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Remarks on intersection numbers and integrable hierarchies. I. Quasi-triviality

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Explicit expressions for quasi-triviality of some scalar non-linear evolutionary PDEs are under consideration.

1. Introduction

Recall that the Korteweg–de Vries (KdV) equation

$$u_t = u u_x + \frac{\epsilon^2}{12} u_{xxx} \tag{1.1}$$

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and the modified KdV (mKdV) equation

$$w_t = \frac{w^2}{2} w_x + \frac{\epsilon^2}{12} w_{xxx} \tag{1.2}$$

are related via a Miura transformation [28]

$$u = \frac{1}{2} w^2 + \sqrt{-1}\, \frac{\epsilon}{2} w_x. \tag{1.3}$$

This means that, substituting (1.3) and its unique formal inverse

$$w = (2u)^{\frac{1}{2}} - \sqrt{-1}\, \frac{\epsilon}{4} \frac{(2u)_x}{2u} + \epsilon^2 \left(\frac{5}{32} \frac{(2u)_x^2}{(2u)^{5/2}} - \frac{1}{8} \frac{(2u)_{xx}}{(2u)^{3/2}} \right) + \mathcal{O}(\epsilon^3) \tag{1.4}$$

in (1.1), one gets (1.2); *vice versa.*

The KdV and mKdV equations are examples of scalar evolutionary PDEs of the form [15]:

$$\begin{aligned} u_t = f(u) u_x &+ \epsilon \big[a_1(u) u_{xx} + a_2(u) u_x^2 \big] \\ &+ \epsilon^2 \big[a_3(u) u_{xxx} + a_4(u) u_{xx} u_x + a_5(u) u_x^3 \big] + \cdots , \end{aligned} \tag{1.5a}$$

$$f(u) \not\equiv 0. \tag{1.5b}$$

Here, ϵ is a parameter, and $f(u)$, $a_1(u)$, $a_2(u)$, $\cdots$ are given smooth functions of u. Following [15, 26], we say that a change of the dependent variable of the form

$$w = W(u) + \sum_{k \geq 1} \epsilon^k W^{[k]}(u; u_x, \ldots, u_k) \tag{1.6}$$

is a *Miura type transformation*, if $W'(u) \not\equiv 0$ and each $W^{[k]}$ is a degree k homogeneous differential polynomial of u. Here, $u_j := \partial_x^j(u)$ is assigned the degree: $\deg u_j = j$, $j \geq 0$. All Miura type transformations form the Miura group.

The $\epsilon \to 0$ limit of equation (1.5)

$$v_t = f(v) v_x, \quad f(v) \not\equiv 0 \tag{1.7}$$

is an evolutionary PDE of hydrodynamic type [13]. The simplest non-trivial example is the dispersionless KdV equation (*aka* the Riemann–Hopf equation or the inviscid Burgers equation):

$$v_t = v v_x. \tag{1.8}$$

This equation is NOT equivalent to the KdV equation (1.1) with $\epsilon \neq 0$ under the Miura group action. However, there exists a remarkable invertible transformation [3, 15, 26]

$$
\begin{aligned}
(1.9)\qquad u = {} & v + \epsilon^2 \partial_x^2 \left(\frac{1}{24} \log v_x \right) \\
& + \epsilon^4 \partial_x^2 \left(\frac{v_{xxxx}}{1152 v_x^2} - \frac{7 v_{xx} v_{xxx}}{1920 v_x^3} + \frac{v_{xx}^3}{360 v_x^4} \right) + \mathcal{O}\left(\epsilon^6\right)
\end{aligned}
$$

transforming (1.1) to (1.8). Such a transformation is called a *quasi-Miura transformation* [15, 26]. We mention that the difference between Miura type and quasi-Miura transformations is simply that the latter allows rational and logarithmic dependence in v_x. We also mention that $\partial_x = \sum_{j\geq 0} v_{j+1} \partial_{v_j} = \sum_{j\geq 0} u_{j+1} \partial_{u_j}$.

A scalar evolutionary PDE (1.5) is called *quasi-trivial* or say *possessing quasi-triviality*, if it can be transformed to its $\epsilon \to 0$ limit via a quasi-Miura transformation.

Liu–Zhang's Theorem ([26]). *For an evolutionary PDE of the form* (1.5) *with $f'(u) \not\equiv 0$, there exists a unique (under some homogeneity condition; see Theorem 4.3 of [26]) quasi-Miura transformation reducing* (1.5) *to its dispersionless limit.*

In this article we consider the following problem.

Problem A. *Give an explicit expression of quasi-triviality of the KdV equation* (1.1).

This problem is *algebraic*, but the solution turns out to be *topological*.

Before presenting a solution to Problem A, we recall some standard notations. For $j \geq 0$, denote $v^{(j)} = v_j := \partial_x^j(v)$. By a partition λ, we mean a non-increasing sequence of non-negative integers $(\lambda_1, \dots, \lambda_{\ell(\lambda)}, 0, \dots)$, where $\ell(\lambda)$ denotes the length of λ, and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell(\lambda)}$ are the nonzero components of λ. The set of all partitions is denoted by $\mathbb{Y}$. Denote by $|\lambda| := \sum_{j=1}^{\ell(\lambda)} \lambda_j$ the weight of λ, by $\mathbb{Y}_k$ the set of partitions of weight k, and by $m_i(\lambda)$ the multiplicity of i in λ, $i \geq 1$. Denote also

$$
(1.10)\qquad m(\lambda)! = \prod_{i\geq 1} m_i(\lambda)!, \quad z_\lambda = m(\lambda)! \prod_{i\geq 1} i^{m_i(\lambda)}.
$$

The partition of 0 is denoted by (0), with $\ell((0)) := 0$ and $|(0)| := 0$. For any $\lambda, \mu \in \mathbb{Y}$, define $\lambda + \mu := (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots)$. Define $\lambda + 1 := \lambda + \left(1^{\ell(\lambda)}\right)$ if $\lambda \neq (0)$, and $(0) + 1 := (0)$. For an arbitrary sequence of indeterminates

$q_1, q_2, \ldots$, denote $q_\lambda := q_{\lambda_1} \cdots q_{\lambda_{\ell(\lambda)}}$ if $\lambda \neq (0)$, and $q_{(0)} := 1$. Introduce also some integers: For any $\lambda, \mu \in \mathbb{Y}$, define

$$Q^{\lambda\mu} := (-1)^{\ell(\lambda)} \sum_{\substack{\mu^1 \in \mathbb{Y}_{\lambda_1}, \ldots, \mu^{\ell(\lambda)} \in \mathbb{Y}_{\lambda_{\ell(\lambda)}} \\ \cup_{q=1}^{\ell(\lambda)} \mu^q = \mu}} \prod_{q=1}^{\ell(\lambda)} \frac{(\lambda_q + \ell(\mu^q))!\,(-1)^{\ell(\mu^q)}}{m(\mu^q)! \prod_{j=1}^{\infty} (j+1)!^{m_j(\mu^q)}}. \tag{1.11}$$

We note that $Q^{\lambda\mu} = 0$ unless $|\lambda| = |\mu|$, and call $\left(Q^{\lambda\mu}\right)_{|\lambda|=|\mu|}$ the *Q-matrices.*

Theorem 1.1. *The quasi-triviality of the KdV equation* (1.1) *has the following expression:*

$$u = v + \epsilon^2 \partial_x^2 \left(M_1(v_x) + \sum_{g=2}^{\infty} \epsilon^{2g-2} M_g(v_x, v_{xx}, \ldots, v_{3g-2}) \right), \tag{1.12}$$

$$M_1(v_x) = \frac{1}{24} \log v_x, \tag{1.13}$$

$$M_g(v_x, v_{xx}, \ldots, v_{3g-2}) = \sum_{\lambda, \mu \in \mathbb{Y}_{3g-3}} \frac{\langle \tau_{\lambda+1} \rangle}{m(\lambda)!} \, Q^{\lambda\mu} \, \frac{v_{\mu+1}}{v_1^{\ell(\mu)+g-1}}, \quad g \geq 2. \tag{1.14}$$

Here, $\langle \tau_{\lambda+1} \rangle$ are the intersection numbers of ψ-classes on the Deligne–Mumford moduli spaces (for the definitions of these numbers see Eq. (3.2)*).*

The proof is in Section 4.

The following three more problems will also be considered.

Problem B. *Give an explicit quasi-triviality of the intermediate Long wave (ILW) equation*

$$u_t = u u_x + \sum_{g \geq 1} \epsilon^{2g} s^{g-1} \frac{|B_{2g}|}{(2g)!} u_{2g+1}. \tag{1.15}$$

Here B_{2g} denote the Bernoulli numbers, defined by $\frac{x}{e^x - 1} =: \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k$.

Problem C. *Give an explicit quasi-triviality of the discrete KdV equation (aka the Volterra lattice equation)*

$$u_t = \frac{1}{\epsilon} \left(e^{u(x+\epsilon)} - e^{u(x-\epsilon)} \right). \tag{1.16}$$

Problem D. *Give an explicit quasi-triviality of the Burgers equation*

$$u_t = u u_x + \epsilon u_{xx}. \tag{1.17}$$

We remark that for an *integrable* PDE of the form (1.5) with $f'(u) \not\equiv 0$, the quasi-triviality of this PDE is the property of the whole corresponding integrable hierarchy.

We will see from solutions to the above problems in Sections 4–5 that the associated essential numbers to each problem (the primitive Hodge integrals for the case of Problems A,B,C, and enumeration of graphs with valencies $\geq$ 3 for the case of Problem D) are *all* contained in a simple nonlinear equation (KdV, ILW, discrete KdV, Burgers, respectively); this is revealed by the deep relationships started by Witten, by Liu–Zhang's theorem and by the so-called inverse Q-matrices (see Definition 2.7 below).

Organization of the paper. In Section 2 we review the topological solution to the Riemann hierarchy. In Section 3 we review the construction of Hodge hierarchy. In Sections 4–5 we give solutions to Problems A–D. Concluding remarks are given in Section 6. A straightforward proof of a technical lemma (Lemma 3.2) is given in Appendix A.

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2. Riemann hierarchy and Q-matrices

The goal of this section is to do some preparations for the later sections. Recall that an evolutionary PDE of the form

$$\begin{aligned} u_s = {} & g(u)u_x + \epsilon\big[b_1(u)u_{xx} + b_2(u)u_x^2\big] \\ & + \epsilon^2\big[b_3(u)u_{xxx} + b_3(u)u_{xx}u_x + b_4(u)u_x^3\big] + \cdots \end{aligned} \tag{2.1}$$

is called an *infinitesimal symmetry* of (1.5) if

$$\partial_s\partial_t u = \partial_t\partial_s u.$$

Following [9], we say equation (1.5) with $f'(u) \not\equiv 0$ is called *integrable*, if it possesses an infinite family of infinitesimal symmetries parameterized by a smooth function of one variable.

The Riemann–Hopf equation (1.8) is integrable: for any smooth function $g(v)$, the PDE

$$v_s = g(v)v_x \tag{2.2}$$

gives an infinitesimal symmetry of (1.8). Let us look at a particular subfamily in these infinitesimal symmetries

$$v_{t_k} = \frac{v^k}{k!} v_x, \quad k \geq 0. \tag{2.3}$$

Observe that equations (2.3) commute pairwise, i.e.,

$$\partial_{t_i} \partial_{t_j} v = \partial_{t_j} \partial_{t_i} v, \quad \forall\, i, j \geq 0. \tag{2.4}$$

Thus (2.3) can be solved together. They are called the *Riemann hierarchy.* The $k = 1$ equation in (2.3) is the Riemann–Hopf equation (1.8). The $k = 0$ equation reads $u_{t_0} = u_x$, so we identify t_0 with x.

We will be interested in solutions of (2.3) in the formal power series ring $\mathbb{C}[[\mathbf{t}]]$. Indeed, consider the initial value problem of (2.3) along with the initial condition:

$$v(x, 0, 0, \dots) = f_0(x) \in \mathbb{C}[[x]], \quad f_0'(0) \neq 0,\ f_0(0) = 0. \tag{2.5}$$

Here, the conditions $f_0'(0) \neq 0$, $f_0(0) = 0$ imply that $f_0(x)$ has a compositional inverse in $\mathbb{C}[[x]]$. (The case $f_0(0) \neq 0$ can also be considered via performing a shift in v; we leave this discussion to the interested readers.) Denote by c_p, $p \geq 0$ the Taylor coefficients of $f_0^{-1}(x)$, i.e.,

$$\sum_{p \geq 0} \frac{c_p}{p!} x^p = f_0^{-1}(x), \quad c_1 \neq 0. \tag{2.6}$$

As explained above, equations (2.3) have a unique solution $v(\mathbf{t})$ in $\mathbb{C}[[\mathbf{t}]]$ with the initial value $v(x, 0, 0, \dots) = f_0(x)$.

Lemma 2.1 ([15]). *The unique solution $v(\mathbf{t})$ satisfies the following equation:*

$$\tilde{t}_0 + \sum_{p \geq 1} \tilde{t}_p \frac{v(\mathbf{t})^p}{p!} = 0, \tag{2.7}$$

where $\tilde{t}_p := t_p - c_p$ ($c_1 \neq 0$). Moreover, solution to (2.7) *in* $\mathbb{C}[[\mathbf{t}]]$ *is unique.*

Equation (2.7) is called the *genus zero Euler–Lagrange equation* for the KdV hierarchy [8, 15].

We now focus on a particular solution to the Riemann hierarchy (2.3), denoted by $v^{\mathrm{top}}(\mathbf{t})$, that is specified by the initial data $f_0(x) = x$. In other words, $c_p = \delta_{p,1}$ is under consideration. The $v^{\mathrm{top}}(\mathbf{t})$ is often called the *topological solution.*

Lemma 2.2. *The $v^{\mathrm{top}}(\mathbf{t})$ has the explicit expression*

$$v^{\mathrm{top}}(\mathbf{t}) = \sum_{k\geq 1} \frac{1}{k} \sum_{\substack{p_1,\dots,p_k\geq 0\\ p_1+\cdots+p_k=k-1}} \frac{t_{p_1}}{p_1!}\cdots\frac{t_{p_k}}{p_k!}. \tag{2.8}$$

Proof. For any solution $v(\mathbf{t})$ in $\mathbb{C}[[\mathbf{t}]]$ to the Riemann hierarchy (2.3), we have

$$v_{t_{k_1}} = \partial_x\left(\frac{v^{k_1+1}}{k_1!\,(k_1+1)}\right), \tag{2.9}$$

$$v_{t_{k_1}t_{k_2}} = \partial_x^2\left(\frac{v^{k_1+k_2+1}}{k_1!\,k_2!\,(k_1+k_2+1)}\right), \tag{2.10}$$

$$v_{t_{k_1}\dots t_{k_N}} = \partial_x^N\left(\frac{v^{k_1+\cdots+k_N+1}}{k_1!\cdots k_N!\,(k_1+\cdots+k_N+1)}\right), \quad \forall\, N\geq 3. \tag{2.11}$$

Here $k_1, k_2, \ldots \geq 0$. The lemma is then proved by noticing $v_m^{\mathrm{top}}(\mathbf{0}) = \delta_{m,1}$. □

Proceed with a simplification of (2.8). Applying ∂_x^m on the both sides of (2.8) we obtain

$$v_m^{\mathrm{top}}(\mathbf{t}) = \sum_{k\geq 1} \sum_{\substack{p_1,\dots,p_k\geq 0\\ p_1+\cdots+p_k=k+m-1}} (k+1)\cdots(k+m-1)\,\frac{t_{p_1}}{p_1!}\cdots\frac{t_{p_k}}{p_k!}, \quad m\geq 1, \tag{2.12}$$

where we recall that $v_m^{\mathrm{top}}(\mathbf{t}) := \partial_x^m\big(v^{\mathrm{top}}(\mathbf{t})\big)$. The following shorthand notations will be used:

(i) Denote $v(\mathbf{t}) = v^{\mathrm{top}}(\mathbf{t})$ unless otherwise specified.

(ii) Denote $v^s = v^s(t_1, t_2, \dots) := v(\mathbf{t})|_{t_0=0}$, and denote $v_j^s = v_j^s(t_1, t_2, \dots) := v_j(\mathbf{t})|_{t_0=0}$, $j\geq 1$.

Obviously $v^s = 0$. The Taylor expansion of $v(\mathbf{t})$ with respect to x then reads

$$v(\mathbf{t}) = v^s + \sum_{j\geq 1} v_j^s\,\frac{x^j}{j!} = \sum_{j\geq 1} v_j^s\,\frac{x^j}{j!}. \tag{2.13}$$

Lemma 2.3. *For $m \geq 1$, the following formula holds true:*

$$v_m^s = \sum_{\mu \in \mathbb{Y}_{m-1}} \frac{(m-1+\ell(\mu))!}{\prod_{j\geq 1}(j+1)!^{m_j(\mu)}} \frac{t_{\mu+1}}{m(\mu)!\,(1-t_1)^{m+\ell(\mu)}}. \tag{2.14}$$

Proof. For $m=1$, we know from (2.12) that $v_1^s = 1/(1-t_1)$. For $m \geq 2$,

$$\begin{aligned} v_m^s &= \sum_{k\geq 1} \sum_{\substack{\lambda\in\mathbb{Y}_{k+m-1}\\ \ell(\lambda)=k}} \frac{(k+m-1)!}{k!} \binom{k}{m_1(\lambda), m_2(\lambda), \cdots} \frac{t_\lambda}{\prod_{i\geq 1} i!^{m_i(\lambda)}} \\ &= \sum_{\substack{m_1,m_2,\cdots\geq 0\\ \sum_{j\geq 1}(j-1)m_j = m-1}} \Big(m_1 + \sum_{j\geq 2} j m_j\Big)! \prod_{j\geq 1} \frac{t_j^{m_j}}{j!^{m_j} m_j!} \\ &= \sum_{\substack{m_2,m_3,\cdots\geq 0\\ \sum_{j\geq 2}(j-1)m_j = m-1}} \Big(\sum_{j\geq 2} j m_j\Big)! \prod_{j\geq 2} \frac{t_j^{m_j}}{j!^{m_j} m_j!} \frac{1}{(1-t_1)^{1+\sum_{j\geq 2} j m_j}}, \end{aligned}$$

where the last equality uses Newton's binomial identity: $(1-x)^{-1-k} = \sum_{s\geq 0}\binom{s+k}{k} x^s$. $\square$

For each partition $\mu \in \mathbb{Y}$, we call the integer

$$L(\mu) := \frac{\big(|\mu| + \ell(\mu)\big)!\,(-1)^{\ell(\mu)}}{m(\mu)!\,\prod_{j\geq 1}(j+1)!^{m_j(\mu)}} = (-1)^{\ell(\mu)} \frac{|\mu+1|!}{z_{\mu+1}} \tag{2.15}$$

the *Lagrange number* associated to μ. The first few Lagrange numbers are

$$L((0)) = 1,\ L((1)) = -1,\ L((2)) = -1,\ L\big((1^2)\big) = 3,\ L((3)) = -6,$$
$$L((2,1)) = 10,\ L\big((1^3)\big) = -15,\ L((n)) = -1,\ L\big((1^n)\big) = (-1)^n (2n-1)!!.$$

Using the Lagrange number we can write formula (2.14) as

$$v_n^s = \sum_{\mu\in\mathbb{Y}_{n-1}} (-1)^{\ell(\mu)} L(\mu) \frac{t_{\mu+1}}{(1-t_1)^{n+\ell(\mu)}}, \quad n \geq 1. \tag{2.16}$$

Lemma 2.4 (Zhou [30]). *The following formulae hold true:*

$$1 - t_1 = \frac{1}{v_1^s}, \tag{2.17}$$

$$-t_k = \sum_{\mu\in\mathbb{Y}_{k-1}} L(\mu) \frac{v_{\mu+1}^s}{(v_1^s)^{1+|\mu+1|}}, \quad k \geq 2. \tag{2.18}$$

We remark that both (2.16) and Lemma 2.4 can also be proved straightforwardly by using the Lagrange inversion (cf. e.g. [22, 25]).

Definition 2.5. *For any two partitions λ, μ, define $Q^{(0)(0)} := 1$, and define*

$$Q^{\lambda\mu} := (-1)^{\ell(\lambda)} \sum_{\substack{\mu^1 \in \mathbb{Y}_{\lambda_1}, \dots, \mu^{\ell(\lambda)} \in \mathbb{Y}_{\lambda_{\ell(\lambda)}} \\ \mu^1 \cup \mu^2 \cup \cdots \cup \mu^{\ell(\lambda)} = \mu}} \prod_{q=1}^{\ell(\lambda)} L(\mu^q). \tag{2.19}$$

For $k \geq 0$, we call $\left(Q^{\lambda\mu}\right)_{|\lambda|=|\mu|=k}$ the Q-matrices.

Lemma 2.6. *The following formula holds true:*

$$t_{\lambda+1} = \sum_{\mu \in \mathbb{Y}_{|\lambda|}} Q^{\lambda\mu} \frac{v^s_{\mu+1}}{(v^s_x)^{l(\mu)+|\lambda+1|}}, \quad \forall\, \lambda \in \mathbb{Y}. \tag{2.20}$$

Proof. For any partition $\lambda \in \mathbb{Y}$, we have

$$\begin{aligned}
t_{\lambda+1} &= (-1)^{\ell(\lambda)} \prod_{q=1}^{l(\lambda)} \sum_{\mu \in \mathbb{Y}_{\lambda_q}} L(\mu) \frac{v^s_{\mu+1}}{(v^s_x)^{1+|\mu+1|}} \\
&= (-1)^{\ell(\lambda)} \sum_{\mu^1 \in \mathbb{Y}_{\lambda_1}, \dots, \mu^{\ell(\lambda)} \in \mathbb{Y}_{\lambda_{\ell(\lambda)}}} \prod_{q=1}^{\ell(\lambda)} L(\mu^q) \frac{v^s_{\mu^q+1}}{(v^s_x)^{l(\mu^q)+\lambda_q+1}} \\
&= \sum_{\mu \in \mathbb{Y}_{|\lambda|}} \frac{v^s_{\mu+1}}{(v^s_x)^{l(\mu)+|\lambda+1|}} (-1)^{\ell(\lambda)} \sum_{\substack{\mu^1 \in \mathbb{Y}_{\lambda_1}, \dots, \mu^{l(\lambda)} \in \mathbb{Y}_{\lambda_{l(\lambda)}} \\ \mu^1 \cup \mu^2 \cup \cdots \cup \mu^{\ell(\lambda)} = \mu}} L(\mu^q).
\end{aligned}$$

The lemma is proved. □

Definition 2.7. *Define $Q_{(0)(0)} := 1$, and define*

$$Q_{\mu\rho} := \sum_{\substack{\rho^1 \in \mathbb{Y}_{\mu_1}, \dots, \rho^{\ell(\mu)} \in \mathbb{Y}_{\mu_{\ell(\mu)}} \\ \rho^1 \cup \rho^2 \cup \cdots \cup \rho^{\ell(\mu)} = \rho}} \prod_{q=1}^{\ell(\mu)} |L(\rho^q)|, \tag{2.21}$$

where μ, ρ are two arbitrary partitions. We call $(Q_{\mu\rho})_{|\mu|=|\rho|}$ the inverse Q-matrices.

Lemma 2.8. *The following formula holds true:*

$$v^s_{\mu+1} = \sum_{\rho\in\mathbb{Y}_{|\mu|}} Q_{\mu\rho}\,\frac{t_{\rho+1}}{(1-t_1)^{\ell(\rho)+|\mu+1|}}, \quad \forall\,\mu\in\mathbb{Y}. \tag{2.22}$$

Proof. For any partition $\mu\in\mathbb{Y}$, we have

$$\begin{aligned}
v_{\mu+1} &= \prod_{q=1}^{l(\mu)} \sum_{\rho\in\mathbb{Y}_{\mu_q}} |L(\rho)|\frac{t_{\rho+1}}{(1-t_1)^{1+|\rho+1|}}\\
&= \sum_{\rho^1\in\mathbb{Y}_{\mu_1},\dots,\rho^{\ell(\mu)}\in\mathbb{Y}_{\mu_{\ell(\mu)}}} \prod_{q=1}^{\ell(\mu)} |L(\rho^q)|\,\frac{t_{\rho^q+1}}{(1-t_1)^{l(\rho^q)+\mu_q+1}}\\
&= \sum_{\rho\in\mathbb{Y}_{|\mu|}} \frac{t_{\rho+1}}{(1-t_1)^{\ell(\rho)+|\mu+1|}} \sum_{\substack{\rho^1\in\mathbb{Y}_{\mu_1},\dots,\rho^{\ell(\mu)}\in\mathbb{Y}_{\mu_{\ell(\mu)}}\\ \rho^1\cup\rho^2\cup\cdots\cup\rho^{\ell(\mu)}=\rho}} |L(\rho^q)|.
\end{aligned}$$

The lemma is proved. □

Lemma 2.9. *We have*

a) $Q^{\lambda\mu}=Q_{\lambda\mu}=0$ *if* $|\lambda|\neq|\mu|$.

b) *The Q-matrices* $\left(Q^{\lambda\mu}\right)_{|\lambda|=|\mu|}$ *and the inverse Q-matrices* $(Q_{\lambda\mu})_{|\lambda|=|\mu|}$ *are upper triangular with respect to the reverse lexicographic ordering.*

c) $Q^{\lambda\mu}$ *are integers and* $Q_{\lambda\mu}$ *are positive integers.*

d) $Q^{\lambda\lambda}=1$, $Q_{\lambda\lambda}=1$, $\forall\,\lambda\in\mathbb{Y}$.

e) $Q^{(n)\,\mu}=-L(\mu)$, $Q_{(n)\,\mu}=|L(\mu)|$, $\forall\mu\in\mathbb{Y}$.

f) $\forall\,k\geq 0$, $(Q_{\lambda\mu})_{|\lambda|=|\mu|=k}\left(Q^{\lambda\mu}\right)_{|\lambda|=|\mu|=k}=I$, *where I denotes the identity matrix.*

Proof. a)–e) are easy consequences of Definition 2.5 and (2.15). Note that $\forall\,\lambda,\rho\in\mathbb{Y}$,

$$\begin{aligned}
t_{\lambda+1} &= \sum_{\mu\in\mathbb{Y}_{|\lambda|}} Q^{\lambda\mu}\,\frac{v^s_{\mu+1}}{(v^s_x)^{l(\mu)+|\lambda|+1}}\\
&= \sum_{\mu,\rho\in\mathbb{Y}_{|\lambda|}} Q^{\lambda\mu}\,Q_{\mu\rho}\frac{1}{(v^s_x)^{l(\mu)+|\lambda+1|}}\,\frac{t_{\rho+1}}{(1-t_1)^{l(\rho)+|\mu+1|}}.
\end{aligned} \tag{2.23}$$

The assertion f) is then proved by noticing that $1-t_1=\frac{1}{v^s_1}$. □

The first several Q-matrices and inverse Q-matrices are given by

$$(Q^{\lambda\mu}) = (1), \quad (Q_{\lambda\mu}) = (1), \quad |\lambda| = |\mu| = 0;$$
$$(Q^{\lambda\mu}) = (1), \quad (Q_{\lambda\mu}) = (1), \quad |\lambda| = |\mu| = 1;$$
$$(Q^{\lambda\mu}) = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}, \quad (Q_{\lambda\mu}) = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \quad |\lambda| = |\mu| = 2;$$
$$(Q^{\lambda\mu}) = \begin{pmatrix} 1 & -10 & 15 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}, \quad (Q_{\lambda\mu}) = \begin{pmatrix} 1 & 10 & 15 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}, \quad |\lambda| = |\mu| = 3;$$
$$(Q^{\lambda\mu}) = \begin{pmatrix} 1 & -15 & -10 & 105 & -105 \\ 0 & 1 & 0 & -10 & 15 \\ 0 & 0 & 1 & -6 & 9 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$
$$(Q_{\lambda\mu}) = \begin{pmatrix} 1 & 15 & 10 & 105 & 105 \\ 0 & 1 & 0 & 10 & 15 \\ 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad |\lambda| = |\mu| = 4;$$
$$(Q^{\lambda\mu}) = \begin{pmatrix} 1 & -21 & -35 & 210 & 280 & -1260 & 945 \\ 0 & 1 & 0 & -15 & -10 & 105 & -105 \\ 0 & 0 & 1 & -3 & -10 & 45 & -45 \\ 0 & 0 & 0 & 1 & 0 & -10 & 15 \\ 0 & 0 & 0 & 0 & 1 & -6 & 9 \\ 0 & 0 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$
$$(Q_{\lambda\mu}) = \begin{pmatrix} 1 & 21 & 35 & 210 & 280 & 1260 & 945 \\ 0 & 1 & 0 & 15 & 10 & 105 & 105 \\ 0 & 0 & 1 & 3 & 10 & 45 & 45 \\ 0 & 0 & 0 & 1 & 0 & 10 & 15 \\ 0 & 0 & 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad |\lambda| = |\mu| = 5.$$

3. Hodge integrals and integrable hierarchies: a short review

Let $\overline{\mathcal{M}}_{g,n}$ be the Deligne–Mumford moduli space of stable algebraic curves of genus g with n distinct marked points. Here the non-negative integers g, n

should satisfy the stability condition

$$2g - 2 + n > 0. \tag{3.1}$$

Denote by $\mathcal{L}_i$, $i = 1, \dots, n$ the i_{th} tautological line bundle on $\overline{\mathcal{M}}_{g,n}$, by $\mathbb{E}_{g,n}$ the Hodge bundle on $\overline{\mathcal{M}}_{g,n}$, by $\psi_i := c_1(\mathcal{L}_i)$ the ψ-class, and by $\lambda_j := c_j(\mathbb{E}_{g,n})$, $j = 1, \dots, g$ the j_{th} Chern class of $\mathbb{E}_{g,n}$. The following integrals

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \lambda_{j_1} \cdots \lambda_{j_m} =: \langle \lambda_{j_1} \cdots \lambda_{j_m} \tau_{k_1} \cdots \tau_{k_n} \rangle_g \tag{3.2}$$

are some rational numbers, called *Hodge integrals of a point.* Here $n, m \geq 0$, $k_1, \dots, k_n \geq 0$, $j_1, \dots, j_m \geq 1$. From the degree-dimension counting, these rational numbers vanish unless

$$j_1 + \cdots + j_m + k_1 + \cdots + k_n = 3g - 3 + n. \tag{3.3}$$

The case $m = 0$ in (3.2) gives the Gromov–Witten (GW) invariants of a point. For this case, the degree-dimension counting reads $k_1 + \cdots + k_n = 3g - 3 + n$. So one could simply write $\langle \tau_{k_1} \cdots \tau_{k_n} \rangle_g$ as $\langle \tau_{k_1} \cdots \tau_{k_n} \rangle$, which are the numbers used in (1.14).

It is appropriate to collect Hodge integrals into generating series. The genus g Hodge potential associated to $\lambda_{i_1} \cdots \lambda_{i_m}$ is defined as the following generating series of Hodge integrals:

$$\mathcal{H}_g(\lambda_{i_1} \cdots \lambda_{i_m}; \mathbf{t}) := \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1, \dots, k_n \geq 0} \big\langle \lambda_{i_1} \cdots \lambda_{i_m} \tau_{k_1} \cdots \tau_{k_n} \big\rangle_g \, t_{k_1} \cdots t_{k_n}, \tag{3.4}$$

where $\mathbf{t} = (t_0, t_1, t_2, \dots)$. Denote by $\mathrm{ch}_r := \mathrm{ch}_r(\mathbb{E}_{g,n})$, $r \geq 0$ components of the Chern character of $\mathbb{E}_{g,n}$. We call the generating series

$$\begin{aligned} \mathcal{H}_g(\mathbf{t}; \mathbf{s}) := & \sum_{m,n=0}^{\infty} \frac{1}{m!\, n!} \\ & \times \sum_{\substack{j_1, \dots, j_m \geq 1 \\ k_1, \dots, k_n \geq 0}} \langle \mathrm{ch}_{2j_1 - 1} \cdots \mathrm{ch}_{2j_m - 1} \tau_{k_1} \cdots \tau_{k_n} \rangle_g \, s_{j_1} \cdots s_{j_m} \, t_{k_1} \cdots t_{k_n} \end{aligned} \tag{3.5}$$

the *genus g Hodge potential.* Here $\mathbf{s} = (s_1, s_2, \dots)$. The restriction $\mathcal{H}_g(\mathbf{t}; \mathbf{0}) =: \mathcal{F}_g(\mathbf{t})$ is called the *genus g GW potential.* We also define the

Hodge potential $\mathcal{H}$ and the GW potential $\mathcal{F}$ by

$$\mathcal{H} = \mathcal{H}(\mathbf{t};\mathbf{s};\epsilon) := \sum_{g\geq 0} \epsilon^{2g-2}\mathcal{H}_g(\mathbf{t};\epsilon), \quad \mathcal{F} = \mathcal{F}(\mathbf{t};\mathbf{s};\epsilon) := \sum_{g\geq 0} \epsilon^{2g-2}\mathcal{F}_g(\mathbf{t};\epsilon).$$

Their exponentials

$$Z_E = Z_E(\mathbf{t};\mathbf{s};\epsilon) := \exp\left(\mathcal{H}(\mathbf{t};\mathbf{s};\epsilon)\right), \quad Z = Z(\mathbf{t};\epsilon) := \exp\left(\mathcal{F}(\mathbf{t};\epsilon)\right)$$

are called the *partition functions* of Hodge integrals and of GW invariants, respectively.

It was conjectured by Witten [29] and proved by Kontsevich [24] that Z is a particular tau-function of the KdV hierarchy (the Witten–Kontsevich (WK) theorem). Z is now also known as the WK tau-function. Define D_k as the following linear differential operators [18]:

$$D_k := \sum_{p\geq 0} t_p \frac{\partial}{\partial t_{p+2k-1}} - \frac{\epsilon^2}{2}\sum_{j=0}^{2k-2}(-1)^j \frac{\partial^2}{\partial t_j \partial t_{2k-2-j}}, \quad k \geq 1. \tag{3.6}$$

Faber–Pandharipande [18] proved that the partition function of Hodge integrals $Z_E(\mathbf{t};\mathbf{s};\epsilon)$ is the unique power series solution to the following linear equations

$$\frac{\partial Z_E}{\partial s_k} = -\frac{B_{2k}}{(2k)!} D_k\big(Z_E\big), \quad k \geq 1 \tag{3.7}$$

along with the initial condition

$$Z_E(\mathbf{t};\mathbf{0};\epsilon) = Z(\mathbf{t};\epsilon). \tag{3.8}$$

This unique solution has the form

$$Z_E(\mathbf{t};\mathbf{s};\epsilon) = \exp\left(-\sum_{k\geq 1}\frac{B_{2k}}{(2k)!} s_k D_k\right)\big(Z(\mathbf{t};\epsilon)\big). \tag{3.9}$$

Lemma 3.1. *The power series Z_E satisfies the following two linear equations:*

$$\text{(string equation)} \quad \sum_{p\geq 0} t_{p+1}\frac{\partial Z_E}{\partial t_p} + \frac{t_0^2}{2\epsilon^2} Z_E + \frac{s_1}{24} Z_E = \frac{\partial Z_E}{\partial t_0}, \tag{3.10}$$

$$\text{(dilaton equation)} \quad \sum_{p\geq 0} t_p\frac{\partial Z_E}{\partial t_p} + \epsilon\frac{\partial Z_E}{\partial \epsilon} + \frac{1}{24} Z_E = \frac{\partial Z_E}{\partial t_0}. \tag{3.11}$$

Proof. We have

$$\langle\gamma\tau_{k_1}\cdots\tau_{k_n}\tau_0\rangle_g = \sum_{j=1}^{n} \left\langle \gamma\, \tau_{k_1-\delta_{1j}}\cdots\tau_{k_n-\delta_{nj}}\right\rangle_g, \tag{3.12}$$

where $\gamma = \lambda_{i_1}\cdots\lambda_{i_\ell}$, $i_1,\dots,i_\ell \geq 1$ ($\gamma := 1$ if $\ell = 0$), and $\langle\gamma\,\tau_{j_1}\cdots\tau_{j_n}\rangle_g := 0$ if $\{j_1,\dots,j_n\}$ contain negative integers. It should be noted that there are two exceptional cases:

(a) $g = 0$, $n = 2$, $\gamma = 1$. We have $\langle\tau_0^3\rangle_0 = 1$.

(b) $g = 1$, $n = 0$, $\gamma = s\,\lambda_1$. We have

$$\langle\lambda_1\tau_0\rangle_1 = \frac{s}{24}. \tag{3.13}$$

They correspond to the terms $\frac{t_0^2}{2\epsilon^2}Z_{\mathbb{E}}$, $\frac{s_1}{24}Z_{\mathbb{E}}$ in (3.10), respectively. This proves the string equation. Similarly, one can show that

$$\langle\gamma\tau_{k_1}\cdots\tau_{k_n}\tau_1\rangle_g = (2g-2+n)\langle\gamma\tau_{k_1}\cdots\tau_{k_n}\rangle_g. \tag{3.14}$$

There is one exceptional case: $g = n = 1$, $\gamma = 1$. We have $\langle\tau_1\rangle_1 = \frac{1}{24}$. This proves (3.11). □

Following [20] (cf. also [21, 30]), we call $\langle\gamma\tau_{k_1}\cdots\tau_{k_n}\rangle_g$ a *primitive* Hodge integral of a point, if $k_1,\dots,k_n \geq 2$.

In [15] Dubrovin–Zhang (DZ) introduced the quasi-triviality approach to construct *the* integrable hierarchy for Gromov–Witten invariants of an arbitrary smooth projective variety X with semisimple quantum cohomology. For the case $X =$ a point, the DZ hierarchy coincides with the KdV hierarchy. The interesting fact is that *one particular solution can contain the full information of the unique integrable system [6, 10–12, 15].* Recently, the integrable hierarchy for Hodge integrals called the *Hodge hierarchy* (*aka* the DZ hierarchy for Hodge integrals) was constructed in [11] (see also [6]) by using the quasi-triviality approach. Let us review the construction. The genus 0 and genus 1 Hodge potentials have the form

$$\mathcal{H}_0(\mathbf{t};\mathbf{s}) = \mathcal{F}_0(\mathbf{t}) = \sum_{n\geq 3}\frac{1}{n(n-1)(n-2)}\sum_{k_1+\cdots+k_n=n-3}\frac{t_{k_1}}{k_1!}\cdots\frac{t_{k_n}}{k_n!}, \tag{3.15}$$

$$\mathcal{H}_1(\mathbf{t};\mathbf{s}) = H_1\big(v^{\mathrm{top}}(\mathbf{t}), v_x^{\mathrm{top}}(\mathbf{t})\big), \tag{3.16}$$

where $H_1(v,v_x) := \frac{1}{24}\log v_x + \frac{1}{24}v$. For higher genera, the following lemma is useful.

Lemma 3.2 (Theorem 1.3 of [11]). *For each $g \geq 2$, there exist a unique element*

$$H_g = H_g(v, v_1, \ldots, v_{3g-2}; s_1, \ldots, s_g)$$

in $C^\infty(v)\big[v_1, v_1^{-1}; v_2, \ldots, v_{3g-2}; s_1, \ldots, s_g\big]$ *satisfying*

$$\mathcal{H}_g(\mathbf{t}; \mathbf{s}) = H_g\big(v(\mathbf{t}), v_x(\mathbf{t}), v_2(\mathbf{t}), \ldots, v_{3g-2}(\mathbf{t}); s_1, \ldots, s_g\big), \tag{3.17}$$

$$\deg H_g = 2g - 2, \tag{3.18}$$

where $v(\mathbf{t}) := v^{\mathrm{top}}(\mathbf{t})$*. Moreover, define* $\overline{\deg}\, v_k := k - 1$ *and* $\overline{\deg}\, s_k := 2k - 1$*,* $k \geq 1$*, then*

$$\overline{\deg}\, H_g \leq 3g - 3. \tag{3.19}$$

A straightforward proof of this lemma by using the string and dilaton equations (cf. Lemma 3.1) is given in Appendix A.

By Lemma 3.2 we know that the change of the dependent variable

$$v \mapsto w = v + \sum_{g \geq 1} \epsilon^{2g} \partial_x^2\big(H_g\big) \tag{3.20}$$

is a quasi-Miura transformation. Here $\partial_x = \sum_{k \geq 0} v_{k+1} \partial_{v_k}$. The Hodge hierarchy is defined as the PDE system obtained from the Riemann hierarchy (2.3) under the quasi-Miura transformation (3.20). Each member of the Hodge hierarchy is proven to have the form (1.5), and so is integrable [6, 11]. We note that the uniqueness part of Lemma 3.2 was implicit in Theorem 1.3 of [11]. (It can be deduced from the statement in Theorem 1.3 of [11] that H_g is independent from the choice of a solution; or, it can be directly proved by using an argument similar to that appears in the proof of Theorem 4.2.4 in the first arXiv preprint version of [12], which uses the "transcendental property" of $v^{\mathrm{top}}(\mathbf{t})$.)

Theorem ([11]). Z_E *is a particular tau-function of the Hodge hierarchy.*

Lemma 3.3. *The following formulae hold true:*

$$\frac{\partial H_g}{\partial v} = 0, \quad \forall\, g \geq 2. \tag{3.21}$$

Proof. Take $v = v(\mathbf{t}) = v^{\mathrm{top}}(\mathbf{t})$. Equation (3.10) implies that for $g \geq 2$, $\sum_{p \geq 0} t_{p+1} \frac{\partial \mathcal{H}_g}{\partial t_p} = \frac{\partial \mathcal{H}_g}{\partial t_0}$. Substituting (3.17) in this equation and using the

Riemann hierarchy we obtain

$$\sum_{k=0}^{3g-2} \frac{\partial H_g}{\partial v_k} \partial_x^{k+1} \left(\sum_{p\geq 0} t_{p+1} \frac{v^{p+1}}{(p+1)!} \right) = \sum_{k=0}^{3g-2} \frac{\partial H_g}{\partial v_k} \partial_x^{k+1}(v). \tag{3.22}$$

Substituting equation (2.7) with $c_p = \delta_{p,1}$ in the above equation we arrive at (3.21). □

We now formulate a theorem providing more accurate expressions for H_g, $g \geq 2$.

Theorem 3.4. *The genus g Hodge potentials have the following expressions:*

$$\mathcal{H}_0(\mathbf{t};\mathbf{s}) = \mathcal{F}_0(\mathbf{t}), \tag{3.23}$$

$$\mathcal{H}_1(\mathbf{t};\mathbf{s}) = M_1(v_x(\mathbf{t})) + \frac{s_1}{24} v(\mathbf{t}), \tag{3.24}$$

$$\begin{aligned} \mathcal{H}_g(\mathbf{t};\mathbf{s}) = {} & M_g(v_x(\mathbf{t}), \ldots, v_{3g-2}(\mathbf{t})) \\ & + \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{1\leq j_1,\ldots,j_m\leq g} s_{j_1} \cdots s_{j_m} v_x(\mathbf{t})^{-g+1+2\sum_{a=1}^m j_a - m} \\ & \times \sum_{\lambda,\mu\in\mathbb{Y}_{3g-3+m-2\sum_{a=1}^m j_a}} \frac{\langle \mathrm{ch}_{2j_1-1}\cdots \mathrm{ch}_{2j_m-1}\, \tau_{\lambda+1}\rangle_g}{m(\lambda)!} \\ & \times Q^{\lambda\mu}\, \frac{v_{\mu+1}(\mathbf{t})}{v_x(\mathbf{t})^{l(\mu)}}, \quad g\geq 2. \end{aligned} \tag{3.25}$$

Here, M_g, $g \geq 1$, are defined in (1.13)–(1.14), *and*

$$v(\mathbf{t}) = \sum_{k\geq 1} \frac{1}{k} \sum_{p_1+\cdots+p_k=k-1} \frac{t_{p_1}}{p_1!} \cdots \frac{t_{p_k}}{p_k!}.$$

Proof. The $g = 0, 1$ cases are already known (cf. (3.15) and (3.16)). For $g \geq 2$, we have

$$\mathcal{H}_g(\gamma;\mathbf{t}) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{k_1,\ldots,k_n\geq 0\\ k_1+\cdots+k_n+\deg\gamma=3g-3+n}} \langle \gamma\, \tau_{k_1}\cdots\tau_{k_n}\rangle_{g,n}\, t_{k_1}\cdots t_{k_n}. \tag{3.26}$$

Here, $\gamma := \mathrm{ch}_{2i_1-1}\cdots\mathrm{ch}_{2i_m-1}$, $i_1,\dots,i_m \ge 1$ ($\gamma := 1$, if $m=0$). By definition we know that

$$\mathcal{H}_g(\mathbf{t};\mathbf{s}) = \sum_{m=0}^{\infty}\frac{1}{m!}\sum_{1\le i_1,\dots,i_m\le g}\mathcal{H}_g(\mathrm{ch}_{2i_1-1}\cdots\mathrm{ch}_{2i_m-1};\mathbf{t})\, s_{i_1}\cdots s_{i_m}. \tag{3.27}$$

According to Lemma 3.2 there exist functions $H_g(\gamma; v_x,\dots,v_{3g-2})$ such that

$$\mathcal{H}_g(\gamma;\mathbf{t}) = H_g\big(\gamma; v_x(\mathbf{t}),\dots,v_{3g-2}(\mathbf{t})\big), \quad g\ge 2, \tag{3.28}$$

where $H_g(\gamma; v_x,\dots,v_{3g-2}) \in C^\infty(v)\big[v_1, v_1^{-1}; v_2,\dots,v_{3g-2}\big]$. Taking $t_0=0$ in equation (3.26) we find

$$\begin{aligned}\mathcal{H}_g(\gamma;0,t_1,t_2,\dots) &= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\substack{k_1,\dots,k_n\ge 1\\ k_1+\cdots+k_n+\deg\gamma=3g-3+n}}\langle\gamma\,\tau_{k_1}\cdots\tau_{k_n}\rangle_{g,n}\, t_{k_1}\cdots t_{k_n}\\ &= \sum_{\substack{m_1,m_2,m_3,\dots\ge 0\\ \sum(i-1)m_i=3g-3-\deg\gamma}}\big\langle\gamma\,\tau_1^{m_1}\tau_2^{m_2}\cdots\big\rangle_g\prod_{i=1}^{\infty}\frac{t_i^{m_i}}{m_i!}.\end{aligned} \tag{3.29}$$

Substituting the dilaton equation (3.11) into (3.29) we obtain

$$\begin{aligned}\mathcal{H}_g(\gamma;0,t_1,t_2,\dots) &= \sum_{\substack{m_1,m_2,m_3,\dots\ge 0\\ \sum(i-1)m_i=3g-3-\deg\gamma}}\frac{(\sum m_i+2g-3)!}{(\sum m_i+2g-3-m_1)!}\\ &\qquad\times\langle\gamma\,\tau_2^{m_2}\tau_3^{m_3}\cdots\rangle_g\prod_{i=1}^{\infty}\frac{t_i^{m_i}}{m_i!}\\ &= \sum_{\substack{m_2,m_3,m_4,\dots\ge 0\\ \sum_{i=2}^{\infty}(i-1)m_i=3g-3-\deg\gamma}}\frac{\langle\gamma\,\tau_2^{m_2}\tau_3^{m_3}\cdots\rangle_g}{(1-t_1)^{\sum_{i=2}^{\infty}m_i+2g-2}}\prod_{i=2}^{\infty}\frac{t_i^{m_i}}{m_i!}\\ &= \sum_{\lambda\in\mathbb{Y}_{3g-3-\deg\gamma}}\frac{\langle\gamma\,\tau_{\lambda+1}\rangle_g}{(1-t_1)^{\ell(\lambda)+2g-2}}\,\frac{t_{\lambda+1}}{m(\lambda)!}.\end{aligned} \tag{3.30}$$

Here we have used Newton's binomial identity $(1-x)^{-1-k} = \sum_{s=0}^{\infty}\binom{s+k}{k}x^s$. Substituting formula (2.20) and (2.17) into (3.30) we obtain

$$\begin{aligned}\mathcal{H}_g(\gamma;0,t_1,t_2,\dots) &= \sum_{\gamma\in\mathbb{Y}_{3g-3-\deg\gamma}} \langle\gamma\,\tau_{\lambda+1}\rangle_g\,(v_x^s)^{2g-2+\ell(\gamma)}\,\frac{(-1)^{\ell(\lambda)}}{m(\lambda)!}\\ &\quad\times\sum_{\mu\in\mathbb{Y}_{|\lambda|}} Q^{\lambda\mu}\,\frac{v^s_{\mu+1}}{(v_x^s)^{l(\mu)+|\lambda+1|}}\\ &= (v_x^s)^{-g+1+\deg\gamma}\sum_{\lambda,\mu\in\mathbb{Y}_{3g-3-\deg\gamma}}\frac{\langle\gamma\,\tau_{\lambda+1}\rangle_g}{m(\lambda)!}\,Q^{\lambda\mu}\,\frac{v^s_{\mu+1}}{(v_x^s)^{l(\mu)}}. \qquad (3.31)\end{aligned}$$

Finally, due to Lemma 3.2 and Lemma 3.3, $\mathcal{H}_g(\gamma;\mathbf{t})$ must have the form

$$\mathcal{H}_g(\gamma;\mathbf{t}) = \sum_{q\geq 0} v_x^{-g+1+\deg\gamma+q}\sum_{\mu\in\mathbb{Y}_{3g-3-\deg\gamma-q}} c_\mu^{g,q}\,\frac{v_{\mu+1}}{v_x^{l(\mu)}}. \qquad (3.32)$$

Taking $t_0=0$ we have

$$\mathcal{H}_g(\gamma;0,t_1,t_2,\dots) = \sum_{q\geq 0}(v_x^s)^{-g+1+\deg\gamma+q}\sum_{\mu\in\mathbb{Y}_{3g-3-\deg\gamma-q}} c_\mu^{g,q}\,\frac{v^s_{\mu+1}}{(v_x^s)^{l(\mu)}}. \qquad (3.33)$$

Comparing equations (3.31) and (3.33) we find

$$c_\mu^{g,q}=0,\ \text{if } q\geq 1;\qquad c_\mu^{g,q} = \sum_{\lambda\in\mathbb{Y}_{3g-3-\deg\gamma}}\frac{\langle\gamma\,\tau_{\lambda+1}\rangle_g}{m(\lambda)!}\,Q^{\lambda\mu},\ \text{if } q=0. \qquad (3.34)$$

The theorem is proved. ☐

Corollary 3.5. *For $g\geq 2$, $H_g(v_x,\dots,v_{3g-2};s_1,\dots,s_g)$ is homogenous of degree $3g-3$ with respect to $\overline{\deg}$.*

It follows from Mumford's relation

$$\Lambda_g^\vee(s)\Lambda_g^\vee(-s) = (-1)^g s^{2g} \qquad (3.35)$$

as well as from the relationship between Schur basis and power sum basis of symmetric functions that the infinite set $\{\lambda_{j_1}\cdots\lambda_{j_n}\}$ and the infinite set $\{\mathrm{ch}_{2i_1-1}\cdots\mathrm{ch}_{2i_m-1}\}$ span the same infinite dimensional vector space. Here $\Lambda_g^\vee(s) := \sum_{i=0}^g(-s)^i\lambda_{g-i}$, $\lambda_0:=1$. Therefore, for any linear combination $\gamma = \sum_n\sum_{j_1,\dots,j_n}A_{j_1,\dots,j_n}\lambda_{j_1}\cdots\lambda_{j_n} = \sum_m\sum_{i_1,\dots,i_m}B_{i_1,\dots,i_m}\mathrm{ch}_{2i_1-1}\cdots\mathrm{ch}_{2i_m-1}$, the function $H_g(\gamma;v,v_x,\dots,v_{3g-2};s_1,\dots,s_g)$ is also defined (via linear combination).

Example 3.6. $\gamma = \lambda_g\lambda_{g-1}\lambda_{g-2}$. *Noting that*

$$\langle\lambda_g\lambda_{g-1}\lambda_{g-2}\rangle_g = \frac{1}{2(2g-2)!}\frac{|B_{2g-2}|}{2g-2}\frac{|B_{2g}|}{2g},$$

we have

$$H_g(\lambda_g\lambda_{g-1}\lambda_{g-2}; v_1, \ldots, v_{3g-2}) = \frac{1}{2(2g-2)!}\frac{|B_{2g-2}|}{2g-2}\frac{|B_{2g}|}{2g}v_x^{2g-2}, \quad g \geq 2.$$

Example 3.7. $\gamma = \lambda_g$. *The λ_g-conjecture proven for example in [18] tells that*

$$\langle\lambda_g\,\tau_{k_1}\cdots\tau_{k_n}\rangle_g = \frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!}\frac{(2g-3+n)!}{k_1!\cdots k_n!}.$$

Therefore,

$$\begin{aligned} &H_g(\gamma; v_1, \ldots, v_{3g-2}) \\ &\quad= \frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!}\,v_x\sum_{\lambda,\mu\in\mathbb{Y}_{2g-3}}\frac{(2g-3+\ell(\lambda))!}{z_{\lambda+1}}\frac{v_{\mu+1}}{v_x^{\ell(\mu)}}Q^{\lambda\mu} \\ &\quad= \frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!}\sum_{\lambda,\mu\in\mathbb{Y}_{2g-3}}(-1)^{\ell(\lambda)}L(\lambda)\,Q^{\lambda\mu}\,\frac{v_{\mu+1}}{v_x^{\ell(\mu)-1}}. \end{aligned}$$

Noting that due to e) and f) of Lemma 2.9, $Q^{\lambda\mu}$ satisfy the following property:

$$\sum_{\lambda\in\mathbb{Y}_{2g-3}}(-1)^{\ell(\lambda)}L(\lambda)\,Q^{\lambda\mu} = \delta_{\mu,(2g-3)}, \quad g \geq 2.$$

Hence we obtain

$$H_g\big(\lambda_g; v_1, \ldots, v_{3g-2}\big) = \frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!}\,v_{2g-2}, \quad g \geq 2.$$

Example 3.8. $\gamma = \lambda_g\lambda_{g-1}$. *Getzler–Pandharipande [19] proved that for $k_1, \ldots, k_n \geq 1$,*

$$\langle\lambda_g\lambda_{g-1}\,\tau_{k_1}\cdots\tau_{k_n}\rangle_g = \frac{(2g+n-3)!}{(2k_1-1)!!\cdots(2k_n-1)!!}\frac{|B_{2g}|}{2^{2g-1}(2g)!}.$$

Therefore we have

$$H_g(\lambda_g\lambda_{g-1}; v_1, \dots, v_{3g-2}) \\ = \frac{|B_{2g}|}{2^{2g-1}(2g)!}\, v_x^g \sum_{\lambda,\mu\in\mathbb{Y}_{g-2}} \frac{(2g-3+\ell(\lambda))!}{\prod_{i\geq 1}(2i+1)!!^{m_i(\lambda)}\, m(\lambda)!}\, Q^{\lambda\mu}\, \frac{v_{\mu+1}}{v_x^{\ell(\mu)}}.$$

4. Solutions to Problems A,B,C

In this section we provide solutions to Problems A,B,C using the Witten–Kontsevich theorem, Buryak's theorem [5], and results of [12], respectively.

A solution to Problem A – Proof of Theorem 1.1. Using Theorem 3.4 it is easy to verify that $H_1(v_x; \mathbf{0}) = M_1(v_x) = \frac{1}{24}\log v_x$, and

$$H_g(v_x, \dots, v_{3g-2}; \mathbf{0}) = M_g(v_x, v_{xx}, \dots, v_{3g-2}), \quad g \geq 2.$$

By using the Witten–Kontsevich theorem and Lemma 3.2, we know that

$$u^{\text{top}}(\mathbf{t};\epsilon) := v^{\text{top}}(\mathbf{t}) + \sum_{g=1}^{\infty} \epsilon^{2g}\partial_x^2 H_g\big(v_x^{\text{top}}(\mathbf{t}), \dots, v_{3g-2}^{\text{top}}(\mathbf{t}); \mathbf{0}\big) \tag{4.1}$$

satisfies the KdV hierarchy. In particular it satisfies the KdV equation (1.1). We then deduce from the transcendental property of $v^{\text{top}}(\mathbf{t})$ that for any solution $v(\mathbf{t})$ to the Riemann hierarchy in $\mathbb{C}[[\mathbf{t}]]$ satisfying $v_x(\mathbf{t}) \neq 0$, the function $u(\mathbf{t};\epsilon)$ defined by (4.1) with v^{top} being replaced by v also satisfies the KdV hierarchy. Theorem 1.1 is proved. □

For $\ell(\lambda) = 1$, it is well known (see for example [17]) that

$$\langle \tau_{3g-2} \rangle = \frac{1}{24^g\, g!}, \quad g \geq 1. \tag{4.2}$$

For $\ell(\lambda) \geq 2$, a recently formula [4] gives

$$\langle \tau_{\lambda+1} \rangle = (-1)^{\ell(\lambda)+1} \prod_{i=1}^{\ell(\lambda)} \operatorname*{res}_{z_i=\infty} dz_i\, z_i^{2\lambda_i+3} \\ \times \left(\sum_{r\in S_{\ell(\lambda)}} \frac{\operatorname{Tr}\big(R(z_{r_1})\cdots R(z_{r_{\ell(\lambda)}})\big)}{\ell(\lambda)\prod_{j=1}^{\ell(\lambda)}(z_{r_j}^2 - z_{r_{j+1}}^2)} + \delta_{\ell(\lambda),2}\frac{z_1^2+z_2^2}{(z_1^2-z_2^2)^2} \right).$$

Here for a permutation $r = [r_1, \dots, r_\ell]$ in S_ℓ, $r_{\ell+1} := r_1$, and

$$(4.3)\quad R(z) = \frac{1}{2}\begin{pmatrix} -\sum_{g=1}^{\infty} \frac{(6g-5)!!}{24^{g-1}\,(g-1)!} z^{-6g+4} & -2\sum_{g=0}^{\infty} \frac{(6g-1)!!}{24^g\, g!} z^{-6g} \\ 2\sum_{g=0}^{\infty} \frac{6g+1}{6g-1}\frac{(6g-1)!!}{24^g\, g!} z^{-6g+2} & \sum_{g=1}^{\infty} \frac{(6g-5)!!}{24^{g-1}\,(g-1)!} z^{-6g+4} \end{pmatrix}.$$

A solution to Problem B. Let $\Lambda_g(s) := \sum_{i=0}^{g} s^i \lambda_i$ be the Chern polynomial of the Hodge bundle. By using Buryak's theorem [5] and Lemma 3.2 we know that

$$\begin{aligned} w(\mathbf{t};\epsilon) &:= \sum_{g=0}^{\infty} \epsilon^{2g}\partial_x^2\Big(\mathcal{H}_g\big(\Lambda(s);\mathbf{t}\big)\Big) \\ (4.4)\qquad &= v^{\mathrm{top}}(\mathbf{t}) + \sum_{g=1}^{\infty} \epsilon^{2g}\partial_x^2\Big(H_g\big(\Lambda(s); v^{\mathrm{top}}(\mathbf{t}), v_x^{\mathrm{top}}(\mathbf{t}), \dots, v_{3g-2}^{\mathrm{top}}(\mathbf{t})\big)\Big) \end{aligned}$$

is a particular solution to an explicit deformation of the intermediate long wave (ILW) hierarchy. We deduce from the transcendental property of $v^{\mathrm{top}}(\mathbf{t})$ that for any solution $v(\mathbf{t})$ to the Riemann hierarchy in $\mathbb{C}[[\mathbf{t}]]$ satisfying $v_x(\mathbf{t}) \neq 0$, the function $w(\mathbf{t};\epsilon)$ defined by (4.4) with v^{top} being replaced by v is a solution to the explicit deformation of the ILW hierarchy. In other words, the composition of the quasi-Miura transformation

$$(4.5)\qquad w = v + \sum_{g=1}^{\infty} \epsilon^{2g}\partial_x^2 H_g(\Lambda(s); v_x, \dots, v_{3g-2}),$$

$$(4.6)\qquad H_1(\Lambda(s); v, v_x) = \frac{1}{24}\log v_x + \frac{s}{24} v,$$

$$\begin{aligned} (4.7)\qquad H_g(\Lambda(s); v_1, \dots, v_{3g-2}) &= \sum_{k=0}^{g} s^g\, v_x^{-g+1+k} \\ &\times \sum_{\lambda,\mu\in\mathbb{Y}_{3g-3-k}} \frac{\langle \lambda_g \tau_{\lambda+1}\rangle_g}{m(\lambda)!}\, Q^{\lambda\mu}\, \frac{v_{\mu+1}}{v_x^{\ell(\mu)}}, \quad g \geq 2 \end{aligned}$$

with the Miura type transformation

$$(4.8)\qquad u = w + \sum_{g\geq 1} \frac{(-1)^g}{2^{2g}(2g+1)!}\epsilon^{2g} s^g w_{2g}$$

gives the quasi-triviality of the ILW equation (equivalently of the whole ILW hierarchy)

$$u_t = uu_x + \sum_{g\geq 1} \epsilon^{2g} s^{g-1} \frac{|B_{2g}|}{(2g)!} u_{2g+1}. \tag{4.9}$$

Here $t = t_1$. In the derivation of (4.7) we used Theorem 3.4 and the relationship

$$\begin{aligned} H_g(v, v_x, \dots, v_{3g-2}; \mathbf{s}) \,&|_{s_k=(2k-2)!s^{2k-1}, k\geq 1} \\ &= H_g(\Lambda(s); v(\mathbf{t}), v_x(\mathbf{t}), v_2(\mathbf{t}), \dots), \quad g \geq 1. \end{aligned} \tag{4.10}$$

We conclude that the above (4.4)–(4.8) give a solution to Problem B in terms of composition of an explicit quasi-Miura transformation with an explicit Miura type transformation.

A solution to Problem C. It was conjectured in [11] that the quasi-Miura map

$$w = v + \sum_{g=1}^{\infty} \epsilon^{2g} \partial_x^2 H_g\big(\Lambda(s)\Lambda(-2s)^2; v_x, v_{xx}, \dots\big) \tag{4.11}$$

with $s = 1$ gives rise to an explicit deformation of the discrete KdV hierarchy. This conjecture was proven in [12]. It says, more precisely, that for $s = 1$ the composition of the following three transformations

$$w = v + \sum_{g=1}^{\infty} \epsilon^{2g} \partial_x^2 H_g(\Lambda(s)\Lambda(-2s)^2; v_x, v_{xx}, \dots), \tag{4.12}$$

$$\begin{aligned} &H_1\big(\Lambda(s)\Lambda(-2s)^2; v_x\big) = \frac{1}{24}\log v_x - \frac{s}{8} v, \\ &H_g\big(\Lambda(s)\Lambda(-2s)^2; v_1, \dots, v_{3g-2}\big) \\ &\quad = \sum_{k=0}^{3g-3} s^k\, v_x^{-g+1+k} \sum_{\substack{k_1+k_2+k_3=k \\ 0\leq k_1,k_2,k_3\leq g}} (-2)^{k_2+k_3} \\ &\quad\quad \times \sum_{\rho,\mu\in\mathbb{Y}_{3g-3-k}} \frac{\langle \lambda_{k_1}\lambda_{k_2}\lambda_{k_3}\tau_{\rho+1}\rangle_g}{m(\rho)!}\, Q^{\rho\mu}\, \frac{v_{\mu+1}}{v_x^{l(\mu)}}, \quad g \geq 2, \end{aligned} \tag{4.13}$$

$$\tilde{w} = \frac{w}{2}, \quad \text{and} \quad u = \tilde{w} + \sum_{k=1}^{\infty} \epsilon^{2k} \frac{3^{2k+2}-1}{(2k+2)!4^{k+1}} \tilde{w}_{2k} \tag{4.14}$$

gives the quasi-triviality of the discrete KdV equation

$$u_t = \frac{1}{\epsilon}\left(e^{u(x+\epsilon)} - e^{u(x-\epsilon)}\right). \tag{4.15}$$

In the derivation of (4.13) we have used Theorem 3.4 and the relationship

$$\begin{aligned} H_g(v_x, \ldots, v_{3g-2}; \mathbf{s})|_{s_k=-(4^k-1)(2k-2)!s^{2k-1}} \\ = H_g\big(\Lambda(s)\Lambda(-2s)^2; v_x, \ldots, v_{3g-2}\big), \quad g \geq 1. \end{aligned} \tag{4.16}$$

5. Quasi-triviality of the Burgers hierarchy

The Burgers hierarchy

$$u_{t_n} = \frac{1}{(n+1)!}\partial_x \circ (\epsilon\,\partial_x + u)^n(u), \quad n \geq 0 \tag{5.1}$$

is an integrable deformation of the Riemann hierarchy, whose first member coincides with the Burgers equation (5). Here, as before we identify t_0 with x. We call a function $\tau = \tau(\mathbf{t}; \epsilon)$ a *viscous tau-function* for the Burgers hierarchy if τ satisfies

$$\tau_{t_n} = \frac{\epsilon^n}{(n+1)!}\,\tau^{(n+1)}, \quad n \geq 0. \tag{5.2}$$

For $\tau(\mathbf{t}; \epsilon)$ being a viscous tau-function for the Burgers hierarchy, one can check that the function $u = u(\mathbf{t}; \epsilon) := \epsilon\,\partial_x \log \tau(\mathbf{t}; \epsilon)$ satisfies (5.1). So we also call τ the *tau-function of* u. On the other hand, for any fixed solution $u = u(\mathbf{t}; \epsilon) \in \mathbb{C}[[\mathbf{t}; \epsilon]]$ to the Burgers hierarchy, the tau-function $\tau \in \mathbb{C}((\epsilon))[[\mathbf{t}]]$ of u exists, and is unique up to multiplying by a non-zero constant (which can depend on ϵ).

The partition function $Z^{1D}(\mathbf{t}; \epsilon)$ of 1D gravity (toy model of quantum field theory)

$$Z^{1D}(\mathbf{t}; \epsilon) := \frac{1}{\sqrt{2\pi\epsilon}} \int_{\mathbb{R}} e^{\frac{1}{\epsilon}\left(-\frac{s^2}{2} + \sum_{n=0}^{\infty} \frac{t_n}{(n+1)!} s^{n+1}\right)} ds \tag{5.3}$$

is known to be a particular viscous tau-function[1] of the Burgers hierarchy. The logarithm of $Z^{1D}(\mathbf{t};\epsilon)$ admits the expansion

$$\log Z^{1D}(\mathbf{t};\epsilon) =: \sum_{g=0}^{\infty} \epsilon^{g-1} \mathcal{F}_g^{1D}(\mathbf{t}). \tag{5.4}$$

By a direct computation of the Gaussian-type integral one obtains

$$Z^{1D}(x,0,0,\cdots;\epsilon) = e^{\frac{x^2}{2\epsilon}}. \tag{5.5}$$

It follows that the initial value of the solution u^{1D} corresponding to $Z^{1D}(\mathbf{t};\epsilon)$ is given by

$$u^{1D}(x,0,0,\cdots;\epsilon) = x. \tag{5.6}$$

The series $v^{1D}(\mathbf{t}) := \partial_x \mathcal{F}_0^{1D}(\mathbf{t})$ satisfies the inviscid Burgers hierarchy (coinciding with the Riemann hierarchy), whose initial value reads

$$v^{1D}(x,0,0,\cdots) = x. \tag{5.7}$$

Therefore $v^{1D}(\mathbf{t}) = v^{\mathrm{top}}(\mathbf{t})$. For $g \geq 1$, the following expressions for $\mathcal{F}_g^{1D}$ are known [30]:

$$\mathcal{F}_1^{1D}(\mathbf{t})\big|_{t_0=0} = \frac{1}{2}\log(1-t_1), \tag{5.8}$$

$$\mathcal{F}_g^{1D}(\mathbf{t})\big|_{t_0=0} = \sum_{\Gamma \in \mathcal{G}^c_{g,\mathrm{val}\geq 3}} \frac{t_{\lambda(\Gamma)+1}}{|\mathrm{Aut}(\Gamma)|\,(1-t_1)^{E(\Gamma)}}, \quad g \geq 2. \tag{5.9}$$

Here the summation is taken over all g-loop connected graphs whose vertices all have valences ≥ 3, and $\lambda(\Gamma) := \big(\mathrm{val}(\mathrm{vertex}_1) - 2, \ldots, \mathrm{val}(\mathrm{vertex}_{V(\Gamma)}) - 2\big)$.

Introduce

$$F_1^{1D}(v_x) := -\frac{1}{2}\log v_x, \tag{5.10}$$

$$F_g^{1D}(v_1,\ldots,v_{2g-2}) := \sum_{\mu \in \mathbb{Y}_{2g-2}} \sum_{\Gamma \in \mathcal{G}^c_{g,\mathrm{val}\geq 3}} \frac{Q^{\lambda(\Gamma)\mu}}{|\mathrm{Aut}(\Gamma)|} \frac{v_{\mu+1}}{v_1^{l(\mu)+g-1}}, \quad g \geq 2. \tag{5.11}$$

[1] $Z^{1D}(\mathbf{t};\epsilon)$ is also a tau-function of the KP hierarchy, where the KP times $T_1 = t_0$, $T_2 = t_1$, $T_3 = t_2$, $\ldots$.

Clearly, we have for $g \geq 2$ that $\deg F_g^{1D} = g-1$, $\overline{\deg} F_g^{1D} = 2g-2$. Applying Lemma 2.6 we obtain that

$$\mathcal{F}_1^{1D}(\mathbf{t})\big|_{t_0=0} = F_1^{1D}\big(v_x^s\big), \tag{5.12}$$

$$\mathcal{F}_g^{1D}(\mathbf{t})\big|_{t_0=0} = F_g^{1D}\big(v_1^s, \dots, v_{2g-2}^s\big), \quad g \geq 2, \tag{5.13}$$

where $v_k^s = v_k(\mathbf{t})|_{t_0=0}$, and we have used the Euler's formula $V(\Gamma) - E(\Gamma) = 1 - g$. Using (181)–(183) of [30] one can derive from (5.12)–(5.13) the following identities:

$$\mathcal{F}_g^{1D}(\mathbf{t}) = F_1^{1D}\big(v_1^{1D}(\mathbf{t})\big), \tag{5.14}$$

$$\mathcal{F}_g^{1D}(\mathbf{t}) = F_g^{1D}\big(v_1^{1D}(\mathbf{t}), \dots, v_{2g-2}^{1D}(\mathbf{t})\big), \quad g \geq 2. \tag{5.15}$$

Using the transcendental property of $v^{1D}(\mathbf{t})$, we arrive at the following solution to Problem D.

Theorem 5.1. *Quasi-triviality of the Burgers hierarchy* (5.1) *has the expression*

$$u = v + \sum_{g=1}^{\infty} \epsilon^g \partial_x F_g^{1D}(v_1, v_2, \dots) = v - \epsilon\left(\frac{v_{xx}}{2v_x}\right) + \mathcal{O}(\epsilon^2), \tag{5.16}$$

where F_g^{1D} are defined explicitly in (5.10)–(5.11).

6. Conclusion

Quasi-triviality of the Hodge hierarchy of a point and primitive Hodge integrals of a point are related via Q-matrices. The relation consists of two parts.

Part a). *From Hodge integrals to quasi-triviality of the Hodge hierarchy.* I.e., one can use primitive Hodge integrals to represent quasi-triviality of the

Hodge hierarchy:

$$
\begin{aligned}
&w = v + \sum_{g\geq 1} \epsilon^{2g} \partial_x^2 H_g, \\
&H_1(v, v_x; \mathbf{s}) = \frac{1}{24}\log(v_x) + \frac{s_1}{24} v, \\
&H_g(v_1, \dots, v_{3g-2}; \mathbf{s}) \\
&\qquad = \sum_{\substack{\phi\in\mathbb{Y} \\ 1\leq \phi_1,\dots,\phi_{\ell(\phi)}\leq g}} \frac{s_\phi}{m(\phi)!} v_x^{-g+1+2|\phi|-\ell(\phi)} \\
&\qquad\quad \times \sum_{\lambda,\mu\in\mathbb{Y}_{3g-3+\ell(\phi)-2|\phi|}} \frac{\langle \mathrm{ch}_{2\phi-1}\, \tau_{\lambda+1}\rangle_g}{m(\lambda)!} Q^{\lambda\mu} \frac{v_{\mu+1}}{v_x^{l(\mu)}}, \quad g\geq 2,
\end{aligned}
$$

where $\mathrm{ch}_{2\phi-1} := \mathrm{ch}_{2\phi_1-1}\cdots \mathrm{ch}_{2\phi_{\ell(\phi)}-1}$; $\mathrm{ch}_{2\phi-1} := 1$ if $\ell(\phi)=0$.

Part b). *From quasi-triviality of the Hodge hierarchy to Hodge integrals.* I.e., one can use quasi-triviality of the Hodge hierarchy to represent primitive Hodge integrals. Write for $g\geq 2$,

$$
\begin{aligned}
H_g(v_1,\dots,v_{3g-2};\mathbf{s}) = &\sum_{\substack{\phi\in\mathbb{Y} \\ 1\leq \phi_1,\dots,\phi_{\ell(\phi)}\leq g}} \frac{s_\phi}{m(\phi)!} v_x^{-g+1+2|\phi|-\ell(\phi)} \\
&\times \sum_{\mu\in\mathbb{Y}_{3g-3+\ell(\phi)-2|\phi|}} c_g^\mu(\mathrm{ch}_{2\phi-1}) \frac{v_{\mu+1}}{v_x^{l(\mu)}}.
\end{aligned}
\tag{6.1}
$$

Then we have $\forall\, \lambda,\mu,\phi\in\mathbb{Y}$,

$$
\big\langle \mathrm{ch}_{2\phi-1}\tau_{\lambda+1}\big\rangle_g = m(\lambda)! \sum_{\lambda\in\mathbb{Y}_{3g-3+\ell(\phi)-2|\phi|}} Q_{\lambda\mu}\, c_g^\mu(\mathrm{ch}_{2\phi-1}), \tag{6.2}
$$

where $g\geq 2$. We could also express Part a) as

$$
c_g^\mu(\mathrm{ch}_{2\phi-1}) = \sum_{\lambda\in\mathbb{Y}_{3g-3+\ell(\phi)-2|\phi|}} Q^{\lambda\mu}\big\langle \mathrm{ch}_{2\phi-1}\tau_{\lambda+1}\big\rangle_g. \tag{6.3}
$$

Appendix A. A straightforward proof of Lemma 3.2

In Theorem 3.4 we express the genus g Hodge potential $\mathcal{H}_g$ ($g\geq 1$) in terms of v_m, $m\geq 0$ with coefficients given by intersection numbers. Here $v=v(\mathbf{t})$ is the topological solution (2.8) to the dispersionless KdV hierarchy, and $v_m = v_m(\mathbf{t}) := \partial_x^m(v(\mathbf{t}))$, $x=t_0$. Recall that our proof in Section 3 uses Lemma 3.2 on the existence of jet-variables representation of $\mathcal{H}_g$; this lemma was proved in [11] based on the known existence of jet-variables representation [15, 16, 30] of the genus g Gromov-Witten potential of *a point* as well as on the uniqueness of the Faber-Pandharipande equations (3.7)–(3.8). In this appendix, to make the results of this paper self-contained, we give a straightforward proof of Lemma 3.2. Recall that [16, 20, 21, 30] Itzykson-Zuber's formal power series are defined by

$$I_0 = I_0(\mathbf{t}) := v(\mathbf{t}), \quad I_k = I_k(\mathbf{t}) := \sum_{n\geq 0} t_{n+k} \frac{I_0^n}{n!} \ (k \geq 1). \tag{A.1}$$

Observing that $I_k = t_k + \cdots$, we know that (A.1) gives an invertible map between the $\mathbf{t}$-variables $t_0, t_1, t_2, \dots$ and the I-variables $I_0, I_1, I_2, \dots$. The inverse map is given explicitly by [30]

$$t_k = \sum_{n=0}^{\infty} \frac{(-1)^n I_0^n}{n!} \, I_{n+k}. \tag{A.2}$$

The following formula, which generalizes Lemma 2.4, was derived in [30]:

$$I_0 = v, \quad I_k = \delta_{k,1} - \sum_{\mu\in\mathbb{Y}_{k-1}} L(\mu) \, \frac{v_{\mu+1}}{v_1^{1+|\mu+1|}} \ (k \geq 1). \tag{A.3}$$

Formula (A.3) can also be obtained by the (generalized) Lagrange inversion (cf. e.g. [22, 25]). Combining (A.3) with (A.2) gives

$$t_k = \delta_{k,1} - \sum_{n=0}^{\infty} \frac{(-1)^n v^n}{n!} \sum_{\mu\in\mathbb{Y}_{n+k-1}} L(\mu) \, \frac{v_{\mu+1}}{v_1^{1+|\mu+1|}}. \tag{A.4}$$

Proof of Lemma 3.2. For $\gamma = \mathrm{ch}_{2i_1-1} \cdots \mathrm{ch}_{2i_m-1}$ with $i_1, \dots, i_m \geq 1$, $m \geq 0$ we have

$$\mathcal{H}_g(\gamma; \mathbf{t}) = \sum_{\substack{m_0, m_1, m_2, m_3, \dots \geq 0 \\ \sum_{i=0}^{\infty} (i-1) m_i = 3g-3-\deg\gamma}} \left\langle \gamma \, \tau_0^{m_0} \tau_1^{m_1} \tau_2^{m_2} \cdots \right\rangle_g \prod_{i=0}^{\infty} \frac{t_i^{m_i}}{m_i!}. \tag{A.5}$$

(γ is defined as 1, if $m=0$.) Substituting the dilaton equation (3.14) in (A.5) we find

(A.6)

$$\begin{aligned}\mathcal{H}_g(\gamma;\mathbf{t}) =& \sum_{\substack{m_0,m_1,m_2,m_3,\dots\geq 0\\ \sum_{i=0}^\infty (i-1)m_i = 3g-3-\deg\gamma}} \frac{(\sum_{i=0}^\infty m_i + 2g-3)!}{(\sum_{i=0}^\infty m_i + 2g-3-m_1)!} \\ & \times \langle \gamma\, \tau_0^{m_0}\tau_2^{m_2}\tau_3^{m_3}\cdots\rangle_g \prod_{i=0}^\infty \frac{t_i^{m_i}}{m_i!} \\ =& \sum_{n=0}^\infty \frac{x^n}{n!} \sum_{\substack{m_2,m_3,m_4,\dots\geq 0\\ \sum_{i=2}^\infty (i-1)m_i = 3g-3-\deg\gamma+n}} \frac{\langle \gamma\, \tau_0^{n}\tau_2^{m_2}\tau_3^{m_3}\cdots\rangle_g}{(1-t_1)^{\sum_{i=2}^\infty m_i + 2g-2+n}} \prod_{i=2}^\infty \frac{t_i^{m_i}}{m_i!}.\end{aligned}$$

Substituting (A.4) in (A.6) and noticing that the dependence of t_1 in $\mathcal{H}_g(\gamma;\mathbf{t})$ is always through $1-t_1$, we find that there exists $\widetilde{H}_g(v,v_1,v_2,v_3,\dots)\in\mathbb{C}[[v,v_1,v_2,v_3,\cdots,v_1^{-1}]]$ such that

$$\mathcal{H}_g(\gamma;\mathbf{t}) = \widetilde{H}_g\big(v(\mathbf{t}),v_1(\mathbf{t}),v_2(\mathbf{t}),\dots\big).$$

Then similarly to the proof of Lemma 3.3 we find that $\partial\widetilde{H}_g/\partial v=0$. Since the dependence on v of $\widetilde{H}_g(v,v_1,v_2,v_3,\dots)$ is *a priori* a power series, we can take $v=0$ when substituting (A.4) in (A.6). So, if we associate to v_m degree $m-1$ for $m\geq 1$, then $\widetilde{H}_g$ has the degree $3g-3-\deg\gamma$. Alternatively, if we assign v_m another degree m for $m\geq 1$, then $\widetilde{H}_g$ has degree $2g-2$. The lemma is proved. $\square$

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