ADV. THEOR. MATH. PHYS.
Volume 25, Number 4, 1025–1093, 2021

Higher-spin kinematics & no ghosts on quantum space-time in Yang-Mills matrix models

Harold C. Steinacker

A classification of bosonic on- and off-shell modes on a cosmological quantum space-time solution of the IIB matrix model is given, which leads to a higher-spin gauge theory. In particular, the no-ghost-theorem is established. The physical on-shell modes consist of 2 towers of higher-spin modes, which are effectively massless but include would-be massive degrees of freedom. The off-shell modes consist of 4 towers of higher-spin modes, one of which was missing previously. The noncommutativity leads to a cutoff in spin, which disappears in the semi-classical limit. An explicit basis allows to obtain the full propagator, which is governed by a universal effective metric. The physical metric fluctuations arise from would-be massive spin 2 modes, which were previously shown to include the linearized Schwarzschild solution. Due to the maximal supersymmetry of the IIB model, this is expected to define a consistent quantum theory in 3+1 dimensions, which includes gravity.

1. Introduction

The starting point of this paper is a recent solution of IKKT-type matrix models with mass term [1], which is naturally interpreted as 3+1-dimensional cosmological FLRW quantum space-time. It was shown that the fluctuation modes around this background include spin-2 metric fluctuations, as well as a truncated tower of higher-spin modes which are organized in a higher-spin gauge theory. The standard Ricci-flat massless graviton modes were found, as well as some additional vector-like and scalar metric modes. The latter was shown to provide the linearized Schwarzschild solution in [2]. However, the fluctuation analysis was not complete. In particular, although general arguments suggest that the model should be free of ghosts, this has not been established up to now.

The present paper provides a complete analysis and classification of all bosonic fluctuation modes which arise on this background in the matrix model. It turns out that in addition to the three towers of (off-shell) higher spin modes found in [1], there is a fourth tower, which is obtained in a coherent way. This provides a full and explicit diagonalization of the gauge-fixed quadratic action for the bosonic matrix fluctuations. Moreover, we classify and find the physical modes (i.e. the gauge-fixed on-shell modes modulo pure gauge modes) and show that the invariant inner product is positive, so that they define a Hilbert space. Since the quadratic action is defined by the same inner product, this amounts to the statement that there are no ghosts, i.e. no physical modes with negative norm. We also compute the inner products for all off-shell modes, which is found to have the same Minkowski structure as in flat space. This allows in principle to write down the full propagator, and should be very useful in a future analysis of perturbative quantization.

Along the way, many useful and surprisingly nice properties of the space-time and its modes are uncovered, including simple on-shell relations which show that the time evolution behaves very much like on commutative space, even in the presence of space-time noncommutativity. Quite generally speaking, even though the organization is rather involved due to the higher-spin structure, the results are remarkably nice and simple.

The origin of higher-spin modes can be understood as follows. The mathematical structure underlying the background under consideration is quantized twistor space $\mathbb{C}P_n^{1,2}$, which is a quantized 6-dimensional coadjoint orbit of $SU(2,2)$ or $SO(4,2)$. Semi-classically, this is an S^2 bundle over the 4-hyperboloid H^4, or over the space-time $\mathcal{M}^{3,1}$. The latter is a projection of H^4 with Minkowski signature, describing a FLRW cosmological space-time with a Big Bounce. This S^2 fiber is quantized and therefore admits only finitely many harmonics, which transmute into higher spin modes on $\mathcal{M}^{3,1}$ due to the twisted bundle structure. All this is automatic on the matrix background under consideration.

For reasons of transparency and simplicity the analysis is performed in the semi-classical Poisson limit, where spacetime is described by a classical manifold carrying extra structure which is underlying the noncommutativity. This case is already very interesting in its own right, and since most computations are based on the Lie-algebraic structures, most steps would go through in the noncommutative case with minor modifications. The classification of modes is literally the same due to the $SO(4,2)$-covariant quantization map $\mathcal{Q}$ (2.13), and the no-ghost result is expected to hold also in the non-commutative case up to the cutoff.

However, there is one complication. Due to the FLRW geometry, the isometry group $SO(3,1)$ of the background comprises space-like translations and rotations, but no boosts. This means that local Lorentz invariance is only partially manifest. The usual 3+1-dimensional tensor fields accordingly decompose into several $SO(3,1)$ sub-sectors. This sub-structure is addressed in section 3.2 which leads to an organization reminiscent of but distinct from primary and secondary fields in CFT. In any case, the underlying $SO(4,2)$ structure group is powerful enough to control the kinematics. There is in fact one advantage, since the absence of ghost is quite transparent as the fields are naturally organized in space-like or radiation gauge. In the end, local Lorentz invariance seems to be effectively respected and all modes propagate in the exact same way, governed by a universal effective metric. This is expected due to the manifest higher spin gauge symmetry, which includes an analog of (modified) volume-preserving diffeos. Nevertheless, the issue of local Lorentz invariance should be clarified further.

The appearance of a higher-spin gauge theory is of course very reminiscent of Vasiliev's higher spin theory [3, 4]. Indeed as elaborated in previous papers [5, 6], the present higher-spin kinematics is clearly related to the higher spin algebras of Vasiliev theory, although further clarification would be desirable. There may also be a close relation with the Yang-Mills higher spin models considered in [7]. However there are clearly significant differences. In particular, the present model is defined by an action and features two scales, and IR scale given by the cosmic curvature and a UV scale where the noncommutativity becomes significant. The separation of these scales is determined by an integer n, and is therefore protected from quantum corrections.

The results of this paper thus provides a solid base for an interacting higher spin gauge theory which appears to include gravity. Although the model is intrinsically noncommutative, it should be viewed in the spirit of field theory. In contrast to holographic approaches space-time arises as a condensation of matrices here, whose dynamical fluctuations are described by an effective (almost-local) field theory. Most importantly, the present model is well suited for quantization, as discussed in the outlook. The present results should allow to study the quantum theory in detail. In particular, it would be very interesting to make contact with the numerical simulations of the IKKT model [8–10], which provide evidence that an expanding 3+1-dimensional space-time indeed arises at the non-perturbative level.

The paper is rather technical and includes all the required details. To make it more accessible, the conceptual considerations are kept in the main text while many technical details are delegated to the appendix. The main results are the classification of modes in sections 5 and 6.3, and the no-ghost theorem in section 6.4. The required background is provided in sections 2 and 3, which should make the paper mostly self-contained. Finally, a disclaimer on mathematical rigour: The use of “Theorem”, “Lemma” etc. should be understood in a semi-rigorous physicist’s sense. The statements are clear-cut and justified with formal proofs, but full mathematical precision is not attempted.

2. Basic definitions and algebraic structures

The theory under consideration [1] is based on the Lie algebra $\mathfrak{so}(4,2)$ generated by M^{ab},

$$[M_{ab}, M_{cd}] = \mathrm{i}\,(\eta_{ac}M_{bd} - \eta_{ad}M_{bc} - \eta_{bc}M_{ad} + \eta_{bd}M_{ac}) \tag{2.1}$$

for $a, b = 0, \ldots, 5$, and a specific class of unitary representations $\mathcal{H}_n$ known as doubletons or minireps [11, 12], labeled by $n \in \mathbb{N}$. These are short discrete series unitary irreps of $\mathfrak{so}(4,2)$, which have the distinctive feature that they remain irreducible if restricted to $SO(4,1) \subset SO(4,2)$. They are also multiplicity-free lowest weight representations. The special case $n = 0$ is excluded.

Fuzzy hyperboloid H^4_n. The fuzzy hyperboloid H^4_n [5, 13] is defined in terms of $SO(4,1)$ vector operators

$$X^a = rM^{a5}, \qquad a = 0, \ldots, 4 \; . \tag{2.2}$$

Here r has dimension length, and $\eta_{ab} = \mathrm{diag}(-1,1,1,1,1,-1)$. Since $\mathcal{H}_n$ remains irreducible for $SO(4,1)$, they satisfy the relations of a 4-dimensional hyperboloid

$$\eta_{ab} X^a X^b = -R^2 \mathbb{1} \; , \qquad R^2 = \frac{r^2}{4}(n^2 - 4) \tag{2.3}$$

where the sum is over $a, b = 0, \ldots, 4$. It is easy to see that the X^a generate the full algebra $\mathrm{End}(\mathcal{H}_n)$, which transforms under $SO(4,2)$ via

$$M^{ab} \triangleright \phi = [M^{ab}, \phi] \; , \qquad \phi \in \mathrm{End}(\mathcal{H}_n) \; . \tag{2.4}$$

The quadratic Casimirs of $SO(4,2)$ and $SO(4,2)$ act on $\phi \in \mathrm{End}(\mathcal{H}_n)$ as

$$\begin{aligned} C^2[\mathfrak{so}(4,2)]\phi &= \frac{1}{2}[M^{ab}, [M_{ab}, \phi]], \qquad a, b = 0, \ldots, 5 \\ C^2[\mathfrak{so}(4,1)]\phi &= \frac{1}{2}[M^{ab}, [M_{ab}, \phi]], \qquad a, b = 0, \ldots, 4 \end{aligned} \tag{2.5}$$

and the $SO(4,1)$- invariant matrix Laplacian on H^4_n

$$\Box_H \phi = [X_a, [X^a, \phi]] = (-C^2[\mathfrak{so}(4,2)] + C^2[\mathfrak{so}(4,1)])\phi \tag{2.6}$$

encodes the geometry of H^4. All indices will be raised or lowered with the appropriate η^{ab} throughout the paper, and latin labels a, b range from 0 to 4 (or possibly 5). In particular, the following $SO(4,1)$- invariant Casimir on $\mathrm{End}(\mathcal{H}_n)$ [1, 5]

$$\begin{aligned} \mathcal{S}^2 &:= \frac{1}{2} \sum_{a,b \neq 5} [M_{ab}, [M^{ab}, .]] + r^{-2}[X_a, [X^a, .]] \\ &= 2C^2[\mathfrak{so}(4,1)] - C^2[\mathfrak{so}(4,2)] \end{aligned} \tag{2.7}$$

can be interpreted as a spin observable on H^4_n, which satisfies

$$[\mathcal{S}^2, \Box_H] = 0 \; . \tag{2.8}$$

Hence $\Box_H$ and $\mathcal{S}^2$ can be simultaneously diagonalized, and $\mathrm{End}(\mathcal{H}_n)$ decomposes into [5]

$$\mathrm{End}(\mathcal{H}_n) = \mathcal{C} = \mathcal{C}^0 \oplus \mathcal{C}^1 \oplus \ldots \oplus \mathcal{C}^n \qquad \text{with} \qquad \mathcal{S}^2|_{\mathcal{C}^s} = 2s(s+1) \ . \tag{2.9}$$

We will see that $\mathcal{C}^0$ describes the space of (scalar) functions on H^4_n, while $\mathcal{C}^s$ describes spin s modes on H^4_n. The origin of this higher spin structure can be understood by noting that $\mathrm{End}(\mathcal{H}_n)$ should be interpreted as *quantized algebra of functions on* $\mathbb{C}P^{1,2}$, which is an equivariant[1] S^2-bundle over H^4. This is best understood in terms of coherent states, which are defined as follows: let

$$|x_0\rangle := |0\rangle \qquad \in \mathcal{H}_n \tag{2.10}$$

be the lowest weight state. This is an optimally localized state[2] at the "south pole" of H^4, with $\langle x_0|X^a|x_0\rangle = x_0 = R(\frac{n}{2}+1, 0, 0, 0, 0)$. Then the coherent state $|x\rangle = g \triangleright |x_0\rangle \in \mathcal{H}_n$ is defined by a rotation $g \in SO(4,1)$ which rotates x_0 into $x \in H^4$. Since the stabilizer group of $x_0 \in H^4$ is $SO(4)$, the expectation values

$$x^a = \langle x|X^a|x\rangle \tag{2.11}$$

span $H^4 \cong SO(4,1)/SO(4)$. However there is a hidden fiber bundle over H^4, which arises from the fact that $\mathcal{H}_n$ is a representation of $\mathfrak{su}(2,2) \cong \mathfrak{so}(4,2) \supset \mathfrak{so}(4,1)$. Then the coherent states sweep out the space

$$\{|p\rangle = g \triangleright |0\rangle, \ g \in SU(2,2)\} \cong SU(2,2)/SU(2,1) = \mathbb{C}P^{1,2} \times U(1) \ . \tag{2.12}$$

Here $\mathbb{C}P^{1,2}$ is a 6-dimensional coadjoint orbit of $SU(2,2)$, which is a S^2 bundle over H^4 via the Hopf map (2.11). The fiber describes in fact a fuzzy S^2_n spanned by the stabilizer $SU(2)_L$ of $x_0 \in H^4$ acting on $|0\rangle$, which spans an $n+1$-dimensional irrep, leading to the truncation in (2.9). For more details we refer to [5]. The extra $U(1)$ is just the phase of the coherent states on $\mathbb{C}P^{1,2}$.

Using these coherent states, we can write down a natural $SO(4,2)$-equivariant quantization map from the classical space of functions on $\mathbb{C}P^{1,2}$

---

[1] i.e. $SO(4,1)$ acts on the entire bundle in a way consistent with the bundle projection.

[2] In a suitable sense, cf. [14], or [15] for a discussion in a similar context.

to the noncommutative or fuzzy functions $End(\mathcal{H}_n)$:

$$\begin{aligned} \mathcal{Q}: \quad & \mathcal{C}(\mathbb{C}P^{1,2}) \to End(\mathcal{H}_n) \\ & \phi(p) \mapsto \hat{\phi} := \int\limits_{\mathbb{C}P^{1,2}} \phi(p) \, |p\rangle \, \langle p| \ . \end{aligned} \tag{2.13}$$

Here $\mathbb{C}P^{1,2}$ is equipped with the canonical $SO(4,2)$-invariant measure. This map is essentially one-to-one up to a cutoff [5], mapping square-integrable functions to Hilbert-Schmidt operators. The inverse map (up to normalization & cutoff) is given by the symbol

$$\hat{\phi} \in End(\mathcal{H}_n) \mapsto \langle p|\hat{\phi}|p\rangle \ = \phi(p) \ \in \ \mathcal{C}(\mathbb{C}P^{1,2}) \ . \tag{2.14}$$

Hence $End(\mathcal{H}_n)$ decomposes into the same unitary irreps as $L^2(\mathbb{C}P^{1,2})$ below the cutoff, and the harmonics on the S^2_n fiber lead to (2.9). Since $\mathcal{Q}$ respects $SO(4,2)$, the generators act as

$$[M^{ab}, \mathcal{Q}(\phi)] = \mathcal{Q}(i\{m^{ab}, \phi(x)\}) \tag{2.15}$$

where $\{m^{ab}, .\}$ implements the $SO(4,2)$ action on $\mathcal{C}(\mathbb{C}P^{1,2})$ via the Poisson bracket arising from the canonical (Kirillov-Kostant-Souriau) symplectic structure. This Poisson bracket is defined through the Lie algebra relations (2.1) for the embedding functions $m^{ab}: \ \mathbb{C}P^{1,2} \hookrightarrow \mathfrak{so}(4,2) \cong \mathbb{R}^{15}$, replacing $[.,.]$ by $i\{.,.\}$. This replacement will be called *semi-classical limit* indicated by $\sim$. In particular, it is easy to see that $M^{ab} = \mathcal{Q}(m^{ab})$ and $X^a = \mathcal{Q}(x^a)$ (up to normalization).

Due to the intertwiner property of $\mathcal{Q}$, most of the (Lie-algebraic) computations carried out at the Poisson level carry over immediately to the full non-commutative (NC) case in $End(\mathcal{H}_n)$. For example, the Casimirs and Laplacian are respected:

$$\begin{aligned} [M^{ab}, [M_{ab}, \mathcal{Q}(\phi)]] &= \mathcal{Q}(-\{M^{ab}, \{M_{ab}, \phi\}\}), \\ \Box_H \mathcal{Q}(\phi) &= \mathcal{Q}(\Box_H \phi) \ , \end{aligned} \tag{2.16}$$

where $\Box_H \phi = -\{x^a, \{x_a, \phi\}\}$ on the rhs is the Laplacian on H^4. Thus even though we will mostly work in the semi-classical case, most of the results carry over immediately to the NC case.

Fuzzy space-time $\mathcal{M}^{3,1}_n$. The main space of interest here is the fuzzy or quantum space-time $\mathcal{M}^{3,1}_n$, which is generated by the $X^\mu, \ \mu = 0, \ldots, 3$, dropping the X^4 generator of H^4_n. Then

$$\eta_{\mu\nu}X^\mu X^\nu = -R^2\mathbb{1} - X_4^2 \ , \tag{2.17}$$

and greek labels μ, ν etc. will run from 0 to 3 throughout the paper. Dropping the X^4 generator corresponds to a projection of H^4 to $\mathbb{R}^{3,1}$, so that $\mathcal{M}^{3,1}_n$ should be interpreted as 2-sheeted hyperboloid, as sketched in figure 1. This

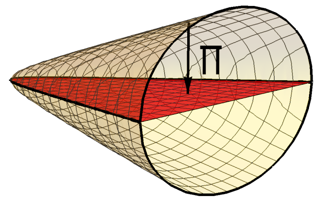

Figure 1: Projection Π from H^4 to $\mathcal{M}^{3,1}$ with Minkowski signature.

interpretation is substantiated via the matrix d'Alembertian

$$\Box\phi = [T_\mu, [T^\mu, \phi]] = (C^2[\mathfrak{so}(4,1)] - C^2[\mathfrak{so}(3,1)])\phi \ , \tag{2.18}$$

which encodes an $SO(3,1)$-invariant d'Alembertian for $\mathcal{M}^{3,1}$ with Lorentzian structure[3], where

$$T^\mu = \frac{1}{R}M^{\mu 4} \ . \tag{2.19}$$

It is easy to see that the X^μ alone generate the full algebra $End(\mathcal{H}_n)$, which can now be interpreted as quantized functions on a S^2-bundle over $\mathcal{M}^{3,1}$. They satisfy the commutation relations

$$[X^\mu, X^\nu] =: i\Theta^{\mu\nu} = -ir^2 M^{\mu\nu} \ . \tag{2.20}$$

It turns out that $\Theta^{\mu\nu}$ is related to T^μ (cf. (3.4)), which satisfy the commutation relations

$$[T^\mu, T^\nu] = -\frac{i}{r^2R^2}\Theta^{\mu\nu}, \qquad [T^\mu, X^\nu] = \frac{i}{R}\eta^{\mu\nu}X_4 \ . \tag{2.21}$$

[3]It is natural to wonder about the Sitter solutions. While this is possible in principle [16, 17], $End(\mathcal{H})$ would imply a non-compact internal fiber and infinitely many dof per unit volume. This is avoided here.

These generators satisfy further constraints due to the special representation $\mathcal{H}_n$. To simplify these relations we will focus on the semi-classical (Poisson) limit $n \to \infty$ from now on, working with commutative functions of $x^\mu \sim X^\mu$ and $t^\mu \sim T^\mu$, but keeping the Poisson or symplectic structure $[.,.] \sim i\{.,.\}$ encoded in $\theta^{\mu\nu}$.

In order to have a well-defined action, we will consider modes on $\mathcal{M}^{3,1}$ which are square-integrable, in the sense that the $SO(4,2)$-invariant inner product is finite,

$$0 < \ \langle\phi, \phi'\rangle := \mathrm{Tr}\phi^\dagger\phi' \ \sim \int_{\mathbb{C}P^{1,2}} \phi^*\phi' \ < \infty \tag{2.22}$$

where functions $\phi \in L^2(\mathbb{C}P^{2,1})$ are identified with operators $End(\mathcal{H}_n)$ via (2.13). The measure is the symplectic volume form $\Omega = \frac{(2\pi)^3}{3!}\omega^{\wedge 3}$ on $\mathbb{C}P^{1,2}$, which is dropped. All integrals in the paper are understood in this sense, unless stated otherwise. Accordingly, $\phi \in L^2(\mathbb{C}P^{1,2})$ belongs to some unitary representation of $SO(4,2)$.

Since $SO(4,2)$ is the conformal group on $\mathbb{R}^{3,1}$, one might hope to apply CFT concepts such as conformal primaries etc. Indeed $\mathcal{H}_n$ is a lowest-weight module with ground state $|0\rangle$ which is an eigenstate of $D = X^4$, whose eigenvalues are raised and lowered with $M^{\mu 5} \pm iM^{\mu 4}$. However, the main object of interest is $End(\mathcal{H}_n) \cong \mathcal{H}_n \otimes \mathcal{H}_n^*$, and the square-integrable modes consists of principal series modules rather than highest or lowest weight modules. Therefore the familiar concepts from CFT are not useful here. Instead we will develop some more suitable structures in section 3.2 which replace these concepts to some extent.

3. Semi-classical structure of $\mathcal{M}^{3,1}$

In the semi-classical limit, the generators x^μ and t^μ satisfy the following constraints [5]

$$x_\mu x^\mu = -R^2 - x_4^2 = -R^2\cosh^2(\eta)\,, \qquad R \sim \frac{r}{2}n \tag{3.1a}$$

$$t_\mu t^\mu = r^{-2}\,\cosh^2(\eta) \tag{3.1b}$$

$$t_\mu x^\mu = 0 \tag{3.1c}$$

which arise from the special properties of $\mathcal{H}_n$. We will interpret $x^\mu : \mathcal{M}^{3,1} \hookrightarrow \mathbb{R}^{3,1}$ as Cartesian coordinate functions. Here η is a global time parameter defined via

$$x^4 = R\sinh(\eta) \tag{3.2}$$

which defines a foliation of $\mathcal{M}^{3,1}$ into space-like surfaces H^3; this will be related to the scale parameter of a FLRW cosmology (3.12) with $k = -1$. Note that η distinguishes the two degenerate sheets of $\mathcal{M}^{3,1}$, cf. figure 1. The t^μ generators clearly describe the S^2 fiber over $\mathcal{M}^{3,1}$, which is space-like due to (3.1c). These generators satisfy the Poisson brackets

$$\begin{aligned} \{x^\mu, x^\nu\} &= \theta^{\mu\nu} = -r^2R^2\{t^\mu, t^\nu\}, \\ \{t^\mu, x^\nu\} &= \frac{x^4}{R}\eta^{\mu\nu} \ . \end{aligned} \tag{3.3}$$

The Poisson tensor $\theta^{\mu\nu}$ can be expressed in terms of t^μ via [5]

$$\theta^{\mu\nu} = \frac{r^2}{\cosh^2(\eta)}\Big(\sinh(\eta)(x^\mu t^\nu - x^\nu t^\mu) + \epsilon^{\mu\nu\alpha\beta}x_\alpha t_\beta\Big) \ , \tag{3.4}$$

and it satisfies the constraints

$$t_\mu\theta^{\mu\alpha} = -\sinh(\eta)x^\alpha, \tag{3.5a}$$

$$x_\mu\theta^{\mu\alpha} = -r^2R^2\sinh(\eta)t^\alpha, \tag{3.5b}$$

$$\eta_{\mu\nu}\theta^{\mu\alpha}\theta^{\nu\beta} = R^2r^2\eta^{\alpha\beta} - R^2r^4t^\alpha t^\beta + r^2x^\alpha x^\beta \tag{3.5c}$$

as well as self-duality relations given in Lemma 9.4.

We observe that due to the relation (3.3), the derivations or Hamiltonian vector fields

$$-i[T^\mu, .] \sim \{t^\mu, .\} \tag{3.6}$$

play the role of momentum generators on $\mathcal{M}^{3,1}$, which satisfy

$$\{t_\mu, \phi\} = \sinh(\eta)\partial_\mu\phi \tag{3.7}$$

for $\phi = \phi(x)$. There is also an $SO(3,1)$-invariant global time-like vector field

$$\tau := x^\mu\partial_\mu. \tag{3.8}$$

3.1. Effective metric and d'Alembertian

In the matrix model framework, the effective metric on any given background is obtained by rewriting the kinetic term in covariant form [1, 18]. For the

$\mathcal{M}^{3,1}$ background under consideration, this is

$$S[\phi] = -\mathrm{Tr}[T^\mu, \phi][T_\mu, \phi] \sim \int d^4x \sqrt{|G|} G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \tag{3.9}$$

and one obtains [1]

$$G^{\mu\nu} = \sinh^{-3}(\eta)\, \gamma^{\mu\nu}, \qquad \gamma^{\alpha\beta} = \eta_{\mu\nu}\theta^{\mu\alpha}\theta^{\nu\beta} = \sinh^2(\eta)\eta^{\alpha\beta} \tag{3.10}$$

dropping some irrelevant constant. This metric can be recognized as $SO(3,1)$-invariant FLRW metric with signature $(-+++)$,

$$\begin{aligned} \mathrm{d}s_G^2 = G_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu &= -R^2\sinh^3(\eta)\mathrm{d}\eta^2 + R^2\sinh(\eta)\cosh^2(\eta)\,\mathrm{d}\Sigma^2 \\ &= -\mathrm{d}t^2 + a^2(t)\mathrm{d}\Sigma^2\,. \end{aligned} \tag{3.11}$$

We can read off the cosmic scale parameter $a(t)$

$$a(t)^2 = R^2\sinh(\eta)\cosh^2(\eta) \overset{t\to\infty}{\sim} R^2\sinh^3(\eta), \tag{3.12}$$

$$\mathrm{d}t = R\sinh(\eta)^{\frac{3}{2}}\mathrm{d}\eta \tag{3.13}$$

which leads to $a(t) \sim \frac{3}{2}t$ for late times. This metric can also be extracted from the "matrix" d'Alembertian (2.18)

$$\Box := [T^\mu, [T_\mu, .]] \;\sim\; -\{t^\mu, \{t_\mu, .\}\} \;= \sinh^3(\eta)\Box_G \tag{3.14}$$

acting on $\phi \in \mathcal{C}^0$, where[4] $\Box_G = -\frac{1}{\sqrt{|G|}}\partial_\mu\big(\sqrt{|G|}\,G^{\mu\nu}\partial_\nu\big)$.

3.2. Higher spin sectors on $\mathcal{M}^{3,1}$ and H^3 substructure

Due to the extra generators t^μ, we obtain explicitly the decomposition (2.9) of the full algebra of functions into sectors $\mathcal{C}^s$ which correspond to spin s

[4]It is interesting to observe that the invariant volume form $d^4x\,\frac{1}{x^4}$ arising from the symplectic volume form [1] does *not* coincide with the Riemannian volume $d^4x\,\sqrt{|G|}$. Accordingly, spin 1 gauge transformations are diffeomorphisms which preserve Ω rather than the Riemannian volume.

harmonics on the S^2 fiber:

$$\text{End}(\mathcal{H}_n) = \mathcal{C} = \mathcal{C}^0 \oplus \mathcal{C}^1 \oplus \ldots \oplus \mathcal{C}^n \qquad \text{with} \quad \mathcal{S}^2|_{\mathcal{C}^s} = 2s(s+1)\,. \tag{3.15}$$

In the semi-classical limit, the $\mathcal{C}^s$ are modules[5] over $\mathcal{C}^0$, which should be viewed as sections of (higher spin) bundles over H^4. More specifically, $\mathcal{C}^s$ can be viewed as totally symmetric traceless space-like rank s tensor fields on $\mathcal{M}^{3,1}$

$$\phi^{(s)} = \phi_{\mu_1\ldots\mu_s}(x) t^{\mu_1}\ldots t^{\mu_s}, \qquad \phi_{\mu_1\ldots\mu_s} x^{\mu_i} = 0 \tag{3.16}$$

due to (3.1). The underlying $\mathfrak{so}(4,2)$ structure provides an $SO(3,1)$ -invariant derivation

$$\begin{aligned} D\phi := \{x^4,\phi\} \; &= r^2R^2\frac{1}{x^4}t^\mu\{t_\mu,\phi\} = -\frac{1}{x^4}x_\mu\{x^\mu,\phi\} \\ &= r^2R\, t^{\mu_1}\ldots t^{\mu_s}t^\mu\, \nabla^{(3)}_\mu \phi_{\mu_1\ldots\mu_s}(x) \end{aligned} \tag{3.17}$$

where $\nabla^{(3)}$ is the covariant derivative along the space-like $H^3 \subset \mathcal{M}^{3,1}$. Hence D relates the different spin sectors in (3.15):

$$D = D^- + D^+ :\; \mathcal{C}^s \;\to \mathcal{C}^{s-1}\oplus\mathcal{C}^{s+1}, \qquad D^\pm\phi^{(s)} = [D\phi^{(s)}]_{s\pm 1} \tag{3.18}$$

where $[.]_s$ denotes the projection to $\mathcal{C}^s$ defined through (3.15). It is easy to see that

$$(D^+)^\dagger = -D^- \tag{3.19}$$

w.r.t. the inner product (2.22). Explicitly, $Dx^\mu = r^2R\,t^\mu$ and $Dt^\mu = R^{-1}\,x^\mu$. In particular, $\mathcal{C}^{(s,0)} \subset \mathcal{C}^s$ is the space of divergence-free traceless space-like rank s tensor fields on $\mathcal{M}^{3,1}$, in radiation gauge.

The $D^\pm$ operators allow to organize the $\mathcal{C}^s$ modes into primals and descendants

$$\boxed{\begin{aligned} \mathcal{C}^{(s,0)} &= \{\phi\in\mathcal{C}^s;\; D^-\phi = 0\} && \ldots\text{primal fields} \\ \mathcal{C}^{(s+k,k)} &= (D^+)^k\mathcal{C}^{(s,0)} && \ldots\text{descendants} \end{aligned}}$$

[5]The module structure also applies in the noncomutative case if $\mathcal{C}^0$ is equipped with the commutative but non-associative pull-back algebra structure, due to (3.46) in [5]. Useful discussions with S. Rangoolam are acknowledged.

cf. [5]. This is somewhat reminiscent of primaries in CFT but the concepts are different. The primals[6] have minimal spin $\mathcal{S}^2$, which is raised and lowered by $D^\pm$; they correspond to divergence-free spin s tensor fields on H^4 in space-like gauge, i.e. tangential to H^3. The descendants are *space-like* derivatives of the primal fields. However they should not be considered as pure gauge fields, and they are part of the physical Hilbert space.

This sub-structure encodes two different concepts on the FRW background, which arise from the presence of a space-like foliation: $\mathcal{S}^2 = 2s(s+1)$ measures the 4-dimensional spin on H^4, while $(s-k)$ measures the 3-dimensional spin of $\mathcal{C}^{(s,k)}$ on H^3. Nevertheless, local Lorentz invariance should be largely restored through gauge invariance, which contains Ω-volume-preserving diffeos. Although these act in a somewhat unusual manner [5], one may expect that they protect the model from pathological Lorentz violation. This will be illustrated by the fact that all modes propagate according to the same effective d'Alembertian $\Box$. In physical terms, an $SO(4,1)$ irrep $\phi^{(s)} \in \mathcal{C}^s$ encodes a series of massless modes $\phi^{(s,k)}$ in radiation gauge with spin $s-k$ for $k = 0, \ldots, s$.

Averaging over S^2. We can interpret the projection $[f(t)]_0$ on the scalar sector $\mathcal{C}^0$ as an averaging or integral over the S^2 fiber described by the t generators,

$$[f(t)]_0 = \frac{1}{4\pi r^{-2}\cosh(\eta)^2}\int_{S^2_t} f(t) \tag{3.20}$$

such that $[1]_0 = 1$. This gives the formula

$$[t^\mu t^\nu]_0 = \frac{\cosh^2(\eta)}{3r^2} P_\perp^{\mu\nu} \tag{3.21}$$

where

$$P_\perp^{\mu\nu} := \eta^{\mu\nu} + \frac{1}{R^2\cosh^2(\eta)} x^\mu x^\nu \tag{3.22}$$

[6] In contrast to primaries in CFT, these are not annihilated by the $M^{\mu 5} - iM^{\mu 4}$ operators which lowers the eigenvalue of D. Primal fields do not have an eigenvalue of D.

is the positive semi-definite projector tangential to the space-like H^3. Furthermore, we have [1]

$$[t^\alpha \theta^{\mu\nu}]_0 = \frac{1}{3}\Big(\sinh(\eta)(\eta^{\alpha\nu}x^\mu - \eta^{\alpha\mu}x^\nu) + x_\beta \varepsilon^{\beta 4\alpha\mu\nu}\Big)\,, \tag{3.23a}$$

$$[t^{\mu_1}\dots t^{\mu_4}]_0 = \frac{3}{5}\big([t^{\mu_1}t^{\mu_2}][t^{\mu_3}t^{\mu_4}]_0 + [t^{\mu_1}t^{\mu_3}][t^{\mu_2}t^{\mu_4}]_0 + [t^{\mu_1}t^{\mu_4}][t^{\mu_2}t^{\mu_3}]_0\big)\,. \tag{3.23b}$$

This also provides a formula for the projection on $\mathcal{C}^1$,

$$[t^\alpha t^\beta t^\gamma]_1 = \frac{3}{5}\Big([t^\alpha t^\beta]_0 t^\gamma + t^\alpha [t^\beta t^\gamma]_0 + t^\beta [t^\alpha t^\gamma]_0\Big)\;. \tag{3.24}$$

The general Wick theorem

$$[t^{\alpha_1}\dots t^{\alpha_{2s}}]_0 = a_{2s}\sum [t^{\alpha_i}t^{\alpha_j}]\dots[t^{\alpha_k}t^{\alpha_l}] \tag{3.25}$$

summing over all contractions can be obtained recursively from Lemma 9.1 in the appendix.

3.3. $\mathcal{C}^s$ and higher spin on H^4

In the previous section, $\mathcal{C}^s$ was identified with space-like spin s tensor fields on $\mathcal{M}^{3,1}$. On the other hand, $\mathcal{C}^s$ can also be identified with totally symmetric, traceless, divergence-free tangential rank s tensor fields $\phi_{a_1\dots a_s}$ on H^4 via [5]

$$\phi^{(s)} = \{x^{a_s}, \dots \{x^{a_1}, \phi_{a_1\dots a_s}\}\dots\} \;\in \mathcal{C}^s\;. \tag{3.26}$$

Conversely, a totally symmetric tensor field on H^4 can be extracted from $\phi^{(s)}$ via

$$\tilde\phi_{a_1 a_2\dots a_s} := \{x_{a_1}, \dots \{x_{a_s}, \phi^{(s)}\}_- \dots\}_- = \mathcal{A}^{(-)}_{a_1}[\dots[\mathcal{A}^{(-)}_{a_s}[\phi^{(s)}]\dots] \quad \in \mathcal{C}^0 \tag{3.27}$$

anticipating the notation (5.8), which is tangential due to $x_a\{x^a,\phi\} = 0$. One can also define intermediate tensor fields such as

$$\phi^{(s)}_{a_s} = \{x^{a_{s-1}}, \dots, \{x^{a_1}, \phi_{a_1\dots a_{s-1}a_s}\}\dots\} \;\in \mathcal{C}^{s-1} \tag{3.28}$$

which are tangential and associated to the underlying irreducible rank s tensor field. Using Lemma 9.3 we obtain

$$-\{x^a, \tilde\phi_a\} = \alpha_1(\Box_H - 4r^2)\phi^{(1)} \tag{3.29}$$

and similarly using (9.9)

$$\begin{aligned}-\{x^{a_1},\tilde{\phi}_{a_1a_2}\} &= \alpha_1(\Box_H - 4r^2)\mathcal{A}^{(-)}_{a_2}[\phi^{(2)}]\\ &= \alpha_1\mathcal{A}^{(-)}_{a_2}[(\Box_H - 2(2+2)r^2)\phi^{(2)}] \end{aligned} \tag{3.30}$$

and in general

$$-\{x^{a_1},\tilde{\phi}_{a_1a_2\ldots a_s}\} = \alpha_1\mathcal{A}^{(-)}_{a_2}[\ldots[\mathcal{A}^{(-)}_{a_s}[(\Box_H - r^2(s^2+s+2))\phi^{(s)}]\ldots] \ . \tag{3.31}$$

Iterating this, we recover (3.26) up to some action of $\Box_H$,

$$(-1)^s\{x^{a_1},\ldots\{x^{a_s}\tilde{\phi}_{a_1a_2\ldots a_s}\}\ldots\} = \mathcal{O}(\Box_H)\phi \tag{3.32}$$

where $\mathcal{O}(\Box_H)$ is a positive and hence invertible operator provided

$$\Box_H > r^2(s^2+s+2) \qquad \text{on } \mathcal{C}^s \ . \tag{3.33}$$

We will see that this is indeed the case for admissible modes, because (3.44) gives

$$r^{-2}\Box_H > s^2+s+9/4 > s^2+s+2 \ . \tag{3.34}$$

Therefore the maps (3.26) and (3.27) are inverse of each other up to normalization.

Relation with higher spin field strength. It is instructive to work out these formulae more explicitly using the tangential derivatives on H^4 [5]

$$\eth^a\phi := \frac{1}{r^2R^2}x_b\{\theta^{ab},\phi\}, \qquad \phi\in\mathcal{C} \ , \tag{3.35}$$

which satisfy

$$\begin{aligned}\{x^a,\cdot\} &= \theta^{ab}\eth_b \qquad x^a\eth_a = 0,\\ \eth^a x^b &= P^{ab} = \eta^{ab} + \frac{1}{R^2}x^ax^b,\\ \eth^a\theta^{cd} &= \frac{1}{R^2}(-\theta^{ac}x^d + \theta^{ad}x^c). \end{aligned} \tag{3.36}$$

It is then straightforward to show (cf. [5])

$$
\begin{aligned}
\phi_{a_2\dots a_s} &= \{x^{a_1}, \phi_{a_1 a_2 \dots a_s}\} = \theta^{a_1 b_1}\eth_{b_1}\phi_{a_1 a_2\dots a_s} \qquad \in \mathcal{C}^1 \\
&\vdots \\
\phi^{(s)} &= \theta^{a_1 b_1}\dots\theta^{a_s b_s}\eth_{b_s}\dots\eth_{b_1}\phi_{a_1 a_2 \dots a_s} =: \theta^{a_1 b_1}\dots\theta^{a_s b_s}\mathcal{F}_{a_1\dots a_d; b_1\dots b_s}
\end{aligned} \tag{3.37}
$$

noting that $\theta^{a_1 b_1}\theta^{a_2 b_2} = r^2 R^2 P^{b_1 b_2}$. Here

$$
\begin{aligned}
\mathcal{F}_{b;a} &= \eth_a \phi_b - \eth_b \phi_a \\
&\vdots \\
\mathcal{F}_{a_1\dots a_d; b_1 \dots b_s} &= \eth_{[b_s}\dots\eth_{b_1}\phi_{a_1 a_2\dots a_s]} \ .
\end{aligned} \tag{3.38}
$$

The last term is a generalization of the curvature or field strength tensor, which has the symmetry of the Young tableau $\begin{array}{|c|c|c|}\hline a&a&a\\ \hline b&b&b\\ \hline\end{array}$. This provides a link with Vasiliev's higher spin theory [3, 4]; see also [5] for further related discussion. However, the realization (3.16) is more transparent.

3.4. Admissible tensor fields and positivity

This section discusses integrability and positivity aspects, and can be skipped at first reading.

In order to have well-defined kinetic energy and similar quantities, we need some refinements of the integrability condition (2.22). Consider for example

$$
0 \leq \int \{x_a, \phi\}\{x^a, \phi\} = \int \phi \Box_H \phi \ . \tag{3.39}
$$

The lhs is positive since $\{x^a, \phi\}$ is tangential to H^4, due to $x_a\{x^a, .\} = 0$. Therefore

$$
\Box_H > 0 \tag{3.40}
$$

must be positive definite. This argument carries over to $End(\mathcal{H}_n)$ (for Hilbert-Schmidt-operators) using $\mathcal{Q}$ (2.13). However, we will need a slightly stronger bound, which can be obtained from group theory. A heuristic argument for such an improved bound is as follows: consider the $SO(4,1)$ invariant expression

(3.41)
$$-\int\{M_{ab},\phi\}\{M^{ab},\phi\} = \int\phi\{M^{ab},\{M_{ab},\phi\}\} = -2\int\phi C^2[\mathfrak{so}(4,1)]\phi$$

for $a,b = 0,\ldots,4$. At the reference point $\xi = (R,0,0,0,0) \in H^4$, the sum on the lhs separates as

(3.42)
$$-\{M_{ab},\phi\}\{M^{ab},\phi\} \stackrel{\xi}{=} 2\sum_a\{M^{a0},\phi\}\{M^{a0},\phi\} \; - \sum_{a,b=1}^4\{M^{ab},\phi\}\{M^{ab},\phi\}\ .$$

The first term is manifestly positive, while the second term is negative and involves the local stabilizer $SO(4)$ acting on ϕ. Hence the second term measures the spin, and we expect heuristically that it contributes $-2s(s+1)$, if we forget about curvature corrections for the moment. This would give the estimate $-C^2[\mathfrak{so}(4,1)] \geq -s(s+1)$ for integrable modes.

The precise statements required are obtained from representation theory for *principal series* of unitary representations. They describe the normalizable fluctuation modes in the present context, corresponding to a continuous basis for square-integrable wavefunctions on the hyperboloids, analogous to plane waves in the flat case. The (bosonic) principal series of unitary representations $\Pi_{\nu,s}$ of $SO(4,1)$ are determined by the spin $s \in \mathbb{N}_0$ and the real ("kinetic") parameter $\nu \in \mathbb{R}$. They can be identified with spin s wavefunctions on H^4. For these representations, the quadratic Casimir satisfies the following bound [19]

$$-C^2[\mathfrak{so}(4,1)] = 9/4 + \nu^2 - s(s+1) > 9/4 - s(s+1) \tag{3.43}$$

assuming[7] $\nu \neq 0$. This is clearly a refined version of the above heuristic argument, and it entails via (2.7) the following bound for $\Box_H$

$$r^{-2}\Box_H\phi^{(s)} = 2s(s+1) - C^2[\mathfrak{so}(4,1)] > s^2 + s + 9/4 \tag{3.44}$$

which is slightly stronger than (3.40). This will imply that the higher spin modes in the present framework are square-integrable over H^4 and form a

[7] Note that ν will *not* play the role of a mass in the present context. The case $\nu = 0$ would correspond to some extreme IR case and is ignored here.

Hilbert space, as discussed below. We will denote modes which satisfy the condition (3.43), i.e. which consist of unitary principal series of $SO(4,1)$, as **admissible modes**[8]. This condition is preserved by $D^{\pm}$ due to (9.12).

It is interesting to compare this with the (bosonic) principal series unitary representations $\Pi_{p,s}$ of $SO(3,1)$, which are determined by the spin $s \in \mathbb{N}_0$ and a kinetic parameter $p \in \mathbb{R}$. They can be identified with spin s wavefunctions on H^3, and satisfy the bound [20, 21]

$$-C^2[\mathfrak{so}(3,1)] = p^2 - s^2 + 1 > -s^2 + 1 \ . \tag{3.45}$$

Even though the conditions (3.43) and (3.45) are a priori independent, they are closely related for on-shell modes here, i.e. modes satisfying the on-shell condition (5.26)

$$0 = \Box = C^2[\mathfrak{so}(4,1)] - C^2[\mathfrak{so}(3,1)] \ . \tag{3.46}$$

Then the 3-dimensional condition (3.45) is slightly stronger than the 4-dimensional condition (3.43) except for $s = 1$. This means that on-shell wave-functions which are square-integrable over some time-slice H^3 are automatically integrable over the entire space-time, which is quite remarkable and helpful for a theory with time evolution.

In particular, it follows that for admissible modes $\phi \in \mathcal{C}^s$, the tensor field $\tilde\phi_{a_1 a_2 \dots a_s}$ defined in (3.27) are square-integrable with positive-definite inner product and form a Hilbert space, since

(3.47)
$$\int \tilde\phi^{a_1 a_2 \dots a_s} \tilde\phi_{a_1 a_2 \dots a_s} = \int (-1)^s \phi \{x^{a_1}, \dots \{x^{a_s} \tilde\phi_{a_1 a_2 \dots a_s}\} \dots \} \ = \int \phi \mathcal{O}(\Box_H) \phi$$

and $\mathcal{O}(\Box_H)$ (3.32) is positive as shown above. For the Minkowski case see Corollary 9.6.

4. Matrix model and higher-spin gauge theory

Now we return to the noncommutative setting, and define a dynamical model for the fuzzy $\mathcal{M}^{3,1}$ space-time under consideration. Consider a Yang-Mills

[8]It is interesting to observe using (2.7) that the admissible modes are precisely those with $C^2[\mathfrak{so}(4,2)] > 9/2$, and it is plausible that those are precisely the principal series irreps of $SO(4,2)$ in $End(\mathcal{H}_n)$. However, this will not be investigated here. There are of course functions (e.g. polynomial functions) which violate these bounds, but they are not normalizable and not considered here.

matrix model with mass term,

$$S[Y] = \frac{1}{g^2}\mathrm{Tr}\Big([Y^\mu, Y^\nu][Y_\mu, Y_\nu] + \frac{6}{R^2}Y^\mu Y_\mu\Big) \ . \tag{4.1}$$

All indices will be raised and lowered with $\eta^{\mu\nu}$ in the following sections. This includes in particular the IKKT or IIB matrix model [22] with mass term, which is best suited for quantization because maximal supersymmetry protects from UV/IR mixing [23]. As observed in [1], $\mathcal{M}^{3,1}$ is indeed a solution of this model[9], through

$$Y^\mu = T^\mu \ . \tag{4.2}$$

Now consider tangential deformations of the above background solution, i.e.

$$Y^\mu = T^\mu + \mathcal{A}^\mu \ , \tag{4.3}$$

where $\mathcal{A}^\mu \in \mathrm{End}(\mathcal{H}_n) \otimes \mathbb{R}^4$ is an arbitrary Hermitian fluctuation. The Yang-Mills action (4.1) can be expanded around the solution as

$$S[Y] = S[T] + S_2[\mathcal{A}] + O(\mathcal{A}^3) \ , \tag{4.4}$$

and the quadratic fluctuations are governed by

$$S_2[\mathcal{A}] = -\frac{2}{g^2}\,\mathrm{Tr}\,\left(\mathcal{A}_\mu\Big(\mathcal{D}^2 - \frac{3}{R^2}\Big)\mathcal{A}^\mu + \mathcal{G}\,(\mathcal{A})^2\right) . \tag{4.5}$$

This involves the vector d'Alembertian on $\mathcal{M}^{3,1}$

$$\mathcal{D}^2\mathcal{A} = (\Box - 2\mathcal{I})\,\mathcal{A} \tag{4.6}$$

(cf. (3.14)) which is an $SO(3,1)$ intertwiner, as well as

$$\mathcal{I}(\mathcal{A})^\mu := -[[Y^\mu, Y^\nu], \mathcal{A}_\nu] = \frac{\mathrm{i}}{r^2R^2}[\Theta^{\mu\nu}, \mathcal{A}_\nu] =: -\frac{1}{r^2R^2}\tilde{\mathcal{I}}(\mathcal{A})^\mu \tag{4.7}$$

[9]This "momentum" embedding via T^μ has some similarity with the ideas in [24] but avoids excessive dof and the associated ghost issues, cf. [25]. The positive mass parameter in (4.1) simply sets the scale of the background. For negative mass parameter, X^μ would be a solution [26], but the fluctuation analysis would be less clear.

using (2.21). As usual in Yang-Mills theories, $\mathcal{A}$ transforms under gauge transformations as

$$\delta_\Lambda \mathcal{A} = -i[T^\mu + \mathcal{A}^\mu, \Lambda] \sim \{t^\mu, \Lambda\} + \{\mathcal{A}^\mu, \Lambda\} \tag{4.8}$$

for any $\Lambda \in \mathcal{C}$, and the scalar ghost mode

$$\mathcal{G}(\mathcal{A}) = -\mathrm{i}[T^\mu, \mathcal{A}_\mu] \sim \{t^\mu, \mathcal{A}_\mu\} \tag{4.9}$$

should be removed to get a meaningful theory. This is achieved by adding a gauge-fixing term $-\mathcal{G}(\mathcal{A})^2$ to the action as well as the corresponding Faddeev-Popov (or BRST) ghost. Then the quadratic action becomes

$$S_2[\mathcal{A}] + S_{g.f} + S_{ghost} = -\frac{2}{g^2}\mathrm{Tr}\,\Big(\mathcal{A}_\mu\Big(\mathcal{D}^2 - \frac{3}{R^2}\Big)\mathcal{A}^\mu + 2\overline{c}\Box c\Big) \tag{4.10}$$

where c denotes the BRST ghost; see e.g. [27] for more details.

5. Fluctuation modes

We should expand the vector modes into higher spin modes according to (3.15), (3.16)

$$\mathcal{A}^\mu = A^\mu(x) + A^\mu_\alpha(x)\, t^\alpha + A^\mu_{\alpha\beta}(x)\, t^\alpha t^\beta + \ldots \;\in\; \mathcal{C}^0 \oplus \mathcal{C}^1 \oplus \mathcal{C}^2 \oplus\; \ldots \tag{5.1}$$

However these are neither irreducible nor eigenmodes of $\mathcal{D}^2$, and the goal of this section is to find explicitly all eigenmodes of $\mathcal{D}^2$. This will be achieved using the $\mathfrak{so}(4,2)$ structure and suitable intertwiners.

Intertwiners. We recall the $SO(3,1)$ intertwiners (3.18)

$$\begin{aligned} D^\pm: \quad \mathcal{C}^{(n)} \otimes \mathbb{R}^4 &\to \mathcal{C}^{(n\pm1)} \otimes \mathbb{R}^4 \\ \mathcal{A}_\mu &\mapsto D^\pm \mathcal{A}_\mu \end{aligned} \tag{5.2}$$

It is easy to show using (9.5) that they satisfy the following intertwiner property for $\mathcal{D}^2$ [2]

$$\begin{aligned} \mathcal{D}^2 D^+ \mathcal{A}^{(s)} &= D^+(\mathcal{D}^2 + \frac{2s+2}{R^2})\mathcal{A}^{(s)}, \\ \mathcal{D}^2 D^- \mathcal{A}^{(s)} &= D^-(\mathcal{D}^2 - \frac{2s}{R^2})\mathcal{A}^{(s)}, \qquad \mathcal{A}^{(s)} \in \mathcal{C}^{(s)}\ . \end{aligned} \tag{5.3}$$

In particular,

$$[\mathcal{D}^2, D^+D^-] = 0 \ . \tag{5.4}$$

We also recall the $SO(3,1)$ intertwiner (4.7)

$$\begin{aligned} \tilde{\mathcal{I}}: \quad \mathcal{C}^s \otimes \mathbb{R}^4 &\to \mathcal{C}^s \otimes \mathbb{R}^4 \\ \mathcal{A}^\mu &\mapsto \{\theta^{\mu\nu}, \mathcal{A}_\nu\} \end{aligned} \tag{5.5}$$

which satisfies

$$[\tilde{\mathcal{I}}, D^\pm] = 0 \ . \tag{5.6}$$

In analogy to the $SO(4,1)$ case discussed in [5], this is related to the total $\mathfrak{so}(3,1)$ Casimir of the vector fields via

$$C^2[\mathfrak{so}(3,1)]^{(4)\otimes(ad)} = C^2[\mathfrak{so}(3,1)]^{(4)} + C^2[\mathfrak{so}(3,1)]^{(ad)} - \frac{2}{r^2}\tilde{\mathcal{I}} \tag{5.7}$$

where (ad) indicates the adjoint action (2.5). Hence $\tilde{\mathcal{I}}$ describes some kind of "spin-orbit" mixing.

5.1. Diagonalization of $\mathcal{D}^2$

In [1], three series of eigenmodes $\mathcal{A}_\mu$ of $\mathcal{D}^2$ were found, of the form

$$\mathcal{A}_\mu^{(g)}[\phi^{(s)}] = \{t_\mu, \phi^{(s)}\} \quad \in \mathcal{C}^s \,, \tag{5.8}$$

$$\mathcal{A}_\mu^{(+)}[\phi^{(s)}] = \{x_\mu, \phi^{(s)}\}\big|_{s+1} \equiv \{x_\mu, \phi^{(s)}\}_+ \quad \in \mathcal{C}^{s+1} \,, \tag{5.9}$$

$$\mathcal{A}_\mu^{(-)}[\phi^{(s)}] = \{x_\mu, \phi^{(s)}\}\big|_{s-1} \equiv \{x_\mu, \phi^{(s)}\}_- \quad \in \mathcal{C}^{s-1} \tag{5.10}$$

for any $\phi^{(s)} \in \mathcal{C}^s$. However there should be another series, and to find it we re-derive the previous results in a more systematic way. We start with the easy observation [1]

$$\mathcal{D}^2 \mathcal{A}_\mu^{(g)}[\phi] = \mathcal{A}_\mu^{(g)}\Big[(\Box + \frac{3}{R^2})\phi\Big] \, . \tag{5.11}$$

This means that $\mathcal{A}_\mu^{(g)}[\phi]$ is an eigenmode of $\mathcal{D}^2$ if $\Box\phi = \lambda\phi$. Using the intertwiner properties (5.3), we obtain new eigenmodes by acting with $D^\pm$. To organize this, observe using the Jacobi identity

$$
\begin{aligned}
D^+\mathcal{A}_\mu^{(g)}[\phi^{(s)}] &= \mathcal{A}_\mu^{(g)}[D^+\phi^{(s)}] + \frac{1}{R}\mathcal{A}_\mu^{(+)}[\phi^{(s)}] \\
D^+D^+\mathcal{A}_\mu^{(g)}[\phi^{(s)}] &= \mathcal{A}_\mu^{(g)}[D^+D^+\phi^{(s)}] + \frac{2}{R}\mathcal{A}_\mu^{(+)}[D^+\phi^{(s)}]
\end{aligned}
\tag{5.12}
$$

etc., and similarly

$$
\begin{aligned}
D^-\mathcal{A}_\mu^{(g)}[\phi^{(s)}] &= \mathcal{A}_\mu^{(g)}[D^-\phi^{(s)}] + \frac{1}{R}\mathcal{A}_\mu^{(-)}[\phi^{(s)}] \\
D^-D^-\mathcal{A}_\mu^{(g)}[\phi^{(s)}] &= \mathcal{A}_\mu^{(g)}[D^-D^-\phi^{(s)}] + \frac{2}{R}\mathcal{A}_\mu^{(-)}[D^-\phi^{(s)}] \ .
\end{aligned}
\tag{5.13}
$$

Using also the intertwiner properties (9.5) between $\Box$ and $D^\pm$ we recover

$$
\mathcal{D}^2\mathcal{A}_\mu^{(+)}[\phi^{(s)}] = \mathcal{A}_\mu^{(+)}\Big[(\Box + \frac{2s+5}{R^2})\phi^{(s)}\Big] , \tag{5.14}
$$

$$
\mathcal{D}^2\mathcal{A}_\mu^{(-)}[\phi^{(s)}] = \mathcal{A}_\mu^{(-)}\Big[(\Box + \frac{-2s+3}{R^2})\phi^{(s)}\Big] . \tag{5.15}
$$

$D^+\mathcal{A}^{(+)}$ does not give a new mode due to (5.12), however $D^+\mathcal{A}_\mu^{(-)}[\phi^{(s)}]$ or $D^-\mathcal{A}_\mu^{(+)}[\phi^{(s)}]$ do. These two modes are linearly dependent modulo $\mathcal{A}^{(+-g)}$ due to the Jacobi identity

$$
\begin{aligned}
&D^+\mathcal{A}_\mu^{(-)}[\phi^{(s)}] + D^-\mathcal{A}_\mu^{(+)}[\phi^{(s)}] \\
&\quad = [D(\{x_\mu, \phi^{(s)}\})]_s = r^2R\{t_\mu, \phi^{(s)}\} + [\{x_\mu, D\phi^{(s)}\}]_s \\
&\quad = r^2R\mathcal{A}^{(g)}[\phi^{(s)}] + \mathcal{A}^{(-)}[D^+\phi^{(s)}] + \mathcal{A}^{(+)}[D^-\phi^{(s)}] \ .
\end{aligned}
\tag{5.16}
$$

Hence either one can be used to represent the new mode (if it is independent). We choose

$$
\boxed{\mathcal{A}_\mu^{(n)}[\phi^{(s)}] := D^+\mathcal{A}_\mu^{(-)}[\phi^{(s)}] \qquad \in \mathcal{C}^s \ .} \tag{5.17}
$$

This provides the following list of eigenmodes of $\mathcal{D}^2$ in $\mathcal{C}^s \otimes \mathbb{R}^4$

$$
\{\mathcal{A}_\mu^{(g)}[\phi^{(s)}],\ \mathcal{A}_\mu^{(+)}[\phi^{(s-1)}],\ \mathcal{A}_\mu^{(-)}[\phi^{(s+1)}],\ \mathcal{A}_\mu^{(n)}[\phi^{(s)}]\} \tag{5.18}
$$

with eigenvalues

$$\mathcal{D}^2\mathcal{A}_\mu^{(+)}[\phi^{(s-1)}] = \mathcal{A}_\mu^{(+)}\Big[\big(\Box + \frac{2s+3}{R^2}\big)\phi^{(s-1)}\Big] , \tag{5.19}$$

$$\mathcal{D}^2\mathcal{A}_\mu^{(-)}[\phi^{(s+1)}] = \mathcal{A}_\mu^{(-)}\Big[\big(\Box + \frac{-2s+1}{R^2}\big)\phi^{(s+1)}\Big] . \tag{5.20}$$

$$\mathcal{D}^2\mathcal{A}_\mu^{(g)}[\phi^{(s)}] = \mathcal{A}_\mu^{(g)}\Big[\big(\Box + \frac{3}{R^2}\big)\phi^{(s)}\Big] \tag{5.21}$$

$$\mathcal{D}^2\mathcal{A}_\mu^{(n)}[\phi^{(s)}] = \mathcal{A}_\mu^{(n)}\Big[\big(\Box + \frac{3}{R^2}\big)\phi^{(s)}\Big] . \tag{5.22}$$

The eigenvalues can be made to coincide upon inserting $D^\pm$ using (9.5), and for any eigenmode of $\Box\phi^{(s)} = m^2\phi^{(s)}$ we obtain 4-tuples of *"regular" eigenmodes* $\tilde{\mathcal{A}}_\mu^{(i)}[\phi^{(s)}] \in \mathcal{C}^s \otimes \mathbb{R}^4$ of $\mathcal{D}^2$

$$\tilde{\mathcal{A}}^{(i)}[\phi] = \begin{pmatrix} \mathcal{A}^{(+)}[D^-\phi] \\ \mathcal{A}^{(-)}[D^+\phi] \\ \mathcal{A}^{(n)}[\phi] \\ r^2R\mathcal{A}^{(g)}[\phi] \end{pmatrix} , \qquad i,j \in \{+,-,n,g\} \tag{5.23}$$

for $\phi = \phi^{(s)}$ dropping the index μ, with the same eigenvalue

$$\boxed{\begin{aligned} \mathcal{D}^2\tilde{\mathcal{A}}^{(+)}[\phi] &= \big(m^2 + \frac{3}{R^2}\big)\tilde{\mathcal{A}}^{(+)}[\phi] \\ \mathcal{D}^2\tilde{\mathcal{A}}^{(-)}[\phi] &= \big(m^2 + \frac{3}{R^2}\big)\, \tilde{\mathcal{A}}^{(-)}[\phi] \\ \mathcal{D}^2\tilde{\mathcal{A}}^{(g)}[\phi] &= \big(m^2 + \frac{3}{R^2}\big)\, \tilde{\mathcal{A}}^{(g)}[\phi] \\ \mathcal{D}^2\tilde{\mathcal{A}}^{(n)}[\phi] &= \big(m^2 + \frac{3}{R^2}\big)\, \tilde{\mathcal{A}}^{(n)}[\phi] . \end{aligned}} \tag{5.24}$$

There is one "special" mode in (5.18) which is not covered by the regular $\tilde{\mathcal{A}}^{(i)}$, namely $\mathcal{A}^{(-)}[\phi^{(s,0)}]$ with

$$\mathcal{D}^2\mathcal{A}^{(-)}[\phi^{(s,0)}] = \mathcal{A}^{(-)}[\big(\Box + \frac{-2s+3}{R^2}\big)\phi^{(s,0)}] . \tag{5.25}$$

We will see that it is orthogonal to all regular modes, and altogether these modes are complete. Hence diagonalizing $\mathcal{D}^2$ is reduced to diagonalizing $\Box$ on $\mathcal{C}^s$. In particular, we obtain the following on-shell modes $\big(\mathcal{D}^2 - \frac{3}{R^2}\big)\mathcal{A} = 0$

$$\begin{aligned} \{\tilde{\mathcal{A}}^{(+)}[\phi^{(s)}], \tilde{\mathcal{A}}^{(-)}[\phi^{(s)}], \tilde{\mathcal{A}}^{(g)}[\phi^{(s)}], \tilde{\mathcal{A}}^{(n)}[\phi^{(s)}]\} \qquad &\text{for} \qquad \Box\phi^{(s)} = 0 \\ \mathcal{A}^{(-)}[\phi^{(s,0)}] \qquad &\text{for} \qquad \big(\Box - \frac{2s}{R^2}\big)\phi^{(s,0)} = 0 . \end{aligned} \tag{5.26}$$

The propagation of all these modes is governed by the effective metric $G_{\mu\nu}$ (3.10) encoded in $\Box$. In particular, we note that the on-shell relation $\Box\phi = 0$ determines $C^2[SO(4,1)]$ via (2.18) for any given $SO(3,1)$ mode, corresponding some irreducible tensor field on the space-like H^3. To put it differently, the state at any given time-slice H^3 completely determines the time evolution, up to forward or backward propagation. This is non-trivial in the NC case, and the time evolution is completely captured by $SO(4,1)$ group theory, even though $\mathcal{M}^{3,1}$ admits only space-like $SO(3,1)$ isometries. Hence we will obtain the standard picture of time evolution even though time does not commute. This would be hard to see in formulations based on higher-derivative star products.

In section 6, we will establish independence and completeness of these modes after dropping $\tilde{\mathcal{A}}^{(n)}[\phi^{(s,s)}]$ (which is not independent) and $\tilde{\mathcal{A}}^{(+)}[\phi^{(s,0)}] \equiv 0$, leading to a ghost-free action and a Hilbert space upon gauge-fixing.

5.2. Diagonalization of $\mathcal{I}$ and eigenmodes

To establish independence of the above modes, we need to distinguish them using some extra observable. Since $\tilde{\mathcal{I}}$ is related to the total $SO(3,1)$ Casimir (5.7) and commutes with both $\Box$ and $\mathcal{D}^2$, we look for a basis of common eigenvectors of $\mathcal{D}^2$ and $\tilde{\mathcal{I}}$ in $\mathcal{C} \otimes \mathbb{R}^4$. Using the above results, it suffices to diagonalize $\tilde{\mathcal{I}}$ on the tuples (5.23), (5.25) of eigenmodes. We can use the relations (9.6)

$$\begin{aligned} \tilde{\mathcal{I}}(\mathcal{A}^{(+)}[\phi^{(s)}]) &= r^2(s+3)\mathcal{A}^{(+)}[\phi^{(s)}] + r^2 R\mathcal{A}^{(g)}[D^+\phi^{(s)}] \\ \tilde{\mathcal{I}}(\mathcal{A}^{(-)}[\phi^{(s)}]) &= r^2(-s+2)\mathcal{A}^{(-)}[\phi^{(s)}] + r^2 R\mathcal{A}^{(g)}[D^-\phi^{(s)}] \end{aligned} \tag{5.27}$$

and (9.13)

$$\begin{aligned} R\tilde{\mathcal{I}}(\mathcal{A}^{(g)}[\phi]) = (s+3)r^2 R\mathcal{A}^{(g)}[\phi] &+ (2s+3)\mathcal{A}^{(-)}[D^+\phi^{(s)}] \\ &+ 2\mathcal{A}^{(+)}[D^-\phi^{(s)}] - (2s+1)\mathcal{A}^{(n)}[\phi] \end{aligned}$$

which gives

$$\begin{aligned} \tilde{\mathcal{I}}(\mathcal{A}^{(n)}[\phi]) &= D^+(\tilde{\mathcal{I}}(\mathcal{A}_\mu^{(-)}[\phi^{(s)}]) \\ &= r^2(-s+2)\mathcal{A}^{(n)}[\phi^{(s)}] + r^2\mathcal{A}^{(+)}[D^-\phi^{(s)}] + r^2 R\mathcal{A}^{(g)}[D^+D^-\phi^{(s)}] \end{aligned} \tag{5.28}$$

using $[\tilde{\mathcal{I}}, D^{\pm}] = 0$. In terms of the $\tilde{\mathcal{A}}^{(i)}$ (5.23), this can be summarized in matrix form as follows

$$\tilde{\mathcal{I}}\begin{pmatrix}\tilde{\mathcal{A}}^{(+)}[\phi]\\ \tilde{\mathcal{A}}^{(-)}[\phi]\\ \tilde{\mathcal{A}}^{(n)}[\phi]\\ \tilde{\mathcal{A}}^{(g)}[\phi]\end{pmatrix} = r^2 \underbrace{\begin{pmatrix} s+2 & 0 & 0 & d_{+-}\\ 0 & -s+1 & 0 & d_{-+}\\ 1 & 0 & -s+2 & d_{+-}\\ 2 & 2s+3 & -(2s+1) & s+3\end{pmatrix}}_{=:I}\begin{pmatrix}\tilde{\mathcal{A}}^{(+)}[\phi]\\ \tilde{\mathcal{A}}^{(-)}[\phi]\\ \tilde{\mathcal{A}}^{(n)}[\phi]\\ \tilde{\mathcal{A}}^{(g)}[\phi]\end{pmatrix} \tag{5.29}$$

for $\phi = \phi^{(s,k)}$. Here we introduce the notation $D^+D^-\phi = r^2 d_{+-}\phi$ and $D^-D^+\phi = r^2 d_{-+}\phi$ assuming that they are diagonalized on $\phi^{(s,k)}$, which is always possible because $D^+D^-, D^+D^-, \Box$ are mutually commuting. To find the eigenvalues of I, we compute

$$\begin{aligned}\det(I - x\mathbb{1}) &= d_{-+}(2s+3)(s-x+2)(s+x-2)\\ &\quad + (s-x+3)(s+x-1)\left(-2d_{+-}s + d_{+-} + s^2 - (x-2)^2\right) .\end{aligned} \tag{5.30}$$

Using the commutation relations (9.70) of $D^{\pm}$ on $\mathcal{C}^{(s,k)}$, this factorizes as

$$\det(I - x\mathbb{1}) = \Big((k-s)^2 - (x-2)^2\Big)\left(-\mathcal{K} - (x-2)^2\right) . \tag{5.31}$$

Here we introduce the useful quantity

$$-\mathcal{K} := s^2 + \frac{4s^2-1}{k(2s-k)}d_{+-} \;= (s+1)^2 + \frac{(2s+1)(2s+3)}{(k+1)(2s-k+1)}d_{-+} \tag{5.32}$$

(for $k = 0$, the second form must be used), which is a measure for the kinetic energy of $\phi^{(s,k)}$ on the time slices H^3. This quantity satisfies the important positivity property

Lemma 5.1. *For all admissible modes ϕ, the following estimate holds*[10]

$$\mathcal{K}\big|_{\phi} > 0 . \tag{5.33}$$

[10]Recall that $\mathcal{K}$ commutes with $\Box_H$. This operator inequality is hence a statement for $\mathcal{K}$ acting on some eigenspace or spectral interval of $\Box_H$.

which is proved in appendix 9.6. We can now read off two "regular" (integer) eigenvalues of $\tilde{\mathcal{I}}$

$$x_\pm = 2 \pm (k-s) \; , \tag{5.34}$$

which essentially measures the spin. The corresponding left eigenvectors $v_\pm \cdot I = x_\pm v_\pm$ are

$$\begin{aligned} v_- &= \begin{pmatrix} -\frac{2k-2s+1}{k(k-2s)} & -\frac{2s+3}{k-2s-1} & -\frac{2s+1}{2s-k} & 1 \end{pmatrix} \\ v_+ &= \begin{pmatrix} -\frac{-2k+2s+1}{k(k-2s)} & -\frac{-2s-3}{k+1} & -\frac{2s+1}{k} & 1 \end{pmatrix} \; . \end{aligned} \tag{5.35}$$

The remaining factor in (5.31) leads to two extra eigenvalues

$$x'_\pm = 2 \pm \sqrt{-\mathcal{K}} \tag{5.36}$$

where $\sqrt{-\mathcal{K}}$ is purely imaginary due to (5.33). The corresponding eigenvectors for $k \neq 0$ are

$$v'_\pm = \Big(\frac{2(s \mp \sqrt{-\mathcal{K}}) - 2s + 1}{s^2 + \mathcal{K}} \, , \frac{3+2s}{s+1 \pm \sqrt{-\mathcal{K}}}, -\frac{1+2s}{s \pm \sqrt{-\mathcal{K}}}, 1\Big) \; . \tag{5.37}$$

Their complexified form is somewhat misleading, and one can replace them by the two real modes

$$\begin{aligned} v'_1 &= \frac{1}{2}(v'_+ + v'_-) = \Big(\frac{1}{s^2+\mathcal{K}}, \frac{(s+1)(2s+3)}{(s+1)^2+\mathcal{K}}, -\frac{s(2s+1)}{s^2+\mathcal{K}}, 1\Big) \\ v'_2 &= \frac{1}{2\sqrt{-\mathcal{K}}}(v'_+ - v'_-) = \Big(-\frac{2}{s^2+\mathcal{K}}, -\frac{(2s+3)}{(s+1)^2+\mathcal{K}}, \frac{(2s+1)}{s^2+\mathcal{K}}, 0\Big) \end{aligned} \tag{5.38}$$

which span the 2-dimensional negative eigenspace of $(\mathcal{I}-2)^2$. More precisely, they satisfy

$$\begin{aligned} (\mathcal{I}-2)v'_1 &= -\mathcal{K}\, v'_2 \; , \\ (\mathcal{I}-2)v'_2 &= v'_1 \; . \end{aligned} \tag{5.39}$$

We will see in section 6 that all the $v_\pm$ and $v'_\pm$ modes are mutually orthogonal w.r.t. the invariant but indefinite inner product, as they must be, and $v_\pm$ have positive norm at least on-shell.

Linear independence and degeneracies. Generically, the 4 vectors above have different eigenvalues of $\tilde{\mathcal{I}}$, and are therefore linearly independent.

Linear dependence can only occur if some of these eigenvalues coincide. Inspecting the above eigenvalues, we have to investigate the following special cases:

- $x_+ = x_-$, which happens if $k = s$. This case will be discussed below.
- $x'_+ = x'_-$, which can only happen for $\mathcal{K} = 0$. This is ruled out by (5.33).
- $x_\pm$ coincide with $x'_\pm$ if $\pm(k - s) = \sqrt{-\mathcal{K}}$. Again this cannot happen since $\mathcal{K} > 0$ (5.33).
- Finally for $k = 0$ and $s = 0$ some of the modes disappear, as discussed below.

Hence except possibly for these special cases, the 4 regular modes $\tilde{\mathcal{A}}^{(i)}$ are linearly independent. This strongly suggests that they provide a complete set of modes, which will be proved in section 6.3.

5.2.1. The primal sector $k = 0$. In this case, we cannot use the above results since $\mathcal{A}^{(+)}[D^-\phi] \equiv 0$, so that there are only 3-tuples of regular modes, supplemented by the special mode $\mathcal{A}^{(-)}[\phi]$. For the 3-tuples, we then have

$$\tilde{\mathcal{I}} \begin{pmatrix} \tilde{\mathcal{A}}^{(-)}[\phi] \\ \tilde{\mathcal{A}}^{(n)}[\phi] \\ \tilde{\mathcal{A}}^{(g)}[\phi] \end{pmatrix} = r^2 \underbrace{\begin{pmatrix} -s+1 & 0 & d_{-+} \\ 0 & -s+2 & 0 \\ 2s+3 & -(2s+1) & s+3 \end{pmatrix}}_{=:I} \begin{pmatrix} \tilde{\mathcal{A}}^{(-)}[\phi] \\ \tilde{\mathcal{A}}^{(n)}[\phi] \\ \tilde{\mathcal{A}}^{(g)}[\phi] \end{pmatrix} \tag{5.40}$$

for $\phi = \phi^{(s,0)}$. To find the eigenvalues of I, we compute

$$\det(I - x\mathbb{1}) = (s + x - 2)\big(-\mathcal{K} - (x-2)^2\big) \tag{5.41}$$

where the 2nd form of $\mathcal{K}$ in (5.32) must be used. This has one "regular" root

$$x_0 = -s + 2 \tag{5.42}$$

with eigenvector

$$v_0 = (0, 1, 0) \tag{5.43}$$

corresponding to $\mathcal{A}^{(n)}$. The two other eigenvectors corresponding to the roots

$$x_\pm = 2 \pm \sqrt{-\mathcal{K}} \tag{5.44}$$

are given by

$$v'_\pm = \left(\frac{2s+3}{s+1\pm\sqrt{-\mathcal{K}}}\ ,\frac{-2s-1}{s\pm\sqrt{-\mathcal{K}}}\ ,1\right)\ . \tag{5.45}$$

It can be checked explicitly using the results of section 6 that these three modes are mutually orthogonal. Again, we can replace the complex modes $v'_\pm$ by 2 real modes

$$\begin{aligned} v'_1 &= \frac{1}{2}(v'_+ + v'_-) = \left(\frac{(2s+3)(s+1)}{(s+1)^2+\mathcal{K}}, -\frac{(2s+1)s}{s^2+\mathcal{K}}, 1\right) \\ v'_2 &= \frac{1}{2\sqrt{-\mathcal{K}}}(v'_+ - v'_-) = \left(-\frac{(2s+3)}{(s+1)^2+\mathcal{K}}, \frac{2s+1}{s^2+\mathcal{K}}, 0\right) \end{aligned} \tag{5.46}$$

which are linearly independent. In addition to the above three modes, there is an extra mode:

Special massless spin s mode $\mathcal{A}_\mu^{(-)}[\phi^{(s,0)}]$. For $k=0$ consider the extra mode

$$v_0^- := \mathcal{A}_\mu^{(-)}[\phi^{(s,0)}]\ . \tag{5.47}$$

This is not contained in the previous modes $v_\pm, v_0$ because $\phi^{(s,0)}$ cannot be written as $D^+\phi'$. Hence it complements the 3 regular modes, so that each $\phi^{(s,0)}$ determines again 4 independent modes. The on-shell condition $\big(\mathcal{D}^2 - \frac{3}{R^2}\big)\mathcal{A}^{(-)}[\phi^{(s,0)}] = 0$ takes the slightly different form $(\Box - \frac{2s}{R^2})\phi^{(s,0)} = 0$, due to (5.15). This mode satisfies

$$x^\mu \mathcal{A}_\mu^{(-)}[\phi^{(s,0)}] = 0 \tag{5.48}$$

due to (9.4) i.e. it is space-like, since x^μ defines the time-like direction (e.g. at a reference point $\xi = (\xi_0, 0, 0, 0)$ on $\mathcal{M}^{3,1}$). Positivity of the inner product then follows immediately, in agreement with the direct computation in section 6. Moreover (9.1) implies that this mode is physical, and we will see that for $s=2$ it provides the 2 standard degrees of freedom of the physical graviton [1].

The case $s=0$. In this case (which implies $k=0$), not only the $\mathcal{A}_\mu^{(+)}$ mode vanishes but also $\mathcal{A}_\mu^{(n)} = 0$, because $\mathcal{A}^{(-)}[\phi^{(0)}] = 0$. The above special mode v_0^- (5.47) also disappears, and only the $v'_\pm$ survive among the above modes, or equivalently $\mathcal{A}_\mu^{(-)}[D^+\phi]$ and $\mathcal{A}_\mu^{(g)}[\phi]$. We will see below that their inner products are non-degenerate, and these 2 modes are complete for $s = 0$. This is consistent with the case of H^4_n studied in [5] and the case of S^4_N in [6], where also two tangential modes were obtained for $s = 0$, and 4 modes for $s \geq 1$.

5.2.2. The scalar sector $k = s \neq 0$. For $k = s \neq 0$, $\phi^{(s,s)} = (D^+)^s \phi^{(0)}$ is the s-fold space-like divergence of a scalar mode[11]. Then the eigenvalues $x_\pm$ of $\tilde{\mathcal{I}}$ and in fact also the corresponding modes $v_\pm$ (5.35) coincide,

$$v_+ = v_- = \Big(\frac{1}{s^2}, \frac{2s+3}{s+1}, -\frac{2s+1}{s}, 1\Big)\ . \tag{5.49}$$

However, we will see in section 9.7 that in fact $v_+ = v_- = 0$ vanishes identically. A substitute can be found by formally taking the limit

(5.50)

$$\begin{aligned} v_{\text{extra}} &:= \lim_{k\to s} \frac{v_+ - v_-}{k-s} \\ &= \lim_{k\to s} \frac{1}{k-s}\Big(\tfrac{2k-2s-1}{k(k-2s)} + \tfrac{2k-2s+1}{k(k-2s)}, \tfrac{2s+3}{k+1} + \tfrac{2s+3}{k-2s-1}, -\tfrac{2s+1}{k} + \tfrac{2s+1}{2s-k}, 0\Big) \\ &= -2\Big(\frac{2}{s^2}, \frac{2s+3}{(s+1)^2}, -\frac{2s+1}{s^2}, 0\Big)\ . \end{aligned}$$

We will see in section 6.2 that v_{extra} has positive norm and is orthogonal to $v'_\pm$, and there are no further scalar modes.

6. Inner product matrix

Now that we have identified the eigenmodes of $\mathcal{D}^2$, we can compute the inner product matrix with respect to (2.22). This will confirm and complete the results of the previous section, and allow to determine the signature of the inner product matrix for all admissible modes. We can then establish a no ghost theorem providing a Hilbert space of physical modes. Moreover, the off-shell results provide all the information needed to obtain the full propagator.

---

[11] Recall that $(D^+)^s\phi^{(0)}$ are *space-like* scalar modes in the sense that the 3-dimensional $SO(3,1)$ spin on H^3 vanishes, since D commute with $SO(3,1)$.

For $\phi, \phi' \in \mathcal{C}^s$ we define the inner product matrix[12]

$$\mathcal{G}^{(i,j)} = \left\langle \tilde{\mathcal{A}}^{(i)}_\mu[\phi'], \tilde{\mathcal{A}}^{\mu(j)}[\phi] \right\rangle, \qquad i,j \in \{+,-,n,g\} \tag{6.1}$$

with the $\tilde{\mathcal{A}}^{(i)}$ defined in (5.23). The matrix elements are computed explicitly in appendix 9.9. They can be evaluated easily e.g. in Mathematica, since the entries $D_\pm$ and $\Box$ mutually commute, and can be simultaneously diagonalized for any fixed mode $\phi^{(s,k)}$. Then the space of modes boils down to 4-dimensional blocks $\tilde{\mathcal{A}}^{(i)}$ which are mutually orthogonal. The metric in the blocks is non-degenerate but indefinite since $\eta_{\mu\nu}$ has Minkowski signature, and one can verify explicitly using the commutation relations (9.69) that $\tilde{\mathcal{I}}$ is hermitian, i.e.

$$(\mathcal{G}\, I^T)^{(i,j)} = (I\, \mathcal{G})^{(i,j)} \tag{6.2}$$

where I is the matrix defined in (5.29). This provides a highly non-trivial consistency check.

Let us discuss the results in detail, assuming first $s \neq k \neq 0$. One can then check explicitly that all modes $\{v_\pm, v'_\pm\}$ (5.35), (5.37) are mutually orthogonal, as they must be. The norm of the vectors $v_\pm$ is obtained (e.g. using Mathematica) as follows

$$\langle v_+, v_+ \rangle = r^4 \frac{2(k-s)^2\left(\mathcal{K} + (s-k)^2\right)}{k^2(k+1)(2s-k)} \left(\mathcal{K} - R^2\Box + s^2 + k - 1\right) . \tag{6.3}$$

Here $\Box$ is understood to act[13] on ϕ resp. ϕ', as resulting from the inner product formulas in section 9.9. The factor $(\mathcal{K} + (s-k)^2)$ is positive since $\mathcal{K} > 0$, due to lemma 5.1. The factor

$$(\mathcal{K} - R^2\Box + s^2 + k - 1) = r^{-2}\Box_H - k(2s-k-1) - 2s - 2 \tag{6.4}$$

(using (9.76)) is positive using the estimate

$$r^{-2}\Box_H \geq k(2s-k-1) + 2s + 2 , \tag{6.5}$$

[12]It is important to observe that the modes are integrable over the entire $\mathcal{M}^{3,1}$, rather than just the space-like H^3. The reason is that we consider the principal series unitary irreps of $SO(4,2)$ in $End(\mathcal{H}_n)$, which correspond to square-integrable tensor fields on H^4. This allows to use invariance relations such as $(D^-)^\dagger = -D^+$. Although semi-classically one could define an inner product based on H^3, this would not make sense in the fully NC case.

[13]recall that $\Box$ commutes with $\mathcal{K}$ and $\Box_H$.

which follows from the admissibility condition (3.33) for $\Box_H$

$$r^{-2}\Box_H > s^2 + s + 2 \geq k(2s-k-1) + 2s + 2 \tag{6.6}$$

which reduces to $(s-k)(s-k-1) \geq 0$. Therefore $\langle v_+, v_+\rangle > 0$, and similarly

$$\langle v_-, v_-\rangle = r^4 \frac{2(k-s)^2\left(\mathcal{K} + (s-k)^2\right)}{k(2s-k+1)(2s-k)^2}\left(\mathcal{K} - R^2\Box + s^2 - k + 2s - 1\right) > 0\,. \tag{6.7}$$

Now consider the $v'_\pm$ modes. Since they are complexified, we refrain from computing their scalar product. The overall signature of $\mathcal{G}^{(i,j)}$ can be determined more easily from the determinant of the full inner product matrix (6.8), which is found to be

$$\begin{aligned}\det(\mathcal{G}^{(i,j)}) = r^{16}\, &\frac{d_{+-}(k+1)(2s-k+1)(k-s)^2}{(4s^2-1)\left(4s(s+2)+3\right)^2}\mathcal{K}\left(\mathcal{K} + (s+1)^2\right)\cdot \\ &\cdot\left((-R^2\Box + \mathcal{K} + s^2 + s - 1)^2 - (s-k)^2\right)\cdot \\ &\cdot\left((R^2\Box + k^2 - 2ks + 1 - s)^2 + \mathcal{K}\right)\,.\end{aligned} \tag{6.8}$$

The first factor in the second line arises from the $v_\pm$ modes, and is positive as shown above. The last factor arises from the $v'_\pm$ modes and is also positive. Using $d_{+-} < 0$ and $\mathcal{K} > 0$ we obtain

Lemma 6.1. *In any 4-dimensional space of modes $\tilde{\mathcal{A}}^{(i)}[\phi]$, $i \in \{+,-,n,g\}$ for admissible $\phi \in \mathcal{C}^{(s,k)}$ with $s \neq k$ and $k \neq 0$, the metric $\mathcal{G}^{(i,j)}$ is non-degenerate with signature* $(+++-)$.

This is the core of the no-ghost theorem, as discussed below. The special cases $k = 0$ and $s = k$ will be discussed separately below. Off-shell, the signature $(+++-)$ should be important e.g. in the context of loop computations and to establish perturbative unitarity and causality statements.

Explicit inner product for v'_{12} modes. We can compute the inner product for the $v'_{1,2}$ modes defined in (5.38). Their inner product is given by

$$\langle v'_i, v'_j\rangle = r^4 \frac{(2s+1)(-\mathcal{K})\big(\mathcal{K}+s^2-k(k-2s)\big)}{(\mathcal{K}+s^2)^2(\mathcal{K}+(s+1)^2)} \begin{pmatrix} a & b \\ b & -\frac{a}{\mathcal{K}} \end{pmatrix},$$

$$a = -\frac{\mathcal{K}+s^2}{2s+1}\big(R^2\Box + k^2 - 2ks + s + 2\big) + s\big(R^2\Box + k^2 - 2ks + 1\big) \ ,$$

$$b = -R^2\Box - 1 - k^2 + 2ks + \frac{\mathcal{K}+s^2}{2s+1} \tag{6.9}$$

noting that

$$k(2s-k)(\mathcal{K}+s^2) = -(4s^2-1)d_{+-} \ . \tag{6.10}$$

Here the 2×2 matrix has negative determinant

$$\det\begin{pmatrix} a & b \\ b & -\frac{a}{\mathcal{K}} \end{pmatrix} = -\frac{1}{\mathcal{K}(2s+1)^2}(\mathcal{K}+s^2)(\mathcal{K}+(s+1)^2)\cdot \\ \cdot\big(\mathcal{K}+(R^2\Box+k^2-2ks-s+1)^2\big) < 0 \ , \tag{6.11}$$

and we recognize the last factor from (6.8). One could now select a canonical basis of two null vectors if desired.

6.1. The primal sector $k = 0$

For primal modes $k=0$, the $\tilde{\mathcal{A}}^{(+)}[\phi^{(s,0)}] = \mathcal{A}^{(+)}[D^-\phi^{(s,0)}] = 0$ mode vanishes, and the inner product matrix simplifies accordingly. One can check again that the 3 eigenmodes $v'_\pm, v_0$ in (5.43), (5.45) are mutually orthogonal, with

$$\langle v_0, v_0\rangle = \frac{r^4 s}{(2s+1)^2}\big(\mathcal{K}+s^2\big)\big(-R^2\Box+\mathcal{K}+s^2-1\big) \ . \tag{6.12}$$

This is again positive for admissible on-shell modes using $\mathcal{K}>0$. The determinant of the inner product matrix for these 3 modes is

$$\det(\mathcal{G}^{(i,j)}) = r^{12}\frac{d_{-+}s}{(2s+1)^2(2s+3)}\cdot \\ \cdot\mathcal{K}\big(\mathcal{K}-R^2\Box+s^2-1\big)\big(\mathcal{K}+(1+R^2\Box-s)^2\big) \ . \tag{6.13}$$

Since $d_{-+}<0$, it follows as in (6.4) ff. that the determinant is negative for all admissible modes.

Now recall the extra special mode $\mathcal{A}_\mu^{(-)}[\phi^{(s,0)}]$ (5.47). It is easy to see from the explicit formulas for the inner products in section 9.9 that this mode is orthogonal to all other modes, and its inner product is positive as already observed in [1]. Therefore we have

Lemma 6.2. *In any 3-dimensional space of modes $\tilde{\mathcal{A}}^{(i)}[\phi]$, $i \in \{-, n, g\}$ for admissible $\phi \in \mathcal{C}^{(s,0)}$ with $s \neq 0$, the metric $\mathcal{G}^{(i,j)}$ is non-degenerate with signature $(++-)$. These 3 modes are orthogonal to $\mathcal{A}^{(-)}[\phi^{(s,0)}]$, which has positive norm.*

Explicit inner product for v'_{12} modes. We can compute the inner product for the $v'_{1,2}$ modes defined in (5.46). Their inner product is given by

$$\begin{aligned} \langle v'_i, v'_j\rangle &= \frac{r^4(2s+1)(-\mathcal{K})}{(\mathcal{K}+(s+1)^2)(\mathcal{K}+s^2)} \begin{pmatrix} a & b \\ b & -\frac{a}{\mathcal{K}} \end{pmatrix}, \\ a &= -\frac{\mathcal{K}+(s+1)^2}{2s+1}\left(R^2\Box+s+2\right)+(s+1)\left(R^2\Box+2\right) \\ b &= -R^2\Box-2+\frac{\mathcal{K}+(s+1)^2}{2s+1} \end{aligned} \tag{6.14}$$

noting that

$$-\mathcal{K} = (s+1)^2+(2s+3)d_{-+} \ . \tag{6.15}$$

The 2×2 matrix again has negative determinant,

$$\det\begin{pmatrix} a & b \\ b & -\frac{a}{\mathcal{K}} \end{pmatrix} = -\frac{1}{\mathcal{K}}(\mathcal{K}+(s+1)^2)(\mathcal{K}+s^2)\left(\mathcal{K}+(1+R^2\Box-s)^2\right) < 0 \ , \tag{6.16}$$

and we recognize the last factor from (6.13). A basis of two null vectors can be found if desired.

The $s=0$ sector. As discussed above there are only two modes $v'_\pm$ in this case, since $\mathcal{A}_\mu^{(n)} \equiv 0$ vanishes identically. The considerations of the $v'_{1,2}$ modes defined in (5.46) goes through, and the determinant of the 2×2 inner product matrix is still given by (6.16) evaluated at $s=0$,

$$\det\begin{pmatrix} a & b \\ b & -\frac{a}{\mathcal{K}} \end{pmatrix} = -(\mathcal{K}+1)\left(\mathcal{K}+(1+R^2\Box)^2\right) < 0 \ . \tag{6.17}$$

Hence the signature is $(+-)$, and we obtain

Lemma 6.3. *In any 2-dimensional space of modes* $\tilde{\mathcal{A}}^{(i)}[\phi],\ i \in \{-, g\}$ *for admissible* $\phi \in \mathcal{C}^{(0,0)}$*, the metric* $\mathcal{G}^{(i,j)}$ *is non-degenerate with signature* $(+-)$.

6.2. The scalar sector $k = s \neq 0$

In this case, (6.8) gives $\det(\mathcal{G}^{(i,j)}) = 0$. This means that there is a null mode, which is of course precisely the mode found in (5.49). In fact we show in section 9.7 that it vanishes identically,

$$v_\pm = (\frac{1}{s^2}, \frac{2s+3}{1+s}, -\frac{2s+1}{s}, 1) \;= v_{\mathrm{null}} = 0\ . \tag{6.18}$$

One can check that the extra mode (5.50)

$$v_{\mathrm{extra}} = -2\left(\frac{2}{s^2}, \frac{2s+3}{(s+1)^2}, -\frac{2s+1}{s^2}, 0\right) \tag{6.19}$$

is orthogonal to both $v'_\pm$, and its inner product is positive,

$$\langle v_{\mathrm{extra}}, v_{\mathrm{extra}}\rangle = \frac{r^4}{s^3(1+s)}\mathcal{K}(\mathcal{K} - \Box R^2 + s^2 + s - 1) > 0\ . \tag{6.20}$$

However, we will see that v_{extra} is not physical. This extra mode also explains why there is only one factor $(k-s)^2$ in $\det(\mathcal{G}^{(i,j)})$ (6.8), which arises from the inner products of either $v_\pm$ (6.3). The $v'_\pm$ modes can again be replaced by $v'_{1,2}$ (5.38), and the inner product in the space spanned by $v'_{1,2}$ has signature $(+-)$, which can be inferred from

$$\det\langle v'_i, v'_j\rangle = -\frac{r^8\mathcal{K}^3\left(\mathcal{K} + (R^2\Box - s(s+1) + 1)^2\right)}{(\mathcal{K}+s^2)^3\left(\mathcal{K} + (s+1)^2\right)} < 0 \tag{6.21}$$

as before, using $s^2(\mathcal{K}+s^2) = -(4s^2-1)d_{+-}$. This means that there are 3 linearly independent modes $\{v'_{1,2}, v_{\mathrm{extra}}\}$ whose metric has signature $(-++)$, and we have established

Lemma 6.4. *In any 3-dimensional space of modes* $\tilde{\mathcal{A}}^{(i)}[\phi],\ i \in \{+, -, g\}$ *for admissible* $\phi \in \mathcal{C}^{(s,s)}$ *with* $s \neq 0$*, the metric* $\mathcal{G}^{(i,j)}$ *is non-degenerate with signature* $(++-)$.

These modes are equivalently spanned by $v_{\mathrm{extra}}, v'_1, v'_2$, while the $\tilde{\mathcal{A}}^{(n)}[\phi]$ mode is a linear combination of these modes via (6.18).

To summarize, we have identified the following scalar modes:

$\mathcal{A} \in \mathcal{C}^0$. The scalar modes $\mathcal{A} \in \mathcal{C}^0$ are given by $\tilde{\mathcal{A}}^{(-)}[\phi]$ and $\mathcal{A}^{(g)}[\phi]$ for $\phi \in \mathcal{C}^0$, with non-degenerate metric with signature $(+-)$. The mode $\mathcal{A}^{(n)}[\phi]$ vanishes identically.

$\mathcal{A} \in \mathcal{C}^s$. The scalar modes $\mathcal{A} \in \mathcal{C}^s$ for $s \neq 0$ are given by $\tilde{\mathcal{A}}^{(-)}[\phi]$ and $\tilde{\mathcal{A}}^{(+)}[\phi]$ and $\tilde{\mathcal{A}}^{(g)}[\phi]$ for $\phi = \phi^{(s,s)} = (D^+)^s\phi^{(0)}$, with non-degenerate metric with signature $(++-)$. We will see that for $s = 1$, the only physical mode in this sector leads to scalar metric perturbations, and in particular to the linearized Schwarzschild solution [2].

6.3. Completeness

Now we want to understand whether the above modes are complete, i.e. if they span the space of all fluctuations $\mathcal{A}$. This will be addressed by counting the number of degrees of freedom (dof), i.e. real scalar fields on $\mathcal{M}^{3,1}$, at each sector $\mathcal{A}_\mu \in \mathcal{C}^s \otimes \mathbb{R}^4$.

$\mathcal{A}_\mu \in \mathcal{C}^0 \otimes \mathbb{R}^4$. This sector clearly contains 4 dof. Among the above modes, only the spin 1 mode $\mathcal{A}^{(-)}[\phi^{(1)}]$ and the spin 0 modes $\mathcal{A}^{(g)}[\phi^{(0)}]$ are in $\mathcal{C}^0 \otimes \mathbb{R}^4$, while $\mathcal{A}^{(n)}[\phi^{(0)}]$ vanishes. It follows from the previous considerations that all these modes are independent. Now $\mathcal{A}_\mu^{(-)}[\phi^{(1)}]$ i.e. $\phi^{(1)}$ encodes the most general space-like vector field on $\mathcal{M}^{3,1}$, cf. (5.48), which amounts to 3 degrees of freedom. Together with the spin 0 mode $\mathcal{A}^{(g)}[\phi^{(0)}]$ we obtain 4 dof, which is precisely the content of $\mathcal{C}^0 \otimes \mathbb{R}^4$. It follows that the above list of modes is complete. These modes are elaborated explicitly in section 9.2.

$\mathcal{A}_\mu \in \mathcal{C}^s \otimes \mathbb{R}^4$. This sector contains $4(2s+1)$ dof. It is convenient to ignore the (s,k) substructure of the $\mathcal{C}^s$ here. Among the above modes, $\mathcal{A}^{(-)}[\phi^{(s+1)}]$, $\mathcal{A}^{(n)}[\phi^{(s)}], \mathcal{A}^{(g)}[\phi^{(s)}]$ and $\mathcal{A}^{(+)}[\phi^{(s-1)}]$ are in $\mathcal{C}^s \otimes \mathbb{R}^4$. If they were all independent, this would provide all the $(2s+3) + 2(2s+1) + (2s-1) = 4(2s+1)$ dof. The above results show that these modes are linearly independent *except* for the scalar sector discussed in section 6.2, which provides only 3 rather than 4 modes due to the relation (6.18). Therefore there must be *one exceptional scalar dof* for each $s \geq 1$,

$$\mathcal{A}^{(ex,s)} \in \mathcal{C}^s \otimes \mathbb{R}^4, \qquad s \geq 1 \ . \tag{6.22}$$

Since none of the regular scalar modes $\tilde{\mathcal{A}}^{(i)}$ is null, we can choose $\mathcal{A}^{(ex,s)}$ to be orthogonal to all $\tilde{\mathcal{A}}^{(i)}$. Due to the explicit form of the $\mathcal{A}^{(i)}$, this implies that the $\mathcal{A}^{(ex,s)}$ can be chosen as follows

$$\{t^\mu, \mathcal{A}_\mu^{(ex,s)}\} = 0 = \{x^\mu, \mathcal{A}_\mu^{(ex,s)}\} , \qquad \mathcal{A}^{(ex,s)} = (D^+)^{s-1}\mathcal{A}^{(ex,1)} . \tag{6.23}$$

Orthogonality implies that this sector is respected by $\mathcal{D}^2$, and the physical constraint is satisfied. Further details are discussed in appendix 9.8, however the explicit form of $\mathcal{A}^{(ex,s)}$ is not known.

Taking these exceptional modes into account, we have recovered all $4(2s+1)$ dof in $\mathcal{C}^s \otimes \mathbb{R}^4$, so that the list of modes is complete. Together with the above lemmas, we have shown

Theorem 6.5. *The $\tilde{\mathcal{A}}^{(i)}[\phi^{(s)}]$ modes (5.23) along with the $\mathcal{A}^{(-)}[\phi^{(s,0)}]$ for all $s \geq 0$ and the exceptional modes $\mathcal{A}^{(ex,s)}$ for $s \geq 1$ span the space of all fluctuations $\mathcal{A}$. A basis is obtained by dropping $\tilde{\mathcal{A}}^{(n)}[\phi^{(s,s)}]$ and $\tilde{\mathcal{A}}^{(+)}[\phi^{(s,0)}]$.*

From a representation theory point of view, we have essentially decomposed the tensor product

$$\mathcal{C} \otimes \mathbb{R}^4 = \oplus(\dots) \tag{6.24}$$

into $SO(3,1)$ irreps. It is natural to expect that that each irrep in $\mathcal{C}$ arises with multiplicity 4 on the rhs, and we have seen that this holds indeed for the regular modes. However for non-compact Lie groups, the appearance of extra modes $\mathcal{A}^{(ex,s)}$ in the tensor product is not too surprising.

6.4. Physical constraint, Hilbert space and no ghost

We first observe that an (admissible, i.e. integrable) fluctuation mode $\mathcal{A}$ satisfies the gauge-fixing condition $\{t^\mu, \mathcal{A}_\mu\} = 0$ if and only if it is orthogonal to all pure gauge modes,

$$\langle \mathcal{A}^{(g)}, \mathcal{A} \rangle = 0 . \tag{6.25}$$

Now consider an on-shell mode $\mathcal{A} \in \mathcal{C}^s$ in some 4-dimensional mode space $\tilde{\mathcal{A}}^{(i)}[\phi]$, $i \in \{+ - ng\}$ determined by some $\phi \in \mathcal{C}^{(s,k)}$ with $\Box\phi = 0$ and $s > k > 0$. Since that 4-dimensional space of modes has signature $(+ + + -)$ due to Lemma 6.1 and $\mathcal{A}^{(g)}$ is null, the gauge-fixing constraint (6.25) leads to a 3-dimensional subspace with signature $(+ + 0)$, which contains $\mathcal{A}^{(g)}$. Then the usual definition

$$\mathcal{H}_{\rm phys} = \{\text{gauge-fixed on-shell modes}\}/_{\{\text{pure gauge modes}\}} \tag{6.26}$$

leads to 2 modes with positive norm. This establishes the generic part of

Theorem 6.6. *The space $\mathcal{H}_{\rm phys}$ (6.26) of admissible solutions of $\big(\mathcal{D}^2 - \frac{3}{R^2}\big)\mathcal{A} = 0$ which are gauge-fixed $\{t^\mu, \mathcal{A}_\mu\} = 0$ modulo pure gauge modes inherits a positive-definite inner product, and forms a Hilbert space.*

Proof. The same argument works for the on-shell modes $\tilde{\mathcal{A}}^{(i)}[\phi] \in \mathcal{C}^s$ with primal $\phi \in \mathcal{C}^{(s,0)}$. For $s \neq 0$ there are 2 physical modes. One is given by a linear combination of the $\tilde{\mathcal{A}}^{(i)}[\phi]$, $i \in \{-ng\}$ which has signature $(+ + -)$ before gauge fixing. In addition there is an extra on-shell physical mode $\mathcal{A}^{(-)}[\phi^{(s,0)}] \in \mathcal{C}^{s-1}$ for $\big(\Box - \frac{2s}{R^2}\big)\phi^{(s,0)} = 0$ (5.26).

For $s = 0$, no physical mode arises from the $\tilde{\mathcal{A}}^{(i)} \in \mathcal{C}^0$ with $i \in \{-g\}$ which has signature $(+-)$ before gauge fixing, due to Lemma 6.3. For the scalar on-shell modes $\tilde{\mathcal{A}}^{(i)}[\phi] \in \mathcal{C}^s$, $i \in \{+ - g\}$ with $\phi \in \mathcal{C}^{(s,s)}$, there is one physical linear combination according to Lemma 6.4. Finally, the exceptional modes $\mathcal{A}^{(ex,s)}$ (6.23) are physical, and their norm is positive because the $2s + 1$ dof in $\mathcal{C}^s \otimes \mathbb{R}^4$ with negative norm are already accounted for by the regular modes as shown above.

The admissibility condition implies square-integrability as discussed in (3.47). Together with the completess theorem 6.5, the statement follows.

$\Box$

Observe that the inner product (6.1) for vector modes is precisely realized in the quadratic action (4.10). Hence the above theorem is tantamount to the statement that the quadratic action is free of ghosts, i.e. physical modes with negative norm. Although the result is established only at the semi-classical (Poisson) level, most of the steps would go through in the non-commutative case using the $\mathfrak{so}(4,2)$-covariant quantization map $\mathcal{Q}$ (2.13), with minor adaptions due to the cutoff. Hence we expect that the theorem holds also in the non-commutative case.

There is no obstacle to determine $\mathcal{H}_{\rm phys}$ explicitly. It turns out that none of the modes $v_\pm$ and $v'_\pm$ satisfy the physical constraint, hence non-trivial combinations are required, and we can just as well us the $\tilde{\mathcal{A}}^{(i)}$ modes. A simplification arises for low spin, since the $\mathcal{A}^{(-)}[\phi^{(2,*)}] \in \mathcal{C}^1$ modes are all physical due to (9.1). This leads to the following sectors of $\mathcal{H}_{\rm phys}$:

The physical modes $\mathcal{A}_\mu \in \mathcal{C}^0$. As explained above, the off-shell modes $\mathcal{A}_\mu \in \mathcal{C}^0$ comprise the spin 1 mode $\mathcal{A}^{(-)}[\phi^{(1)}]$ and the spin 0 modes $\mathcal{A}^{(g)}[\phi^{(0)}]$

are in $\mathcal{C}^0 \otimes \mathbb{R}^4$. These modes are elaborated explicitly in section 9.2. Among these, only the spin 1 modes $\mathcal{A}^{(-)}[\phi^{(1,0)}]$ are physical, and

$$\mathcal{H}_{\rm phys} \cap \mathcal{C}^0 = \{\mathcal{A}^{(-)}[\phi] \text{ for } \phi \in \mathcal{C}^{(1,0)},\ (\Box - \frac{2}{R^2})\phi = 0\}\ . \tag{6.27}$$

These modes satisfy $\partial^\mu \mathcal{A}_\mu = 0 = x^\mu \mathcal{A}_\mu$, and describe a spin 1 Yang-Mills (or Maxwell) field.

The physical modes $\mathcal{A}_\mu \in \mathcal{C}^1$. They arise from the 12 off-shell modes $\mathcal{A}^{(-)}[\phi^{(2)}]$, $\mathcal{A}^{(n)}[\phi^{(1)}]$, $\mathcal{A}^{(g)}[\phi^{(1)}]$ and $\mathcal{A}^{(ex,1)}$ modulo the relation (9.90). Among these, all $\mathcal{A}^{(-)}[\phi^{2)}]$ are physical due to (9.1), and so is the exceptional scalar mode $\mathcal{A}^{(ex,1)}$, whose on-shell condition is not known explicitly. We claim that there are no further physical states in this sector, so that

$$\begin{aligned}\mathcal{H}_{\rm phys} \cap \mathcal{C}^1 &= \{\mathcal{A}^{(-)}[\phi] \text{ for } \phi \in \mathcal{C}^{(2,*)},\ (\Box - \frac{4}{R^2})\phi = 0\} \\ &\cup \{\mathcal{A}^{(ex,1)};\ (\mathcal{D}^2 - \frac{3}{R^2})\mathcal{A}^{(ex,1)} = 0\}\ .\end{aligned} \tag{6.28}$$

They satisfy $\{t^\mu, \mathcal{A}_\mu\} = 0$, and $x^\mu \mathcal{A}_\mu[\phi^{(2,0)}] = 0$. To see this, note that $\mathcal{A}^{(n)}[\phi^{(1,0)}]$ is in the same tuple of primal spin 1 modes as $\tilde{\mathcal{A}}^{(-)}[\phi^{(1,0)}]$ and $\tilde{\mathcal{A}}^{(g)}[\phi^{(1,0)}]$ which contains only one physical mode due to Lemma 6.2, given by $\mathcal{A}^{(-)}[D^+\phi^{(1,0)}] = \tilde{\mathcal{A}}^{(-)}[\phi^{(1,0)}]$. Note that the on-shell condition in (6.28) for $\phi = D^+\phi^{(1,*)}$ is equivalent to $\Box\phi^{(1,*)} = 0$ due to (9.5). Similarly, $\mathcal{A}^{(+)}[\phi^{(0)}] \sim \tilde{\mathcal{A}}^{(+)}[D^+\phi^{(0)}]$ is in the same tuple of scalar modes as $\tilde{\mathcal{A}}^{(-)}[D^+\phi^{(0)}]$ and $\tilde{\mathcal{A}}^{(g)}[D^+\phi^{(0)}]$, and due to Lemma 6.4 only $\tilde{\mathcal{A}}^{(-)}[D^+\phi^{(0)}] = \mathcal{A}^{(-)}[D^+D^+\phi^{(0)}]$ is physical. Again the on-shell condition in (6.28) for $\phi = D^+D^+\phi^{(0)}$ is equivalent to $\Box D^+\phi^{(0)} = 0$.

The modes in (6.28) govern the linearized gravity sector, as discussed below.

The physical modes $\mathcal{A}_\mu \in \mathcal{C}^s$ with $s \geq 2$. For the regular modes, the physical constraint $\{t^\mu, \mathcal{A}_\mu\} = 0$ must be solved directly. To determine $\mathcal{H}_{\rm phys}$, we can drop any contribution from $\mathcal{A}^{(g)}$. The simplest case is the mode (5.26) which is always physical due to (9.3),

$$\{\mathcal{A}^{(-)}[\phi^{(s+1,0)}] \text{ for } (\Box - \frac{2(s+1)}{R^2})\phi^{(s+1,0)} = 0\} \ \subset\ \mathcal{H}_{\rm phys} \cap \mathcal{C}^s\ . \tag{6.29}$$

All other modes are contained in some regular $\tilde{\mathcal{A}}^{(i)}$ tuple, and we need to work a bit harder. The gauge fixing constraint for the $\mathcal{A}^{(\pm)}$ modes is given in (9.3), and for the $\mathcal{A}^{(n)}$ mode it is

$$\begin{aligned}\{t^\mu, \mathcal{A}_\mu^{(n)}[\phi^{(s)}]\} &= \{t^\mu, D^+\mathcal{A}_\mu^{(-)}[\phi^{(s)}]\} \\ &= D^+\{t^\mu, \mathcal{A}_\mu^{(-)}[\phi^{(s)}]\} - \frac{1}{R}\{x^\mu, \mathcal{A}_\mu^{(-)}[\phi^{(s)}]\}_+ \\ &= \frac{-s+2}{R} D^+D^-\phi^{(s)} - \frac{1}{R}\{x^\mu, \mathcal{A}_\mu^{(-)}[\phi^{(s)}]\}_+ \\ &= \frac{1}{R}\Big((-s+3)D^+D^- + \alpha_s(\Box_H - 2r^2(s+1))\Big)\phi^{(s)} \end{aligned} \tag{6.30}$$

using (9.50), consistent with (9.106). We should hence determine all on-shell linear combinations

$$\mathcal{A}_\mu^{(\mathrm{phys})}[\phi] = c_+\tilde{\mathcal{A}}^{(+)}[\phi] + c_-\tilde{\mathcal{A}}^{(-)}[\phi] + c_n\mathcal{A}^{(n)}[\phi], \qquad \Box\phi = 0 \tag{6.31}$$

for $\phi \in \mathcal{C}^s$ which satisfy the gauge-fixing constraint

$$\begin{aligned} 0 &= R\{t^\mu, \mathcal{A}_\mu^{(\mathrm{phys})}[\phi]\} \\ &= \Big(c_+(s+2)D^+D^- + c_-(-s+1)D^-D^+ \\ &\qquad + c_n\big(\alpha_s(\Box_H - 2r^2(s+1)) + (-s+3)D^+D^-\big)\Big)\phi\ . \end{aligned} \tag{6.32}$$

Replacing $\Box_H$ on-shell using (9.76) allows to recast this into a 3-dimensional constraint on H^3, but does not lead to a simple expression. The first two terms are non-trivial since $s \geq 2$.

Consider first the primal tuple $\tilde{\mathcal{A}}^{(i)}[\phi^{(s,0)}]$ for $i = -, n, g$. This contains one physical mode due to Lemma 6.2, which we can choose to be a linear combination with $c_n = 1$,

$$\{c_-\tilde{\mathcal{A}}^{(-)}[\phi^{(s,0)}] + \tilde{\mathcal{A}}^{(n)}[\phi^{(s,0)}] \ \text{ for } \Box\phi^{(s,0)} = 0 \text{ and } (6.32)\} \ \subset\ \mathcal{H}_{\mathrm{phys}} \cap \mathcal{C}^s \tag{6.33}$$

where c_- is determined by solving the above constraint. Next, the scalar tuple $\tilde{\mathcal{A}}^{(i)}[\phi^{(s,s)}]$ for $i = +, -, g$ contains also one physical mode due to Lemma 6.4, which we can choose to be

$$\{\tilde{\mathcal{A}}^{(-)}[\phi^{(s,s)}] + c_+\tilde{\mathcal{A}}^{(+)}[\phi^{(s,s)}] \ \text{ for } \Box\phi^{(s,s)} = 0 \text{ and } (6.32)\} \ \subset\ \mathcal{H}_{\mathrm{phys}} \cap \mathcal{C}^s\ . \tag{6.34}$$

Next, the generic tuple $\tilde{\mathcal{A}}^{(i)}[\phi^{(s,k)}]$ for $i = +, -, n, g$ and $s \neq k \neq 0$ contains two physical modes due to Lemma 6.1, which we can choose to be

$$\begin{aligned} &\{\tilde{\mathcal{A}}^{(-)}[\phi^{(s,k)}] + c_+ \tilde{\mathcal{A}}^{(+)}[\phi^{(s,k)}] \ \text{ for } \Box\phi^{(s,k)} = 0 \text{ and } (6.32)\} \ \subset \ \mathcal{H}_{\text{phys}} \cap \mathcal{C}^s \\ &\{c_- \tilde{\mathcal{A}}^{(-)}[\phi^{(s,k)}] + \tilde{\mathcal{A}}^{(n)}[\phi^{(s,k)}] \ \text{ for } \Box\phi^{(s,k)} = 0 \text{ and } (6.32)\} \ \subset \ \mathcal{H}_{\text{phys}} \cap \mathcal{C}^s \end{aligned} \tag{6.35}$$

the first of which was found in [1]. Finally, the exceptional scalar modes $\mathcal{A}^{(ex,s)}$ are always physical, upon imposing the on-shell condition

$$\{\mathcal{A}^{(ex,s)};\ (\mathcal{D}^2 - \frac{3}{R^2})\mathcal{A}^{(ex,s)} = 0\} \quad \subset \ \mathcal{H}_{\text{phys}} \cap \mathcal{C}^s \tag{6.36}$$

This completes the list of physical modes.

Discussion. To summarize, the model contains generically 2 physical modes parametrized by $\phi^{(s)} \in \mathcal{C}^s$ with $\Box\phi^{(s)} = 0$ for each spin $s \geq 2$, up to the exceptional cases discussed above. The $\phi^{(s)}$ are "would-be massive" spin s modes, i.e. they contain the $2s+1$ dof of massive spin s multiplets with vanishing mass parameter, and they decompose further into a series of irreducible massless spin $s-k$ modes (in radiation gauge) as discussed in section 3.2. These modes mix under the higher-spin gauge transformations. It is hence plausible that some of these modes become massive in the interacting theory, which remains to be clarified.

Furthermore, we recall that at least for the regular modes, the above Hilbert space is determined uniquely be the wavefunction on any space-like slide H^3. More precisely, the 4-dimensional Casimir $C^2[\mathfrak{so}(4,1)]$ is determined on-shell by the space-like Casimir $C^2[\mathfrak{so}(3,1)]$. These statements apply also in the fully noncommutative case, resulting in a picture which is quite close to the usual setup in field theory.

7. Metric fluctuation modes

To illustrate the physical relevance of the above results, we briefly discuss how metric fluctuations arise from the above modes, elaborating on [1]. The effective metric for functions of $\mathcal{M}^{3,1}$ on a perturbed background $Y = T + \mathcal{A}$ can be extracted from the kinetic term in (3.9), which defines the bi-derivation

$$\begin{aligned} \gamma: \quad & \mathcal{C} \times \mathcal{C} \ \to \quad \mathcal{C} \\ & (\phi, \phi') \mapsto \{Y^\alpha, \phi\}\{Y_\alpha, \phi'\}\ . \end{aligned} \tag{7.1}$$

Specializing to $\phi = x^\mu, \phi' = x^\nu$ we obtain the coordinate form

$$\gamma^{\mu\nu} = \overline{\gamma}^{\mu\nu} + \delta_\mathcal{A}\gamma^{\mu\nu} + [\{\mathcal{A}^\alpha, x^\mu\}\{\mathcal{A}_\alpha, x^\nu\}]_0 \tag{7.2}$$

whose linearized contribution in $\mathcal{A}$ is given by

$$\delta_\mathcal{A}\gamma^{\mu\nu} = \sinh(\eta)\{\mathcal{A}^\mu, x^\nu\}_0 + (\mu \leftrightarrow \nu) \ . \tag{7.3}$$

The projection on $\mathcal{C}^0$ ensures that this is the metric for functions on $\mathcal{M}^{3,1}$. Clearly only $\mathcal{A} \in \mathcal{C}^1$ can contribute to $\delta_\mathcal{A}\gamma^{\mu\nu}$, which we assume henceforth. To evaluate this explicitly, it is convenient to consider the following rescaled graviton mode:

$$h^{\mu\nu}[\mathcal{A}] := \{\mathcal{A}^\mu, x^\nu\}_0 + (\mu \leftrightarrow \nu), \qquad h[\mathcal{A}] = 2\{\mathcal{A}^\mu, x_\mu\}_0 \ . \tag{7.4}$$

Including the conformal factor in (3.10), this leads to the effective metric fluctuation [2]

$$\delta G^{\mu\nu} = \beta^2\big(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\, h\big) \ . \tag{7.5}$$

Let us discuss the mode content of $h^{\mu\nu}[\mathcal{A}]$. Recall that the 12 off-shell dof in $\mathcal{A}_\mu = \mathcal{A}_{\mu;\alpha}t^\alpha \in \mathcal{C}^1$ are realized by $\mathcal{A}^{(-)}[\phi^{(2)}]$, $\mathcal{A}^{(n)}[\phi^{(1)}], \mathcal{A}^{(g)}[\phi^{(1)}]$, $\mathcal{A}^{(+)}[\phi^{(0)}]$ and $\mathcal{A}^{(ex,1)}$. Hence the 10 dof of the most general off-shell metric fluctuations are provided by $\mathcal{A}^{(-)}[\phi^{(2)}]$, $\mathcal{A}^{(g)}[\phi^{(1)}]$, and the scalar modes $\mathcal{A}^{(+)}[\phi^{(0)}]$ and $\mathcal{A}^{(ex,1)}$. The 6 physical metric fluctuations[14] arise from $\mathcal{A}^{(-)}[\phi^{(2)}]$ and $\mathcal{A}^{(ex,1)}$. According to the results of section 6.4, the 5 physical would-be massive modes $\mathcal{A}^{(-)}[\phi^{(2)}]$ decompose into the massless graviton $\mathcal{A}^{(-)}[\phi^{(2,0)}]$, one massless vector mode $\mathcal{A}^{(-)}[\phi^{(2,1)}]$, and one scalar mode $\mathcal{A}^{(-)}[\phi^{(2,2)}]$. The vector field can be extracted by

$$\begin{aligned}\{t_\mu, h^{\mu\nu}\} &= \{t_\mu, \{\mathcal{A}^\mu, x^\nu\}_-\} + \{t_\mu, \{\mathcal{A}^\nu, x^\mu\}_-\}\\ &= \{\{t_\mu, \mathcal{A}^\mu\}, x^\nu\}_- - \frac{2}{R}D^-\mathcal{A}^\nu\\ &\overset{phys}{=} -\frac{2}{R}D^-\mathcal{A}^\nu\end{aligned} \tag{7.6}$$

[14] In particular, even though $\mathcal{A}^{(n)}[\phi^{(1)}]$ encodes off-shell dof of a space-like 2-form in (5.1), it is not physical. However the 2-form may be determined by the metric modes arising from $\mathcal{A}^{(-)}[\phi^{(2,1)}]$, which are physical.

using the Jacobi identity and (9.1), which vanishes for the $\mathcal{A}^{(-)}[\phi^{(2,0)}]$ mode. Together with

$$(7.7)\quad \{t_\nu, \{t_\mu, h^{\mu\nu}\}\} = \{t_\nu, \{\{t_\mu, \mathcal{A}^\mu\}, x^\nu\}_-\} - \frac{2}{R}\{t_\nu, D^-\mathcal{A}^\nu\} \stackrel{phys}{=} -\frac{1}{R^2}h$$

we obtain

$$(7.8)\qquad \{t_\nu, \{t_\mu, h^{\mu\nu}\}\} + \frac{1}{R^2}h = 0 \qquad \text{for physical } \mathcal{A}\ .$$

This constraint is satisfied by the physical scalar metric mode arising from $\mathcal{A}^{(-)}[D^+D^+\phi]$, which underlies the linearized Schwarzschild solution [2].

8. Conclusions and outlook

The results of this paper demonstrate that the model under consideration defines a consistent and ghost-free higher spin gauge theory in 3+1 dimensions, at least at the linearized level. It leads to truncated towers of higher-spin modes, which include spin 2 fluctuation modes of the effective metric leading to Ricci-flat metric perturbations as shown in [1], and the linearized Schwarzschild solution as shown in [2]. Since it is defined in terms of a maximally supersymmetric Yang-Mills matrix model, it is plausible that this defines in fact a consistent quantum theory which includes gravity. The crucial feature in contrast to standard Yang-Mills theories is that space-time is not put in by hand, but emerges in the semi-classical limit from the background solution given in terms of 3+1 large (in fact infinite) matrices.

Let us briefly discuss briefly the quantization of the model. Even though the noncommutative space has only finitely many degrees of freedom per volume, it is not automatic that the theory is finite and approximately local, because of UV/IR mixing [23]. It is well-known that in NC field theories, the UV degrees of freedom are dominated by string-like modes, which have both IR and UV properties and violate the Wilsonian paradigm. In order to have a good locality *and* UV behavior, their contributions in loops must cancel. It is also known that in 4 dimensions, sufficient cancellations occur basically only in the maximally supersymmetric $\mathcal{N}=4$ case [28–30]. But this is precisely what happens in the IKKT matrix model. In fact, one can view the present model as noncommutative $\mathcal{N}=4$ SYM [31] with $\mathfrak{hs}$ - valued gauge fields, where $\mathfrak{hs}$ is the *finite* higher-spin-like “algebra” generated by $\theta^{\mu\nu}$ or t^μ. This suggests that the theory should be UV finite at all loops, and it is manifest from the generic formulas in [22, 27] that the one-loop effective action is indeed finite, cf. [32]. Of course the argument is not fully justified since $\mathfrak{hs}$ is not a standard Lie algebra but includes some x-dependence; nevertheless the similarity with $\mathcal{N} = 4$ SYM suggests that the present model might provide a UV-finite quantum theory including spin 2. This is certainly intriguing, and vindicates more detailed investigations.

Although the model is not yet sufficiently developed, it is tempting to compare and relate it with other approaches to quantum gravity. Conformal or quadratic gravity (cf. [33] and references therein) is reminiscent of Yang-Mills theory and is renormalizable [34], but contains ghosts. A similar issue may be expected in asymptotic safety scenarios [35]. In contrast, we have seen that the present model does *not* contain ghosts, as the fundamental degrees of freedom are different and arise from matrix fluctuations. String theory in its conventional formulation can claim to provide 9+1-dimensional (quantum) gravity, however compactification to 3+1 dimensions leads to a lack of predictivity known as the landscape problem. This is avoided in the IKKT model, which can be viewed as different, constructive approach to string theory. Hence the present matrix model and the type of background under consideration may provide the basis for a consistent and useful 3+1-dimensional quantum theory including gravity, however it remains to be seen whether the resulting physics is viable.

Acknowledgements. I would like to thank C-S Chu, C. Iazeolla, H. Kawai, J. Nishimura and E. Skvortsov for useful discussions, and I am grateful to E. Delay and W. Schlag for pointing me to the appropriate mathematical literature. This work was supported by the Austrian Science Fund (FWF) grant P32086-N27.

9. Appendix

9.1. Useful identities for the vector modes $\mathcal{A}$

We recall the following gauge-fixing identities for the vector modes $\mathcal{A}^\mu$

$$\begin{aligned}\{t^\mu, \mathcal{A}_\mu^{(+)}[\phi^{(s)}]\} &= \frac{s+3}{R} D^+\phi^{(s)}\,, \\ \{t^\mu, \mathcal{A}_\mu^{(-)}[\phi^{(s)}]\} &= \frac{-s+2}{R} D^-\phi^{(s)}\end{aligned} \tag{9.1}$$

for $\phi^{(s)} \in \mathcal{C}^s$, which follow from (A.34) in [1]

$$\begin{aligned} R\{t^\mu, \{x_\mu, \phi^{(s)}\}\} &= \frac{1}{2}\Big(\frac{1}{2}\mathcal{S}^2 - s(s+1) + 4\Big) D\phi^{(s)} \\ &= (s+3)D^+\phi^{(s)} + (-s+2)D^-\phi^{(s)} \\ R\{x_\mu, \{t^\mu, \phi^{(s)}\}\} &= (s-1)D^+\phi^{(s)} - (s+2)D^-\phi^{(s)} \ . \end{aligned} \tag{9.2}$$

In particular, we note

$$\begin{aligned} \{t^\mu, \mathcal{A}^{(+)}_\mu[D^-\phi^{(s)}]\} &= \frac{s+2}{R} D^+D^-\phi^{(s)} \,, \\ \{t^\mu, \mathcal{A}^{(-)}_\mu[D^+\phi^{(s)}]\} &= \frac{-s+1}{R} D^-D^+\phi^{(s)} \,. \end{aligned} \tag{9.3}$$

The time component of $\mathcal{A}^{(\pm)}$ along the vector field τ (3.8) can be obtained using (3.17) as

$$x^\mu \mathcal{A}^{(\pm)}_\mu[\phi^{(s)}] = -x_4 D^\pm \phi^{(s)} \ . \tag{9.4}$$

Intertwiner relations for $\Box$ and $\boldsymbol{\mathcal{A}}$. The following relations were shown in [1]

$$\begin{aligned} \Box D^-\phi^{(s)} &= D^-\Big(\Box - \frac{2s}{R^2}\Big)\phi^{(s)} \\ \Box D^+\phi^{(s)} &= D^+\Big(\Box + \frac{2s+2}{R^2}\Big)\phi^{(s)} \\ \Box D^+D^-\phi^{(s)} &= D^+D^-\Box\phi^{(s)} \end{aligned} \tag{9.5}$$

as well as

$$\begin{aligned} \tilde{\mathcal{I}}(\mathcal{A}^{(+)}_\mu[\phi^{(s)}]) &= r^2(s+3)\mathcal{A}^{(+)}_\mu[\phi^{(s)}] + r^2 R\{t_\mu, D^+\phi^{(s)}\} \\ \tilde{\mathcal{I}}(\mathcal{A}^{(-)}_\mu[\phi^{(s)}]) &= r^2(-s+2)\mathcal{A}^{(-)}_\mu[\phi^{(s)}] + r^2 R\{t_\mu, D^-\phi^{(s)}\} \end{aligned} \tag{9.6}$$

and

$$\begin{aligned} \mathcal{D}^2\mathcal{A}^{(+)}_\mu[\phi^{(s)}] &= \mathcal{A}^{(+)}_\mu\Big[\Big(\Box + \frac{2s+5}{R^2}\Big)\phi^{(s)}\Big] \\ \mathcal{D}^2\mathcal{A}^{(-)}_\mu[\phi^{(s)}] &= \mathcal{A}^{(-)}_\mu\Big[\Big(\Box + \frac{-2s+3}{R^2}\Big)\phi^{(s)}\Big] \,. \end{aligned} \tag{9.7}$$

Since $\mathcal{D}^2\mathcal{A} = (\Box + \frac{2}{r^2R^2}\tilde{\mathcal{I}})\mathcal{A}$, these two relations can be combined to obtain

$$\Box\mathcal{A}^{(\pm)}_\mu[\phi^{(s)}] = \mathcal{A}^{(\pm)}_\mu\Big[\Big(\Box - \frac{1}{R^2}\Big)\phi^{(s)}\Big] - \frac{2}{R}\mathcal{A}^{(g)}_\mu[D^\pm\phi^{(s)}] \ . \tag{9.8}$$

Intertwiner relations for $\Box_H$ and $\mathcal{A}$. The $SO(4,1)$ intertwiner relation

$$
\begin{aligned}
r^2 C^2[\mathfrak{so}(4,1)]^{(\mathrm{full})}\mathcal{A}_a[\phi^{(s)}] &= -(\Box_H + 2\mathcal{I}^{(5)} - r^2(\mathcal{S}^2+4))\mathcal{A}_a[\phi^{(s)}] \\
&= \mathcal{A}_a[r^2 C^2[\mathfrak{so}(4,1)]\phi^{(s)}]
\end{aligned}
$$

(cf. (D.30) in [5]) can be used to derive several useful identities for $\Box_H$. In particular for $\mathcal{A}_a = \mathcal{A}_a^{(\pm)}[\phi^{(s)}] = \{x_a, \phi^{(s)}\}_\pm$ and $a = 0, \ldots, 4$, one obtains

$$
\begin{aligned}
\Box_H \mathcal{A}_a^{(-)}[\phi^{(s)}] &= \mathcal{A}_a^{(-)}[(\Box_H - 2r^2 s)\phi^{(s)}] \\
\Box_H \mathcal{A}_a^{(+)}[\phi^{(s)}] &= \mathcal{A}_a^{(+)}[(\Box_H + 2r^2(s+1))\phi^{(s)}]
\end{aligned}
\tag{9.9}
$$

using

$$
\begin{aligned}
\mathcal{I}^{(5)}\mathcal{A}_a^{(-)}[\phi^{(s)}] &= r^2(2-s)\mathcal{A}_a^{(-)}[\phi^{(s)}] \quad \text{and} \\
\mathcal{I}^{(5)}\mathcal{A}_a^{(+)}[\phi^{(s)}] &= r^2(s+3)\mathcal{A}_a^{(+)}[\phi^{(s)}],
\end{aligned}
$$

cf. (5.48) in [5]. This implies for $a = 4$

$$
\begin{aligned}
\Box_H D^- \phi^{(s)} &= D^-((\Box_H - 2r^2 s)\phi^{(s)}) \\
\Box_H D^+ \phi^{(s)} &= D^+((\Box_H + 2r^2(s+1))\phi^{(s)}) \\
\Box_H D^+ D^- \phi^{(s)} &= D^+ D^- \Box_H \phi^{(s)} \ .
\end{aligned}
\tag{9.10}
$$

These are completely analogous to the relation for $\Box$ (9.5), and can also be checked directly. It is also easy to see (e.g. using their expression in terms of Casimirs) that

$$
[\Box_H, \Box] = 0 = [\Box_H, \mathcal{D}^2] \ . \tag{9.11}
$$

Together with (3.44), we also obtain

$$
\begin{aligned}
(-C^2[\mathfrak{so}(4,1)] + (s+1)(s+2))D^+\phi^{(s)} &= (\Box_H - r^2(s+1)(s+2))D^+\phi^{(s)} \\
&= D^+(\Box_H - r^2 s(s+1))\phi^{(s)} \\
&= D^+(-C^2[\mathfrak{so}(4,1)] + s(s+1))\phi^{(s)}
\end{aligned}
\tag{9.12}
$$

an similarly for D^-, which means via (3.43) that $D^\pm$ preserves admissible modes.

Evaluation of $\tilde{\mathcal{I}}(\mathcal{A}^{(g)})$. Consider for $\phi \in \mathcal{C}^{(s)}$

$$\begin{aligned}
\tilde{\mathcal{I}}(\mathcal{A}^{(g)}[\phi]) &= \{\theta^{\mu\nu}, \{t_\nu, \phi\}\} = \{\{x^\mu, x^\nu\}, \{t_\nu, \phi\}\} \\
&= -\{\{x^\nu, \{t_\nu, \phi\}\}, x^\mu\} - \{\{t_\nu, \phi\}, x^\mu\}, x^\nu\} \\
&= -\{\{x^\nu, \{t_\nu, \phi\}\}, x^\mu\} + \{\{\phi, x^\mu\}, t_\nu\}, x^\nu\} + \{\{x^\mu, t_\nu\}, \phi\}, x^\nu\} \\
&= -\frac{1}{R}\{(s-1)D^+\phi - (s+2)D^-\phi, x^\mu\} \\
&\quad - \{\{\mathcal{A}_\mu^{(+)}[\phi], t_\nu\}, x^\nu\} - \{\{\mathcal{A}_\mu^{(-)}[\phi], t_\nu\}, x^\nu\} - \frac{1}{R}\{D\phi, x^\mu\} \\
&= \frac{s}{R}\mathcal{A}^{(-)\mu}[D^+\phi] - \frac{(s+1)}{R}\mathcal{A}^{(+)\mu}[D^-\phi] \\
&\quad + \frac{(s+3)}{R}D^-\mathcal{A}_\mu^{(+)}[\phi] - \frac{(s-2)}{R}D^+\mathcal{A}_\mu^{(-)}[\phi]
\end{aligned}$$

using (9.2). Using the definition of $\mathcal{A}_\mu^{(n)}$ and (5.16), this gives

$$\begin{aligned}
R\,\tilde{\mathcal{I}}(\mathcal{A}^{(g)}[\phi]) &= (s+3)r^2 R\mathcal{A}^{(g)} + (2s+3)\mathcal{A}^{(-)\mu}[D^+\phi] \\
&\quad + 2\mathcal{A}^{(+)\mu}[D^-\phi] - (2s+1)\mathcal{A}_\mu^{(n)}[\phi] \ .
\end{aligned} \tag{9.13}$$

9.2. Explicit vector modes $\mathcal{A} \in \mathcal{C}^0$

We give explicitly the fluctuation modes discussed in section 6.3. For $\phi^{(1)} = \phi_\alpha t^\alpha$ we have

$$\begin{aligned}
\mathcal{A}_\mu^{(-)}[\phi^{(1)}] &= \{x^\mu, \phi_\alpha t^\alpha\}_0 = \partial_\nu\phi_\alpha[\theta^{\mu\nu}t^\alpha]_0 + \phi_\alpha\{x^\mu, t^\alpha\} \\
&= \frac{1}{3}\sinh(\eta)(x^\mu\partial^\alpha\phi_\alpha - (\tau+3)\phi_\mu) + \frac{1}{3}x_\beta\varepsilon^{\beta4\alpha\mu\nu}\partial_\nu\phi_\alpha
\end{aligned} \tag{9.14}$$

using (3.23). The last term is the 3-dimensional rotation on H^3. This vector field separates into the space-like divergence-free field

$$\begin{aligned}
\mathcal{A}_\mu^{(-)}[\phi^{(1,0)}] &= -\frac{1}{3}\sinh(\eta)(\tau+3)\phi_\mu + \frac{1}{3}x_\beta\varepsilon^{\beta4\alpha\mu\nu}\partial_\nu\phi_\alpha, \\
\partial^\mu\mathcal{A}_\mu &= 0 = x^\mu\mathcal{A}_\mu
\end{aligned} \tag{9.15}$$

(hence in radiation gauge) using (9.4), and the scalar mode

$$\begin{aligned}
\mathcal{A}_\mu^{(-)}[D\phi] &= \frac{r^2R}{3}\sinh(\eta)(x_\mu\partial^\alpha\partial_\alpha\phi - (\tau+3)\partial_\mu\phi), \\
\partial^\mu\mathcal{A}_\mu &= -\frac{1}{R^2\sinh^2(\eta)}x^\mu\mathcal{A}_\mu
\end{aligned} \tag{9.16}$$

using (9.1) for $\phi \in \mathcal{C}^0$, which is neither space-like nor divergence-free[15]. The remaining mode in $\mathcal{C}^0$ is the pure gauge mode

$$\mathcal{A}_\mu^{(g)}[\phi^{(0)}] = \{t_\mu, \phi^{(0)}\} = \sinh(\eta)\partial_\mu\phi^{(0)} \ . \tag{9.17}$$

This illustrates the sub-structure of tensor fields resulting from the reduced $SO(3,1)$ covariance. The only physical mode in this sector is $\mathcal{A}_\mu^{(-)}[\phi^{(1,0)}]$, which corresponds to a massless vector field.

9.3. Wick theorem for averaging over S^2

Lemma 9.1.

$$[t^{\alpha_1} \dots t^{\alpha_{2s}}]_0 = b_{2s} \sum_{i<j} [t^{\alpha_i} t^{\alpha_j}][t \dots t]_0, \qquad b_{2s} = \frac{3}{s(2s+1)} \tag{9.18}$$

i.e. $b_2 = 1, \quad b_4 = \frac{3}{10}, \quad b_6 = \frac{1}{7}$ etc.

Proof. The structure of the rhs follows from the fact that all totally symmetric $SO(3,1)$-invariant tensors are obtained from $\eta^{\alpha\beta}$. The constants b_{2s} can be determined either using a recursive combinatorial argument by contracting with $\eta_{\alpha_1\alpha_2}$, or implicitly & recursively from

$$\begin{aligned}
[t_3^{2s}]_0 &= \frac{1}{2} 2s(2s-1) b_{2s} [t_3 t_3][t_3^{2s-2}]_0 = \dots = \frac{1}{2^s} b_{2s} b_{2s-2} \dots b_2 (2s)! [t_3 t_3]^s \\
&= 3^s \frac{1}{2^s} [t_3 t_3]^s \frac{(2s)!}{s!(2s+1)(2s-1)\dots 1} \\
&= 3^s [t_3 t_3]^s \frac{(2s)! s!}{s!(2s+1)!} = 3^s [t_3 t_3]^s \frac{1}{2s+1} = \frac{\cosh^s(\eta)}{r^{2s}} \frac{1}{2s+1} \qquad (9.19)
\end{aligned}$$

at the reference point ξ. Taking into account the local radius of S^2, this agrees with (3.20)

$$\frac{1}{4\pi} \int_{S^2} \cos(\vartheta)^{2s} 2\pi \sin(\vartheta) d\vartheta = \frac{1}{2} \int_{-1}^{1} du u^{2s} = \frac{1}{2} \frac{1}{2s+1} [u^{2s+1}]_{-1}^{1} = \frac{1}{2s+1} \ . \tag{9.20}$$

□

We will also need the following variant of Wicks theorem:

---

[15]Incidentally, the explicit form of $\mathcal{A}_\mu^{(-)}[D\phi]$ shows that the scalar mode $\mathcal{A}_\mu^{(\tau)}[\phi] = x_\mu \phi$ as discussed in [2] is a linear combination of $\mathcal{A}_\mu^{(-)}$ and $\mathcal{A}_\mu^{(g)}$.

Lemma 9.2.

$$[t^{\alpha_1}\dots t^{\alpha_{s+1}}]_{s-1} = c_{s+1}\sum_{i<j}[t^{\alpha_i}t^{\alpha_j}][t\dots t]_{s-1}, \qquad c_{s+1} = \frac{3}{2s+1} \tag{9.21}$$

summing over all contractions, where $[.]_{s-1}$ *denotes the projection on* $\mathcal{C}^{s-1}$*.*

Proof. The constants c_{s+1} can be determined by contracting with $\eta_{\alpha_1\alpha_2}$:

(9.22)

$$\begin{aligned}
&[(t^\mu t_\mu)t^{\alpha_3}\dots t^{\alpha_{s+1}}]_{s-1} \\
&\quad = c_{s+1}\Big([t^\mu t_\mu][t^{\alpha_3}\dots t^{\alpha_{s+1}}]_{s-1} + \sum_i [t^\mu t][t_\mu t\dots t]_{s-1} + \sum_j [tt^\mu][t_\mu\dots t]_{s-1}\Big) \\
&\quad = c_{s+1}\frac{\cosh^2(\eta)}{r^2}\Big([t^{\alpha_3}\dots t^{\alpha_{s+1}}]_{s-1} + 2(s-1)\frac{1}{3}[t^{\alpha_3}\dots t^{\alpha_{s+1}}]_{s-1}\Big)
\end{aligned}$$

noting that no contractions can occur in the last term, and using (3.21)

$$[t^\alpha t^\mu]_0 t^\mu = \frac{\cosh^2(\eta)}{3r^2}P_\perp^{\alpha\mu}t_\mu = \frac{\cosh^2(\eta)}{3r^2}t_\alpha \tag{9.23}$$

as well as $t^\mu t_\mu = \frac{\cosh^2(\eta)}{r^2}$. Thus

$$\begin{aligned}
[t^{\alpha_3}\dots t^{\alpha_{s+1}}]_{s-1} &= c_{s+1}\Big([t^{\alpha_3}\dots t^{\alpha_{s+1}}]_{s-1} + \frac{2}{3}(s-1)[t^{\alpha_3}\dots t^{\alpha_{s+1}}]_{s-1}\Big) \\
&= c_{s+1}\big(1+\frac{2}{3}(s-1)\big)[t^{\alpha_3}\dots t^{\alpha_{s+1}}]_{s-1}
\end{aligned} \tag{9.24}$$

which implies (9.21). □

9.4. Computation of α_s

We want to show the useful formula

Lemma 9.3.

$$-\{x^a,\{x_a,\phi^{(s)}\}_-\}_+ = \alpha_s(\Box_H - 2r^2(s+1))\phi^{(s)}, \qquad \alpha_s = \frac{s}{2s+1}\ . \tag{9.25}$$

This formula was derived in [5] using the representation (3.26) for $s=1$, and for general s based on an indirect argument; however α_s was not yet found for $s > 2$. The structure of the formula is not surprising, since the lhs is a $SO(4,1)$-invariant 2nd order derivation on $\mathcal{C}^s$, which can only be $\Box_H$ up to some constants. Here we provide a direct proof, using the result for $s = 1$.

Proof. For $s = 1$, the formula (9.25) was proved in [5] for $\phi = \{x^a, \phi_a\}$ for any tangential divergence-free vector field ϕ_a, and it is not hard to see that all $\phi \in \mathcal{C}^1$ can be written in this way[16]. Using this result for $\phi = f\theta^{ab}$ as well as [5]

$$\Box_H = -r^2 R^2 \eth^d \eth_d \tag{9.26}$$

where $\eth$ is defined in (3.35), we obtain

$$\Box_H(f\theta^{ab}) = -r^2 R^2 f\theta^{ab}\eth^d\eth_d f - 2r^2 f\theta^{ab} - 2r^2 R^2(\eth^d\theta^{ab})\eth_d f \ . \tag{9.27}$$

On the other hand,

$$\begin{aligned} -\{x^c, \{x_c, f\theta^{ab}\}_-\}_+ &= -2r^2 f\theta^{ab} - r^2R^2(\eth^d\theta^{ab})\eth_d f - \{x^c, [\theta^{ab}\theta^{cd}]_0\eth_d f\} \\ &\stackrel{!}{=} \frac{1}{3}(\Box_H - 4r^2)(f\theta^{ab}) \\ &= -\frac{1}{3}r^2R^2\theta^{ab}\eth^d\eth_d f - 2r^2 f\theta^{ab} - \frac{2}{3}r^2R^2(\eth^d\theta^{ab})\eth_d f \end{aligned} \tag{9.28}$$

which gives the useful formula

$$-\{x^c, [\theta^{ab}\theta^{cd}]_0\eth_d f\} = \frac{1}{3}r^2R^2\big(- \theta^{ab}\eth^d\eth_d f + (\eth^d\theta^{ab})\eth_d f\big) \ . \tag{9.29}$$

Now consider the following constant modes

$$\phi^{(s)} = \phi_{a_1\ldots a_s;b_1\ldots b_s}\theta^{a_1b_1}\ldots\theta^{a_sb_s} \qquad \in \mathcal{C}^s \tag{9.30}$$

where $\phi_{a_1\ldots a_s;b_1\ldots b_s} \in \mathbb{C}$ are traceless with the symmetry of a Young diagram $\begin{array}{|c|c|c|}\hline a & a & a \\ \hline b & b & b \\ \hline\end{array}$. Then

$$-\{x^a, \phi^{(s)}\}_- = -s\phi_{a_1\ldots a_s;b_1\ldots b_s}\theta^{a_1b_1}\ldots\{x^a, \theta^{a_sb_s}\} \tag{9.31}$$

[16]For example, it suffices to show this for polynomial functions on $\mathbb{C}P^{1,2}$, for which the representation $\phi = \{x^a, \phi_a\}$ can be shown using Young diagrams along the lines in [6]. It is also easy to see that $\{x^a, \phi_a\} = 0$ for $\phi_a = \eth_a\phi$. For more details we refer to [5].

and

$$
\begin{aligned}
-\{x^a, \{x_a, \phi^{(s)}\}_-\}_+ &= -s\phi_{a_1\dots a_s;b_1\dots b_s}\theta^{a_1b_1}\dots\{x^a,\{x_a,\theta^{a_sb_s}\}\}_+ \\
&= -2r^2 s\phi^{(s)}
\end{aligned} \tag{9.32}
$$

since $-\{x^a, \{x_a, \theta^{bc}\}\} = -2r^2\theta^{bc}$. It is easy to see that this coincides with

$$
\Box_H \phi^{(s)} = -2r^2 s\phi^{(s)} \tag{9.33}
$$

because $\phi_{a_1\dots a_s;b_1\dots b_s}$ is traceless. Therefore

$$
-\{x^a, \{x_a, \phi^{(s)}\}_-\}_+ = \alpha_s(\Box_H - 2r^2(s+1))\phi^{(s)} = -2r^2(2s+1)\alpha_s\phi^{(s)} \tag{9.34}
$$

and we obtain

$$
\alpha_s = \frac{s}{2s+1} \ . \tag{9.35}
$$

Now consider general (non-constant) modes in $\mathcal{C}^s$ for $s \geq 2$. They are spanned by modes obtained by multiplying the above constant modes $\phi^{(s)}$ with some functions:

$$
f(x)\phi^{(s)} = f(x)\phi_{a_1\dots a_s;b_1\dots b_s}\theta^{a_1b_1}\dots\theta^{a_sb_s} \qquad \in \mathcal{C}^s \ . \tag{9.36}
$$

Then

$$
\begin{aligned}
-\{x^c, f\phi^{(s)}\} &= -s\phi_{a_1\dots a_s;b_1\dots b_s} f\theta^{a_1b_1}\dots\{x^c,\theta^{a_sb_s}\} \\
&\quad - \phi_{a_1\dots a_s;b_1\dots b_s}\theta^{a_1b_1}\dots\theta^{a_sb_s}\theta^{cd}\eth_d f
\end{aligned} \tag{9.37}
$$

(note that there is no factor s in the second term), and

$$
\begin{aligned}
-\{x^c, f\phi^{(s)}\}_- &= -s\phi_{a_1\dots a_s;b_1\dots b_s} f\theta^{a_1b_1}\dots\{x^c,\theta^{a_sb_s}\} \\
&\quad - sc_{s+1}\phi_{a_1\dots a_s;b_1\dots b_s}\theta^{a_1b_1}\dots[\theta^{a_sb_s}\theta^{cd}]_0\eth_d f
\end{aligned} \tag{9.38}
$$

using tracelessness, where (9.21)

$$
c_{2s+1} = \frac{3}{2s+1} \ . \tag{9.39}
$$

Now consider first

$$\begin{aligned}
-\{x_c,\{x^c,f\phi^{(s)}\}\} &= -s\phi_{a_1\ldots a_s;b_1\ldots b_s} f\theta^{a_1b_1}\ldots \underbrace{\{x_c,\{x^c,\theta^{a_sb_s}\}\}}_{2r^2\theta^{a_sb_s}} \\
&\quad - 2s\phi_{a_1\ldots a_s;b_1\ldots b_s}\theta^{a_1b_1}\ldots\{x_c,\theta^{a_sb_s}\}\theta^{cd}\eth_d f \\
&\quad - s(s-1)\phi_{a_1\ldots a_s;b_1\ldots b_s} f\,\underbrace{\{x_c,\theta^{a_1b_1}\}\ldots\{x^c,\theta^{a_sb_s}\}}_{0} \\
&\quad - \phi_{a_1\ldots a_s;b_1\ldots b_s}\theta^{a_1b_1}\ldots\theta^{a_sb_s}\{x_c,\theta^{cd}\eth_d f\} \\
&= -r^2R^2\phi^{(s)}\eth^d\eth_d f - 2sr^2 f\phi^{(s)} \\
&\quad - 2s\phi_{a_1\ldots a_s;b_1\ldots b_s}\theta^{a_1b_1}\ldots\{x_c,\theta^{a_sb_s}\}\theta^{cd}\eth_d f \\
&= -r^2R^2\phi^{(s)}\eth^d\eth_d f - 2sr^2 f\phi^{(s)} \\
&\quad - 2sr^2R^2\phi_{a_1\ldots a_s;b_1\ldots b_s}\theta^{a_1b_1}\ldots(\eth^d\theta^{a_sb_s})\eth_d f
\end{aligned}$$

using

$$\{x_c,\theta^{cd}\eth_d f\} = \{x_c,\theta^{cd}\}\eth_d f + \theta^{cd}\{x_c,\eth_d f\} = r^2R^2\eth^d\eth_d f \tag{9.40}$$

since $x^d\eth_d = 0$. We observe that the last term is in $\mathcal{C}^s$. That formula could be obtained simply from (9.26), but the intermediate steps are useful here. The first terms also arise in

$$\begin{aligned}
-\{x_c,\{x^c,f\phi^{(s)}\}_-\}_+ &= -s\phi_{a_1\ldots a_s;b_1\ldots b_s} f\theta^{a_1b_1}\ldots \underbrace{\{x_c,\{x^c,\theta^{a_sb_s}\}\}}_{2r^2\theta^{a_sb_s}} \\
&\quad - s\phi_{a_1\ldots a_s;b_1\ldots b_s}\theta^{a_1b_1}\ldots\{x_c,\theta^{a_sb_s}\}\theta^{cd}\eth_d f \\
&\quad - sc_{s+1}\phi_{a_1\ldots a_s;b_1\ldots b_s}\theta^{a_1b_1}\ldots\{x_c,[\theta^{a_sb_s}\theta^{cd}]_0\eth_d f\} \\
&\quad - s(s-1)c_{s+1}\phi_{a_1\ldots a_s;b_1\ldots b_s}\{x_c,\theta^{a_1b_1}\}\ldots[\theta^{a_sb_s}\theta^{cd}]_0\eth_d f \\
&= -2sr^2 f\phi^{(s)} - sr^2R^2\phi_{a_1\ldots a_s;b_1\ldots b_s}\theta^{a_1b_1}\ldots(\eth^d\theta^{a_sb_s})\eth_d f \\
&\quad - sc_{s+1}\phi_{a_1\ldots a_s;b_1\ldots b_s}\theta^{a_1b_1}\ldots\{x_c,[\theta^{a_sb_s}\theta^{cd}]_0\eth_d f\} \\
&= -r^2 s\phi_{a_1\ldots a_s;b_1\ldots b_s}\theta^{a_1b_1}\ldots\theta^{a_{s-1}b_{s-1}} \\
&\qquad \Big(2f\theta^{a_sb_s} + R^2(\eth^d\theta^{a_sb_s})\eth_d f + \frac{1}{r^2}c_{s+1}\{x_c,[\theta^{a_sb_s}\theta^{cd}]_0\eth_d f\}\Big)\,.
\end{aligned} \tag{9.41}$$

Here we observe that the term proportional to $s(s-1)$ vanishes since

$$\begin{aligned}
&\phi_{a_1\ldots a_s;b_1\ldots b_s}\{x^c,\theta^{a_1b_1}\}\ldots[\theta^{a_sb_s}\theta^{cd}]_0 \\
&\quad = r^2\phi_{a_1\ldots a_s;b_1\ldots b_s}(\eta^{a_1c}x^{b_1} - \eta^{b_1c}x^{a_1})\ldots[\theta^{a_sb_s}\theta^{cd}]_0 \\
&\quad = r^2\phi_{a_1\ldots a_s;b_1\ldots b_s}([\theta^{a_sb_s}\theta^{a_1d}]_0x^{b_1} - [\theta^{a_sb_s}\theta^{b_1d}]_0x^{a_1})\cdots = 0
\end{aligned} \tag{9.42}$$

as it involves a contraction or the irreducible tensors with η_{ab} or ε_{abcde} due to (3.23). Now we can reduce the term in brackets using the $s = 1$ result (9.29). This gives

$$
\begin{aligned}
-\{x_c, \{x^c, f\phi^{(s)}\}_-\}_+ &= -\frac{r^2 s}{2s+1}\phi_{a_1\ldots a_s;b_1\ldots b_s}\theta^{a_1b_1}\ldots\theta^{a_{s-1}b_{s-1}} \\
&\quad \Big(2(2s+1)f\theta^{a_sb_s} + 2sR^2(\eth^d\theta^{a_sb_s})\eth_d f + R^2\theta^{ab}\eth^d\eth_d f\Big) \\
&\overset{!}{=} \frac{s}{2s+1}(\Box_H - 2r^2(s+1))(f\phi^{(s)}) \\
&= \frac{r^2 s}{2s+1}\Big(- R^2\phi^{(s)}\eth^d\eth_d f - 2(2s+1)f\phi^{(s)} \\
&\qquad - 2sR^2\phi_{a_1\ldots a_s;b_1\ldots b_s}\theta^{a_1b_1}\ldots(\eth^d\theta^{a_sb_s})\eth_d f\Big)
\end{aligned}
$$

using (9.27), which proves (9.25). □

It is quite instructive to check the $s = 1$ case explicitly for $\phi^{(1)} = x^p M^{ab}$:

$$
\Box_H\phi^{(1)} = -6r^2\phi^{(1)} + 2(\theta^{ap}x^b - \theta^{bp}x^a) \ . \tag{9.43}
$$

Now

$$
\begin{aligned}
\{x^c, x^p M^{ab}\}_- &= -x^p\{M^{ab}, x^c\} + [\theta^{cp}M^{ab}]_0 \\
&= -x^p(\eta^{ac}x^b - \eta^{bc}x^a) - \frac{R^2}{3}\big(P_\perp^{ca}P_\perp^{pb} - P_\perp^{cb}P_\perp^{pa} + \frac{1}{R}\varepsilon^{capbe}x_e\big) \qquad (9.44)
\end{aligned}
$$

hence

$$
\begin{aligned}
&\{x_c, \{x^c, x^p M^{ab}\}_-\} \\
&\quad = -\{x_a, x^p x^b\} + \{x_b, x^p x^a\} - \frac{R^2}{3}\big(\{x_a, P_\perp^{pb}\} - \{x_b, P_\perp^{pa}\} + \frac{1}{R}\varepsilon^{capbe}\theta_{ce}\big) \\
&\quad = -\frac{4}{3}\big(2x^p\theta^{ab} + x^b\theta^{ap} - x^a\theta^{bp}\big) - \frac{1}{3}R\varepsilon^{abpce}\theta_{ce} \\
&\quad = -\frac{4}{3}\big(2x^p\theta^{ab} + x^b\theta^{ap} - x^a\theta^{bp}\big) - \frac{2}{3}(\theta^{ab}x^p + \theta^{bp}x^a + \theta^{pa}x^b) \\
&\quad = \frac{10}{3}r^2\phi^{(1)} - \frac{2}{3}x^b\theta^{ap} + \frac{2}{3}x^a\theta^{bp} \\
&\quad = -\frac{1}{3}(\Box_H - 4r^2)\phi^{(1)}
\end{aligned}
$$

using the self-duality relations in Lemma 9.4:

Lemma 9.4. *θ^{ab} satisfies the following self-duality relations*

$$\varepsilon^{abpce}\theta_{ce}x^p = 2R\theta^{ab} \tag{9.45}$$

$$\theta^{ab} = \frac{1}{2R}\varepsilon^{abcde}x_c\theta_{de} \tag{9.46}$$

$$\varepsilon^{abpce}\theta_{ce} = \frac{2}{R}(\theta^{ab}x^p + \theta^{bp}x^a + \theta^{pa}x^b) \tag{9.47}$$

where the indices of $\theta_{ce} = \eta_{cc'}\eta_{ee'}\theta^{c'e'}$.

Proof. The first relation is already known [5], and the second relation reduces to the first at the reference point $\xi = (R, 0, 0, 0, 0)$. Now consider the third relation. The rhs is totally antisymmetric. At the reference point we can use $\theta^{0a} \sim \xi_b\theta^{ba} = 0$, so that the lhs vanishes if all 3 indices abp are tangential at ξ. If one is transversal, say $a = 0$, this reduces to

$$\varepsilon^{0bpce}\theta_{ce} = \frac{2}{R}\theta^{bp}x^0 = 2\theta^{bp} \tag{9.48}$$

which is correct using (9.45). As a check, contracting (9.47) with ε_{abprs} gives

$$\varepsilon_{abprs}\varepsilon^{abpce}\theta_{ce} = \frac{6}{R}\theta^{ab}x^p\varepsilon_{abprs} = 12\,\theta_{rs}\ . \tag{9.49}$$

□

As a corollary, we obtain

Corollary 9.5.

$$\begin{aligned} -\{x^\mu, \{x_\mu, \phi^{(s,k)}\}_-\}_+ &= \Big(\alpha_s(\Box_H - 2r^2(s+1)) + D^+D^-\Big)\phi^{(s,k)}, \\ &= \alpha_s(\Box_H - 2r^2(s+1))\phi^{(s,0)}, \qquad k = 0 \end{aligned} \tag{9.50}$$

Proof. This follows from the above noting that $\{x^4, \{x_4, \phi\}_-\}_+ = D^+D^-\phi$ and $D^-\phi^{(s,0)} = 0$. □

Corollary 9.6. *The totally symmetric tensor field $\phi_{\mu_1\ldots\mu_s}(x)$ associated to $\phi^{(s,0)}$ via (3.27) is square-integrable, space-like and divergence-free with positive inner product if $\phi^{(s,0)}$ is admissible. It is proportional to the tensor field in (3.16).*

For generic higher-spin modes such as $\mathcal{A}_\mu^{(-)}[\phi^{(s,k)}]$, positivity should not be expected, since they are in general not physical.

Proof. To see this, we first note that $\{x_\mu, \phi^{(s,0)}\}_- \in \mathcal{C}^{(s-1,0)}$, because

$$D^-\{x_\mu, \phi^{(s,0)}\}_- = 0 \ . \tag{9.51}$$

More generally,

$$\phi_{\mu_1\ldots\mu_l} := \mathcal{A}_{\mu_1}^{(-)}[\ldots[\mathcal{A}_{\mu_l}^{(-)}[\phi^{(s,0)}]\ldots] \ \in \ \mathcal{C}^{(s-l,0)} \tag{9.52}$$

for any l, so that

$$\begin{aligned}\{x^\mu, \{x_\mu, \phi_{\mu_1\ldots\mu_l}\}_-\}_+ &= \{x^a, \{x_a, \phi_{\mu_1\ldots\mu_l}\}_-\}_+ \\ &= \alpha_s(\Box_H - 2r^2(s-l+1))\phi_{\mu_1\ldots\mu_{l1}} \ .\end{aligned} \tag{9.53}$$

Then in the computation of the inner product (3.47) goes through with indices in $0,\ldots,3$. Space-like and divergence-free follows as in (9.15). The relation with (3.16) follows from irreducibility. $\Box$

9.5. Algebraic relations for $D^\pm$, $\Box_H$ and $\mathcal{K}$

We can derive a relation between $\Box$ and $\Box_H$ as follows: consider

$$\begin{aligned}-D^2\Box\phi &= \frac{2}{R^2}\{x_\mu, \{x^\mu, \phi\}\} + \frac{2}{R}(\{t_\mu, \{x^\mu, D\phi\}\} \\ &\quad + \{x_\mu, \{t^\mu, D\phi\}\}) + \{t_\mu, \{t^\mu, D^2\phi\}\} - 2r^2\Box\phi \\ &= \frac{2}{R^2}\{x_\mu, \{x^\mu, \phi\}\} + \frac{4}{R}\{t_\mu, \{x^\mu, D\phi\}\} \\ &\quad - \frac{8}{R^2}D^2\phi - \Box(D^2\phi) - 2r^2\Box\phi \\ &= -\frac{2}{R^2}\Box_H\phi + \Box D^2\phi - \frac{2}{R^2}D^2\phi - 2D\Box D\phi - 2r^2\Box\phi\end{aligned} \tag{9.54}$$

using $D^2x^\mu = r^2x^\mu$ and the identity

$$2R\{t^\mu, \{x_\mu, \phi^{(s)}\}\} = (R^2\Box + 4)D\phi^{(s)} - R^2D(\Box\phi^{(s)}) \tag{9.55}$$

which is proved in (A.36) in [1]. Hence

$$\boxed{2\Box_H = R^2(D^2\Box + \Box D^2 - 2D\Box D - 2r^2\Box) - 2D^2 \ .} \tag{9.56}$$

Writing $D = D^+ + D^-$ this can be written using (9.5) as

$$\boxed{\Box_H \phi^{(s)} = \Big(- R^2 r^2 \Box + (2s-1) D^+ D^- - (2s+3) D^- D^+ \Big) \phi^{(s)} \ .} \tag{9.57}$$

This is a very useful relation, which can be checked easily e.g. for $\phi = x^a$. It will allow to evaluate the inner products of the fluctuation modes. It also allows to express $D^- D^+$ in terms of $D^+ D^-$ and the Box operators. Since $D^- D^+$ commutes with both $\Box$ and $\Box_H$, we obtain

$$[D^- D^+, D^+ D^-] = 0 \ . \tag{9.58}$$

In particular, this gives

$$\Box_H \phi^{(s,0)} = \Big(- R^2 r^2 \Box - (2s+3) D^- D^+ \Big) \phi^{(s,0)} \ . \tag{9.59}$$

Now consider

$$\begin{aligned} \Box_H D^+ \phi^{(s,0)} &= D^+ (\Box_H + 2r^2(s+1)) \phi^{(s,0)} \\ &= D^+ \Big(- R^2 r^2 \Box - (2s+3) D^- D^+ + 2r^2(s+1) \Big) \phi^{(s)} \\ &= \Big(- (2s+3) D^+ D^- - R^2 r^2 \Box + 4r^2(s+1) \Big) D^+ \phi^{(s,0)} \ . \end{aligned} \tag{9.60}$$

Combining this with (9.57) for $D^+ \phi^{(s,0)}$ gives

$$\begin{aligned} &\Big(- (2s+3) D^+ D^- + 4r^2(s+1) \Big) D^+ \phi^{(s,0)} \\ &\qquad = \Big((2s+1) D^+ D^- - (2s+5) D^- D^+ \Big) D^+ \phi^{(s,0)} \end{aligned} \tag{9.61}$$

hence

$$D^+ D^- (D^+ \phi^{(s,0)}) = \Big(\frac{2s+5}{4(s+1)} D^- D^+ + r^2 \Big) D^+ \phi^{(s,0)} \ . \tag{9.62}$$

These are effectively commutation relations between D^+ and D^- on $\mathcal{C}^{(s+1,1)}$. For the general case, we make the ansatz

$$\boxed{D^+ D^- \big((D^+)^k \phi^{(s,0)} \big) = \big(\tilde{a}_k D^- D^+ + \tilde{b}_k \big) \big((D^+)^k \phi^{(s,0)} \big) \ .} \tag{9.63}$$

The constants are determined recursively by considering

$$\begin{aligned}\Box_H(D^+)^k\phi^{(s,0)} &= D^+\big(\Box_H + 2r^2(s+k)\big)(D^+)^{k-1}\phi^{(s,0)}\\ &= D^+\big(-R^2r^2\Box + (2s+2k-3)D^+D^-\\ &\qquad -(2s+2k+1)D^-D^+ + 2r^2(s+k)\big)(D^+)^{k-1}\phi^{(s,0)}\\ &= \Big(-R^2r^2\Box + \big((2(s+k)-3)\tilde{a}_{k-1} - 2(s+k)-1\big)D^+D^-\\ &\qquad + (2(s+k)-3)\tilde{b}_{k-1} + 4r^2(s+k)\Big)(D^+)^k\phi^{(s,0)} \ . \end{aligned} \tag{9.64}$$

On the other hand, the lhs can be written using (9.57) as

$$\begin{aligned}\Box_H(D^+)^k\phi^{(s,0)} = \Big(&-R^2r^2\Box + (2(s+k)-1)D^+D^-\\ &-(2(s+k)+3)D^-D^+\Big)(D^+)^k\phi^{(s,0)}\end{aligned} \tag{9.65}$$

and combining these we obtain

$$\begin{aligned}&\big((2(s+k)-3)\tilde{a}_{k-1} - 4(s+k)\big)D^+D^-\\ &\quad + (2s+2k-3)\tilde{b}_{k-1} + 2r^2(2s+2k)\\ &\qquad = -(2(s+k)+3)D^-D^+\end{aligned} \tag{9.66}$$

acting on $(D^+)^k\phi^{(s,0)}$. Comparing with (9.63), we obtain two recursion relations

$$\begin{aligned}\tilde{a}_k &= -\frac{2(s+k)+3}{(2(s+k)-3)\tilde{a}_{k-1} - 4(s+k)}\\ \tilde{b}_k &= -\frac{(2(s+k)-3)\tilde{b}_{k-1} + 4r^2(s+k)}{(2(s+k)-3)\tilde{a}_{k-1} - 4(s+k)}\end{aligned} \tag{9.67}$$

with

$$\tilde{a}_0 = 0 = \tilde{b}_0 \ . \tag{9.68}$$

This is solved by the remarkably simple general formula[17]

$$\boxed{\begin{aligned}\tilde{b}_k &= kr^2\frac{2s+k}{2s+2k-1}\\ \tilde{a}_k &= \frac{k}{k+1}\,\frac{2s+2k+3}{2s+2k-1}\,\frac{2s+k}{2s+k+1} \ .\end{aligned}}$$

[17]A random change of the recursion would lead to a complete mess here, which strongly indicates that we are on the right track.

Now we change notation as follows:

$$D^+D^-\phi^{(s,k)} = \big(a_{s,k}D^-D^+ + b_{s,k}\big)\phi^{(s,k)}, \qquad \phi^{(s,k)} = (D^+)^k\phi^{(s-k,0)} \ . \tag{9.69}$$

Comparing with the above we see that $a_{s,k} = \tilde{a}_k|_{s\to s-k}$ and $a_{s,k} = \tilde{a}_k|_{s\to s-k}$, and therefore

$$\boxed{\begin{aligned} b_{s,k} &= r^2k\,\frac{2s-k}{2s-1} \\ a_{s,k} &= \frac{k}{k+1}\,\frac{2s+3}{2s-1}\,\frac{2s-k}{2s-k+1} \ . \end{aligned}}$$

We also note the inverse relation

$$\begin{aligned} D^-D^+ &= \frac{1}{a_{s,k}}D^+D^- - \frac{b_{s,k}}{a_{s,k}} \\ &= \frac{k+1}{k}\,\frac{2s-1}{2s+3}\,\frac{2s-k+1}{2s-k}D^+D^- - r^2(k+1)\frac{2s-k+1}{2s+3}, \quad k \geq 1 \end{aligned} \tag{9.70}$$

which however only makes sense for $k \geq 1$.

Relations for $\mathcal{K}$. As a consequence, we obtain

$$\begin{aligned} D^+D^-\phi^{(s,k)} &= D^+(D^-D^+)\phi^{(s-1,k-1)} \\ &= D^+\Big(\frac{k}{k-1}\,\frac{2s-k}{2s-k-1}\,\frac{2s-3}{2s+1}\,D^+D^- - r^2k\frac{2s-k}{2s+1}\Big)\phi^{(s-1,k-1)} \end{aligned} \tag{9.71}$$

for $k \geq 2$, which gives

$$\begin{aligned} &\frac{(2s+1)(2s-1)}{k(2s-k)}D^+D^-\phi^{(s,k)} \\ &\quad = D^+\Big(\frac{(2s-1)(2s-3)}{(k-1)(2s-k-1)}\,D^+D^- - r^2(2s-1)\Big)\phi^{(s-1,k-1)} \ . \end{aligned}$$

Comparing with the definition (5.32) of $\mathcal{K}$

$$\begin{aligned} -r^2\mathcal{K} &= r^2s^2 + \frac{4s^2-1}{k(2s-k)}D^+D^- \\ &= r^2(s+1)^2 + \frac{(2s+1)(2s+3)}{(k+1)(2s-k+1)}D^-D^+ \end{aligned} \tag{9.72}$$

we obtain

$$[\mathcal{K}, D^+] = 0 \qquad \text{on } \mathcal{C}^{(s,k)},\ k \geq 1 \ . \tag{9.73}$$

For $k = 1$, we can write

$$\begin{aligned} \mathcal{K}\phi^{(s,1)} &= \mathcal{K}D^+\phi^{(s-1,0)} = -\big((2s+1)D^+D^- + s^2\big)D^+\phi^{(s-1,0)} \\ &= -D^+\big((2s+1)D^-D^+ + s^2\big)\phi^{(s-1,0)} = D^+\mathcal{K}\phi^{(s-1,0)} \end{aligned} \tag{9.74}$$

using the second form in (5.32) of $\mathcal{K}$. It follows that

$$\boxed{[\mathcal{K}, D^\pm] = 0} \tag{9.75}$$

without any restrictions. In particular, diagonalizing the space-like Laplacian D^+D^- on $\phi^{(s,k)} = (D^+)^k\phi^{(s-k,0)}$ is equivalent to diagonalizing it on $\phi^{(s-k,0)}$.

Evaluation of $\Box_H$ and positivity. We can use the above results to show

$$\Box_H\phi^{(s,k)} = r^2\Big(-R^2\Box + \mathcal{K} + (s+1)^2 + k(2s-k)\Big)\phi^{(s,k)} \tag{9.76}$$

which is obtained from (9.57) using the relations (9.69). This provides an on-shell relation between the Laplacians on H^3 and H^4. Moreover, we recall that $\Box_H$ is manifestly positive, and satisfies the bound $\Box_H > r^2(s^2+s+2)$ using the admissibility condition (3.33). Then (9.76) gives

$$\Big(-R^2\Box + \mathcal{K} + s - 1 + k(2s-k)\Big)\phi^{(s,k)} > 0 \ . \tag{9.77}$$

This is useful to establish the signature $(+++-)$ of off-shell modes in section 6.

9.6. Positivity of the space-like Laplacian $\mathcal{K}$

Now we show lemma 5.1, which states that $\mathcal{K} > 0$ for admissible ϕ. To get some insight, recall from [1] that for scalar fields $\phi \in \mathcal{C}^0$,

$$-D^-D^+\phi = \frac{r^2R^2}{3}\cosh^2(\eta)\Delta^{(3)}\phi \ . \tag{9.78}$$

Here $\Delta^{(3)} = -\nabla^{(3)\alpha}\nabla^{(3)}_\alpha$ is the space-like Laplacian on H^3 w.r.t. the induced metric, extended to symmetric tensor fields $\phi_{\mu_1\ldots\mu_s}(x)$. The lhs is related to $\mathcal{K}$ (9.72) by a factor and a shift. Clearly $\Delta^{(3)} > 0$ for square-integrable functions, but the required bound $\mathcal{K} > 0$ is slightly stronger.

Proof. Using (9.75), it suffices to show $\mathcal{K} > 0$ for $k = 0$, which is the statement

$$-D^- D^+ \; > \; r^2 \frac{(s+1)^2}{2s+3} \qquad \text{on} \;\; \mathcal{C}^{(s,0)} \; . \tag{9.79}$$

Recall that the tenssor field encoded in $\phi^{(s)} = \phi_{\mu_1 \ldots \mu_s} t^{\mu_1} \ldots t^{\mu_s} \in \mathcal{C}^{(s,0)}$ is divergence-free. Then

$$\begin{aligned} D^- D^+ \phi^{(s)} &= r^2 R D^- \left(\nabla^{(3)}_\alpha \phi_{\mu_1 \ldots \mu_s} t^{\mu_1} \ldots t^{\mu_s} t^\alpha \right) \\ &= r^4 R^2 \left[\nabla^{(3)}_\beta \; \nabla^{(3)}_\alpha \phi_{\mu_1 \ldots \mu_s} t^{\mu_1} \ldots t^{\mu_s} t^\alpha t^\beta \right]_s \\ &= \frac{c_{s+2}}{3} r^2 R^2 \cosh^2(\eta) \Big(\nabla^{(3)\alpha} \nabla^{(3)}_\alpha \phi_{\mu_1 \ldots \mu_s} t^{\mu_1} \ldots t^{\mu_s} \\ &\qquad + s \nabla^{(3)\mu_1} \nabla^{(3)}_\alpha \phi_{\mu_1 \ldots \mu_s} t^{\mu_2} \ldots t^{\mu_s} t^\alpha \Big) \\ &= \frac{r^2}{2s+3} \left(-R^2 \cosh^2(\eta) \Delta^{(3)} \phi^{(s)} - s(s+1) \phi^{(s)} \right) \end{aligned} \tag{9.80}$$

using Lemma 9.2, noting that $\nabla^{(3)} P^{\mu\nu}_\perp = 0$ and

$$\begin{aligned} [\nabla^{(3)}_{\mu_1}, \nabla^{(3)}_\alpha] \phi^{\mu_1 \ldots \mu_s} &= (R^{(3)}_{\mu_1 \alpha})^{\mu_1}_{\ \nu} \phi^{\nu \mu_2 \ldots \mu_s} + \sum_{j \neq 1} (R^{(3)}_{\mu_1 \alpha})^{\mu_j}_{\ \nu} \phi^{\mu_1 \nu \ldots \mu_s'} \\ &= -\frac{1}{R^2 \cosh^2(\eta)} \\ &\quad \cdot \Big(2 P^\perp_{\alpha\nu} \phi^{\nu \mu_2 \ldots \mu_s} + \sum_{j \neq 1} \left(\phi^{\mu_j \alpha \ldots \mu_s'} - P^\perp_{\mu_1 \nu} \phi^{\mu_1 \nu \ldots \alpha \ldots \mu_s'} \right) \Big) \\ &= -\frac{1}{R^2 \cosh^2(\eta)} \Big(2 \phi^{\alpha \mu_2 \ldots \mu_s} + (s-1) \phi^{\alpha \mu_2 \ldots \mu_s'} \Big) \\ &= -\frac{1}{R^2 \cosh^2(\eta)} (s+1) \phi^{\alpha \mu_2 \ldots \mu_s} \end{aligned} \tag{9.81}$$

using space-like gauge and tracelessness of $\phi^{\mu_1 \cdots \mu_s}$. Here

$$R^{(3)}_{\mu\nu;\alpha\beta} = -\frac{1}{\rho^2} \left(P^\perp_{\mu\alpha} P^\perp_{\nu\beta} - P^\perp_{\mu\beta} P^\perp_{\nu\alpha} \right), \qquad R^{(3)}_{\mu\alpha} = -\frac{2}{\rho^2} P^\perp_{\mu\alpha} \tag{9.82}$$

are the Riemann and Ricci tensors on H^3 with radius $\rho = R \cosh(\eta)$, and $P^\perp_{\mu\nu}$ is the tangential projector (3.22) on H^3. Now (9.79) follows using results of Delay (remark 6.2 in [36]) and Lee (Proposition E in [37]), which essentially state that the spectrum[18] of $\rho^2\Delta^{(3)}$ on rank s symmetric square-integrable[19] tensor fields on H^3 is given by $[s+1,\infty)$, i.e. $\rho^2\Delta^{(3)}|_{\phi_{\mu_1\cdots\mu_s}} > s+1$. $\square$

9.7. Proof of $v_{\text{null}} = 0$

In this section we prove that the null vector (6.18) which arises in the scalar sector actually vanishes,

$$v_{\text{null}} = 0 \ . \tag{9.83}$$

To see this, we need some identities.

Proposition 9.7. *The following identities hold*

$$\begin{aligned}
&(2s+1)D^+D^-\mathcal{A}^{(+)}[\phi^{(s)}] \\
&\quad = \mathcal{A}^{(+)}\big[\big((2s-1)D^+D^- - (2s+3)D^-D^+ + r^2(2s+1)\big)\phi^{(s)}\big] \\
&\qquad + (2s+5)D^-\mathcal{A}^{(+)}[D^+\phi^{(s)}] - 2r^2R\mathcal{A}^{(g)}[D^+\phi^{(s)}] \ ,
\end{aligned} \tag{9.84}$$

$$\begin{aligned}
&(2s+1)D^+\mathcal{A}^{(n)}[\phi^{(s)}] \\
&\quad = 2\mathcal{A}^{(+)}[\big(D^+D^- - D^-D^+\big)\phi^{(s)}] - (2s+5)\mathcal{A}^{(-)}[D^+D^+\phi^{(s)}] \\
&\qquad + 2(2s+3)\mathcal{A}^{(n)}[D^+\phi^{(s)}] - 2r^2R\mathcal{A}^{(g)}[D^+\phi^{(s)}] \ .
\end{aligned} \tag{9.85}$$

Proof. We can rewrite the lhs of the first relation using (9.57) and use the intertwiner properties (9.8) and (9.9) to get

$$\begin{aligned}
&(2s+1)D^+D^-\mathcal{A}^{(+)}[\phi^{(s)}] \\
&\quad = \Big(R^2r^2\Box + \Box_H + (2s+5)D^-D^+\Big)\mathcal{A}^{(+)}[\phi^{(s)}] \\
&\quad = \mathcal{A}^{(+)}[(R^2r^2\Box + \Box_H + r^2(2s+1))\phi^{(s)}] \\
&\qquad + (2s+5)D^-\mathcal{A}^{(+)}[D^+\phi^{(s)}] - 2r^2R\mathcal{A}^{(g)}[D^+\phi^{(s)}] \\
&\quad = \mathcal{A}^{(+)}\big[\big((2s-1)D^+D^- - (2s+3)D^-D^+ + r^2(2s+1)\big)\phi^{(s)}\big] \\
&\qquad + (2s+5)D^-\mathcal{A}^{(+)}[D^+\phi^{(s)}] - 2r^2R\mathcal{A}^{(g)}[D^+\phi^{(s)}] \ .
\end{aligned} \tag{9.86}$$

---

[18]In [37], the result is established only for the essential spectrum, but we assume that this is not a significant restriction. I am grateful for useful communications with Erwann Delay and Wilhelm Schlag.

[19]Since $\phi_{\mu_1\ldots\mu_s}$ is square-integrable on H^4 being a principal series irrep of $SO(4,1)$ (see also corollary 9.6), it is also square-integrable on almost all H^3 by Fubini's theorem, and for sufficiently smooth wavefunctions this should hold for all H^3. However, there should be a better way to justify this.

Then (9.85) is obtained using (5.16) twice. □

Now we can prove (9.83). Consider first

***s* = 1 Case.** We start with the easy observation

$$D^- \mathcal{A}^{(+)}_\mu[\phi] = r^2 R\{t^\mu, \phi\} + \mathcal{A}^{(-)}_\mu[D\phi] \tag{9.87}$$

for $\phi \in \mathcal{C}^0$. Acting with D^+, this gives

$$D^+ D^- \mathcal{A}^{(+)}_\mu[\phi] = r^2 \mathcal{A}^{(+)}_\mu[\phi] + r^2 R \mathcal{A}^{(g)}_\mu[D\phi] + \mathcal{A}^{(n)}_\mu[D\phi] \ , \tag{9.88}$$

and using (9.84) for the lhs leads to

$$-3\mathcal{A}^{(+)}[D^- D\phi] + 5D^- \mathcal{A}^{(+)}[D\phi] = 3r^2 R \mathcal{A}^{(g)}_\mu[D\phi] + \mathcal{A}^{(n)}_\mu[D\phi] \ . \tag{9.89}$$

Writing $D\phi = \phi^{(1,1)}$ and replacing $D^-\mathcal{A}^{(+)}$ using (5.16), we obtain

$$\begin{aligned} v^{(s=1)}_{\text{null}} &\equiv 2r^2 R \mathcal{A}^{(g)}[\phi^{(1,1)}] + 5\mathcal{A}^{(-)}[D^+\phi^{(1,1)}] \\ &\quad + 2\mathcal{A}^{(+)}[D^-\phi^{(1,1)}] - 6\mathcal{A}^{(n)}_\mu[\phi^{(1,1)}] = 0\,. \end{aligned} \tag{9.90}$$

This is precisely the null vector in (6.18) for $s = 1$, which is thus shown to vanish identically.

Generic *s*. Acting with D^+ on $v^{(s)}_{\text{null}}$ (6.18) and assuming inductively that it vanishes, we obtain with (9.84) and (9.85) after some straightforward calculations

$$\begin{aligned} 0 = sD^+ v^{(s)}_{\text{null}} &= \frac{1}{s}\mathcal{A}^{(+)}[D^+D^-\phi^{(s,s)}] + \frac{s(2s+3)}{1+s}\mathcal{A}^{(n)}[D^+\phi^{(s,s)}] \\ &\quad - (2s+1)D^+\mathcal{A}^{(n)}[\phi^{(s,s)}] + sr^2 R D^+ \mathcal{A}^{(g)}[\phi^{(s,s)}] \\ &= \mathcal{A}^{(+)}\big[\big(\frac{1-2s}{s}D^+D^- + 2D^-D^+ + r^2 s\big)\phi^{(s,s)}\big] \\ &\quad + (2s+5)\mathcal{A}^{(-)}[D^+D^+\phi^{(s,s)}] \\ &\quad - (2s+3)\frac{s+2}{1+s}\mathcal{A}^{(n)}[D^+\phi^{(s,s)}] + (s+2)r^2 R \mathcal{A}^{(g)}[D^+\phi^{(s,s)}] \ . \end{aligned} \tag{9.91}$$

Now we can use the commutation relations (9.69) in the form

$$\frac{2s-1}{s}D^+D^-\phi^{(s,s)} = \Big(\frac{s(2s+3)}{(s+1)^2}D^-D^+ + r^2 s\Big)\phi^{(s,s)} \tag{9.92}$$

and $\phi^{(s+1,s+1)} = D^+\phi^{(s,s)}$, which leads to

$$0 = (s+2)v^{(s+1)}_{\text{null}} \ . \tag{9.93}$$

9.8. Exceptional scalar modes

It was shown in section 6.2 that the regular scalar modes $\tilde{\mathcal{A}}_\mu^{(i)}[\phi^{(s,s)}]$ for $s \geq 1$ span only a 3-dimensional space, which implies that there is one missing scalar mode for each $s \geq 1$. Here we show how to determine this missing mode for $s = 1$. The remaining modes then arise as in (6.23).

A relation for $s = 0$. As a preparation, recall that $\mathcal{A}_\mu^{(-)}[\phi^{(1)}]$ and $\mathcal{A}_\mu^{(g)}[\phi]$ are complete in $C^0 \otimes \mathbb{R}^4$. Therefore there must be a relation

$$0 = x_\mu \phi + \tilde{\mathcal{A}}_\mu^{(g)}[\tilde{\phi}] + \mathcal{A}_\mu^{(-)}[D\phi'] \ . \tag{9.94}$$

By acting with $\{t^\mu, .\}$ and $\{x^\mu, .\}$, this implies

$$\begin{aligned} 0 &= \{t^\mu, x_\mu \phi\} + \{t^\mu, \tilde{\mathcal{A}}_\mu^{(g)}[\tilde{\phi}]\} + \{t^\mu, \mathcal{A}_\mu^{(-)}[D\phi']\} \\ &= \sinh(\eta)(\tau + 4)\phi - \Box\tilde{\phi} + \frac{1}{R} D^- D\phi' \ . \end{aligned} \tag{9.95}$$

Similarly,

$$\begin{aligned} 0 &= \{x^\mu, x_\mu \phi\} + \{x^\mu, \tilde{\mathcal{A}}_\mu^{(g)}[\tilde{\phi}]\} + \{x^\mu, \tilde{\mathcal{A}}_\mu^{(-)}[D\phi']\} \\ &= -R\sinh(\eta) D\phi + \{\{x^\mu, t_\mu\}, \tilde{\phi}\} + \{t_\mu, \{x^\mu, \tilde{\phi}\}\} \\ &\quad - \big(\alpha_1(\Box_H - 4r^2) + D^+ D^-\big) D\phi' \\ &= D\Big(- R\sinh(\eta)\phi + \frac{2}{R}\tilde{\phi} - \big(\alpha_1(\Box_H - 2r^2) + D^- D\big)\phi'\Big) \end{aligned} \tag{9.96}$$

implies

$$\frac{2}{R}\tilde{\phi} = R\sinh(\eta)\phi + \big(\alpha_1(\Box_H - 2r^2) + D^- D\big)\phi' \ . \tag{9.97}$$

These two equations can be solved for ϕ' and $\tilde{\phi}$, for given (admissible, generic) ϕ. It follows that

$$\begin{aligned} 0 &= D(x_\mu \phi) + D\tilde{\mathcal{A}}_\mu^{(g)}[\tilde{\phi}] + D\mathcal{A}_\mu^{(-)}[D\phi'] \\ &= r^2 R t_\mu \phi + x_\mu D\phi + \tilde{\mathcal{A}}_\mu^{(g)}[D\tilde{\phi} + r^2 R D\phi'] \\ &\quad + \frac{1}{R}\tilde{\mathcal{A}}_\mu^{(+)}[\tilde{\phi}] + \mathcal{A}_\mu^{(-)}[D^+ D\phi']. \end{aligned} \tag{9.98}$$

Exceptional mode for $s = 1$. Similar to (9.94), we make the ansatz[20]

$$\mathcal{A}_\mu^{(ex,1)}[\phi] = t_\mu D^+\phi + \tilde{\mathcal{A}}_\mu^{(g)}[D\tilde{\phi}] + \tilde{\mathcal{A}}_\mu^{(+)}[D\phi_+] + \tilde{\mathcal{A}}_\mu^{(-)}[D\phi_-] \tag{9.99}$$

for $\phi, \tilde{\phi}, \phi'_\pm \in \mathcal{C}^0$. By definition, these are orthogonal to all $\mathcal{A}_\mu^{(i)}$ modes, which amounts to

$$0 = D^- \mathcal{A}_\mu^{(ex,1)} = \{t^\mu, \mathcal{A}_\mu^{(ex,1)}\} = \{x^\mu, \mathcal{A}_\mu^{(ex,1)}\} \ . \tag{9.100}$$

These 3 equations can be solved for $\tilde{\phi}, \phi'_\pm$, for any given (admissible, generic) ϕ. The same constraints follow for all $\mathcal{A}_\mu^{(ex,s)}$ by acting with D^+. In particular, the $\mathcal{A}_\mu^{(ex,s)}$ are physical. We refrain from studying these in detail, since no nice general formula was found.

9.9. Inner products of $\mathcal{A}$ modes.

Here we derive the explicit formulas for the inner products (6.1) for all $\tilde{\mathcal{A}}^{(i)}[\phi]$ modes for $\phi = \phi^{(s,k)}$ and $\phi' = \phi'^{(s,k)}$. First, it is clear (by invariance) that the modes are orthogonal unless the spin quantum numbers $s = s'$ and $k = k'$ coincide. Assuming this, we obtain

$$\int \mathcal{A}_\mu^{(g)}[\phi'] \mathcal{A}^{(g)\mu}[\phi] = \int \phi' \Box \phi \tag{9.101a}$$

(9.101b)

$$\int \mathcal{A}_\mu^{(g)}[\phi'] \mathcal{A}^{(+)\mu}[D^-\phi] = -\frac{s+2}{R} \int \phi' D^+ D^- \phi$$

(9.101c)

$$\int \mathcal{A}_\mu^{(g)}[\phi'] \mathcal{A}^{(-)\mu}[D^+\phi] = \frac{s-1}{R} \int \phi' D^- D^+ \phi$$

$$\begin{aligned} \int \mathcal{A}_\mu^{(-)}[D^+\phi'] \mathcal{A}^{(+)\mu}[D^-\phi] &= -\int D^- D^+ \phi' \, D^+ D^- \phi \\ &= -\int \phi' \, D^- D^+ D^+ D^- \phi \end{aligned} \tag{9.101d}$$

$$\begin{aligned} \int \mathcal{A}_\mu^{(+)}[D^-\phi'] \mathcal{A}^{(+)\mu}[D^-\phi^{(s)}] &= \int D^-\phi' \big((1-\alpha_{s-1})\Box_H \\ &\qquad + 2\alpha_{s-1} r^2 s + D^- D^+\big) D^-\phi \\ &= -\int \phi' \big(\frac{s}{2s-1}(\Box_H - 2r^2) \end{aligned}$$

[20] Using (9.98), this is equivalent to the ansatz $\mathcal{A}_\mu^{(ex,1)} = x_\mu D\phi + \sum \tilde{\mathcal{A}}^{(i)}$.

$$
+ D^+D^-\big)D^+D^-\phi \tag{9.101e}
$$

$$
\int \mathcal{A}_\mu^{(-)}[D^+\phi']\mathcal{A}^{(-)\mu}[D^+\phi] = \int D^+\phi'\big(\alpha_{s+1}(\Box_H - 2r^2(s+2))
$$

$$
+ D^+D^-\big)D^+\phi \tag{9.101f}
$$

$$
= -\int \phi'\big(\alpha_{s+1}(\Box_H - 2r^2) + D^-D^+\big)D^-D^+\phi \tag{9.101g}
$$

using (9.10) in the last two relations, and $\alpha_s = \frac{s}{2s+1}$ (9.25). The inner products with the $\mathcal{A}^{(n)}$ modes is obtained as follows:

$$
\begin{aligned}
\int \mathcal{A}_\mu^{(n)}[\phi']\mathcal{A}^{(-)\mu}[D^+\phi] &= \int D^+\mathcal{A}_\mu^{(-)}[\phi']\mathcal{A}^{(-)\mu}[D^+\phi] \\
&= -\int \mathcal{A}_\mu^{(-)}[\phi']D^-\mathcal{A}^{(-)\mu}[D^+\phi] \\
&= -\int \mathcal{A}_\mu^{(-)}[\phi']\mathcal{A}^{(-)\mu}[D^-D^+\phi] \\
&= -\int \phi'\big(\alpha_s(\Box_H - 2r^2(s+1)) + D^+D^-\big)D^-D^+\phi \ .
\end{aligned} \tag{9.102}
$$

Here we used

$$
D^-\mathcal{A}^{(-)}[\phi] = \mathcal{A}^{(-)}[D^-\phi] \ , \qquad D^+\mathcal{A}^{(+)}[\phi] = \mathcal{A}^{(+)}[D^+\phi] \ . \tag{9.103}
$$

Next,

$$
\begin{aligned}
\int \mathcal{A}_\mu^{(n)}[\phi']\mathcal{A}^{(+)\mu}[D^-\phi] &= \int D^+\mathcal{A}_\mu^{(-)}[\phi']\mathcal{A}^{(+)\mu}[D^-\phi] \\
&= \int \Big(-D^-\mathcal{A}_\mu^{(+)}[\phi'] + r^2R\{t_\mu,\phi'\} \\
&\qquad + \mathcal{A}_\mu^{(-)}[D^+\phi'] + \mathcal{A}_\mu^{(+)}[D^-\phi']\Big)\mathcal{A}^{(+)\mu}[D^-\phi] \\
&= \int \mathcal{A}_\mu^{(+)}[\phi']\mathcal{A}^{(+)\mu}[D^+D^-\phi] \\
&\quad + \int \Big(r^2R\mathcal{A}_\mu^{(g)}[\phi'] + \mathcal{A}_\mu^{(-)}[D^+\phi'] \\
&\qquad + \mathcal{A}_\mu^{(+)}[D^-\phi']\Big)\mathcal{A}^{(+)\mu}[D^-\phi] \\
&= \int \phi'\Big(\frac{1}{1-4s^2}\Box_H + r^2\,\frac{-2s^2+s+2}{4s^2-1} - D^+D^-\Big)D^+D^-\phi
\end{aligned} \tag{9.104}
$$

using (5.16) along with the previous inner products, and

$$\int \mathcal{A}_\mu^{(+)}[\phi']\mathcal{A}^{(+)\mu}[\phi^{(s)}] = \int \phi' \big(\frac{s+1}{2s+1}\Box_H + 2r^2\frac{s(s+1)}{2s+1} + D^-D^+\big)\phi^{(s)} \; . \tag{9.105}$$

Next,

$$\begin{aligned}\int \mathcal{A}_\mu^{(n)}[\phi']\mathcal{A}^{(g)\mu}[\phi] &= -\int \mathcal{A}_\mu^{(-)}[\phi']D^-\{t^\mu,\phi\}\\ &= -\int \frac{1}{R}\mathcal{A}_\mu^{(-)}[\phi']\mathcal{A}_\mu^{(-)}[\phi] + \mathcal{A}_\mu^{(-)}[\phi']\mathcal{A}^{(g)\mu}[D^-\phi]\\ &= \frac{1}{R}\int \phi'\Big(\alpha_s(-\Box_H + r^2 2(s+1)) + (s-3)D^+D^-\Big)\phi\end{aligned} \tag{9.106}$$

and finally

$$\begin{aligned}&\int \mathcal{A}_\mu^{(n)}[\phi']\mathcal{A}^{(n)\mu}[\phi] = \int D^+\mathcal{A}_\mu^{(-)}[\phi']D^+\mathcal{A}_\mu^{(-)}[\phi]\\ &= -\int \mathcal{A}_\mu^{(-)}[\phi']D^-D^+\mathcal{A}_\mu^{(-)}[\phi]\\ &= -\frac{1}{2s+1}\int \mathcal{A}_\mu^{(-)}[\phi']\Big((2s-3)D^+D^- - \Box_H - R^2r^2\Box\Big)\mathcal{A}_\mu^{(-)}[\phi]\\ &= \frac{2s-3}{2s+1}\int D^-\mathcal{A}_\mu^{(-)}[\phi']D^-\mathcal{A}_\mu^{(-)}[\phi]\\ &\quad + \frac{1}{2s+1}\int \mathcal{A}_\mu^{(-)}[\phi'](\Box_H + R^2r^2\Box)\mathcal{A}_\mu^{(-)}[\phi]\\ &= \frac{2s-3}{2s+1}\int \mathcal{A}_\mu^{(-)}[D^-\phi']\mathcal{A}_\mu^{(-)}[D^-\phi]\\ &\quad + \frac{1}{2s+1}\int \mathcal{A}_\mu^{(-)}[\phi']\Big(\mathcal{A}_\mu^{(-)}[(\Box_H + r^2R^2\Box - r^2(2s+1))\phi]\\ &\quad - 2Rr^2\mathcal{A}^{(g)\mu}[D^-\phi^{(s)}]\Big)\\ &= \frac{2s-3}{2s+1}\int \phi'\big(-\frac{s-1}{2s-1}(\Box_H - 4r^2 s)D^+D^- - D^+D^+D^-D^-\big)\phi\\ &\quad + \frac{1}{2s+1}\int \phi'\big(\alpha_s(\Box_H - 2r^2(s+1)) + D^+D^-\big)\\ &\quad \cdot (\Box_H + r^2R^2\Box - r^2(2s+1))\phi\\ &\quad + r^2\frac{2(s-2)}{2s+1}\int \phi' D^+D^-\phi\end{aligned} \tag{9.107}$$

using (9.57), (9.8) ff, and the previous relations with (9.1) in the last step. To rewrite $D^+D^+D^-D^-$ we need to specify $\phi = \phi^{(s,k)}$. Then we obtain using (9.69)

$$\begin{aligned}\int \mathcal{A}_\mu^{(n)}[\phi']\mathcal{A}^{(n)\mu}[\phi] = &-\frac{2s-3}{2s+1}\int \phi'\Big(\big(\frac{s-1}{2s-1}(\Box_H - 4r^2 s) + b_{s-1,k-1}\big)D^+D^- \\ &+ a_{s-1,k-1}D^+D^-D^+D^-\Big)\phi \\ &+ \frac{1}{2s+1}\int \phi'\big(\frac{s}{2s+1}(\Box_H - 2r^2(s+1)) + D^+D^-\big) \\ &\cdot (\Box_H + r^2R^2\Box - r^2(2s+1))\phi \\ &+ r^2\frac{2(s-2)}{2s+1}\int \phi' D^+D^-\phi \ . \end{aligned} \tag{9.108}$$

This can be used to perform the computations in section 6, and a non-trivial consistency check is provided by (6.2).

References

[1] M. Sperling and H. C. Steinacker, *Covariant cosmological quantum space-time, higher-spin and gravity in the IKKT matrix model*, *JHEP* **07** (2019) 010, [arXiv:1901.03522].

[2] H. C. Steinacker, *Scalar modes and the linearized Schwarzschild solution on a quantized FLRW space-time in Yang-Mills matrix models*, *Class. Quant. Grav.* **36** (2019), no. 20 205005, [arXiv:1905.07255].

[3] M. A. Vasiliev, *Consistent equation for interacting gauge fields of all spins in (3+1)-dimensions*, *Phys. Lett.* **B243** (1990) 378–382.

[4] V. E. Didenko and E. D. Skvortsov, *Elements of Vasiliev theory*, arXiv:1401.2975.

[5] M. Sperling and H. C. Steinacker, *The fuzzy 4-hyperboloid H_n^4 and higher-spin in Yang–Mills matrix models*, *Nucl. Phys.* **B941** (2019) 680–743, [arXiv:1806.05907].

[6] M. Sperling and H. C. Steinacker, *Higher spin gauge theory on fuzzy S_N^4*, *J. Phys.* **A51** (2018), no. 7 075201, [arXiv:1707.00885].

[7] L. Bonora, M. Cvitan, P. Dominis Prester, S. Giaccari, and T. Stemberga, *HS in flat spacetime. YM-like models*, arXiv:1812.05030.

[8] J. Nishimura and A. Tsuchiya, *Complex Langevin analysis of the space-time structure in the Lorentzian type IIB matrix model*, *JHEP* **06** (2019) 077, [arXiv:1904.05919].

[9] T. Aoki, M. Hirasawa, Y. Ito, J. Nishimura, and A. Tsuchiya, *On the structure of the emergent 3d expanding space in the Lorentzian type IIB matrix model*, arXiv:1904.05914.

[10] S.-W. Kim, J. Nishimura, and A. Tsuchiya, *Expanding (3+1)-dimensional universe from a Lorentzian matrix model for superstring theory in (9+1)-dimensions*, *Phys. Rev. Lett.* **108** (2012) 011601, [arXiv:1108.1540].

[11] G. Mack, *All unitary ray representations of the conformal group SU(2,2) with positive energy*, *Commun. Math. Phys.* **55** (1977) 1.

[12] S. Fernando and M. Gunaydin, *Minimal unitary representation of SU(2,2) and its deformations as massless conformal fields and their supersymmetric extensions*, *J. Math. Phys.* **51** (2010) 082301, [arXiv:0908.3624].

[13] K. Hasebe, *Non-Compact Hopf Maps and Fuzzy Ultra-Hyperboloids*, *Nucl. Phys.* **B865** (2012) 148–199, [arXiv:1207.1968].

[14] A. M. Perelomov, *Generalized coherent states and their applications.* Springer, Berlin Heidelberg, 1986.

[15] L. Schneiderbauer and H. C. Steinacker, *Measuring finite Quantum Geometries via Quasi-Coherent States*, *J. Phys.* **A49** (2016), no. 28 285301, [arXiv:1601.08007].

[16] J.-P. Gazeau and F. Toppan, *A Natural fuzzyness of de Sitter space-time*, *Class. Quant. Grav.* **27** (2010) 025004, [arXiv:0907.0021].

[17] M. Buric, D. Latas, and L. Nenadovic, *Fuzzy de Sitter Space*, *Eur. Phys. J.* **C78** (2018), no. 11 953, [arXiv:1709.05158].

[18] H. Steinacker, *Emergent Geometry and Gravity from Matrix Models: an Introduction*, *Class. Quant. Grav.* **27** (2010) 133001, [arXiv:1003.4134].

[19] J. Dixmier, *Représentations intégrables du groupe de de sitter*, *Bulletin de la Societe mathematique de France* **89** (1961) 9–41.

[20] M. A. Naimark, *Linear representations of the Lorentz group*, vol. 63. Elsevier, 2014.

[21] S. Browne and D. Šijački, *On the irreducible representations of the lorentz group*, *Annals of Physics* **99** (1976), no. 1 92–126.

[22] N. Ishibashi, H. Kawai, Y. Kitazawa, and A. Tsuchiya, *A Large N reduced model as superstring*, *Nucl. Phys.* **B498** (1997) 467–491, [hep-th/9612115].

[23] S. Minwalla, M. Van Raamsdonk, and N. Seiberg, *Noncommutative perturbative dynamics*, *JHEP* **02** (2000) 020, [hep-th/9912072].

[24] M. Hanada, H. Kawai, and Y. Kimura, *Describing curved spaces by matrices*, *Prog. Theor. Phys.* **114** (2006) 1295–1316, [hep-th/0508211].

[25] K. Sakai, *A note on higher spin symmetry in the IIB matrix model with the operator interpretation*, arXiv:1905.10067.

[26] H. C. Steinacker, *Quantized open FRW cosmology from Yang-Mills matrix models*, *Phys. Lett.* **B782** (2017) 2018, [arXiv:1710.11495].

[27] D. N. Blaschke and H. Steinacker, *On the 1-loop effective action for the IKKT model and non-commutative branes*, *JHEP* **10** (2011) 120, [arXiv:1109.3097].

[28] A. Matusis, L. Susskind, and N. Toumbas, *The IR / UV connection in the noncommutative gauge theories*, *JHEP* **12** (2000) 002, [hep-th/0002075].

[29] I. Jack and D. R. T. Jones, *Ultraviolet finiteness in noncommutative supersymmetric theories*, *New J. Phys.* **3** (2001) 19, [hep-th/0109195].

[30] M. Hanada and H. Shimada, *On the continuity of the commutative limit of the 4d N=4 non-commutative super Yang–Mills theory*, *Nucl. Phys.* **B892** (2015) 449–474, [arXiv:1410.4503].

[31] H. Aoki, N. Ishibashi, S. Iso, H. Kawai, Y. Kitazawa, and T. Tada, *Noncommutative Yang-Mills in IIB matrix model*, *Nucl. Phys.* **B565** (2000) 176–192, [hep-th/9908141].

[32] H. C. Steinacker, *Emergent gravity on covariant quantum spaces in the IKKT model*, *JHEP* **12** (2016) 156, [arXiv:1606.00769].

[33] A. Salvio, *Quadratic Gravity*, *Front.in Phys.* **6** (2018) 77, [arXiv:1804.09944].

[34] K. Stelle, *Renormalization of higher-derivative quantum gravity*, *Physical Review D* **16** (1977), no. 4 953.

[35] M. Reuter and F. Saueressig, *Quantum Einstein Gravity*, *New J. Phys.* **14** (2012) 055022, [arXiv:1202.2274].

[36] E. Delay, *Essential spectrum of the lichnerowicz laplacian on two tensors on asymptotically hyperbolic manifolds*, *Journal of Geometry and Physics* **43** (2002), no. 1 33–44.

[37] J. Lee and A. M. Society, *Fredholm Operators and Einstein Metrics on Conformally Compact Manifolds.* No. Bd. 13 in Fredholm operators and Einstein metrics on conformally compact manifolds. American Mathematical Society, 2006.

Faculty of Physics, University of Vienna
Boltzmanngasse 5, A-1090 Vienna, Austria
E-mail address: harold.steinacker@univie.ac.at