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Symplectic coarse-grained dynamics: chalkboard motion in classical and quantum mechanics

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In the usual approaches to mechanics (classical or quantum) the primary object of interest is the Hamiltonian, from which one tries to deduce the solutions of the equations of motion (Hamilton or Schrödinger). In the present work we reverse this paradigm and view the motions themselves as being the primary objects. This is made possible by studying arbitrary phase space motions, not of points, but of (small) ellipsoids with the requirement that the symplectic capacity of these ellipsoids is preserved. This allows us to guide and control these motions as we like. In the classical case these ellipsoids correspond to a symplectic coarse graining of phase space, and in the quantum case they correspond to the "quantum blobs" we defined in previous work, and which can be viewed as minimum uncertainty phase space cells which are in a one-to-one correspondence with Gaussian pure states.

1. Introduction

In traditional classical and quantum mechanics it is assumed that the Hamiltonian function (or its quantization) is given and one thereafter sets out to solve the corresponding dynamical equations (Hamilton or Schrödinger). In the present paper we reverse this paradigm by considering the primary objects as being motions, classical or quantum. These motions are not defined by their actions on points, but rather on ellipsoids with constant symplectic capacity, as motivated by our discussion above. We will see that there is a great latitude in choosing these motions, justifying our use of the metaphor "chalkboard motion": these motions can be compared to chalk drawings on a blackboard leaving a continuous succession of thick points. The surprising fact, which originally motivated our study, is that chalkboard motions are indeed Hamiltonian, but this in a very simple and unexpected way. We will be able to construct such motions at will: exactly as when one stands in front of a blackboard and uses a piece of chalk to make a drawing – except that in our case the blackboard is infinite and multidimensional, and the drawing consists of paths left by moving ellipsoids.

As we will see, these constructs allow us to define a quantum phase space, obtained from the usual Euclidean phase space using a coarse-graining by minimum uncertainty ellipsoids (we have dubbed these ellipsoids "quantum blobs" elsewhere [31]). We will then be able to define a "chalkboard motion" in this quantum phase space by quantizing the classical chalkboard motions; this will again give us great latitude in "piloting" and controlling at each step these quantum motions. As we will see, this procedure has many advantages, in particular that of conceptual and computational simplicity. Admittedly, the term "quantum phase space" is usually perceived as a red herring in physics: some physicists argue that there can't be any phase space in quantum mechanics, since the notion of a well-defined point does not make sense because of the uncertainty principle. Dirac himself dismissed in 1945 in a letter to Moyal even the suggestion that quantum mechanics can be expressed in terms of classical-valued phase space variables (see Curtright *et al.* [16] for a detailed account of the Dirac–Moyal discussion). Still, most theoretical physicists use phase space techniques every day when they work with the Wigner functions of quantum states: these functions are defined on the classical phase space $\mathbb{R}^n_x \times \mathbb{R}^n_p$, and this does not lead to any contradictions: the datum of the Wigner function $W\psi$ of a state ψ is both mathematically and physically equivalent to the datum of the state itself. There are in truth many phase space approaches to quantum mechanics; see for instance [7, 55, 56] for various and sometimes conflicting points of view.

1.1. Introductory example

Let us consider the disk $D(\varepsilon) : x^2 + p^2 \leq \varepsilon^2$ in the phase plane $\mathbb{R}_x \times \mathbb{R}_p$. We smoothly deform this disk into an ellipse; such a deformation is represented by a family of real 2×2 matrices

$$S_t = \begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix} \tag{1}$$

and the disk thus becomes after time t the ellipse $D_t(\varepsilon) = S_t D(\varepsilon)$ represented by

$$(c_t^2 + d_t^2)x^2 + (a_t^2 + b_t^2)p^2 - 2(a_t c_t + b_t d_t)px \leq \varepsilon^2.$$

Assume now that the ellipses $S_t D(\varepsilon)$ all have the same area $\pi\varepsilon^2$; the family (S_t) must then consist of symplectic matrices, *i.e.* $\det S_t = a_t d_t - b_t c_t = 1$. This constraint implies that

$$(a_t^2 + b_t^2)(c_t^2 + d_t^2) = 1 + \mu_t^2$$

where we have set $\mu_t = a_t c_t + b_t d_t$ so we can rewrite the equation of $D_t(\varepsilon)$ in the form

$$\frac{1 + \mu_t^2}{a_t^2 + b_t^2} x^2 + (a_t^2 + b_t^2)p^2 - 2\mu_t px \leq \varepsilon^2$$

which shows that the ellipse $D_t(\varepsilon)$ can be obtained from the disk $D(\varepsilon)$ using, instead of S_t, the family of lower triangular matrices

$$R_t = \begin{pmatrix} \lambda_t^{-1} & 0 \\ \mu_t \lambda_t^{-1} & \lambda_t \end{pmatrix} \quad \textit{with} \quad \begin{cases} \lambda_t = 1/\sqrt{a_t^2 + b_t^2} \\ \mu_t = (a_t c_t + b_t d_t)/(a_t^2 + b_t^2) \end{cases} \tag{2}$$

or, equivalently,

$$R_t = \begin{pmatrix} 1 & 0 \\ \mu_t & 0 \end{pmatrix} \begin{pmatrix} \lambda_t^{-1} & 0 \\ 0 & \lambda_t \end{pmatrix}. \tag{3}$$

This shows, in particular, that any ellipse can be obtained from a disk with the same area and center using only a coordinate rescaling and a shear. All this actually becomes much more obvious if one recalls that every symplectic matrix can be factorized as a product of a shear, a rescaling, and a rotation: this is called the "Iwasawa factorization", which we will study in detail in Section 4.2. This implies that there is considerable redundancy when we let the S_t act on circular disks centered at the origin; in fact we could replace S_t with any product

$$S_t = \begin{pmatrix} 1 & 0 \\ \mu_t & 0 \end{pmatrix} \begin{pmatrix} \lambda_t^{-1} & 0 \\ 0 & \lambda_t \end{pmatrix} \begin{pmatrix} \cos\alpha_t & \sin\alpha_t \\ -\sin\alpha_t & \cos\alpha_t \end{pmatrix}$$

where α_t is a smoothly varying angle. We next note that the matrices S_t and R_t being symplectic, the families (S_t) and (R_t) can both be interpreted as Hamiltonian flows. These flows are generated by the quadratic time-dependent Hamiltonian functions

$$H_S(x,p,t) = \frac{1}{2}(a_t\dot{b}_t - \dot{a}_t b_t)p^2 + \frac{1}{2}(\dot{d}_t c_t - \dot{c}_t d_t)x^2 - (a_t\dot{d}_t - b_t\dot{c}_t)px \tag{4}$$

$$H_R(x,p,t) = \frac{1}{2}[2(\dot{\lambda}_t\lambda_t^{-1})\dot{\mu}_t - \mu_t]x^2 + (\dot{\lambda}_t\lambda_t^{-1})px \tag{5}$$

where the dots $\dot{}$ signify differentiation $\frac{d}{dt}$ with respect to t. (We invite the reader who is wondering by what magic we have obtained these two formulas to have a sneak preview of Sections 2.3 and 5.2.) The main observation we now make is that the Hamiltonian H_R lacks any term in p^2; it does not produce any kinetic energy. It is a simple (but dull) exercise to show that while the Hamilton equations of motion corresponding to respectively (4) and (5) are different they lead to the same deformation of the initial disk $D(\varepsilon)$. Here is an elementary example: consider the family of symplectic matrices

$$S_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}; \tag{6}$$

it corresponds to a particle with mass one freely moving along the x-axis and it is thus the flow of the elementary Hamiltonian $H_S = \frac{1}{2}p^2$. The reduced Hamiltonian H_R, generating the flow (R_t), is given by

$$H_R = -\frac{1}{2(1+t^2)}x^2 + \frac{tpx}{1+t^2}. \tag{7}$$

Let us now go one step further: while deforming the disk as just described, we simultaneously move its center along an arbitrary curve $z_t = (x_t, p_t)$ starting from the origin at time $t = 0$. Assuming that z_t is continuously differentiable we can view it as a Hamiltonian trajectory, and this in many ways. The simplest choice is to take the "translation Hamiltonian" $H = p\dot{x}_t - x\dot{p}_t$ whose associated Hamilton equations are $\dot{x} = \dot{x}_t$ and $\dot{p} = \dot{p}_t$. As time elapses, the disk is being stretched and deformed while moving along z_t, and at time t it has become the ellipsoid $T(z_t)R_tD(\varepsilon)$ where $T(z_t)$ is the translation

$z \longmapsto z + z_t$. An absolutely not obvious fact is that the motion of this ellipsoid is *always* Hamiltonian! In fact, it corresponds to the inhomogeneous quadratic Hamiltonian function

$$H = H_S(x,p,t) + (a_t\dot{x}_t + b_t\dot{p}_t)p - (c_t\dot{x}_t + d\dot{p}_t) \tag{8}$$

where the first term H_S is given by formula (4) and represents the deformation, while the second term corresponds to the motion. It is not difficult to see that we can actually recover *all* time-dependent Hamiltonian functions of the type

$$H = \alpha_t x^2 + \beta_t p^2 + \gamma_t px + \delta_t p + \varepsilon_t x$$

that is, all affine Hamiltonian flows by using "chalkboard motions"!

Let us now see what these manipulations become at the quantum level. Setting $\varepsilon = \sqrt{\hbar}$ the disk $D(\sqrt{\hbar}) = D(0, \sqrt{\hbar})$ corresponds to the Wigner ellipsoid of the standard Gaussian state

$$\phi_0(x) = (\pi\hbar)^{-1/4} e^{-x^2/2\hbar}.$$

Now, to the family (S_t) one associates canonically a family of unitary operators $(\widehat{S}_t)$ (these are the metaplectic operators familiar from quantum optics), and the deformation of $D(\sqrt{\hbar})$ by S_t corresponds to the action of the metaplectic operator $\widehat{S}_t$ on ϕ_0. Now in general the new function $\widehat{S}_t\phi_0$ is rather cumbersome to calculate. For instance, returning to the simple case of free motion (6) the function $\widehat{S}_t\phi_0$ is given by the integral

$$\widehat{S}_t\phi_0(x) = \left(\tfrac{1}{2\pi i\hbar t}\right)^{1/2} \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}\frac{(x-x')^2}{2t}} \phi_0(x')dx' \tag{9}$$

(this formula actually holds for any ϕ_0, not just the standard coherent state); after some calculations involving the Fresnel formula for Gaussian integrals one finds that

$$\widehat{S}_t\phi_0(x) = \frac{1}{(\pi\hbar)^{1/4}} \frac{1}{\sqrt{1+it}} \exp\left(-\frac{x^2}{2(1+it)\hbar}\right) \tag{10}$$

which is well-known in the literature on coherent states [53]. If we now replace as in the geometric discussion above (S_t) with (R_t) we will have to replace $(\widehat{S}_t)$ with the corresponding family $(\widehat{R}_t)$ of metaplectic operators. It turns out that these are quite generally obtained from formula (3) by

$$\widehat{R}_t\phi_0(x) = e^{\frac{i}{2\hbar}tx^2}(1+t^2)^{-1/2}\phi_0((1+t^2)^{-1/2}x)$$

which leads to

$$\widehat{R}_t\phi_0(x) = \frac{i^{\phi(t)}}{(\pi\hbar)^{1/4}\sqrt{1+t^2}}\exp\left(-\frac{x^2}{2\hbar(1+t^2)}\right). \tag{11}$$

This is exactly formula (10) up to $i^{\phi(t)}$ where $\phi(t)$ is a phase coming from the argument of $1+it$. We have thus recovered the propagation formula for the standard coherent state without any calculation of integrals at all. In fact, this procedure works as well for arbitrary families (S_t): if S_t is given by (1) with $a_t d_t - b_t c_t = 1$ and then applying the corresponding metaplectic operator $\widehat{S}_t$ to ϕ_0 yields [53]

$$\widehat{S}_t\phi_0(x) = \frac{1}{(\pi\hbar)^{1/4}}\frac{i^{\phi(t)}}{\sqrt{a_t+ib_t}}\exp\left(-\frac{(d_t-ic_t)x^2}{2(a_t+ib_t)\hbar}\right). \tag{12}$$

We obtain the same result by applying the metaplectic version of (3), and this immediately yields

$$\widehat{R}_t\phi_0(x) = e^{-\frac{i}{2\hbar}\mu_t x^2}\lambda_t^{-1/2}\phi_0(\lambda_t x)$$

where $\lambda_t = (a_t^2+b_t^2)^{-1/2}$ and $\mu_t = a_t c_t + b_t d_t$ and this is seen to coincide with the expression above after some trivial calculations, that is we have

$$\widehat{S}_t\phi_0 = \widehat{R}_t\phi_0. \tag{13}$$

It should now be remarked that the time evolution of a Gaussian is – as is the evolution of any wavefunction – governed by a Schrödinger equation (at least in a nonrelativistic setting); for instance for the free particle considered above the function $\psi = \widehat{S}_t\phi_0$ satisfies

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2}\frac{\partial^2\psi}{\partial x^2}$$

as can be verified by a direct calculation using the explicit formula (10); the operator $\widehat{H} = -(\hbar^2/2)(\partial^2/\partial x^2)$ appearing in this equation is in fact the quantization of the free-particle Hamiltonian function $H = p^2/2$. However we have seen above (*cf.* the equality (13)) that we also have $\psi = \widehat{R}_t\phi_0$; since the operators $\widehat{R}_t$ correspond to the flow (R_t) generated by the Hamiltonian function (7), it seems plausible that the function ψ should satisfy the

Schrödinger equation corresponding to the quantization $\widehat{H}_R$ of this function, that is

$$i\hbar\frac{\partial\psi}{\partial t} = -t\left(\frac{2+t^2}{1+t^2}\right)x^2 - \frac{i\hbar t}{1+t^2}\frac{1}{2}\left(x\frac{\partial\psi}{\partial x} + x\frac{\partial\psi}{\partial x}\right).$$

It turns out that this guess is correct.

1.2. Why ellipsoids are so useful in classical and quantum mechanics

Mathematical points do not have any operative meaning in physics, be it classical or quantum. Points live in the Platonic realm and are epistemologically inaccessible. As Gazeau [23] jokingly notes "... *nothing is mathematically exact from the physical point of view*". What is however accessible to us are gross approximations, like the chalk dots on a blackboard to take one naive example. On a slightly more elaborate level, ellipsoids are good candidates as substitutes for points. Consider for instance a particle moving in the configuration space $\mathbb{R}^n_x$. We perform a succession of simultaneous position and momentum measurements and find a cloud of points concentrated in a small region of the phase space $\mathbb{R}^n_x \times \mathbb{R}^n_p$. Position and momentum measurements lead to a cloud of points, and we can use a method familiar from multivariate statistical analysis to associate to our cloud a phase space ellipsoid. It works as follows: after having eliminated possible outliers, we associate a phase space ellipsoid Ω of minimum volume containing the convex hull of the remaining set of points. This is the "John–Löwner ellipsoid" [5, 48, 52, 60] which plays an extremely important role not only in statistics [28, 61], but also in many other related and unrelated disciplines (*e.g.* convex geometry and optimization, optimal design, computational geometry, computer graphics, and pattern recognition). Since (ideally) the precision of measurements can be arbitrarily increased, the volume of Ω can become as small as we want: for every $\varepsilon > 0$ we can make a sequence of measurements such that $\mathrm{Vol}(\Omega) < \varepsilon$. What about quantum systems? We again perform measurements leading to a plot of points in phase space. But while we can arbitrarily decrease the volume of the John–Löwner ellipsoid Ω by increasing the precision of measurements in the classical case, this does not work out in quantum mechanics because of the uncertainty principle (for the precision limits in quantum metrology see the recent review article [41]). Suppose in fact that we project Ω on the planes of conjugate variables $\mathcal{P}_j = \mathcal{P}(x_j, p_j)$. We thus obtain n ellipses $\omega_1, \omega_2, \ldots, \omega_n$ contained in the planes $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ and the Heisenberg inequalities imply that we must have

$$\text{Area}(\omega_j) \gtrsim \hbar. \tag{14}$$

This does not however lead to any estimate on the volume of the ellipsoid Ω as one is tempted to believe by inference from the classical case [58]; in particular it does *not* imply that phase space is coarse grained by minimum cells with volume $\sim \hbar^n$. Suppose indeed the volume of the smallest John–Löwner ellipsoid Ω is, say,

$$\text{Vol}(\Omega) = \frac{\pi^n}{n!}\hbar^n \tag{15}$$

which is the volume of a phase space ball with radius $\sqrt{\hbar}$. The projection of this ball on each of the planes $\mathcal{P}_j$ of conjugate variables is a disk with area $\pi\hbar = \frac{1}{2}h$ in conformity with (14). However, if we choose for Ω any ellipsoid

$$\Omega : \sum_{j=1}^{n} \frac{x_j^2 + p_j^2}{R_j^2} \leq 1 \tag{16}$$

the equality (15) will still hold provided that $R_1^2 R_2^2 \cdots R_n^2 = \hbar^2$ but for this it is not necessary at all that the projections of Ω on all the planes $\mathcal{P}_j$ have area at least $\pi R_j^2 = \frac{1}{2}h$. For instance, in the case $n = 2$ the ellipsoid

$$\frac{x_1^2 + p_1^2}{N\hbar} + \frac{x_2^2 + p_2^2}{N^{-1}\hbar} \leq 1 \tag{17}$$

indeed has volume $\pi^2\hbar^2/2$ for every value of $N > 0$ and its projection on the $\mathcal{P}_1$ plane has area $N\pi\hbar$ but its projection on the $\mathcal{P}_2$ plane has area $N^{-1}\pi\hbar$ so the uncertainty principle is violated for large N. This discussion shows that the volume condition $\text{Vol}(\Omega) \sim \hbar^n$ is not sufficient to describe a phase space coarse graining; it does not allow us to tell whether a phase space ellipsoid (or more general phase space domains) is in compliance with the most basic feature of quantum mechanics, the uncertainty principle (which it would be better to call the *indeterminacy principle*).

1.3. Notation and terminology

We will equip $\mathbb{R}^{2n} = T^*\mathbb{R}$ with the standard symplectic structure $\sigma = \sum_{j=1}^{n} dp_j \wedge dx_j$ that is

$$\sigma(z, z') = p \cdot x' - p' \cdot x$$

where $z = (x, p)$, $z' = (x', p')$. In matrix notation $\sigma(z, z') = (z')^T Jz$ where

$$J = \begin{pmatrix} 0_{n\times n} & I_{n\times n} \\ -I_{n\times n} & 0_{n\times n} \end{pmatrix}.$$

The symplectic group of $\mathbb{R}^{2n}$ is denoted by $\mathrm{Sp}(n)$; it consists of all linear automorphisms of $\mathbb{R}^{2n}$ such that $S^*\sigma = \sigma$, that is $\sigma(Sz, Sz') = \sigma(z, z')$ for all $z, z' \in \mathbb{R}^{2n}$. Working in the canonical basis $\mathrm{Sp}(n)$ is identified with the group of all real $2n \times 2n$ matrices S such that $S^T JS = J$ (or, equivalently, $SJS^T = J$).

2. Hamiltonian flows

We review some results from symplectic mechanics, that is Hamiltonian mechanics expressed in the language of symplectic geometry; it is mainly concerned with geometric properties and ultimately cooks down to the study of symplectic isotopies. See for instance [22, 64] for the implementation of symplectic algorithms allowing the numerical resolution of Hamiltonian systems.

2.1. Symplectic matrices

It is convenient to fix a symplectic basis of $\mathbb{R}^{2n}$ and to identify the symplectic automorphisms of $(\mathbb{R}^{2n}, \sigma)$ with their matrices in that basis. We will mainly work in the canonical basis (which is both orthogonal and symplectic) and write symplectic matrices in block-form

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{18}$$

where the submatrices A, B, C, D all have same dimension $n \times n$. The condition $S^T JS = J$ can then be expressed as conditions on these submatrices; for instance

$$A^T D - C^T B = I_{\mathrm{d}}, \quad A^T C = CA^T, \quad B^T D = D^T B \tag{19}$$

or

$$AD^T - BC^T = I_{\mathrm{d}}, \quad AB^T = BA^T, \quad CD^T = DC^T. \tag{20}$$

Also, the inverse of S is explicitly given by

$$S^{-1} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix}. \tag{21}$$

See [26], §2.1, for details and additional material.

The symplectic group is generated by the set of all matrices J, V_{-P} and M_L where

$$V_{-P} = \begin{pmatrix} I & 0 \\ P & I \end{pmatrix}, \quad M_L = \begin{pmatrix} L^{-1} & 0 \\ 0 & L^T \end{pmatrix} \tag{22}$$

with $P = P^T$ and $\det L \neq 0$.

2.2. The group Ham(n)

Let I_T be the closed interval $[-T, T]$ with $T > 0$ (or $T = \infty$). A function $H \in C^\infty(\mathbb{R}^{2n} \times I_T, \mathbb{R})$ will be called a "Hamiltonian"; whenever necessary it will be convenient to assume that there exists a compact subset K of $\mathbb{R}^{2n}$ such that the support of H satisfies $\operatorname{supp} H(\cdot, t)$ is contained in K for all $t \in I_T$. This apparently restrictive assumption avoids problems arising with Hamiltonians defined on open manifolds which can have bad behavior at infinity, as discussed in [57], §1.3 (for instance, solutions to Hamilton's equations can blow up at finite time). When applied, this assumption guarantees that the flow generated by the Hamiltonian vector field $X_H = J\partial_z H$ exists for all $t \in I_T$. Some afterthought shows that this condition is after all not too stringent. Assume for instance that we want to deal with the standard physical Hamiltonian

$$H(x, p, t) = \frac{1}{2}|p|^2 + V(x, t).$$

The latter is never compactly supported. But for all (or most) practical purposes we want to study the Hamilton equations $\dot{x} = p$, $\dot{p} = -\partial_x V(x, t)$ in a bounded domain D of phase space. Choosing a compactly supported cutoff function $\chi \in C_0^\infty(\mathbb{R}^{2n})$ such that $\chi(z) = 1$ for $z \in \overline{D}$ the solutions $t \longmapsto (x, p)$ of the Hamilton equations for χH with initial value $z_0 \in D$ are just those of the initial problem $\dot{x} = p$, $\dot{p} = -\partial_x V(x, t)$ as long as (x, p) remains in D.

Let D be an open subset of $\mathbb{R}^{2n}$. A diffeomorphism $f : D \longrightarrow \mathbb{R}^{2n}$ is called a symplectomorphism (or "canonical transformation" [4]) if $f^*\sigma = f$, that is, the Jacobian matrix $Df(z)$ is symplectic at every $z \in D$:

$$(Df(z)^T)JDf(z) = Df(z)J(Df(z)^T) = J. \tag{23}$$

The symplectomorphisms of $D = \mathbb{R}^{2n}$ form a subgroup $\mathrm{Symp}(n)$ of the group $\mathrm{Diff}(n)$ of all diffeomorphisms of $\mathbb{R}^{2n}$: this easily follows from the relations (23) above using the chain rule. The symplectic group $\mathrm{Sp}(n)$ is a subgroup of $\mathrm{Symp}(n)$.

Let H be a Hamiltonian in the sense above. The Hamiltonian vector field $X_H = J\partial_z H$ is compactly supported and hence complete, so that Hamilton's equations

$$\dot{z}(t) = J\partial_z H(z(t), t), \quad z(0) = z_0 \tag{24}$$

have a unique solution for every choice of initial point $z_0 \in \mathbb{R}^{2n}$. The time-dependent flow (f_t^H) generated by X_H is the family of diffeomorphisms $z_0 \longmapsto z(t) = f_t^H(z_0)$ associating to z_0 the solution $z(t)$ at time $t \in I_T$. Each f_t^H is a symplectomorphism of M: $f_{t,t'}^H \in \mathrm{Symp}(n)$:

$$(Df_t^H(z))^T JDf_t^H(z) = Df_t^H(z)J(Df_t^H(z)^T) = J \tag{25}$$

where $Df_t^H(z)$ is the Jacobian matrix of f_t^H calculated at z.

A remarkable fact is that composition and inversion of Hamiltonian flows also yield Hamiltonian flows [26, 29, 33, 45, 57]. Let (f_t^H) and (f_t^K) be determined by two Hamiltonian functions $H = H(z,t)$ and $K = K(z,t)$; we have the following composition and inversion rules:

Proposition 1. *(i) Let H and K be Hamiltonians; we have:*

$$f_t^H f_t^K = f_t^{H\#K} \quad \textit{with} \quad H\#K(z,t) = H(z,t) + K((f_t^H)^{-1}(z), t) \tag{26}$$

$$(f_t^H)^{-1} = f_t^{\bar{H}} \quad \textit{with} \quad \bar{H}(z,t) = -H(f_t^H(z), t) \tag{27}$$

$$f_t^{H+K} = f_t^H f_t^{K'} \quad \textit{with } K'(z,t) = K(f_t^H(z), t). \tag{28}$$

(ii) The composition law # defined by (26) is associative: if L is a third Hamiltonian then

$$(H\#K)\#L = H\#(K\#L). \tag{29}$$

(iii) For every $g \in \mathrm{Symp}(n)$ we have the conjugation property

$$g^{-1} f_t^H g = f_t^{H\circ g}. \tag{30}$$

Proof. The proofs of these formulas are based on the transformation property $X_{g_*H} = g_*X_H$ of Hamiltonian vector fields for $g \in \mathrm{Symp}(n)$ [26, 33, 45, 57]. The associativity (29) follows from the associativity of the composition of mappings in a same space: $(f_t^H f_t^K) f_t^L = f_t^H (f_t^K f_t^L)$ hence $f_t^{(H\#K)\#L} = f_t^{H\#(K\#L)}$. □

Notice that even when H and K are time-independent Hamiltonians the functions $H\#K$ and $\bar{H}$ are generally time-dependent. Formula (30) is often expressed in physics by saying that "Hamilton's equations are covariant under canonical transformations" [4].

Let $f \in \mathrm{Diff}(n)$ such that $f = f_{t_0}^H$ for some Hamiltonian function H and time t_0. We will say that f is a *Hamiltonian symplectomorphism.* (Rescaling time if necessary one can always assume $t_0 = 1$.)

As immediately follows from formulas (26), (27), and (30) Hamiltonian symplectomorphisms form a normal subgroup $\mathrm{Ham}(n)$ of $\mathrm{Symp}(n)$. A somewhat surprising fact is that every continuous path of Hamiltonian symplectomorphisms passing through the identity is the phase flow of a Hamiltonian function; this was first proved by Banyaga [6] in the very general context of symplectic manifolds; see [33, 64] for elementary proofs:

Proposition 2. *Let (f_t) be a smooth one-parameter family of Hamiltonian symplectomorphisms such that $f_0 = I_{\mathrm{d}}$. Then (f_t) is the flow determined by the Hamiltonian function*

$$H(z,t) = -\int_0^1 \sigma(\dot{f}_t f_t^{-1}(\lambda z), z)\, d\lambda \tag{31}$$

where $\dot{f}_t = df_t/dt$.

We will call a smooth path (f_t), $t \in I_T$, in $\mathrm{Ham}(n)$ joining the identity to some element $f \in \mathrm{Ham}(n)$ a *Hamiltonian isotopy.*

2.3. Time-dependent quadratic Hamiltonians

Quadratic Hamiltonian functions are widely used both in theoretical and practical situations. They are easy to manipulate in the time-independent case and lead to rich structures (see the excellent review by Combescure and Robert [14]). Here we study general quadratic Hamiltonians with time-dependent coefficients.

Let $S \in \mathrm{Sp}(n)$. Since $\mathrm{Sp}(n)$ is arcwise connected we can find a C^1 path $t \longmapsto S_t$, $t \in I_T$ in $\mathrm{Sp}(n)$ joining the identity to S. We will write S_t in $n \times n$

block-matrix form

$$S_t = \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix}, \quad S_0 = I_{\mathrm{d}}. \tag{32}$$

Proposition 3. *(i)* (S_t) *is the phase flow determined by the quadratic Hamiltonian*

$$H(z,t) = -\frac{1}{2} J\dot{S}_t S_t^{-1} z^2; \tag{33}$$

(ii) the latter is explicitly given by

$$\begin{aligned} H = {} & \tfrac{1}{2}(\dot{D}_t C_t^T - \dot{C}_t D_t^T)x^2 - (\dot{D}_t A_t^T - \dot{C}_t B_t^T)p\cdot x \\ & + \tfrac{1}{2}(\dot{B}_t A_t^T - \dot{A}_t B_t^T)p^2. \end{aligned} \tag{34}$$

Proof. *(i)* In view of formula (31) we have

$$H(z,t) = -\int_0^1 \sigma\left(\dot{S}_t S_t^{-1}(\lambda z), z\right) d\lambda; \tag{35}$$

formula (33) follows since

$$\sigma\left(\dot{S}_t S_t^{-1}(\lambda z), z\right) = \lambda J\dot{S}_t S_t^{-1} z \cdot z.$$

(ii) The inverse of S_t is given by

$$S_t^{-1} = \begin{pmatrix} D_t^T & -B_t^T \\ -C_t^T & A_t^T \end{pmatrix} \tag{36}$$

and hence

$$J\dot{S}_t S_t^{-1} = \begin{pmatrix} \dot{C}_t D_t^T - \dot{D}_t C_t^T & \dot{D}_t A_t^T - \dot{C}_t B_t^T \\ \dot{B}_t C_t^T - \dot{A}_t D_t^T & \dot{A}_t B_t^T - \dot{B}_t A_t^T \end{pmatrix}; \tag{37}$$

formula (34) now follows from (33). □

In particular, if $S_t = e^{tX}$ with $X \in \mathfrak{sp}(n)$ (the symplectic Lie algebra) one recovers the usual formula $H = -\frac{1}{2}JXz^2$.

The group $U(n) = \mathrm{Sp}(2n) \cap O(2n, \mathbb{R})$ of symplectic rotations can be identified with the unitary group $U(n, \mathbb{C})$ via the embedding

$$U(n, \mathbb{C}) \ni X + iY \longmapsto \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \in U(n);$$

notice that X and Y must satisfy the conditions

$$XX^T + YY^T = I_\mathrm{d}, \quad XY^T - YX^T = 0 \tag{38}$$

$$X^T X + Y^T Y = I_\mathrm{d}, \quad X^T Y - Y^T X = 0. \tag{39}$$

Suppose that the symplectic isotopy consists of symplectic rotations

$$U_t = \begin{pmatrix} X_t & Y_t \\ -Y_t & X_t \end{pmatrix}, \quad U_0 = I_\mathrm{d}. \tag{40}$$

Proposition 3 implies:

Corollary 4. *(i) The family (U_t) of symplectic rotations (40) is the flow generated by the quadratic Hamiltonian*

$$H_U = \tfrac{1}{2} Z_t x^2 + \tfrac{1}{2} Z_t p^2 \tag{41}$$

where

$$Z_t = \dot{Y}_t X_t^T - \dot{X}_t Y_t^T = Z^T. \tag{42}$$

(ii) Conversely, every every Hamiltonian (41) such that $Z = Z^T$ generates a flow (U_t) with $U_t \in U(n)$ and thus uniquely determines X_t, Y_t such that (42) holds.

Proof. (i) In view of formula (33) (U_t) is generated by the Hamiltonian $H_U(z, t) = -\frac{1}{2} J \dot{U}_t U_t^{-1} z^2$. Noting that J commutes with both U_t and $\dot{U}_t$ it follows that $H_U(Jz, t) = H_U(z, t)$ and formula (41) follows from (34) since the cross terms are $(\dot{X} X^T + \dot{Y} Y^T) px = 0$. The matrix Z_t is symmetric: we have $X_t Y_t^T - Y_t X_t^T = 0$ hence, differentiating with respect to t, $\dot{Y}_t X_t^T - \dot{X}_t Y_t^T = X_t \dot{Y}_t^T - Y_t \dot{X}_t^T$. *(ii)* Let (U_t) be the flow determine by H_U. Since $U_t \in \mathrm{Sp}(n)$ for all t it is sufficient to show that in addition $U_t J = J U_t$.

Let $D_t = \begin{pmatrix} Z_t & 0 \\ 0 & Z_t \end{pmatrix}$. Since $\dot{U}_t = JD_tU_t$ we have

$$\frac{d}{dt}(U_tJ) = \left(\frac{d}{dt}U_t\right) J = JD_t(U_tJ);$$

similarly, since $D_tJ = JD_t$

$$\frac{d}{dt}(JU_t) = (-JD_tJ)JU_t = JD_t(JU_t)$$

hence U_tJ and JU_t satisfy the same first order differential equation with same initial value $U_0J = JU_0 = J$ so we must have $U_tJ = JU_t$. □

The method described above extends to the case of affine symplectic isotopies without difficulty:

Proposition 5. *Let (S_t) be a symplectic isotopy in $\mathrm{Sp}(n)$ and $I_T \ni t \longmapsto z_t$ a C^1 path in $\mathbb{R}^{2n}$ with $z_0 = 0$. (i) The affine symplectic isotopy (f_t) defined by $f_t = S_tT(z_t)$ is the phase flow determined by the Hamiltonian*

$$H(z,t) = -\frac{1}{2}J\dot{S}_tS_t^{-1}z^2 + \sigma(z, S_t\dot{z}_t) \tag{43}$$

and that defined by $g_t = T(z_t)S_t$ is

$$H(z,t) = -\frac{1}{2}J\dot{S}_tS_t^{-1}(z - z_t) + \sigma(z, \dot{z}_t). \tag{44}$$

(ii) Conversely, every Hamiltonian function

$$H(z,t) = \frac{1}{2}M(t)z^2 + m(t)z \tag{45}$$

with $M(t) = M(t)^T$ and $m(t) \in \mathbb{R}^{2n}$ depending continuously on $t \in I_T$ can be rewritten in the form (43) (or (44)).

Proof. (i) Formula (43) follows from formula (35) using the product formula (26) for Hamiltonian flows with $H = -\frac{1}{2}J\dot{S}_tS_t^{-1}z^2$ and $K = \sigma(z, \dot{z}_t)$, and noticing that $\sigma\left(S_t^{-1}z, \dot{z}_t\right) = \sigma(z, S_t\dot{z}_t)$. Formula (44) is proven likewise swapping H and K. *(ii)* Let (S_t) be the flow determined by the homogeneous part $H_0(z,t) = \frac{1}{2}M(t)z^2$ of $H(z,t)$. We have $\dot{S}_t = JM(t)S_t$ hence

$H_0(z,t) = -\frac{1}{2}J\dot{S}_t S_t^{-1}z^2$. Set now

$$z_t = \int_0^t S_{t'}^{-1} Jm(t')dt', \tag{46}$$

that is $\dot{z}_t = S_t^{-1}Jm(t)$; the Hamilton equations for (45) are

$$\dot{z}(t) = JM(t)z(t) + Jm(t) = \dot{S}_t S_t^{-1}z(t) + S_t\dot{z}_t$$

and are solved by $z(t) = S_t T(z_t)z(0)$; it follows that the flow (f_t) determined by H is given by $f_t = S_t T(z_t)$ and we can thus rewrite H as (43). □

Assume for instance that the coefficients M and m are time-independent and $\det M \neq 0$; the solution of Hamilton's equations

$$\dot{z}(t) = JMz(t) + Jm, \quad z(0) = z_0$$

are given by

$$z(t) = e^{tJM}z_0 + (JM)^{-1}(e^{tJM} - I)Jm. \tag{47}$$

(If M fails to be invertible, this formula remains formally correct, expanding e^{tJM} in a Taylor series [12].)

3. Symplectic capacities

We denote by $B^{2n}(z_0, R)$ the ball $|z - z_0| \leq R$ in $\mathbb{R}^{2n}$; we write $B^{2n}(0,R) = B^{2n}(R)$. Denoting by $T(z_0)$ the translation $z \longmapsto z + z_0$ we have $B^{2n}(z_0, R) = T(z_0)B^{2n}(R)$.

3.1. Definition of a symplectic capacity

It is well-known that Hamiltonian flows are volume preserving [4, 57]; this property is an easy consequence of the fact that any Hamiltonian flow consists of symplectomorphisms and hence preserves the successive powers σ, $\sigma \wedge \sigma, \ldots, \sigma^{\wedge n}$ of the symplectic form. This property is however not characteristic of Hamiltonian flows, because any flow generated by a divergence-free vector fields has this property. It however turns out that there exist quantities whose preservation is characteristic of Hamiltonian flows (and, more generally, of symplectomorphisms). These are the symplectic capacities of subsets of phase space.

A (normalized, or intrinsic) symplectic capacity on $(\mathbb{R}^{2n}, \sigma)$ assigns to every $\Omega \subset \mathbb{R}^{2n}$ a number $c(\Omega) \geq 0$, or $+\infty$, and must satisfy the following axioms [45]:

(SC1): Monotonicity: *If* $\Omega \subset \Omega'$ *then* $c(\Omega) \leq c(\Omega')$;

(SC2): Symplectic invariance: *If* $f \in \mathrm{Symp}(n)$ *then* $c(f(\Omega)) = c(\Omega)$;

(SC3): Conformality: *If* $\lambda \in \mathbb{R}$ *then* $c(\lambda\Omega) = \lambda^2 c(\Omega)$;

(SC4): Non-triviality: *We have* $c(B^{2n}(R)) = \pi R^2 = c(Z_j^{2n}(R))$ *where* $Z_j^{2n}(R)$ *is the cylinder* $x_j^2 + p_j^2 \leq R^2$ in $\mathbb{R}^{2n}$.

The archetypical example of a symplectic capacity is the "Gromov width" [45] defined by

$$c_{\min}(\Omega) = \sup_{f \in \mathrm{Symp}(n)} \{\pi R^2 : f(B^{2n}(R)) \subset \Omega\}; \tag{48}$$

that $c_{\min}$ indeed satisfies axiom (SC4) is a non-trivial property, equivalent to Gromov's symplectic non-squeezing theorem [38]: if there exists $f \in \mathrm{Symp}(n)$ such that $f(B^{2n}(R)) \subset Z_j^{2n}(r)$ then $R \leq r$ (see [26, 27, 37] for discussions of Gromov's result). As the notation suggests, $c_{\min}$ is the smallest of all symplectic capacities: $c_{\min} \leq c \leq c_{\max}$ where

$$c_{\max}(\Omega) = \inf_{f \in \mathrm{Symp}(n)} \{\pi R^2 : f(\Omega) \subset Z_j^{2n}(R)\} \tag{49}$$

so that we have $c_{\min} \leq c \leq c_{\max}$ for every symplectic capacity c on $(\mathbb{R}^{2n}, \sigma)$.

Let c be a symplectic capacity on the phase plane $\mathbb{R}^2$. Then $c(\Omega) = \mathrm{Area}(\Omega)$ for every connected and simply connected surface Ω. In higher dimensions the symplectic capacity can be finite while the volume is infinite: for instance the symplectic capacity of a cylinder $Z_j^{2n}(R)$ is finite, whereas its volume is infinite. It follows in fact from the monotonicity and non-triviality properties of a symplectic capacity that

$$B^{2n}(R) \subset \Omega \subset Z_j^{2n}(R) \implies c(\Omega) = \pi R^2$$

so Ω can have arbitrarily large volume (even infinite). Symplectic capacities are not related to volume when $n > 1$; for instance if Ω and Ω' are disjoint we do not in general have $c(\Omega \cup \Omega') = c(\Omega) + c(\Omega')$.

There exist infinitely many symplectic capacities, but they all agree on phase space ellipsoids. The symplectic capacity of an ellipsoid

$$\Omega = \{z : M(z - z_0)^2 \leq R^2\}$$

($M = M^T > 0$) is calculated as follows. Recall Williamson's symplectic diagonalization theorem [26, 45]: for every positive-definite symmetric real $2n \times 2n$ matrix M there exists $S \in \mathrm{Sp}(n)$ such that

$$S^T M S = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix} \tag{50}$$

where $\Lambda = \mathrm{diag}(\lambda_1^\sigma, \ldots, \lambda_j^\sigma)$ is the diagonal matrix whose diagonal entries λ_j^σ are the symplectic eigenvalues of M: $\lambda_j^\sigma > 0$ and the numbers $\pm i\lambda_j$ are the eigenvalues of JM (that these eigenvalues are indeed of the type $\pm i\lambda_j$ follows from the fact that JM has the same eigenvalues as the antisymmetric matrix $M^{1/2}JM^{1/2}$). This allows us to put the equation of the ellipsoid Ω in the diagonal form

$$\sum_j \lambda_j^\sigma((x_j - x_{0,j})^2 + (p_j - p_{0,j})^2) \leq R^2$$

and it is then easy to see [26, 29, 37], using Gromov's non-squeezing theorem that

$$c_{\min}(\Omega) = c_{\max}(\Omega) = \pi R^2/\lambda_{\max} \tag{51}$$

where $\lambda_{\max}$ is the largest symplectic eigenvalue of M. It follows that $c(\Omega) = \pi R^2/\lambda_{\max}$ for every symplectic capacity c.

3.2. A continuity property

Let Ω and Ω' be two nonempty compact subsets of $\mathbb{R}^{2n}$. The numbers

$$d_1(\Omega, \Omega') = \sup_{z\in\Omega} d(z, \Omega'), \quad d_2(\Omega, \Omega') = \sup_{z'\in\Omega'} d(z', \Omega)$$

are called, respectively, the directed Hausdorff distance from Ω to Ω' and from Ω' to Ω. The number

$$d_{\mathrm{H}}(\Omega, \Omega') = \max(d_1(\Omega, \Omega'), d_2(\Omega, \Omega')) \tag{52}$$

is called the *Hausdorff distance* of Ω and Ω'. (In some texts one defines this distance as the sum $d_1(\Omega, \Omega') + d_2(\Omega, \Omega')$; both choices of course lead to the same topology since the metrics are equivalent). The Hausdorff distance is a metric on the set $\mathcal{K}(2n)$ of all nonempty compact subsets of $\mathbb{R}^{2n}$ and $(\mathcal{K}(2n), d_{\mathrm{H}})$ is a complete metric space [21]. We have $d(\Omega, \Omega') < \infty$ since Ω

and Ω' are bounded. If $0 \in \Omega$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that [2]

$$d_{\mathrm{H}}(\Omega, \Omega') < \delta \Longrightarrow (1-\varepsilon)\Omega \subset \Omega' \subset (1+\varepsilon)\Omega. \tag{53}$$

Using (53) one shows that

Proposition 6. *Let c be a symplectic capacity on $(\mathbb{R}^{2n}, \sigma)$. The restriction of c to the set of all convex compact subsets equipped with the Hausdorff distance is continuous. That is, for every $\varepsilon > 0$ there exists $\delta > 0$ such that if Ω and Ω' are convex and compact then*

$$d_H(\Omega, \Omega') < \delta \Longrightarrow |c(\Omega) - c(\Omega')| < \varepsilon \tag{54}$$

(see *e.g.* [54], p.376).

The Hausdorff distance is used in pattern recognition and computer vision, and also plays an essential role in medical imaging [63].

3.3. Moving phase space ellipsoids

We now give a fundamental characterization of symplectomorphisms initially due to Ekeland and Hofer [20].

Let Ω be a phase space ellipsoid as above; in [20, 45] it is proven that the only C^1 mappings that preserve the symplectic capacities of all ellipsoids in $(\mathbb{R}^{2n}, \sigma)$ are either symplectic or antisymplectic (an antisymplectic mapping $F : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ is such that $F^*\sigma = -\sigma$; if F is linear and identified with its matrix this means that $F^T JF = -J$). In [17] we have proven a refinement of this result in the linear case, by showing that it is sufficient to consider a particular class of ellipsoids, called symplectic balls. By definition, a symplectic ball is the image of a phase space ball by an element of the inhomogeneous symplectic group $\mathrm{ISp}(n)$. It is thus an ellipsoid

$$B_S^{2n}(z_0, R) = T(z_0)SB^{2n}(R) \tag{55}$$

or, equivalently

$$B_S^{2n}(z_0, R) = ST(S^{-1}z_0)B^{2n}(R). \tag{56}$$

As follows from axioms (SC2) and (SC4) characterizing symplectic capacities, a symplectic ball has symplectic capacity

$$c(B_S^{2n}(z_0, R)) = c(B^{2n}(R)) = \pi R^2.$$

The proof of our refinement relies on the following algebraic result:

Lemma 7. *Let $F \in GL(2n,\mathbb{R})$. If $F^T M_L F \in \mathrm{Sp}(n)$ for every symplectic matrix*

$$M_L = \begin{pmatrix} L^{-1} & 0 \\ 0 & L \end{pmatrix}, \quad L = L^T > 0 \tag{57}$$

then F is either symplectic or antisymplectic: $F^T J F = \pm J$.

The proof of this lemma is rather long and technical, we therefore refer to [17], §1.2, for a detailed argument. The particular symplectic matrices (57) will play an essential role in Section 4.2 where we study the pre-Iwasawa factorization of general symplectic matrices. Notice that the matrices (57) do not form a subgroup of $\mathrm{Sp}(n)$: we have $M_L M_{L'} = M_{L'L}$ but in general $L'L$ is not symmetric if L and L' are.

Proposition 8. *(i) Assume that $K \in GL(2n,\mathbb{R})$ takes the symplectic ball $B_S^{2n}(z_0,R)$ to a symplectic ball $B_{S'}^{2n}(z_0',R)$ with the same radius. Then, K is either symplectic or antisymplectic. (ii) More generally, if K takes every ellipsoid in $\mathbb{R}^{2n}$ to an ellipsoid with the same symplectic capacity, then K is symplectic or antisymplectic.*

Proof. (i) Since translations are symplectomorphisms we can assume $z_0 = 0$ so that the symplectic ball is just $SB^{2n}(R)$ and thus defined by the inequality $|S^{-1}z| \leq R$. It follows that its image by K is the set of all $z \in \mathbb{R}^{2n}$ such that $|(KS)^{-1}z| \leq R$ that is

$$((KS)^{-1})^T (KS)^{-1} z^2 \leq R^2.$$

If $KSB^{2n}(R)$ is a symplectic ball we must thus have

$$((KS)^{-1})^T (KS)^{-1} = (K^T)^{-1}(SS^T)^{-1}K^{-1} \in \mathrm{Sp}(n).$$

Taking $F = K^{-1}$ then in view of Lemma 7 the matrix F and hence K must be either symplectic or antisymplectic. □

In the nonlinear case we have ([20], Thm. 4):

Proposition 9. *Let $f : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ be a C^1 diffeomorphism such that $c(f(\Omega)) = c(\Omega)$ for every ellipsoid $\Omega \subset \mathbb{R}^{2n}$. Then either $f^*\sigma = \sigma$ or $f^*\sigma = -\sigma$, that is f is either a symplectomorphism or an anti-symplectomorphism.*

In [20] (Thm. 5) Ekeland and Hofer prove the following nonlinear version of Proposition 9 using a mild differentiability requirement:

Proposition 10. *(i) Let $(f_t)_{t\in I_T}$ be a family of C^1 diffeomorphisms $\mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ such that $f_0 = I_{\mathrm{d}}$ and $c(f_t(\Omega)) = c(\Omega)$ for every ellipsoid $\Omega \subset \mathbb{R}^{2n}$ and $t \in I_T$. Then (f_t) is a Hamiltonian flow. (ii) If the f_t are affine mappings then H is a quadratic polynomial of the type (45).*

Proof. (i) Let $z \in \mathbb{R}^{2n}$; we have

$$(Df_t(z))^T J Df_t(z) = \pm J$$

for all $t \in I_T$. Since $Df_0(z) = z$ and the mapping $t \longmapsto Df_t(z)$ is continuous the only possible choice is

$$(Df_t(z))^T J Df_t(z) = J$$

and hence the f_t are symplectomorphisms; the conclusion now follows from Proposition 2; the Hamiltonian is given by

$$H(z,t) = -\int_0^1 \sigma(\dot{f}_t f_t^{-1}(\lambda z), z) d\lambda. \tag{58}$$

(ii) Immediately follows using Proposition 5. □

We urge the Reader to note that the assumption that symplectic capacities of ellipsoids – and not volumes – are preserved is essential. As soon as $n > 1$ the conclusions of Proposition 10 for families of mappings which are volume preserving are no longer true. Consider a divergence-free vector field on $\mathbb{R}^{2n}$; by Liouville's theorem [4] the flow it generates is certainly volume-preserving, but has no reason in general to preserve symplectic capacities. The properties above are of a topological nature, and show that general volume-preserving mappings cannot be approximated in the C^0 topology by symplectomorphisms. In this context we remark [33] that Katok [51] has shown that given two subsets Ω and Ω' with the same volume, then for every $\varepsilon > 0$ there exists $f \in \mathrm{Symp}(n)$ such that $\mathrm{Vol}(f(\Omega) \setminus \Omega') < \varepsilon$. Thus, an arbitrarily large part of Ω can be symplectically embedded inside Ω' – but not all of it! This again shows how different the notions of volume conservation and symplectic capacity conservation are.

4. Symplectic actions on ellipsoids

We introduce here the notion of local symplectic automorphisms; the terminology will be justified in Section 6.1 where we will show that these automorphisms are the projections on $\mathrm{Sp}(n)$ of local metaplectic operators, *i.e.* those which preserve the supports of functions or tempered distributions.

4.1. The local symplectic group

Let ℓ be a Lagrangian subspace of the symplectic phase space $(\mathbb{R}^{2n}, \sigma)$: $\dim \ell = n$ and σ vanishes identically on ℓ. Such a maximal isotropic subspace is also called a "Lagrangian plane". One proves ([26], §2.2) that for every $S \in \mathrm{Sp}(n)$ there exist $S_1, S_2 \in \mathrm{Sp}(n)$ such that $S_1\ell \cap \ell = S_2\ell \cap \ell = 0$ and $S = S_1 S_2$. Choosing for ℓ the momentum space $0 \times \mathbb{R}^n$ the symplectic matrices S_1, S_2 are of the type

$$S_j = \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix}, \quad \det B_j \neq 0 \tag{59}$$

with $n \times n$ blocks. (Such symplectic matrices are called *free.* We will return to them in Section 6.1.) A straightforward calculation leads to the factorization

$$S_j = V_{-D_j B_j^{-1}} M_{B_j^{-1}} J V_{-B_j^{-1} A_j} \tag{60}$$

where we define, for $P = P^T$ and $\det L \neq 0$,

$$V_{-P} = \begin{pmatrix} I & 0 \\ P & I \end{pmatrix}, \quad M_L = \begin{pmatrix} L^{-1} & 0 \\ 0 & L^T \end{pmatrix} \tag{61}$$

(DB^{-1} and $B^{-1}A$ are indeed symplectic due to the constraints imposed on the blocks A, B, C, D by the conditions $S^T J S = S J S^T = J$). It follows that the set of all matrices V_{-P} and M_L together with the standard symplectic matrix J generate $\mathrm{Sp}(n)$. Notice that these matrices obey the product formulas

$$V_{-P} V_{-P'} = V_{-(P+P')}, \quad M_L M_{L'} = M_{L'L}. \tag{62}$$

Let $\mathrm{St}(\ell)$ be the stabilizer of ℓ in $\mathrm{Sp}(n)$: it is the subgroup of all $S \in \mathrm{Sp}(n)$ such that $S\ell = \ell$. Of special importance for us is the case $\ell = 0 \times \mathbb{R}^n$, we will write $\mathrm{Sp}_0(n) = \mathrm{St}(0 \times \mathbb{R}^n)$. It consists of all symplectic block matrices with upper corner $B = 0$. Since

$$V_{-P}M_L = \begin{pmatrix} L^{-1} & 0 \\ PL^{-1} & L^T \end{pmatrix}, \quad M_L V_{-P} = \begin{pmatrix} L^{-1} & 0 \\ L^T P & L^T \end{pmatrix}$$

the group $\mathrm{Sp}_0(n)$ is generated by the symplectic matrices V_{-P} and M_L. It is thus the extension of the group $\{M_L : \det L \neq 0\}$ of symplectic rescalings by the group of symplectic shears $\{V_{-P} : P = P^T\}$. Using the obvious identities

$$M_L V_{-P} = V_{-L^T PL} M_L, \quad V_{-P} M_L = M_L V_{-(L^{-1})^T PL^{-1}} \tag{63}$$

and

$$(V_{-P}M_L)^{-1} = V_{-(L^{-1})^T PL^{-1}} M_{L^{-1}} \tag{64}$$

we see that the group $\mathrm{Sp}_0(n)$ in fact simply consists of all products $V_{-P}M_L$ (or $M_L V_{-P}$). In particular, if $S = V_{-P}M_L$ and $S' = V_{-P'}M_{L'}$ we have

$$S'S^{-1} = V_{-P'+(L^{-1}L')^T P(L^{-1}L')} M_{L^{-1}L'}. \tag{65}$$

The affine (or inhomogeneous) extension [12]

$$\mathrm{ISp}(n) = \mathrm{Sp}(n) \ltimes \mathbb{R}^{2n} \tag{66}$$

of the symplectic group consists of all products $ST(z) = T(Sz)S$ where $T(z)$ is the translation operator in $\mathbb{R}^{2n}$. Every element of $\mathrm{ISp}(n)$ can be written as either a product $ST(z_0)$, or a product $T(z_0)S$. For each element of $\mathrm{ISp}(n)$ this factorization is unique: for example if $ST(z_0) = S'T(z_0')$ then $(S')^{-1}S = T(z_0' - z_0)$ which is only possible if $z_0' = z_0$ and hence $S' = S$. We call the subgroup

$$\mathrm{ISp}_0(n) = \mathrm{Sp}_0(n) \ltimes \mathbb{R}^{2n} \tag{67}$$

the "local inhomogeneous symplectic group". It consists of all affine symplectic transformations of the type

$$T(z_0)V_{-P}M_L = V_{-P}M_L T(M_{L^{-1}}V_P z_0). \tag{68}$$

The main formulas are recapitulated in the table below:

$V_{-P}V_{-P'} = V_{-(P+P')}$	$M_L M_{L'} = M_{L'L}$
$M_L V_{-P} = V_{-L^T PL} M_L$	$V_{-P}M_L = M_L V_{-(L^{-1})^T PL^{-1}}$
$(V_{-P}M_L)^{-1} = V_{-(L^{-1})^T PL^{-1}} M_{L^{-1}}$	$T(z_0)V_{-P}M_L = V_{-P}M_L T(M_{L^{-1}}V_P z_0)$

4.2. The pre-Iwasawa factorization

We have seen in formula (60) that every symplectic matrix

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{69}$$

with $\det B \neq 0$ can be factorized as $S = V_{-P}M_L JV_{-P}$. The pre-Iwasawa factorization generalizes this result; it says that every $S \in \mathrm{Sp}(n)$ can be written as a product of an element of a subgroup of the local symplectic group $\mathrm{Sp}_0(n)$ and of a symplectic rotation. More precisely, writing $S \in \mathrm{Sp}(n)$ in block-matrix form ($n \times n$ blocks) there exist unique matrices $P = P^T$ and $L = L^T > 0$ and $U_{X,Y} \in U(n)$ such that

$$S = \begin{pmatrix} I & 0 \\ P & I \end{pmatrix} \begin{pmatrix} L^{-1} & 0 \\ 0 & L \end{pmatrix} \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} = V_{-P}M_L U_{X,Y}. \tag{70}$$

These matrices are given by

$$P = (CA^T + DB^T)(AA^T + BB^T)^{-1} = P^T \tag{71}$$

$$L = (AA^T + BB^T)^{-1/2} = L^T > 0 \tag{72}$$

$$X = (AA^T + BB^T)^{-1/2}A, \quad Y = (AA^T + BB^T)^{-1/2}B. \tag{73}$$

The proof of these formulas is purely computational; see [10, 19, 26, 65]. It is clear that $L = L^T$ is positive definite; that P is also symmetric follows from the fact that the relation $S = V_{-P}M_L U$ implies that V_{-P} is symplectic which requires that $P = P^T$. The uniqueness follows from the observation that if $V_{-P}M_L U_{X,Y} = V_{-P'}M_{L'}U_{X',Y'}$ then

$$M_{L'}^{-1} V_{P-P'} M_L = U_{X',Y'} U_{X,Y}^{-1} = U_{X'',Y''}$$

and thus

$$\begin{pmatrix} L'L^{-1} & 0 \\ (L')^{-1}(P - P')L & (L')^{-1}L \end{pmatrix} = \begin{pmatrix} X'' & Y'' \\ -Y'' & X'' \end{pmatrix}$$

hence $P = P'$ and $L' = L$ since $L, L' > 0$.

Notice that, as a particular case, any dilation $M_K = \begin{pmatrix} K^{-1} & 0 \\ 0 & K^T \end{pmatrix}$, $\det K \neq 0$, has the pre-Iwasawa factorization

$$\begin{pmatrix} (K^TK)^{-1/2} & 0 \\ 0 & (K^TK)^{1/2} \end{pmatrix} \begin{pmatrix} (K^TK)^{1/2}K^{-1} & 0 \\ 0 & (K^TK)^{1/2}K^{-1} \end{pmatrix}. \tag{74}$$

Summarizing:

The symplectic matrix (69) has a unique factorization
$S = RU$ where $R \in \mathrm{Sp}_0(n)$ and $U \in U(n)$ are given by:
$R = \begin{pmatrix} (AA^T + BB^T)^{1/2} & 0 \\ (CA^T + DB^T)(AA^T + BB^T)^{-1/2} & (AA^T + BB^T)^{-1/2} \end{pmatrix}$
$U = \begin{pmatrix} (AA^T + BB^T)^{-1/2}A & (AA^T + BB^T)^{-1/2}B \\ -(AA^T + BB^T)^{-1/2}B & (AA^T + BB^T)^{-1/2}A \end{pmatrix}$

When writing $S = RU$ we will call R and U respectively the *local* and the *unitary* components of S. They are uniquely defined.

4.3. Iwasawa factorization of a quadratic Hamiltonian

Let (S_t) be a symplectic isotopy and

$$H(z,t) = -\frac{1}{2}J\dot{S}_tS_t^{-1}z^2 \tag{75}$$

the associated Hamiltonian. Writing

$$S_t = \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix}$$

the pre-Iwasawa factorization yields $S_t = R_tU_t$ where $R_t = V_{-P_t}M_{L_t}$ is given by

$$R_t = \begin{pmatrix} L_t^{-1} & 0 \\ P_tL_t^{-1} & L_t \end{pmatrix} = \begin{pmatrix} L_t^{-1} & 0 \\ Q_t & L_t \end{pmatrix}, \tag{76}$$

the symmetric $n \times n$ matrices P_t and L_t being calculated using the formulas (71) and (72):

$$P_t = (C_t A_t^T + D_t B_t^T)(A_t A_t^T + B_t B_t^T)^{-1} \tag{77}$$

$$L_t = (A_t A_t^T + B_t B_t^T)^{-1/2} \tag{78}$$

$$Q_t = (C_t A_t^T + D_t B_t^T)(A_t A_t^T + B_t B_t^T)^{-1/2}. \tag{79}$$

Similarly the symplectic rotations

$$U_t = \begin{pmatrix} X_t & Y_t \\ -Y_t & X_t \end{pmatrix} \tag{80}$$

are given by

$$X_t = (A_t A_t^T + B_t B_t^T)^{-1/2} A_t, \quad Y_t = (A_t A_t^T + B_t B_t^T)^{-1/2} B_t. \tag{81}$$

The families (V_{-P_t}), (M_{L_t}), and (U_t) are symplectic isotopies in their own right; they correspond to Hamiltonians that we will denote by H_V, H_L, H_U. It is easy to find explicit expressions for these Hamiltonians, and to show that the Hamiltonian function (75) determining the symplectic isotopy (S_t) can be written as a sum of Hamiltonians. This is what we call the "Iwasawa sum":

Proposition 11. *(i) The symplectic isotopies (V_{-P_t}), (M_{L_t}), and (U_t) are the flows determined by the Hamiltonians:*

$$H_V(z,t) = \frac{1}{2}\dot{P}_t x^2, \quad H_L(z,t) = -\dot{L}_t L_t^{-1} x \cdot p \tag{82}$$

and

$$H_U(z,t) = \tfrac{1}{2}(\dot{Y}_t X_t^T - \dot{X}_t Y_t^T)x^2 + \tfrac{1}{2}(\dot{Y}_t X_t^T - \dot{X}_t Y_t^T)p^2. \tag{83}$$

(ii) The Hamiltonian function H can be written

$$H(z,t) = H_V(z,t) + H_L(V_{P_t} z, t) + H_U(M_{L_t^{-1}} V_{P_t} z, t) \tag{84}$$

(iii) We also have

$$H(z,t) = H_R(z,t) + H_U(R_t^{-1} z, t) \tag{85}$$

where

$$H_R(z,t) = \frac{1}{2}(\dot{L}_t Q_t^T - \dot{Q}_t L_t)x^2 - \dot{L}_t L_t^{-1} p \cdot x. \tag{86}$$

Proof. (i) The formulas (82) immediately follow from (75) replacing (S_t) with (V_{-P_t}) and (M_{L_t}), respectively. Notice that $\dot{L}_t L_t^{-1}$ is symmetric since $L_t > 0$ is: $\dot{L}_t L_t^{-1} = \frac{d}{dt} \operatorname{Log} L_t$. Formula (83) is obtained by writing

$$H_U(z,t) = -\frac{1}{2} J \dot{U}_t U_t^{-1} z^2$$

(see Corollary 4). *(ii)* In view of formula (29) in Proposition 1 $(S_t) = (V_{-P_t} M_{L_t} U_t)$ is the flow determined by $H = H_V \# H_L \# H_U$ hence (84). *(iii)* Formula (85) is obtained in a similar fashion writing $(S_t) = (R_t U_t)$ and using the equality $H = H_R \# H_U$. Formula (86) follows from the equality $H_R = -\frac{1}{2} J \dot{R}_t R_t^{-1}$ and using formula (93). □

Notice that formula (86) can be rewritten

$$H_R = \frac{1}{2} N(t) x^2 - \dot{L}_t L_t^{-1} p \cdot x \tag{87}$$

where the symmetric matrix $N(t)$ is given by

$$N(t) = \dot{L}_t L_t^{-1} P_t + P_t \dot{L}_t L_t^{-1} - \dot{P}_t. \tag{88}$$

Notice that, conversely, every Hamiltonian of the type

$$H_0 = \frac{1}{2} N(t) x^2 - K(t) p \cdot x$$

with $N(t) = N(t)^T$ and $K(t) = K(t)^T$ leads to a flow in $\operatorname{Sp}_0(n)$: the corresponding Hamilton equations are

$$\begin{aligned} \dot{x}(t) &= -K(t) x(t) \\ \dot{p}(t) &= -N(t) x(t) + K(t) p(t) \end{aligned}$$

and the corresponding symplectic isotopy is of the type (76) since the equation $\dot{x}(t) = -K(t) x(t)$ contains no term $p(t)$.

5. Chalkboard motions and their shadows

We now apply the notions developed in the previous sections to what we call "chalkboard motion" in phase space, and thereafter study the orthogonal projections (or "shadows") of these motions on subspaces. Chalkboard motion essentially consists in moving and distorting an ellipsoid in phase space while preserving its symplectic capacity (or area in the case $n = 1$).

5.1. The action of $\mathrm{ISp}_0(n)$ on symplectic balls

From now on the real number $\varepsilon > 0$ has the vocation to be a small radius. For instance, in our applications to quantum mechanics we will choose $\varepsilon = \sqrt{\hbar}$; the need for "smallness" in classical considerations will actually only be necessary in Section 5.3 where we study nonlinear evolution. We call *symplectic ball* the image of a ball $B^{2n}(\varepsilon)$ by some element of $\mathrm{ISp}(n)$. A symplectic ball can always be written

$$B_S^{2n}(z_0, \varepsilon) = T(z_0)SB^{2n}(\varepsilon)$$

for some $S \in \mathrm{Sp}(n)$ and z_0 will be called the center of $B_S^{2n}(z_0, \varepsilon)$. Let $\mathrm{SBall}_\varepsilon(2n)$ be the set of all symplectic balls in $(\mathbb{R}^{2n}, \varepsilon)$ with the same radius ε (equivalently, with the same symplectic capacity $\pi\varepsilon^2$); we have a natural transitive action

$$\mathrm{ISp}(n) \times \mathrm{SBall}_\varepsilon(2n) \longrightarrow \mathrm{SBall}_\varepsilon(2n).$$

It turns out that the restriction

$$\mathrm{ISp}_0(n) \times \mathrm{SBall}_\varepsilon(2n) \longrightarrow \mathrm{SBall}_\varepsilon(2n)$$

of this action to the local inhomogeneous symplectic group $\mathrm{ISp}_0(n)$ is also transitive:

Proposition 12. *(i) Every symplectic ball $B_S^{2n}(z_0, \varepsilon)$ can be obtained from the ball $B^{2n}(\varepsilon)$ using the local subgroup $\mathrm{ISp}_0(n)$ of $\mathrm{ISp}(n)$. In fact, for every $S \in \mathrm{Sp}(n)$ there exist unique $P = P^T$, $L = L^T$, and $z_0 \in \mathbb{R}^{2n}$ such that*

$$B_S^{2n}(z_0, \varepsilon) = T(z_0)V_P M_L B^{2n}(\varepsilon). \tag{89}$$

(ii) More generally, if $S = V_P M_L$ and $S' = V_{P'} M_{L'}$ then

$$B_{S'}^{2n}(z_0', \varepsilon) = S(P, L, P', L', z_0, z_0')B_S^{2n}(z_0, \varepsilon) \tag{90}$$

with $S(P, L, P', L', z_0, z_0') \in \mathrm{ISp}_0(n)$ given by

$$S(P, L, P', L', z_0, z_0') = T(z_0' - Rz_0)R \tag{91}$$

where $R \in \mathrm{ISp}_0(n)$ is the product

$$R = V_{P' - (L'L^{-1})^T P L^{-1} L'} M_{L^{-1}L'}. \tag{92}$$

Proof. *(i)* Using a pre-Iwasawa factorization we can find unique $P = P^T$, $L = L^T > 0$ given by (71), (72), and a symplectic rotation $U_{X,Y} \in U(n)$ such that $S = V_P M_L U_{X,Y}$; formula (89) follows since by rotational symmetry, $U_{X,Y} B^{2n}(\varepsilon) = B^{2n}(\varepsilon)$ and thus

$$\begin{aligned} Q_S^{2n}(z_0) &= T(z_0) V_P M_L U_{X,Y} B^{2n}(\varepsilon) \\ &= T(z_0) V_P M_L B^{2n}(\varepsilon). \end{aligned}$$

The uniqueness of a transformation $T(z_0)V_P M_L \in \mathrm{ISp}_0(n)$ such that (89) holds is easily verified: suppose that

$$T(z_0)V_P M_L B^{2n}(\varepsilon) = T(z_0')V_{P'} M_{L'} B^{2n}(\varepsilon)$$

then there exists $U \in U(n)$ such that

$$T(z_0)V_P M_L U = T(z_0')V_{P'} M_{L'}.$$

This implies that we must have $z_0 = z_0'$, applying both sides to $z = 0$. But then $V_P M_L U = V_{P'} M_{L'}$ which is only possible if U is the identity, so we have $V_P M_L = V_{P'} M_{L'}$ which implies $P = P'$ and $L = L'$. *(ii)* Let $B_S^{2n}(\varepsilon)$ $= SB^{2n}(\varepsilon)$ and $B_{S'}^{2n}(\varepsilon) = S'B^{2n}(\varepsilon)$ be two symplectic balls centered at 0; we thus have $B_{S'}^{2n}(\varepsilon) = S'S^{-1}B_S^{2n}(\varepsilon)$. Taking $S = V_P M_L$ and $S' = V_{P'} M_{L'}$ we have, in view of formula (65),

$$S'S^{-1} = V_{P' - (L'L^{-1})^T P L^{-1} L'} M_{L^{-1}L'}$$

proving (90) for $z_0 = z_0' = 0$. The case of arbitrary centers z_0 and z_0' readily follows: assume that $B_S^{2n}(z_0, \varepsilon)$ and $B_{S'}^{2n}(z_0', \varepsilon)$ are centered at z_0 and z_0', respectively. We have $B_S^{2n}(z_0, \varepsilon) = T(z_0)SB^{2n}(\varepsilon)$ hence

$$\begin{aligned} B_{S'}^{2n}(z_0', \varepsilon) &= T(z_0')S'(T(z_0)S)^{-1}B_S^{2n}(z_0, \varepsilon) \\ &= T(z_0' - S'S^{-1}z_0)S'S^{-1}B_S^{2n}(z_0, \varepsilon). \end{aligned}$$

Choosing $S = V_P M_L$ and $S' = V_{P'} M_{L'}$ as above we are done. □

5.2. Linear and affine chalkboard motions

It will be convenient to write (76) as above in the form

(93) $$R_t = \begin{pmatrix} L_t^{-1} & 0 \\ Q_t & L_t \end{pmatrix} \quad \textit{with} \quad Q_t = P_t L_t^{-1}.$$

Recall that the symplectic isotopy (R_t) is determined by a Hamiltonian which does not contain any kinetic term.

Let $(T(z_t)S_t)$ be a chalkboard motion; then

$$T(z_t)S_tB^{2n}(\varepsilon) = T(z_t)R_tB^{2n}(\varepsilon)$$

which shows that this motion is entirely determined by the trajectory of the center of $B^{2n}(\varepsilon)$ and the flow determined by a reduced Hamiltonian of the type (86)–(87). This immediately follows from the Iwasawa factorization $S_t = R_tU_t$ since balls centered at the origin are invariant under the action of the group $U(n)$.

More generally:

Proposition 13. *Let $B^{2n}_S(a,\varepsilon) = T(a)SB^{2n}(\varepsilon)$ be a symplectic ball. We have*

$$(T(z_t)S_t)B^{2n}_S(a,\varepsilon) = B^{2n}_{R_tS}(a_t,\varepsilon) \tag{94}$$

where $a_t = S_ta$ and (R_t) is a symplectic isotopy in $\mathrm{Sp}_0(n)$ defined as follows: let H be the quadratic Hamiltonian generating (S_t), that is $H = -\frac{1}{2}J\dot{S}_tS_t^{-1}z^2$. Then (R_t) is the reduced flow determined by $H \circ R$ where R is the local part in the pre-Iwasawa factorization $S = RU$.

Proof. Let $S = RU$ be the pre-Iwasawa factorization of S. Since $S_tT(a) = T(S_ta)S_t$ and $UB^{2n}(\varepsilon) = B^{2n}(\varepsilon)$ we have

$$\begin{aligned}(T(z_t)S_t)B^{2n}_S(a,\varepsilon) &= T(z_t)S_tT(a)RUB^{2n}(\varepsilon)\\ &= T(z_t + a_t)S_tRB^{2n}(\varepsilon)\\ &= T(z_t + a_t)R(R^{-1}S_tR)B^{2n}(\varepsilon).\end{aligned}$$

In view of the conjugation formula (30), $(S'_t) = (R^{-1}S_tR)$ is the symplectic isotopy generated by the quadratic Hamiltonian $H \circ S$; the latter is given by

$$H \circ R(z,t) = -\frac{1}{2}JR^{-1}\dot{S}_tS_t^{-1}Rz^2.$$

We now apply the pre-Iwasawa factorization to S'_t and write $S'_t = R'_t U'_t$ so that

$$\begin{aligned}(T(z_t)S_t)B_S^{2n}(a,\varepsilon) &= T(z_t+a_t)RR'_tU'_tB^{2n}(\varepsilon)\\ &= T(z_t+a_t)RR'_tB^{2n}(\varepsilon)\\ &= T(z_t+a_t)(RR'_tR^{-1})RB^{2n}(\varepsilon)\\ &= T(z_t+a_t)(RR'_tR^{-1})SB^{2n}(\varepsilon).\end{aligned}$$

hence the equality (94) with $R_t = RR'_tR^{-1}$. □

5.3. The nonlinear case: nearby orbit approximation

Assume now that we have a method allowing us to displace and deform any ellipsoid $\Omega = T(z_0)FB^{2n}(\varepsilon)$, $F \in GL(2n,\mathbb{R})$, in such a way that its symplectic capacity remains constant. More explicitly, we make the assumption that there exists a C^1 curve $t \longmapsto z_t$ starting from z_0 and a family (g_t) of C^1 diffeomorphisms satisfying $g_0 = I_{\mathrm{d}}$ together with the equilibrium condition

(95) $$g_t(0) = 0 \quad \textit{for} \quad t \in I_T.$$

At time $t \in I_T$ the ellipsoid Ω becomes a (usually not elliptic) set

$$\Omega_t = T(z_t)g_tT(-z_0)\Omega$$

such that $c(\Omega_t) = c(\Omega)$; we thus have $\Omega_t = f_t(\Omega)$ where the symplectomorphisms f_t are defined by

(96) $$f_t = T(z_t)g_tT(-z_0) = T(z_t - z_0)T(z_0)g_tT(-z_0)$$

and it follows from Proposition 10 that this motion must be Hamiltonian: (f_t) is a symplectic isotopy generated by a Hamiltonian function H which we determine now.

Proposition 14. *The symplectic isotopy (f_t) defined by*

(97) $$f_t = T(z_t)g_tT(-z_0)$$

is the Hamiltonian flow determined by the Hamiltonian

(98) $$H(z,t) = H_2(z - z_t, t) + \sigma(z, \dot{z}_t)$$

where

$$H_2(z,t) = -\int_0^1 \sigma(\dot{g}_t g_t^{-1}(\lambda z), z)d\lambda \tag{99}$$

is the Hamiltonian function generating (g_t).

Proof. Let us write

$$f_t = T(z_t - z_0)T(z_0)g_t T(-z_0). \tag{100}$$

We first remark that $t \longmapsto T(z_t - z_0)$ is the flow determined by the translation Hamiltonian $H_1(z,t) = \sigma(z, \dot{z}_t)$. The symplectic isotopy (g_t) is determined by (99) (Proposition 2), and in view of the conjugation property (30) the flow $t \longmapsto T(z_0)g_t T(-z_0)$ is thus determined by $H_2 \circ T(-z_0)$. Formula (98) now follows from the product property (26) of Hamiltonian flows. □

Assuming that the radius ε is small it makes sense to replace the symplectic isotopy (g_t) with its linearization (g_t^0) around its equilibrium point $z = 0$ (Arnol'd [4], §5.22), that is, we take

$$g_t^0(z) = g_t(0) + Dg_t(0)z = S_t^0 z \tag{101}$$

where $S_t^0 = Dg_t(0)$ is the Jacobian matrix of g_t calculated at the origin. A classical result (see *e.g.* [26], §2.3.2) tells us that $t \longmapsto S_t^0$ satisfies the "variational equation"

$$\frac{d}{dt}S_t^0 = JD_z^2 H(g_t(0), t)S_t^0 = JD_z^2 H(0,t)S_t^0$$

and hence $(g_t^0) = (S_t^0)$ is the flow determined by the quadratic Hamiltonian function

$$H_2^0(z,t) = \frac{1}{2}D_z^2 H(0,t)z^2. \tag{102}$$

With this approximation the symplectomorphisms (97) are replaced with

$$f_t^0 = T(z_t - z_0)T(z_0)S_t^0 T(-z_0) \tag{103}$$

and (f_t^0) is the flow determined by the Hamiltonian

$$H^0(z,t) = \frac{1}{2}D_z^2 H(0,t)(z - z_t)^2 + \sigma(z, \dot{z}_t). \tag{104}$$

We remark that $H_2^0(z,t)$ is obtained from the "exact" Hamiltonian (99) by truncating the Taylor series of H_2 at $z=0$ by dropping third order terms and above: noting that $\partial_z H(0,t)=0$ since 0 is an equilibrium point we have

$$H_2(z,t)=H_2(0,t)+\tfrac{1}{2}D_z^2H_2(0,t)z^2+O(z^3) \tag{105}$$

and the term $H_2(0,t)$ can be neglected. Similarly, dismissing the terms $H_2(0,t)$ and $\sigma(z_t,\dot{z}_t)$,

$$H(z,t)=\tfrac{1}{2}D_z^2H_2(0,t)(z-z_t)^2+\sigma(z-z_t,\dot{z}_t)+O((z-z_t)^3) \tag{106}$$

so our method is closely related to the so-called "nearby orbit method" popularized by researchers working in semiclassical approximations [13, 44, 47, 53]. In this method one expands the Hamiltonian around an orbit and truncates the Taylor series in order to get a more tractable problem.

A natural question arises at this point: in view of (106) we have $f_t^H(z_0)=f_t^{H_0}(z_0)=z_t$. What can we say about the difference $f_t^H(z_1)-f_t^{H_0}(z_1)$ for an arbitrary point $z_1\in\Omega$? Intuitively, the smaller the radius ε is, the better will $f_t^H(z_1)$ approximate $f_t^H(z_0)$ (at least for not too big times t). Let us briefly discuss this without going too much into the theory of the stability of Hamiltonian systems, for which there exists an immense literature. See the recent preprint by Hong Qin [46] for new results in the case of periodic orbits.

5.4. A recalibration procedure

We now briefly discuss an option which, to the best of our knowledge, has not been explored yet, and leads to open question. Recall that we discussed in the Introduction the John–Löwner ellipsoid. As before we start with a symplectic ball $B_S^{2n}(\varepsilon)$, which we suppose centered at the origin for simplicity. We displace $B_S^{2n}(\varepsilon)$ along a curve (z_t) in phase space while deforming it using a symplectic isotopy (g_t) consisting of an arbitrary family of symplectomorphisms starting from the identity at time $t=0$. If we assume that this deformation is sufficiently "gentle" and preserves the convexity, a natural idea is to replace $\Omega_t=g_t(B_S^{2n}(\varepsilon))$ with an ellipsoid $\widetilde{\Omega}_t$ having the same symplectic capacity $\pi\varepsilon^2$ as Ω_t: $c(\widetilde{\Omega}_t)=c(\widetilde{\Omega}_t)$ for some choice of symplectic capacity c. It turns out that there exists a unique maximum volume ellipsoid containing Ω_t, and by dilation one can obtain a minimum volume ellipsoid containing $g_t(B_S^{2n}(\varepsilon))$. Now, volume is not related to symplectic capacity (except in the case $n=1$ where both notions coincide with area for connected simply connected surfaces), and it is not known whether one can construct a "minimum (or maximum) capacity ellipsoid" which is the symplectic analogue of the John–Löwner ellipsoid. However, we can do the following [5, 39, 40]. Among the ellipsoids circumscribing Ω_t, there exists a unique one $\Omega_t^{\min}$ with minimum volume (the Löwner ellipsoid) and similarly, among the ellipsoids inscribed in Ω_t, there exists a unique one $\Omega_t^{\max}$ of maximum volume (the John ellipsoid) and we have

$$\frac{1}{n}\Omega_t^{\min} \subset \widetilde{\Omega}_t \subset \Omega_t^{\min}, \quad \Omega_t^{\max} \subset \widetilde{\Omega}_t \subset n\Omega_t^{\max}.$$

In case Ω_t is symmetric (*i.e.* $\Omega_t = -\Omega_t$) the coefficients $1/n$ and n can be changed into $1/\sqrt{n}$ and $\sqrt{n}$. Also note that this also works when Ω_t fails to be a convex body. It suffices to replace Ω_t with its convex hull.

5.5. The shadow of a chalkboard motion

We now study the projection ("shadow") of the chalkboard motion on the "configuration space" $X = \mathbb{R}^n_x$. For this the following lemma about projections of ellipsoids will be helpful:

Lemma 15. *Let $\Omega = \{Mz^2 \leq \varepsilon^2; z \in \mathbb{R}^{2n}\}$ and assume that $M = M^T > 0$ is given in $n \times n$ block form by*

$$M = \begin{pmatrix} M_{XX} & M_{XP} \\ M_{PX} & M_{PP} \end{pmatrix} \tag{107}$$

(thus $M_{XP} = M_{PX}^T$). Let Π be the orthogonal projection $\mathbb{R}^{2n} \longrightarrow \mathbb{R}^n_x \times \{0\}$. We have

$$\Pi\Omega = \{x \in \mathbb{R}^n : (M/M_{PP})x^2 \leq \varepsilon^2\} \tag{108}$$

where the $n \times n$ matrix

$$M/M_{PP} = M_{XX} - M_{XP}M_{PP}^{-1}M_{PX} \tag{109}$$

is the Schur complement of the block M_{PP} of M.

Proof. Recall [66] that if $M > 0$ then $M/M_{PP} > 0$ and hence the ellipsoid (108) is nondegenerate. Set $Q(z) = Mz^2 - \varepsilon^2$; the hypersurface $\partial\Omega : Q(z) =$

0 bounding Ω is defined by

$$M_{XX}x^2 + 2M_{PX}x \cdot p + M_{PP}p^2 = \varepsilon^2. \tag{110}$$

The normal vectors to the boundary of $\Pi_X\Omega$ must stay in X hence the constraint $\partial_z Q(z) = 2Mz \in \mathbb{R}^n \times \{0\}$, which is equivalent to $M_{PX}x + M_{PP}p = 0$, that is to $p = -M_{PP}^{-1}M_{PX}x$. Inserting p in (110) shows that $\Pi_X\Omega$ is bounded by $\Sigma_X : (M/M_{PP})x^2 = \varepsilon^2$ which yields (108). □

This result easily allows us to find the orthogonal projection of a symplectic ball on the x-space $\mathbb{R}^n \times \{0\}$ (or on the p-space $\{0\} \times \mathbb{R}^n$); we will more generally consider the projection of a chalkboard motion:

Proposition 16. *Let $(T(z_t)S_t)$ be a symplectic isotopy with*

$$S_t = \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix}, \quad S_0 = I_{\mathrm{d}}. \tag{111}$$

The orthogonal projection of the symplectic ball

$$B^{2n}_{S_t}(z_t, \varepsilon) = T(z_t)S_t B^{2n}(\varepsilon)$$

on the configuration space $\mathbb{R}^n \times \{0\}$ is the ellipsoid

$$\Pi B^{2n}_{S_t}(z_t, \varepsilon) = T(x_t, 0)(A_tA_t^T + B_tB_t^T)^{1/2}B^n(\varepsilon). \tag{112}$$

Proof. It is no restriction to assume $z_t = 0$ since translations project to translations in the first component. We have $S_tB^{2n}(\varepsilon) = R_tB^{2n}(\varepsilon)$ where (R_t) is the symplectic isotopy in $\mathrm{ISp}_0(n)$ given by (93), that is

$$R_t = \begin{pmatrix} L_t^{-1} & 0 \\ Q_t & L_t \end{pmatrix}, \tag{113}$$

the matrices Q_t and L_t being given by

$$Q_t = (C_tA_t^T + D_tB_t^T)(A_tA_t^T + B_tB_t^T)^{-1/2} \tag{114}$$

$$L_t = (A_tA_t^T + B_tB_t^T)^{-1/2}. \tag{115}$$

The ellipsoid $R_tB^{2n}(\varepsilon)$ is the set of all $z \in \mathbb{R}^{2n}$ such that $(R_tR_t^T)^{-1}z^2 \leq \varepsilon$ hence the matrix M in (107) is given by

$$M = \begin{pmatrix} QQ^T + L^2 & -QL^{-1} \\ -L^{-1}Q^T & L^{-2} \end{pmatrix}$$

and the Schur complement M/M_{PP} is then just $L_t^2 = (A_t A_t^T + B_t B_t^T)^{-1}$. Formula (112) follows. □

Formula (112) perfectly illustrates that the phenomenon of "spreading" is not, *per se*, a quantum phenomenon as many physicists still believe (this was in fact remarked on a long time ago by Littlejohn [53]). In fact spreading will always occur provided that $A_t A_t^T + B_t B_t^T$ is not constant, that is equal to I_{d} for all t. For instance, in the case $n = 1$ we would have $A_t^2 + B_t^2 = 1$ and the phase space motion would be a rotation leaving the disk $|z| \leq \varepsilon$ invariant.

Let us next briefly consider the case of subsystems, obtained by orthogonal projection on a smaller phase space. In the quantum case such projections intervene in the study of entanglement. Let again $\Omega = FB^{2n}(\varepsilon)$ be a non-degenerate ellipsoid centered at the origin; setting $M = (FF^T)^{-1}$ this ellipsoid is the set of all $z \in \mathbb{R}^{2n}$ such that $Mz^2 \leq R^2$; M is a symmetric and positive definite matrix. Consider now the splitting $\mathbb{R}^{2n_A} \oplus \mathbb{R}^{2n_B}$ of $\mathbb{R}^{2n}$ with $n_A + n_B = n$. The spaces $\mathbb{R}^{2n_A} \equiv \mathbb{R}^{n_A}_{x_A} \times \mathbb{R}^{n_A}_{p_A}$ and $\mathbb{R}^{2n_B} \equiv \mathbb{R}^{n_B}_{x_B} \times \mathbb{R}^{n_B}_{p_B}$ are viewed as the phase spaces of two subsystems A and B. We will write the matrix of M as

$$M = \begin{pmatrix} M_{AA} & M_{AB} \\ M_{BA} & M_{BB} \end{pmatrix} \tag{116}$$

where the blocks M_{AA}, M_{AB}, M_{BA}, M_{BB} have, respectively, dimensions $n_A \times n_A$, $n_A \times n_B$, $n_B \times n_A$, $n_B \times n_B$. Since M is positive definite and symmetric (written $M > 0$) the blocks M_{AA} and M_{BB} are symmetric and positive definite (and hence invertible) and $M_{BA} = M_{AB}^T$.

Abbondandolo and Matveyev [1] (also see [2]) have shown that for $S \in \mathrm{Sp}(n)$.

$$\mathrm{Vol}_{2n_A} \Pi_A S(B^{2n}(R) \geq \frac{(\pi R^2)^{n_A}}{n_A!} \tag{117}$$

for every $R > 0$. This inequality can be seen as an interpolation between Gromov's theorem and Liouville's theorem on the conservation of volume for $2 \leq n_A \leq n-1$. In [18] we have proven the following improvement of Abbondandolo and Matveyev's result:

Proposition 17. *Let Π_A be the orthogonal projection $\mathbb{R}^{2n_A} \times \mathbb{R}^{2n_B} \longrightarrow \mathbb{R}^{2n_A}$. There exists $S_A \in \mathrm{Sp}(n_A)$ such that for every $R > 0$ the projected* *ellipsoid* $\Pi_A(S(B_R^{2n}))$ *contains the symplectic ball* $S_A(B_R^{2n_A})$*:*

$$\Pi_A(S(B_R^{2n})) \supset S_A(B_R^{2n_A})) \ . \tag{118}$$

More generally,

$$\Pi_A(S(B_R^{2n}(z_0))) \supset \{\Pi_A(Sz_0)\} + S_A(B_R^{2n_A}) \ . \tag{119}$$

We have equality in (118), (119) if and only if $S = S_A \oplus S_B$ *for some* $S_B \in \mathrm{Sp}(n_B)$*.*

The proof of this result makes use of the properties of the projection operator using Williamson's symplectic diagonalization theorem and is thus conceptually somewhat simpler than the method of Abbondandolo and Matveyev which uses the Wirtinger inequality from the theory of differential forms. We mention that Abbondandolo and Benedetti [3] have very recently improved and extended the scope of their results using the theory of Zoll contact forms.

6. Quantum blobs and the Wigner transform

The symplectic group and its double covering, the metaplectic group, are the keystones of mechanics in both their Hamiltonian and quantum formulations. Loosely speaking one can say that the passage from the symplectic group to its metaplectic representation is the shortest bridge between classical and quantum mechanics [33], because it provides us automatically with the Weyl quantization of quadratic Hamiltonians without any recourse to physical arguments (also see the discussion in [30]).

6.1. The local metaplectic group

The symplectic group $\mathrm{Sp}(n)$ is a connected matrix Lie group, contractible to its compact subgroup $U(n)$ and has covering groups of all orders. The metaplectic group $\mathrm{Mp}(n)$ is a unitary representation in $L^2(\mathbb{R}^n)$ of the double cover of $\mathrm{Sp}(n)$ by unitary operators acting on square integrable functions (see [26], Chapter 7 for a detailed study and construction of $\mathrm{Mp}(n)$). To every $S \in \mathrm{Sp}(n)$ the metaplectic representation associates two unitary operators $\pm\widehat{S} \in \mathrm{Mp}(n)$ on $L^2(\mathbb{R}^n)$. This representation is entirely determined by its action on the generators of $\mathrm{Mp}(n)$ since the covering projection $\pi^{\mathrm{Mp}} : \mathrm{Mp}(n) \longrightarrow \mathrm{Sp}(n)$ is a group epimorphism. This correspondence is summarized in the table below:

$\widehat{J}\psi(x) = \left(\frac{1}{2\pi i\hbar}\right)^{n/2} \int e^{-\frac{1}{\hbar}x\cdot x'}\psi(x')d^n x'$	$\overset{\pi^{\mathrm{Mp}}}{\longrightarrow}$	$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$
$\widehat{V}_P\psi(x) = e^{-\frac{i}{2\hbar}Px\cdot x}\psi(x)$	$\overset{\pi^{\mathrm{Mp}}}{\longrightarrow}$	$V_P = \begin{pmatrix} I & 0 \\ -P & I \end{pmatrix}$
$\widehat{M}_{L,m}\psi(x) = i^m\sqrt{\lvert\det L\rvert}\psi(Lx)$	$\overset{\pi^{\mathrm{Mp}}}{\longrightarrow}$	$M_L = \begin{pmatrix} L^{-1} & 0 \\ 0 & L^T \end{pmatrix}$

the integer m in $\widehat{M}_{L,m}$ ("Maslov index") being chosen so that $\arg\det L = m\pi$ $(\bmod 2\pi)$. Observe that these operators (and hence every $\widehat{S} \in \mathrm{Mp}(n)$) can be extended into continuous operators acting on the Schwartz space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions. Needless to say there are other ways to introduce the metaplectic group. A good way is to use generalized Fourier transforms and the apparatus of generating functions (see [26], Chapter 7). It works as follows: assume that

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \det B \neq 0$$

is a free symplectic matrix; to S we associate its generating function

$$W(x, x') = \frac{1}{2}DB^{-1}x^2 - B^{-1}x \cdot x' + \frac{1}{2}B^{-1}Ax'^2.$$

This function has the property that

$$(x, p) = S(x', p') \Longleftrightarrow \begin{cases} p = \partial_x W(x, x') \\ p' = -\partial_{x'} W(x, x') \end{cases}$$

as can be checked by a direct calculation. Now, exactly every element of $\mathrm{Sp}(n)$ is the product of two free symplectic matrices, every $\widehat{S} \in \mathrm{Mp}(n)$ can be written (non uniquely) as a product of two Fourier integral operators of the type $\widehat{S}_{W,m}$ where

$$\widehat{S}_{W,m}\psi(x) = \left(\tfrac{1}{2\pi i\hbar}\right)^{n/2} i^m\sqrt{\lvert\det B^{-1}\rvert}\int e^{\frac{i}{\hbar}W(x,x')}\psi(x')d^n x' \tag{120}$$

where m is an integer $\bmod 4$ corresponding to a choice of $\arg\det B^{-1}$. We notice that $\widehat{S}_{W,m}$ can be simply expressed in terms of the unitary operators $\widehat{J}$, $\widehat{V}_P$ and $\widehat{M}_{L,m}$ defined above: a simple inspection of the formula above shows that (*cf.* formula (60) in Section 4.1)

$$\widehat{S}_{W,m} = \widehat{V}_{-DB^{-1}}\widehat{M}_{B^{-1},m}\widehat{J}\widehat{V}_{-B^{-1}A}. \tag{121}$$

We now define the *local metaplectic group*: it is the subgroup $\mathrm{Mp}_0(n)$ of $\mathrm{Mp}(n)$ generated by the operators $\widehat{M}_{L,m}$ and $\widehat{V}_P$. This group actually consists of all products $\widehat{V}_P\widehat{M}_{L,m}$ (or $\widehat{M}_{L,m}\widehat{V}_P$) as follows from the formulas (*cf.* (63) and (64)):

$$\widehat{M}_{L,m}\widehat{V}_P = \widehat{V}_{L^TPL}\widehat{M}_{L,m}, \quad \widehat{V}_P\widehat{M}_{L,m} = \widehat{M}_{L,m}\ \widehat{V}_{(L^{-1})^TPL^{-1}} \tag{122}$$

and, using the relations $(\widehat{V}_P)^{-1} = \widehat{V}_{-P}$, $(\widehat{M}_{L,m})^{-1} = \widehat{M}_{L^{-1},-m}$,

$$(\widehat{V}_P\widehat{M}_{L,m})^{-1} = \widehat{V}_{(L^{-1})^TPL^{-1}}\widehat{M}_{L^{-1},-m}. \tag{123}$$

Combining these relations we get the following formula, which is the metaplectic analogue of (65):

$$\widehat{S'}\widehat{S}^{-1} = \widehat{V}_{P'-(L^{-1}L')^TP(L^{-1}L')}\widehat{M}_{L^{-1}L',-m+m'}. \tag{124}$$

The local symplectic group $\mathrm{Sp}_0(n)$ is the image of the local metaplectic group $\mathrm{Mp}_0(n)$ by the covering projection $\pi^{\mathrm{Mp}} : \mathrm{Mp}(n) \longrightarrow \mathrm{Sp}(n)$ as $\pi^{\mathrm{Mp}}(\widehat{V}_P) = V_P$ and $\pi^{\mathrm{Mp}}(\widehat{M}_{L,m}) = M_L$. We use the denomination "local metaplectic group" because the products $\widehat{V}_P\widehat{M}_{L,m}$ are the only local operators in $\mathrm{Mp}(n)$: in harmonic analysis an operator is said to be "local" if it does not increase the supports of the functions to which it is applied. For instance the modified Fourier transform $\widehat{J} \in \mathrm{Mp}(n)$ is not local since, for example, $\widehat{J}\psi$ cannot be of compact support if $\psi \neq 0$ because $\widehat{J}\psi$ is an analytic function in view of Paley–Wiener's theorem. More generally, the Fourier integral operators (120) are never local as can be seen by letting them act on a Dirac δ distribution.

We defined the inhomogeneous symplectic group $\mathrm{ISp}(n)$ as being the group generated by $\mathrm{Sp}(n)$ and the translations $T(z_0) : z \longmapsto z_0$. Similarly, we define the inhomogeneous metaplectic group $\mathrm{IMp}(n)$ as the group of unitary operators generated by $\mathrm{Mp}(n)$ and the Heisenberg–Weyl operators $\widehat{T}(z_0)$, defined by [26, 29, 53]

$$\widehat{T}(z_0)\psi(x) = e^{\frac{i}{\hbar}(p_0x - \frac{1}{2}p_0x_0)}\psi(x - x_0).$$

These operators satisfy the Weyl relations

$$\widehat{T}(z_0+z_1)=e^{-\frac{i}{2\hbar}\sigma(z_0,z_1)}\widehat{T}(z_0)\widehat{T}(z_1) \tag{125}$$

$$\widehat{T}(z_0)\widehat{T}(z_1)=e^{\frac{i}{\hbar}\sigma(z_0,z_1)}\widehat{T}(z_1)\widehat{T}(z_0). \tag{126}$$

The inhomogeneous metaplectic group $\mathrm{IMp}(n)$ consists of all products

$$\widehat{S}\widehat{T}(z_0)=\widehat{T}(Sz_0)\widehat{S}. \tag{127}$$

We will denote by $\mathrm{IMp}_0(n)$ the subgroup of $\mathrm{IMp}(n)$ generated by $\mathrm{Mp}_0(n)$ and the Heisenberg–Weyl operators; it consists of all products $\widehat{S}\widehat{T}(z_0)$ or $\widehat{T}(z_0)\widehat{S}$ with $\widehat{S}\in\mathrm{Mp}_0(n)$: we have

$$\widehat{T}(z_0)\widehat{V}_P\widehat{M}_{L,m}=\widehat{V}_P\widehat{M}_{L,m}\widehat{T}[M_{L^{-1}}V_{-P}z_0] \tag{128}$$

which is the metaplectic version of (68).

6.2. Wigner transforms of Gaussians

There is an immense literature about Gaussians and their Wigner transforms. See for instance [15, 53, 62].

The Wigner transform of a function $\psi\in L^2(\mathbb{R}^n)$ is the function $W\psi\in L^2(\mathbb{R}^{2n})$ defined by

$$W\psi(x,p)=\left(\tfrac{1}{2\pi\hbar}\right)^n\int e^{-\frac{i}{\hbar}py}\psi(x+\tfrac{1}{2}y)\psi^*(x-\tfrac{1}{2}y)d^ny. \tag{129}$$

This function is covariant under the action of the inhomogeneous groups $\mathrm{ISp}(n)$ and $\mathrm{IMp}(n)$ in the sense that

$$W(\widehat{S}\psi)(z)=W\psi(S^{-1}z),\quad W(\widehat{T}(z_0)\psi)(z)=W\psi(T(z_0)z) \tag{130}$$

for all $\widehat{S}\in\mathrm{Mp}(n)$ with projection $S\in\mathrm{Sp}(n)$ (see for instance [35, 53]). Of particular interest to us are the Wigner transforms of non-degenerate Gaussian functions ("generalized squeezed coherent states")

$$\phi_{X,Y}(x)=(\pi\hbar)^{-n/4}(\det X)^{1/4}e^{-\frac{1}{2\hbar}(X+iY)x^2} \tag{131}$$

(X and Y real symmetric $n\times n$ matrices, $X>0$); one has the well-known formula [8, 26, 29, 35, 53]

$$W\phi_{X,Y}(z)=(\pi\hbar)^{-n}e^{-\frac{1}{\hbar}Gz^2} \tag{132}$$

where G is the symplectic symmetric positive definite matrix

$$G = \begin{pmatrix} X + YX^{-1}Y & YX^{-1} \\ X^{-1}Y & X^{-1} \end{pmatrix} \in \mathrm{Sp}(n). \tag{133}$$

Noting that we can write $G = S_{X,Y}^T S_{X,Y}$ where

$$S_{X,Y} = M_{X^{-1/2}} V_{-Y} = \begin{pmatrix} X^{1/2} & 0 \\ X^{-1/2}Y & X^{-1/2} \end{pmatrix} \in \mathrm{Sp}_0(n) \tag{134}$$

we thus have

$$G = (M_{X^{-1/2}} V_{-Y})^T M_{X^{-1/2}} V_{-Y}. \tag{135}$$

Notice that when $X = I_{\mathrm{d}}$ and $Y = 0$ the Gaussian $\phi_{X,Y}$ reduces to the "standard coherent state" [53]

$$\phi_0(x) = (\pi\hbar)^{-n/4} e^{-|x|^2/2\hbar} \tag{136}$$

whose Wigner transform is simply

$$W\phi_0(z) = (\pi\hbar)^{-n} e^{-|z|^2/\hbar}. \tag{137}$$

More generally we will consider Gaussians centered at an arbitrary point; we define the Gaussian $\phi_{X,Y}^{z_0}$ centered at z_0 as

$$\phi_{X,Y}^{z_0} = \widehat{T}(z_0)\phi_{X,Y} \tag{138}$$

where $\widehat{T}(z_0)$ is the Heisenberg–Weyl operator, and we have, using the translational covariance of the Wigner transform (second formula (130))

$$W\phi_{X,Y}^{z_0}(z) = (\pi\hbar)^{-n} e^{-\frac{1}{\hbar}G(z-z_0)^2}.$$

Setting $\Sigma^{-1} = \frac{2}{\hbar}G$ we can rewrite the Gaussian (132) as

$$W\phi_{X,Y}(z) = (2\pi)^{-n}\sqrt{\det \Sigma^{-1}}\, e^{-\frac{1}{2}\Sigma^{-1}z^2}$$

which immediately leads to the following statistical interpretation: the $2n \times 2n$ matrix

$$\Sigma = \frac{\hbar}{2}\begin{pmatrix} X^{-1} & -X^{-1}Y \\ -YX^{-1} & X + YX^{-1}Y \end{pmatrix}$$

is the covariance matrix [25, 26, 28, 53] of the Gaussian state $\phi_{X,Y}$.

The connection with the uncertainty principle is the following. We write the covariance matrix in traditional form

$$\Sigma = \begin{pmatrix} \Delta(x,x) & \Delta(x,p) \\ \Delta(p,x) & \Delta(p,p) \end{pmatrix} \tag{139}$$

where $\Delta(x,x) = (\Delta(x_j,x_k))_{1\leq j,k\leq n}$ and so on. Since G is symplectic and symmetric positive definite so is its inverse and the $n\times n$ blocks $\Delta(x,x) = \Delta(x,x)^T$, $\Delta(x,p) = \Delta(p,x)^T$, and $\Delta(p,p) = \Delta(p,p)^T$ must satisfy the relation

$$\Delta(x,x)\Delta(p,p) - \Delta(x,p)^2 = \tfrac{1}{4}\hbar^2 I_{\mathrm{d}}$$

as follows from the conditions (19) or (20) on the blocks of a symplectic matrix. The latter implies that we must have

$$(\Delta x_j)^2(\Delta p_j)^2 = \Delta(x_j,p_j)^2 + \tfrac{1}{4}\hbar^2, \quad 1\leq j\leq n \tag{140}$$

which means that the so-called Robertson–Schrödinger inequalities are saturated (*i.e.*become equalities). Notice that in particular we have the textbook Heisenberg inequalities $\Delta x_j \Delta p_j \geq \frac{1}{2}\hbar$, which are a weaker form of the Robertson–Schrödinger uncertainty principle; they lack any symplectic invariance property, and should therefore be avoided in any precise discussion of the uncertainty principle (see the discussions in [26, 27, 37]).

We will denote by $\mathrm{Gauss}(n)$ the set of all Gaussian functions (138); we will identify that set with the set of all symplectic balls with radius $R = \sqrt{\hbar}$. We will write $\mathrm{Gauss}_0(n)$ for the subset consisting of Gaussians centered at the origin.

6.3. Quantum blobs

We have introduced the notion of "quantum blob" in [27, 31, 37]. A quantum blob $Q_S^{2n}(z_0)$ is a symplectic ball with radius $\sqrt{\hbar}$:

$$Q_S^{2n}(z_0) = T(z_0)SB^{2n}(\sqrt{\hbar}) \tag{141}$$

and thus has symplectic capacity $\pi\hbar$. In view of Gromov's non-squeezing theorem, every ellipsoid Ω in $\mathbb{R}^{2n}$ with symplectic capacity $c(\Omega)\geq \pi\hbar$ contains a quantum blob. It is easy to see that

$$\textstyle\bigcap_{S\in \mathrm{Sp}(n)} Q_S^{2n}(z_0) = \{z_0\} \tag{142}$$

(it is sufficient to assume $z_0 = 0$ and $S = M_{\lambda I_{\mathrm{d}}}$ with arbitrary $\lambda \neq 0$).

Quantum blobs are minimum uncertainty phase space ellipsoids as follows from the discussion in the previous subsection where we showed that Gaussians saturate the Robertson–Schrödinger principle (140):

Proposition 18. *Let* Σ *be the covariance matrix (139) of a coherent state (138). The covariance ellipsoid*

$$\Omega_\Sigma = \{z \in \mathbb{R}^{2n} : \tfrac{1}{2}\Sigma^{-1}z^2 \leq 1\} \tag{143}$$

is a quantum blob.

Proof. Since $\Sigma^{-1} = \frac{2}{\hbar}G$, the covariance ellipsoid Σ is equivalently determined by the inequality $Gz^2 \leq \hbar$; in view of the factorization (135) of G we thus have

$$\Omega_\Sigma = V_Y M_{X^{1/2}} B^{2n}(\sqrt{\hbar}) = Q^{2n}_{V_Y M_{X^{1/2}}}(0). \tag{144}$$ □

The ellipsoid (143) is called the *Wigner ellipsoid* in some texts.

All quantum blobs can be built from the elementary quantum blob $B^{2n}(\sqrt{\hbar})$, which is the covariance ellipsoid of the standard coherent state (136):

Every quantum blob $Q^{2n}_S(z_0)$ can be generated from the ball $B^{2n}(\sqrt{\hbar})$ using the local subgroup $\mathrm{ISp}_0(n)$ of $\mathrm{ISp}(n)$. In fact (Proposition 12), for every $S \in \mathrm{Sp}(n)$ there exist unique $P = P^T$, $L = L^T$, and $z_0 \in \mathbb{R}^{2n}$ such that

$$Q^{2n}_S(z_0) = T(z_0)V_P M_L B^{2n}(\sqrt{\hbar}). \tag{145}$$

More generally, it immediately follows from Proposition 12 that:

Proposition 19. *The group* $\mathrm{Sp}_0(n)$ *acts transitively on* $\mathrm{Quant}_0(2n)$ *and* $\mathrm{ISp}_0(n)$ *acts transitively on* $\mathrm{Quant}(2n)$. *Explicitly, if* $S = V_P M_L$ *and* $S' = V_{P'} M_{L'}$ *then*

$$Q^{2n}_{S'}(z_0') = S(P, L, P', L', z_0, z_0')Q^{2n}_S(z_0) \tag{146}$$

with $S(P, L, P', L', z_0, z_0') \in \mathrm{ISp}_0(n)$ *being given by (90), (91), and (92).*

We will denote by $\mathrm{Quant}(2n)$ the set of all quantum blobs in $\mathbb{R}^{2n}$; the subset of $\mathrm{Quant}(2n)$ consisting of quantum blobs $Q^{2n}_S(0)$ centered at 0 will be denoted $\mathrm{Quant}_0(2n)$. The set $\mathrm{Quant}(2n)$ plays the role of a *quantum phase space*. It will be equipped with the topology induced by the Hausdorff distance (52).

6.4. The correspondence between quantum blobs and Gaussians

Consider again the standard coherent state (136):

$$\phi_0(x) = (\pi\hbar)^{-n/4} e^{-|x|^2/2\hbar}$$

and let $\widehat{S} \in \mathrm{Mp}(n)$ be a metaplectic operator with projection

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

on $\mathrm{Sp}(n)$. One can calculate $\widehat{S}\phi_0$ as follows [13, 53]: one first assumes that S is a free symplectic matrix (i.e. $\det B \neq 0$) so that $\widehat{S}$ is a Fourier integral operator (120); a tedious but straightforward calculation of Gaussian integrals then yields the explicit formula

$$\widehat{S}\phi_0(x) = (\pi\hbar)^{-n/4} K \exp\left(\frac{i}{2\hbar}\Gamma x^2\right) \tag{147}$$

where K and Γ are defined by

$$K = (\det(A + iB))^{-1/2} \quad \text{and} \quad \Gamma = (C + iD)(A + iB)^{-1}; \tag{148}$$

the argument of $\det(A + iB) \neq 0$ depends on the choice of the operator $\widehat{S} \in \mathrm{Mp}(n)$ with projection $S \in \mathrm{Sp}(n)$ (there are several ways of proving that $A + iB$ is invertible and that Γ is symmetric; see for instance [14, 29, 53]). Now, this can be considerably simplified if one uses local metaplectic operators. We begin by remarking that if $S = U \in U(n)$ then we have

$$U = \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix},$$

the blocks X and Y satisfying the conditions (38), (39). It follows that $(C + iD)(A + iB)^{-1} = -i$ and, since $X + iY \in U(n, \mathbb{C})$ that $|\det(A + iB)| = |\det(X + iY)| = 1$. Formulas (147)–(148) thus lead to

$$\widehat{U}\phi_0 = i^{\gamma}\phi_0 \tag{149}$$

where γ is a real phase associated to a choice of argument of $\det(X + iY)$. The standard Gaussian ϕ_0 is thus an eigenfunction of every metaplectic operator arising from a symplectic rotation. This observation allows a considerable simplification in the derivation of formula (147):

Proposition 20. *Let $\widehat{S} \in \mathrm{Mp}(n)$ have projection $S = V_{-P} M_L U$ on $\mathrm{Sp}(n)$ (pre-Iwasawa factorization). Then*

$$\widehat{S}\phi_0 = i^{\gamma} \widehat{V}_{-P} \widehat{M}_{L,m} \phi_0 \tag{150}$$

with $m \in \{0, 2\}$, that is, explicitly,

$$\widehat{S}\phi_0(x) = \frac{i^{m+\gamma}}{(\pi\hbar)^{n/4}} \sqrt{\det L} e^{-\frac{1}{2\hbar}(iP - L^2)x^2} \exp\left(-\frac{1}{2\hbar}(iP - L^2)x^2\right) \tag{151}$$

with $P = P^T$, $L = L^T > 0$ being given by

$$P = (CA^T + DB^T - I_{\mathrm{d}})(AA^T + BB^T)^{-1} \tag{152}$$

$$L = (AA^T + BB^T)^{-1/2}. \tag{153}$$

Proof. We have $\widehat{S} = \widehat{V}_{-P} \widehat{M}_{L,m} \widehat{U}$ with $m \in \{0, 2\}$ (because $\det L > 0$). In view of formula (149) we have

$$\widehat{S}\phi_0 = i^{\gamma} \widehat{V}_{-P} \widehat{M}_{L,m} \phi_0$$

which is (150). Formula (151) follows using the expressions (71) and (72) for the matrices P and L. □

It requires a modest number of matrix calculations to verify that (151) is equivalent to (147). The argument goes as follows: one rewrites Γ in (147) as

$$\Gamma = (C + iD)(A^T - iB^T) \left[(A + iB)(A^T - iB^T)\right]^{-1}$$

and one expands the products taking into account the relations (19)–(20) satisfied by the matrices A, B, C, D, which leads to

$$\Gamma = (C + iD)(A^T - iB^T) \left[(A + iB)(A^T - iB^T)\right]^{-1}.$$

We leave the computational details to the reader.

The result above allows us to prove that there is a natural bijection

$$\mathrm{Quant}(2n) \overset{\approx}{\longleftrightarrow} \mathrm{Gauss}(n)$$

allowing to construct a commutative diagram

$$\begin{array}{ccc} \mathrm{IMp}_0(n) \times \mathrm{Gauss}(n) & \longrightarrow & \mathrm{Gauss}(n) \\ \downarrow & & \downarrow \\ \mathrm{ISp}_0(n) \times \mathrm{Quant}(2n) & \longrightarrow & \mathrm{Quant}(2n). \end{array}$$

Let us study these properties in detail. We begin by proving the correspondence between quantum blobs and Gaussians.

Proposition 21. *The natural mapping*

$$Q_S^{2n}(z_0) \longmapsto \phi_{X,Y}^{z_0} = \widehat{T}(z_0)\widehat{S}\phi_0, \tag{154}$$

where $T(z_0)S$ *is the projection of* $\widehat{T}(z_0)\widehat{S} \in \mathrm{IMp}_0(n)$ *on* $\in \mathrm{ISp}_0(n)$ *is a bijection*

$$\mathrm{Quant}(2n) \longrightarrow \mathrm{Gauss}(n). \tag{155}$$

Proof. Since $T(z_0)S$ is uniquely determined the mapping (155) is well-defined. To prove that it it is a bijection it suffices to note that the relation $\phi_{X,Y}^{z_0} = \widehat{T}(z_0)\widehat{S}\phi_0$ unambiguously determines $\widehat{T}(z_0)\widehat{S}$ and hence also $T(z_0)S$. □

The following statement is the quantum analogue of part *(ii)* of Proposition 12.

Proposition 22. *The local inhomogeneous metaplectic group* $\mathrm{IMp}_0(n)$ *acts transitively on the Gaussian phase space* $\mathrm{Gauss}(n)$*. In fact, for any two Gaussians* $\phi_{X,Y}^{z_0}$ *and* $\phi_{X',Y'}^{z_0'}$ *we have*

$$\phi_{X',Y'}^{z_0'} = e^{\frac{i}{\hbar}\chi(z_0,z_0')}\widehat{T}(z_0'')\widehat{V}_{P''}\widehat{M}_{L'',0}\phi_{X,Y}^{z_0} \tag{156}$$

with

$$\chi(z_0, z_0') = \tfrac{1}{2}\sigma(z_0', -Rz_0) \tag{157}$$

$$L'' = X^{-1/2}X'^{1/2}, \quad P'' = Y' - L''Y(L'')^T \tag{158}$$

$$z_0'' = z_0' - V_{P''}M_{L''}z_0. \tag{159}$$

Proof. We have $\phi_{X,Y}^{z_0} = \widehat{T}(z_0)\widehat{S}\phi_0$ and $\phi_{X',Y'}^{z_0'} = \widehat{T}(z_0')\widehat{S}'\phi_0$ with $\widehat{S} = \widehat{V}_Y\widehat{M}_{X^{1/2},0}$ and $\widehat{S}' = \widehat{V}_{Y'}\widehat{M}_{X'^{1/2},0}$; hence

$$\phi_{X',Y'}^{z_0'} = \widehat{T}(z_0')\widehat{S}'(\widehat{T}(z_0)\widehat{S})^{-1}\phi_{X,Y}^{z_0} = \widehat{T}(z_0')\widehat{S}'\widehat{S}^{-1}\widehat{T}(-z_0)\phi_{X,Y}^{z_0}.$$

Using successively formulas (127) and (125) we have

$$\widehat{T}(z_0')\widehat{S}'\widehat{S}^{-1}\widehat{T}(-z_0) = e^{\frac{i}{2\hbar}\sigma(z_0',-S'S^{-1}z_0)}\widehat{T}(z_0' - S'S^{-1}z_0)\widehat{S}'\widehat{S}^{-1}.$$

Using formulas (65) and (124) with $P = Y$, $P' = Y'$, $L = X^{1/2}$, and $L' = X'^{1/2}$, we get

$$\widehat{S}'\widehat{S}^{-1} = \widehat{V}_{Y'-(X^{-1/2}X'^{1/2})^T Y(X^{-1/2}X'^{1/2})}\widehat{M}_{X^{-1/2}X'^{1/2},0}$$

and the projection of $\widehat{S}'\widehat{S}^{-1}$ on $\mathrm{ISp}_0(n)$ is

$$S'S^{-1} = V_{Y'-(X^{-1/2}X'^{1/2})^T Y(X^{-1/2}X'^{1/2})}M_{X^{-1/2}X'^{1/2}},$$

hence formulas (157) and (158). □

7. Discussion and perspectives

The theory of "chalkboard motion" we have outlined in this paper might have applications to several important topics in mathematics and mathematical physics. In particular:

- *Celestial mechanics*: recent work of Scheeres and collaborators shown the important role played by techniques from symplectic topology in guidance and control theory (see for instance [59] which uses symplectic capacities to study spacecraft trajectory uncertainty). The approach outlined in the present work could certainly be used with success in analyzing planetary motions since we do not have to solve directly complicated Hamilton equations arising in, say, the many-body problem, but rather control the trajectories at every step. Numerical algorithms such as symplectic integrators could certainly be easily implemented here following the work of Feng and Qin [22] or Wang [64];
- *Entropy*: in [49, 50] Kalogeropoulos applies techniques for symplectic topology (the non-squeezing theorem) to the study of various notions of entropy in the context of thermodynamics; this approach seems to be very promising; we will return in a future work to the applications of chalkboard motion to these important questions where coarse-graining methods have played historically an important role;
- *Semiclassical methods*: the nearby orbit method we have used to study chalkboards motions both from a classical and quantum perspective originate from robust techniques which have been used for decades in semiclassical mechanics to approximate non-linear motions;

- *Collective motions*: An interesting property of Hamiltonian symplectomorphisms due to Boothby [11] is "N-fold transitivity": given two arbitrary sets $\{z_1, \dots, z_N\}$ and $\{z'_1, \dots, z'_N\}$ of N distinct points in $\mathbb{R}^{2n}$, Boothby proved that there exists $f \in \mathrm{Ham}(n)$ such that $z'_j = f(z_j)$ for every $j \in \{1, \dots, N\}$. A natural question would then be "given N disjoint symplectic balls $B_S^{2n}(z_1, \varepsilon), \dots, B_S^{2n}(z_N, \varepsilon)$ at time $t = 0$ can we find a Hamiltonian flow taking these balls to a new configuration of disjoint symplectic balls $B_{S'}^{2n}(z'_1, \varepsilon), \dots, B_{S'}^{2n}(z'_N, \varepsilon)$ at some later time t? The difficulty here comes from the fact that in the course of this collective motion the ellipsoids might "collide" and have a non-empty intersection: conservation of symplectic capacity (and even volume) has nothing to do with conservation of shape, so two adjacent initially non-intersecting ellipsoids might very well intersect after a while, being stretched and sheared. So an answer to this question might require strong limitations on the type of chalkboard motion we can choose;
- *Study of subsystems of a Hamiltonian system*: Proposition 17 says that the orthogonal projection of a symplectic phase space ball on a phase space with a smaller dimension also contains a symplectic ball with the same radius. In the quantum case, these symplectic balls are just quantum blobs. This projection result is the key to the applications of the theory of chalkboard motion to subsystems. We are currently investigating the topic; for some preliminary results see our preprint [36].

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