrm{exc},s})_{\mathrm{ferm}} \tag{55}$$

with

$$\begin{aligned}
(c_{\mathrm{exc},s})_{\mathrm{bos}} = \;& \left.\begin{array}{l} \alpha_0(s) B_{ab} \wedge e^a \wedge e^b + \alpha_1(s) B^{a_1}{}_{a_2} \wedge B^{a_2}{}_{a_3} \wedge B^{a_3}{}_{a_1} \\ + \alpha_2(s) B_{b_1 a_1 \cdots a_4} \wedge B^{b_1}{}_{b_2} \wedge B^{b_2 a_1 \cdots a_4} \\ + \alpha_4(s) \epsilon_{\alpha_1 \cdots \alpha_6 b_1 \cdots b_5} B^{a_1 a_2 a_3}{}_{c_1 c_2} \wedge B^{a_4 a_5 a_6 c_1 c_2} \wedge B^{b_1 \cdots b_5} \end{array}\right\} c_{\mathrm{fib},s} \\
& - \alpha_3(s) \epsilon_{\alpha_1 \cdots \alpha_5 b_1 \cdots b_5 c} B^{a_1 \cdots a_5} \wedge B^{b_1 \cdots b_5} \wedge e^c
\end{aligned} \tag{56}$$

and

$$\begin{aligned}
(c_{\mathrm{exc},s})_{\mathrm{ferm}} = -\tfrac{1}{2}\overline{\eta}_\alpha \wedge \psi^\beta \wedge \Big(& \beta_1(s)(\Gamma_a)^\alpha{}_\beta \, e^a + \beta_2(s)(\Gamma_{ab})^\alpha{}_\beta \, B^{ab} \\
& + \beta_3(s)(\Gamma_{a_1 \cdots a_5})^\alpha{}_\beta \, B^{a_1 \cdots a_5}\Big),
\end{aligned} \tag{57}$$

for analytic functions α_i, β_j of the parameter $s \in \mathbb{R} \setminus \{0\}$ with the following zeros

$$\begin{aligned}
&\alpha_0(s) \neq 0 \\
&\alpha_1(s) = 0 \quad \Leftrightarrow \quad s = -3 \\
&\alpha_2(s) = 0 \quad \Leftrightarrow \quad s = -6 \\
&\alpha_3(s) = 0 \quad \Leftrightarrow \quad s = -6 \\
&\alpha_4(s) = 0 \quad \Leftrightarrow \quad s = -6 \\
&\beta_1(s) = 0 \quad \Leftrightarrow \quad s = -3/2 \\
&\beta_2(s) = 0 \quad \Leftrightarrow \quad s = -3 \\
&\beta_3(s) = 0 \quad \Leftrightarrow \quad s = -6.
\end{aligned} \tag{58}$$

□

Now by Prop. 4.14 we may ask whether the spherical T-duality correspondence of the M5-brane cocycle from Example 4.3 transfers along the comparison map

$$\begin{tikzcd}
\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s} \arrow[rr, "\mathrm{comp}_s"] \arrow[dr, "\pi_{\mathrm{exc},s}"'] & & \mathfrak{m}2\mathfrak{brane} \arrow[dl, "\mathrm{hofib}(\mu_{M2})"] \\
& \mathbb{R}^{10,1|\mathbf{32}} &
\end{tikzcd}$$

to an isomorphism on the M5-brane twisted cohomology of the super exceptional super tangent spacetime of Def. 4.16. By Theorem 3.23 this requires analysis of the C-cohomology of the decomposed C-field:

Proposition 4.15 (C-cohomology of decomposed supergravity C–field). *For $s \in \mathbb{R} \setminus \{0, -3/2, -3, -6\}$, the C-cohomology (Def. 3.20) of the decomposed C-field $c_{\mathrm{exc},s} \in \mathrm{CE}(\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s})^{\mathrm{Spin}(10,1)}$ (Prop. 4.14) is spanned by elements which are wedge products $f(\psi) \wedge \mathrm{vol}_{528}$ of elements generated from $\overline{\psi} \wedge \psi$ with the volume form vol_{528} (expression (48)) of the 528-dimensional exceptional superspacetime.*

Proof. Write n_{bos} and n_{ferm} for the numbers of bosonic or fermionic generators, respectively, in a wedge product. Then consider the degrees

$$\deg_0 := \tfrac{1}{2} n_{\mathrm{ferm}} \qquad \text{and} \qquad \deg_1 := n_{\mathrm{bos}} - \tfrac{1}{2} n_{\mathrm{ferm}} \,.$$

With respect to this bigrading the two summands of the decomposed C-field (55) have the following bidegrees:

$$c_{\mathrm{exc},s} = \underbrace{(c_{\mathrm{exc},s})_{\mathrm{bos}}}_{(\deg_1,\deg_0)=(3,0)} + \underbrace{(c_{\mathrm{exc},s})_{\mathrm{ferm}}}_{(0,1)}.$$

Therefore, the $c_{\mathrm{exc},s}\wedge$-complex is the direct sum of three total complexes of the following three double complexes

$$\Big(\mathrm{CE}(\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s})^{\mathrm{Spin}(10,1)}|_{\deg_1 \bmod 3=\epsilon}, d_0 := (c_{\mathrm{exc},s})_{\mathrm{ferm}}\wedge\,,\; d_1 := (c_{\mathrm{exc},s})_{\mathrm{bos}}\wedge\Big)$$

for the off-set

$$\epsilon \in \{0,1,2\}.$$

Since these bicomplexes are concentrated in a half plane, we may compute the $c_{\mathrm{exc},s}$-cohomology by the corresponding double complex spectral sequences (see page 682 for illustration)

$$\begin{aligned} E_2^{\bullet,\bullet} &= H^\bullet_{(c_{\mathrm{exc},s})_{\mathrm{bos}}}\Big(H^\bullet_{(c_{\mathrm{exc},s})_{\mathrm{ferm}}}\big(\mathrm{CE}(\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s})^{\mathrm{Spin}(10,1)}|_\epsilon\big)\Big) \\ &\Longrightarrow\; H^\bullet_{c_{\mathrm{exc},s}}\Big(\mathrm{CE}(\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s})^{\mathrm{Spin}(10,1)}|_\epsilon\Big) \end{aligned} \tag{59}$$

Now in the Spin(10, 1)-invariant subalgebra the fermions always appear paired, as a linear combination of the 528 degree-2 elements which are quadratic in the gravitino field

$$(\overline{\psi}\wedge\psi) := \Big(\overline{\psi}_\alpha\wedge\psi^\beta(\Gamma_a)^\alpha{}_\beta,\; \overline{\psi}_\alpha\wedge\psi^\beta(\Gamma_{a_1a_2})^\alpha{}_\beta,\; \overline{\psi}_\alpha\wedge\psi^\beta(\Gamma_{a_1\cdots a_5})^\alpha{}_\beta\Big), \tag{60}$$

as well as the 528 degree-2 elements which are products of a gravitino field with the auxiliary fermion η,

$$(\mathrm{dp}_A) := \Big(\overline{\eta}_\alpha\wedge\psi^\beta(\Gamma_a)^\alpha{}_\beta,\; \overline{\eta}_\alpha\wedge\psi^\beta(\Gamma_{a_1a_2})^\alpha{}_\beta,\; \overline{\eta}_\alpha\wedge\psi^\beta(\Gamma_{a_1\cdots a_5})^\alpha{}_\beta\Big).$$

Under the assumption on s, indeed all these 528 elements are non-vanishing in $(c_{\mathrm{exc},s})_{\mathrm{ferm}}$, by (58).

In terms of these quadratic fermionic elements, the fermionic part (57) of the decomposed C-field has the simple form

$$(c_{\mathrm{exc},s})_{\mathrm{ferm}} = \mathrm{dp}_A\wedge\mathrm{dx}^A$$

where

$$(\mathrm{dx}^A) := (e^a, B^{a_1a_2}, B^{a_1\cdots a_5}) \tag{61}$$

denotes the collection of all the bosonic generators, which may be thought of as the 528-vielbein on the exceptional spacetime.

The C-cohomology of such "odd symplectic forms" $\mathrm{dp}_A \wedge \mathrm{dx}^A$ has been computed in [Šev05] (there called H-cohomology), and in our case it is spanned by the terms proportional to

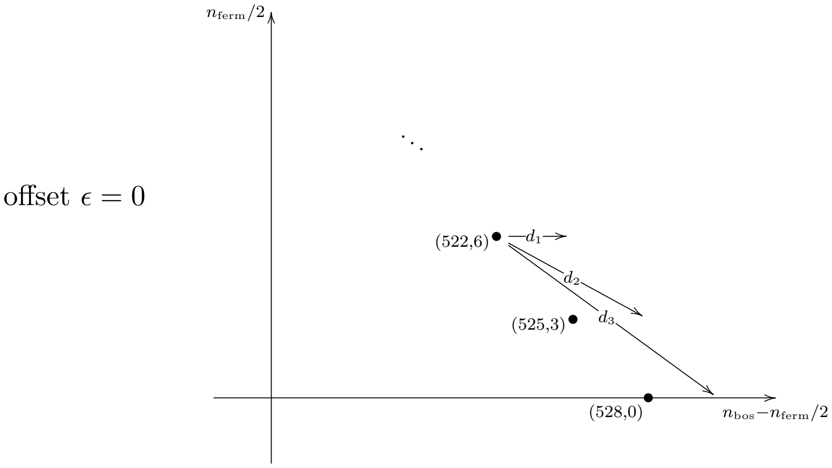
$$f(\psi) \wedge \bigwedge_A \mathrm{dx}^A = f(\psi) \wedge \mathrm{vol}_{528}\,, \tag{62}$$

where $f(\psi)$ is any element generated from the elements $\overline{\psi} \wedge \psi$ from expression (60) alone and where vol_{528} is the exceptional volume form (48). A homotopy operator that witnesses the vanishing of the C-cohomology away from these elements is

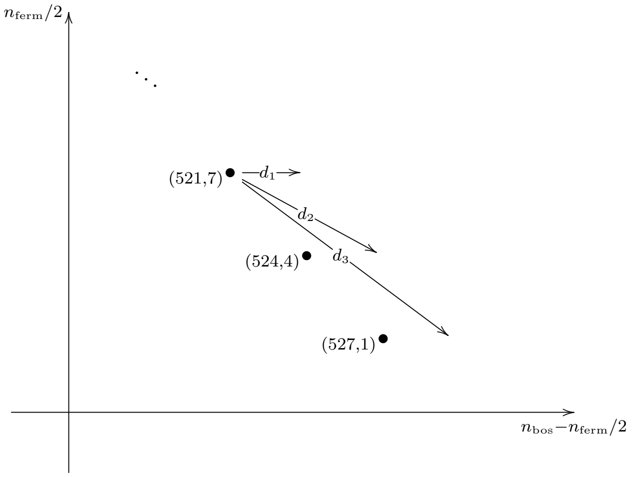
$$\partial_{\mathrm{dp}_A} \partial_{\mathrm{dx}^A}\,.$$

This manifestly commutes with the $\mathrm{Spin}(10,1)$-action in our case, so that the statement passes to the $\mathrm{Spin}(10,1)$-invariant algebra.

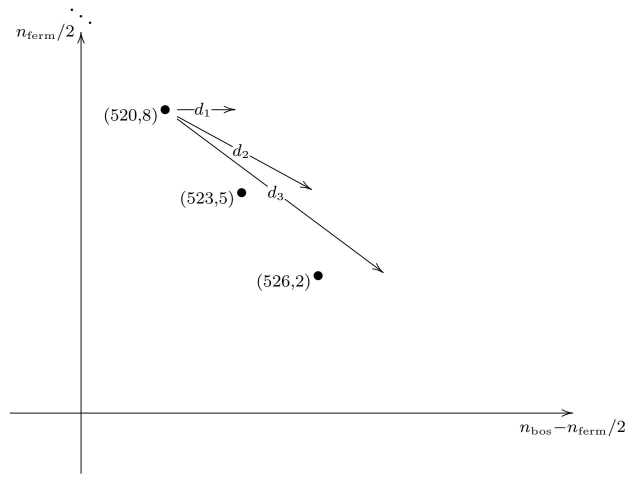
This C-cohomology of $(c_{\mathrm{exc},s})_{\mathrm{ferm}} \wedge$ constitutes the first page of the three spectral sequences, respectively. Inspection of the degrees shows that none of the higher differentials can be non-vanishing, hence that the spectral sequences already collapse on this page. This implies the claim. □



The E_1-pages of the three spectral sequences (59) which jointly compute the C-cohomology of the decomposed supergravity C-field $c_{\mathrm{exc},s}$ (Prop. 4.14). The fat dots indicate the non-vanishing C-offset $\epsilon = 1$



offset $\epsilon = 2$



cohomology classes (62) of the fermionic component $(c_{\mathrm{exc},s})_{\mathrm{ferm}}$; they are all represented by a wedge product of bi-fermions $\psi \wedge \overline{\psi}$ with the bosonic volume form vol_{528} of the exceptional tangent super-spacetime (48). Some higher differentials are shown in order to visualize that the spectral sequences all collapse already on the E_1-page.

As a consequence we are able to identify spherical T-duality in exceptional geometry:

Definition 4.16 (M5-twisted cohomology of super exceptional tangent super spacetime). Let $s \in \mathbb{R} \setminus \{0\}$ with $\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s} \xrightarrow{\mathrm{comp}_s} \mathbb{R}^{10,1|\mathbf{32}}$ the corresponding super exceptional tangent super spacetime (Def. 4.13).

(i) Write

$$\widetilde{\mu}_{{}_{M5,s}} := (\mathrm{comp}_s)^*(\widetilde{\mu}_{{}_{M5}}) = 2\mu_{{}_{M5}} + c_{\mathrm{exc},s} \wedge \mu_{{}_{M2}} \in \mathrm{CE}\big(\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s}\big) \tag{63}$$

for the pullback of the M5-brane cocycle

$$\widetilde{\mu}_{{}_{M5}} := 2\mu_{{}_{M5}} + c \wedge \mu_{{}_{M2}} \in \mathrm{CE}\left(\mathbb{R}^{10,1|\mathbf{32}}\right)$$

(expressions (43)), for which the C-field factor c is replaced by the decomposed C-field $c_{\mathrm{exc},s}$ from Prop. 4.14.

(ii) This induces the corresponding 6-periodic $\tilde{\mu}_{{}_{M5}}$-twisted Spin(10, 1)-invariant cohomology groups

$$H^{\bullet + \tilde{\mu}_{M5,s}}\big(\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s}/\mathrm{Spin}(10,1)\big) := H^{\bullet}\big(\mathrm{CE}\big(\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s/s'}\big), d + \tilde{\mu}_{{}_{M5,s}}\big)$$

of the super exceptional super spacetime via Def. 2.9. We write

$$\begin{aligned}\tilde{H}^{\bullet + \tilde{\mu}_{M5,s}}&\big(\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s}/\mathrm{Spin}(10,1)\big)\\ &:= H^{\bullet}\Big(\mathrm{CE}\big(\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s/s'}\big)/\big\langle f(\psi)\wedge \mathrm{vol}_{528}, f(\psi) d\mathrm{vol}_{528}\big\rangle, d + \tilde{\mu}_{{}_{M5,s}}\Big)\end{aligned} \tag{64}$$

for the corresponding cohomology groups after quotienting out the subcomplex spanned by the $c_{\mathrm{exc},s}$-cohomology, hence, by Prop. 4.15, the multiples of wedge products of Spin(10, 1)-invariants in the ψ-generators with the exceptional volume form vol_{528} (Def. 48).

Proposition 4.17 (Spherical T-duality on exceptional super spacetime). *For $s \in \mathbb{R} \setminus \{0, -3/2, -3, -6\}$ we have a spherical T-duality isomorphism for decomposed form fields (Theorem 3.23) between the corresponding exceptional superspacetimes $\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s}$, $\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s'}$ (Def. 4.13) in that we have an isomorphism $\mathcal{T}_{\mathrm{exc},s}$ of $\tilde{\mu}_{{}_{M5,s}}$-twisted cohomology groups according to Def. 4.16 fitting into the diagram below*

$$\begin{tikzcd}
H^{(\bullet+3)+\tilde{\mu}_{M5}}\big(\mathfrak{m2brane}/\mathrm{Spin}(10,1)\big) \arrow[d, "(\mathrm{comp}_s)^*"'] & & H^{\bullet+\tilde{\mu}_{M5}}\big(\mathfrak{m2brane}/\mathrm{Spin}(10,1)\big) \arrow[ll, "\mathcal{T}", "\simeq"'] \arrow[d, "(\mathrm{comp}_s)^*"] \\
\tilde{H}^{(\bullet+3)+\tilde{\mu}_{M5,s}}\big(\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s}/\mathrm{Spin}(10,1)\big) & & \tilde{H}^{\bullet+\tilde{\mu}_{M5,s}}\big(\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s}/\mathrm{Spin}(10,1)\big) \arrow[ll, "\mathcal{T}_{\mathrm{exc},s}", "\simeq"'] \\
{[-\omega_{\mathrm{w}} + c_{\mathrm{exc},s}\wedge\omega_{\mathrm{nw}})]} & & {[\omega_{\mathrm{nw}} + c_{\mathrm{exc},s}\wedge\omega_{\mathrm{w}}].} \arrow[ll, mapsto]
\end{tikzcd}$$

Proof. The result is obtained by using the result of Prop. 4.15 in the statement of Theorem 3.23. ☐

Remark 4.18 (M5-twisted cocycles from heterotic Green-Schwarz mechanism). Proposition 4.17 is a higher/M-theoretic analog of the string theoretic T-duality isomorphism of [FSS16b, Prop. 6.4], recalled in the top line of (40). It it now desireable to obtain also the corresponding analog of the bottom line in (40), namely examples of nontrivial $\tilde{\mu}_{M5,s}$-twisted and $\mathrm{Spin}(10,1)$-invariant cocycles. Due to the nature of the twisted differential $d + \tilde{\mu}_{M5,s}$ with its ingredients from (41), (45), (57) and (63), the cocycle condition for $d + \tilde{\mu}_{M5,s}$ translates to a complicated-looking higher-order condition on spinor identities. Here we will leave its solution as an open mathematical problem. Nevertheless we now offer an informal argument that nontrivial such cocycles do exist at least in degree 2 or 3 mod 6. Namely, in terms of string/M-theory, this open problem is the question for the M-theoretic analog of the D-brane charges μ_{Dp} in (40) when the role of the fundamental type II string $\mu_{F1}^{\mathrm{IIA/B}}$ is taken by the M5-brane $\tilde{\mu}_{M5,s}$ (as indicated in the table at the beginning of Section 3.10, see [Sa10] for proposals and further discussion). In view of this, there is the following "physics proof" of the existence of non-trivial cocycles:

(i) Recall that the twisted cocycle condition in Prop. 4.17, by Def. 2.9, reads

$$\begin{aligned} d\mu_\bullet &= 0, \\ d\mu_{\bullet+6} &= -\,\tilde{\mu}_{M5} \wedge \mu_n. \end{aligned}$$

By the logic of the torsion constraints of supergravity, reviewed in Section 1, this must correspond to an equation of motion for field strength super differential forms as

$$\begin{aligned} dF_\bullet &= 0 \\ dF_{\bullet+6} &= -\,H_7 \wedge F_\bullet\,, \end{aligned}$$

where H_7 is the 7-form flux to which the 5-brane couples, while the nature of the fluxes $F_\bullet$ and $F_{\bullet+6}$ is to be determined.

(ii) But for degrees $\bullet = 2 \bmod 6$ it was recognized in [Sa09, Section 3] that the effective background equations of motion in heterotic string theory, whose Yang-Mills sector may be rewritten as

$$(65) \qquad d\underbrace{F_2}_{\substack{\text{gauge field}\\ \text{magnetic flux}}} = 0 \qquad \text{and} \qquad d\underbrace{F_8}_{\substack{\text{gauge field}\\ \text{electric flux}}} = -H_7 \wedge F_2\,,$$

imply that the heterotic gauge field strength F_2 and its Hodge dual F_8 jointly constitute a single cocycle (in the sense discussed in Section 2.3)

$$\mathcal{F}_{2\,\mathrm{mod}\,6} := (F_2, F_8) \in H_{\mathrm{dR}}^{2\,\mathrm{mod}\,6 + H_7}(X) \otimes \mathfrak{g}$$

in H_7-twisted de Rham cohomology (with coefficients in the gauge Lie algebra), where H_7 is the flux form to which the NS5-brane couples.

(iii) In [Sa09] it was furthermore observed that this is directly analogous to how the RR-fields in type II string theory, i.e. the fluxes corresponding to the cocycles $\mu_{{}_{Dp}}$ (40) to which the D-branes couple, as in Section 4.1, jointly constitute a cocycle in H_3-twisted de Rham cohomology. But under the lift of heterotic string theory to heterotic M-theory [HoWi96a, HoWi96b] the H_7 in (65) is identified with the flux corresponding to $\tilde{\mu}_{{}_{M5}}$ [FSS15d].

(iv) In conclusion this means that from perturbative string theory we expect non-trivial twisted cocycles in Prop. 4.17 to exist in degree 2 mod 6 (or in degree 3 mod 6 if they are subject to double dimensional reduction [FSS16b, Section 3]) and to correspond to the M-theoretic lift of the heterotic gauge field and its magnetic dual. Of course this is part of the open question for the M-theoretic origin of the gauge fields — see Remark 4.22 — and certainly deserves to be discussed elsewhere.

(v) There may be more twisted cocycles, of course. Indeed, Prop. 4.17 predicts that if $\mathcal{F}_{2\,\mathrm{mod}\,6} = (F_2, F_8)$ is a non-trivial twisted cocycle in degree 2 mod 6, then there must also be a, presently mysterious, spherical-T-dual twisted cocycle

$$\mathcal{F}_{5\,\mathrm{mod}\,6} := \mathcal{T}(F_2, F_8)$$

in degree 5 mod 6.

Finally it is curious to note that the derivation in [Sa09, Section 3] shows that the degree-7 twisted cocycle relation (65) is really a cohomological incarnation of the Green-Schwarz mechanism in heterotic string theory, the origin of all string theoretic grand unification.

Remark 4.19 (M5-brane twisted cocycles involving the exceptional volume element?). The restriction to the "reduced" twisted cohomology groups $\tilde{H}^{\bullet+\mu_{M5,s}}$ (Prop. 4.15) from Def. 4.16 means that the spherical T-duality on exceptional superspace from Prop. 4.17 would be somewhat "blurred" on those Spin(10, 1)-invariant cocycles $\tilde{\mu}_{M5}$-twisted cocycles, if any, which involve summands that are multiples $f(\psi) \wedge \mathrm{vol}_{528}$ of the exceptional volume form vol_{528} and Spin-invariant combinations $f(\psi)$ of the super-vielbein. It seems plausible that in fact no such cocycles exist; we will revisit this elsewhere.

4.6. Exceptional generalized supergeometry

Having established spherical T-duality on $\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s}$ (in Prop. 4.17) we discuss the relevance of Prop. 4.14 for the exceptional generalized geometry of the supergravity C-field (see Remark 4.7). In fact, Prop. 4.17 will be seen as establishing a duality on the exceptional moduli space of C-field configurations.

First notice the "moduli problem" for C-field configurations: The (2,2)-component of the C-field strength G_4 on super-spacetime is constrained to equal the M2-brane super cocycle (41) in every super tangent space. However, under the identification of super tangent spaces with $\mathbb{R}^{10,1|\mathbf{32}}$ established by the super vielbein $(E^A) := (e^a, \psi^\alpha)$ [BST87, (145)]

$$(G_4)_{(2,2)} = \underbrace{\tfrac{i}{2}(\Gamma_{ab})^\alpha{}_\beta \overline{\psi}_\alpha \wedge \psi^\beta \wedge e^a \wedge e^b}_{\mu_{M2}}$$

this still leaves, even in the absence of any bosonic flux, hence even if

$$(G_4)_{4,0} = 0\,,$$

a moduli space of possible C-field configurations. That is, the C-field itself is, of course, locally a differential 3-form potential C for G_4

$$dC = \mu_{M2}\,.$$

Here we are identifying via Prop. 2.21

$$\mu_{M2} \in \mathrm{CE}(\mathbb{R}^{10,1|\mathbf{32}}) \simeq \Omega^4_{\mathrm{LI}}(\mathbb{R}^{10,1|\mathbf{32}}) \hookrightarrow \Omega^4(\mathbb{R}^{10,1|\mathbf{32}})$$

with a super-differential 4-form on $\mathbb{R}^{10,1}$ that happens to be left-invariant (hence invariant under super-translations).

Now, of course, the actual C-field $C \in \Omega^3(\mathbb{R}^{10,1|\mathbf{32}})$ is never left-invariant, because if it were then $dC = \mu_{{}_{M2}}$ would mean that $[\mu_{{}_{M2}}] = 0 \in H^\bullet(\mathfrak{g}) \simeq H^\bullet_{\mathrm{dR,LI}}(\mathbb{R}^{10,1|\mathbf{32}})$, which is not the case. But, in fact, the C-field itself is to be thought of as part of the connection data on a 2-gerbe whose curvature 4-form is $\mu_{{}_{M2}}$, and the transformation properties for this connection data is more flexible in that it allows invariance up to higher gauge transformation. According to [SSS09, Section 2.2.2],[FSS12, Def. 4.3.6] such 2-gerbe connections may be encoded by transgression elements as in Def. 2.19. For the M2-cocycle $\mu_{{}_{M2}}$ such a transgression element is precisely what Prop. 4.14 establishes: The transgression element is the decomposed C-field (54) in the D'Auria-Fré algebra [D'AF82, Section 6], [BAIPV04, Section 3].

Consequently, Example 2.22 says that the transgression of $\mu_{{}_{M2}}$ (41) on $\mathbb{R}^{10,1|\mathbf{32}}$ allows to obtain well-behaved C-field configurations by pullback along linear sections σ of the exceptional super spacetime regarded as a bundle over 11d super-spacetime:

$$\begin{tikzcd}
\textbf{Moduli space} & \mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s} \arrow[dd, "\pi_{\mathrm{exc},s}"] & & \\
\textbf{Classifying map} & & \Longrightarrow & d\underbrace{(\sigma^* c_{\mathrm{exc},s})}_{\substack{\text{C-field}\\ \text{configuration}}} = \mu_{{}_{M2}}\,. \\
\textbf{Spacetime} & \mathbb{R}^{10,1|\mathbf{32}} \arrow[uu, bend left, "\sigma"] & &
\end{tikzcd}$$

Therefore:

1) *Each of the fermionic extensions* $\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s}$ *(Def. 4.13) of the exceptional super-spacetime* $\mathbb{R}^{10,1|\mathbf{32}}$ *(Def. 4.5), for each value of* $s \in \mathbb{R} \setminus \{0\}$*, serves as a moduli space for C-field configurations.*

2) *The decomposed C-field* $c_{\mathrm{exc},s} \in \mathrm{CE}(\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s})$ *(Prop. 4.14) is the corresponding universal field on the moduli space, whose pullback along classifying maps* σ *yield the actual C-field configurations on super-Minkowski spacetime.*

We illustrate how this works with the following basic examples.

Example 4.20 (C-fields via splitting of exceptional tangent bundle). Let $C \in \Omega^3(\mathbb{R}^{10,1|\mathbf{32}})$ be a bosonic differential 3-form, with components $C = C_{a_1 a_2 a_3} dx^a \wedge dx^b \wedge dx^c$ then a section is obtained by contraction of vectors in C

(66)
$$\begin{tikzcd}
v + \iota_v C & \mathbb{R}^{10,1} \oplus \wedge^2(\mathbb{R}^{10,1}) \oplus \wedge^5(\mathbb{R}^{10,1}). \\
v \arrow[u, mapsto] & \mathbb{R}^{10,1} \arrow[u, "\sigma_C"']
\end{tikzcd}$$

Pullback along this section of the generators

$$B_{ab} \in \mathrm{CE}(\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s}) \simeq \Omega^\bullet_{\mathrm{LI}}(\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s},$$

from Prop. 4.6 yields the corresponding "wrapping modes" of C, namely

$$\sigma_C^* B_{ab} = C_{abc} dx^c,$$

and hence the pullback of the decomposed C-field (55) has bosonic component (56) which at $s = -3$ (see (58)) reproduces the 3-form C:

$$\begin{aligned}
\sigma_C^*(c_{\mathrm{exc},-3})_{\mathrm{bos}} &= \sigma_C^*(B_{ab} \wedge e^a \wedge e^c) \\
&= C.
\end{aligned}$$

This is the way that C-field configurations have been proposed to be encoded by *exceptional generalized geometry* in [H07b, PW08, Bar12]; see also Remark 4.7.

However, notice that this looks different at different values of s:

Example 4.21 (Decomposed bosonic C-field at different values of s). For other values of s, the very same section σ_C (66) induces *different* C-field configurations via pullback. This is due to the second summand $\propto$ ${B^a}_b \wedge {B^b}_c \wedge {B^c}_a$ in (56), namely a linear combination of C with the 3-form

$$\sigma_C^*({B^a}_b \wedge {B^b}_c \wedge {B^c}_a) \propto \mathrm{CS}(A),$$

where on the right we have the (flat) Chern-Simons form for C regarded as an $\mathfrak{so}(10,1)$-valued differential 1-form A,

$$(A_\mu)^a{}_b := C_\mu{}^a{}_b\,.$$

Remark 4.22 (Non-abelian gauge degrees of freedom in M-theory). It is a famous open problem that the following two facts are superficially incompatible:

1) On the one hand, M-theory must contain avatars of nonabelian gauge fields, since these are seen in its string theoretic weak coupling limit in various guises.
2) On the other hand, its low-energy-limit in the form of 11d supergravity theory only contains, with the C-field, an abelian, albeit higher-, gauge field.

More specifically, various anomaly-cancellation arguments (see [FSS15a] for review and homotopy-theoretic discussion in line with the present perspective) show that the supergravity C-field ought to contain a contribution that is locally given by the Chern-Simons 3-form of non-abelian gauge field, which plain 11d supergravity knows nothing about such a Chern-Simons summand. However, notice that in the perspective on M-theory via super L_∞-homotopy theory, the equations of motion of 11d supergravity are the consequence of just one of several super tangent space-wise super L_∞-algebraic structures, namely of the torsion constraint (1), which implies the equations of motion of 11d supergravity by [CL94, Ho97]. But, in addition, there is also the constraint (3) on the M2-brane's WZW term. If one demands that this constrained be solved by super L_∞-algebraic means, namely by transgression (Def. 2.19), then Example 4.21 shows that this makes the Chern-Simons term of a nonabelian gauge field appear as a summand of the C-field. See [FSS14] for details on how the latter arises in the context of multiple M5-brane theories.

Finally, we observe that the perspective of exceptional generalized geometry allows us to explain the "meaning" of the extra fermionic generators η^α from (53), which was the cause of some concern in [ADR16]:

Example 4.23 (Interpreting the extra fermionic generator via heterotic M-theory). A linear section of the exceptional super-tangent bundles with non-vanishing component in the extra fermion generators is given in particular by a choice of fermion (χ^α) via

$$\begin{tikzcd}
v + (v^a (\Gamma_a)^\alpha{}_\beta \chi^\beta) \\
v \arrow[u, mapsto, "\sigma_\chi"']
\end{tikzcd}$$

Pullback along this section of the extra fermionic generators from (53)

$$\eta^\alpha \in \mathrm{CE}(\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s}) \simeq \Omega^\bullet_{\mathrm{LI}}(\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s})$$

yields

$$\sigma_\chi^*(\eta^\alpha) = dx^a (\Gamma_a)^\alpha{}_\beta \chi^\beta$$

and hence takes the following value on the fermionic component (57) of the decomposed C-field:

$$\sigma_\chi^*((c_{\mathrm{exc},s})_{\mathrm{ferm}}) \propto \overline{\psi}_\alpha (\Gamma_{ab})^\alpha{}_\beta \chi^\beta e^a \wedge e^b \,.$$

After an identification of bulk fermions with Hořava-Witten-boundary fermions as in [ES03], this is the form in which the dilatino has to appear as a summand in the supergravity C-field in heterotic M-theory, according to [ADR86, (4.21)].

Note that this works because our identification of the bosonic part of the D'Auria-Fré (DF) algebra (Def. 4.13) with the exceptional tangent bundle of [H07b] immediately implies that we have to interpret the extra fermionic component of the DF algebra as providing the supersymmetrization of the exceptional generalized geometry proposal for M-theory.

4.7. Spherical T-duality of M5-branes over 7d super-spacetime

Under dimensional reduction, the non-wrapping part of the M2/M5-brane supercocycle remains an M2/M5-brane supercocycle in lower dimensions with higher supersymmetry. In some lower dimensions it remains even at minimal supersymmetry. This is the case notably in $d = 7$. (Notice that the $N = 2$ supergravity in seven dimensions [TvN83][SS83] can be obtained from eleven dimensions in a consistent way [Ka14].)

The direct analog of the spherical T-duality for M5-branes from Example 4.3 holds in 7d . Instead of repeating the whole applicable discussion in detail, we encapsulate this in the following.

Remark 4.24. **(i)** The cochains on minimal 7d super-Minkowski spacetime

$$\mu^{7d}_{M2}, \quad \mu^{7d}_{M5} \in \mathrm{CE}\big(\mathbb{R}^{6,1|\mathbf{16}}\big)$$

satisfy the analog of the relation (42) (see also [ADR16, expression (4.36)]):

$$d\mu^{7d}_{M2} = 0 \qquad \text{and} \qquad d\mu^{7d}_{M5} = -\tfrac{1}{2}\mu^{7d}_{M2} \wedge \mu^{7d}_{M2}\,, \tag{67}$$

hence constitute a rational 4-sphere valued supercocycle

$$\big(\mu^{7d}_{M2},\, \mu^{7d}_{M5}\big) \;:\; \mathbb{R}^{6,1|\mathbf{16}} \longrightarrow \mathfrak{l}(S^4).$$

(ii) Consequently, the discussion in Section 4.3 applies and provides an example of topological spherical T-duality (Def. 3.10) for super M5-branes on the corresponding M2-extended super-spacetimes just as in Prop. 4.4

$$\begin{tikzcd}
b^6\mathbb{R} & \mathfrak{m2brane}^{7d} \arrow[l, "\widetilde{\mu}^{7d}_{M5}"'] \arrow[dr, "\pi^A:=\mathrm{hofib}(\mu^{7d}_{M2})"'] & & \mathfrak{m2brane}^{7d} \arrow[r, "\widetilde{\mu}^{7d}_{M5}"] \arrow[dl, "\pi^B:=\mathrm{hofib}(\mu^{7d}_{M2})"] & b^6\mathbb{R} \\
 & & \mathbb{R}^{6,1|\mathbf{16}} \arrow[dl, "\mu^{7d}_{M2}"] \arrow[dr, "\mu^{7d}_{M2}"'] & & \\
 & b^3\mathbb{R} & & b^3\mathbb{R} &
\end{tikzcd}$$

$$\big(\mathfrak{m2brane}, \widetilde{\mu}^{7d}_{M5}\big) \xleftrightarrow{\quad\mathcal{T}\quad} \big(\mathfrak{m2brane}, \widetilde{\mu}^{7d}_{M5}\big).$$

(iii) Moreover, there is again an exceptional superspacetime $\mathbb{R}^{6,1|\mathbf{16}}_{\mathrm{exc},s}$ as in Def. 4.13 over which the 7d C-field has a transgression element and decomposes as in Prop. 4.14; this is due to [ADR16, Section 4]. Accordingly there is a 7d analog of spherical T-duality on exceptional spacetime as in Prop. 4.17.

4.8. Parity isomorphism on brane charges over exceptional M-theory spacetime

In this section we prove a "parity isomorphism" for $\widetilde{\mu}_{M5}$-twisted cohomology on exceptional superspacetime (Prop. 4.27 below), different from but akin to the spherical T-duality isomorphism in Prop. 4.17. Before we construct the isomorphism in Prop. 4.26 below, first we survey some background on the role that such an isomorphism may be expected to play in M-theory.

The parity symmetry appears at the level of supergravity as the transformation $C_3 \mapsto -C_3$ together with an odd number of space or time reflections [DLP86]. In M-theory, parity acts by orientation-reversal, together with $G_4 \to -G_4$.

The considerations of parity in M-theory affect the description of some of its objects. For instance, the usual requirement that the fivebrane worldvolume be oriented can be relaxed, by virtue of M-theory conserving parity [Wi97]. For the membrane, the theories describing multiple membranes should preserve parity, which places constraints on the structure constants appearing in the Lagrangian [BLMP13]. Chern-Simons-matter theories describing fractional M2-branes [ABJ08], arising from backgrounds $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ with torsion class in $H^4(S^7/\mathbb{Z}_k;\mathbb{Z}) \cong \mathbb{Z}_k$, lead to symmetries of the form $U(n+\ell)_k \times U(n)_{-k}$, where k is the Chern-Simons level. The parity symmetry acting as $C_3 \mapsto -C_3$ takes k to $-k$ and then $G_4 \mapsto k - G_4$, so that the rank ℓ goes to $\ell - k$ in structure groups of level $-k$ [ABJ08].

From global geometric and topological perspective, M-theory is parity invariant, and so should in principle be formulated in a way that makes sense on unoriented, and possibly orientable manifolds (see [Mo04]). How to formulate M-theory on such manifolds? The parity problem was discussed in [DFM07] [Mo04] with the purpose of extending the E_8-model to unoriented manifolds, via orientation double covers and Deck transformations. The authors point out (see particularly the summary discussion in [Mo04]) that this problem is still unsatisfactorily addressed and hence a proper formulation is still lacking.

Now we consider the degree four field to be captured by an E_8 bundle over spacetime, as in [Wi97][DMW03] [DFM07], characterized by an integral degree four class a. In terms of this, the quantization condition $a = G_4 + \frac{1}{2}\lambda$ holds, where $\lambda = \frac{1}{2}p_1$ is the first Spin characteristic class. The parity transformation acts on the degree four classes as

$$a \mapsto \lambda - a \qquad \text{and} \qquad G_4 \mapsto -G_4.$$

In this global formulation, one has

- When the first Spin characteristic class $\lambda = \frac{1}{2}p_1$ is zero, i.e. on String manifolds, then this is simply $a \mapsto -a$. Note that considering such higher structures in the rational setting has been discussed extensively in [SW].
- When considering M-theory on a circle, $Y^{11} = S^1 \times X^{10}$, the parity symmetry acts by reflection of the S^1 factor together with $a \mapsto \lambda - a$.
- If $G_4 = 0$ in cohomology then the corresponding configuration would be parity-invariant.

Observe that when we take our spacetimes to be a String manifold (as done in [Sa09][Sa11]) then the parity transformation acts simply by a minus sign on both a and G_4. We could then take the degree four configurations to correspond to two E_8 bundles, related by a parity transformation. We will aim to find a home of this transformation in the context of higher rational T-duality.

Example 4.25 (Parity as rational T-duality of E_8 bundles). Since the homotopy group of E_8 are concentrated in degrees $(3, 15, \ldots)$, the group E_8 has the same homotopy type as $K(\mathbb{Z}, 3)$ up to degree 14, and hence the classifying spaces BE_8 and $K(\mathbb{Z}, 4)$ have the same homotopy type up to dimension 15. We now make the further observation that E_8 and $SU(2)$ have the same rational homotopy and cohomology in the above range. This means that overall we have the identifications

$$\text{“}E_8 \simeq_{14,\mathbb{Q}} K(\mathbb{Q}, 3) \simeq_{\mathbb{Q}} SU(2) \simeq_{\mathbb{Q}} S^3\text{”}.$$

Consequently, within equivalence in rational homotopy theory we are free to view our E_8 bundle over an 11-dimensional base space as a 3-sphere bundle. This is then summarized as follows

$$\begin{tikzcd}
S^3_{\mathbb{Q}} \simeq K(\mathbb{Q}, 3) \simeq_{14,\mathbb{Q}} E_8 \arrow[r] & E \arrow[dr, "\pi"] & & E' \arrow[dl, "\pi'"'] & E_8 \simeq_{\mathbb{Q}} K(\mathbb{Q}, 3) \simeq S^3_{\mathbb{Q}} \arrow[l] \\
& & Y & &
\end{tikzcd}$$

Taking the class of the bundle E to be a and the class of the bundle E' to be $-a$ then puts the two bundles as a parity dual pair, which fits into our discussion of T-duality for rational sphere bundle as a special case. A parity-invariant formulation of the E_8 model is given in [DFM07] by passing to the orientation double cover Y_d of spacetime Y and declaring the C-field to be the parity invariant cocycle on Y_d. Hence we could use Y_d in place of Y.

We now return to our supercocycles and study the effect of parity on them.

Proposition 4.26 (Reflection automorphism on exceptional tangent super-spacetime). *There is an action of* $\mathbb{Z}/2$ *on* $\mathrm{CE}(\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s})$ *where the non-trivial element* $\rho \in \mathbb{Z}/2$ *acts dually by* [6]

---

[6] Beware that this is saying that the $B_{a_1 a_2}$-generator picks up a sign precisely if its indices do *not* take the value 10.

$$\rho^* \ : \ \left\{ \begin{array}{rl} e^a \longmapsto & \left\{ \begin{array}{cl} -e^a & \mid \ a = 10 \\ e^a & \mid \ \mathit{otherwise} \end{array} \right. \\ B_{a_1 a_2} \longmapsto & \left\{ \begin{array}{cl} -B_{a_1 a_2} & \mid \ a_1, a_2 \neq 10 \\ B_{a_1 a_2} & \mid \ \mathit{otherwise} \end{array} \right. \\ B_{a_1 \cdots a_5} \longmapsto & \left\{ \begin{array}{cl} -B_{a_1 \cdots a_5} & \mid \ \mathit{one\ of\ the}\ a_i = 10 \\ B_{a_1 \cdots a_5} & \mid \ \mathit{otherwise} \end{array} \right. \\ \psi \longmapsto & \ \ \Gamma_{10} \psi \\ \eta \longmapsto & -\Gamma_{10} \eta \end{array} \right.$$

Proof. It is clear that ρ^* is an isomorphism in the underlying graded algebra. What needs to be checked is that it does respect the differentials. Hence first we need to show that under ρ^* the bispinorial expressions $\overline{\psi}\Gamma^a\psi$, $\overline{\psi}\Gamma^{a_1 a_2}\psi$, $\overline{\psi}\Gamma^{a_1 \cdots a_5}\psi$ pick up the same signs as the corresponding elements e^a, $B^{a_1 a_2}$ and $B^{a_1 \cdots a_5}$ do. This is equivalent to saying that their contractings are preserved by ρ^*. Using the identities $\overline{\psi} = \psi^\dagger \Gamma_0$, $(\Gamma_{10})^\dagger = -\Gamma_{10}$, and $\Gamma_{10}\Gamma_{10} = -1$ we establish ρ-invariance of the supercocycles as follows:

$$\begin{aligned} \rho(\overline{\psi}\Gamma_a\psi \wedge e^a) &= \sum_{0 \leq a \leq 9} \overline{\Gamma_{10}\psi}\Gamma_a \Gamma_{10}\psi \wedge e^a + \overline{\Gamma_{10}\psi}\Gamma_{10}\Gamma_{10}\psi \wedge (-e^{10}) \\ &= \sum_{0 \leq a \leq 9} \psi^\dagger(-\Gamma_{10})\Gamma_0 \Gamma_a \Gamma_{10}\psi \wedge e^a \\ &\quad + \psi^\dagger(-\Gamma_{10})\Gamma_0\Gamma_{10}\Gamma_{10}\psi \wedge (-e^{10}) \\ &= +\overline{\psi}\Gamma_a\psi \wedge e^a, \\ \rho(\tfrac{1}{2}\overline{\psi}\Gamma_{a_1 a_2}\psi \wedge B^{a_1 a_2}) &= \sum_{0 \leq a_1 < a_2 \leq 9} \overline{\Gamma_{10}\psi}\Gamma_{a_1 a_2}\Gamma_{10}\psi \wedge (-B^{a_1 a_2}) \\ &\quad + \sum_{0 \leq a \leq 9} \overline{\Gamma_{10}\psi}\Gamma_{a,10}\psi \wedge B^{a,10} \\ &= \sum_{0 \leq a_1 < a_2 \leq 9} \psi^\dagger(-\Gamma_{10})\Gamma_0\Gamma_{a_1 a_2}\Gamma_{10}\psi \wedge (-B^{a_1 a_2}) \\ &\quad + \sum_{0 \leq a \leq 9} \psi^\dagger(-\Gamma_{10})\Gamma_0\Gamma_{a,10}\psi \wedge B^{a,10} \\ &= +\tfrac{1}{2}\overline{\psi}\Gamma_{a_1 a_2}\psi \wedge B^{a_1 a_2}, \end{aligned}$$

and

$$\begin{aligned}
\rho(\tfrac{1}{5!}\overline{\psi}\Gamma_{a_1\cdots a_5}\psi\wedge B^{a_1\cdots a_5}) &= \sum_{0\le a_1<\cdots<a_5\le 9}\overline{\Gamma_{10}\psi}\Gamma_{a_1\cdots a_5}\Gamma_{10}\psi\wedge(B^{a_1\cdots a_5})\\
&\quad+\sum_{0\le a_1<\cdots<a_4\le 9}\overline{\Gamma_{10}\psi}\Gamma_{a_1\cdots a_4,10}\psi\wedge B^{a_1\cdots a_4,10}\\
&= \sum_{0\le a_1<\cdots<a_5\le 9}\psi^\dagger(-\Gamma_{10})\Gamma_0\Gamma_{a_1\cdots a_5}\Gamma_{10}\psi\wedge(B^{a_1\cdots a_5})\\
&\quad+\sum_{0\le a_1<\cdots<a_4\le 9}\psi^\dagger(-\Gamma_{10})\Gamma_0\Gamma_{a_1\cdots a_4,10}\psi\wedge B^{a_1\cdots a_4,10}\\
&= +\tfrac{1}{5!}\overline{\psi}\Gamma_{a_1\cdots a_5}\psi\wedge B^{a_1\cdots a_5}.
\end{aligned}$$

Similarly, the following computation shows that d and ρ^* commute

$$\begin{aligned}
d\rho^*(\overline{\eta}) &= d(\overline{-\Gamma_{10}\eta})\\
&= d\eta^\dagger\Gamma_{10}\Gamma_0\\
&= -d\overline{\eta}\Gamma_{10}\\
&= -\overline{\psi}\Gamma_a\Gamma_{10}e^a-\overline{\psi}\Gamma_{a_1a_2}\Gamma_{10}B^{a_1a_2}-\overline{\psi}\Gamma_{a_1\cdots a_5}\Gamma_{10}B^{a_1\cdots a_5}\\
&= -\psi^\dagger\Gamma_0(-\Gamma_{10})\Gamma_a\rho^*(e^a)-\psi^\dagger\Gamma_0(-\Gamma_{10})\Gamma_{a_1a_2}\rho^*(B^{a_1a_2})\\
&\quad-\psi^\dagger\Gamma_0(-\Gamma_{10})\Gamma_{a_1\cdots a_5}\rho^*(B^{a_1\cdots a_5})\\
&= \overline{\rho^*\psi}\Gamma_a\rho^*(e^a)+\overline{\rho^*\psi}\Gamma_{a_1a_2}\rho^*(B^{a_1a_2})+\overline{\rho^*\psi}\Gamma_{a_1\cdots a_5}\rho^*(B^{a_1\cdots a_5})\\
&= \rho^*\left(d\overline{\eta}\right).
\end{aligned}$$

□

We use this result to determine the effect on the M-brane supercocycles.

Proposition 4.27 (Parity symmetry of decomposed supergravity C-field). *Under the reflection automorphism ρ from Prop. 4.26 we have that*

(i) *the decomposed supergravity C-field (Prop. 4.14) changes sign: $\rho^*(c_{\mathrm{exc},s})=-c_{\mathrm{exc},s}$;*

(ii) *the decomposed M5-brane cocycle (63) is invariant: $\rho^*(\tilde{\mu}_{M5,s})=\tilde{\mu}_{M5,s}$.*

Proof. **(i)** The first statement follows by inspection. For instance, the transformation of the first summand $B_{ab}\wedge e^a\wedge e^b$ of the bosonic component

$(c_{\mathrm{exc},s})_{\mathrm{bos}}$ may be computed as follows:

$$\begin{aligned}\rho^*(B_{a_1a_2}\wedge e^{a_1}\wedge e^{a_2}) &= 2\sum_{0\leq a_1<a_2\leq 9}(-B_{a_1a_2})\wedge e^{a_1}\wedge e^{a_2}\\ &\quad+2\sum_{0\leq a\leq 9}B_{a10}\wedge e^{a}\wedge(-e^{10})\\ &= -B_{a_1a_2}\wedge e^{a_1}\wedge e^{a_2},\end{aligned}$$

while the transformation of the second summand may be computed as

$$\begin{aligned}&\rho^*\left(B^a{}_b\wedge B^b{}_c\wedge B^c{}_a\right)\\ &=\sum_{0\leq a,b\leq 9}(-B^a{}_b)\left(\sum_{0\leq c\leq 9}(-B^b{}_c)\wedge(-B^c{}_a)+B^b{}_{10}\wedge B^{10}{}_a\right)+\text{cyclic}\\ &=-B^a{}_b\wedge B^b{}_c\wedge B^c{}_a,\end{aligned}$$

and similarly for the other bosonic summands.

For the fermionic term $(c_{\mathrm{exc},s})_{\mathrm{ferm}}$ we already checked at the beginning of the proof of Prop. 4.26 that it becomes invariant under ρ^* if we replaced the factor of η by ψ. But the transformations of η and ψ under ρ^* are the same up to a minus sign.

(ii) Regarding the second statement, by the same reasoning as in proof of Prop. 4.26, we have

$$\rho^*(\mu_{{}_{M5}})=\rho^*(\tfrac{1}{5!}\overline{\psi}\Gamma_{a_1\cdots a_5}\psi\wedge e^{a_1}\wedge\cdots\wedge e^{a_5})=\mu_{{}_{M5}}$$

and

$$\rho^*(\mu_{{}_{M2}})=\rho^*(\tfrac{i}{2!}\overline{\psi}\Gamma_{a_1a_2}\psi\wedge e^{a_1}\wedge e^{a_2})=-\mu_{{}_{M2}}\,.$$

Hence the statement follows from the previous one:

$$\rho^*(\tilde{\mu}_{{}_{M5,s}})=\rho^*(2\mu_{{}_{M5}}+c_{\mathrm{exc},s}\wedge\mu_{{}_{M2}})=\tilde{\mu}_{{}_{M5,s}}\,.$$

□

Indeed, this is compatible with the results of parity in the topological sector in [DFM07]. As a direct consequence, we can establish the following.

Proposition 4.28 (Parity as an isomorphism on twisted cohomology). *For $s\in\mathbb{R}\setminus\{0\}$, the $\mathbb{Z}/2$-action (4.26) on the exceptional superspacetime $\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s}$ induces an isomorphism of its $\tilde{\mu}_{{}_{M5,s}}$-twisted cohomology*

(Def. 2.9):

$$H^{\bullet+\tilde{\mu}_{M5,s}}(\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s}) \xrightarrow[\simeq]{\rho^*} H^{\bullet+\tilde{\mu}_{M5,s}}(\mathbb{R}^{10,1|\mathbf{32}}_{\mathrm{exc},s}) \; .$$

Curiously, observe the following interesting effect on the top cocycle.

Proposition 4.29. *The 528-volume element (48) is invariant under the reflection automorphism of Prop. 4.26:*

$$\rho^*(\mathrm{vol}_{528}) = \mathrm{vol}_{528}$$

Proof. The factors in vol_{528} that change sign under ρ^* are 1. those e^a for which $a = 10$, of which there is one, which is an odd number; 2. those $B_{a_1 a_2}$ with $a_1 < a_2$ for which $a_1 \neq 10$ and $a_2 \neq 10$, of which there are $\binom{10}{2} = 45$, which is also an odd number; 3. those $B_{a_1 \cdots a_5}$ with $a_1 < \cdots < a_5$ for which $a_5 = 10$, of which there are $\binom{10}{4} = 210$, which is an even number.

In total this means that under ρ^* the element vol_{528} picks up an even number of signs, hence is fixed. ☐

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