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The $\mathcal{N} = 2$ supersymmetric Calogero-Sutherland model and its eigenfunctions

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In a recent work, we have initiated the theory of $\mathcal{N} = 2$ symmetric superpolynomials. As far as the classical bases are concerned, this is a rather straightforward generalization of the $\mathcal{N} = 1$ case. However this construction could not be generalized to the formulation of Jack superpolynomials. The origin of this obstruction is unraveled here, opening the path for building the desired Jack extension. Those are shown to be obtained from the non-symmetric Jack polynomials by a suitable symmetrization procedure and an appropriate dressing by the anticommuting variables. This construction is substantiated by the characterization of the $\mathcal{N} = 2$ Jack superpolynomials as the eigenfunctions of the $\mathcal{N} = 2$ supersymmetric version of the Calogero-Sutherland model, for which, as a side result, we demonstrate the complete integrability by displaying the explicit form of four towers of mutually commuting (bosonic) conserved quantities. The $\mathcal{N} = 2$ Jack superpolynomials are orthogonal with respect to the analytical scalar product (induced by the quantum-mechanical formulation) as well as a new combinatorial scalar product defined on a suitable deformation of the power-sum basis.

1. Introduction

The study of the $\mathcal{N}=2$ symmetric superpolynomials has been initiated in [1]. Let us review briefly what is meant by this program.

The construction amounts to extending the classical symmetric polynomials to functions depending on not only $x_1,\dots,x_N$ but also on two extra independent sets of anticommuting variables $\theta_1,\dots,\theta_N$ and $\phi_1,\dots,\phi_N$. We require the variables in each set to anticommute among themselves:

$$\theta_i\theta_j = -\theta_j\theta_i\,, \qquad \phi_i\phi_j = -\phi_j\phi_i, \tag{1}$$

and also with each other:

$$\theta_i\phi_j = -\phi_j\theta_i. \tag{2}$$

Equivalently, we consider the ring $\mathbb{C}(x_1,\dots,x_N)\otimes\bigwedge\big(\mathbb{C}(\theta_1,\dots,\theta_n,\phi_1,\dots,\phi_N)\big)$. This addition of the anticommuting variables is understood in the context of superspace: the variables ϕ_i and θ_i are attached to the bosonic variable x_i. Therefore, the symmetry requirement imposed on polynomials is the invariance under the interchange of two triplets $(x_i,\phi_i,\theta_i)\leftrightarrow(x_{\sigma(i)},\phi_{\sigma(i)},\theta_{\sigma(i)})$ where σ belongs to S_N, the symmetric group on N elements. We call the resulting objects $\mathcal{N}=2$ symmetric superpolynomials and denote their ring as Π^N.

A detailed analysis of the $\mathcal{N}=2$ supersymmetric version of the classical bases $m_\lambda, e_\lambda, h_\lambda$ and p_λ (λ being a partition), was presented in [1]. Take for instance the power-sum basis. It is a multiplicative basis built out of four constituents:

$$p_n=\sum_{i=1}^N x_i^n,\quad \overline{p}_r=\sum_{i=1}^N \phi_i\,x_i^r,\quad \underline{p}_r=\sum_{i=1}^N \theta_i\,x_i^r,\quad \underline{\overline{p}}_r=\sum_{i=1}^N \phi_i\theta_i\,x_i^r \tag{3}$$

with $n\geq 1$ and $r\geq 0$.

Symmetric $\mathcal{N}=2$ superpolynomials are labelled by $\mathcal{N}=2$ superpartitions. The occurrence of four types of power-sums suggests that the superpartitions are composed of four partitions. The splitting of these four types into two bosonic and two fermionic ones further entails that two of these partitions – that associated to the product of the $\overline{p}_r$'s and that associated to the product of the $\underline{p}_r$'s – have distinct parts. The superpartition Λ will be written as

$$\Lambda=(\underline{\overline{\Lambda}};\overline{\Lambda};\underline{\Lambda};\Lambda^s) \tag{4}$$

where $\underline{\overline{\Lambda}}$ and Λ^s are usual partitions while $\overline{\Lambda}$ and $\underline{\Lambda}$ are partitions without repeated parts. For instance, we have for $N=2$:

$$\begin{aligned} p_{(;1;1;)}&=\overline{p}_1\underline{p}_1=(\phi_1x_1+\phi_2x_2)(\theta_1x_1+\theta_2x_2),\\ p_{(2;;;)}&=\underline{\overline{p}}_2=\phi_1\theta_1x_1^2+\phi_2\theta_2x_2^2. \end{aligned} \tag{5}$$

There is a natural extension of the combinatorial scalar product defined in terms of the power-sums which preserves the dual nature of the extension of m_λ and h_λ.

However, the ultimate objective of this generalization of the theory of symmetric polynomials is to construct the $\mathcal{N}=2$ Jack superpolynomials. We expect those to be defined by directly extending the $\mathcal{N}=0,1$ definition to the $\mathcal{N}=2$ case, namely, in terms of two conditions: triangularity in the monomial basis and orthogonality. The scalar product with respect to which we expect the yet-to-be-defined $\mathcal{N}=2$ Jack superpolynomials to be orthogonal is the α-deformation of the power-sums scalar product just alluded to. However, in [1], we have indicated the difficulty of obtaining the Jack deformation of the classical bases along those lines.

Let us pinpoint the source of the problem. We have considered in [1] the characterization of the superpartitions that label the symmetric superpolynomials by three numbers: the degree of the polynomial, denoted n, and the number of ϕ_i and θ_j factors in the monomial of anticommuting variables that decorate each term in the expression of the superpolynomial in a given sector. Let us denote these numbers m_ϕ and m_θ. Now what is the problem with this description? Take the simple monomial (still for $N = 2$):

$$m_{(;1;1;)} = \phi_1 x_1 \theta_2 x_2 + \phi_2 x_2 \theta_1 x_1 \tag{6}$$

(The construction of the monomial is explained below.) Its decomposition in terms of power sums is easily found to be

$$m_{(;1;1;)} = \overline{p}_1 \underline{p}_1 - \underline{\overline{p}}_2. \tag{7}$$

This preserves the sector $m_\phi = m_\theta = 1$. However, it mixes for instance the sectors corresponding to the product of θ_1 and ϕ_1 to the sector corresponding to $\theta_1\phi_1$. According to our earlier attempts, this mixing seems to prevent the introduction of a consistent dominance ordering, which in turn implies the impossibility of using the triangularity requirement for defining the Jack superpolynomials.

Heuristically, the cure for this problem is clear: the separation of the superpartition into four blocks suggests the characterization of each sector by four numbers, n, $\underline{\overline{m}}$, $\overline{m}$ and $\underline{m}$, the latter three counting respectively the number of factors $\phi_i\theta_i$ (i.e., paired with the same indices), ϕ_j and θ_k. Equivalently, $\underline{\overline{m}}$, $\overline{m}$ and $\underline{m}$ stand respectively for the length of $\underline{\overline{\Lambda}}$, $\overline{\Lambda}$ and $\underline{\Lambda}$. This refinement of the characterization of the fermionic sector is indeed a necessary requirement for the successful construction of Jack superpolynomials. But can we figure out a firmer argument for the necessity of four entries specifying a given sector?

The physics of integrable N–body problems provide such a foundation. Recall that the usual Jack polynomials are the eigenfunctions (with the ground-state contribution factored out) of the Calogero-Sutherland model. Their $\mathcal{N} = 1$ extension is similarly related to the supersymmetric version of the CS model (referred to as the sCS model). We thus require the $\mathcal{N} = 2$ Jack superpolynomials to be eigenfunctions of the $\mathcal{N} = 2$ supersymmetric extension of the CS model (to be dubbed, for short, the s^{2}CS model), first introduced in [22]. This model is shown here to be integrable, as expected, by displaying four towers of N bosonic mutually commuting conservation law. This naturally implies a characterization of the sectors by four quantum numbers.

But this simple cure (refinement of the fermionic sector) entails the replacement of the power-sum basis by an alternative one that does not lead to sector mixing. This is one of the key technical point of our new construction and the new basis, called quasi-power-sums, is not multiplicative. The combinatorial scalar product is now defined with respect to this new basis.

In this way, we have succeeded in constructing the $\mathcal{N} = 2$ Jack superpolynomials orthogonal with respect to this new combinatorial scalar product. But there is more: their construction from an appropriate symmetrization of the non-symmetric Jack polynomials, taylor made to render them s^{2}CS eigenfunctions, implies their orthogonality with respect to an analytic scalar product. This is compatible with their physical interpretation as wavefunctions.

The outline of the article is a follows. In Section 2, we derive the $\mathcal{N} = 2$ supersymmetric Calogero-Sutherland model using the formalism of [22] and following the construction of the $\mathcal{N} = 0, 1$ cases. In Section 3, we introduce the space of $\mathcal{N} = 2$ symmetric superfunctions and provide two simples bases: the monomial symmetric functions and the quasi-power sums. We also present superpartitions, the combinatorial objects which naturally index the bases, as well as the dominance ordering on superpartitions. In Section 4, we introduce the $\mathcal{N} = 2$ Jack superpolynomials from the non-symmetric Jack polynomials. We then construct $4N$ quantities built out of Dunkl operators that have those polynomials as eigenfuntions, a result that implies the integrability of the $\mathcal{N} = 2$ supersymmetric Calogero-Sutherland model. We then show that if a triangularity condition is imposed, it suffices to consider only 4 commuting quantities, one of them being the Hamiltonian, to characterize the $\mathcal{N} = 2$ Jack superpolynomials. In Section 5, we present two scalar products, dubbed analytic and combinatorial, with respect to which the $\mathcal{N} = 2$ Jack superpolynomials are orthogonal. But in order to not overburden the text, only an outline of the proofs of the orthogonality are presented. In Section 6, we give conjectures for the norm (with respect to the combinatorial scalar product) and the evaluation of the $\mathcal{N} = 2$ Jack superpolynomials. Finally, we discuss in Appendix A how our construction of the $\mathcal{N} = 2$ Jack superpolynomials from the non-symmetric Jack polynomials is only of one many possible constructions.

2. $\mathcal{N} = 2$ supersymmetric Calogero-Sutherland model

Before defining the $\mathcal{N} = 2$ version of the Calogero-Sutherland model, we introduce the $\mathcal{N} = 0$ and $\mathcal{N} = 1$ versions. The construction of the $\mathcal{N} = 2$ version will mimic very particularly that of the $\mathcal{N} = 1$ case.

The Calogero-Sutherland (CS) model [6, 21] describes a system of N identical particles (of mass $m = 1$) lying on a circle of circumference L and interacting pairwise:

$$H^{(\mathcal{N}=0)} = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \left(\frac{\pi}{L}\right)^2 \beta(\beta-1) \sum_{1\leq i<j\leq N} \frac{1}{\sin^2(\pi x_{ij}/L)}, \tag{8}$$

where $x_{ij} = x_i - x_j$ and $p_j = -i\partial/\partial x_j$ (we set $\hbar = 1$).

In the $\mathcal{N} = 1$ version of the CS model, every particle coordinate x_j is paired with an anticommuting variable θ_j. In this case, the Hamiltonian is built out of two anticommuting charges Q and $Q^\dagger$ (with $\theta_j^\dagger = \partial/\partial\theta_j$) defined in terms of a prepotential W as

$$Q = \sum_j \theta_j \big(p_j - i\partial_{x_j} W(x)\big). \tag{9}$$

Explicitly, the Hamiltonian is obtained as follows

$$H^{(\mathcal{N}=1)} = \frac{1}{2}\{Q, Q^\dagger\}, \tag{10}$$

where $W(x)$ is determined by the requirement

$$H^{(\mathcal{N}=1)}\Big|_{\theta_j=0} = H^{(\mathcal{N}=0)}. \tag{11}$$

This fixes $W(x)$ to be

$$W(x) = \sum_{1\leq i<j\leq N} \frac{\beta}{2} \ln\left(\frac{1}{\sin^2(\frac{\pi}{L}x_{ij})}\right), \tag{12}$$

and the resulting Hamiltonian reads

$$H^{(\mathcal{N}=1)} = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \left(\frac{\pi}{L}\right)^2 \sum_{1\leq i<j\leq N} \frac{\beta(\beta-1+\theta_{ij}\theta_{ij}^\dagger)}{\sin^2(\pi x_{ij}/L)}, \tag{13}$$

with $\theta_{ij} = \theta_i - \theta_j$.

For the $\mathcal{N}=2$ extension, in which case x_j is then paired with two independent anticommuting variables, θ_j and ϕ_j, we need two supercharges Q_1 and Q_2 realizing the algebra

$$\{Q_a, Q_b^\dagger\} = 2\delta_{ab}H, \quad \{Q_a, Q_b\} = 0, \quad \{Q_a^\dagger, Q_b^\dagger\} = 0, \quad a, b = 1, 2 \tag{14}$$

where

$$H \equiv H^{(\mathcal{N}=2)}. \tag{15}$$

As noted in [22], the supercharges are now expressed in terms of two prepotentials: $W^{[0]}$ (the previous $W(x)$) and $W^{[1]}$:[1]

$$Q_1 = \sum_j \theta_j \left(p_j - iW_j^{[0]}(x) - i\sum_{k,l=1}^N W_{jkl}^{[1]}(x)\phi_k\phi_l^\dagger \right) \tag{16}$$

$$Q_2 = \sum_j \phi_j \left(p_j - iW_j^{[0]}(x) - i\sum_{k,l=1}^N W_{jkl}^{[1]}(x)\theta_k\theta_l^\dagger \right), \tag{17}$$

where we have introduced the notation

$$W_i^{[0]} := \partial_i W^{[0]}, \quad W_{ijk}^{[1]} := \partial_i\partial_j\partial_k W^{[1]}. \tag{18}$$

It is readily seen that when the ϕ variables are set equal to zero, Q_1 reduces to Q while Q_2 vanishes.

Under the assumption that $W^{[1]} = \sum_{i<j} w(x_{ij})$, the conditions (14) lead to the following Hamiltonian (we refer to [22] for the details of this analysis)

(19)
$$H_{\mathrm{s^2CS}} = \frac{1}{2}\sum_i p_i^2 + \beta\left(\frac{\pi}{L}\right)^2 \sum_{i<j} \frac{1}{\sin^2 \frac{\pi}{L}x_{ij}} \left(\beta - (1-\phi_{ij}\phi_{ij}^\dagger)(1-\theta_{ij}\theta_{ij}^\dagger)\right).$$

This is thus the candidate $\mathcal{N}=2$ version of the supersymmetric CS model (s^{2}CS for short). As it will be shown below, this is precisely the form of the Hamiltonian that would result from an exchange formalism projected onto

[1]Due to the presence of four charges, $Q_{1,2}$ and $Q_{1,2}^\dagger$, the model is said to have four supersymmetries in [22]. Our point of view is that there are two independent charges, hence the $\mathcal{N}=2$ qualifier.

the space of symmetric superfunctions.

This Hamiltonian has the same ground state as the $\mathcal{N}=0$ and $\mathcal{N}=1$ versions, namely

$$\psi_0(x)=\Delta^\beta(x)=\prod_{j<k}\left[\sin\left(\frac{\pi x_{jk}}{L}\right)\right]^\beta. \tag{20}$$

The ground-state energy is

$$E_0=\left(\frac{\pi\beta}{L}\right)^2\frac{N(N^2-1)}{6}. \tag{21}$$

Any excited-state wavefunction will be of the form

$$\psi(x,\theta,\phi)=\psi_0(x)\varphi(x,\theta,\phi),$$

with $\varphi(x,\theta,\phi)$ a polynomial in its variables.

Upon the change of variables $z_i=e^{2\pi i x_i/L}$, the Hamiltonian becomes

$$H=2\left(\frac{\pi}{L}\right)^2\left(\sum_i(z_i\partial_i)^2-2\sum_{i<j}\frac{z_iz_j}{(z_{ij})^2}\beta\left(\beta-(1-\phi_{ij}\phi_{ij}^\dagger)(1-\theta_{ij}\theta_{ij}^\dagger)\right)\right). \tag{22}$$

It is convenient to factor out the contribution of the ground-state by a conjugation operation and perform a rescaling to get rid of the above prefactor, defining thereby the new Hamiltonian

$$\mathcal{H}=\frac{1}{2}\left(\frac{L}{\pi}\right)^2\Delta^{-\beta}(H-E_0)\Delta^\beta \tag{23}$$

A simple computation yields

$$\begin{aligned}\mathcal{H}=&\sum_i(z_i\partial_i)^2+\beta\sum_{i<j}\frac{z_i+z_j}{z_{ij}}(z_i\partial_i-z_j\partial_j)\\&-2\beta\sum_{i<j}\frac{z_iz_j}{z_{ij}^2}(1-(1-\phi_{ij}\phi_{ij}^\dagger)(1-\theta_{ij}\theta_{ij}^\dagger)).\end{aligned} \tag{24}$$

To demonstrate the integrability of this model and to study its eigenfunctions, we first need to introduce the space of symmetric superfunctions.

3. Superpartitions and the space of symmetric superfunctions

3.1. Symmetric superfunctions

One obvious symmetry of the Hamiltonian (24) is its invariance under the simultaneous exchange of the triplet of variables, that is, under $(z_i,\theta_i,\phi_i)\leftrightarrow(z_j,\theta_j,\phi_j)$ for all i,j. This is the defining property of the $\mathcal{N}=2$ symmetric superfunctions. Let us define the following operators:

$$K_{ij}\,:\,z_i\longleftrightarrow z_j,\quad \overline{\kappa}_{ij}\,:\,\phi_i\longleftrightarrow\phi_j,\quad \underline{\kappa}_{ij}\,:\,\theta_i\longleftrightarrow\theta_j. \tag{25}$$

The operator that produces the simultaneous exchange of the three types of variables is thus

$$\mathcal{K}_{ij}:=K_{ij}\overline{\kappa}_{ij}\underline{\kappa}_{ij} \tag{26}$$

with the following action on a superfunction

$$\mathcal{K}_{ij}f(z_i,z_j,\theta_i,\theta_j,\phi_i,\phi_j)=f(z_j,z_i,\theta_j,\theta_i,\phi_j,\phi_i). \tag{27}$$

Accordingly, a superfunction $f(z,\theta,\phi)$ in N (triplets of) variables is said to be symmetric if and only if

$$\mathcal{K}_{ij}f(z,\theta,\phi)=f(z,\theta,\phi)\quad\forall\quad i,j=1,\dots,N. \tag{28}$$

We will denote the space of symmetric superfunctions in the $3N$ variables (x_i,θ_i,ϕ_i) by Π^N:

$$f\in\Pi^N\iff\mathcal{K}_{ij}f=f\quad\forall\quad i,j=1,\dots,N. \tag{29}$$

This space is graded by four numbers and each set of those four numbers defines a sector. To define those sectors, we must first introduce some notation.

We first define the fermionic sector, denoted M, which is itself characterized by three numbers:

$$M:=(\underline{\overline{m}},\overline{m},\underline{m}). \tag{30}$$

These numbers are defined as follows:

1) $\underline{\overline{m}}$ is the degree of the polynomial in the doublet of variables $\phi_i\theta_i$;

2) $\overline{m}$ is the degree of the polynomial in the variables ϕ_j that do not form a doublet with a variable θ_j;

3) $\underline{m}$ is the degree of the polynomial in the variables θ_j that do not form a doublet with a variable ϕ_j.

For example, taking $N = 4$, the following superpolynomial is in the $M = (1, 1, 2)$ fermionic sector:

$$
\begin{aligned}
(31) \qquad & (\phi_1\theta_1\phi_2 + \phi_2\theta_2\phi_1)\theta_3\theta_4(z_3 - z_4) + \text{permutations} \\
& \quad = (\phi_1\theta_1)(\phi_2)(\theta_3\theta_4)z_3 + \cdots
\end{aligned}
$$

Focusing on the sole term written on the right-hand side, we see that we have only one doublet of variables ϕ and θ with the same index ($\underline{\overline{m}} = 1$), one variable ϕ that do not form a doublet with a θ of the same index ($\overline{m} = 1$) and two θ variables that do not form a doublet with a ϕ of the same index ($\underline{m} = 2$). The subspace of symmetric superfunctions (in N variables) in the fermionic sector M will de denoted $\Pi^N_{(M)}$.

It is convenient to introduce the following partial sums over the three numbers that define the fermionic sector

$$
(32) \qquad M_1 = \underline{\overline{m}}, \quad M_2 := \underline{\overline{m}} + \overline{m}, \quad M_3 := \underline{\overline{m}} + \overline{m} + \underline{m}.
$$

We then introduce the M-fermion monomial

$$
(33) \qquad [\phi;\theta]_M = \prod_{i=1}^{M_1} \phi_i\theta_i \prod_{j=M_1+1}^{M_2} \phi_j \prod_{k=M_2+1}^{M_3} \theta_k,
$$

with the understanding that the product is 1 if the upper bound of the product is lower than the lower bound. The projector onto the monomial term $[\phi;\theta]_M$ is

$$
(34) \qquad \mathcal{P}^M = [\phi;\theta]_M \Big([\phi;\theta]_M\Big)^\dagger.
$$

For instance,

$$
\begin{aligned}
(35) \qquad & \mathcal{P}^{(1,1,2)}[(\phi_1\theta_1\phi_2 + \phi_2\theta_2\phi_1)\theta_3\theta_4(z_3 - z_4) + \text{permutations}] \\
& \quad = \phi_1\theta_1\phi_2\theta_3\theta_4(z_3 - z_4).
\end{aligned}
$$

To recover the full symmetric superpolynomial from the projected term (e.g., the term on the right-hand side of the previous equality), we need to sum over the permutations of the symmetric group S_N that mix the elements of the different fermionic subsectors, that is, over the elements of $S_{(M)}$ defined as

$$S_{(M)} := S_N/(S_{M_1} \times S_{]M_1,M_2]} \times S_{]M_2,M_3]} \times S_{]M_3,N]}) \tag{36}$$

where the following notation has been used

$$S_{]j,j+k]} := S_{\{j+1,\dots,j+k\}} \tag{37}$$

(so that $S_N = S_{]0,N]}$). We can thus characterize a superfunction f of $\Pi^N_{(M)}$ with the condition

$$\sum_{\omega \in S_{(M)}} \mathcal{K}_\omega \mathcal{P}^M f = f. \tag{38}$$

We finally define the subspace $\Pi^N_{(n|M)}$ as the set of polynomials f in Π^N that have degree n in the variables z and that belong to the fermionic sector M.

The following proposition shows that this characterization of the superpolynomials in terms of the three numbers defining the fermionic sector is sound.

Proposition 1. *Let us introduce the three operators*

$$\underline{\overline{F}} = \sum_{i=1}^N \phi_i\theta_i(\phi_i\theta_i)^\dagger, \quad \overline{F} = \sum_{i=1}^N \phi_i\phi_i^\dagger - \underline{\overline{F}}, \quad \underline{F} = \sum_{i=1}^N \theta_i\theta_i^\dagger - \underline{\overline{F}}. \tag{39}$$

Then, for a function $f \in \Pi^N_{(M)}$, we have

$$\underline{\overline{F}}\, f = \underline{\overline{m}} f, \qquad \overline{F}\, f = \overline{m} f, \qquad \underline{F}\, f = \underline{m} f. \tag{40}$$

The proof is reported at the end of the section.

Since these three operators commute with the s²CS Hamiltonian, their eigenvalues partly characterize its eigenfunctions.

3.2. Interlude: rederivation of the s^{2}CS Hamiltonian

The $\mathcal{N}=0$ Hamiltonian can be recovered from the exchange formalism, where

$$H^{\text{exch}} = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \left(\frac{\pi}{L}\right)^2 \sum_{1\le i<j\le N} \frac{\beta(\beta - K_{ij})}{\sin^2(\pi x_{ij}/L)}, \tag{41}$$

when the latter is restricted to the space of symmetric functions $f(x)$ such that $K_{ij}f(x) = f(x)$, i.e.,

$$H^{(\mathcal{N}=0)} = H^{\text{exch}}|_{\Pi^N} \tag{42}$$

Similarly, the $\mathcal{N}=1$ sCS Hamiltonian is recovered from

$$H^{(\mathcal{N}=1)} = H^{\text{exch}}|_{\Pi^N} \tag{43}$$

where now the restriction is on the space of symmetric superfunctions $f(x,\theta)$ such that $K_{ij}f(x,\theta) = \kappa_{ij}f(x,\theta)$, where

$$\kappa_{ij} = 1 - \theta_{ij}\theta_{ij}^\dagger \tag{44}$$

In the same way, the s^{2}CS Hamiltonian constructed previously (cf. eq. (19)) is easily recovered form

$$H^{(\mathcal{N}=2)} = H^{\text{exch}}|_{\Pi^N} \tag{45}$$

where in this case the restriction is on the space of $\mathcal{N}=2$ symmetric superfunctions $f(x,\phi,\theta)$ such that $K_{ij}f(x,\phi,\theta) = \overline{\kappa}_{ij}\underline{\kappa}_{ij}f(x,\phi,\theta)$. This observation will be crucial when we study the conserved quantities of the s^{2}CS model.

3.3. Superpartitions

Bases of the space of symmetric superpolynomials, to be introduced shortly, are labeled by superpartitions [1]. A superpartition Λ is a set of four partitions, written as,

$$\Lambda = (\,\overline{\underline{\Lambda}};\, \overline{\Lambda};\, \underline{\Lambda};\, \Lambda^s) \tag{46}$$

with restrictions on the constituent partitions: $\overline{\underline{\Lambda}}$ is a standard partition in which 0's are allowed and contribute to the length of the partition. Both $\overline{\Lambda}$

and $\underline{\Lambda}$ are partitions with distinct parts that can contain one zero which, if present, also contributes to the length. Finally, $\overset{s}{\Lambda}$ is a standard partition (for which zeros are ignored). Or, more explicitly:

$$\begin{aligned}
&\underline{\overline{\Lambda}}_1 \geq \underline{\overline{\Lambda}}_2 \geq \cdots \geq \underline{\overline{\Lambda}}_{\underline{\overline{m}}} \geq 0 \\
&\overline{\Lambda}_1 > \overline{\Lambda}_2 > \cdots > \overline{\Lambda}_{\overline{m}} \geq 0 \\
&\underline{\Lambda}_1 > \underline{\Lambda}_2 > \cdots > \underline{\Lambda}_{\underline{m}} \geq 0 \\
&\overset{s}{\Lambda}_1 \geq \overset{s}{\Lambda}_2 \geq \cdots \geq \overset{s}{\Lambda}_{\ell(\overset{s}{\Lambda})} > 0, \qquad (47)
\end{aligned}$$

where $\ell(\lambda)$ is the length of the partition λ. The length of the superpartition, $\ell(\Lambda)$, is the sum of the length of the constituent partitions,

$$\ell(\Lambda) = \ell(\underline{\overline{\Lambda}}) + \ell(\overline{\Lambda}) + \ell(\underline{\Lambda}) + \ell(\overset{s}{\Lambda}). \tag{48}$$

A superpartition is said to belong to the M-fermion sector, with $M = (\underline{\overline{m}}, \overline{m}, \underline{m})$, if

$$\ell(\underline{\overline{\Lambda}}) = \underline{\overline{m}}, \quad \ell(\overline{\Lambda}) = \overline{m}, \quad \ell(\underline{\Lambda}) = \underline{m}. \tag{49}$$

With the M_i's defined in (32), a superpartition takes the form

$$\Lambda = (\Lambda_1, \ldots, \Lambda_{M_1}; \Lambda_{M_1+1}, \ldots, \Lambda_{M_2}; \Lambda_{M_2+1}, \ldots, \Lambda_{M_3}; \Lambda_{M_3+1}, \ldots, \Lambda_N) \tag{50}$$

with the understanding that

$$\begin{aligned}
&\Lambda_i \in \underline{\overline{\Lambda}} \text{ for } i \in \{1, \ldots, M_1\} \qquad (51) \\
&\Lambda_i \in \overline{\Lambda} \text{ for } i \in \{M_1+1, \ldots, M_2\} \\
&\Lambda_i \in \underline{\Lambda} \text{ for } i \in \{M_2+1, \ldots, M_3\} \\
&\Lambda_i \in \overset{s}{\Lambda} \text{ for } i \in \{M_3+1, \ldots, N\}. \qquad (52)
\end{aligned}$$

Finally, the bosonic degree of a superpartition is defined to be the sum of all its entries and is written $|\Lambda|$:

$$|\Lambda| = \sum_{i=1}^{\ell(\Lambda)} \Lambda_i. \tag{53}$$

Therefore, a superpartition of bosonic degree n in the M-fermion sector is said to be part of the $(n|M) = (n|\underline{\overline{m}}, \overline{m}, \underline{m})$ sector. For instance,

$$\begin{aligned} &\Lambda = (3,2,2,0,0;\ 1,0;\ 3,1;\ 2,1,1) \in (16|5,2,2), \\ &\Omega = (1,0;\ 5,2,1;\ 5,1;\ 4,1) \in (20|2,3,2). \end{aligned} \tag{54}$$

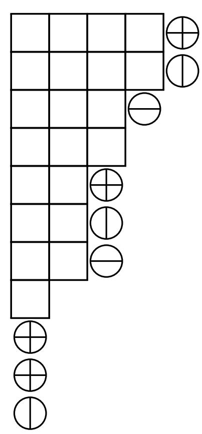
3.4. Diagrammatic representation of superpartitions

The superpartition $\Lambda = (\underline{\overline{\Lambda}}; \overline{\Lambda}; \underline{\Lambda}; \Lambda^s)$ can also be written as a standard partition where the parts are marked according to which constituent partition they belong: with overbars, underbars, both overbars and underbars, and unmarked. If there are parts which are equal, we use the ordering $\underline{\overline{a}}, \overline{a}, \underline{a}, a$. Here is an example:

$$\Lambda = (4,2,0;4,2,0;3,2,0;3,1) = (\underline{\overline{4}}, \overline{4}, \underline{3}, 3, \underline{\overline{2}}, \overline{2}, \underline{2}, 1, \underline{\overline{0}}, \underline{\overline{0}}, \overline{0}) \tag{55}$$

This notation suggests the following diagrammatic representation. As usual, every part is represented by a row with as many boxes as its numerical value. If the part is marked, we add a circle of a given type the end of the row: a $\varobar$ if the part is overlined, a $\ominus$ if the part is underlined and a $\oplus$ if the part is overlined and underlined. We add the above ordering convention: when there are more than one circle in a column, the ordering, from top to bottom, is $\oplus$, $\varobar$ and $\ominus$. Here is the diagrammatic representation of the above example:

(56) $\Lambda = (\underline{\overline{4}}, \overline{4}, \underline{3}, 3, \underline{\overline{2}}, \overline{2}, \underline{2}, 1, \underline{\overline{0}}, \underline{\overline{0}}, \overline{0}) \quad \longleftrightarrow$



Note that there cannot be two circles of any type in the same row.

3.5. Ordering on superpartitions

We now introduce the dominance ordering on superpartitions. To formulate it, we first need to a introduce a few concepts. For a composition $\eta \in \mathbb{Z}_{\geq 0}^N$, let η^+ be the partition obtained by reordering the entries of η in weakly decreasing order. Considering Λ as a composition, that is, by replacing its semicolons by commas, we define

$$\Lambda^{[0]} = \Lambda^+ \tag{57}$$

Also, for $1 \leq m \leq N$, let $\eta + 1^m$ stand for the composition $(\eta_1 + 1, \ldots, \eta_m + 1, \eta_{m+1}, \ldots, \eta_N)$. This allows to define

$$\Lambda^{[k]} = (\Lambda + 1^{M_k})^+, \quad k = 1, 2, 3. \tag{58}$$

In the diagrammatic representation defined in the previous subsection, $\Lambda^{[0]}$ correspond to the partition whose diagram is that of Λ without its circles, $\Lambda^{[1]}$ correspond to the partition whose diagram is that of Λ where every $\oplus$ is replaced by a box, $\Lambda^{[2]}$ correspond to the partition whose diagram is that of Λ where every $\oplus$ or $\varobar$ is replaced by a box, and $\Lambda^{[3]}$ correspond to the partition whose diagram is that of Λ where every circle is replaced by a box. For instance, using the example given in (56), we have

$$\begin{aligned} &\Lambda^{[0]} = (4,4,3,3,2,2,2,1), \quad \Lambda^{[1]} = (5,4,3,3,3,2,2,1,1,1), \\ &\Lambda^{[2]} = (5,5,3,3,3,3,2,1,1,1,1), \quad \Lambda^{[3]} = (5,5,4,3,3,3,3,1,1,1,1)\,. \end{aligned} \tag{59}$$

Note that it is then obvious that there is a bijective correspondence between $(\Lambda^{[0]}, \Lambda^{[1]}, \Lambda^{[2]}, \Lambda^{[3]})$ and Λ.

The ordering on superpartitions can now be defined as

$$\Lambda \geq \Omega \iff \Lambda^{[k]} \geq \Omega^{[k]} \quad \forall \quad k = 0, 1, 2, 3. \tag{60}$$

where the ordering on partitions is the standard dominance ordering [17]

$$\lambda \geq \mu \quad \iff \quad |\lambda| = |\mu| \quad \text{and} \quad \sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \mu_i \quad \forall\, k \tag{61}$$

Example 2. Consider the three superpartitions:

$$\Lambda = (0,0;\ 1,0;\ 0;\ 2)$$
$$\Omega = (1,0;\ 1,0;\ 0;\ 1)$$
$$\Gamma = (0,0;\ 2,1;\ 0;\ ) \tag{62}$$

We have

$$\begin{array}{ll} \Lambda^{[0]} = (2,1) & \Lambda^{[1]} = (2,1,1,1) \\ \Lambda^{[2]} = (2,2,1,1,1) & \Lambda^{[3]} = (2,2,1,1,1,1) \\ \Omega^{[0]} = (1,1,1) & \Omega^{[1]} = (2,1,1,1) \\ \Omega^{[2]} = (2,2,1,1,1) & \Omega^{[3]} = (2,2,1,1,1,1) \\ \Gamma^{[0]} = (2,1) & \Gamma^{[1]} = (2,1,1,1) \\ \Gamma^{[2]} = (3,2,1,1) & \Gamma^{[3]} = (3,2,1,1,1) \end{array} \tag{63}$$

which gives

$$\begin{array}{l} \Lambda^{[0]} > \Omega^{[0]}, \quad \Lambda^{[1]} = \Omega^{[1]}, \quad \Lambda^{[2]} = \Omega^{[2]} \text{ and } \Lambda^{[3]} = \Omega^{[3]} \implies \Lambda > \Omega \\ \Gamma^{[0]} = \Lambda^{[0]}, \quad \Gamma^{[1]} = \Lambda^{[1]}, \quad \Gamma^{[2]} > \Lambda^{[2]} \text{ and } \Gamma^{[3]} > \Lambda^{[3]} \implies \Gamma > \Lambda \end{array} \tag{64}$$

3.6. Two bases of symmetric superpolynomials

We now introduce two bases of superpolynomials that will be central in our construction of the eigenfunctions of the s^{2}CS model.

Definition 3. Let $z^\Lambda = z_1^{\Lambda_1} z_2^{\Lambda_2} \cdots$. To every Λ in the M-fermion sector, we associate a monomial symmetric polynomial defined as [2]

$$m_\Lambda = \frac{1}{f_\Lambda} \sum_{\omega \in S_N} \mathcal{K}_\omega [\phi; \theta]_M z^\Lambda \tag{65}$$

where

$$f_\Lambda = f_{\underline{\bar\Lambda}} f_{\Lambda^s} \quad \text{with} \qquad f_\lambda = n_\lambda(0)!\, n_\lambda(1)! \cdots \tag{66}$$

[2]Note that this basis differs sightly from the one presented in [1] since we use a different ordering on $[\phi;\theta]_M$ (compare eq. (27) of [1] with eq. (33) above). This minor redefinition is more in line with the symmetrization of the non-symmetric Jack polynomials that plays a pivotal role in our construction.

In the last equation, $n_\lambda(i)$ stands for the multiplicity of the part i in the partition λ (the part 0 being considered only for the partition $\underline{\overline{\Lambda}}$).

Note that $1/f_\Lambda$ is a normalization factor that can be avoided by restricting the summation to distinct permutations.

Proposition 4 ([1]). *The monomial symmetric functions $\{m_\Lambda\}_\Lambda$ for all superpartitions Λ in the M-fermionic sector and length at most N form a basis of $\Pi^N_{(M)}$.*

Example 5. Here are some examples of the monomial symmetric functions:

$$\begin{aligned}
m_{(\,;1,0;2;\,)} &= \phi_1\phi_2\theta_3(z_1-z_2)z_3^2+\phi_2\phi_3\theta_1(z_2-z_3)z_1^2\\
&\quad+\phi_1\phi_3\theta_2(z_1-z_3)z_2^2,\\
m_{(0,0;1;\,;1)} &= \phi_1\theta_1\phi_2\theta_2\phi_3z_3z_4+\text{ distinct permutations},\\
m_{(2,1;1,0;0;3,1,1)} &= \phi_1\theta_1\phi_2\theta_2\phi_3\phi_4\theta_5z_1^2z_2^1z_3^1z_4^0z_5^0z_6^3z_7^1z_8^1\\
&\quad+\text{distinct permutations}\ ,
\end{aligned}\tag{67}$$

where in the first example we set $N=3$ while in the two other examples the number of variables in unspecified.

Now, a key step in the construction of the Jack superpolynomials relies on the introduction of a new basis that can be viewed as a deformation of the super power-sums introduced in [1].

Definition 6. To every Λ in the M-fermion sector, we associate a symmetric function q_Λ, dubbed the quasi-power sums, defined as

$$q_\Lambda = p_{\underline{\overline{\Lambda}}}\, m_{(;\,\overline{\Lambda};\,\underline{\Lambda};)}\, p_{\Lambda^s}\tag{68}$$

where

$$p_{\underline{\overline{\Lambda}}} = \prod_i\left(\sum_{k=1}^N \phi_k\theta_k z_k^{\underline{\overline{\Lambda}}_i}\right)\tag{69}$$

and where p_λ stands for the usual power sums:

$$p_\lambda = p_{\lambda_1}\cdots p_{\lambda_\ell}\qquad\text{with}\qquad p_n=\sum_{i=1}^N z_i^n.\tag{70}$$

Example 7. We give some examples of the q_Λ polynomials

$$
\begin{aligned}
q_{(\,;\,1,0;\,2;\,)} &= m_{(\,;\,1,0;\,2;\,)}\\
q_{(0,0;\,1;\,;\,1)} &= (\phi_1\theta_1+\phi_2\theta_2+\cdots)^2(\phi_1 z_1+\phi_2 z_2+\cdots)\\
&\quad\times(z_1+z_2+\cdots)\\
q_{(1,0;\,2,1;\,1,0;\,2,2)} &= (\phi_1\theta_1 z_1+\phi_2\theta_2 z_2+\cdots)(\phi_1\theta_1+\phi_2\theta_2+\cdots)\\
&\quad\times m_{(\,;\,2,1;\,1,0;\,)}(z_1^2+z_2^2+\cdots)^2
\end{aligned}
\tag{71}
$$

We stress that this new basis is not multiplicative due to the non-multiplicative character of the factor $m_{(;\,\bar{\Lambda};\,\underline{\Lambda};)}$.

3.7. Proof of Proposition 1

Having introduced the monomial basis, we are now in position to prove Proposition 1.

Proof. Only the proof of the first relation in (40) will be presented since the other two are similar. The proposition is proven by direct calculation. Applying $\underline{\overline{F}}$ on an arbitrary monomial $m_\Lambda \in \Pi^N_{(M)}$ we have

$$
\underline{\overline{F}}\, m_\Lambda = \sum_{i=1}^N \phi_i\theta_i(\phi_i\theta_i)^\dagger {\sum_{\omega\in S_N}}' \mathcal{K}_\omega[\phi;\theta]_M z^\Lambda \tag{72}
$$

where the prime indicates that we sum only over distinct permutations. Now, since $(\sum_i \phi_i\theta_i(\phi_i\theta_i)^\dagger)$ is S_N-invariant, we can move it through $\mathcal{K}_\omega$ to get

$$
\underline{\overline{F}}\, m_\Lambda = {\sum_{\omega\in S_N}}' \mathcal{K}_\omega \sum_{i=1}^N \phi_i\theta_i(\phi_i\theta_i)^\dagger[\phi;\theta]_M z^\Lambda \tag{73}
$$

Let us now focus on the second sum:

$$
\begin{aligned}
&\sum_{i=1}^N \phi_i\theta_i(\phi_i\theta_i)^\dagger[\phi;\theta]_M\\
&\qquad= \sum_{i=1}^N \phi_i\theta_i(\phi_i\theta_i)^\dagger \phi_1\theta_1\cdots\phi_{\overline{\underline{m}}}\theta_{\overline{\underline{m}}}\phi_{M_1+1}\cdots\phi_{M_2}\theta_{M_2+1}\cdots\theta_{M_3}
\end{aligned}
\tag{74}
$$

It is clear that $\phi_i\theta_i(\phi_i\theta_i)^\dagger\phi_j\theta_j = \delta_{ij}\phi_j\theta_j + (1-\delta_{ij})\phi_j\theta_j\phi_i\theta_i(\phi_i\theta_i)^\dagger$. We can thus write

$$\sum_{i=1}^{N}\phi_i\theta_i(\phi_i\theta_i)^\dagger[\phi;\theta]_M = \sum_{i=1}^{\underline{\overline{m}}}[\phi;\theta]_M = \underline{\overline{m}}\,[\phi;\theta]_M. \tag{75}$$

Substituting this back into (73) gives us

$$\underline{\overline{F}}\,m_\Lambda = \underline{\overline{m}}\,m_\Lambda \tag{76}$$

Now, since any $f \in \Pi^N_{(M)}$ has a unique decomposition on the monomial basis, we have

$$\underline{\overline{F}}\,f = \underline{\overline{F}}\sum_\Lambda c_\Lambda m_\Lambda = \sum_\Lambda c_\Lambda \underline{\overline{F}} m_\Lambda = \sum_\Lambda c_\Lambda \underline{\overline{m}} m_\Lambda = \underline{\overline{m}}\,f \tag{77}$$

and the result holds. □

4. Conserved quantities of the s^{2}CS model and its eigenfunctions

4.1. Eigenfunctions of the s^{2}CS model in terms of the non-symmetric Jack polynomials

The construction of the s^{2}CS eigenfunctions that is presented below is a direct generalization of that worked out in the CS and the sCS models when formulated in terms of the non-symmetric Jack polynomials. We thus start with a brief review of these two known cases after summarizing the properties of the non-symmetric Jack polynomials.

When discussing Jack polynomials and their generalizations, we will comply with the standard notation and use instead the parameter α defined as

$$\alpha = 1/\beta. \tag{78}$$

4.1.1. Brief review of the non-symmetric Jack polynomials. The non-symmetric Jack polynomials [4, 19], denoted E_η, are indexed by a composition $\eta \in \mathbb{Z}^N_{\geq 0}$ and defined as follows: E_η is the unique polynomial of the form

$$E_\eta(z) = z^\eta + \sum_{\nu \prec \eta} a_{\eta,\nu} z^\nu \tag{79}$$

which simultaneously diagonalizes all Dunkl operators

$$\mathcal{D}_i E_\eta = \widehat{\eta}_i E_\eta, \tag{80}$$

where the Dunkl operators $\mathcal{D}_i$ is defined as

$$\mathcal{D}_i = \alpha z_i \partial_i + \sum_{j<i} \frac{z_i}{z_{ij}}(1 - K_{ij}) + \sum_{j>i} \frac{z_j}{z_{ij}}(1 - K_{ij}) - (i-1). \tag{81}$$

The ordering $\prec$ in (79) is the Bruhat order on weak compositions, that is:

$$\nu \prec \eta \iff \nu^+ < \eta^+ \quad \text{or} \quad \nu^+ = \eta^+ \text{ and } w_\nu > w_\eta \tag{82}$$

where w_η is the unique permutation of minimal length such that $\eta = w_\eta \eta^+$ (w_η permutes the entries of η) and where the Bruhat order on the symmetric group S_N is such that $w_\nu > w_\eta$ iff w_η can be written using reduced decompositions as a subword of w_ν. The eigenvalues $\widehat{\eta}_i$ are given by

$$\begin{aligned} \widehat{\eta}_i = \alpha \eta_i - \big(&\#\{j = 1, \dots, i-1 | \eta_j \geq \eta_i\} \\ &+ \#\{j = i+1, \dots, N | \eta_j > \eta_i\}\big). \end{aligned} \tag{83}$$

For instance, for $\eta = (6, 2, 3, 5, 2, 7, 3, 2)$, we have

$$\widehat{\eta}_1 = 6\alpha - 1 \quad \text{and} \quad \widehat{\eta}_5 = 2\alpha - (4+2). \tag{84}$$

The action of the Dunkl operators on monomials is triangular in the Bruhat order. To be more specific, we have

$$\mathcal{D}_i x^\eta = \hat{\eta}_i \, x^\eta + \sum_{\nu \prec \eta} * \, z^\nu \tag{85}$$

where the expansion coefficients are represented by $*$ for simplicity.

In the remainder of this subsection, we collect some relations that will later be useful, relations that describe the action of $K_{i,i+1}$ on both the Dunkl operators and the E_η.

The Dunkl operators satisfy the degenerate Hecke relations :

$$\begin{aligned} &K_{i,i+1}\mathcal{D}_{i+1}-\mathcal{D}_iK_{i,i+1}=-1, \quad \text{and} \\ &K_{j,j+1}\mathcal{D}_i=\mathcal{D}_iK_{j,j+1} \text{ for } i\neq j,\, j+1. \end{aligned} \tag{86}$$

Finally, the non-symmetric Jack polynomials have the following property (see for instance [15])

$$K_{i,i+1}E_\eta=\begin{cases} \frac{1}{\delta_{i,\eta}}E_\eta+(1-\frac{1}{\delta_{i,\eta}})E_{K_{i,i+1}\eta} & \eta_i>\eta_{i+1} \\ E_\eta & \eta_i=\eta_{i+1} \\ \frac{1}{\delta i,\eta}E_\eta+E_{K_{i,i+1}\eta} & \eta_i<\eta_{i+1} \end{cases}, \tag{87}$$

where

$$\delta_{i,\eta}=\widehat{\eta}_i-\widehat{\eta}_{i+1}. \tag{88}$$

4.1.2. Construction of the Jack polynomials in terms of E_η. It is well known that the (symmetric) Jack polynomials can be constructed out of the non-symmetric ones by a direct symmetrization process

$$P_\lambda^{(\alpha)}(z)=\frac{1}{f_\lambda}\sum_{\omega\in S_N}K_\omega E_{\lambda^R}(z) \tag{89}$$

where f_λ was defined in (66) and where λ^R is the composition obtained by reordering the entries of λ in a weakly increasing way, that is, given the partition $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_N)$ (note that λ may contain a string of zeros at the end), we have

$$\lambda^R=(\lambda_N,\dots,\lambda_2,\lambda_1)\,. \tag{90}$$

We stress that any composition η that rearranges to λ could have been used instead of λ^R. The only difference would be that the normalization factor would not necessarily be given by f_λ.

4.1.3. Construction of the $\mathcal{N}=1$ Jack superpolynomials in terms of E_η. The $\mathcal{N}=1$ Jack superpolynomials can also be defined by a similar symmetrization of the non-symmetric Jack polynomials (suitably dressed with θ-terms). In this case, the superpartition Λ is of the form $\Lambda=(\Lambda^a;\Lambda^s)$ where the parts of Λ^a are distinct and the partition Λ^s is an ordinary partition (with possibly zeros at the end). The Jack superpolynomial $P_\Lambda^{(\alpha)}$ – in the fermionic sector m – takes the form

$$P_\Lambda^{(\alpha)}(z,\theta) = \frac{(-1)^{\binom{m}{2}}}{f_{\Lambda^s}} \sum_{\omega \in S_N} \mathcal{K}_\omega \, \theta_1 \cdots \theta_m \, E_{\Lambda^R}(z) \tag{91}$$

where $\Lambda^R = ((\Lambda^a)^R, (\Lambda^s)^R)$ and where $\mathcal{K}_{ij}$ stands for $\mathcal{K}_{ij} = K_{ij}\kappa_{ij}$, with κ_{ij} defined in (44). It was shown that the $P_\Lambda^{(\alpha)}(z,\theta)$'s are the eigenfunctions of the sCS model [9].

4.1.4. Construction of the $\mathcal{N} = 2$ Jack superpolynomials in terms of E_{η}.

By analogy, the candidate $\mathcal{N} = 2$ Jack superpolynomials are constructed as follows.

Definition 8. The $\mathcal{N} = 2$ Jack superpolynomials, in the M-fermionic sector, are given by

$$P_\Lambda^{(\alpha)}(z,\phi,\theta) = \frac{(-1)^{\binom{\overline{m}}{2}+\binom{\underline{m}}{2}}}{f_\Lambda} \sum_{\omega \in S_N} \mathcal{K}_\omega [\phi;\theta]_M E_{\Lambda^R}(z) \tag{92}$$

where Λ^R is the composition defined as follow

$$\Lambda^R = ((\underline{\overline{\Lambda}})^R, (\overline{\Lambda})^R, (\underline{\Lambda})^R, (\Lambda^s)^R). \tag{93}$$

4.2. Sekiguchi operators

We will construct four families of conserved quantities in involution using Sekiguchi operators. The first is the usual Sekiguchi operator

$$S^{[0]}(u,\alpha) = \prod_{i=1}^{N} (\mathcal{D}_i + u) \tag{94}$$

The other three, $S^{[k]}(u,\alpha)$, are defined as

$$S^{[k]}(u,\alpha) = \sum_M \frac{1}{|S_{(M)}|} \sum_{\omega \in S_N} \mathcal{K}_\omega \mathcal{P}^M \prod_{i=1}^{M_k} (\mathcal{D}_i + \alpha + u) \prod_{j=M_k+1}^{N} (\mathcal{D}_i + u), \tag{95}$$

where $k = 1, 2, 3$.

Proposition 9. *We have that* $P_\Lambda^{(\alpha)}(z,\phi,\theta)$ *given in* (92) *is a common eigenfunction of the operators* $S^{[k]}(u,\alpha)$. *More precisely, let*

$$\varepsilon_\lambda(u,\alpha)=\prod_{i=1}^{N}(\alpha\lambda_i+1-i+u) \tag{96}$$

where λ *is a partition (with possibly zeros at the end). Then*

$$S^{[k]}(u,\alpha)P_\Lambda^{(\alpha)}=\varepsilon_{\Lambda^{[k]}}(u,\alpha)P_\Lambda^{(\alpha)} \quad \text{for} \quad k=0,1,2,3 \tag{97}$$

Proof. The proof follows most of the steps of the proof of Proposition 1 of [12]. It is easy to verify, using (86), that for all $i=1,\dots,N-1$ we have

$$K_{i,i+1}(\mathcal{D}_i+c)(\mathcal{D}_{i+1}+c)=(\mathcal{D}_i+c)(\mathcal{D}_{i+1}+c)K_{i,i+1}\,, \tag{98}$$

where c is an arbitrary constant. Hence $K_\omega S^{[0]}(u,\alpha)=S^{[0]}(u,\alpha)K_\omega$ for all $\omega\in S_N$. And since $S^{[0]}(u,\alpha)$ only acts on the variables z, we also have $\mathcal{K}_\omega[\phi;\theta]_M S^{[0]}(u,\alpha)=S^{[0]}(u,\alpha)\mathcal{K}_\omega[\phi;\theta]_M$ for all $\omega\in S_N$. Therefore, using (92), to prove the $S^{[0]}(u,\alpha)$ case we simply need to show that

$$\left(\prod_{i=1}^{N}(\mathcal{D}_i+u)\right)E_{\Lambda^R}(z)=\varepsilon_{\Lambda^{[0]}}(u,\alpha)\,E_{\Lambda^R}(z). \tag{99}$$

Similarly, we will now show that to prove the remaining cases, it suffices to prove that

$$\left(\prod_{i=1}^{M_k}(\mathcal{D}_i+\alpha+u)\prod_{j=M_k+1}^{N}(\mathcal{D}_j+u)\right)E_{\Lambda^R}(z)=\varepsilon_{\Lambda^{[k]}}(u,\alpha)\,E_{\Lambda^R}(z)\,. \tag{100}$$

for $k=1,2,3$.

As observed before, $\mathcal{P}^M\mathcal{K}_\omega[\phi;\theta]_M$ is non-zero only if $\omega\in S_{(M)}$. Using (92) again (forgetting the multiplicative factor), we get

$$\begin{aligned} &S^{[k]}(u,\alpha)\sum_{\omega\in S_N}\mathcal{K}_\omega[\phi;\theta]_M E_{\Lambda^R}(z)\\ &\qquad=\frac{1}{|S_{(M)}|}\sum_{\sigma\in S_N}\mathcal{K}_\sigma\left(\prod_{i=1}^{M_k}(\mathcal{D}_i+\alpha+u)\prod_{j=M_k+1}^{N}(\mathcal{D}_j+u)\right)\\ &\qquad\quad\times\sum_{\omega\in S_{(M)}}\mathcal{K}_\omega[\phi;\theta]_M E_{\Lambda^R}(z) \end{aligned} \tag{101}$$

From (98), we can deduce that $\prod_{i=1}^{M_k}(\mathcal{D}_i+\alpha+u)\prod_{j=M_k+1}^{N}(\mathcal{D}_j+u)$ commutes with $\mathcal{K}_\omega[\phi;\theta]_M$ for every $\omega\in S_{(M)}$. Hence

$$
\begin{aligned}
S^{[k]}(u,\alpha)&\sum_{\omega\in S_N}\mathcal{K}_\omega[\phi;\theta]_M E_{\Lambda^R}(z)\\
&=\sum_{\sigma\in S_N}\mathcal{K}_\sigma[\phi;\theta]_M\left(\prod_{i=1}^{M_k}(\mathcal{D}_i+\alpha+u)\prod_{j=M_k+1}^{N}(\mathcal{D}_j+u)\right)E_{\Lambda^R}(z)
\end{aligned}
\tag{102}
$$

and, as claimed, (100) implies the remaining statements in the proposition.

We have left to prove expressions (99) and (100). Let $\eta=\Lambda^R$ and suppose that $\eta_i=r$. It is easy to get from (83) that the eigenvalue $\hat{\eta}_i$ of $\mathcal{D}_i$ is

$$
\begin{aligned}
\hat{\eta}_i=\alpha r&-\#\{\text{rows of }\Lambda^{[0]}\text{ of size larger than }r\}\\
&-\#\{\text{rows of }\Lambda^{R}\text{ of size }r\text{ above row }i\}\,.
\end{aligned}
\tag{103}
$$

Therefore, letting

$$
\begin{aligned}
j_i=&\#\{\text{rows of }\Lambda^{[0]}\text{ of size larger than }r\}\\
&+\#\{\text{rows of }\Lambda^{R}\text{ of size }r\text{ above row }i\}+1
\end{aligned}
\tag{104}
$$

we have $\{j_1,\dots,j_N\}=\{1,\dots,N\}$, $\Lambda^{[0]}_{j_i}=r$, and $\hat{\eta}_i=\alpha\Lambda^{[0]}_{j_i}+1-j_i$, which gives (99).

Continuing with the same notation, we suppose that i belongs to $\{1,\dots,m\}$ and that there are ℓ rows of size r in $(\eta_1,\dots,\eta_m)$. Then $\eta_i=r$ belongs to the ℓ highest rows of size r in η, and thus, by (104), $\Lambda^{[0]}_{j_i}$ is also one of the ℓ highest rows of size r in $\Lambda^{[0]}$. Hence, in this case

$$
\hat{\eta}_i+\alpha=\alpha\Lambda^{[0]}_{j_i}+1-j_i+\alpha=\alpha\Lambda^{[k]}_{j_i}+1-j_i\,.
\tag{105}
$$

If i does not belong to $\{1,\dots,m\}$, then $\eta_i=r$ does not belong to the ℓ highest rows of size r in η, and we have

$$
\hat{\eta}_i=\alpha\Lambda^{[0]}_{j_i}+1-j_i=\alpha\Lambda^{[k]}_{j_i}+1-j_i
\tag{106}
$$

and (100) follows. □

From this proposition, we will later conclude that the Jack superpolynomials expand triangularly in the monomial basis. But we first need to establish that the Sekiguchi operators act triangularly on the monomial basis.

Proposition 10. *We have that*

$$S^{[k]}(u,\alpha)\, m_\Lambda = \varepsilon_{\Lambda^{[k]}}(u,\alpha)\, m_\Lambda + \text{lower terms} \tag{107}$$

Proof. For any weak composition η, define $\eta^{[0]}$ to be η^+ and $\eta^{[k]}$ to be $(\eta + 1^{M_k})^+$. It was shown in [5] that if z^ν occurs in the expansion of $Y_i z^\eta$, where Y_i is a Cherednik operator [7, 8, 18] (whose precise form is not needed here) then $\nu^{[0]} \leq \eta^{[0]}$ and $\nu^{[k]} \leq \eta^{[k]}$. Since $\mathcal{D}_i$ can be obtained as a limit of Y_i, that is, since [14]

$$\mathcal{D}_i = \lim_{q=t^\alpha, t\to 1} \frac{1-Y_i}{1-q}, \tag{108}$$

the result also holds for $\mathcal{D}_i$. Using (85), which gives the triangularity in the Bruhat order, we thus have

$$\mathcal{D}_i z^\eta = \hat{\eta}_i z^\eta + \sum_{\eta<\gamma} * \, z^\nu \tag{109}$$

for certain coefficients $*$, where $\eta \leq \gamma$ iff $\nu^{[k]} \leq \eta^{[k]}$ for $k=0,1,2,3$. We then have that

$$\mathcal{D}_i z^\eta = \hat{\eta}_i z^\eta + \sum_{\eta<\gamma} * \, z^\nu \tag{110}$$

where the order is now

$$\eta \leq \gamma \iff \nu^{[k]} \leq \eta^{[k]} \text{ with } k=0,1,2,3. \tag{111}$$

We finish the proof by showing that (the other cases are similar)

$$S^{[1]}(u,\alpha)\, m_\Lambda = \varepsilon_{\Lambda^{[1]}}(u,\alpha)\, m_\Lambda + \text{lower terms} \tag{112}$$

It is easy to see that, up to a sign $(-1)^\xi$, we have

$$m_\Lambda = \frac{1}{f_\Lambda} \sum_{\omega\in S_N} \mathcal{K}_\omega [\phi;\theta]_M z^\Lambda = (-1)^\xi \frac{1}{f_\Lambda} \sum_{\omega\in S_N} \mathcal{K}_\omega [\phi;\theta]_M z^{\Lambda^R} \tag{113}$$

since the permutation γ that sends Λ to Λ^R is such that $\mathcal{K}_\gamma[\phi;\theta]_M = (-1)^\xi[\phi;\theta]_M \mathcal{K}_\gamma$. After acting with the projector $\mathcal{P}^M$ contained in $S^{[1]}(u,\alpha)$, we then obtain

$$S^{[1]}(u,\alpha)\,m_\Lambda = \frac{(-1)^\xi}{f_\Lambda|S_{(M)}|}\sum_{\omega\in S_N}\mathcal{K}_\omega\prod_{i=1}^{M_1}(\mathcal{D}_i+\alpha+u) \times \prod_{j=M_1+1}^{N}(\mathcal{D}_i+u)\sum_{\sigma\in S_{(M)}}\mathcal{K}_\sigma[\phi;\theta]_M z^{\Lambda^R} \tag{114}$$

Now, K_σ commutes with the two products of $\mathcal{D}_i$'s since $\sigma\in S_{(M)}$, which gives

$$S^{[1]}(u,\alpha)\,m_\Lambda = \frac{(-1)^\xi}{f_\Lambda}\sum_{\omega\in S_N}\mathcal{K}_\omega[\phi;\theta]_M\prod_{i=1}^{M_1}(\mathcal{D}_i+\alpha+u)\prod_{j=M_1+1}^{N}(\mathcal{D}_i+u)z^{\Lambda^R} \tag{115}$$

Using (100) and (110) , this gives

$$\begin{aligned} S^{[1]}(u,\alpha)\,m_\Lambda &= \frac{(-1)^\xi}{f_\Lambda}\sum_{\omega\in S_N}\mathcal{K}_\omega[\phi;\theta]_M\left(\varepsilon_{\Lambda^{[1]}}(u,\alpha)z^{\Lambda^R}+\sum_{\eta<\Lambda^R} * z^\eta\right)\\ &= \frac{1}{f_\Lambda}\sum_{\omega\in S_N}\mathcal{K}_\omega[\phi;\theta]_M\left(\varepsilon_{\Lambda^{[1]}}(u,\alpha)z^{\Lambda}+\sum_{\eta<\Lambda^R} * z^\eta\right) \end{aligned} \tag{116}$$

Now, η corresponds to a unique superpartition $\Omega = w_\eta\eta$, where $w_\eta\in S_{(M)}$. This correspondence is easily seen to be such that $\eta<\nu$ iff the corresponding superpartitions Ω and Γ are such that $\Omega<\Gamma$. The previous equation then immediately implies (112). □

Proposition 11. *The Jack superpolynomials are unitriangularly related to the monomials, that is,*

$$P_\Lambda^{(\alpha)} = m_\Lambda + \sum_{\Omega<\Lambda} d_{\Lambda\Omega}(\alpha)\,m_\Omega \tag{117}$$

Proof. The proof that follows is basically the proof of the triangularity in Proposition 7 of [5]. We include it for completeness.

Suppose that there exists a term m_Ω such that $\Omega\not\leq\Lambda$ in $P_\Lambda^{(\alpha)}$ and suppose that Ω is maximal among those superpartitions. Then, by Proposition 10, the coefficient of m_Ω in $S^{[k]}(u,\alpha)J_\Lambda^\alpha$ is equal to $d_{\Lambda\Omega}\varepsilon_{\Omega^{[k]}}(u,\alpha)$ for $k=0,1,2,3$. On the other hand, Proposition 9 tells us that the coefficient of m_Ω in in $S^{[k]}(u,\alpha)J^\alpha_\Lambda$ is equal to $d_{\Lambda\Omega}\varepsilon_{\Lambda^{[k]}}(u,\alpha)$, again for $k=0,1,2,3$. But this gives that

$$\begin{aligned} \varepsilon_{\Omega^{[0]}}(u,\alpha) &= \varepsilon_{\Lambda^{[0]}}(u,\alpha), \quad \varepsilon_{\Omega^{[1]}}(u,\alpha) = \varepsilon_{\Lambda^{[1]}}(u,\alpha), \\ \varepsilon_{\Omega^{[2]}}(u,\alpha) &= \varepsilon_{\Lambda^{[2]}}(u,\alpha), \quad \varepsilon_{\Omega^{[3]}}(u,\alpha) = \varepsilon_{\Lambda^{[3]}}(u,\alpha), \end{aligned} \tag{118}$$

which is a contradiction since $\Lambda \neq \Omega$ ($\varepsilon_{\Lambda^{[0]}}(u,\alpha)$, $\varepsilon_{\Lambda^{[1]}}(u,\alpha)$, $\varepsilon_{\Lambda^{[2]}}(u,\alpha)$ and $\varepsilon_{\Lambda^{[3]}}(u,\alpha)$ uniquely determine Λ since they uniquely determine $\Lambda^{[0]}$, $\Lambda^{[1]}$, $\Lambda^{[2]}$ and $\Lambda^{[3]}$). □

Given that the monomials form a basis of $\Pi^N_{(M)}$, we have immediately.

Corollary 12. *The Jack superpolynomials $\{P^{(\alpha)}_\Lambda\}_\Lambda$ for all superpartitions Λ in the M-fermionic sector and length at most N form a basis of $\Pi^N_{(M)}$.*

For $i=1,\ldots,N$, let $H^{[0]}_i$ be the coefficient of u^{N-i} in $S^{[0]}(u,\alpha)$. Let also $H^{[1]}_i$ be the coefficient of u^{N-i} in $S^{[1]}(u,\alpha)$, and similarly for $H^{[2]}_i$ and $H^{[3]}_i$. The previous corollary together with Proposition 9 imply that these $4N$ operators are in involution. We will see in the next subsection how the integrability of the s^{2}CS model is then immediate since the s^{2}CS Hamiltonian $\mathcal{H}$ can be taken to be one of those operators.

Corollary 13. *The $4N$ quantities $H^{[0]}_i$, $H^{[1]}_j$, $H^{[2]}_k$ and $H^{[3]}_\ell$, for $i,j,k,\ell = 1,\ldots,N$ mutually commute when restricted to the space of symmetric superpolynomials, that is,*

$$[H^{[k]}_i, H^{[l]}_j]f = 0 \quad \forall \quad k,l=0,1,2,3 \quad \text{and} \quad i,j=1,\ldots,N \tag{119}$$

whenever f belongs to $\Pi^N_{(M)}$.

4.3. Complete characterization of the eigenfunctions from a minimal set of commuting operators and integrability of the s^{2}CS model

The Jack polynomials are fully characterized by being (1) triangular in the monomial basis, and (2) eigenfunctions of the CS Hamiltonian. Similarly, the $\mathcal{N}=1$ Jack superpolynomials are completely characterized by the triangularity condition and that they diagonalize both the sCS Hamiltonian and another conservation law $\mathcal{I}$. For the $\mathcal{N}=2$ version, the condition (2) amounts to diagonalizing the s^{2}CS Hamiltonian together with three extra conservation laws (that we shall denote $\mathcal{I}_{[1]}$, $\mathcal{I}_{[2]}$ and $\mathcal{I}_{[3]}$). This implies in passing that the s^{2}CS model is integrable.

We first prove that if we impose the unitriangularity, then the operators $H_2^{[i]}$ for $i = 0, 1, 2, 3$ are sufficient to characterize the Jack superpolynomials. The eigenvalue of those operators is the coefficient of u^{N-2} in $\varepsilon_{\Lambda^{[k]}}(u, \alpha)$ for $k = 0, 1, 2, 3$. In general, we have that

$$\varepsilon_\lambda^{(2)}(\alpha) := \varepsilon_\lambda(u, \alpha)\Big|_{u^2} = e_2(\alpha\lambda_1, \alpha\lambda_2 - 1, \ldots, \alpha\lambda_N + 1 - N) \tag{120}$$

where $e_2(x_1, x_2, x_3, \ldots) = x_1x_2 + x_1x_3 + x_2x_3 + \cdots$ is an elementary symmetric function.

The following lemma will prove useful.

Lemma 14. *If $\mu < \lambda$ then $\varepsilon_\lambda^{(2)}(\alpha) \neq \varepsilon_\mu^{(2)}(\alpha)$.*

Proof. It is a known easy lemma (see for instance [20]) that we prove for completeness. Suppose that λ and ν are two partitions such that $\lambda_i = \nu_i + 1$ and $\lambda_j = \nu_j - 1$ for $i < j$. Comparing their quadratic terms in α, we get

$$\left(\varepsilon_\nu^{(2)}(\alpha) - \varepsilon_\lambda^{(2)}(\alpha)\right)\Big|_{\alpha^2} = 1 + \nu_i - \nu_j > 0 \tag{121}$$

since ν is a partition. When $\mu < \lambda$, it is well-known [17] that one can go from λ to μ using steps such as those we just used to go from λ to ν. Hence

$$\left(\varepsilon_\mu^{(2)}(\alpha) - \varepsilon_\lambda^{(2)}(\alpha)\right)\Big|_{\alpha^2} > 0 \tag{122}$$

and we can conclude that $\varepsilon_\lambda^{(2)}(\alpha) \neq \varepsilon_\mu^{(2)}(\alpha)$. □

Proposition 15. *The Jack superpolynomial $P_\Lambda^{(\alpha)}$ is uniquely defined by the following two conditions:*

$$\begin{aligned} &(1):\ P_\Lambda^{(\alpha)} = m_\Lambda + \text{ lower terms} \\ &(2):\ P_\Lambda^{(\alpha)} \text{ diagonalizes the operators } H_2^{[0]},\ H_2^{[1]},\ H_2^{[2]} \text{ and } H_2^{[3]} \end{aligned} \tag{123}$$

Proof. This is again a standard proof that we include for completeness. Let $P_\Lambda^{(\alpha)}$ and $\tilde{P}_\Lambda^{(\alpha)}$ be two polynomials that obey the two conditions (123). Then

$$P_\Lambda^{(\alpha)} - \tilde{P}_\Lambda^{(\alpha)} = \sum_{\Omega<\Lambda} c_{\Lambda\Omega}(\alpha)\, m_\Omega \tag{124}$$

for some coefficients $c_{\Lambda\Omega}(\alpha)$. Assume that Γ is maximal among the Ω's such that $c_{\Lambda\Omega}(\alpha) \neq 0$. Since $\Lambda \neq \Gamma$, we have that $\Lambda^{[k]} \neq \Gamma^{[k]}$ for either $k = 0,1,2,3$. Suppose without loss of generality that $\Lambda^{[0]} \neq \Gamma^{[0]}$. Applying $H_2^{[0]}$ on both sides of the previous equation then gives, from our assumptions,

$$\begin{aligned} \varepsilon^{(2)}_{\Lambda^{[0]}}(\alpha) \left(P_\Lambda^{(\alpha)} - \tilde{P}_\Lambda^{(\alpha)} \right) &= \varepsilon^{(2)}_{\Lambda^{[0]}}(\alpha) \sum_{\Omega<\Lambda} c_{\Lambda\Omega}(\alpha)\, m_\Omega \\ &= H_2^{[0]} \sum_{\Omega<\Lambda} c_{\Lambda\Omega}(\alpha)\, m_\Omega\,, \end{aligned} \tag{125}$$

where we used the fact, on the l.h.s., that the eigenvalue of $P_\Lambda^{(\alpha)}$ and $\tilde{P}_\Lambda^{(\alpha)}$ needs to be $\varepsilon^{(2)}_{\Lambda^{[0]}}(\alpha)$ from Proposition 10. The coefficient of m_Γ is then, by maximality and Proposition 10, such that

$$\varepsilon^{(2)}_{\Lambda^{[0]}}(\alpha)\, c_{\Lambda\Gamma}(\alpha) = \varepsilon^{(2)}_{\Gamma^{[0]}}(\alpha)\, c_{\Lambda\Gamma}(\alpha) \tag{126}$$

Since $\Gamma^{[0]} < \Lambda^{[0]}$, we have from Lemma 14 that $\varepsilon^{(2)}_{\Lambda^{[0]}}(\alpha) \neq \varepsilon^{(2)}_{\Gamma^{[0]}}(\alpha)$. This leads to the contradiction that $c_{\Lambda\Gamma}(\alpha) = 0$ from which we deduce that $c_{\Lambda\Omega}(\alpha) = 0$ for all Ω. Hence $P_\Lambda^{(\alpha)} = \tilde{P}_\Lambda^{(\alpha)}$ and the proof is complete. □

We will now show that we can replace the four operators in the previous proposition by simpler ones. Let

$$\mathcal{I}_{[k]} = \sum_M \frac{1}{|S_{(M)}|} \sum_{\omega \in S_N} \mathcal{K}_\omega \mathcal{P}^M (\mathcal{D}_1 + \cdots + \mathcal{D}_{M_k}). \tag{127}$$

Theorem 16. *The Jack superpolynomials $P_\Lambda^{(\alpha)}$ are uniquely defined by the following two conditions:*

$$\begin{aligned} &(1):\ P_\Lambda^{(\alpha)} = m_\Lambda + \textit{ lower terms} \\ &(2):\ P_\Lambda^{(\alpha)} \textit{ diagonalizes the operators } \mathcal{H},\ \mathcal{I}_{[1]},\ \mathcal{I}_{[2]} \textit{ and } \mathcal{I}_{[3]} \end{aligned} \tag{128}$$

where $\mathcal{H}$ is the Hamiltonian defined in (24).

Proof. Let

$$D := H_1^{[0]} = \mathcal{D}_1 + \cdots + \mathcal{D}_N \tag{129}$$

whose eigenvalue on $P_\Lambda^{(\alpha)}$ is $\sum_i(\alpha\Lambda_i^{[0]} + 1 - i) = \alpha|\Lambda| - N(N-1)/2$. The eigenvalue of D is thus constant on basis elements of the same degree (which is the case we are considering). From (45) we have that the relation

$$\alpha^2\mathcal{H} = D^2 - 2H_2^{[0]} + (N-1)D + \frac{N(N^2-1)}{6} \tag{130}$$

which is valid in the $\mathcal{N} = 0$ case[3] also holds in the $\mathcal{N} = 2$ case. Hence $\mathcal{H}$ can replace $H_2^{[0]}$ in Proposition 15 without affecting the validity of the proposition.

It is also straightforward to check that

$$\begin{aligned} \mathcal{D}_1 + \cdots + \mathcal{D}_m \\ &= \left(\prod_{j=1}^{N}(\mathcal{D}_i + u) - \prod_{i=1}^{m}(\mathcal{D}_i + \alpha + u) \prod_{j=m+1}^{N}(\mathcal{D}_i + u) \right)\Bigg|_{u^2} \\ &\quad + m(\mathcal{D}_1 + \cdots + \mathcal{D}_N) + m(m-1)/2 \end{aligned} \tag{131}$$

which implies that

$$\mathcal{I}_{[k]} = H_2^{[0]} - H_2^{[k]} + M_k D + M_k(M_k - 1)/2 \tag{132}$$

for $k = 1, 2, 3$. Since the eigenvalue of D are constant on basis elements of the same degree (these numbers are already encoded in Λ), we have that $\mathcal{I}_{[k]}$ can also replace $H_2^{[k]}$ in Proposition 15 without affecting the validity of the proposition. $\square$

From (129) and (130), the s^{2}CS Hamiltonian $\mathcal{H}$ can be taken to be one of the $4N$ independent commuting quantities $H_i^{[k]}$ for $i = 1, \ldots, N$ and $k = 0, 1, 2, 3$ (see Corollary 13). Hence, we have the following.

Corollary 17. *The s^{2}CS model is integrable.*

For completeness, we give the eigenvalues of $\mathcal{H}$, $\mathcal{I}_{[1]}$, $\mathcal{I}_{[2]}$ and $\mathcal{I}_{[3]}$.

[3]This is a well know result that can be compared for instance with (3.27) in [16] for lack of a better reference

Proposition 18. *We have that*

$$\mathcal{H}\, P^{\alpha}_{\Lambda} = \epsilon_{\Lambda}(\alpha)\, P^{\alpha}_{\Lambda} \quad \text{and} \quad \mathcal{I}_{[k]}\, P^{\alpha}_{\Lambda} = \epsilon^{[k]}_{\Lambda}(\alpha)\, P^{\alpha}_{\Lambda} \tag{133}$$

for $k=1,2,3$. *The eigenvalues are given explicitly as*

$$\begin{aligned} \epsilon_{\Lambda}(\alpha) &= \sum_{i=1}^{N} \big[\alpha^2 (\Lambda^{[0]}_i)^2 + \alpha(N+1-2i)\Lambda^{[0]}_i\big] \\ \text{and} \quad \epsilon^{[k]}_{\Lambda}(\alpha) &= \sum_{i:\Lambda^{[k]}_i \neq \Lambda^{[0]}_i} (\alpha \Lambda^{[k]}_i + 1 - i) \end{aligned} \tag{134}$$

with $k=1,2,3$.

Proof. Straightforward using (130), (132) and the known eigenvalues of $H^{[0]}_2, H^{[k]}_2$ and D. □

5. Two orthogonality characterizations of the Jack superpolynomials

5.1. Relation with Jack polymomials with prescribed symmetries and the analytic scalar product

The relation between the Jack superpolynomials and the non-symmetric Jack polynomials implies that we can define a scalar product with respect to which the Jack superpolynomials are orthogonal. We outline the details of this implication in the present section.

It proves convenient to introduce as an intermediate step the relation between $P^{(\alpha)}_{\Lambda}(z,\phi,\theta)$ and the Jack polynomials with prescribed symmetry.

Definition 19. The Jack polynomials with prescribed symmetry (the prescription being SAAS where S and A stand respectively for symmetry and antisymmetry), are defined as

$$\mathscr{E}_{\Lambda} = \frac{(-1)^{\binom{\overline{m}}{2}+\binom{\underline{m}}{2}}}{f_{\Lambda}} \sum_{\underline{\overline{\alpha}},\, \overline{\alpha},\, \underline{\alpha},\, \gamma} (-1)^{\ell(\overline{\alpha})+\ell(\underline{\alpha})} \mathcal{K}_{\underline{\overline{\alpha}}} \mathcal{K}_{\overline{\alpha}} \mathcal{K}_{\underline{\alpha}} \mathcal{K}_{\gamma} E_{\Lambda^R} \tag{135}$$

where f_Λ was defined in (66) and where we used the compact notation

$$\sum_{\underline{\overline{\alpha}}, \overline{\alpha}, \underline{\alpha}, \gamma} \equiv \sum_{\substack{\underline{\overline{\alpha}} \in S_{M_1} \\ \overline{\alpha} \in S_{]M_1, M_2]} \\ \underline{\alpha} \in S_{]M_2, M_3]} \\ \gamma \in S_{]M_3, N]}} \tag{136}$$

Explicitly, the SAAS prescription means that the symmetrization is taken independently with respect to the $\overline{\underline{m}}$ first and last $N - M_3$ variables while the antisymmetrization is taken independently with respect to the variables in position $M_1 + 1, \ldots, M_2$ and $M_2 + 1, \ldots, M_3$. This entails the following corollary of Definition 8:

Corollary 20. *The Jack superpolynomials* (92) *can equivalently be written as*

$$P_\Lambda^{(\alpha)}(z, \phi, \theta) = \sum_{\omega \in S_{(M)}} \mathcal{K}_\omega [\phi; \theta]_M \mathscr{E}_\Lambda(z) \tag{137}$$

where we recall that $S_{(M)}$ was defined in (36).

Proof. The proof is obtained by direct calculation using Definition 19:

$$\begin{aligned} \sum_{\omega \in S_{(M)}} \mathcal{K}_\omega [\phi; \theta]_M \mathscr{E}_\Lambda &= \frac{(-1)^{\binom{\overline{m}}{2} + \binom{\underline{m}}{2}}}{f_\Lambda} \\ &\quad \times \sum_{\substack{\omega \in S_{(M)} \\ \underline{\overline{\alpha}}, \overline{\alpha}, \underline{\alpha}, \gamma}} \mathcal{K}_\omega [\phi; \theta]_M (-1)^{\ell(\overline{\alpha}) + \ell(\underline{\alpha})} \mathcal{K}_{\underline{\overline{\alpha}}} \mathcal{K}_{\overline{\alpha}} \mathcal{K}_{\underline{\alpha}} \mathcal{K}_\gamma E_{\Lambda^R} && (138) \\ &= \frac{(-1)^{\binom{\overline{m}}{2} + \binom{\underline{m}}{2}}}{f_\Lambda} \sum_{\substack{\omega \in S_{(M)} \\ \underline{\overline{\alpha}}, \overline{\alpha}, \underline{\alpha}, \gamma}} \mathcal{K}_\omega \mathcal{K}_{\underline{\overline{\alpha}}} \mathcal{K}_{\overline{\alpha}} \mathcal{K}_{\underline{\alpha}} \mathcal{K}_\gamma [\phi; \theta]_M E_{\Lambda^R} && (139) \end{aligned}$$

In the last equation, passing the factor $[\phi; \theta]_M$ through the permutations $\mathcal{K}_{\overline{\alpha}}, \mathcal{K}_{\underline{\alpha}}$ produces a sign $(-1)^{\ell(\overline{\alpha}) + \ell(\underline{\alpha})}$. Then, the combined sum over all the permutations is exactly the sum over all permutations of S_N. The last line then matches the definition of $P_\Lambda^{(\alpha)}$ given in (92). □

Definition 21. The analytic scalar product in the M-fermion sector is defined as

$$\begin{aligned}(140)\qquad \langle A(z,\theta,\phi)|B(z,\theta,\phi)\rangle_\alpha &= \prod_{i=1}^{N}\left(\oint \frac{dz_i}{2\pi i z_i}\right)\\ &\times \prod_{i=1}^{N}\int d\phi_i d\theta_i \prod_{k\neq l}\left(1-\frac{z_k}{z_l}\right)^{\frac{1}{\alpha}}[A(z,\theta,\phi)]^{\ddagger}\,B(z,\theta,\phi)\end{aligned}$$

where the $\ddagger$ operation on the z_i variables acts as $z_i^{\ddagger}={z_i}^{-1}$ while on the anticommuting variables it is defined such that

$$(141)\qquad \prod_{i\in I}\phi_i\theta_i\prod_{j\in J}\phi_j\prod_{l\in L}\theta_l\left(\prod_{i\in I}\phi_i\theta_i\prod_{j\in J}\phi_j\prod_{l\in L}\theta_l\right)^{\ddagger}=\theta_N\phi_N\cdots\theta_1\phi_1.$$

The non-symmetric Jack polynomials are known to be orthogonal with respect to the scalar product (141) (see [19]) in the case where $M_1=M_2=M_3=0$. It then easily follows that the Jack polynomials with prescribed symmetry are also orthogonal with respect to that scalar product. We have indeed

$$(142)\qquad \langle \mathscr{E}_\Lambda|\mathscr{E}_\Omega\rangle_\alpha=\delta_{\Lambda\Omega}\,c_\Lambda(\alpha)$$

where $c_\Lambda(\alpha)$ is a non-zero constant belonging to $\mathbb{Q}(\alpha)$ (see [2] for an explicit formula).

Lemma 22. *The sJacks are orthogonal with respect to the scalar product* (141) *and have the following norm:*

$$(143)\qquad \langle P_\Lambda^{(\alpha)}|P_\Omega^{(\alpha)}\rangle_\alpha=\delta_{\Lambda\,\Omega}\,\frac{N!}{\underline{\overline{m}}!\,\overline{m}!\,\underline{m}!\,(N-M_3)!}\,c_\Lambda(\alpha)$$

where $c_\Lambda(\alpha)$ is the norm of the Jack polynomials with prescribed symmetry.

Proof. We know that any two symmetric superpolynomials that are not in the same sector will be de facto orthogonal. So Λ and Ω must belong to the same sector M. We thus have

$$\langle P_\Lambda^{(\alpha)} | P_\Omega^{(\alpha)} \rangle_\alpha = \langle \sum_{\omega \in S_{(M)}} \mathcal{K}_\omega [\phi;\theta]_M \mathscr{E}_\Lambda \big| \sum_{\sigma \in S_{(M)}} \mathcal{K}_\sigma [\phi;\theta]_M \mathscr{E}_\Omega \rangle_\alpha \tag{144}$$

Now, using the adjoint $([\phi;\theta]_M)^\dagger (\mathcal{K}_\sigma)^{-1}$ of $\mathcal{K}_\sigma[\phi;\theta]_M$, we obtain

$$\langle P_\Lambda^{(\alpha)} | P_\Omega^{(\alpha)} \rangle_\alpha = \langle \sum_{\omega,\sigma \in S_{(M)}} ([\phi;\theta]_M)^\dagger (\mathcal{K}_\sigma)^{-1} \mathcal{K}_\omega [\phi;\theta]_M \mathscr{E}_\Lambda | \mathscr{E}_\Omega \rangle_\alpha \tag{145}$$

Here, we see that $([\phi;\theta]_M)^\dagger (\mathcal{K}_\sigma)^{-1} \mathcal{K}_\omega [\phi;\theta]_M$ will be 0 unless $([\phi;\theta]_M)^\dagger$ and $(\mathcal{K}_\sigma)^{-1} \mathcal{K}_\omega [\phi;\theta]_M$ have the exact same fermionic content, that is, unless $\omega = \sigma$. We then get

$$\langle P_\Lambda^{(\alpha)} | P_\Omega^{(\alpha)} \rangle_\alpha = \sum_{\omega \in S_{(M)}} \langle \mathscr{E}_\Lambda | \mathscr{E}_\Omega \rangle_\alpha \tag{146}$$

$$= \frac{N!}{\underline{\overline{m}}!\, \overline{m}!\, \underline{m}!\, (N - M_3)!} \langle \mathscr{E}_\Lambda | \mathscr{E}_\Lambda \rangle_\alpha \delta_{\Lambda,\Omega} \tag{147}$$

□

Summing up, the Jack superpolynomials given in Definition 8 are equivalently characterized as follows:

Theorem 23. *The superpolynomials* $\{P_\Lambda^{(\alpha)}\}_\Lambda$ *are defined by the two conditions:*

$$\begin{aligned} &(1): \ P_\Lambda^{(\alpha)} = m_\Lambda + \text{ lower terms} \\ &(2): \ \langle P_\Lambda^{(\alpha)} | P_\Omega^{(\alpha)} \rangle_\alpha \propto \delta_{\Lambda\Omega}. \end{aligned} \tag{148}$$

Proof. The triangularity was shown in Proposition 11. Given that the Gram-Schmidt process constructs a unique basis from any total order compatible with the order on superpartitions, the theorem follows immediately. □

5.2. Combinatorial scalar product

We first give two simple examples of sJacks constructed from (92), or equivalently, as eigenfunctions of the s^{2}CS four basic conservation laws, expressed both in the monomial basis and the quasi-power sums.

$$
\begin{aligned}
P^{(\alpha)}_{(1;2,1,0;;)} &= m_{(1;2,1,0;;)} + \frac{1}{\alpha+2} m_{(0;2,1,0;;1)} \\
&= \frac{1}{(2+\alpha)} q_{(0;2,1,0;;1)} - \frac{1}{(2+\alpha)} q_{(0;3,1,0;;)} + \frac{1+\alpha}{2+\alpha} q_{(1;2,1,0;;)} \\
P^{(\alpha)}_{(0;3,1,0;;)} &= m_{(0;3,1,0;;)} + \frac{1}{\alpha+1} m_{(0;2,1,0;;1)} + \frac{1}{\alpha+1} m_{(1;2,1,0;;)} \\
&= \frac{1}{(1+\alpha)} q_{(0;2,1,0;;1)} + \frac{\alpha}{1+\alpha} q_{(0;3,1,0;;)}
\end{aligned} \tag{149}
$$

We now introduce a combinatorial scalar product defined directly in terms of the quasi-power sums. It is a natural (albeit non-trivial) extension of the $\mathcal{N}=1$ scalar product for super power-sums [10].

Definition 24. The scalar product is defined on the q_Λ basis as

$$\langle\!\langle q_\Lambda | q_\Omega \rangle\!\rangle_\alpha = \alpha^{\overline{m}+\ell(\Lambda^s)}\, \xi_{\underline{\bar{\Lambda}}}\, z_{\Lambda^s} \delta_{\Lambda\Omega} \tag{150}$$

with

$$z_\lambda = \prod_i i^{n_\lambda(i)} n_\lambda(i)! \quad \text{and} \quad \xi_{\underline{\bar{\Lambda}}} = \prod_i n_{\underline{\bar{\Lambda}}}(i)! \tag{151}$$

We can check that the $P^{(\alpha)}$'s given in (149) are orthogonal with respect to the scalar product defined in (150):

$$
\begin{aligned}
\langle\!\langle P^{(\alpha)}_{(1;2,1,0;;)} | P^{(\alpha)}_{(0;3,1,0;;)} \rangle\!\rangle_\alpha &= \frac{1}{(\alpha+2)(\alpha+1)} ||q_{(0;2,1,0;;1)}||^2 \\
&\quad - \frac{\alpha}{(\alpha+2)(\alpha+1)} ||q_{(0;3,1,0;;)}||^2 \\
&= \frac{1}{(\alpha+2)(\alpha+1)} \alpha^2 - \frac{\alpha}{(\alpha+2)(\alpha+1)} \alpha \\
&= 0
\end{aligned} \tag{152}
$$

Claim 25. *The superpolynomials $\{P^{(\alpha)}_\Lambda\}_\Lambda$ are defined by the two conditions:*

$$
\begin{aligned}
&(1):\ P^{(\alpha)}_\Lambda = m_\Lambda + \text{ lower terms} \\
&(2):\ \langle\!\langle P^{(\alpha)}_\Lambda | P^{(\alpha)}_\Omega \rangle\!\rangle_\alpha \propto \delta_{\Lambda\Omega}
\end{aligned} \tag{153}
$$

where the scalar product is given in (150).

In order to prove the claim, it suffices to prove that the Jack superpolynomials are orthogonal with respect to the combinatorial scalar product (150). As we will briefly outline, the main ingredient is a symmetry property of the $4N$ commuting quantities $H_n^{[k]}$ with respect to a reproductive kernel of the scalar product (150). This symmetry property, which is equivalent to the self-adjointness of the operators $H_n^{[k]}$ with respect to the scalar product (150), then implies the orthogonality of the superpolynomials $\{P_\Lambda^{(\alpha)}\}_\Lambda$. This result is given as a claim since it is presented without a complete proof. We only detail the original part of the argument, which is the formulation of the kernel.

To appropriately describe the reproductive kernel, it is convenient to define auxiliary variables. Let the variables $(\eta, \tilde{\phi}, \tilde{\theta})$ obey the relations

$$\{\tilde{\phi}_i, \tilde{\phi}_j\} = \{\tilde{\theta}_i, \tilde{\theta}_j\} = \{\tilde{\phi}_i, \tilde{\theta}_j\} = 0 \tag{154}$$

$$[\eta_i, \eta_j] = 0 \tag{155}$$

$$\tilde{\phi}_k\tilde{\theta}_k = 0, \quad \eta_i\eta_i = 0 \tag{156}$$

$$\tilde{\phi}_i\eta_i = \tilde{\theta}_i\eta_i = 0 \tag{157}$$

We then introduce the space $\tilde{\Pi}_N$ of symmetric superfunctions in the $4N$ auxiliary variables $(z_i, \tilde{\phi}_i, \tilde{\theta}_i, \eta_i)$, where $\tilde{f}$ belongs to $\tilde{\Pi}_N$ if and only if it is invariant under the simultaneous exchange of the quartet of variables $(z_i, \tilde{\phi}_i, \tilde{\theta}_i, \eta_i) \longleftrightarrow (z_j, \tilde{\phi}_j, \tilde{\theta}_j, \eta_j)$ for any $i, j \in \{1, \ldots, N\}$. It is easy to see that if $\tilde{f}(z, \tilde{\phi}, \tilde{\theta}, \eta) \in \tilde{\Pi}_N$, then we can obtain a function $f(z, \theta, \phi) \in \Pi_N$ by doing the substitution

$$f(z, \phi, \theta) = \Big[\tilde{f}(z, \tilde{\phi}, \tilde{\theta}, \eta)\Big]_{\substack{\tilde{\phi}_i \to \phi_i \\ \tilde{\theta}_i \to \theta_i \\ \eta_i \to \phi_i\theta_i}} \tag{158}$$

In particular, it is not too difficult to show that

$$q_\Lambda(z, \phi, \theta) = \Big[p_\Lambda(z, \tilde{\phi}, \tilde{\theta}, \eta)\Big]_{\substack{\tilde{\phi}_i \to \phi_i \\ \tilde{\theta}_i \to \theta_i \\ \eta_i \to \phi_i\theta_i}} \tag{159}$$

where $p_\Lambda = p_{\overline{\underline{\Lambda}}} p_{\overline{\Lambda}} p_{\underline{\Lambda}} p_{\Lambda^s}$ is defined in the obvious way with for instance (compare with (69))

$$p_{\overline{\underline{\Lambda}}}(z, \tilde{\phi}, \tilde{\theta}, \eta) = \prod_i \left(\sum_{k=1}^N \eta_k z_k^{\overline{\underline{\Lambda}}_i} \right) \tag{160}$$

The auxiliary kernel $K^A(z,\tilde\phi,\tilde\theta,\eta;y,\tilde\psi,\tilde\tau,\zeta)\equiv K^A(\tilde Z,\tilde Y;\alpha)$ is defined to be the bi-symmetric formal power series

$$K^A(\tilde Z,\tilde Y;\alpha)=\prod_{ij}\frac{1}{(1-z_iy_j)^{1/\alpha}}\prod_{ij}\left(1+\frac{\alpha^{-1}\eta_i\zeta_j}{1-z_iy_j}\right)\times\prod_{ij}\left(1+\frac{\tilde\phi_i\tilde\psi_j}{1-z_iy_j}\right)\prod_{ij}\left(1+\frac{\tilde\theta_i\tilde\tau_j}{1-z_iy_j}\right). \tag{161}$$

It is then straightforward to show that

$$K^A(\tilde Z,\tilde Y;\alpha)=\sum_{\Lambda\in\mathrm{SPar}}\frac{1}{\alpha^{\underline{\overline{m}}+\ell(\Lambda^s)}\xi_\Lambda z_\Lambda}p_\Lambda^\top(z,\tilde\theta,\tilde\phi,\eta)p_\Lambda(y,\tilde\tau,\tilde\psi,\zeta) \tag{162}$$

Taking this result from the auxiliary world back to our world, we get the following.

Proposition 26. *Let $K(Z,Y;\alpha)=K(z,\phi,\theta;y,\tau,\psi;\alpha)$ be given by*

$$K(Z,Y;\alpha)=\Big[K^A(\tilde Z,\tilde Y;\alpha)\Big]_{\substack{\tilde\phi_i\to\phi_i,\,\tilde\tau_i\to\tau_i\\ \tilde\theta_i\to\theta_i,\,\tilde\psi_i\to\psi_i\\ \eta_i\to\phi_i\theta_i,\,\zeta_i\to\psi_i\tau_i}} \tag{163}$$

We then have

$$K(Z,Y;\alpha)=\sum_{\Lambda\in SPar}\frac{1}{\alpha^{\underline{\overline{m}}+\ell(\Lambda^s)}\xi_\Lambda z_\Lambda}q_\Lambda^\top(z,\theta,\phi,\eta)q_\Lambda(y,\tau,\psi,\zeta) \tag{164}$$

which implies that $K(Z;Y;\alpha)$ is a reproducing kernel of the scalar product (150)*, that is,*

$$\langle\!\langle K(Z,Y;\alpha)^\top|f(Z)\rangle\!\rangle_\alpha=f(Y)\quad\forall\quad f\in\Pi_N, \tag{165}$$

The main step to prove Claim 25 is then to show that the $4N$ commuting quantities $H_n^{[k]}(Z)$ are symmetric with respect to the reproducing kernel $K(Z,Y;\alpha)$, that is

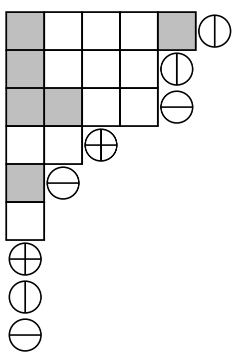
$$H_n^{[k]}(Z)K(Z;Y;\alpha)=H_n^{[k]}(Y)K(Z;Y;\alpha) \tag{166}$$

This can be done using the methods described in [11]. The orthogonality of the Jack superpolynomials, and thus also the claim, then follows from standard arguments in symmetric function theory (see for instance [17]).

6. Norm and evaluation

6.1. The combinatorial norm

In order to present our conjectured expression for the norm of the Jack superpolynomial with respect to the scalar product (150), we have to refine the description of the diagrams introduced in Subsection 3.4. We first divide the set of boxes in a diagram into fermionic and bosonic boxes. The fermionic boxes are the boxes that have a $\ominus$ both at the end of their row and at the end of their column or a $\obar$ both at the end of their row and at the end of their column. The bosonic boxes are then simply the boxes that are not fermionic. We further subdivide the set of bosonic boxes into four subsets defined as follows. Let $B_k\Lambda$ with $k = 0, 1, 2, 3$ be the subset of bosonic boxes that are in rows which end with a box, a $\oplus$, a $\obar$, or a $\ominus$ respectively. We illustrate this definition with an example in which the fermionic boxes are indicated in gray and the boxes in the sets $B_k\Lambda$ are identified by their coordinates (i, j) (i-th row and j-th column):

(167) 

$$
\begin{aligned}
B_0\Lambda &= \{(6,1)\}\\
B_1\Lambda &= \{(4,1),(4,2)\}\\
B_2\Lambda &= \{(1,2),(1,3),(1,4),(2,2),(2,3),(2,4)\}\\
B_3\Lambda &= \{(3,3),(3,4)\}
\end{aligned}
$$

We are now in position to present the conjectured norm of the Jack superpolynomials.

Conjecture 27. *The superpolynomial $P_\Lambda^{(\alpha)}$ has the following norm with respect to the combinatorial scalar product* (150)*:*

$$
j_\Lambda := \langle\!\langle P_\Lambda^{(\alpha)} | P_\Lambda^{(\alpha)} \rangle\!\rangle_\alpha = \frac{\alpha^{M_3}}{\xi_{\underline{\bar{\Lambda}}}} \prod_{i=0}^{3} \prod_{s\in B_i\Lambda} \frac{\ell_{\Lambda^{[i-1]}}(s) + (a_{\Lambda^{[0]}}(s)+1)\alpha}{\ell_{\Lambda^{[i]}}(s) + 1 + a_{\Lambda^{[3]}}(s)\alpha}, \tag{168}
$$

with the convention that $\Lambda^{[-1]} \equiv \Lambda^{[3]}$ and where, for a partition λ and its conjugate λ' (obtained by interchanging rows and columns), we have that *the arm-length and the leg-length are respectively given by*

$$a_\lambda(s) = \lambda_i - j \quad \textit{and} \quad \ell_\lambda(s) = \lambda'_j - i, \qquad \textit{where } s = (i,j). \tag{169}$$

This conjecture has been tested for every superpartition of the following sectors:

$$\begin{matrix} (1|1,1,1), & (2|1,1,1), & (2|2,1,1), \\ (3|2,1,0), & (3|2,1,1), & (3|2,1,2), \\ (3|2,2,0), & (3|2,2,1), & (3|2,2,2), \\ (4|2,2,2), & (5|3,2,2), & (6|2,2,2). \end{matrix} \tag{170}$$

These sectors all together contain 171 superpartitions. As further evidence of the validity of the conjecture, we stress that the norm has the correct reduction for $\mathcal{N}=1$ superpartitions. Let us make this explicit. In the case where there is only one type of circles, we need to replace $\Lambda^{(3)}$ by $\Lambda^{(1)}$ (so that now $\Lambda^{(-1)} = \Lambda^{(1)}$) and to restrict the product to $i = 0, 1$. Thus, the $\mathcal{N}=1$ special case of (171) reads (with $M_3 = m$)

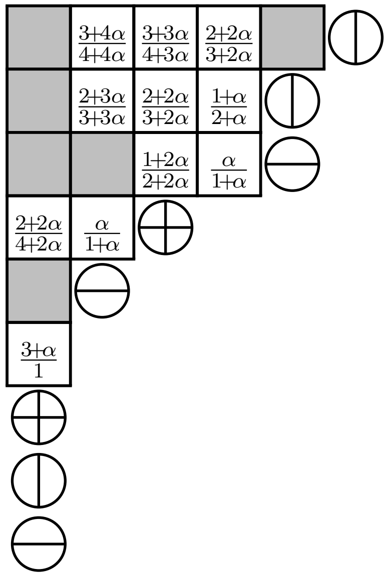
$$\begin{aligned} j_\Lambda &\overset{\mathcal{N}=1}{=} \alpha^m \prod_{s \in B_0\Lambda} \frac{\ell_{\Lambda^{[1]}}(s) + (a_{\Lambda^{[0]}}(s)+1)\alpha}{\ell_{\Lambda^{[0]}}(s) + 1 + a_{\Lambda^{[1]}}(s)\alpha} \\ &\qquad \times \prod_{s \in B_1\Lambda} \frac{\ell_{\Lambda^{[0]}}(s) + (a_{\Lambda^{[0]}}(s)+1)\alpha}{\ell_{\Lambda^{[1]}}(s) + 1 + a_{\Lambda^{[1]}}(s)\alpha}. \end{aligned} \tag{171}$$

Note that the boxes in $B_1\Lambda$ belong to rows that end with a circle and because they are bosonic they cannot have a circle in their column. Therefore, for the boxes in $B_1\Lambda$, we have that $\ell_{\Lambda^{[0]}}(s) = \ell_{\Lambda^{[1]}}(s)$ so that we can rewrite j_Λ under the compact form:

$$j_\Lambda \overset{\mathcal{N}=1}{=} \alpha^m \prod_{s \in B_0\Lambda \cup B_1\Lambda} \frac{\ell_{\Lambda^{[1]}}(s) + (a_{\Lambda^{[0]}}(s)+1)\alpha}{\ell_{\Lambda^{[0]}}(s) + 1 + a_{\Lambda^{[1]}}(s)\alpha} \tag{172}$$

which is precisely the formula given in [13, Eq. (18)].

Back to the general $\mathcal{N}=2$ case, we see that each bosonic box contributes to a factor in the conjectural expression for the norm. Here is an example, where the contribution of each bosonic box is written within the box:



The norm is the product of all theses contributions times the prefactor $\alpha^{M_3}/\xi_{\underline{\bar{\Lambda}}} = \alpha^8/1$, which gives

$$j_\Lambda = \alpha^8 \left[\frac{\alpha^2(3+\alpha)(1+2\alpha)(2+3\alpha)(3+4\alpha)}{2(2+\alpha)^2(3+2\alpha)^2(4+3\alpha)} \right]. \tag{173}$$

6.2. Evaluation

The evaluation of the Jack polynomials $J_\lambda^{(\alpha)}$ refer to its explicit expression (in terms of λ, α and the number of variables N) when all variables x_i are set equal to 1. In the $\mathcal{N}=1$ case, because there is a part of the superpolynomial that is antisymmetric in the x_i's, setting $x_i = 1$ for all i makes the polynomial vanish if its fermionic sector m is greater than 1. Note that the fermionic variables θ_i are not set to a definite value. The proper way to do the evaluation is by:

1) removing the monomial prefactor $\theta_1 \cdots \theta_m$,
2) dividing the result by the Vandermonde determinant in the variables $x_1, \ldots, x_m$, and
3) setting all the variables $x_i = 1$.

In the resulting combinatorial expression for the evaluation, the contributing boxes are those of the skew diagram $\Lambda^{[1]}/\delta_m$ where δ_m is the fermionic core, defined as $\delta_m = (m, m-1, \dots, 1)$.

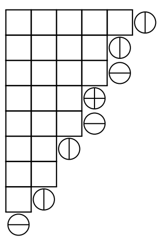
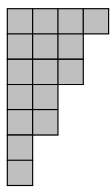
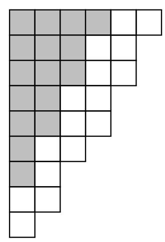
The procedure for the evaluation of the $\mathcal{N} = 2$ superpolynomial is a direct generalization of the $\mathcal{N} = 1$ case. We first introduce the fermionic core

$$\delta_{\overline{m},\underline{m}} = (\overline{m}, \overline{m}-1, \dots, 1) \cup (\underline{m}, \underline{m}-1, \dots, 1) \tag{174}$$

and then define the skew diagram

$$\Delta\Lambda = \Lambda^{[3]}/\delta_{\overline{m},\underline{m}}, \tag{175}$$

i.e., the set of boxes of $\Lambda^{[3]}$ that are not in $\delta_{\overline{m},\underline{m}}$. Here is an example

(176) $\Lambda =$  $\delta_{4,3} =$  $\Delta\Lambda =$ 

with the understanding that $\Delta\Lambda$ is the set of white boxes in the last diagram.

For a superpolynomial $F(x, \theta, \phi)$ in the M-fermionic sector, with $N \geq M_3$, we define its evaluation as

$$E_{N,M}\left[F(x,\theta,\phi)\right] := \left[\frac{[\phi;\theta]_M^\dagger F(x,\theta,\phi)}{V_M(x)}\right]_{x_1=x_2=\cdots=x_N=1}, \tag{177}$$

where

$$V_M(x) = \prod_{M_1<i<j\leq M_2}(x_i - x_j)\prod_{M_2<k<l\leq M_3}(x_k - x_l), \tag{178}$$

is a product of Vandermonde determinants in the variables $x_{M_1+1}, \dots, x_{M_2}$ and $x_{M_2+1}, \dots, x_{M_3}$ respectively. We are now in position to formulate our conjectural expression for the evaluation.

Conjecture 28. *The evaluation of the Jack superpolynomial $P_\Lambda^{(\alpha)}$, for $N \geq \ell(\Lambda)$, is*

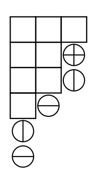
$$E_{N,M}\left[P_\Lambda^{(\alpha)}\right] = \binom{N-\overline{m}-\underline{m}}{\overline{m}}^{-1} \times \frac{\prod_{s\in\Delta\Lambda}\big(N-\ell'_{\Lambda^{[3]}}(s)+\alpha\, a'_{\Lambda^{[3]}}(s)\big)}{\xi_{\underline{\overline{\Lambda}}}\prod_{i=0}^{3}\prod_{s\in B_i\Lambda}\big(\ell_{\Lambda^{[i]}}(s)+1+\alpha a_{\Lambda^{[3]}}(s)\big)}, \tag{179}$$

where, for a partition λ,

$$a'_\lambda(s) = j-1 \quad \textit{and} \quad \ell'_\lambda(s) = i-1, \qquad \textit{with} \quad s=(i,j). \tag{180}$$

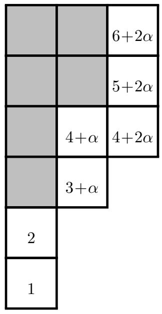
This conjecture has been tested for $N = 3,4,5,6$ in each of the following sectors (with the understanding that the evaluation only makes sense whenever $N \geq \ell(\Lambda)$): $(1|1,0,1)$, $(2|1,0,1)$, $(3|2,0,1)$, $(3|2,2,1)$, $(3|2,2,2)$, $(n|0,0,\overline{m})$ with $n = 1\dots4$ and $\overline{m} = 1\dots3$. It was also tested for $(4|0,0,3)$ with $N = 7$. These sectors represent together 120 superpartitions. Note also that this formula is *de facto* correct for any sector $(n|\overline{m},0,0)$ or $(n|0,\underline{m},0)$ since, in these cases, it reduces to the $\mathcal{N}=1$ evaluation formula presented in [13].

We illustrate the formula for the Jack superpolynomial in the fermionic sector $M = (1,2,2)$ indexed by the superpartition

(181) $\Lambda =$ 

We will consider the case $N = 6 = \ell(\Lambda)$. First, we have $\xi_{\underline{\overline{\Lambda}}} = 1$ and the binomial coefficient is $\binom{N-\overline{m}-\underline{m}}{\overline{m}} = 2$. We next compute the product in the numerator. Here $\delta_{2,2} = (2,\overline{2},1,1)$. The contribution of each box that belongs to $\Delta\Lambda$ is given explicitly in the following diagram

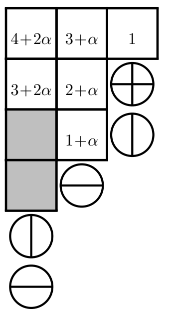
(182)



$$\Longrightarrow \prod_{s\in\Delta\Lambda} (N - \ell'_{\Lambda^{[3]}}(s) + \alpha\, a'_{\Lambda^{[3]}}(s)) = 8(2+\alpha)(3+\alpha)^2(4+\alpha)(5+2\alpha).$$

For the denominator, we have

(183)



$$\Longrightarrow \prod_{i=0}^{3} \prod_{s\in B_i\Lambda} (\ell_{\Lambda^{[i]}}(s) + 1 + \alpha a_{\Lambda^{[3]}}(s)) = 2(1+\alpha)(2+\alpha)^2(3+\alpha)(3+2\alpha).$$

Collecting these contributions gives

$$E_{6,(1,2,2)}[P^{(\alpha)}] = \frac{2(3+\alpha)(4+\alpha)(5+2\alpha)}{(1+\alpha)(2+\alpha)(3+2\alpha)}. \tag{184}$$

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Appendix A. Different prescribed symmetries

The construction of the Jack superpolynomials in terms of the Jack polynomials with prescribed symmetry was discussed in Section 5.1. The prescribed symmetry underlying our construction is of type SAAS – cf. Definition 19. Although this ordering of the (anti)-symmetrization operation appears to be rather natural (up to the trivial permutation of A_ϕ and A_θ which amounts to a simple relabeling of variables), one can ask wether this is the only possibility for defining the $\mathcal{N}=2$ version of the Jacks.

For stability reasons when the number of variables is set to infinity, it is natural to let the symmetrization associated to the unmarked entries in a superpartition be to the right. Given the aforementioned trivial permutation between A_ϕ and A_θ we actually only have to consider the alternatives ASA or AAS for the first three constituent partitions with the understanding that

$$S_{\phi\theta}A_\phi A_\theta \longrightarrow \begin{cases} M=(\underline{\overline{m}},\overline{m},\underline{m}) \\ \Lambda=(\,\underline{\overline{\Lambda}};\,\overline{\Lambda};\,\underline{\Lambda};\,\Lambda^{\circledast}) \end{cases} \tag{A.1}$$

$$A_\phi S_{\phi\theta} A_\theta \longrightarrow \begin{cases} M=(\overline{m},\underline{\overline{m}},\underline{m}) \\ \Lambda=(\,\overline{\Lambda};\,\underline{\overline{\Lambda}};\,\underline{\Lambda};\,\Lambda^{\circledast}) \end{cases} \tag{A.2}$$

$$A_\phi A_\theta S_{\phi\theta} \longrightarrow \begin{cases} M=(\overline{m},\underline{m},\underline{\overline{m}}) \\ \Lambda=(\,\overline{\Lambda};\,\underline{\Lambda};\,\underline{\overline{\Lambda}};\,\Lambda^{\circledast}) \end{cases} \tag{A.3}$$

with the partial sums M_i being changed accordingly. To each case, there corresponds a specific dominance ordering. Accordingly, the ordering of symbols in the diagrammatic representation of superpartitions must be coherent with the choice of symmetrization, that is

$$\text{SAA:}\ \begin{matrix}\oplus\\ \obar\\ \ominus\end{matrix}\ ,\quad \text{ASA:}\ \begin{matrix}\obar\\ \oplus\\ \ominus\end{matrix}\ ,\quad \text{AAS:}\ \begin{matrix}\obar\\ \ominus\\ \oplus\end{matrix} \tag{A.4}$$

Let us introduce a compact notation to cover these alternative constructions. Let

$$\Lambda^{(1)}=\underline{\overline{\Lambda}},\quad \Lambda^{(2)}=\overline{\Lambda},\quad \Lambda^{(3)}=\underline{\Lambda},\quad \Lambda^{(4)}=\Lambda^{\circledast} \tag{A.5}$$

and σ be an element of S_3, so that

$$\sigma(\Lambda)=(\Lambda^{(\sigma(1))};\Lambda^{(\sigma(2))};\Lambda^{(\sigma(3))};\Lambda^{(4)}) \tag{A.6}$$

The reverse superpartition would now read

$$(\sigma(\Lambda))^R=((\Lambda^{(\sigma(1))})^R;(\Lambda^{(\sigma(2))})^R;(\Lambda^{(\sigma(3))})^R;(\Lambda^{(4)})^R) \tag{A.7}$$

Finally, with

$$\Xi_j^{(i)}=\begin{cases} \phi_j\theta_j & \text{for } i=1 \\ \phi_j & \text{for } i=2 \\ \theta_j & \text{for } i=3 \end{cases} \tag{A.8}$$

the modified version of $[\phi;\theta]_M$ is

$$\text{(A.9)}\qquad [\phi;\theta]_M^\sigma = \left[\prod_{i=1}^{M_1}\Xi^{\sigma(1)}\right]\left[\prod_{j=M_1+1}^{M_2}\Xi^{\sigma(2)}\right]\left[\prod_{k=M_2+1}^{M_3}\Xi^{\sigma(3)}\right]$$

The polynomial with σ(SAA) symmetry and labelled by $\sigma(\Lambda)$ would the be given by

$$\text{(A.10)}\qquad P_{\sigma(\Lambda)}^{(\alpha)} = \frac{(-1)^{\binom{2}{\overline{m}}+\binom{2}{\underline{m}}}}{f_\Lambda}\sum_{\omega\in S_N}\mathcal{K}_\omega[\phi;\theta]_M^\sigma E_{(\sigma(\Lambda))^R}$$

Are these proper candidates for $\mathcal{N}=2$ versions of the Jack superpolynomials? Remarkably, it seems that it is indeed the case given the following properties/conjectures:

1) These polynomials are still eigenfunction of the s^{2}CMS Hamiltonian.
2) By construction, they are still orthogonal with respect to the analytic scalar product (141).
3) They appear to be also orthogonal with respect to the combinatorial scalar product (150).
4) The expression for the norm given in Conjecture 27 is still valid if the role of the $B_i\Lambda$'s is changed according to the permutation of the constituent partitions $\underline{\overline{\Lambda}}$, $\overline{\Lambda}$ and $\underline{\Lambda}$. (This version of the conjecture has been tested for every permutation of (SAA) for all the cases listed in eq. (170).)

Sector(Λ)	P^{α}_{Λ}
$(1\vert 1,1,1)$	$P^{\alpha}_{(0;0;0;1)} = m_{(0;0;0;1)}$ $P^{\alpha}_{(0;0;1;)} = m_{(0;0;1;)} - \frac{1}{(\alpha+3)} m_{(0;0;0;1)}$ $P^{\alpha}_{(0;1;0;)} = m_{(0;1;0;)} - \frac{1}{(\alpha+2)} m_{(0;0;1;)} + \frac{1}{(\alpha+2)} m_{(0;0;0;1)}$ $P^{\alpha}_{(1;0;0;)} = m_{(1;0;0;)} + \frac{1}{(\alpha+1)} m_{(0;1;0;)} - \frac{1}{(\alpha+1)} m_{(0;0;1;)} + \frac{1}{(\alpha+1)} m_{(0;0;0;1)}$
$(2\vert 2,2,1)$	$P^{\alpha}_{(0,0;0;1;1)} = m_{(0,0;0;1;1)} - \frac{2}{(\alpha+5)} m_{(0,0;0;0;1,1)}$ $P^{\alpha}_{(0,0;0;0;2)} = m_{(0,0;0;0;2)} + \frac{1}{(\alpha+1)} m_{(1,0;0;0;1)} + \frac{1}{(\alpha+1)} m_{(0,0;1;0;1)} - \frac{1}{(\alpha+1)} m_{(0,0;0;1;1)} + \frac{2}{(\alpha+1)} m_{(0,0;0;0;1,1)}$ $P^{\alpha}_{(0,0;0;2;)} = m_{(0,0;0;2;)} + \frac{1}{(\alpha+1)} m_{(1,0;0;1;)} - \frac{1}{2(\alpha+2)}^{-1} m_{(0,0;0;0;2)} - \frac{1}{2(\alpha+2)(\alpha+1)} m_{(1,0;0;0;1)} - \frac{1}{(\alpha+1)} m_{(0,0;1;1;)} - \frac{1}{2(\alpha+2)(\alpha+1)} m_{(0,0;1;0;1)} + \frac{2\,\alpha+5}{2(\alpha+2)(\alpha+1)} m_{(0,0;0;1;1)} - \frac{1}{(\alpha+2)(\alpha+1)} m_{(0,0;0;0;1,1)}$
$(3\vert 2,2,0)$	$P^{\alpha}_{(1,0;2,0;;)} = m_{(1,0;2,0;;)} + \frac{2}{(\alpha+2)} m_{(1,1;1,0;;)} + \frac{2}{(\alpha+2)} m_{(0,0;2,1;;)} + \frac{2}{(\alpha+2)} m_{(0,0;2,0;;1)} + \frac{\alpha+4}{(\alpha+2)^2} m_{(1,0;1,0;;1)} + \frac{4}{(\alpha+2)^2} m_{(0,0;1,0;;1,1)}$ $P^{\alpha}_{(0,0;3,0;;)} = m_{(0,0;3,0;;)} + \frac{1}{(2\,\alpha+1)} m_{(2,0;1,0;;)} + \frac{2}{(2\,\alpha+1)} m_{(1,0;2,0;;)} + \frac{2}{(\alpha+1)(2\,\alpha+1)} m_{(1,1;1,0;;)} + \frac{1}{(2\,\alpha+1)} m_{(0,0;2,1;;)} + \frac{2}{(2\,\alpha+1)} m_{(0,0;2,0;;1)} + \frac{1}{(2\,\alpha+1)} m_{(0,0;1,0;;2)} + \frac{2}{(\alpha+1)(2\,\alpha+1)} m_{(1,0;1,0;;1)} + \frac{2}{(\alpha+1)(2\,\alpha+1)} m_{(0,0;1,0;;1,1)}$
$(5\vert 1,3,2)$	$P^{\alpha}_{(0;2,1,0;2,1;)} = m_{(0;2,1,0;2,1;)} - \frac{1}{(\alpha+3)} m_{(0;2,1,0;2,0;1)} - \frac{\alpha+2}{(\alpha+3)(2\,\alpha+5)} m_{(0;2,1,0;1,0;2)} + \frac{1}{(\alpha+3)(2\,\alpha+5)} m_{(1;2,1,0;1,0;1)} + \frac{2}{(\alpha+3)(2\,\alpha+5)} m_{(0;2,1,0;1,0;1,1)}$

Table A1: Sample of small degree Jack superpolynomials expanded in the monomial basis.

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