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Transgression of D-branes

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Closed strings can be seen either as one-dimensional objects in a target space or as points in the free loop space. Correspondingly, a B-field can be seen either as a connection on a gerbe over the target space, or as a connection on a line bundle over the loop space. Transgression establishes an equivalence between these two perspectives. Open strings require D-branes: submanifolds equipped with vector bundles twisted by the gerbe. In this paper we develop a loop space perspective on D-branes. It involves bundles of simple Frobenius algebras over the branes, together with bundles of bimodules over spaces of paths connecting two branes. We prove that the classical and our new perspectives on D-branes are equivalent. Further, we compare our loop space perspective to Moore-Segal/Lauda-Pfeiffer data for open-closed 2-dimensional topological quantum field theories, and exhibit it as a smooth family of reflection-positive, colored knowledgable Frobenius algebras.

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1. Introduction

The motivation for this article comes from the Lagrangian approach to 2-dimensional field theories, where we consider smooth maps $\phi : \Sigma \to M$ from a Riemann surface Σ to a smooth manifold M. If the surface Σ has a boundary, then we specify a family of submanifolds $Q_i \subseteq M$ (the "D-branes") and require that ϕ map each boundary component to one of these [Pol96].

The usual sigma model action functional on the space of maps ϕ requires a metric g and a hermitian line bundle gerbe $\mathcal{G}$ with connection over M, the "B-field" [Alv85, Gaw88, Mur96]. In the case that Σ has boundaries, the D-branes Q_i have to be equipped with $\mathcal{G}|_{Q_i}$-twisted vector bundles $\mathcal{E}_i$ with connections, known as "twisted Chan-Paton gauge fields" in string theory, [Kap00, GR02, CJM02]. The twisted vector bundles make up the famous relation between D-branes and twisted K-theory [FW99, FH00].

In Wess-Zumino-Witten (WZW) models, for instance, M is a compact Lie group G and the branes Q_i are conjugacy classes of G [AS99]. The metric is induced by the Killing form, and the bundle gerbe $\mathcal{G}$ represents the "level" via its Dixmier-Douady class $[\mathcal{G}] \in \mathrm{H}^3(G, \mathbb{Z}) \cong \mathbb{Z}$. The bundle gerbe $\mathcal{G}$ and the twisted vector bundles can be constructed explicitly in a Lie-theoretic way, for all compact simple Lie groups [GR02, Mei02, GR03, Gaw05].

Especially in the discussion of WZW models it is common to consider, instead of the bundle gerbe $\mathcal{G}$, a hermitian line bundle with connection over the loop group LG. One advantage of this perspective is, for example, that one can (at least at an informal level) study the geometric quantization of the model. In fact, every smooth manifold M has a corresponding *transgression* map to its free loop space $LM = C^\infty(S^1, M)$ [Gaw88, Bry93, Wal10]

$$\left\{ \begin{array}{c} \text{Hermitian line bundle gerbes} \\ \text{with connection over } M \end{array} \right\} \to \left\{ \begin{array}{c} \text{Hermitian line bundles} \\ \text{with connection over } LM \end{array} \right\}.$$

In good cases, for instance in WZW models with simply-connected target groups, this map induces an equivalence of the corresponding categories, so that the loop space perspective is equivalent to the original setting.

In cases of more general topology, transgression is neither injective nor surjective, but the equivalence can be re-established by changing the range of transgression [Wal16]. Two changes are necessary: the first change is to add *fusion products*, a structure defined for a line bundle $\mathcal{L}$ over the loop space LM [ST05, Wal16]. It consists of fiber-wise isomorphisms

$$\mathcal{L}|_{\gamma_1\cup\gamma_2} \otimes \mathcal{L}|_{\gamma_2\cup\gamma_3} \to \mathcal{L}|_{\gamma_1\cup\gamma_3},$$

where $(\gamma_1, \gamma_2, \gamma_3)$ is a triple of paths in M with a common initial point and a common end-point, and $\gamma_i \cup \gamma_j$ denotes the loop that first follows γ_i and then returns along the reverse of γ_j. The second change is to impose various conditions on the connection on $\mathcal{L}$, most importantly the condition that it is *superficial*. This is a property that can be considered for connections on bundles over mapping spaces $C^\infty(K, M)$, where K is a compact manifold (here, $K = S^1$ or $K = [0, 1]$ later). If $\Gamma : [0, 1] \to C^\infty(K, M)$ is a smooth path in such a mapping space, then there is a corresponding "adjoint" map $\Gamma^\vee : [0, 1] \times K \to M$, and superficiality requires certain conditions for the parallel transport along Γ depending on the rank of the differential of $\Gamma^\vee$. We refer to [Wal16, Sec. 2] or Appendix A.2 for more details. Taking fusion and superficial connections into account, transgression provides a complete loop space perspective to connections on bundle gerbes ("B-fields") for arbitrary smooth manifolds M.

In this article we establish an analogous loop space perspective to D-branes and their twisted Chan-Paton gauge fields. To that end, we provide a transgression map for $\mathcal{G}|_{Q_i}$-twisted vector bundles $\mathcal{E}_i$ supported on submanifolds Q_i and prove that it establishes an equivalence between the appropriate categories. Due to the twist represented by the bundle gerbe $\mathcal{G}$, transgression of D-branes will be defined relative to the above-mentioned transgression of bundle gerbes.

First results in this direction have been obtained by Brylinski [Bry93], Gawędzki-Reis [GR02] and Gawędzki [Gaw05]. In the following, we say that a $\mathcal{G}$*-brane* is a pair $(Q, \mathcal{E})$ of a submanifold $Q \subseteq M$ and a $\mathcal{G}|_Q$-twisted vector bundle $\mathcal{E}$ with connection. In [GR02, Gaw05] it is described how to associate to a pair of two $\mathcal{G}$-branes $(Q, \mathcal{E})$ and $(Q', \mathcal{E}')$ a hermitian vector bundle $\mathcal{R}$ with connection over the space P of paths connecting Q with Q'. In Section 4.2 we give an alternative, but equivalent description of this vector bundle in terms of a modern, bicategorical treatment of bundle gerbes. We also clarify some regularity aspects of the vector bundle $\mathcal{R}$ that have not been treated so far. More precisely, we work in the convenient setting of diffeology, which provides a simple and yet powerful framework to study geometry over spaces

of loops and paths. Furthermore, we exhibit in Section 4.3 a new property of the connection on $\mathcal{R}$, which is analogous to the superficiality for line bundles over loop spaces mentioned above.

While this construction of the vector bundle $\mathcal{R}$ over P is the basis of our transgression map for D-branes, the picture is incomplete in essentially two aspects. First, it seems that the vector bundle $\mathcal{R}$ has lost its relation to the twist $\mathcal{G}$, or rather its loop space analogue, the transgressed hermitian line bundle $\mathcal{L}$. In Section 4.4 we discover an isomorphism

$$\mathcal{L}|_{\gamma_1\cup\gamma_2} \otimes \mathcal{R}|_{\gamma_2} \to \mathcal{R}|_{\gamma_1}, \tag{1}$$

where γ_1, γ_2 are paths with a common initial point and a common end point. It satisfies the axiom of an action with respect to the fusion product on $\mathcal{L}$; therefore, we will call it the *fusion representation* of $\mathcal{L}$ on $\mathcal{R}$. Second, we discover in Section 4.5 a homomorphism

$$\mathcal{R}_{23}|_{\gamma_{23}} \otimes \mathcal{R}_{12}|_{\gamma_{12}} \to \mathcal{R}_{13}|_{\gamma_{23}\star\gamma_{12}} \tag{2}$$

relating the vector bundles $\mathcal{R}_{ij}$ formed from three $\mathcal{G}$-branes $(Q_1, \mathcal{E}_1)$, $(Q_2, \mathcal{E}_2)$ and $(Q_3, \mathcal{E}_3)$, which lifts the concatenation of a path γ_{12} between Q_1 and Q_2 and a path γ_{23} from Q_2 to Q_3 to the path $\gamma_{23} \star \gamma_{12}$ between Q_1 and Q_3. The homomorphism of (2) is accompanied by additional structure rendering it compatible with identity paths and path reversal, and satisfies several compatibility conditions. We call it the *lifted path concatenation.*

We assemble the data obtained by transgression abstractly into a category $\mathrm{LBG}(M, Q)$ of *loop space brane geometry* (LBG), depending on a target space M and a family $Q = \{Q_i\}_{i\in I}$ of D-branes. Roughly, the objects are tuples consisting of a line bundle $\mathcal{L}$ over LM with superficial connection, a fusion product on $\mathcal{L}$, vector bundles $\mathcal{R}_{ij}$ over the spaces P_{ij} of paths connecting Q_i with Q_j, fusion representations for each of these vector bundles, and a lifted path concatenation relating them to each other. A precise definition is given in Section 2.2. On the other side, the basis of transgression was a bicategory $\mathrm{TBG}(M, Q)$ of *target space brane geometry* (TBG), whose objects consist of a bundle gerbe $\mathcal{G}$ over M with connection and $\mathcal{G}|_{Q_i}$-twisted vector bundles $\mathcal{E}_i$ with connection. A precise definition of that bicategory and some background about bundle gerbes is given in Section 2.1. In Section 4.8 we prove that transgression furnishes a functor

$$\mathscr{T} : \mathrm{h}_1\mathrm{TBG}(M, Q) \to \mathrm{LBG}(M, Q),$$

where h_1 denotes the 1-truncation of a bicategory (identify 2-isomorphic 1-morphisms). In Section 5 we construct in a natural way a functor $\mathscr{R}$ in the opposite direction called *regression*. Our main result is the following theorem.

Theorem 1. *Transgression and regression form an equivalence between target space brane geometry and loop space brane geometry,*

$$\mathrm{h}_1\mathrm{TBG}(M,Q) \cong \mathrm{LBG}(M,Q).$$

The proof of Theorem 1 is carried out in Section 6 by constructing explicitly natural isomorphisms $\mathscr{R} \circ \mathscr{T} \cong \mathrm{id}$ and $\mathscr{T} \circ \mathscr{R} \cong \mathrm{id}$. Theorem 1 means that the Lagrangian approach to 2-dimensional topological field theories with D-branes can be pursued in an equivalent way either via TBG or via LBG.

An interesting aspect of the equivalence of Theorem 1 is the presence of algebra bundles over the D-branes, arising in both perspectives. In TBG, algebra bundles A_i arise as endomorphism bundles of the $\mathcal{G}|_{Q_i}$-twisted vector bundles [STV14]. They are well-known for their relation to twisted K-theory [Kar12]. In LBG, algebra bundles $\mathcal{A}_i$ are obtained by restricting the vector bundles $\mathcal{R}_{ii}$ to constant paths in Q_i, and turning these restrictions into algebra bundles using the lifted path concatenation of (2), see Section 3.1. We prove the following result.

Theorem 2.

(a) The algebra bundles $\mathcal{A}_i$ arising from loop space brane geometry are bundles of simple Frobenius algebras.

(b) Transgression and regression induce isomorphisms $A_i \cong \mathcal{A}_i$ between the algebra bundles that arise independently from target space and loop space brane geometry.

At first sight, it seems that (b) implies (a) because endomorphism algebras are automatically simple Frobenius algebras. However, we prove (a) loop-space-intrinsically without any reference to endomorphism bundles, see Proposition 3.4. In fact, we use (a) in the construction of the regression functor $\mathscr{R}$ so that it really is an independent result. Part (b) is proved in Section 6.1.

In Section 3.2 we also discover the following algebraic feature of the vector bundles $\mathcal{R}_{ij}$ over the spaces P_{ij} of paths connecting Q_i with Q_j.

Theorem 3.

(a) The vector bundles $\mathcal{R}_{ij}$ are bundles of bimodules over the algebra bundles $\mathrm{ev}_0^*\mathcal{A}_i$ and $\mathrm{ev}_1^*\mathcal{A}_j$, where $\mathrm{ev}_0 : P_{ij} \to Q_i$ and $\mathrm{ev}_1 : P_{ij} \to Q_j$ are the end point evaluations.

(b) Lifted path concatenation induces connection-preserving isomorphisms $\mathcal{R}_{jk} \otimes_{\mathcal{A}_j} \mathcal{R}_{ij} \cong \mathcal{R}_{ik}$.

Our motivation for exhibiting these algebraic features is the *functorial perspective* to topological field theories [Seg04, ST04, Lur09], where field theories are regarded as functors from certain categories of bordisms to certain algebraic categories. The bordisms are equipped with smooth maps to M in order to incorporate a target space. Bunke-Turner-Willerton have shown that 2-dimensional topological functorial field theories are equivalent to bundle gerbes with connection over M [BTW04]. In an upcoming article [BW21] we will extend this result to an equivalence between TBG and functorial field theories with D-branes, whose values will be given by the algebraic data provided in Theorem 3.

In the setting of functorial field theories, *quantum* theories are defined on bordisms without a map to a target space. Equivalently, a quantum field theory is one with $M = \{*\}$. Quantization is supposed to correspond to the pushforward to a point in a suitable generalized cohomology theory [ST04]. For quantum field theories, it is common to describe functorial field theories by algebraic structure obtained from a presentation of the bordism category in terms of generators and relations. For the closed sector and two dimensions, this leads to commutative Frobenius algebras [Abr96]. For quantum theories with D-branes, the algebraic structure has been determined by Lazaroiu, Moore-Segal, and Lauda-Pfeiffer [Laz01, MS06, LP08], of which the last reference termed it a *colored knowledgable Frobenius algebra* (the "colors" are the brane indices $i \in I$). One can now perform an interesting consistency check between classical and quantum field theories: the restriction of a classical field theory to a point; here, this means putting $M = Q_i = \{*\}$ for all brane indices i. Of course, this will not give "the" quantization of the original theory, but it does give "a" quantum theory and thus should fit into that framework. We show that our LBG passes this consistency check in the sense that there is a naturally defined faithful functor

$$\mathcal{F} : \mathrm{LBG}(*, \{*\}_{i\in I}) \to \text{K-Frob}^{(I)}$$

to the category of I-colored knowledgable Frobenius algebras. Our final result determines the image of this functor. We show that all colored knowledgable Frobenius algebras in its image are equipped with an additional structure that we call a *positive reflection structure*. In our upcoming article [BW21] we will show that it is equivalent – under the correspondence to topological quantum field theories – to a positive reflection structure in the sense of functorial field theories [FH21]. We prove the following.

Theorem 4. *The functor $\mathcal{F}$ induces an equivalence between loop space brane geometries of a point and reflection-positive, I-colored knowledgable Frobenius algebras whose underlying Frobenius algebra is $\mathbb{C}$.*

Thus, LBG is a target space family of reflection-positive, colored knowledgable structures on the Frobenius algebra $\mathbb{C}$. All details and the proof of Theorem 4 are given out in Section 3.3.

This paper is organized in the following way. In Section 2 we give precise definitions of the categories $\mathrm{TBG}(M, Q)$ and $\mathrm{LBG}(M, Q)$, and we recall some relevant aspects about bundle gerbes. In Section 3 we derive the algebraic structures induced by LBG as stated in Theorem 2 (a) and Theorem 3 and discuss the reduction to the work of Moore-Segal [MS06] and Lauda-Pfeiffer [LP08]. In Sections 4 and 5 we construct the transgression and regression functors $\mathscr{T}$ and $\mathscr{R}$, respectively. In Section 6 we prove Theorem 1 and Theorem 2 (b). We conclude with an appendix providing technical background material, in particular about diffeological vector bundles, superficial connections on path spaces, and bundles of algebras and bimodules. For the benefit of the reader we include a small table of notation on page 1194.

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2. Brane geometries

In the following, a *target space* will be a pair (M, Q) of a connected smooth manifold M and a family $Q = \{Q_i\}_{i \in I}$ of submanifolds $Q_i \subseteq M$. We work over a fixed target space; but all definitions and constructions will be natural under maps between target spaces, i.e. smooth maps $f : M \to M'$ such that $f(Q_i) \subseteq Q'_i$.

2.1. Target space brane geometry

As motivated in Section 1, an object in the bicategory $\mathrm{TBG}(M,Q)$ is a pair $(\mathcal{G},\mathcal{E})$, consisting of a hermitian line bundle gerbe $\mathcal{G}$ with connection over M and a family $\mathcal{E}=\{\mathcal{E}_i\}_{i\in I}$ of $\mathcal{G}|_{Q_i}$-twisted vector bundles $\mathcal{E}_i$ with connections.

We recall some minimal facts to explain these notions. Hermitian line bundle gerbes with connection over M form a bicategory $\mathcal{G}rb^\nabla(M)$ that can be defined in the following very elegant way [NS11]. Let $\mathcal{HVB}un^\nabla(M)$ denote the symmetric monoidal category of hermitian vector bundles equipped with (unitary) connections, with the morphisms connection-preserving, unitary bundle morphisms. The assignment $M\mapsto\mathcal{HVB}un^\nabla(M)$ forms a sheaf of symmetric monoidal categories over the site of smooth manifolds (with surjective submersions as coverings). We consider the following bicategory $\mathcal{T}riv\mathcal{G}rb^\nabla(M)$:

(a) Its objects are 2-forms $B\in\Omega^2(M)$, which we will denote by $\mathcal{I}_B$ in this context.

(b) The Hom-category $\mathcal{H}om(\mathcal{I}_{B_1},\mathcal{I}_{B_2})$ is by definition the full subcategory $\mathcal{HVB}un^\nabla(M)^{B_2-B_1}$ of $\mathcal{HVB}un^\nabla(M)$ on the objects (E,∇) with

$$\frac{1}{\mathrm{rk}(E)}\,\mathrm{tr}(\mathrm{curv}(\nabla))=B_2-B_1. \tag{3}$$

(c) Composition is the tensor product in $\mathcal{HVB}un^\nabla(M)$.

The assignment $M\mapsto\mathcal{T}riv\mathcal{G}rb^\nabla(M)$ is a presheaf of bicategories. The sheaf $\mathcal{G}rb^\nabla$ is by definition its sheafification, i.e.

$$\mathcal{G}rb^\nabla:=(\mathcal{T}riv\mathcal{G}rb^\nabla)^+.$$

The objects $\mathcal{I}_B\in\mathcal{T}riv\mathcal{G}rb^\nabla(M)\subseteq\mathcal{G}rb^\nabla(M)$ are now called *trivial* bundle gerbes. Spelling this out results exactly in the usual notion of bundle gerbes [Mur96, MS00, Ste00] with the bicategorical structure described in [Wal07]. In particular, 1-morphisms in $\mathcal{G}rb^\nabla(M)$ have a well-defined rank, and a 1-morphism is a 1-*iso*morphism ("stable isomorphism") if and only if it has rank one [Wal07, Prop. 3].

We will use a slightly generalized version of this bicategory, obtained by performing the sheafification in the site of diffeological spaces (with respect to the Grothendieck topology subductions), and then restricting again to smooth manifolds. The occurring vector bundles have to be treated as vector bundles over diffeological spaces (see Appendix A.1). This generalization

results in an equivalent bicategory, see [Wal16, Section 3.1] for more information. It is necessary because the regression functor defined in Section 5 takes values in the generalized version of bundle gerbes.

If $\mathcal{G}$ is a bundle gerbe with connection over M, then a *$\mathcal{G}$-twisted vector bundle with connection* is a 2-form $\omega \in \Omega^2(M)$ together with a 1-morphism $\mathcal{E} : \mathcal{G} \to \mathcal{I}_\omega$ in $\mathcal{G}rb^\nabla(M)$. This gives precisely the usual notion of a "bundle gerbe module with connection" introduced in [CJM02], see [Wal07]. A twisted vector bundle of rank one is called a *trivialization* of $\mathcal{G}$.

We return to the definition of the bicategory $\mathrm{TBG}(M, Q)$ of target space brane geometry, whose objects are pairs $(\mathcal{G}, \mathcal{E})$ of a hermitian line bundle gerbe $\mathcal{G}$ with connection over M and a family $\mathcal{E} = \{\mathcal{E}_i\}_{i\in I}$ of $\mathcal{G}|_{Q_i}$-twisted vector bundles $\mathcal{E}_i$ with connections. The 1-morphisms between $(\mathcal{G}, \mathcal{E})$ and $(\mathcal{G}', \mathcal{E}')$ are pairs $(\mathcal{A}, \psi)$, consisting of a 1-isomorphism $\mathcal{A} : \mathcal{G} \to \mathcal{G}'$ in $\mathcal{G}rb^\nabla(M)$ and a family $\psi = \{\psi_i\}_{i\in I}$ of 2-morphisms $\psi_i : \mathcal{E}_i \Rightarrow \mathcal{E}'_i \circ \mathcal{A}|_{Q_i}$ in $\mathcal{G}rb^\nabla(Q_i)$. The 2-morphisms between $(\mathcal{A}, \psi)$ and $(\mathcal{A}', \psi')$ are 2-morphisms $\varphi : \mathcal{A} \Rightarrow \mathcal{A}'$ in $\mathcal{G}rb^\nabla(M)$ such that the diagram

$$\begin{tikzcd}
\mathcal{E}_i \arrow[rr, Rightarrow, "\psi_i"] \arrow[dr, Rightarrow, "\psi'_i"'] & & \mathcal{E}'_i \circ \mathcal{A}|_{Q_i} \arrow[dl, Rightarrow, "\mathrm{id}_{\mathcal{E}'_i} \circ \varphi|_{Q_i}"] \\
& \mathcal{E}'_i \circ \mathcal{A}'|_{Q_i} &
\end{tikzcd}$$

of 2-morphisms in $\mathcal{G}rb^\nabla(Q_i)$ is commutative for all $i \in I$.

In the remainder of this subsection we discuss an operation on 1-morphisms between bundle gerbes that will be used frequently throughout this article. Namely, for bundle gerbes $\mathcal{G}$, $\mathcal{H} \in \mathcal{G}rb^\nabla(M)$ there is a functor

$$\Delta : \mathcal{H}om(\mathcal{G}, \mathcal{I}_{\omega_2}) \times \mathcal{H}om(\mathcal{G}, \mathcal{I}_{\omega_1})^{op} \to \mathcal{HVB}un^\nabla(M)^{\omega_2-\omega_1}, \tag{4}$$

that can be seen as an enriched version of an internal hom, see [BSS18] and [Bun17, Thm. 3.6.3], and [BS17] for a less technical overview. It is defined in the following way. We first consider the functor $()^* : \mathcal{H}om(\mathcal{G}, \mathcal{H})^{op} \to \mathcal{H}om(\mathcal{H}, \mathcal{G})$, which on the level of the presheaf $\mathcal{T}riv\mathcal{G}rb^\nabla(M)$ is just the dualization $E \mapsto E^*$. Since this is a morphism of presheaves, it survives the sheafification and induces the claimed functor $()^*$. Now, Δ is defined by

$$\begin{aligned}
\mathcal{H}om(\mathcal{G}, \mathcal{I}_{\omega_2}) \times \mathcal{H}om(\mathcal{G}, \mathcal{I}_{\omega_1})^{op} &\xrightarrow{\mathrm{id}\times()^*} \mathcal{H}om(\mathcal{G}, \mathcal{I}_{\omega_2}) \times \mathcal{H}om(\mathcal{I}_{\omega_1}, \mathcal{G}) \\
&\xrightarrow{\circ} \mathcal{H}om(\mathcal{I}_{\omega_1}, \mathcal{I}_{\omega_2}) \cong \mathcal{HVB}un^\nabla(M)^{\omega_2-\omega_1}.
\end{aligned}$$

Since the functor Δ is quite important in this article, we spell out its definition in terms of the explicit definition of bundle gerbes. Suppose $\mathcal{G}$ consists of a surjective submersion $Y \to M$, a hermitian line bundle L with connection over $Y^{[2]} := Y \times_M Y$, and a bundle gerbe product μ over $Y^{[3]}$. Twisted vector bundles $\mathcal{E}_i : \mathcal{G} \to \mathcal{I}_{\omega_i}$ $(i = 1, 2)$ consist of hermitian vector bundles E_i with connection over Y, and of connection-preserving, unitary bundle isomorphisms

$$\zeta_i : L \otimes \mathrm{pr}_2^* E_i \to \mathrm{pr}_1^* E_i$$

over $Y^{[2]}$, satisfying a compatibility condition with the bundle gerbe product μ over $Y^{[3]}$, see [Wal07]. In order to define the vector bundle $\Delta(\mathcal{E}_2, \mathcal{E}_1)$, we consider the vector bundle $E_1^* \otimes E_2$ over Y, and over $Y^{[2]}$ the bundle isomorphism $\tilde{\zeta}$ defined by

$$\begin{aligned} (5) \quad \mathrm{pr}_2^*(E_1^* \otimes E_2) &\cong \mathrm{pr}_2^* E_1^* \otimes L^* \otimes L \otimes \mathrm{pr}_2^* E_2 \\ &\xrightarrow{\zeta_1^{tr-1} \otimes \zeta_2} \mathrm{pr}_1^* E_1^* \otimes \mathrm{pr}_1^* E_2 \cong \mathrm{pr}_1^*(E_1^* \otimes E_2). \end{aligned}$$

Here we have used the coevaluation isomorphisms between a complex line bundle and its dual, and $(..)^{tr}$ denotes the transpose of a linear map. The conditions for ζ_1 and ζ_2 imply a cocycle condition for $\tilde{\zeta}$ over $Y^{[3]}$. Since $\tilde{\zeta}$ is connection-preserving and unitary, $E_1^* \otimes E_2$ descends to a hermitian vector bundle $\Delta(\mathcal{E}_2, \mathcal{E}_1)$ with connection over M; this gives the definition of Δ.

Remark 2.1. We observe the following features of the functor Δ:

(a) For three twisted vector bundles $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ we have a morphism

$$\Delta(\mathcal{E}_3, \mathcal{E}_2) \otimes \Delta(\mathcal{E}_2, \mathcal{E}_1) \to \Delta(\mathcal{E}_3, \mathcal{E}_1)$$

in $\mathcal{HVB}un^\nabla(M)$, which is an isomorphism if and only if $\mathcal{E}_2$ is invertible. It is induced by a 2-morphism $\mathcal{E}_2^* \circ \mathcal{E}_2 \Rightarrow \mathrm{id}_{\mathcal{I}_{\omega_2}}$, which is in turn induced by the evaluation $E_2 \otimes E_2^* \to \mathbb{C}$ of dual vector bundles.

(b) For two twisted vector bundles $\mathcal{E}_i : \mathcal{G} \to \mathcal{I}_{\omega_i}$ we have an isomorphism $\Delta(\mathcal{E}_1, \mathcal{E}_2) \to \Delta(\mathcal{E}_2, \mathcal{E}_1)^*$ in $\mathcal{HVB}un^\nabla(M)$, induced by the canonical isomorphism $(\mathcal{E}_1 \circ \mathcal{E}_2^*)^* = \mathcal{E}_2 \circ \mathcal{E}_1^*$.

(c) For two twisted vector bundles $\mathcal{E}_i : \mathcal{G} \to \mathcal{I}_{\omega_i}$ and a 1-morphism $\mathcal{A} : \mathcal{G}' \to \mathcal{G}$ in $\mathcal{G}rb^\nabla(M)$ we have a morphism

$$\Delta(\mathcal{E}_1 \circ \mathcal{A}, \mathcal{E}_2 \circ \mathcal{A}) \to \Delta(\mathcal{E}_1, \mathcal{E}_2)$$

in $\mathcal{HVB}un^{\nabla}(M)$, which is an isomorphism if and only if $\mathcal{A}$ is invertible. It is induced from the 2-morphism $\mathcal{A} \circ \mathcal{A}^* \Rightarrow \mathrm{id}_{\mathcal{G}}$ and a 2-isomorphism $(\mathcal{E}_2 \circ \mathcal{A})^* \Rightarrow \mathcal{A}^* \circ \mathcal{E}_2^*$, which is in turn induced by the canonical isomorphism $(A \circ E_2)^* \cong E_2^* \otimes A^*$.

Remark 2.2. If $\mathcal{E}$ is a $\mathcal{G}$-twisted vector bundle, we obtain a hermitian vector bundle $\mathrm{End}(\mathcal{E}) := \Delta(\mathcal{E}, \mathcal{E})$ with connection. Remark 2.1 (a) provides a connection-preserving bundle morphism

$$\mathrm{End}(\mathcal{E}) \otimes \mathrm{End}(\mathcal{E}) \to \mathrm{End}(\mathcal{E}), \tag{6}$$

which endows the fibers of $\mathrm{End}(\mathcal{E})$ with (associative, unital, complex) algebra structures. The fact that $\mathrm{End}(\mathcal{E})$ carries a connection for which (6) is connection-preserving, together with the fact that all fibers are simple algebras, assures that $\mathrm{End}(\mathcal{E})$ is an algebra bundle, see Lemmas A.13 and A.14. For an explicit construction, one notices that the descent isomorphism $\tilde{\zeta}$ of (5) is an isomorphism between algebra bundles [STV14]. In relation to twisted K-theory the same algebra bundle has been constructed in [Kar12].

Remark 2.3. Applying the previous remark to a TBG object $(\mathcal{G}, \mathcal{E})$, we obtain bundles of central simple algebras $\mathrm{End}(\mathcal{E}_i)$ over the branes Q_i. The assignment of these bundles is functorial with respect to TBG morphisms. Indeed, if $(\mathcal{A}, \psi)$ is a 1-morphism, then we obtain a bundle morphism

$$\Delta(\mathcal{E}_i, \mathcal{E}_i) \xrightarrow{\Delta(\psi_i, \psi_i^{-1})} \Delta(\mathcal{E}_i' \circ \mathcal{A}, \mathcal{E}_i' \circ \mathcal{A}) \longrightarrow \Delta(\mathcal{E}_i', \mathcal{E}_i')$$

by Remark 2.1 (c). Following Remark 2.1 it is easy to check that this bundle morphism preserves the algebra structures. Furthermore, one can check that it only depends on the 2-isomorphism class of $(\mathcal{A}, \psi)$. Finally, the composition of TBG morphisms induces the composition of algebra bundle homomorphisms. Summarizing, for every brane index $i \in I$ we have constructed a functor $(\mathcal{G}, \mathcal{E}) \mapsto \mathrm{End}(\mathcal{E}_i)$ from $\mathrm{h}_1\mathrm{TBG}(M, Q)$ to the category of central simple algebra bundles over Q_i.

Remark 2.4. We recall the following facts about trivializations of bundle gerbes, described in [Wal16, Lemma 3.2.3]. Suppose $\mathcal{G} \in \mathcal{G}rb^{\nabla}(M)$ and $s : M \to Y$ is a smooth section into the surjective submersion of $\mathcal{G}$. Then, we obtain a 1-morphism $\mathcal{T}_s : \mathcal{G} \to \mathcal{I}_{s^*B}$ in $\mathcal{G}rb^{\nabla}(M)$, where B is the curving of $\mathcal{G}$. Explicitly, it is defined by the hermitian line bundle $T_s := (\mathrm{id}_Y, s \circ \pi)^* L$ over Y and the isomorphism $\sigma_s := (\mathrm{pr}_1, \mathrm{pr}_2, s \circ \pi)^* \mu$, where L is the line

bundle of $\mathcal{G}$, and μ is its bundle gerbe product. We call $\mathcal{T}_s$ the *trivialization associated to the section* s. If $s' : M \to Y$ is another section, and $\tilde{s} : M \to L$ is a parallel unit-length section along $(s, s') : M \to Y^{[2]}$, then we obtain a 2-isomorphism $\psi_{\tilde{s}} : \mathcal{T}_s \Rightarrow \mathcal{T}_{s'}$ in $\mathcal{G}rb^{\nabla}(M)$, given fiber-wise by

$$\begin{aligned} T_s|_y = L_{y,s(\pi(y))} \xrightarrow{\mathrm{id}\otimes\tilde{s}} L_{y,s(\pi(y))} \otimes L_{s(\pi(y)),s'(\pi(y))} \\ \xrightarrow{\mu} L_{y,s'(\pi(y))} = T_{s'}|_y \ . \end{aligned}$$

Now, consider a 1-morphism $\mathcal{E} : \mathcal{G} \to \mathcal{I}_\rho$ in $\mathcal{G}rb^{\nabla}(M)$, consisting of a vector bundle E over Y and a bundle isomorphism ζ over $Y^{[2]}$, and a section $s : M \to Y$. Then, there is a connection-preserving, unitary bundle isomorphism $\varphi_s : \Delta(\mathcal{E}, \mathcal{T}_s) \to s^*E$, which is induced by

$$T_s|_y^* \otimes E|_y = L|_{y,s(\pi(y))}^* \otimes E|_y \xrightarrow{\tilde{\mu}\otimes\mathrm{id}} L|_{s(\pi(y)),y} \otimes E|_y \xrightarrow{\zeta|_{s(\pi(y)),y}} E|_{s(\pi(y))}.$$

Here, $\tilde{\mu} : L|_{y_1,y_2}^* \to L|_{y_2,y_1}$ is induced from the bundle gerbe product. In the presence of a second section s' and a parallel unit-length section $\tilde{s}$ of L along (s, s') we obtain from the definitions a commutative diagram

$$\begin{tikzcd} \Delta(\mathcal{E}, \mathcal{T}_{s'}) \arrow[r, "\varphi_{s'}"] \arrow[d, "\Delta(\mathrm{id},\psi_{\tilde{s}})"'] & s'^*E \arrow[d, "\zeta(\tilde{s}\otimes-)"] \\ \Delta(\mathcal{E}, \mathcal{T}_s) \arrow[r, "\varphi_s"'] & s^*E. \end{tikzcd}$$

2.2. Loop space brane geometry

We let $LM := C^\infty(S^1, M)$ be the loop space of M, considered as a Fréchet manifold or, equivalently, as a diffeological space. We let $PM \subseteq C^\infty([0,1], M)$ be the subset of paths with sitting instants, considered as a diffeological space. Further, we let $P_{ij} \subseteq PM$ be the subspace of paths γ with $\gamma(0) \in Q_i$ and $\gamma(1) \in Q_j$. Vector bundles and connections on diffeological spaces are discussed in Appendix A.1; essentially, they can be treated just like vector bundles over smooth manifolds.

We now describe the objects of the category $\mathrm{LBG}(M, Q)$, making the announcements of Section 1 precise. An object is a septuple $(\mathcal{L}, \lambda, \mathcal{R}, \phi, \chi, \epsilon, \alpha)$ consisting of the following structure:

(1) A hermitian line bundle $\mathcal{L}$ over LM with a superficial connection.
The definition of "superficial" is [Wal16, Definition 2.2.1]. We recall that a *thin homotopy* between loops $\tau, \tau' : S^1 \to M$ is a path

$h : [0,1] \to LM$ such that the differential of its adjoint map $h^\vee : [0,1] \times S^1 \to M$ has rank less than two everywhere. If τ, τ' are thin homotopic, then the parallel transport of a superficial connection along a thin homotopy h between τ and τ' is independent of the choice of h, and hence determines a *canonical* unitary isomorphism $d_{\tau,\tau'} : \mathcal{L}|_\tau \to \mathcal{L}|_{\tau'}$, independent of the thin homotopy.

(2) A fusion product λ on $\mathcal{L}$, i.e. unitary isomorphisms

$$\lambda_{\gamma_1,\gamma_2,\gamma_3} : \mathcal{L}|_{\gamma_1\cup\gamma_2} \otimes \mathcal{L}|_{\gamma_2\cup\gamma_3} \to \mathcal{L}|_{\gamma_1\cup\gamma_3}$$

for all triples $(\gamma_1, \gamma_2, \gamma_3)$ of paths in M with a common initial point and a common end point, forming a connection-preserving unitary bundle isomorphism over the 3-fold fibre product $PM^{[3]}$ of $\mathrm{ev} : PM \to M \times M$ with itself. The fusion product is required to be associative over $PM^{[4]}$ and to be symmetrizing with respect to the connection ([Wal16, Def. 2.1.5]); this latter property only plays a minor role in the present article, though.
The fusion product induces a parallel, unit-length section $PM \to \mathcal{L} : \gamma \mapsto \nu_\gamma$ along the inclusion of "flat" loops $PM \to LM : \gamma \mapsto \gamma \cup \gamma$, which is neutral with respect to fusion [Wal16, Lemma 2.1.4]. Further, we obtain a connection-preserving, unitary isomorphism $\tilde{\lambda}_{\gamma_1,\gamma_2} : \mathcal{L}|_{\gamma_1\cup\gamma_2} \to \mathcal{L}^*|_{\gamma_2\cup\gamma_1}$ that we will often combine with the identification $\mathcal{L}^* \cong \overline{\mathcal{L}}$ between the dual and the complex conjugate line bundle, induced by the hermitian metric on $\mathcal{L}$.

(3) A family $\mathcal{R} = \{\mathcal{R}_{ij}\}_{i,j\in I}$ of hermitian vector bundles $\mathcal{R}_{ij}$ over P_{ij} with superficial connections pt_{ij}.
We refer to Definition A.11 in Appendix A.2 for the definition of "superficial" for connections on vector bundles over path spaces. Similarly to (1), a superficial connection determines – via parallel transport – a canonical unitary isomorphism $d_{\gamma,\gamma'} : \mathcal{R}_{ij}|_\gamma \to \mathcal{R}_{ij}|_{\gamma'}$ between the fibers of $\mathcal{R}_{ij}$ over all pairs of fixed-ends thin homotopic paths $\gamma, \gamma' \in P_{ij}$.

(4) A family $\phi = \{\phi_{ij}\}_{i,j\in I}$ of unitary isomorphisms

$$\phi_{ij}|_{\gamma_1,\gamma_2} : \mathcal{L}|_{\gamma_1\cup\gamma_2} \otimes \mathcal{R}_{ij}|_{\gamma_2} \to \mathcal{R}_{ij}|_{\gamma_1}$$

for all $\gamma_1, \gamma_2 \in P_{ij}$ with common initial point and common end point. These have to form connection-preserving bundle isomorphisms ϕ_{ij} over $P_{ij}^{[2]} := P_{ij} \times_{Q_i\times Q_j} P_{ij}$.

(5) A family $\chi = \{\chi_{ijk}\}_{i,j,k\in I}$ of linear maps

$$\chi_{ijk}|_{\gamma_{12},\gamma_{23}} : \mathcal{R}_{jk}|_{\gamma_{23}} \otimes \mathcal{R}_{ij}|_{\gamma_{12}} \to \mathcal{R}_{ik}|_{\gamma_{23}\star\gamma_{12}}$$

for all composable paths γ_{12} and γ_{23}. These have to form a connection-preserving bundle morphism χ_{ijk} over $P_{jk} \times_{Q_j} P_{ij}$.

(6) A family $\epsilon = \{\epsilon_i\}_{i\in I}$ of parallel sections $\epsilon_i : Q_i \to \mathcal{R}_{ii}$ along the inclusion $x \mapsto \mathsf{c}_x$ of constant paths.

(7) A family $\alpha = \{\alpha_{ij}\}_{i,j\in I}$ of unitary isomorphisms

$$\alpha_{ij}|_\gamma : \mathcal{R}_{ij}|_\gamma \to \overline{\mathcal{R}_{ji}}|_{\overline{\gamma}}$$

for all $\gamma \in P_{ij}$, where $\overline{\gamma}$ denotes the reversed path. These have to form a connection-preserving bundle isomorphism over P_{ij}.

This structure has to satisfy the following axioms:

(LBG1) The isomorphisms ϕ_{ij} are compatible with the fusion product λ; that is, the diagram

$$\begin{tikzcd}
\mathcal{L}|_{\gamma_1\cup\gamma_2} \otimes \mathcal{L}|_{\gamma_2\cup\gamma_3} \otimes \mathcal{R}_{ij}|_{\gamma_3} \arrow[r, "\mathrm{id}\otimes\phi_{ij}|_{\gamma_2,\gamma_3}"] \arrow[d, "\lambda_{\gamma_1,\gamma_2,\gamma_3}\otimes\mathrm{id}"'] & \mathcal{L}|_{\gamma_1\cup\gamma_2} \otimes \mathcal{R}_{ij}|_{\gamma_2} \arrow[d, "\phi_{ij}|_{\gamma_1,\gamma_2}"] \\
\mathcal{L}|_{\gamma_1\cup\gamma_3} \otimes \mathcal{R}_{ij}|_{\gamma_3} \arrow[r, "\phi_{ij}|_{\gamma_1,\gamma_3}"'] & \mathcal{R}_{ij}|_{\gamma_1}
\end{tikzcd}$$

is commutative for all $(\gamma_1, \gamma_2, \gamma_3) \in P_{ij}^{[3]}$. We say that ϕ_{ij} is a *fusion representation* of $(\mathcal{L}, \lambda)$ on $\mathcal{R}_{ij}$.

(LBG2) The maps χ_{ijk} are associative up to parallel transport along a reparameterization; that is, the diagram

$$\begin{tikzcd}
& \mathcal{R}_{kl}|_{\gamma_{34}} \otimes \mathcal{R}_{jk}|_{\gamma_{23}} \otimes \mathcal{R}_{ij}|_{\gamma_{12}} \arrow[dl, "\chi_{jkl}|_{\gamma_{23},\gamma_{34}}\otimes\mathrm{id}"'] \arrow[dr, "\mathrm{id}\otimes\chi_{ijk}|_{\gamma_{12},\gamma_{23}}"] & \\
\mathcal{R}_{jl}|_{\gamma_{34}\star\gamma_{23}} \otimes \mathcal{R}_{ij}|_{\gamma_{12}} \arrow[d, "\chi_{ijl}|_{\gamma_{12},\gamma_{34}\star\gamma_{23}}"'] & & \mathcal{R}_{kl}|_{\gamma_{34}} \otimes \mathcal{R}_{ik}|_{\gamma_{23}\star\gamma_{12}} \arrow[d, "\chi_{ikl}|_{\gamma_{23}\star\gamma_{12},\gamma_{34}}"] \\
\mathcal{R}_{il}|_{(\gamma_{34}\star\gamma_{23})\star\gamma_{12}} \arrow[rr, "d"'] & & \mathcal{R}_{il}|_{\gamma_{34}\star(\gamma_{23}\star\gamma_{12})}.
\end{tikzcd}$$

is commutative for all triples $\gamma_{12} \in P_{ij}$, $\gamma_{23} \in P_{jk}$ and $\gamma_{13} \in P_{ik}$ of composable paths, and d is the canonical isomorphism of (3). We say that χ is a *lifted path concatenation* on $\mathcal{R}$.

(LBG3) Fusion representation and lifted path concatenation are compatible in the following sense. Suppose $\gamma_{12}, \gamma'_{12} \in P_{ij}$ and $\gamma_{23}, \gamma'_{23} \in P_{jk}$ connect three points in the following way:

$$\begin{tikzcd} x \arrow[r, bend left, "\gamma_{12}"] \arrow[r, bend right, "\gamma'_{12}"'] & y \arrow[r, bend left, "\gamma_{23}"] \arrow[r, bend right, "\gamma'_{23}"'] & z. \end{tikzcd}$$

Then, the diagram

$$\begin{tikzcd} \mathcal{L}|_{\gamma_{23}\cup\gamma'_{23}} \otimes \mathcal{L}|_{\gamma_{12}\cup\gamma'_{12}} \otimes \mathcal{R}_{jk}|_{\gamma'_{23}} \otimes \mathcal{R}_{ij}|_{\gamma'_{12}} \arrow[r, "\lambda'\otimes\chi_{ijk}"] \arrow[d, "\mathrm{id}\otimes braid\otimes\mathrm{id}"'] & \mathcal{L}|_{(\gamma_{23}\star\gamma_{12})\cup(\gamma'_{23}\star\gamma'_{12})} \otimes \mathcal{R}_{ik}|_{\gamma'_{23}\star\gamma'_{12}} \arrow[dd, "\phi_{ik}"] \\ \mathcal{L}|_{\gamma_{23}\cup\gamma'_{23}} \otimes \mathcal{R}_{jk}|_{\gamma'_{23}} \otimes \mathcal{L}|_{\gamma_{12}\cup\gamma'_{12}} \otimes \mathcal{R}_{ij}|_{\gamma'_{12}} \arrow[d, "\phi_{jk}\otimes\phi_{ij}"'] & \\ \mathcal{R}_{jk}|_{\gamma_{23}} \otimes \mathcal{R}_{ij}|_{\gamma_{12}} \arrow[r, "\chi_{ijk}"'] & \mathcal{R}_{ik}|_{\gamma_{23}\star\gamma_{12}} \end{tikzcd}$$

has to be commutative. The isomorphism λ' on the top of this diagram is given by

$$\begin{tikzcd} \mathcal{L}|_{\gamma_{23}\cup\gamma'_{23}} \otimes \mathcal{L}|_{\gamma_{12}\cup\gamma'_{12}} \arrow[r, "d\otimes d"] & \mathcal{L}|_{(\overline{\gamma'_{23}}\star\gamma_{23})\cup\mathsf{c}_y} \otimes \mathcal{L}|_{\mathsf{c}_y\cup(\gamma'_{12}\star\overline{\gamma_{12}})} \arrow[d, "\lambda"] & \\ & \mathcal{L}|_{(\overline{\gamma'_{23}}\star\gamma_{23})\cup(\gamma'_{12}\star\overline{\gamma_{12}})} \arrow[r, "d"'] & \mathcal{L}|_{(\gamma_{23}\star\gamma_{12})\cup(\gamma'_{23}\star\gamma'_{12})}. \end{tikzcd}$$

(LBG4) The sections ϵ_i provide units (up to reparameterization) for the lifted path concatenation, i.e.

$$\chi_{iij}|_{\mathsf{c}_x,\gamma}(v, \epsilon_i(x)) = d_{\gamma,\gamma\star\mathsf{c}_x}(v) \quad \text{and} \quad \chi_{ijj}|_{\gamma,\mathsf{c}_y}(\epsilon_j(y), v) = d_{\gamma,\mathsf{c}_y\star\gamma}(v)$$

for all paths $\gamma \in P_{ij}$ with $x := \gamma(0)$ and $y := \gamma(1)$, and all $v \in \mathcal{R}_{ij}|_\gamma$. We say that ϵ_i is a *lifted constant path.*

(LBG5) The isomorphisms α_{ij} satisfy the following compatibility condition with the hermitian metric h_{ij} on $\mathcal{R}_{ij}$ and the lifted path concatenation: for $\gamma \in P_{ij}$ and elements $v, w \in \mathcal{R}_{ij}|_\gamma$ we have

$$\begin{aligned} &h_{ii}(\chi_{iji}|_{\gamma,\overline{\gamma}}(\alpha_{ij}(w)\otimes v), d_{\mathsf{c}_x,\overline{\gamma}\star\gamma}(\epsilon_i(x))) \\ &\qquad = h_{ij}(v,w) = h_{jj}(d_{\mathsf{c}_x,\gamma\star\overline{\gamma}}(\epsilon_j(x)), \chi_{jij}|_{\overline{\gamma},\gamma}(w\otimes\alpha_{ij}(v))). \end{aligned}$$

We will say that α_{ij} is a *lifted path reversal.*

(LBG6) Lifted constant paths are invariant under the lifted path reversal:

$$\alpha_{ii}|_{\mathsf{c}_x}(\epsilon_i(x)) = \epsilon_i(x).$$

(LBG7) Lifted path reversal is involutive: $\alpha_{ji} \circ \alpha_{ij} = \mathrm{id}_{\mathcal{R}_{ij}}$.

(LBG8) Lifted path reversal is an anti-homomorphism with respect to lifted path concatenation: the diagram

$$\begin{tikzcd}
\mathcal{R}_{jk}|_{\gamma_2} \otimes \mathcal{R}_{ij}|_{\gamma_1} \arrow[r, "\chi_{ijk}|_{\gamma_1,\gamma_2}"] \arrow[d, "\alpha_{jk}\otimes\alpha_{ij}"'] & \mathcal{R}_{ik}|_{\gamma_2\star\gamma_1} \arrow[dd, "\alpha_{ik}"] \\
\overline{\mathcal{R}_{kj}}|_{\overline{\gamma_2}} \otimes \overline{\mathcal{R}_{ji}}|_{\overline{\gamma_1}} \arrow[d, "braid"'] & \\
\overline{\mathcal{R}_{ji}}|_{\overline{\gamma_1}} \otimes \overline{\mathcal{R}_{kj}}|_{\overline{\gamma_2}} \arrow[r, "\chi_{kji}|_{\overline{\gamma_2},\overline{\gamma_1}}"'] & \overline{\mathcal{R}_{ki}}|_{\overline{\gamma_1}\star\overline{\gamma_2}}
\end{tikzcd}$$

is commutative for all $\gamma_1 \in P_{ij}$ and $\gamma_2 \in P_{jk}$.

(LBG9) Lifted path reversal intertwines the fusion representation (up to reparameterization), in the sense that the diagram

$$\begin{tikzcd}
\mathcal{L}|_{\gamma_1\cup\gamma_2} \otimes \mathcal{R}_{ij}|_{\gamma_2} \arrow[r, "\phi_{ij}"] \arrow[d, "\tilde{\lambda}\otimes\alpha_{ij}"'] & \mathcal{R}_{ij}|_{\gamma_1} \arrow[dd, "\alpha_{ij}"] \\
\overline{\mathcal{L}}|_{\gamma_2\cup\gamma_1} \otimes \overline{\mathcal{R}}_{ji}|_{\overline{\gamma_2}} \arrow[d, "d\otimes\mathrm{id}"'] & \\
\overline{\mathcal{L}}|_{\overline{\gamma_1}\cup\overline{\gamma_2}} \otimes \overline{\mathcal{R}}_{ji}|_{\overline{\gamma_2}} \arrow[r, "\phi_{ji}"'] & \overline{\mathcal{R}_{ji}}|_{\overline{\gamma_1}}
\end{tikzcd}$$

is commutative, where $\tilde{\lambda}$ was explained in (2).

(LBG10) The following normalization condition holds between the lifted path concatenation and the hermitian metric h_{ij} on $\mathcal{R}_{ij}$: for all $\gamma \in P_{ij}$ with $x := \gamma(0)$ and $y := \gamma(1)$, every orthonormal basis $(v_1, ..., v_n)$ of $\mathcal{R}_{ij}|_\gamma$ and every $v \in \mathcal{R}_{ii}|_{\mathsf{c}_x}$:

$$\sum_{k=1}^{n} \chi_{jij}|_{\overline{\gamma},\gamma\star\mathsf{c}_x}(\chi_{iij}|_{\mathsf{c}_x,\gamma}(v_k \otimes v) \otimes \alpha_{ij}(v_k)) = h_{ii}(\epsilon_i(x), v) \cdot d_{\mathsf{c}_y,\gamma\star\mathsf{c}_x\star\overline{\gamma}}(\epsilon_j(y)).$$

Loop space brane geometry as defined above forms a groupoid $\mathrm{LBG}(M, Q)$ in a natural way: a morphism between $(\mathcal{L}, \lambda, \mathcal{R}, \phi, \chi, \epsilon, \alpha)$ and

$(\mathcal{L}', \lambda', \mathcal{R}', \phi', \chi', \epsilon', \alpha')$ is a pair (φ, ξ) consisting of a connection-preserving unitary isomorphism $\varphi : \mathcal{L} \to \mathcal{L}'$ that preserves the fusion products λ and λ', and of a family $\xi = \{\xi_{ij}\}_{i,j\in I}$ of connection-preserving, unitary bundle isomorphisms $\xi_{ij} : \mathcal{R}_{ij} \to \mathcal{R}'_{ij}$ that are compatible with the remaining structure in the obvious way. In the remainder of this subsection we deduce some direct consequences from the axioms of LBG in the following remarks.

Remark 2.5. From (LBG1) one can easily deduce that the section ν in $\mathcal{L}$ described in (2) is neutral for the fusion representation, i.e. $\phi_{ij}|_{\gamma,\gamma}(\nu_\gamma, v) = v$ for all $v \in \mathcal{R}_{ij}|_\gamma$.

Remark 2.6. In the context of field theories, (LBG10) will be responsible for the Cardy condition, see Section 3.3. Here we remark the following consequences of (LBG10) for the rank of the vector bundles $\in \mathcal{R}_{ij}$:

(a) We have

$$\|\epsilon_i(x)\| = \sqrt[4]{\mathrm{rk}(\mathcal{R}_{ii})},$$

where the norm is formed fibrewise using the hermitian metric on $\mathcal{R}_{ii}$. Indeed, we set $i = j$, $\gamma = \mathsf{c}_x$ and $v = \epsilon_i(x)$ in (LBG10), and obtain using (LBG4):

$$\sum_{k=1}^{n} \chi_{iii}|_{\mathsf{c}_x,\mathsf{c}_x}(v_k \otimes \alpha_{ii}(v_k)) = h_{ii}(\epsilon_i(x), \epsilon_i(x)) \cdot \epsilon_i(x).$$

Inserting this into $h_{ii}(\epsilon_i(x), -)$ and using (LBG5) gives

$$\sum_{k=1}^{n} h_{ii}(v_k, v_k) = h_{ii}(\epsilon_i(x), \epsilon_i(x))^2.$$

Since $(v_1, ..., v_n)$ is an orthonormal basis, we have the claim.

(b) Putting $v = \epsilon_i(x)$ in (LBG10) we obtain using (a)

$$\sum_{k=1}^{n} \chi_{jij}|_{\overline{\gamma},\gamma}(v_k \otimes \alpha_{ij}(v_k)) = \sqrt{\mathrm{rk}(\mathcal{R}_{ii})} \cdot d_{\mathsf{c}_y,\gamma\star\overline{\gamma}}(\epsilon_j(y)). \tag{7}$$

We insert (7) into $h_{jj}(-, d_{\mathsf{c}_y,\gamma\star\overline{\gamma}}(\epsilon_j(y)))$ and obtain

$$\sum_{k=1}^{n} h_{jj}(\chi_{jij}|_{\overline{\gamma},\gamma}(v_k \otimes \alpha_{ij}(v_k)), d_{\mathsf{c}_y,\gamma\star\overline{\gamma}}(\epsilon_j(y))) = \sqrt{\mathrm{rk}(\mathcal{R}_{ii})} \cdot h_{jj}(\epsilon_j(y), \epsilon_j(y)).$$

With (LBG5) the left hand side sums up to $n = \mathrm{rk}(\mathcal{R}_{ij})$, so that we obtain:

$$\mathrm{rk}(\mathcal{R}_{ij})^2 = \mathrm{rk}(\mathcal{R}_{ii}) \cdot \mathrm{rk}(\mathcal{R}_{jj}).$$

Remark 2.7. Let $L_i := \{\gamma \in P_{ii} \mid \gamma(0) = \gamma(1)\}$, coming with an injective smooth map $L_i \to LM$ which we usually drop from notation (the image consists of loops based in Q_i with sitting instants at $1 \in S^1$). Consider the map $L_i \to P_{ii} \times_{Q_i \times Q_i} P_{ii} : \gamma \mapsto (\mathsf{c}_x, \gamma)$, where $x := \gamma(0)$. The pullback of the fusion representation along this map is a connection-preserving bundle isomorphism

$$\phi_{ii}|_{\mathsf{c}_x,\gamma} : \mathcal{L}_{\mathsf{c}_x \cup \gamma} \otimes \mathcal{R}_{ii}|_{\mathsf{c}_x} \to \mathcal{R}_{ii}|_\gamma.$$

Using the superficial connection on $\mathcal{L}$ to identify $\mathcal{L}_{\mathsf{c}_x \cup \gamma} \cong \mathcal{L}_\gamma$, and the lifted constant path $\epsilon_i(x)$ we obtain a connection-preserving bundle morphism

$$\iota_i : \mathcal{L} \to \mathcal{R}_{ii}|_{L_i}$$

over L_i. We call it the *string opening morphism*; in the context of topological field theories it will describe the phenomenon that a closed string opens up in the presence of a D-brane [BW21]. In order to describe the converse phenomenon, we define a smooth, fiber-wise linear map $\theta_i : \mathsf{c}^*\mathcal{R}_{ii} \to \mathbb{C}$ by $\theta_i(v) := h_{ii}(\epsilon_i(x), v)$, where $v \in \mathcal{R}_{ii}|_{\mathsf{c}_x}$. We will reveal it in Section 3.1 as the trace of a certain Frobenius algebra. Now, the inverse of ϕ_{ii} in combination with θ_i gives a connection-preserving bundle morphism

$$\iota_i^* : \mathcal{R}_{ii}|_{L_i} \to \mathcal{L}$$

over L_i, which is called the *string closing morphism*.

3. Algebraic structures in loop space brane geometry

In this section we study LBG intrinsically, without relation to TBG. We show that LBG induces bundles of simple Frobenius algebras over the branes, together with bundles of bimodules over spaces of paths connecting the branes. By an algebra we always mean a complex, associative, unital, finite-dimensional algebra, and algebra homomorphisms are assumed to be unital.

3.1. Frobenius algebra bundles over the branes

Let $(\mathcal{L}, \lambda, \mathcal{R}, \phi, \chi, \epsilon, \alpha) \in \mathrm{LBG}(M, Q)$. A vector bundle over Q_i is obtained by putting $\mathcal{A}_i := \mathsf{c}^*\mathcal{R}_{ii}$, where $\mathsf{c} : Q_i \to P_{ii}$ associates to each point $x \in Q_i$

the constant path c_x at that point. The pullback of χ_{iii} along $Q_i \to P_{ii} \times_{Q_i} P_{ii} : x \mapsto (\mathsf{c}_x, \mathsf{c}_x)$ gives a bundle morphism $\mu_i : \mathcal{A}_i \otimes \mathcal{A}_i \to \mathcal{A}_i$. We consider condition (LBG2) restricted to a triple of constant paths, $(\mathsf{c}_x, \mathsf{c}_x, \mathsf{c}_x)$. The reparameterization at the bottom of the diagram is trivial in this case, and we obtain that μ_i is associative. Further, the lifted constant paths $\epsilon_i : Q_i \to \mathcal{R}_{ii}$ induce a smooth section of $\mathcal{A}_i$ that by (LBG4) provides a unit for each fiber.

Lemma 3.1. *$\mathcal{A}_i$ is a bundle of simple algebras over Q_i.*

Proof. So far we have constructed an algebra structure on the vector bundle $\mathcal{A}_i$, see Appendix A.3. The connection on $\mathcal{R}_{ii}$ induces a connection on $\mathcal{A}_i$, for which the multiplication μ_i is connection-preserving. By Lemma A.13 the algebra structure is local. We will show in Corollary 3.19 that all algebras $\mathcal{A}_i|_x$ are simple, so that $\mathcal{A}_i$ is a genuine algebra bundle (Lemma A.14). □

We continue writing $a \cdot b := \mu_i(a \otimes b)$ and $1 := \epsilon_i(x)$ for short. Further, we consider the following additional structures: first, the lifted path reversal induces an isomorphism $\mathsf{c}^*\alpha_{ii} : \mathcal{A}_i \to \overline{\mathcal{A}_i}$ that is unital (LBG6), involutive (LBG7) and anti-multiplicative (LBG8); we write $a^* := \mathsf{c}^*\alpha_{ii}(a)$. With these operations, $\mathcal{A}_i$ becomes a bundle of involutive algebras. Second, the hermitian metric h_{ii} on $\mathcal{R}_{ii}$ induces a hermitian metric $\langle -, - \rangle : \mathcal{A}_i \times \mathcal{A}_i \to \mathbb{C}$ on $\mathcal{A}_i$. The induced norm satisfies $\|1\| = \sqrt[4]{\mathrm{rk}(\mathcal{A}_i)}$ by Remark 2.6 (a). In particular, $\mathcal{A}_i$ is not a bundle of C*-algebras unless $\mathrm{rk}(\mathcal{A}_i) = \mathrm{rk}(\mathcal{R}_{ii}) = 1$.

Remark 3.2. Since units in algebras are uniquely determined, we obtain that the lifted constant path is fiber-wise uniquely determined. More precisely, if $(\mathcal{L}, \lambda, \mathcal{R}, \phi, \chi, \epsilon, \alpha)$ and $(\mathcal{L}, \lambda, \mathcal{R}, \phi, \chi, \epsilon', \alpha')$ are LBG objects, then $\epsilon = \epsilon'$.

Using the involution we can turn the hermitian metric into a bilinear product,

$$\sigma_i : \mathcal{A}_i \times \mathcal{A}_i \to \mathbb{C} : (v, w) \mapsto \langle v^*, w \rangle .$$

Lemma 3.3. *The bilinear product σ_i is fiber-wise non-degenerate, symmetric, and invariant, i.e. $\sigma_i(v \cdot w, x) = \sigma_i(v, w \cdot x)$ for all $v, w, x \in \mathcal{A}_i$.*

Proof. It is fiber-wise non-degenerate because h_{ii} is so and because α_{ii} is an isomorphism. The unitarity of α_{ii} implies that $\langle a^*, b^* \rangle = \langle b, a \rangle$; this shows that σ_i is symmetric. For the invariance, we use (LBG5) to check:

$$\langle (v \cdot w)^*, x \rangle = \langle x^* \cdot (v \cdot w)^*, 1 \rangle = \langle (w \cdot x)^* \cdot v^*, 1 \rangle = \langle v^*, w \cdot x \rangle ,$$

which proves the claim. □

This shows that $\mathcal{A}_i|_x$ is a symmetric Frobenius algebra for each $x \in Q_i$, and together with Lemma 3.1 we have the following result:

Proposition 3.4. *$\mathcal{A}_i$ is a bundle of simple, symmetric Frobenius algebras over Q_i.*

Remark 3.5. The trace $\mathcal{A}_i \to \mathbb{C}$ corresponding to the bilinear form σ_i is $a \mapsto \sigma_i(a, 1)$. In terms of the hermitian metric, it is $a \mapsto \langle a^*, 1\rangle = \langle 1, a\rangle$, so that it is precisely the map θ_i defined in Remark 2.7. From this it is easy to see that it respects the involution: $\theta_i(a^*) = \overline{\theta_i(a)}$. It is non-degenerate in the sense that its kernel contains no non-trivial left ideals; this follows from the non-degeneracy of σ_i, see [Koc03, Lemma 2.2.4].

3.2. Bimodule bundles and Morita equivalences

In this section we show that the vector bundles $\mathcal{R}_{ij}$ are bundles of bimodules over the algebra bundles defined in Section 3.1. For this purpose, we define bundle morphisms

$$\lambda_{ij} : \mathrm{ev}_1^* \mathcal{A}_j \otimes \mathcal{R}_{ij} \to \mathcal{R}_{ij} \quad \text{and} \quad \rho_{ij} : \mathcal{R}_{ij} \otimes \mathrm{ev}_0^* \mathcal{A}_i \to \mathcal{R}_{ij}$$

over P_{ij} in the following way. For a path $\gamma \in P_{ij}$ with $x := \gamma(0)$ and $y := \gamma(1)$, we let $\lambda_{ij}|_\gamma$ be defined by

$$\mathcal{A}_j|_y \otimes \mathcal{R}_{ij}|_\gamma \xrightarrow{\chi_{ijj}|_{\gamma, \mathsf{c}_y}} \mathcal{R}_{ij}|_{\mathsf{c}_y \star \gamma} \xrightarrow{d_{\mathsf{c}_y \star \gamma, \gamma}} \mathcal{R}_{ij}|_\gamma,$$

and we let $\rho_{ij}|_\gamma$ be defined by

$$\mathcal{R}_{ij}|_\gamma \otimes \mathcal{A}_i|_x \xrightarrow{\chi_{iij}|_{\mathsf{c}_x, \gamma}} \mathcal{R}_{ij}|_{\gamma \star \mathsf{c}_x} \xrightarrow{d_{\gamma \star \mathsf{c}_x, \gamma}} \mathcal{R}_{ij}|_\gamma.$$

Lemma 3.6. *The bundle morphisms λ_{ij} and ρ_{ij} define commuting left and right actions of $\mathcal{A}_j$ and $\mathcal{A}_i$ on $\mathcal{R}_{ij}$.*

Proof. We consider the following diagram; the commutativity of its outer square corresponds to the statement that left and right actions commute:

$$\begin{tikzcd}
\mathcal{A}_j|_y \otimes \mathcal{R}_{ij}|_\gamma \otimes \mathcal{A}_i|_x \arrow[r, "\mathrm{id}\otimes\chi_{iij}|_{\mathsf{c}_x,\gamma}"] \arrow[dd, "\chi_{ijj}|_{\gamma,\mathsf{c}_y}\otimes\mathrm{id}"'] & \mathcal{A}_j|_y \otimes \mathcal{R}_{ij}|_{\gamma\star\mathsf{c}_x} \arrow[r, "\mathrm{id}\otimes d"] \arrow[d, "\chi_{ijj}|_{\gamma\star\mathsf{c}_x,\mathsf{c}_y}"'] & \mathcal{A}_j|_y \otimes \mathcal{R}_{ij}|_\gamma \arrow[d, "\chi_{ijj}|_{\gamma,\mathsf{c}_y}"] \\
& \mathcal{R}_{ij}|_{\mathsf{c}_y\star(\gamma\star\mathsf{c}_x)} \arrow[r, "d"'] & \mathcal{R}_{ij}|_{\mathsf{c}_y\star\gamma} \arrow[dd, "d"] \\
\mathcal{R}_{ij}|_{\mathsf{c}_y\star\gamma} \otimes \mathcal{A}_i|_x \arrow[r, "\chi_{iij}|_{\mathsf{c}_x,\mathsf{c}_y\star\gamma}"] \arrow[d, "d\otimes\mathrm{id}"'] & \mathcal{R}_{ij}|_{(\mathsf{c}_y\star\gamma)\circ\mathsf{c}_x} \arrow[ur, "d"] \arrow[d, "d"] & \\
\mathcal{R}_{ij}|_\gamma \otimes \mathcal{A}_i|_x \arrow[r, "\chi_{iij}|_{\mathsf{c}_x,\gamma}"'] & \mathcal{R}_{ij}|_{\gamma\circ\mathsf{c}_x} \arrow[r, "d"'] & \mathcal{R}_{ij}|_\gamma
\end{tikzcd}$$

The diagram in the upper left corner is the pentagon diagram of (LBG2). The two rectangular diagrams commute because χ_{iij} is connection-preserving, and the diagram in the lower right corner is commutative because the isomorphism d is independent of the ways the thin homotopy is performed.

Associativity of the left action follows from (LBG2) in a very similar way, and the associativity of the right action is seen analogously. The fact that the actions are unital can easily be deduced from (LBG4). □

Lemma 3.7. *The bundle morphisms λ_{ij} and ρ_{ij} are connection-preserving.*

Proof. For λ_{ij}, we have to show that for each path $\Gamma : \gamma \to \gamma'$ in P_{ij} there is a commutative diagram

$$\begin{tikzcd}
\mathcal{A}_j|_y \otimes \mathcal{R}_{ij}|_\gamma \arrow[r, "\chi_{ijj}|_{\gamma,\mathsf{c}_y}"] \arrow[d, "pt_\eta\otimes pt_{ij}|_\Gamma"'] & \mathcal{R}_{ij}|_{\mathsf{c}_y\star\gamma} \arrow[r, "d_{\mathsf{c}_y\circ\gamma,\gamma}"] \arrow[d, "pt_{ij}|_{\Gamma_\eta}"] & \mathcal{R}_{ij}|_\gamma \arrow[d, "pt_{ij}|_\Gamma"] \\
\mathcal{A}_j|_{y'} \otimes \mathcal{R}_{ij}|_{\gamma'} \arrow[r, "\chi_{ijj}|_{\gamma',\mathsf{c}_y}"'] & \mathcal{R}_{ij}|_{\mathsf{c}'_y\star\gamma'} \arrow[r, "d_{\mathsf{c}_{y'}\circ\gamma',\gamma'}"'] & \mathcal{R}_{ij}|_{\gamma'}
\end{tikzcd}$$

Here, $\eta \in PQ_j$ is the path formed by the end points of Γ, i.e. $\eta(t) := \Gamma(t)(1)$, and $\Gamma_\eta \in PPM$ is defined by $\Gamma_\eta(t) := \mathsf{c}_{\eta(t)} \star \Gamma(t)$. Further, we have set $y := \gamma(1)$ and $y' := \gamma'(1)$. The diagram on the left is commutative because χ_{ijj} is connection-preserving. The diagram on the right is commutative because the connection on $\mathcal{R}_{ij}$ is superficial. Indeed, we note that the obvious homotopy $h \in PPPM$ between Γ_η and Γ fixes the paths of end points

(namely, $t \mapsto \Gamma(t)(0)$ and η). Further, the adjoint map $h^\vee : [0,1]^3 \to M$ factors through $\Gamma^\vee : [0,1]^2 \to M$, which implies that its rank is less or equal than two. Then, the diagram on the right is precisely an instance of the property Definition A.11 (ii) of a superficial connection. The discussion for ρ_{ij} is analogous. $\square$

Proposition 3.8. *The vector bundle $\mathcal{R}_{ij}$ is bundle of $\mathrm{ev}_1^*\mathcal{A}_j$-$\mathrm{ev}_0^*\mathcal{A}_i$-bimodules.*

Proof. By now we have equipped $\mathcal{R}_{ij}$ with a bimodule structure $(\lambda_{ij}, \rho_{ij})$. As a consequence of Lemmas 3.6, 3.7 and A.15 this bimodule structure is local. We show below (Remark 3.12 (c)) that the bimodule structure is faithfully balanced, so that Lemma A.16 implies that $\mathcal{R}_{ij}$ is a bundle of bimodules. $\square$

Remark 3.9.

(a) $\mathcal{R}_{ii}|_{\mathsf{c}_x}$ is the identity $\mathcal{A}_i|_x$-$\mathcal{A}_i|_x$-bimodule.

(b) Lemma 3.7 implies that parallel transport along a fixed-ends path Γ in P_{ij} is an intertwiner of $\mathrm{ev}_1^*\mathcal{A}_j$-$\mathrm{ev}_0^*\mathcal{A}_i$-bimodules. In particular, the canonical isometries $d_{\gamma,\gamma'} : \mathcal{R}_{ij}|_\gamma \to \mathcal{R}_{ij}|_{\gamma'}$ for a pair (γ, γ') of thin fixed-ends homotopic paths are intertwiners. If a path Γ does not fix the endpoints, then parallel transport is an intertwiner with respect to the parallel transport in $\mathcal{A}_i$ and $\mathcal{A}_j$ along the paths of end points, respectively.

(c) The lifted path reversal $\alpha_{ij} : \mathcal{R}_{ij} \to \overline{\mathcal{R}_{ji}}$ exchanges left and right actions under the involutions of the algebra bundles. More precisely, for a path $\gamma \in P_{ij}$ with $x := \gamma(0)$ and $y := \gamma(1)$ the diagram

$$\begin{tikzcd}
\mathcal{A}_j|_y \otimes \mathcal{R}_{ij}|_\gamma \arrow[r, "\mathsf{c}^*\alpha_{ii}\otimes\alpha_{ij}"] \arrow[d, "\lambda_{ij}"'] & \overline{\mathcal{A}_j|_y} \otimes \overline{\mathcal{R}_{ij}|_\gamma} \arrow[r, "braid"] & \overline{\mathcal{R}_{ji}}|_\gamma \otimes \overline{\mathcal{A}_j|_y} \arrow[d, "\rho_{ij}"] \\
\mathcal{R}_{ij}|_\gamma \arrow[rr, "\alpha_{ij}"'] & & \overline{\mathcal{R}_{ji}}|_\gamma
\end{tikzcd}$$

is commutative. This follows from (LBG8) and the fact that α_{ij} is connection-preserving.

Next, we consider the space $P_{jk} \times_{Q_j} P_{ij}$ of composable paths between three D-branes, equipped with the projections p_{ij} to P_{ij} and p_{jk} to P_{jk}, the composition c to P_{ik}, and the projections p_i, p_j, p_k to the end points of the paths.

Proposition 3.10. *Lifted path concatenation χ_{ijk} induces an isomorphism*

$$p_{jk}^*\mathcal{R}_{jk} \otimes_{p_j^*\mathcal{A}_j} p_{ij}^*\mathcal{R}_{ij} \cong c^*\mathcal{R}_{ik}$$

of bundles of $p_k^*\mathcal{A}_k$-$p_i^*\mathcal{A}_i$-bimodules over $P_{jk} \times_{Q_j} P_{ij}$.

Proof. In order to see that χ_{ijk} is well-defined on $p_{jk}^*\mathcal{R}_{jk} \otimes_{p_j^*\mathcal{A}_j} p_{ij}^*\mathcal{R}_{ij}$ it suffices to show that it vanishes on elements of the form $\rho_{jk}|_{\gamma'}(w \otimes a) \otimes v - w \otimes \lambda_{ij}|_\gamma(a \otimes v)$, where $a \in \mathcal{A}_j|_y$, $v \in \mathcal{R}_{ij}|_\gamma$ and $w \in \mathcal{R}_{jk}|_{\gamma'}$, and $\gamma(1) = y = \gamma'(0)$. This follows from the definitions of λ_{ij} and ρ_{jk} and (LBG2) via a direct calculation.

Similarly one checks that χ_{ijk} is an intertwiner for both actions. In order to show that it is an isomorphism we construct an inverse map in the fiber over a point $(\gamma', \gamma) \in P_{jk} \times_{Q_j} P_{ij}$. Let $(v_1, ..., v_n)$ be an orthonormal basis of $\mathcal{R}_{ij}|_\gamma$. For $x \in \mathcal{R}_{ik}|_{\gamma'\star\gamma}$ we consider the element

$$\begin{aligned}\psi(x) := \frac{1}{\sqrt{\mathrm{rk}(\mathcal{R}_{jj})}} \sum_{l=1}^n d_{(\gamma'\star\gamma)\star\overline{\gamma},\gamma'} \big(\chi_{jik}|_{\overline{\gamma},\gamma'\star\gamma}(x \otimes \alpha_{ij}(v_l))\big) \otimes v_l \\ \in \mathcal{R}_{jk}|_{\gamma'} \otimes \mathcal{R}_{ij}|_\gamma.\end{aligned}$$

Using the fact that χ_{ijk} is connection-preserving and (LBG2), we compute

$$\begin{aligned}\chi_{ijk}|_{\gamma,\gamma'}(\psi(x)) = \frac{1}{\sqrt{\mathrm{rk}(\mathcal{R}_{jj})}} \sum_{l=1}^n d_{(\gamma'\star\gamma)\star(\overline{\gamma}\star\gamma),\gamma'\star\gamma} \\ \times \big(\chi_{iik}|_{\overline{\gamma}\star\gamma,\gamma'\star\gamma}(x \otimes \chi_{iji}|_{\gamma,\overline{\gamma}}(\alpha_{ij}(v_l) \otimes v_l))\big).\end{aligned}$$

With (7) the latter becomes

$$\chi_{ijk}|_{\gamma,\gamma'}(\psi(x)) = d_{(\gamma'\star\gamma)\star(\overline{\gamma}\star\gamma),\gamma'\star\gamma}(\chi_{iik}|_{\overline{\gamma}\star\gamma,\gamma'\star\gamma}(x \otimes d_{\mathsf{c}_x,\overline{\gamma}\star\gamma}(\epsilon_i(x)))).$$

Via (LBG4) this is equal to x. Conversely, we compute $\psi(\chi_{ijk}|_{\gamma,\gamma'}(w \otimes v))$ for $v \in \mathcal{R}_{ij}|_\gamma$ and $w \in \mathcal{R}_{jk}|_{\gamma'}$. Using the definitions, the fact that χ_{ijk} is connection-preserving, and (LBG2), we obtain

$$\psi(\chi_{ijk}|_{\gamma,\gamma'}(w \otimes v)) = \frac{1}{\sqrt{\mathrm{rk}(\mathcal{R}_{jj})}} \sum_{l=1}^n \rho_{jk}\big(w \otimes d_{\gamma\star\overline{\gamma},\mathsf{c}_y}(\chi_{jij}|_{\overline{\gamma},\gamma}(v \otimes \alpha_{ij}(v_l)))\big) \otimes v_l.$$

Under the quotient by the $\mathcal{A}_j$-action, the right hand side is identified with

$$\psi(\chi_{ijk}|_{\gamma,\gamma'}(w \otimes v)) = \frac{1}{\sqrt{\mathrm{rk}(\mathcal{R}_{jj})}} \sum_{l=1}^n w \otimes \lambda_{ij}(d_{\gamma\star\overline{\gamma},\mathsf{c}_y}(\chi_{jij}|_{\overline{\gamma},\gamma}(v \otimes \alpha_{ij}(v_l))) \otimes v_l).$$

Again, from the definitions, the fact that χ_{ijk} is connection-preserving, and (LBG2) we obtain

$$\psi(\chi_{ijk}|_{\gamma,\gamma'}(w\otimes v)) = \frac{1}{\sqrt{\mathrm{rk}(\mathcal{R}_{jj})}}\sum_{l=1}^{n} w\otimes d_{\gamma\star(\overline{\gamma}\star\gamma),\gamma} \\ \times\big(\chi_{iij}|_{\overline{\gamma}\star\gamma,\gamma}(v\otimes\chi_{iji}|_{\gamma,\overline{\gamma}}(\alpha_{ij}(v_l)\otimes v_l))\big).$$

Now we use (7) to obtain

$$\psi(\chi_{ijk}|_{\gamma,\gamma'}(w\otimes v)) = w\otimes d_{\gamma\star(\overline{\gamma}\star\gamma),\gamma}(\chi_{iij}|_{\overline{\gamma}\star\gamma,\gamma}(v\otimes d_{\mathsf{c}_x,\overline{\gamma}\star\gamma}(\epsilon_i(x)))).$$

Finally, by (LBG4), the latter is equal to $w\otimes v$. □

Corollary 3.11. *The bimodule bundles $\mathcal{R}_{ij}$ and $\mathcal{R}_{ji}$ are invers to each other, in the sense that there exist bimodules isomorphisms*

$$\mathcal{R}_{ji}|_{\overline{\gamma}}\otimes_{\mathcal{A}_j|_y}\mathcal{R}_{ij}|_\gamma\cong\mathcal{A}_i|_x \quad \textit{and} \quad \mathcal{R}_{ij}|_\gamma\otimes_{\mathcal{A}_i|_x}\mathcal{R}_{ji}|_{\overline{\gamma}}\cong\mathcal{A}_j|_y,$$

for every $\gamma\in P_{ij}$ with $x:=\gamma(0)$ and $y:=\gamma(1)$, forming bundle isomorphisms over P_{ij}. In particular, the bimodule $\mathcal{R}_{ij}|_\gamma$ establishes a Morita equivalence.

Proof. Proposition 3.10 provides bimodule isomorphisms

$$\mathcal{R}_{ji}|_{\overline{\gamma}}\otimes_{\mathcal{A}_i|_x}\mathcal{R}_{ij}|_\gamma\cong\mathcal{R}_{ii}|_{\overline{\gamma}\star\gamma} \quad \text{and} \quad \mathcal{R}_{ij}|_\gamma\otimes_{\mathcal{A}_j|_y}\mathcal{R}_{ji}|_{\overline{\gamma}}\cong\mathcal{R}_{jj}|_{\gamma\star\overline{\gamma}},$$

and by Remark 3.9 (b) we have bimodule isomorphisms $\mathcal{R}_{ii}|_{\overline{\gamma}\star\gamma}\cong\mathcal{R}_{ii}|_{\mathsf{c}_x}=\mathcal{A}_i|_x$ and $\mathcal{R}_{jj}|_{\gamma\star\overline{\gamma}}\cong\mathcal{R}_{jj}|_{\mathsf{c}_y}=\mathcal{A}_j|_y$. □

Remark 3.12.

(a) Since all algebras $\mathcal{A}_i|_x$ are simple, it is clear that they are all Morita equivalent (to $\mathbb{C}$); the point is that LBG provides a consistent choice of these Morita equivalences, parameterized by paths.

(b) Inverses of invertible bimodules are unique up to unique intertwiners, and a canonical choice is to take the complex conjugate vector space with swapped left and right actions. Remark 3.9 (c) shows that $\alpha_{ij}:\mathcal{R}_{ij}\to\overline{\mathcal{R}_{ji}}$ is that unique intertwiner. In particular, if $(\mathcal{L},\lambda,\mathcal{R},\phi,\chi,\epsilon,\alpha)$ and $(\mathcal{L},\lambda,\mathcal{R},\phi,\chi,\epsilon,\alpha')$ are LBG objects, then $\alpha=\alpha'$.

(c) By Lemma 3.6 we have morphisms between algebra bundles

$$\mathrm{ev}_1^*\mathcal{A}_j \to \mathrm{End}_{\mathrm{ev}_0^*\mathcal{A}_i}(\mathcal{R}_{ij}) \quad \text{and} \quad \mathrm{ev}_0^*\mathcal{A}_i \to (\mathrm{End}_{\mathrm{ev}_1^*\mathcal{A}_j}(\mathcal{R}_{ij}))^{op}$$

over P_{ij}, where $\mathrm{End}_{\mathrm{ev}_0^*\mathcal{A}_i}(\mathcal{R}_{ij})$ and $(\mathrm{End}_{\mathrm{ev}_1^*\mathcal{A}_j}(\mathcal{R}_{ij}))^{op}$ are bundles of endomorphisms of right $\mathrm{ev}_0^*\mathcal{A}_i$-module bundles and of left $\mathrm{ev}_1^*\mathcal{A}_j$-module bundles, respectively. Since the bimodule $\mathcal{R}_{ij}$ establishes a Morita equivalence, it is faithfully balanced; i.e., the above algebra bundle homomorphisms are isomorphisms.

The following statement combines the bimodules structure with the fusion representation, and will be useful in Section 5.

Lemma 3.13. *The fusion representation commutes with the bimodule actions of Lemma 3.6; more precisely, the diagrams*

$$\begin{tikzcd}
\mathcal{A}_j|_y \otimes (\mathcal{L}|_{\gamma_1\cup\gamma_2} \otimes \mathcal{R}_{ij}|_{\gamma_2}) \arrow[r, "\mathrm{id}\otimes\phi_{ij}|_{\gamma_1,\gamma_2}"] \arrow[d, "braid\otimes\mathrm{id}"'] & \mathcal{A}_j|_y \otimes \mathcal{R}_{ij}|_{\gamma_1} \arrow[dd, "\lambda_{ij}|_{\gamma_1}"] \\
\mathcal{L}|_{\gamma_1\cup\gamma_2} \otimes (\mathcal{A}_j|_y \otimes \mathcal{R}_{ij}|_{\gamma_2}) \arrow[d, "\mathrm{id}\otimes\lambda_{ij}|_{\gamma_2}"'] & \\
\mathcal{L}|_{\gamma_1\cup\gamma_2} \otimes \mathcal{R}_{ij}|_{\gamma_2} \arrow[r, "\phi_{ij}|_{\gamma_1,\gamma_2}"'] & \mathcal{R}_{ij}|_{\gamma_1}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{L}|_{\gamma_1\cup\gamma_2} \otimes \mathcal{R}_{ij}|_{\gamma_2} \otimes \mathcal{A}_i|_x \arrow[r, "\phi_{ij}|_{\gamma_1,\gamma_2}\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\rho_{ij}|_{\gamma_2}"'] & \mathcal{R}_{ij}|_{\gamma_1} \otimes \mathcal{A}_i|_x \arrow[d, "\rho_{ij}|_{\gamma_1}"] \\
\mathcal{L}|_{\gamma_1\cup\gamma_2} \otimes \mathcal{R}_{ij}|_{\gamma_2} \arrow[r, "\phi_{ij}|_{\gamma_1,\gamma_2}"'] & \mathcal{R}_{ij}|_{\gamma_1}
\end{tikzcd}$$

are commutative for all $(\gamma_1, \gamma_2) \in P_{ij}^{[2]}$ *with* $x := \gamma_1(0) = \gamma_2(0)$ *and* $y := \gamma_1(1) = \gamma_2(1)$.

Proof. For the first diagram, we insert $k = j$, $\gamma_{12} := \gamma_1$, $\gamma'_{12} := \gamma_2$, $\gamma_{23} := \gamma'_{23} := \mathsf{c}_y$ into (LBG3) and use the section ν_{c_y} into $\mathcal{L}_{\gamma_{23}\cup\gamma'_{23}}$. As ν_{c_y} is neutral with respect to the fusion product and the fusion representation (Remark 2.5) we obtain from (LBG3) the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{L}|_{\gamma_1\cup\gamma_2} \otimes \mathcal{R}_{jj}|_{\mathsf{c}_y} \otimes \mathcal{R}_{ij}|_{\gamma_2} \arrow[r, "d\otimes\chi_{ijj}|_{\gamma'_{12},\mathsf{c}_y}"] \arrow[d, "braid\otimes\mathrm{id}"'] & \mathcal{L}|_{(\mathsf{c}_y\star\gamma_1)\cup(\mathsf{c}_y\star\gamma_2)} \otimes \mathcal{R}_{ij}|_{\mathsf{c}_y\star\gamma_2} \arrow[dd, "\phi_{ij}|_{\mathsf{c}_y\star\gamma_1,\mathsf{c}_y\star\gamma_2}"] \\
\mathcal{R}_{jj}|_{\mathsf{c}_y} \otimes \mathcal{L}|_{\gamma_1\cup\gamma_2} \otimes \mathcal{R}_{ij}|_{\gamma_2} \arrow[d, "\mathrm{id}\otimes\phi_{ij}|_{\gamma_1,\gamma_2}"'] & \\
\mathcal{R}_{jj}|_{\mathsf{c}_y} \otimes \mathcal{R}_{ij}|_{\gamma_1} \arrow[r, "\chi_{ijj}|_{\gamma_1,\mathsf{c}_y}"'] & \mathcal{R}_{ij}|_{\mathsf{c}_y\star\gamma_1}.
\end{tikzcd}$$

Using the obvious reparameterizations and the superficial connection, as well as the fact that ϕ_{ij} is connection-preserving, this shows the commutativity of the first diagram. The second diagram follows analogously. □

3.3. Reduction to the point

One important insight of the Stolz-Teichner programme [ST04] is that *classical* field theories (with a target space) and *quantum* field theories fit into the same framework of functorial field theories, in such a way that the target space of a quantum theory is a point. Under a conjectural identification between certain types of field theories and generalized cohomology theories, quantization is the pushforward to the point in that cohomology theory.

As we will show in [BW21], our LBG is precisely the data for a 2-dimensional open-closed topological field theory with target space M. On the other hand, Lazaroiu, Lauda-Pfeiffer and Moore-Segal determined the data for a 2-dimensional open-closed topological *quantum* field theory [Laz01, MS06, LP08], resulting in a structure called an I-colored knowledgable Frobenius algebra in [LP08]. In this subsection we prove that both data are consistent in the sense that in the case $M = \{*\}$ our LBG reduces to an I-colored knowledgable Frobenius algebra (with additional structure and properties). Loosely speaking, LBG is a family of I-colored knowledgable Frobenius algebras. We start with the following simple reduction of LBG to a point:

Lemma 3.14. *Consider the target space* (M, Q) *with* $M = \{*\}$ *and* $Q = \{*\}_{i\in I}$. *Then, the category* $\mathrm{LBG}(M, Q)$ *is canonically equivalent to a category* $\mathrm{LBG}^{(I)}$ *defined as follows. An object is a septuple* $(\mathcal{L}, \lambda, \mathcal{R}, \phi, \chi, \epsilon, \alpha)$ *consisting of the following structure:*

(1*) *A complex inner product space* $\mathcal{L}$ *together with a unitary isomorphism* $\lambda : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$ *such that* $(\mathcal{L}, \lambda)$ *is a commutative algebra.*

(2*) *A family* $\mathcal{R} = \{\mathcal{R}_{ij}\}_{i,j\in I}$ *of complex inner product spaces.*

(3*) *A family* $\phi = \{\phi_{ij}\}_{i,j\in I}$ *of unitary isomorphisms* $\phi_{ij} : \mathcal{L} \otimes \mathcal{R}_{ij} \to \mathcal{R}_{ij}$, *forming a representation of* $(\mathcal{L}, \lambda)$ *on* $\mathcal{R}_{ij}$.

(4*) *A family* $\chi = \{\chi_{ijk}\}_{i,j,k\in I}$ *of associative, linear maps* $\chi_{ijk} : \mathcal{R}_{jk} \otimes \mathcal{R}_{ij} \to \mathcal{R}_{ik}$, *and a family* $\epsilon = \{\epsilon_i\}_{i\in I}$ *of elements* $\epsilon_i \in \mathcal{R}_{ii}$ *that are neutral with respect to* χ_{iij} *and* χ_{ijj}.

(5*) *A family* $\alpha = \{\alpha_{ij}\}_{i,j\in I}$ *of unitary isomorphisms* $\alpha_{ij} : \mathcal{R}_{ij} \to \overline{\mathcal{R}_{ji}}$ *that are unital* ($\alpha_{ii}(\epsilon_i) = \epsilon_i$), *involutive* ($\alpha_{ji} \circ \alpha_{ij} = \mathrm{id}_{\mathcal{R}_{ij}}$) *and anti-multiplicative, i.e.* $\chi_{kji}(\alpha_{ij}(v) \otimes \alpha_{jk}(w)) = \alpha_{ik}(\chi_{ijk}(w \otimes v))$ *for all* $v \in \mathcal{R}_{ij}$, $w \in \mathcal{R}_{jk}$.

This structure has to satisfy the following axioms:

(LBG1*) *The following diagram is commutative:*

$$\begin{tikzcd}
\mathcal{L} \otimes \mathcal{L} \otimes \mathcal{R}_{jk} \otimes \mathcal{R}_{ij} \arrow[r, "\lambda\otimes\chi_{ijk}"] \arrow[d, "\mathrm{id}\otimes braid\otimes\mathrm{id}"'] & \mathcal{L} \otimes \mathcal{R}_{ik} \arrow[dd, "\phi_{ik}"] \\
\mathcal{L} \otimes \mathcal{R}_{jk} \otimes \mathcal{L} \otimes \mathcal{R}_{ij} \arrow[d, "\phi_{jk}\otimes\phi_{ij}"'] & \\
\mathcal{R}_{jk} \otimes \mathcal{R}_{ij} \arrow[r, "\chi_{ijk}"'] & \mathcal{R}_{ik}
\end{tikzcd}$$

(LBG2*) *For any elements* $v, w \in \mathcal{R}_{ij}$ *we have*

$$h_{ii}(\chi_{iji}(\alpha_{ij}(w) \otimes v), \epsilon_i) = h_{ij}(v, w) = h_{jj}(\epsilon_j, \chi_{jij}(w \otimes \alpha_{ij}(v))).$$

(LBG3*) *The following diagram, where $\tilde{\lambda}$ is defined as in (2), is commutative:*

$$\begin{tikzcd} \mathcal{L} \otimes \mathcal{R}_{ij} \arrow[r, "\phi_{ij}"] \arrow[d, "\tilde{\lambda}\otimes\alpha_{ij}"'] & \mathcal{R}_{ij} \arrow[d, "\alpha_{ij}"] \\ \overline{\mathcal{L}} \otimes \overline{\mathcal{R}}_{ji} \arrow[r, "\phi_{ji}"'] & \overline{\mathcal{R}_{ji}} \end{tikzcd}$$

(LBG4*) *For every orthonormal basis $(v_1, ..., v_n)$ of $\mathcal{R}_{ij}$ and every $v \in \mathcal{R}_{ii}$ we have*

$$\sum_{k=1}^{n} \chi_{jij}(\chi_{iij}(v_k \otimes v) \otimes \alpha_{ij}(v_k)) = h_{ii}(\epsilon_i, v) \cdot \epsilon_j.$$

Finally, a morphism in $\mathrm{LBG}^{(I)}$ *is a pair (φ, ξ), consisting of a unitary algebra isomorphism $\varphi : \mathcal{L} \to \mathcal{L}'$ and of a family $\xi = \{\xi_{ij}\}_{i,j\in I}$ of unitary isomorphisms $\xi_{ij} : \mathcal{R}_{ij} \to \mathcal{R}'_{ij}$ that are compatible with the remaining structure in the obvious way.* □

Remark 3.15. The unit-length section ν of (2) yields a unit-length element $1 \in \mathcal{L}$ that is neutral with respect to λ. By definition, the map $\tilde{\lambda} : \mathcal{L} \to \mathcal{L}^*$ mentioned in (2) is given by

$$\tilde{\lambda}(\ell)(\ell') \cdot 1 = \lambda(\ell \otimes \ell').$$

Using that $\{1\}$ is an orthonormal basis of $\mathcal{L}$ and λ is unitary, one can show that $\tilde{\lambda}$ (under the isomorphism $\mathcal{L}^* \cong \overline{\mathcal{L}}$) is given by

$$\tilde{\lambda} : \mathcal{L} \to \overline{\mathcal{L}} : \ell \mapsto h(\ell, 1) \cdot 1.$$

We find that $\tilde{\lambda}$ is a unital, involutive, algebra homomorphism. The map $\mathbb{C} \to \mathcal{L} : z \mapsto z \cdot 1$ is a unitary algebra isomorphism, and under this algebra homomorphism, $\tilde{\lambda}$ becomes complex conjugation.

Our goal is to compare the category $\mathrm{LBG}^{(I)}$ with the following category, which in turn is equivalent to the category of topological open-closed quantum field theories with boundary labels in I, valued in the symmetric monoidal category of finite-dimensional complex vector spaces [LP08, Theorem 5.6] and [MS06]. The relationship to quantum field theories will be further elaborated in [BW21].

Definition 3.16. Let I be a set. An *I-colored knowledgable Frobenius algebra* is a septuple $(\mathcal{L}, \mathcal{R}, \chi, \epsilon, \theta, \iota, \iota^*)$ consisting of the following structure:

(CFa1) $\mathcal{L}$ is a commutative Frobenius algebra whose trace will be denoted by ϑ.

(CFa2) $\mathcal{R} = \{\mathcal{R}_{ij}\}_{i,j\in I}$ is a family of finite-dimensional complex vector spaces.

(CFa3) $\chi = \{\chi_{ijk}\}_{i,j,k\in I}$ is a family of linear maps $\chi_{ijk} : \mathcal{R}_{jk} \otimes \mathcal{R}_{ij} \to \mathcal{R}_{ik}$ satisfying an associativity condition for four indices, and $\epsilon = \{\epsilon_i\}_{i\in I}$ is a family of elements $\epsilon_i \in \mathcal{R}_{ii}$ that are neutral with respect to χ. In particular, $\mathcal{R}_{ii}$ is an algebra.

(CFa4) $\theta = \{\theta_i\}_{i\in I}$ is a family of linear maps $\theta_i : \mathcal{R}_{ii} \to \mathbb{C}$.

(CFa5) $\iota = \{\iota_i\}_{i\in I}$ is a family of algebra homomorphisms $\iota_i : \mathcal{L} \to \mathcal{R}_{ii}$ which are central in the sense that $\chi_{iij}(v \otimes \iota_i(\ell)) = \chi_{ijj}(\iota_j(\ell) \otimes v)$ for all $\ell \in \mathcal{L}$ and $v \in \mathcal{R}_{ij}$.

(CFa6) $\iota^* = \{\iota_i^*\}_{i\in I}$ is a family of linear maps $\iota_i^* : \mathcal{R}_{ii} \to \mathcal{L}$ which are adjoint to ι_i in the sense that $\vartheta(\ell \cdot \iota_i^*(v)) = \theta_i(\chi_{iii}(\iota_i(\ell) \otimes v))$ for all $v \in \mathcal{R}_{ii}$ and $\ell \in \mathcal{L}$.

This structure defines a pairing σ_{ij} by

$$\mathcal{R}_{ji} \otimes \mathcal{R}_{ij} \xrightarrow{\chi_{iji}} \mathcal{R}_{ii} \xrightarrow{\theta_i} \mathbb{C} \tag{8}$$

which is supposed to satisfy the following three axioms:

(CFa7) σ_{ij} is non-degenerate, i.e. the induced map $\Phi_{ij} : \mathcal{R}_{ij} \to \mathcal{R}_{ji}^*$ is bijective.

(CFa8) σ_{ij} and σ_{ji} are symmetric: $\sigma_{ij}(a \otimes b) = \sigma_{ji}(b \otimes a)$ for all $b \in \mathcal{R}_{ij}$ and $a \in \mathcal{R}_{ji}$.

(CFa9) If $(v_1, ..., v_n)$ is a basis of $\mathcal{R}_{ij}$, $(v^1, ..., v^n)$ is the dual basis of $\mathcal{R}_{ji}$ with respect to σ_{ij}, and $v \in \mathcal{R}_{ii}$, then

$$(\iota_j \circ \iota_i^*)(v) = \sum_{k=1}^{n} \chi_{jij}(\chi_{iij}(v_k \otimes v) \otimes v^k).$$

Note that (CFa7) and (CFa8) imply that $\mathcal{R}_{ii}$ is a symmetric Frobenius algebra. A *homomorphism* between I-colored knowledgable Frobenius algebras is a pair (φ, ξ), consisting of a Frobenius algebra homomorphism $\varphi : \mathcal{L} \to \mathcal{L}'$

and of a family $\xi = \{\xi_{ij}\}_{i,j\in I}$ of linear maps $\xi_{ij} : \mathcal{R}_{ij} \to \mathcal{R}'_{ij}$ that respect products, units, and traces in the obvious sense, and satisfy $\iota'_i \circ \varphi = \xi_{ii} \circ \iota_i$ and $\iota'^*_i \circ \xi_{ii} = \varphi \circ \iota^*_i$. The category of I-colored knowledgable Frobenius algebras is denoted by K-Frob$^{(I)}$.

Remark 3.17. Definition 3.16 was described first in [MS06, Section 2.2], where (CFa9) was related to the Cardy condition. The terminology "I-colored knowledgable Frobenius algebra" was coined in [LP08, Def. 5.1] for a slightly different but equivalent structure – the equivalence is established by an "I-colored version" of the well-known equivalent ways to define a Frobenius algebra (one by a non-degenerate inner product and the other one by a co-product, see [Koc03, Prop. 2.3.22 & 2.3.24]).

Moore and Segal prove the following theorem [MS06, Theorem 2]; their proof applies without changes.

Theorem 3.18. *Let $(\mathcal{L}, \mathcal{R}, \chi, \epsilon, \theta, \iota, \iota^*)$ be an I-colored knowledgable Frobenius algebra with $\mathcal{L}$ one-dimensional. Then, the Frobenius algebra $\mathcal{R}_{ii}$ is simple, for every $i \in I$.* □

In the following we construct a functor

$$\mathcal{F} : \mathrm{LBG}^{(I)} \to \text{K-Frob}^{(I)}. \tag{9}$$

Suppose $(\mathcal{L}, \lambda, \mathcal{R}, \phi, \chi, \epsilon, \alpha)$ is an object in LBG$^{(I)}$. Thus, $\mathcal{L}$ is a commutative algebra. The hermitian metric h on $\mathcal{L}$ induces a non-degenerate trace $\vartheta(\ell) := h(1, \ell)$, making $\mathcal{L}$ into a commutative Frobenius algebra; this is (CFa1). The vector spaces $\mathcal{R}_{ij}$, products χ_{ijk}, and units ϵ_i give (CFa2) and (CFa3). The trace θ_i is the one given in Remark 3.5, i.e. $\theta_i(v) := h_{ii}(\epsilon_i, v)$; this gives (CFa4). The linear map $\iota_i : \mathcal{L} \to \mathcal{R}_{ii}$ is the string opening morphism of Remark 2.7, i.e. $\iota_i(\ell) := \phi_{ii}(\ell \otimes \epsilon_i)$. In order to check (CFa5) we have to prove that ι_i has the following properties:

(a) It is an algebra homomorphism. This follows by reducing (LBG1*) to $i = j = k$, inserting $\epsilon_i \in \mathcal{R}_{ii}$, and using that $\chi_{iii}(\epsilon_i \otimes \epsilon_i) = \epsilon_i$.

(b) It is unital. This follows from Remark 2.5.

(c) It is central. This follows by reducing (LBG1*) to the cases $i = j$ and $j = k$, and by using the neutrality of ϵ_i.

The linear map $\iota_i^* : \mathcal{R}_{ii} \to \mathcal{L}$ is the string closing morphism, i.e. $\iota_i^*(v) := 1 \cdot \theta_i(v)$. In order to check (CFa6) we show that it is adjoint to ι_i:

$$\begin{aligned}
\vartheta(\iota_i^*(v) \cdot \ell) &= h_{ii}(\epsilon_i, h(1,\ell) \cdot v) = h_{ii}(\epsilon_i, \alpha_{ii}(\phi_{ii}(h(\ell,1) \cdot 1 \otimes \alpha_{ii}(v)))) \\
&\overset{\text{Remark 3.15}}{\downarrow} \\
&= h_{ii}(\epsilon_i, \alpha_{ii}(\phi_{ii}(\tilde{\lambda}(\ell) \otimes \alpha_{ii}(v)))) \\
&\overset{(\text{LBG3}^*)}{\downarrow} \\
&= h_{ii}(\epsilon_i, \phi_{ii}(\ell \otimes v)) \\
&\overset{(\text{LBG1}^*)}{\downarrow} \\
&= h_{ii}(\epsilon_i, \chi_{iii}(v \otimes \phi_{ii}(\ell \otimes \epsilon_i))) = \theta_i(\chi_{iii}(v \otimes \iota_i(\ell))).
\end{aligned}$$

Now it remains to verify the axioms that involve the pairings σ_{ij}. For (CFa7), the non-degeneracy of σ_{ij}, we show that the corresponding map Φ_{ij} coincides with the isomorphism α_{ij} under the isomorphism $()^\flat : \overline{\mathcal{R}_{ji}} \to \mathcal{R}_{ji}^*$, i.e. we show that $\Phi_{ij}(v) = \alpha_{ij}(v)^\flat$ for all $v \in \mathcal{R}_{ij}$. To that end, let $w \in \mathcal{R}_{ji}$. We compute:

$$\begin{aligned}
\Phi_{ij}(v)(w) &= h_{ii}(\epsilon_i, \chi_{iji}(w \otimes v)) \qquad\qquad \overset{(\text{LBG2}^*)}{\downarrow} \\
&= h_{ii}(\chi_{iji}(\alpha_{ij}(v) \otimes \alpha_{ji}(w))), \epsilon_i) = h_{ij}(\alpha_{ji}(w), v) \\
&= \overline{h_{ji}(w, \alpha_{ij}(v))} = \alpha_{ij}(v)^\flat(w).
\end{aligned}$$

The next calculation verifies (CFa8), the symmetry between σ_{ij} and σ_{ji}:

$$\begin{aligned}
\sigma_{ij}(w \otimes v) &= h_{ii}(\chi_{iji}(\alpha_{ij}(v) \otimes \alpha_{ji}(w)), \epsilon_i) \\
&\overset{(\text{LBG2}^*)}{\downarrow} \\
&= h_{jj}(\epsilon_j, \chi_{jij}(v \otimes w)) = \sigma_{ji}(v \otimes w).
\end{aligned}$$

Finally, we verify the Cardy condition (CFa9). Suppose $(v_1, ..., v_n)$ is an orthonormal basis of $\mathcal{R}_{ij}$ with respect to h_{ij}, and suppose $(v^1, ..., v^n)$ is the dual basis of $\mathcal{R}_{ji}$ with respect to σ_{ij}. We have already seen that $\Phi_{ij} = ()^\flat \circ \alpha_{ij}$; this means that $v^k = \alpha_{ij}(v_k)$ for all $k = 1, ..., n$. We get for $v \in \mathcal{R}_{ii}$:

$$\iota_j(\iota_i^*(v)) = \iota_j(1 \cdot \theta_i(v)) = \theta_i(v) \cdot \epsilon_j = h_{ii}(\epsilon_i, v) \cdot \epsilon_j.$$

Now, (LBG4*) implies (CFa9). This finishes the proof that $(\mathcal{L}, \mathcal{R}, \chi, \epsilon, \theta, \iota, \iota^*)$ is an I-colored knowledgable Frobenius algebra.

In order to define the functor $\mathcal{F}$ on the level of morphisms, we consider a morphism (φ, ξ) in $\mathrm{LBG}^{(I)}$. We claim that the same maps $\varphi : \mathcal{L} \to \mathcal{L}'$ and $\xi_{ij} : \mathcal{R}_{ij} \to \mathcal{R}'_{ij}$ form a morphism of the corresponding objects in $\text{K-Frob}^{(I)}$. First of all, the unitarity of φ implies that $\vartheta' \circ \varphi = \vartheta$, i.e. φ is a Frobenius algebra homomorphism. Second, by assumption, ξ_{ij} respects products and units, and since ξ_{ij} is unitary, it also respects the traces θ_i. Proving the identities $\iota'_i \circ \varphi = \xi_{ii} \circ \iota_i$ and $\iota'^*_i \circ \xi_{ii} = \varphi \circ \iota^*_i$ is straightforward. This completes the definition of the functor $\mathcal{F}$.

Now we are in position to deliver the remaining part of the proof of Proposition 3.4, namely that the algebras $\mathcal{A}_i|_x$, for $x \in Q_i$ and $i \in I$, are simple. Restricting a given LBG object to x yields an object in $\mathrm{LBG}^{(I)}$ with $\mathcal{L} := \mathcal{L}|_{\mathsf{c}_x}$ and $\mathcal{R}_{ii} := \mathcal{R}_{ii}|_{c_x} = \mathcal{A}_i|_x$. Under the functor $\mathcal{F}$, it becomes an I-colored knowledgable Frobenius algebra with $\mathcal{L}$ one-dimensional. Applying Theorem 3.18, we obtain:

Corollary 3.19. *The algebras $\mathcal{A}_i|_x$ obtained from LBG in Section 3.1 are all simple.*

We continue studying the functor $\mathcal{F}$. It is obviously faithful, but neither full (i.e., non-isomorphic LBG may give isomorphic I-colored knowledgable Frobenius algebras), nor essentially surjective. In order to properly understand these phenomena, we lift the functor $\mathcal{F}$ to a new category $\text{RPK-Frob}^{(I)}$, where the I-colored knowledgable Frobenius algebras are equipped with so-called *positive reflection structures*, which we introduce next. We will discuss in [BW21] the relation to positive reflection structures in functorial field theories, in the sense of Freed-Hopkins [FH21].

Definition 3.20. A *reflection structure* on an I-colored knowledgable Frobenius algebra $(\mathcal{L}, \mathcal{R}, \chi, \epsilon, \theta, \iota, \iota^*)$ is a pair $(\tilde{\lambda}, \alpha)$ consisting of an involutive algebra isomorphism $\tilde{\lambda} : \mathcal{L} \to \overline{\mathcal{L}}$ and of a family $\alpha = \{\alpha_{ij}\}_{i,j \in I}$ of involutive (i.e., $\alpha_{ji} \circ \alpha_{ij} = \mathrm{id}$), anti-multiplicative isomorphisms $\alpha_{ij} : \mathcal{R}_{ij} \to \overline{\mathcal{R}_{ji}}$ such that the conditions

(10)
$$\vartheta(\tilde{\lambda}(\ell)) = \overline{\vartheta(\ell)}, \quad \alpha_{ii}(\epsilon_i) = \epsilon_i, \quad \theta_i(\alpha_{ii}(v)) = \overline{\theta_i(v)} \quad \text{and} \quad \alpha_{ii} \circ \iota_i = \iota_i \circ \tilde{\lambda}$$

are satisfied for all $i \in I$. A reflection structure is called *positive* if the sesquilinear pairings

$$(v, w) \mapsto \sigma_{ij}(\alpha_{ji}^{-1}(v) \otimes w) \quad \text{and} \quad (\ell, \ell') \mapsto \vartheta(\tilde{\lambda}^{-1}(\ell) \cdot \ell')$$

on $\mathcal{R}_{ij}$ and $\mathcal{L}$, respectively, are positive-definite for all $i, j \in I$.

We remark that the last equation in (10) implies the analogous condition $\tilde{\lambda} \circ \iota_i^* = \iota_i^* \circ \alpha_{ii}$ for ι^*, due to the adjointness in (CFa6) and the non-degeneracy of θ. A homomorphism (φ, ξ) between I-colored knowledgable Frobenius algebras is called *reflection-preserving*, if $\tilde{\lambda}' \circ \varphi = \varphi \circ \tilde{\lambda}$ and $\alpha'_{ij} \circ \xi_{ij} = \xi_{ji} \circ \alpha_{ij}$. The category of I-colored knowledgable Frobenius algebras with positive reflection structures is denoted by RPK-Frob$^{(I)}$. There is an obvious forgetful functor RPK-Frob$^{(I)} \to$ K-Frob$^{(I)}$. Next, we construct a lift of the functor $\mathcal{F}$ to RPK-Frob$^{(I)}$,

$$\begin{tikzcd} & \text{RPK-Frob}^{(I)} \arrow[d] \\ \text{LBG}^{(I)} \arrow[ur, bend left, "\widetilde{\mathcal{F}}"] \arrow[r, "\mathcal{F}"'] & \text{K-Frob}^{(I)}. \end{tikzcd}$$

In order to define the lift $\widetilde{\mathcal{F}}$, we have to define positive reflection structures on the I-colored knowledgable Frobenius algebras in the image of $\mathcal{F}$. The isomorphisms $\tilde{\lambda}$ and α_{ij} are the given ones. Remark 3.15 shows that $\tilde{\lambda}$ has the required properties; the properties of α_{ij} follow directly from (5*). The compatibility condition (10) follows from the definition of ι and (LBG3*). The pairing reproduces exactly the given hermitian metric on $\mathcal{R}_{ij}$,

$$\begin{aligned} (v, w) \mapsto \theta_i(\chi_{iji}(\alpha_{ij}(v) \otimes w)) &= h_{ii}(\epsilon_i, \chi_{iji}(\alpha_{ij}(v) \otimes w)) \\ &\overset{\text{(LBG2*)}}{\downarrow} \\ &= h_{ji}(\alpha_{ij}(w), \alpha_{ij}(v)) = h_{ij}(v, w), \end{aligned}$$

and is hence positive definite. Similarly, the pairing on $\mathcal{L}$ reproduces the metric h:

$$\begin{aligned} (\ell, \ell') \mapsto \vartheta(\tilde{\lambda}(\ell) \cdot \ell') = h(1, \tilde{\lambda}(\ell) \cdot \ell') &= \overline{h(\tilde{\lambda}(\ell), 1)} \cdot h(1, \ell') \\ &= h(\ell, 1) \cdot h(1, \ell') = h(\ell, \ell'). \end{aligned}$$

Finally, we observe that the morphisms of LBG$^{(I)}$ respect the reflection structure by definition. This completes the definition of the lift $\widetilde{\mathcal{F}}$.

Proposition 3.21. *The lifted functor $\widetilde{\mathcal{F}}$ is full and faithful, and surjective onto those objects with $\mathcal{L} \cong \mathbb{C}$ as Frobenius algebras (with* $\mathrm{id}_{\mathbb{C}}$ *as the trace on $\mathbb{C}$). In particular, it induces an equivalence*

$$\text{LBG}^{(I)} \cong \text{RPK-Frob}^{(I)}{}_{\mathbb{C}},$$

where RPK-Frob$^{(I)}{}_{\mathbb{C}}$ *is the full subcategory on all objects with* $\mathcal{L} = \mathbb{C}$ *as Frobenius algebras.*

Proof. It is obvious that $\widetilde{\mathcal{F}}$ is faithful. In order to show that it is full, we consider two objects $(\mathcal{L}, \lambda, \mathcal{R}, \phi, \chi, \epsilon, \alpha)$ and $(\mathcal{L}', \lambda', \mathcal{R}', \phi', \chi', \epsilon', \alpha')$ in LBG$^{(I)}$, and consider a morphism (φ, ξ) between the corresponding I-colored knowledgable Frobenius algebras with reflection structures. The compatibility of φ and ξ_{ij} with the involutions $\tilde{\lambda}$ and α_{ij} follows because (φ, ξ) is reflection-preserving. It remains to show that φ and ξ_{ij} are unitary, and that the fusion representations are preserved. We have seen above that the metrics h and h_{ij} are determined by the traces, the multiplications, and the involutions, which are all preserved by φ and ξ_{ij}; this implies unitarity. Concerning the fusion representation, (LBG1*) implies that $\phi_{ij} = \chi_{ijj} \circ (\iota_j \otimes \mathrm{id}_{\mathcal{R}_{ij}})$. Thus, the fusion representation ϕ_{ij} is determined by the product χ_{ijj} and the algebra homomorphism ι_j. Since these are preserved by ξ_{ij} and φ, the fusion representation is preserved. This completes the proof that $\widetilde{\mathcal{F}}$ is full.

Now we assume that $(\mathcal{L}, \mathcal{R}, \chi, \epsilon, \theta, \iota, \iota^*)$ is an I-colored knowledgable Frobenius algebra with a positive reflection structure $(\tilde{\lambda}, \alpha_{ij})$, such that $\mathcal{L} \cong \mathbb{C}$ as Frobenius algebras. First of all, we remark that if $\mathcal{L} \cong \mathbb{C}$, preservation of traces implies that there is only one such isomorphism, namely the trace ϑ itself. In particular, $\vartheta : \mathcal{L} \to \mathbb{C}$ is a unital algebra isomorphism. We equip $\mathcal{L}$ with the sesquilinear form $h(\ell, \ell') := \vartheta(\tilde{\lambda}(\ell) \cdot \ell')$. Using that ϑ is involutive and an algebra homomorphism, we get $h(\ell, \ell') = \overline{\vartheta(\ell)} \cdot \vartheta(\ell')$; this shows that h is a hermitian metric and that the product λ of $\mathcal{L}$ is unitary. We equip $\mathcal{R}_{ij}$ with the sesquilinear form $h_{ij}(v, w) := \theta_i(\chi_{iji}(\alpha_{ji}^{-1}(v) \otimes w))$. Using that α_{ij} is involutive, anti-multiplicative and compatible with θ_i, we see that h_{ij} is hermitian. It is non-degenerate, since the pairing σ_{ij} is non-degenerate and α_{ij} is an isomorphism. Finally, it is positive-definite since the reflection structure is positive. The involutions α_{ij} are unitary with respect to the metrics h_{ij} and $\overline{h_{ji}}$: we have

$$\begin{aligned} h_{ji}(\alpha_{ij}(v), \alpha_{ij}(w)) &= \sigma_{ji}(v \otimes \alpha_{ij}(w)) \\ &\overset{\text{(CFa8)}}{=} \sigma_{ij}(\alpha_{ij}(w) \otimes v) = h_{ij}(w, v) = \overline{h_{ij}(v, w)}. \end{aligned}$$

We define the fusion representation by $\phi_{ij} := \chi_{ijj} \circ (\iota_j \otimes \mathrm{id}_{\mathcal{R}_{ij}})$. This is a representation because χ_{ijk} are associative and ι_i is an algebra homomorphism. Further, ϕ_{ij} is unitary:

$$
\begin{aligned}
&h_{ij}(\phi_{ij}(\ell \otimes v), \phi_{ij}(\ell' \otimes v')) \\
&\qquad \overset{(10)}{=} \theta_i(\chi_{iji}(\chi_{jii}(\iota_i(\tilde{\lambda}(\ell)) \otimes \alpha_{ji}^{-1}(v)) \otimes \chi_{ijj}(\iota_j(\ell') \otimes v'))) \\
&\qquad \text{(CFa5)} \\
&\qquad \downarrow \\
&\qquad = \theta_i(\chi_{iii}(\chi_{iii}(\iota_i(\tilde{\lambda}(\ell)) \otimes \iota_i(\ell')) \otimes \chi_{iji}(\alpha_{ji}^{-1}(v) \otimes v'))) \\
&\qquad = \theta_i(\chi_{iii}(\iota_i(\tilde{\lambda}(\ell) \cdot \ell') \otimes \chi_{iji}(\alpha_{ji}^{-1}(v) \otimes v'))) \\
&\qquad \text{(CFa6)} \\
&\qquad \downarrow \\
&\qquad = \vartheta(\tilde{\lambda}(\ell) \cdot \ell' \cdot \iota_i^*(\chi_{iji}(\alpha_{ji}^{-1}(v) \otimes v'))) \\
&\vartheta \text{ is an algebra homomorphism} \\
&\qquad \downarrow \\
&\qquad = \vartheta(\tilde{\lambda}(\ell) \cdot \ell') \cdot \vartheta(1 \cdot \iota_i^*(\chi_{iji}(\alpha_{ji}^{-1}(v) \otimes v'))) \\
&\qquad \text{(CFa6)} \\
&\qquad \downarrow \\
&\qquad = h(\ell, \ell') \cdot \theta_i(\chi_{iii}(\epsilon_i \otimes \chi_{iji}(\alpha_{ji}^{-1}(v) \otimes v'))) \\
&\qquad = h(\ell, \ell') \cdot h_{ij}(v, v').
\end{aligned}
$$

Now we have provided the structure (1*) to (5*) of an object of $\mathrm{LBG}^{(I)}$. It remains to check the axioms. Axioms (LBG1*) and (LBG3*) follow from the definition of ϕ_{ij}, the centrality of ι_i in (CFa5) and its compatibility with α in (10). The first part of (LBG2*) follows from the associativity of χ_{ijk} and its compatibility with α_{ij}, and the second part additionally from the symmetry of the pairing σ_{ij} in (CFa8). Finally, for Axiom (LBG4*) we have seen above that $v^k = \alpha_{ij}(v_k)$ for a basis $(v_1, ..., v_n)$ of $\mathcal{R}_{ij}$ and its dual basis $(v^1, ..., v^n)$ with respect to σ_{ij}. With (CFa9), it thus remains to prove that

$$h_{ii}(\epsilon_i, v) \cdot \epsilon_j = (\iota_j \circ \iota_i^*)(v).$$

To see this, we first note that $\vartheta(\iota_i^*(v)) = \theta_i(v)$ due to the adjointness in (CFa6). Since ϑ is a unital algebra isomorphism, this implies $\iota_i^*(v) = \theta_i(v) \cdot 1$. Then, we obtain $(\iota_j \circ \iota_i^*)(v) = \theta_i(v) \cdot \epsilon_j$, which coincides with $h_{ii}(\epsilon_i, v) \cdot \epsilon_j$. Summarizing, we have constructed an object of $\mathrm{LBG}^{(I)}$, which is (by construction) sent by the functor $\widetilde{\mathcal{F}}$ to the I-colored knowledgable Frobenius algebra with reflection structure we started with. □

Proposition 3.21 shows that by restriction of LBG to a point one obtains all reflection-positive open-closed topological quantum field theories whose

bulk algebra is $\mathbb{C}$. This will be further investigated in our upcoming paper [BW21].

4. Transgression

In this section we construct our transgression functor $\mathscr{T} : \mathrm{TBG}(M, Q) \to \mathrm{LBG}(M, Q)$. In Sections 4.1 to 4.7 we describe its action on the level of objects, i.e. the transgression of a TBG object $(\mathcal{G}, \mathcal{E})$. In Section 4.8 we treat the morphisms, and in Section 4.9 we consider the situation where the bundle gerbe $\mathcal{G}$ is trivial.

As announced in the introduction, we will treat analytical regularity over loop spaces and path spaces in the framework of diffeology. The reason is that we use extensively the concatenation of arbitrary paths, whenever they have a common end point. This requires sitting instants (i.e., the map $\gamma : [0, 1] \to M$ is constant around $\{0\}$ and $\{1\}$). Spaces of paths with sitting instants are not manifolds in any way; this forces us to use diffeological spaces. An introduction to diffeology can be found in [IZ13, BH11]. A systematic application to spaces of paths and loops in the context of parallel transport has been pursued in [Bae07, SW13], and in the context of transgression in [Wal12, Wal16]. For the sake of self-containedness, we have included an Appendix A.1 about vector bundles over diffeological spaces.

4.1. The line bundle over the loop space

The transgression of gerbes with connection to the loop space was described first by Gawędzki [Gaw88] (in terms of Deligne cohomology) and Brylinski [Bry93] (in terms of Dixmier-Douady sheaves of groupoids). A transgression for bundle gerbes was described by Gawędzki and Reis in [GR02]. An adaption to bundle gerbes of Brylinski's transgression was described in [Wal10], and then extended in [Wal16] to include fusion products. In the following we recall this approach briefly.

Let $\mathcal{G}$ be a bundle gerbe with connection over M. We define a principal U(1)-bundle $L\mathcal{G}$ over LM in the following way. The fiber $L\mathcal{G}|_\tau$ over a loop τ is the set of 2-isomorphism classes of trivializations $\mathcal{T} : \tau^*\mathcal{G} \to \mathcal{I}_0$, i.e. 1-isomorphisms in $\mathcal{G}rb^\nabla(S^1)$. This set is a torsor over the group of isomorphism classes of principal U(1)-bundles with connection over S^1, which can be identified canonically with U(1) by taking the holonomy around the base S^1; this establishes the U(1)-action on $L\mathcal{G}|_\tau$. The total space of $L\mathcal{G}$ is the disjoint union of the fibers $L\mathcal{G}|_\tau$. A diffeology on $L\mathcal{G}$ is defined as follows. A map $\tilde{c} : U \to L\mathcal{G}$ is a plot if the projection $c := \pi \circ \tilde{c} : U \to LM$ is a plot

of LM, and every point $u \in U$ has an open neighborhood $u \in W \subseteq U$ and a trivialization $\mathcal{T}$ of $(c^\vee)^*\mathcal{G}$ over $W \times S^1$ such that $\tilde{c}(w) = [i_w^*\mathcal{T}] \in L\mathcal{G}|_\tau$ for all $w \in W$. Here $c^\vee : U \times S^1 \to M$ is the map $c^\vee(u,z) := c(u)(z)$, and $i_w : S^1 \to W \times S^1$ is defined by $i_w(t) := (w,t)$. It is proved in [Wal16, Sec. 4.1] that this makes $L\mathcal{G}$ into a diffeological principal U(1)-bundle over LM. By Lemma A.2, the associated vector bundle $\mathcal{L} := L\mathcal{G} \times_{\mathrm{U}(1)} \mathbb{C}$ with respect to the standard representation of U(1) on $\mathbb{C}$ is a hermitian line bundle over LM.

The fusion product λ on $\mathcal{L}$ is defined first on the principal U(1)-bundle $L\mathcal{G}$ as described in [Wal16, Sec. 4.2]. We consider $(\gamma_1, \gamma_2, \gamma_3) \in PM^{[3]}$ and the corresponding loops $\tau_{ab} := \gamma_a \cup \gamma_b$, for $a, b = 1, 2.3$. Let $\mathcal{T}_{ab} : \tau_{ab}^*\mathcal{G} \to \mathcal{I}_0$ be trivializations for $ab \in \{12, 23, 13\}$. We work with $S^1 = \mathbb{R}/\mathbb{Z}$, and consider the two maps $\iota_1, \iota_2 \colon [0,1] \to S^1$ defined by $\iota_1(t) := \frac{1}{2}t$ and $\iota_2(t) := 1 - \frac{1}{2}t$. Suppose there exist 2-isomorphisms $\phi_1 \colon \iota_1^*\mathcal{T}_{12} \Rightarrow \iota_1^*\mathcal{T}_{13}$, $\phi_2 \colon \iota_2^*\mathcal{T}_{12} \Rightarrow \iota_1^*\mathcal{T}_{23}$ and $\phi_3 \colon \iota_2^*\mathcal{T}_{23} \Rightarrow \iota_2^*\mathcal{T}_{13}$ over the interval $[0,1]$ such that $\phi_1|_0 = \phi_3|_0 \bullet \phi_2|_0$ and $\phi_1|_1 = \phi_3|_1 \bullet \phi_2|_1$, where $\bullet$ denotes the vertical composition of 2-morphisms in $\mathcal{G}rb^\nabla$. Then, we set

$$\lambda|_{\gamma_1,\gamma_2,\gamma_3}(\mathcal{T}_{12} \otimes \mathcal{T}_{23}) := \mathcal{T}_{13}.$$

It is proved in [Wal16, Sec. 4.2] that this sufficiently characterizes λ as a bundle morphism. On the associated bundle $\mathcal{L}$, the fusion product is then defined by

$$\lambda|_{\gamma_1,\gamma_2,\gamma_3}([\mathcal{T}_{12}, z_{12}] \otimes [T_{23}, z_{23}]) := [\lambda(\mathcal{T}_{12} \otimes T_{23}), z_{12} \cdot z_{23}].$$

Remark 4.1. Pullback along $\mathsf{rev} : S^1 \to S^1 : t \mapsto 1 - t$ defines a bundle morphism $L\mathcal{G} \to L\mathcal{G}^*$ that covers the loop reflection $\widetilde{\mathsf{rev}} : LM \to LM : \tau \mapsto \tau \circ \mathsf{rev}$. We note that $\widetilde{\mathsf{rev}}(\gamma_1 \cup \gamma_2) = \gamma_2 \cup \gamma_1$. On the associated line bundle, $\widetilde{\mathsf{rev}}$ induces the bundle morphism

$$\tilde{\lambda}_{\gamma_1,\gamma_2} : \mathcal{L}_{\gamma_1 \cup \gamma_2} \to \overline{\mathcal{L}}|_{\gamma_2 \cup \gamma_1} : [\mathcal{T}, z] \mapsto [\mathsf{rev}^*\mathcal{T}, \overline{z}],$$

which is part of the LBG structure (2).

A connection ω on $L\mathcal{G}$ is defined in [Wal16, Sec. 4.3]. Here, we describe its parallel transport τ^ω, combining the definition of the 1-form ω [Wal16, Sec. 4.3] with the derivation of the parallel transport described in [Wal12, Def. 3.2.9]. If $\Gamma \in PLM$ is a path, $\mathcal{T} : (\Gamma^\vee)^*\mathcal{G} \to \mathcal{I}_\rho$ is a trivialization, and

$\mathcal{T}_0$ and $\mathcal{T}_1$ are its restrictions to $\{0\} \times S^1$ and $\{1\} \times S^1$, respectively, then

$$\tau_\Gamma^\omega(\mathcal{T}_0) := \mathcal{T}_1 \cdot \exp\left(\int_{[0,1]\times S^1} \rho\right).$$

The connection ω on $L\mathcal{G}$ is superficial [Wal16, Cor. 4.3.3] and symmetrizes the fusion product [Wal16, Prop. 4.3.5]. We obtain an associated connection pt on the associated line bundle $\mathcal{L}$, see Lemma A.7. This completes the construction of LBG structures (1) and (2) from a bundle gerbe $\mathcal{G}$ with connection over M.

4.2. The vector bundle over the path space

In this section we construct the vector bundles $\mathcal{R}_{ij}$ over the path spaces P_{ij}, from a given TBG object $(\mathcal{G}, \mathcal{E})$. It is different from a construction of Gawędzki and Reis [GR02], but results into an isomorphic bundle. We use the following notation: if $p \in M$ is a point, we denote by $\mathcal{G}|_p$ the pullback of $\mathcal{G}$ along the map $\{*\} \to M : * \mapsto p$.

The fiber of $\mathcal{R}_{ij}$ over a path $\gamma \in P_{ij}$ with $x := \gamma(0)$ and $y := \gamma(1)$ is defined as follows. Let $\mathcal{T} : \gamma^*\mathcal{G} \to \mathcal{I}_0$ be a trivialization. Over the point we have the two 1-morphisms $\mathcal{T}|_0 : \mathcal{G}|_x \to \mathcal{I}_0$ and $\mathcal{E}_i|_x : \mathcal{G}|_x \to \mathcal{I}_0$, so that we obtain a hermitian vector bundle $\Delta(\mathcal{E}_i|_x, \mathcal{T}|_0)$ with connection over the point, i.e. a complex inner product space; see Section 2.1. Similarly we obtain a complex inner product space $\Delta(\mathcal{E}_j|_y, \mathcal{T}|_1)$. We define

$$\mathcal{R}_{ij}|_\gamma(\mathcal{T}) := \mathrm{Hom}\big(\Delta(\mathcal{E}_i|_x, \mathcal{T}|_0), \Delta(\mathcal{E}_j|_y, \mathcal{T}|_1)\big),$$

which is a vector space of dimension $\mathrm{rk}(\mathcal{E}_i) \cdot \mathrm{rk}(\mathcal{E}_j)$, and comes equipped with the complex inner product $h_{ij}(\varphi, \psi) := \mathrm{tr}(\varphi^* \circ \psi)$, where φ^* denotes the adjoint map.

Remark 4.2. Under the identification $\mathrm{Hom}(V, W) = V^* \otimes W$, this inner product is the one induced from the inner products on V and W. The induced norm is the Frobenius norm.

We show that the vector spaces $\mathcal{R}_{ij}|_\gamma(\mathcal{T})$ and $\mathcal{R}_{ij}|_\gamma(\mathcal{T}')$ associated to two trivializations are canonically isomorphic. If $\psi : \mathcal{T} \Rightarrow \mathcal{T}'$ is a 2-isomorphism

in $\mathcal{G}rb^{\nabla}([0,1])$, then we have unitary isomorphisms

$$\begin{aligned}\psi_0 &:= \Delta(\mathrm{id}, \psi|_0) : \Delta(\mathcal{E}_i|_x, \mathcal{T}'|_0) \to \Delta(\mathcal{E}_i|_x, \mathcal{T}|_0)\\ \psi_1 &:= \Delta(\mathrm{id}, \psi|_1) : \Delta(\mathcal{E}_j|_y, \mathcal{T}'|_1) \to \Delta(\mathcal{E}_j|_y, \mathcal{T}|_1),\end{aligned}$$

combining into a unitary isomorphism

$$r_\psi : \mathcal{R}_{ij}|_\gamma(\mathcal{T}') \to \ \mathcal{R}_{ij}|_\gamma(\mathcal{T}) : \varphi \mapsto \psi_1 \circ \varphi \circ \psi_0^{-1}.$$

This isomorphism is in fact independent of ψ: since $[0,1]$ is connected, any other 2-isomorphism $\psi' : \mathcal{T} \Rightarrow \mathcal{T}'$ satisfies $\psi' = \psi \cdot z$ for a constant $z \in \mathrm{U}(1)$; and hence we have $r_\psi = r_{\psi'}$. Moreover, since $[0,1]$ is contractible and 1-dimensional, any two trivializations over $[0,1]$ are 2-isomorphic.

Alternatively, we give another description of r_ψ. The two trivializations $\mathcal{T}$ and $\mathcal{T}'$ determine a hermitian line bundle $L := \Delta(\mathcal{T}, \mathcal{T}')$ with connection over $[0,1]$. By Remark 2.1 (a) we have canonical isomorphisms

$$\begin{aligned}\Delta(\mathcal{E}_i|_x, \mathcal{T}|_0) \otimes L|_0 &= \Delta(\mathcal{E}_i|_x, \mathcal{T}|_0) \otimes \Delta(\mathcal{T}|_0, \mathcal{T}'|_0) \cong \Delta(\mathcal{E}_i|_x, \mathcal{T}'|_0)\\ \Delta(\mathcal{E}_j|_y, \mathcal{T}|_1) \otimes L|_1 &= \Delta(\mathcal{E}_j|_y, \mathcal{T}|_1) \otimes \Delta(\mathcal{T}|_1, \mathcal{T}'|_1) \cong \Delta(\mathcal{E}_j|_y, \mathcal{T}'|_1)\end{aligned}$$

that together yield an isomorphism

$$L|_0^* \otimes \mathcal{R}_{ij}|_\gamma(\mathcal{T}) \otimes L|_1 \cong \mathcal{R}_{ij}|_\gamma(\mathcal{T}').$$

Any parallel unit-length section σ in L then determines an isomorphism

$$\tilde{\sigma} : \mathcal{R}_{ij}|_\gamma(\mathcal{T}) \to \mathcal{R}_{ij}|_\gamma(\mathcal{T}').$$

This isomorphism is again independent of σ: if σ' is another parallel unit-length section, we have $\sigma' = \sigma \cdot z$ for a constant $z \in \mathrm{U}(1)$. This shows that $\tilde{\sigma} = \tilde{\sigma}'$.

A parallel unit-length section σ in $\Delta(\mathcal{T}, \mathcal{T}')$ is the same as a 2-isomorphism $\psi : \mathcal{T} \Rightarrow \mathcal{T}'$, and under this correspondence we have $r_\psi = \tilde{\sigma}$. Thus, we have two descriptions of the same, canonical isomorphism, which we will denote by $r_{\mathcal{T},\mathcal{T}'}$. On the class of pairs $(\mathcal{T}, \varphi)$, where $\varphi \in \mathcal{R}_{ij}|_\gamma(\mathcal{T})$, we define the relation $(\mathcal{T}', \varphi') \sim (\mathcal{T}, r_{\mathcal{T},\mathcal{T}'}(\varphi'))$. This is an equivalence relation, and $\mathcal{R}_{ij}|_\gamma$ is, by definition, the set of equivalence classes. It is a complex inner product space of dimension $\mathrm{rk}(\mathcal{E}_i) \cdot \mathrm{rk}(\mathcal{E}_j)$. We note that the choice of

any trivialization $\mathcal{T}$ determines a unitary isomorphism

$$\mathcal{R}_{ij}|_\gamma(\mathcal{T}) \to \mathcal{R}_{ij}|_\gamma : \varphi \mapsto [(\mathcal{T}, \varphi)].$$

We will shortly need to consider the vector spaces $\mathcal{R}_{ij}$ in smooth families, and we thus prepare the following notation and Lemma 4.3 below. Suppose U is a smooth manifold, possibly with boundary, and $f : U \to P_{ij}$ is a smooth map. By definition of the diffeology on P_{ij}, this means that $f^\vee : U \times [0,1] \to M$ is smooth. For $u \in U$ and $t \in [0,1]$ we will use the maps two maps $i_u : [0,1] \to U \times [0,1]$ and $j_t : U \to U \times [0,1]$ defined by $i_u(t) := j_t(u) := (u,t)$. If $\mathcal{T} : (f^\vee)^*\mathcal{G} \to \mathcal{I}_\rho$ is a trivialization, then we define the hermitian vector bundles

$$\mathcal{V}_\mathcal{T} := \Delta((\mathrm{ev}_0 \circ f)^*\mathcal{E}_i, j_0^*\mathcal{T}) \quad \text{and} \quad \mathcal{W}_\mathcal{T} := \Delta((\mathrm{ev}_1 \circ f)^*\mathcal{E}_j, j_1^*\mathcal{T})$$

with connection over U. In terms of this notation, we have $\mathcal{R}_{ij}|_{f(u)}(i_u^*\mathcal{T}) = \mathrm{Hom}(\mathcal{V}_\mathcal{T}, \mathcal{W}_\mathcal{T})|_u$ for all $u \in U$. We need the following result about a change of trivialization in smooth families.

Lemma 4.3. *Let $\mathcal{T} : (f^\vee)^*\mathcal{G} \to \mathcal{I}_\rho$ and $\mathcal{T}' : (f^\vee)^*\mathcal{G} \to \mathcal{I}_{\rho'}$ be two trivializations. If U is contractible, then there exist the following:*

- *a 1-form $\eta \in \Omega^1(U \times [0,1])$ such that $\rho - \rho' = \mathrm{d}\eta$ and $i_u^*\eta = 0$ for all $u \in U$*
- *a connection-preserving bundle isomorphism*

$$\tilde\sigma : \mathrm{Hom}(\mathcal{V}_\mathcal{T}, \mathcal{W}_\mathcal{T}) \otimes \mathbb{C}_{j_1^*\eta - j_0^*\eta} \to \mathrm{Hom}(\mathcal{V}_{\mathcal{T}'}, \mathcal{W}_{\mathcal{T}'})$$

over U, such that $\tilde\sigma|_u = r_{i_u^*\mathcal{T}, i_u^*\mathcal{T}'}$ for all $u \in U$. Here, $\mathbb{C}_\omega$ denotes the trivial line bundle equipped with the connection induced by a 1-form ω.

Proof. We use a U-family version of the canonical isomorphism $r_{\mathcal{T},\mathcal{T}'}$. We consider the hermitian line bundle $L := \Delta(\mathcal{T}, \mathcal{T}')$ over $U \times [0,1]$ with connection of curvature $\rho - \rho'$. By Remark 2.1 (a) we have an isomorphism

$$\begin{aligned}\mathcal{V}_\mathcal{T} \otimes j_0^*L &= \Delta((\mathrm{ev}_0 \circ c)^*\mathcal{E}_i, j_0^*\mathcal{T}) \otimes \Delta(j_0^*\mathcal{T}, j_0^*\mathcal{T}') \\ &\cong \Delta((\mathrm{ev}_0 \circ c)^*\mathcal{E}_i, j_0^*\mathcal{T}') = \mathcal{V}_{\mathcal{T}'}\end{aligned}$$

A similar construction works for $\mathcal{W}_\mathcal{T}$ and $\mathcal{W}_{\mathcal{T}'}$, and together we obtain a bundle isomorphism

$$j_0^*L^* \otimes \mathrm{Hom}(\mathcal{V}_\mathcal{T}, \mathcal{W}_\mathcal{T}) \otimes j_1^*L \cong \mathrm{Hom}(\mathcal{V}_{\mathcal{T}'}, \mathcal{W}_{\mathcal{T}'}).$$

Now, L may not have a parallel unit-length section. However, it always admits a unit-length section σ that is parallel along i_u. Indeed, since U is contractible, there exists a smooth unit-length section $\sigma_0 : U \to i_0^* L$. We define $\sigma(u,t) := pt_{\gamma_{u,t}}(\sigma_0(u))$, where $\gamma_{u,t}$ is the path $\tau \mapsto i_u(\tau)$ restricted to $[0,t]$. Since the parallel transport depends smoothly on the path, this gives a smooth section of L, as desired. We denote by $\eta \in \Omega^1(U \times [0,1])$ its covariant derivative, so that $\mathrm{d}\eta = \rho - \rho'$. Therefore, σ determines the claimed connection-preserving bundle isomorphism $\tilde{\sigma}$, as claimed. Since σ is parallel along i_u, we have $i_u^*\eta = 0$ and $\tilde{\sigma}|_u = r_{i_u^*\mathcal{T}, i_u^*\mathcal{T}'}$. □

We are now in position to assemble the fibers $\mathcal{R}_{ij}|_\gamma$ into a diffeological vector bundle. Its total space $\mathcal{R}_{ij}$ is the disjoint union of the fibers $\mathcal{R}_{ij}|_\gamma$ for $\gamma \in P_{ij}$, equipped with the obvious projection $\pi : \mathcal{R}_{ij} \to P_{ij}$. Let $U \subseteq \mathbb{R}^k$ be open. We define a map $\tilde{c} : U \to \mathcal{R}_{ij}$ to be a plot of $\mathcal{R}_{ij}$ if the following conditions hold:

(a) The composition $c := \pi \circ \tilde{c} : U \to P_{ij}$ is a plot of P_{ij}.

(b) Every point $u \in U$ has an open neighborhood $W \subseteq U$, a trivialization $\mathcal{T} : (c|_W^\vee)^*\mathcal{G} \to \mathcal{I}_\rho$, and a smooth section τ into the bundle $\mathrm{Hom}(\mathcal{V}_\mathcal{T}, \mathcal{W}_\mathcal{T})$ over W, such that $\tilde{c}(w) = [i_w^*\mathcal{T}, \tau(w)]$ for all $w \in W$.

It is straightforward to show that this indeed defines a diffeology.

Proposition 4.4. *$\mathcal{R}_{ij}$ is a hermitian vector bundle over P_{ij}.*

Proof. Let $c : U \to P_{ij}$ be a plot and let $u \in U$. We choose a contractible open neighborhood $W \subseteq U$ of u. Since $W \times [0,1]$ is contractible, there exists a trivialization $\mathcal{T} : (c|_W^\vee)^*\mathcal{G} \to \mathcal{I}_\rho$, and since W is contractible, there exists a trivialization

$$\tau : W \times \mathbb{C}^k \to \mathrm{Hom}(\mathcal{V}_\mathcal{T}, \mathcal{W}_\mathcal{T}),$$

where $k = \mathrm{rk}(\mathcal{E}_i) \cdot \mathrm{rk}(\mathcal{E}_j)$. We define a local trivialization ϕ of $\mathcal{R}_{ij}$ by the formula

$$\phi : W \times \mathbb{C}^k \to W \times_{P_{ij}} \mathcal{R}_{ij} : (w,v) \mapsto (w, [i_w^*\mathcal{T}, \tau(w,v)]).$$

This map is smooth (by definition of the plots) and fiber-wise linear. We have to show that it is a diffeomorphism.

The inverse map of ϕ can be described in the following way. Given $(w, [\mathcal{T}_w, \varphi]) \in W \times_{P_{ij}} \mathcal{R}_{ij}$ with a trivialization $\mathcal{T}_w : c(w)^*\mathcal{G} \to \mathcal{I}_0$ and $\varphi \in$

$\mathcal{R}_{ij}|_{c(w)}(\mathcal{T}_w)$, we have

$$\phi^{-1}(w, [\mathcal{T}_w, \varphi]) = \tau^{-1}(r_{\iota_w^*\mathcal{T}, \mathcal{T}_w}(\varphi)).$$

In order to show that ϕ^{-1} is smooth, we consider a plot of $W \times_{P_{ij}} \mathcal{R}_{ij}$, which is a pair $(f, \tilde{c})$ of a smooth map $f : U' \to W$ and a plot $\tilde{c} : U' \to \mathcal{R}_{ij}$ such that $c \circ f = \pi \circ \tilde{c} =: c'$. Since $\tilde{c}$ is a plot, there exists (possibly after replacing U' by a smaller subset) a trivialization $\mathcal{T}' : (c'^{\vee})^*\mathcal{G} \to \mathcal{I}_{\rho'}$ and a smooth section τ' of $\mathrm{Hom}(\mathcal{V}_{\mathcal{T}'}, \mathcal{W}_{\mathcal{T}'})$ such that $\tilde{c}(u) = [i_u^*\mathcal{T}, \tau'(u)]$ for all $u \in U'$. Now, we have to show that the map $U' \to W \times \mathbb{C}^k$ given by

$$u \mapsto \phi^{-1}(f(u), [i_u^*\mathcal{T}', \tau'(u)]) = \tau^{-1}(r_{i_u^*\mathcal{T}, i_u^*\mathcal{T}'}(\tau'(u)))$$

is smooth. Indeed, by Lemma 4.3 it is the composition of smooth bundle morphisms. □

4.3. The superficial connection

We define a connection on the vector bundle $\mathcal{R}_{ij}$, in the sense of Definition A.5. A similar definition of that connection has been sketched in [GR02]. Let $\Gamma \in PP_{ij}$ be a path in P_{ij}, and let $\gamma_s := \Gamma(s) \in P_{ij}$. Thus, Γ is a path from γ_0 to γ_1. We consider the adjoint map $\Gamma^{\vee} : [0,1]^2 \to M$ (i.e. $\Gamma^{\vee}(s,t) := \gamma_s(t)$) and choose a trivialization $\mathcal{T} : (\Gamma^{\vee})^*\mathcal{G} \to \mathcal{I}_{\rho}$. Let $\mathcal{T}_s := \mathcal{T}|_{\{s\}\times[0,1]}$; these are trivializations of $(\gamma_s^{\vee})^*\mathcal{G}$. Further, let $\mathcal{T}^0 := \mathcal{T}|_{[0,1]\times\{0\}}$ and $\mathcal{T}^1 := \mathcal{T}|_{[0,1]\times\{1\}}$, which are trivializations along the paths of end points. In the notation of the previous subsection, we consider the hermitian vector bundles $\mathcal{V}_{\mathcal{T}^0}$ and $\mathcal{W}_{\mathcal{T}^1}$ with connection over $[0,1]$, and the corresponding Hom-bundle $\mathrm{Hom}(\mathcal{V}_{\mathcal{T}^0}, \mathcal{W}_{\mathcal{T}^1})$ over $[0,1]$ with its induced connection. Now we have

$$\mathcal{R}_{ij}|_{\gamma_0}(\mathcal{T}_0) = \mathrm{Hom}(\mathcal{V}_{\mathcal{T}^0}, \mathcal{W}_{\mathcal{T}^1})|_0 \quad \text{and} \quad \mathcal{R}_{ij}|_{\gamma_1}(\mathcal{T}_1) = \mathrm{Hom}(\mathcal{V}_{\mathcal{T}^0}, \mathcal{W}_{\mathcal{T}^1})|_1.$$

Then, we define

$$pt_{ij}|_{\Gamma} : \mathcal{R}_{ij}|_{\gamma_0}(\mathcal{T}_0) \to \mathcal{R}_{ij}|_{\gamma_1}(\mathcal{T}_1) : \varphi \mapsto \exp\left(\int_{[0,1]^2} \rho\right) \cdot pt(\varphi), \tag{11}$$

where pt is the parallel transport in $\mathrm{Hom}(\mathcal{V}_{\mathcal{T}^0}, \mathcal{W}_{\mathcal{T}^1})$ along the linear path in $[0,1]$ from 0 to 1.

Proposition 4.5. (11) *defines a superficial, unitary connection pt_{ij} on $\mathcal{R}_{ij}$.*

Proof. The proof is split into five parts.

Part I: the definition of $pt_{ij}|_\Gamma$ is independent of the choice of the trivialization $\mathcal{T}$. In order to prove this, let $\mathcal{T}' : (\Gamma^\vee)^*\mathcal{G} \to \mathcal{I}_{\rho'}$ be another trivialization. We consider a 1-form $\eta \in \Omega^1([0,1]^2)$ and a connection-preserving bundle isomorphism

$$\tilde{\sigma} : \mathrm{Hom}(\mathcal{V}_\mathcal{T}, \mathcal{W}_\mathcal{T}) \otimes \mathbb{C}_{j_1^*\eta - j_0^*\eta} \to \mathrm{Hom}(\mathcal{V}_{\mathcal{T}'}, \mathcal{W}_{\mathcal{T}'})$$

over U, as in Lemma 4.3. Thus, the parallel transport pt in $\mathrm{Hom}(\mathcal{V}_\mathcal{T}, \mathcal{W}_\mathcal{T})$ and pt' in $\mathrm{Hom}(\mathcal{V}_{\mathcal{T}'}, \mathcal{W}_{\mathcal{T}'})$ differ by the parallel transport of $\mathbb{C}_{j_1^*\eta - j_0^*\eta}$, i.e.,

$$pt'(\tilde{\sigma}|_0(\varphi)) = \tilde{\sigma}|_1(pt(\varphi)) \cdot \exp\left(\int_{[0,1]} j_1^*\eta - j_0^*\eta\right)$$

for all $\varphi \in \mathrm{Hom}(\mathcal{V}_\mathcal{T}, \mathcal{W}_\mathcal{T})|_0$. Due to Stokes' Theorem, and the properties $\rho' - \rho = \mathrm{d}\eta$ and $i_0^*\eta = i_1^*\eta = 0$ of η, we can write this as

$$pt'(\tilde{\sigma}|_0(\varphi)) \cdot \exp\left(\int_{[0,1]^2} \rho'\right) = \tilde{\sigma}|_1(pt(\varphi)) \cdot \exp\left(\int_{[0,1]^2} \rho\right).$$

Since $\tilde{\sigma}|_s$ establishes the isomorphism $r_{\mathcal{T}_s, \mathcal{T}'_s}$ between $\mathcal{R}_{ij}|_{\gamma_s}(\mathcal{T}_s)$ and $\mathcal{R}_{ij}|_{\gamma_s}(\mathcal{T}'_s)$; this shows the claimed independence.

Part II is the verification of A.5 (a): $pt_{ij}|_\Gamma$ depends only on the thin homotopy class of Γ. In order to prove this, we suppose $\Gamma_1, \Gamma_2 \in PP_{ij}$ are thin homotopic (Definition A.4 for $X = P_{ij}$). Thus, there exists $h \in PPP_{ij}$ such that

(a) $h(0) = \Gamma_1$ and $h(1) = \Gamma_2$

(b) $h(r)(0) = \Gamma_1(0) = \Gamma_2(0)$ and $h(r)(1) = \Gamma_1(1) = \Gamma_2(1)$ for all $r \in [0,1]$

(c) $h^\vee : [0,1]^2 \to P_{ij} : (r,s) \mapsto h(r)(s)$ has rank one.

Here, a smooth map f between smooth manifolds is said to have rank k if $\mathrm{rk}(\mathrm{d}f_x) \leq k$ for all points x in its domain. We let $H : [0,1]^3 \to M$ denote the map $H(r,s,t) = h(r)(s)(t) = h^\vee(r,s)(t)$; which by (c) has rank two. We choose a trivialization $\mathcal{T} : H^*\mathcal{G} \to \mathcal{I}_\rho$; since H has rank two, $H^*\mathcal{G}$ is flat and hence $\mathrm{d}\rho = 0$.

The restriction of H to the $(t = 0)$ face of the cube $[0,1]^3$ has rank one due to (c). By Theorem C.1 there exists a parallel trivialization $\mathcal{T}'_{t=0}$, a hermitian vector bundle E_0 with flat connection, and a 2-isomorphism

$\mathcal{T}'_{t=0} \otimes E_0 \cong H|^*_{t=0}\mathcal{E}_i$. Thus,

$$\mathcal{V}_{\mathcal{T}} = \Delta(H|^*_{t=0}\mathcal{E}_i, \mathcal{T}_{t=0}) \cong \Delta(\mathcal{T}'_{t=0} \otimes E_0, \mathcal{T}_{t=0}) \cong \Delta(\mathcal{T}'_{t=0}, \mathcal{T}_{t=0}) \otimes E_0. \tag{12}$$

We note that the line bundle $\Delta(\mathcal{T}'_{t=0}, \mathcal{T}_{t=0})$ carries a connection of curvature $-\rho_{t=0}$. Analogously, $\mathcal{W}_{\mathcal{T}} \cong \Delta(\mathcal{T}'_{t=1}, \mathcal{T}_{t=1}) \otimes E_1$, for another flat hermitian vector bundle E_1 and a parallel trivialization $\mathcal{T}'_{t=1}$. Computing the holonomy of the vector bundle $\mathrm{Hom}(\mathcal{V}_{\mathcal{T}}, \mathcal{W}_{\mathcal{T}})$ over $[0,1]^2$ around the boundary, we obtain

$$pt_{r=1} \circ pt_{s=0} = pt_{s=1} \circ pt_{r=0} \cdot \exp\left(\int_{[0,1]^2} -\rho_{t=0} - \rho_{t=1}\right). \tag{13}$$

Here and in the following, the orientation on the faces of the cube $[0,1]^3$ are always the ones induced on the boundary by the standard orientation on $[0,1]^3$.

Next, we consider the $(s=0)$ face of the cube $[0,1]^3$, where H is constant in r due to (b). Let $\mathcal{T}_{s=0} := \mathcal{T}|_{[0,1]\times\{0\}\times[0,1]}$ be the restriction of $\mathcal{T}$ to that face, and let $\rho_{s=0}$ be its 2-form. For $p : [0,1]^2 \to [0,1]^2 : (r,t) \mapsto (0,t)$ we have another trivialization $\mathcal{T}'_{s=0} := p^*\mathcal{T}|_{\{0\}\times\{0\}\times[0,1]}$ with vanishing 2-form, since p factors through $[0,1]$. Applying Lemma 4.3 to the pair $(\mathcal{T}_{s=0}, \mathcal{T}'_{s=0})$, we obtain a 1-form $\eta_0 \in \Omega^1([0,1]^2)$ such that $\rho_{s=0} = \mathrm{d}\eta_0$ and $i_r^*\eta_0 = 0$ for all $r \in [0,1]$, and a connection-preserving bundle isomorphism

$$\tilde{\sigma}_0 : \mathrm{Hom}(\mathcal{V}_{\mathcal{T}_{s=0}}, \mathcal{W}_{\mathcal{T}_{s=0}}) \otimes \mathbb{C}_{j_1^*\eta_0 - j_0^*\eta_0} \to \mathrm{Hom}(\mathcal{V}_{\mathcal{T}'_{s=0}}, \mathcal{W}_{\mathcal{T}'_{s=0}})$$

over $[0,1]$. The parallel transport from 0 to 1 gives the identity

$$\tilde{\sigma}_0|_1 \circ pt_{s=0} = pt'_{s=0} \circ \tilde{\sigma}_0|_0 \ \cdot\ \exp\left(\int_0^1 j_0^*\eta_0 - j_1^*\eta_0\right).$$

In this identity, we used the fact that over each point $r \in [0,1]$ $\tilde{\sigma}_0$ restricts to the canonical isomorphism

$$r_{\mathcal{T}_{s,r=0}, \mathcal{T}'_{s,r=0}} : \mathcal{R}_{ij}|_{\gamma_0}(\mathcal{T}_{s,r=0}) \to \mathcal{R}_{ij}|_{\gamma_0}(\mathcal{T}'_{s,r=0}),$$

see Lemma 4.3. Likewise, the parallel transport $pt'_{s=0}$ in $\mathrm{Hom}(\mathcal{V}_{\mathcal{T}'_{s=0}}, \mathcal{W}_{\mathcal{T}'_{s=0}})$ coincides with the canonical isomorphism $r_{\mathcal{T}'_{s,r=0}, \mathcal{T}'_{s=0,r=1}}$, since it describes a trivialization of $\Delta(\mathcal{T}'_{s,r=0}, \mathcal{T}'_{s=0,r=1})$. Finally, we use the properties of η_0 and

Stokes' Theorem, and obtain

$$pt_{s=0} = r_{\mathcal{T}_{s,r=0},\mathcal{T}_{s=0,r=1}} \cdot \exp\left(\int_{[0,1]^2} \rho_{s=0}\right). \tag{14}$$

The $(s=1)$ face is treated analogously, just that we get $-\rho_{s=1}$ under the integral.

Now we are in position to show that the two parallel transports $pt_{ij}|_{\Gamma_1}$ and $pt_{ij}|_{\Gamma_2}$ coincide, under the canonical isomorphisms r. Indeed, for $\varphi \in \mathcal{R}_{ij}|_{\gamma_0}(\mathcal{T}_{s,r=0})$ we have:

$$\begin{aligned}
&pt_{ij}|_{\Gamma_2}(r_{\mathcal{T}_{s,r=0},\mathcal{T}_{s=0,r=1}}(\varphi)) \\
&\quad = pt_{r=1}(r_{\mathcal{T}_{s,r=0},\mathcal{T}_{s=0,r=1}}(\varphi)) \cdot \exp\left(\int_{[0,1]^2} -\rho_{r=1}\right) \\
&\quad \overset{(14)}{=} pt_{r=1}(pt_{s=0}(\varphi)) \cdot \exp\left(\int_{[0,1]^2} -\rho_{s=0} - \rho_{r=1}\right) \\
&\quad \overset{(13)}{=} pt_{s=1}(pt_{r=0}(\varphi)) \cdot \exp\left(\int_{[0,1]^2} -\rho_{t=0} - \rho_{t=1} - \rho_{s=0} - \rho_{r=1}\right) \\
&\quad \overset{(14)}{=} r_{\mathcal{T}_{s=1,r=0},\mathcal{T}_{s,r=1}}(pt_{r=0}(\varphi)) \cdot \exp\left(\int_{[0,1]^2} -\rho_{t=0} - \rho_{t=1} - \rho_{s=0} - \rho_{s=1} - \rho_{r=1}\right) \\
&\quad = r_{\mathcal{T}_{s=1,r=0},\mathcal{T}_{s,r=1}}(pt_{r=0}(\varphi)) \cdot \exp\left(\int_{[0,1]^2} \rho_{r=0}\right) \\
&\quad = r_{\mathcal{T}_{s=1,r=0},\mathcal{T}_{s,r=1}}(pt_{ij}|_{\Gamma_1}(\varphi))
\end{aligned}$$

In the last-but-one step we have used Stokes' Theorem for the closed 2-form ρ.

Part III is the verification of A.5 (b): parallel transport is compatible with path concatenation; this follows directly from the definition.

Part IV is the verification of A.5 (c): pt_{ij} is compatible with local trivializations. Let $c : U \to P_{ij}$ be a plot and let $\phi : W \times \mathbb{C}^k \to W \times_{P_{ij}} \mathcal{R}_{ij}$ be a local trivialization with $W \subseteq U$. We can assume that $\phi(w,v) = (w, [\iota_w^* \mathcal{T}, \tau(w,v)])$, where $\mathcal{T} : (c^\vee)^* \mathcal{G} \to \mathcal{I}_\rho$ is a trivialization over $[0,1] \times W$, and τ is a bundle isomorphism $\tau : W \times \mathbb{C}^k \to \mathrm{Hom}(\mathcal{V}_\mathcal{T}, \mathcal{W}_\mathcal{T})$. We let $\omega_\tau \in \Omega^1(W, \mathfrak{gl}(\mathbb{C}^k))$ be the corresponding connection 1-form, i.e. it induces the unique connection on $W \times \mathbb{C}^k$ such that τ is connection-preserving. This

means that for a path $\gamma \in PW$ we have

$$\tau(\gamma(1), \exp(\omega_\tau)(\gamma) \cdot v) = pt_\gamma(\tau(\gamma(0), v)), \tag{15}$$

where $\exp(\omega_\tau)(\gamma) \in \mathrm{GL}(\mathbb{C}^k)$ is the path-ordered exponential of ω_τ along γ. Then we define

$$\omega_\phi := \omega_\tau + \int_{[0,1]} \rho \in \Omega^1(W, \mathfrak{gl}(\mathbb{C}^k)), \tag{16}$$

with the addition performed under the diagonal embedding $\mathbb{R} \subseteq \mathfrak{gl}(\mathbb{C}^k)$. We note that

$$\exp(\omega_\phi)(\gamma) = \exp(\omega_\tau)(\gamma) \cdot \exp\left(\int_{[0,1]^2} (\mathrm{id}_{[0,1]} \times \gamma)^* \rho\right). \tag{17}$$

Now we consider the relevant diagram, whose commutativity is to check:

$$\begin{tikzcd}
\mathbb{C}^k \arrow[r, "\exp(\omega_\phi)(\gamma)"] \arrow[d, "\phi|_{\gamma(0)}"'] & \mathbb{C}^k \arrow[d, "\phi|_{\gamma(1)}"] \\
\mathcal{R}_{ij}|_{c(\gamma(0))} \arrow[r, "pt_{ij}|_{c\circ\gamma}"'] & \mathcal{R}_{ij}|_{c(\gamma(1))}.
\end{tikzcd}$$

Clockwise, using (15) and (17) we obtain the map

$$v \mapsto \left[\iota_{\gamma(1)}^* \mathcal{T}, \exp\left(\int_{[0,1]^2} (\mathrm{id} \times \gamma)^* \rho\right) \cdot pt_\gamma(\tau(\gamma(0), v))\right].$$

Counter-clockwise, we first have $v \mapsto [\iota_{\gamma(0)}^* \mathcal{T}, \tau(\gamma(0), v)]$ and then obtain, by definition of pt_{ij}, precisely the same result.

Part V: the connection pt_{ij} is superficial (Definition A.11). For condition A.11 (i), we may equivalently show that a thin, fixed-ends loop $\Gamma : S^1 \to P_{ij}$ has trivial holonomy, $pt_{ij}|_\Gamma = \mathrm{id}$. Here, by a thin loop we mean that its adjoint $\Gamma^\vee : S^1 \times [0,1] \to M$ has rank one. Since $\Gamma^\vee|_{S^1 \times \{0\}}$ and $\Gamma^\vee|_{S^1 \times \{1\}}$ are constant and all paths have sitting instants, we can extend $\Gamma^\vee$ constantly to discs glued along their boundary to $S^1 \times \{0\}$ and $S^1 \times \{1\}$. By Theorem C.1 (a) there exists a trivialization $\mathcal{T} : (\Gamma^\vee)^* \mathcal{G} \to \mathcal{I}_0$. Thus, $pt_{ij}|_\Gamma$ is determined completely by the holonomy of $\mathrm{Hom}(\mathcal{V}_\mathcal{T}, \mathcal{W}_\mathcal{T})$ around $S^1 = \partial D^2$. The vector bundles $\mathcal{V}_\mathcal{T}$ and $\mathcal{W}_\mathcal{T}$ are defined over D^2 and are flat by Theorem C.1 (b); hence, their holonomy vanishes.

For condition A.11 (ii), we consider a rank-two-homotopy $h \in PPP_{ij}$ between paths $\Gamma_1 = h(0)$ and $\Gamma_2 = h(1)$. Let $\mathcal{T} : (h^\vee)^*\mathcal{G} \to \mathcal{I}_\rho$ be a trivialization. Due to condition A.10 (c), $h^\vee$ has rank two, so that $(h^\vee)^*\mathcal{G}$ is flat and $\mathrm{d}\rho = 0$. Over the $(t = 0)$ face, $h^\vee$ is constant in r due to A.10 (b), and hence is of rank one. Using Theorem C.1, there exists a trivialization $\mathcal{T}_0 : (h^\vee)|^*_{t=0}\mathcal{G} \to \mathcal{I}_0$ and a flat hermitian vector bundle E_0 over $[0,1]^2$ such that $\mathcal{V}_\mathcal{T} \cong \Delta(\mathcal{T}_0, \mathcal{T}_{t=0}) \otimes E_0$, see (12). Since $\Delta(\mathcal{T}_0, \mathcal{T}_{t=0})$ is a hermitian line bundle with connection of curvature $-\rho|_{t=0}$, we have

$$\mathrm{Hol}_{\mathcal{V}_\mathcal{T}}(\partial[0,1]^2) = \exp\left(\int_{[0,1]^2} -\rho|_{t=0}\right). \tag{18}$$

We treat the $(t = 1)$ face analogously, producing the same formula for $\mathcal{W}_\mathcal{T}$ and $\rho|_{t=1}$. Now we are in position to prove condition A.11 (ii); we need to check that

$$pt_{ij}|_{h_1} \circ pt_{ij}|_{\gamma_1} = pt_{ij}|_{\gamma_2} \circ pt_{ij}|_{h_0}.$$

Substituting our above findings, we obtain the integral of ρ over four faces of the cube, as well as the holonomy of $\mathrm{Hom}(\mathcal{V}_\mathcal{T}, \mathcal{W}_\mathcal{T})$ around $\partial[0,1]^2$. Using (18) the latter provides integrals of ρ over the remaining faces. All together, these integrals vanish due to $\mathrm{d}\rho = 0$ by Stokes' Theorem. □

We conclude the discussion of the connection pt_{ij} with the following result. It coincides with a corresponding claim in [GR02] (for rank one D-branes).

Lemma 4.6. *The curvature of the connection pt_{ij} on $\mathcal{R}_{ij}$ satisfies*

$$\frac{1}{\mathrm{rk}(\mathcal{R}_{ij})}\,\mathrm{tr}(\mathrm{curv}(pt_{ij})) = \int_{[0,1]} \mathrm{ev}^*\mathrm{curv}(\mathcal{G}) + \mathrm{ev}_1^*\omega_j - \mathrm{ev}_0^*\omega_i.$$

Proof. According to Remark A.9, we have to compute $\mathrm{tr}(\mathrm{d}\omega_\phi)$, where ω_ϕ is the 1-form encountered in the preceding proof in (16), which in turn was obtained from a trivialization $\mathcal{T} : (c^\vee)^*\mathcal{G} \to \mathcal{I}_\rho$. For the first summand in (16) we have to compute $\mathrm{tr}(\mathrm{d}\omega_\tau)$, i.e. the trace of the curvature of the vector bundle $\mathrm{Hom}(\mathcal{V}_\mathcal{T}, \mathcal{W}_\mathcal{T})$. From the definition of the functor Δ and the vector bundles $\mathcal{V}_\mathcal{T}$ and $\mathcal{W}_\mathcal{T}$ we see that

$$\frac{1}{\mathrm{rk}(\mathcal{E}_i)}\mathrm{tr}(\mathrm{curv}(\mathcal{V}_\mathcal{T})) = (\mathrm{ev}_0 \circ c)^*\omega_i - j_0^*\rho \quad \text{and}$$
$$\frac{1}{\mathrm{rk}(\mathcal{E}_j)}\mathrm{tr}(\mathrm{curv}(\mathcal{W}_\mathcal{T})) = (\mathrm{ev}_0 \circ c)^*\omega_j - j_1^*\rho.$$

Thus, we obtain

$$\frac{1}{\mathrm{rk}(\mathcal{E}_i)\mathrm{rk}(\mathcal{E}_j)}\mathrm{tr}(\mathrm{curv}(\mathrm{Hom}(\mathcal{V}_\mathcal{T},\mathcal{W}_\mathcal{T}))) = c^*(\mathrm{ev}_1^*\omega_j - \mathrm{ev}_0^*\omega_i) - j_1^*\rho + j_0^*\rho$$

in $\in \Omega^2(W)$. For the second summand in (16) we recall that integration over a fiber with boundary satisfies a version of Stokes' Theorem,

$$\mathrm{d}\int_{[0,1]}\rho = \int_{[0,1]}\mathrm{d}\rho + j_1^*\rho - j_0^*\rho.$$

We have $\mathrm{d}\rho = (c^\vee)^*\mathrm{curv}(\mathcal{G})$ and obtain – under the sum of (16) – the claimed formula. □

4.4. Fusion representation

We equip the vector bundle $\mathcal{R}_{ij}$ over P_{ij} with a fusion representation of the line bundle $\mathcal{L}$ over LM. We start by constructing isomorphisms

$$\phi_{ij}|_{\gamma_1,\gamma_2} : \mathcal{L}|_\tau \otimes \mathcal{R}_{ij}|_{\gamma_2} \to \mathcal{R}_{ij}|_{\gamma_1}$$

for $(\gamma_1,\gamma_2) \in P_{ij}^{[2]} := P_{ij} \times_{Q_i\times Q_j} P_{ij}$, with $\tau := \gamma_1 \cup \gamma_2$. Let $\mathcal{T}$ be a trivialization of $\tau^*\mathcal{G}$. We recall that $S^1 = \mathbb{R}/\mathbb{Z}$ and consider the two maps $\iota_1,\iota_2\colon [0,1] \to S^1$ defined by $\iota_1(t) := \frac{1}{2}t$ and $\iota_2(t) := 1 - \frac{1}{2}t$. Then, $\mathcal{T}_1 := \iota_1^*\mathcal{T}$ is a trivialization of $\gamma_1^*\mathcal{G}$, and $\mathcal{T}_2 := \iota_2^*\mathcal{T}$ is a trivialization of $\gamma_2^*\mathcal{G}$. We note that

$$\mathcal{R}_{ij}|_{\gamma_2}(\mathcal{T}_2) = \mathrm{Hom}(\Delta(\mathcal{E}_i|_x,\mathcal{T}|_0),\Delta(\mathcal{E}_j|_y,\mathcal{T}|_{\frac{1}{2}})) = \mathcal{R}_{ij}|_{\gamma_1}(\mathcal{T}_1),$$

and define:

$$\phi_{ij}|_{\gamma_1,\gamma_2} : \mathcal{L}|_\tau \otimes \mathcal{R}_{ij}|_{\gamma_2}(\mathcal{T}_2) \to \mathcal{R}_{ij}|_{\gamma_1}(\mathcal{T}_1) : [\mathcal{T},z]\otimes\varphi \mapsto z\varphi.$$

Lemma 4.7. *This defines a connection-preserving, unitary bundle isomorphism ϕ_{ij}.*

Proof. (1) The map $\phi_{ij}|_{\gamma_1,\gamma_2}$ is independent of the choice of the trivialization $\mathcal{T}$. Indeed, if $\mathcal{T}'$ is another trivialization of $\tau^*\mathcal{G}$, let P be a hermitian line bundle with connection over S^1 such that $\mathcal{T}' \cong \mathcal{T}\otimes P$. Let σ_1 and σ_2 be parallel unit-length sections into ι_1^*P and ι_2^*P, inducing 2-isomorphisms $\psi_1 : \mathcal{T}_1 \Rightarrow \mathcal{T}_1'$ and $\psi_2 : \mathcal{T}_2 \Rightarrow \mathcal{T}_2'$. These differ over the endpoints

by numbers $a_0, a_1 \in \mathrm{U}(1)$, say $\psi_1|_0 = \psi_2|_0 \cdot a_0$ and $\psi_1|_1 = \psi_2|_1 \cdot a_1$. We obtain $\mathrm{Hol}_P(\mathrm{id}_{S^1}) = a_0 a_1^{-1}$, and $[\mathcal{T}, z] = [\mathcal{T}', a_1 a_0^{-1} \cdot z]$ in $\mathcal{L}|_\tau$. Further, we have (in the notation of Section 4.2)

$$r_{\psi_2}(\varphi) = (\psi_2)_1 \circ \varphi \circ (\psi_2)_0^{-1} = a_1^{-1} a_0 \cdot (\psi_1)_1 \circ \varphi \circ (\psi_1)_0^{-1} = a_1^{-1} a_0 \cdot r_{\psi_1}(\varphi).$$

We obtain

$$(a_1 a_0^{-1} \cdot z) \cdot r_{\psi_2}(\varphi) = r_{\psi_1}(z\varphi);$$

this shows the independence.

(2) The map ϕ_{ij} is smooth. Consider a plot $c : U \to P_{ij}^{[2]}$. We consider an open subset $W \subseteq U$ with a trivialization $\mathcal{T} : (\tilde{c}^\vee)^*\mathcal{G} \to \mathcal{I}_\rho$, where $\tilde{c} : W \to LM$ is the induced plot of LM, i.e., $\tilde{c} := \cup \circ c$. We obtain induced trivializations $\mathcal{T}_1 := (\iota_1 \times \mathrm{id}_W)^*\mathcal{T}$ and $\mathcal{T}_2 := (\iota_2 \times \mathrm{id}_W)^*\mathcal{T}$. After choosing a trivialization τ of the vector bundle $\mathrm{Hom}(\mathcal{V}_\mathcal{T}, \mathcal{W}_\mathcal{T})$ over W, we obtain local trivializations ϕ_1 of $\mathrm{pr}_1^*\mathcal{R}_{ij}$ and ϕ_2 of $\mathrm{pr}_2^*\mathcal{R}_{ij}$ over $P_{ij}^{[2]}$, defined by $\phi_1(w, v) := (w, [\iota_w^*\mathcal{T}_1, \tau(w, v)])$ and $\phi_2(w, v) := (w, [\iota_w^*\mathcal{T}_2, \tau(w, v)])$, according to the proof of Proposition 4.4. Further, we have a local trivialization ϕ of $\cup^*\mathcal{L}$ defined (see Lemma A.2) by $\phi(w, v) := (w, [\iota_w^*\mathcal{T}, v])$. By definition of the isomorphism ϕ_{ij}, we have a commutative diagram

$$\begin{tikzcd}
((W \times \mathbb{C}) \otimes (W \times \mathbb{C}^k) \arrow[r] \arrow[d, "\phi\otimes\phi_2"'] & W \times \mathbb{C}^k \arrow[d, "\phi_1"] \\
(W \times_{P_{ij}^{[2]}} \cup^*\mathcal{L}) \otimes (W \times_{P_{ij}^{[2]}} \mathrm{pr}_2^*\mathcal{R}_{ij}) \arrow[r, "\phi_{ij}"'] & W \times_{P_{ij}^{[2]}} \mathrm{pr}_1^*\mathcal{R}_{ij}
\end{tikzcd}$$

whose top arrow is pointwise scalar multiplication. This shows that ϕ_{ij} is smooth.

(3) The bundle morphism ϕ_{ij} is connection-preserving. Let $\Gamma = (\Gamma_1, \Gamma_2)$ be a path in $P_{ij}^{[2]}$, and let $\tau := \Gamma_1 \cup \Gamma_2$ be the induced path in LM. Let $\mathcal{T} : (\tau^\vee)^*\mathcal{G} \to \mathcal{I}_\rho$ be a trivialization over $[0, 1] \times S^1$. Let $\mathcal{T}_1 : (\Gamma_1^\vee)^*\mathcal{G} \to \mathcal{I}_{\rho_1}$ and $\mathcal{T}_2 : (\Gamma_2^\vee)^*\mathcal{G} \to \mathcal{I}_{\rho_2}$ be the trivializations obtained by pullback of $\mathcal{T}$ along the maps $\mathrm{id} \times \iota_1, \mathrm{id} \times \iota_2 : [0, 1]^2 \to [0, 1] \times S^1$, respectively. Let $\mathcal{T}_k(s)$ be the restriction of $\mathcal{T}_k$ to $\{s\} \times [0, 1]$, for $s \in \{0, 1\}$. By construction, we have $\mathcal{V}_{\mathcal{T}_1} = \mathcal{V}_{\mathcal{T}_2}$ and $\mathcal{W}_{\mathcal{T}_1} = \mathcal{W}_{\mathcal{T}_2}$, as well as

$$\int_{[0,1]\times S^1} \rho + \int_{[0,1]^2} \rho_2 = \int_{[0,1]^2} \rho_1.$$

From the definitions of the parallel transport in $\mathcal{L}$ and $\mathcal{R}_{ij}$, we conclude that the diagram

$$\begin{tikzcd}
\mathcal{L}|_{\tau(0)} \otimes \mathcal{R}_{ij}|_{\Gamma_2(0)}(\mathcal{T}_2(0)) \arrow[r, "\phi_{ij}|_{\Gamma_1(0),\Gamma_2(0)}"] \arrow[d, "pt_\tau \otimes pt_{ij}|_{\Gamma_2}"'] & \mathcal{R}_{ij}|_{\Gamma_1(0)}(\mathcal{T}_1(0)) \arrow[d, "pt_{ij}|_{\Gamma_1}"] \\
\mathcal{L}|_{\tau(1)} \otimes \mathcal{R}_{ij}|_{\Gamma_2(1)}(\mathcal{T}_2(1)) \arrow[r, "\phi_{ij}|_{\Gamma_1(1),\Gamma_2(1)}"'] & \mathcal{R}_{ij}|_{\Gamma_1(1)}(\mathcal{T}_1(1))
\end{tikzcd}$$

is commutative; this shows that ϕ_{ij} is connection-preserving. $\square$

Next we show that the first LBG axiom is satisfied.

Lemma 4.8. *Axiom (LBG1) is satisfied: ϕ_{ij} is a fusion representation.*

Proof. We consider a triple $(\gamma_1, \gamma_2, \gamma_3) \in PM^{[3]}$ and the corresponding loops $\tau_{ab} := \gamma_a \cup \gamma_b$. Using trivializations $\mathcal{T}_{ab} : \tau_{ab}^*\mathcal{G} \to \mathcal{I}_0$, and 2-isomorphisms $\phi_1 \colon \iota_1^*\mathcal{T}_{12} \Rightarrow \iota_1^*\mathcal{T}_{13}$, $\phi_2 \colon \iota_2^*\mathcal{T}_{12} \Rightarrow \iota_1^*\mathcal{T}_{23}$ and $\phi_3 \colon \iota_2^*\mathcal{T}_{23} \Rightarrow \iota_2^*\mathcal{T}_{13}$, (LBG1) implies the commutativity of the following diagram:

$$\begin{tikzcd}
\mathcal{L}|_{\gamma_1\cup\gamma_2} \otimes \mathcal{L}|_{\gamma_2\cup\gamma_3} \otimes \mathcal{R}_{ij}|_{\gamma_3}(\iota_2^*\mathcal{T}_{23}) \arrow[r, "\mathrm{id}\otimes\phi_{ij}|_{\gamma_2,\gamma_3}"] \arrow[dd, "\lambda_{\gamma_1,\gamma_2,\gamma_3}\otimes\mathrm{id}"'] & \mathcal{L}_{\gamma_1\cup\gamma_2} \otimes \mathcal{R}_{ij}|_{\gamma_2}(\iota_1^*\mathcal{T}_{23}) \arrow[d, "\mathrm{id}\otimes r_{\phi_2}"] \\
& \mathcal{L}_{\gamma_1\cup\gamma_2} \otimes \mathcal{R}_{ij}|_{\gamma_2}(\iota_2^*\mathcal{T}_{12}) \arrow[d, "\phi_{ij}|_{\gamma_1,\gamma_2}"] \\
\mathcal{L}|_{\gamma_1\cup\gamma_3} \otimes \mathcal{R}_{ij}|_{\gamma_3}(\iota_2^*\mathcal{T}_{23}) \arrow[d, "\mathrm{id}\otimes r_{\phi_3}^{-1}"'] & \mathcal{R}_{ij}|_{\gamma_1}(\iota_1^*\mathcal{T}_{12}) \arrow[d, "(r_{\phi_1})^{-1}"] \\
\mathcal{L}_{\gamma_1\cup\gamma_3} \otimes \mathcal{R}_{ij}|_{\gamma_3}(\iota_2^*\mathcal{T}_{13}) \arrow[r, "\phi_{ij}|_{\gamma_1,\gamma_3}"'] & \mathcal{R}_{ij}|_{\gamma_1}(\iota_1^*\mathcal{T}_{13}).
\end{tikzcd}$$

In this diagram, the bundle morphisms ϕ_{ij}, ϕ_{jk} and ϕ_{ik} are just scalar multiplication. We may additionally assume that $\lambda([\mathcal{T}_{12}, z_{12}] \otimes [\mathcal{T}_{23}, z_{23}]) = [\mathcal{T}_{13}, z_{12}z_{23}]$, reducing the commutativity of the diagram to the equation $r_{\phi_1}^{-1} \circ r_{\phi_2} = r_{\phi_3}^{-1}$. By definition of the fusion product and of the morphisms r_{ϕ_a}, this equation follows from the identities between the 2-isomorphisms ϕ_1, ϕ_2 and ϕ_3. $\square$

4.5. Lifted path concatenation

We equip the vector bundle $\mathcal{R}_{ij}$ over P_{ij} with a lifted path concatenation, and start by constructing a linear map

$$\chi_{ijk}|_{\gamma_{12},\gamma_{23}} : \mathcal{R}_{jk}|_{\gamma_{23}} \otimes \mathcal{R}_{ij}|_{\gamma_{12}} \to \mathcal{R}_{ik}|_{\gamma_{23}\star\gamma_{12}}$$

for all $(\gamma_{23}, \gamma_{12}) \in P_{jk} \times_{Q_j} P_{ij}$ and $i, j, k \in I$. Let $x := \gamma_{12}(0)$, $y := \gamma_{12}(1) = \gamma_{23}(0)$ and $z := \gamma_{23}(1)$. Consider a trivialization $\mathcal{T}$ of $(\gamma_{23} \star \gamma_{12})^*\mathcal{G}$, and let trivializations $\mathcal{T}_{12}$ of $\gamma_{12}^*\mathcal{G}$ and $\mathcal{T}_{23}$ of $\gamma_{23}^*\mathcal{G}$ be defined via restriction, i.e. $\mathcal{T}_{12} := \iota_1^*\mathcal{T}$ and $\mathcal{T}_{23} := \iota_2^*\mathcal{T}$, where $\iota_1, \iota_2 : [0,1] \to [0,1]$ are defined by $\iota_1(t) := \frac{1}{2}t$ and $\iota_2(t) := \frac{1}{2} + \frac{1}{2}t$. We find

$$\begin{aligned}
\mathcal{R}_{ij}|_{\gamma_{12}}(\mathcal{T}_{12}) &= \mathrm{Hom}(\Delta(\mathcal{E}_i|_x, \mathcal{T}|_0), \Delta(\mathcal{E}_j|_y, \mathcal{T}|_{\frac{1}{2}})) \\
\mathcal{R}_{jk}|_{\gamma_{23}}(\mathcal{T}_{23}) &= \mathrm{Hom}(\Delta(\mathcal{E}_j|_y, \mathcal{T}|_{\frac{1}{2}}), \Delta(\mathcal{E}_k|_z, \mathcal{T}|_1)) \\
\mathcal{R}_{ik}|_{\gamma_{23}\circ\gamma_{12}}(\mathcal{T}) &= \mathrm{Hom}(\Delta(\mathcal{E}_i|_x, \mathcal{T}|_0), \Delta(\mathcal{E}_k|_z, \mathcal{T}|_1)).
\end{aligned}$$

Thus, we define

$$\mathcal{R}_{jk}|_{\gamma_{23}}(\mathcal{T}_{23}) \otimes \mathcal{R}_{ij}|_{\gamma_{12}}(\mathcal{T}_{12}) \to \mathcal{R}_{ik}|_{\gamma_{23}\star\gamma_{12}}(\mathcal{T})$$

simply as the composition of linear maps, i.e., $\varphi_{23} \otimes \varphi_{12} \mapsto \varphi_{23} \circ \varphi_{12}$.

Lemma 4.9. *This defines a connection-preserving bundle morphism*

$$\chi_{ijk} : \mathrm{pr}_1^*\mathcal{R}_{jk} \otimes \mathrm{pr}_2^*\mathcal{R}_{ij} \to \star^*\mathcal{R}_{ik}.$$

Proof. (1) Independence of the choice of the trivialization is straightforward to see and left out for brevity.

(2) In order to see that χ_{ijk} is a smooth bundle morphism we represent it in local trivializations. Consider a plot $c : U \to P_{jk} \times_{Q_j} P_{ij}$, and let c_{12}, c_{23} and c_{13} be the pointwise projections to P_{ij} and P_{jk}, and the pointwise concatenation, respectively. We restrict to a contractible open subset $W \subseteq U$ such that there is a trivialization $\mathcal{T} : (c_{13}^\vee)^*\mathcal{G} \to \mathcal{I}_\rho$. We induce trivializations $\mathcal{T}_{12}$ and $\mathcal{T}_{23}$ of $(c_{12}^\vee)^*\mathcal{G}$ and $(c_{23}^\vee)^*\mathcal{G}$ by pullback along $\mathrm{id}_W \times \iota_1$ and $\mathrm{id}_W \times \iota_2$, respectively. We then consider the vector bundles

$$\begin{aligned}
&\mathcal{V}_1 := \Delta((\mathrm{ev}_0 \circ c_{13})^*\mathcal{E}_i, j_0^*\mathcal{T}), \quad \mathcal{V}_2 := \Delta((\mathrm{ev}_{\frac{1}{2}} \circ c_{13})^*\mathcal{E}_j, j_{\frac{1}{2}}^*\mathcal{T}) \\
&\text{and} \quad \mathcal{V}_3 := \Delta((\mathrm{ev}_1 \circ c_{13})^*\mathcal{E}_i, j_1^*\mathcal{T})
\end{aligned}$$

over W. We claim that there exist trivializations τ_{ab} of the Hom-bundles $\mathrm{Hom}(\mathcal{V}_a, \mathcal{V}_b)$ such that the diagram

$$\begin{tikzcd}
(W \times \mathbb{C}^{n_3 \times n_2}) \otimes (W \times \mathbb{C}^{n_2 \times n_1}) \arrow[r, "\cdot"] \arrow[d, "\tau_{23} \otimes \tau_{12}"'] & W \times \mathbb{C}^{n_3 \times n_1} \arrow[d, "\tau_{13}"] \\
\mathrm{Hom}(\mathcal{V}_2, \mathcal{V}_2) \otimes \mathrm{Hom}(\mathcal{V}_1, \mathcal{V}_2) \arrow[r, "\circ"'] & \mathrm{Hom}(\mathcal{V}_1, \mathcal{V}_3)
\end{tikzcd}$$

is commutative, with pointwise matrix multiplication in the top row. These can be found by choosing trivializations of the bundles $\mathcal{V}_a$ separately, and then inducing trivializations of the Hom-bundles via $\mathrm{Hom}(\mathcal{V}_a, \mathcal{V}_b) = \mathcal{V}_a^* \otimes \mathcal{V}_b$. According to the proof of Proposition 4.4, the gerbe trivializations $\mathcal{T}_{ab}$ and the bundle isomorphisms τ_{ab} induce local trivializations ϕ_{jk} of $c_{23}^*\mathcal{R}_{jk}$, ϕ_{ij} of $c_{12}^*\mathcal{R}_{ij}$, and ϕ_{ik} of $c_{13}^*\mathcal{R}_{ik}$. By construction, the diagram

$$\begin{tikzcd}
W \times (\mathbb{C}^{n_k \times n_j} \otimes \mathbb{C}^{n_j \times n_i}) \arrow[r, "\cdot"] \arrow[d, "\phi_{jk} \otimes \phi_{ij}"'] & W \times \mathbb{C}^{n_k \times n_i} \arrow[d, "\phi_{ik}"] \\
W \times_{P_{jk} \times_{Q_j} P_{ij}} (\mathrm{pr}_{jk}^* \mathcal{R}_{jk} \otimes \mathrm{pr}_{ij}^* \mathcal{R}_{ij}) \arrow[r, "\mathrm{id} \times \chi_{ijk}"'] & W \times_{P_{jk} \times_{Q_j} P_{ij}} (\star^* \mathcal{R}_{ik})
\end{tikzcd}$$

is commutative. This shows that χ_{ijk} is smooth under local trivializations, and hence smooth.

(3) The bundle morphism χ_{ijk} is connection-preserving. Indeed, consider a path $(\Gamma_{23}, \Gamma_{12})$ in $P_{jk} \times_{Q_j} P_{ij}$, and let Γ be its pointwise concatenation. Let $\mathcal{T} : (\Gamma^\vee)^*\mathcal{G} \to \mathcal{I}_\rho$ be a trivialization, and let $\mathcal{T}_{12} : (\Gamma_{12}^\vee)^*\mathcal{G} \to \mathcal{I}_{\rho_{12}}$ and $\mathcal{T}_{23} : (\Gamma_{23}^\vee)^*\mathcal{G} \to \mathcal{I}_{\rho_{23}}$ be the pullbacks of $\mathcal{T}$ under $(\mathrm{id} \times \iota_1)$ and $(\mathrm{id} \times \iota_2)$, respectively. We consider the hermitian vector bundles with connections

$$\begin{aligned}
&\mathcal{U} := \Delta((\mathrm{ev}_0 \circ \Gamma)^*\mathcal{E}_i, j_0^*\mathcal{T}), \quad \mathcal{V} := \Delta((\mathrm{ev}_{\frac{1}{2}} \circ \Gamma)^*\mathcal{E}_j, j_{\frac{1}{2}}^*\mathcal{T}) \\
&\text{and} \quad \mathcal{W} := \Delta((\mathrm{ev}_1 \circ \Gamma)^*\mathcal{E}_k, j_1^*\mathcal{T})
\end{aligned}$$

over $[0,1]$, so that

$$\begin{aligned}
&\mathcal{R}_{ij}|_{\Gamma_{12}(s)}(\mathcal{T}_{12}|_{\{s\}\times[0,1]}) = \mathrm{Hom}(\mathcal{U}, \mathcal{V})|_s, \\
&\mathcal{R}_{jk}|_{\Gamma_{23}(s)}(\mathcal{T}_{23}|_{\{s\}\times[0,1]}) = \mathrm{Hom}(\mathcal{V}, \mathcal{W})|_s \\
&\text{and} \quad \mathcal{R}_{ik}|_{\Gamma(s)}(\mathcal{T}|_{\{s\}\times[0,1]}) = \mathrm{Hom}(\mathcal{U}, \mathcal{W})|_s,
\end{aligned}$$

and lifted path concatenation is the composition

$$\mathrm{Hom}(\mathcal{V}, \mathcal{W}) \otimes \mathrm{Hom}(\mathcal{U}, \mathcal{V}) \to \mathrm{Hom}(\mathcal{U}, \mathcal{W}). \tag{19}$$

It is elementary to see that (19) is a connection-preserving bundle morphism over $[0,1]$. Further, we find

$$\int_{[0,1]^2} (\Gamma^\vee)^* \rho = \int_{[0,1]^2} (\Gamma_{12}^\vee)^* \rho_{12} + \int_{[0,1]^2} (\Gamma_{23}^\vee)^* \rho_{23}$$

for the integrals of the 2-form of the trivializations. These results prove the claim. □

Now that we have established the lifted path concatenation in $\mathcal{R}_{ij}$, we are in position to show that the next two LBG axioms are satisfied.

Lemma 4.10. *Axiom (LBG2) is satisfied: the maps χ_{ijk} are associative up to reparameterization.*

Proof. In order to prove the pentagon diagram of (LBG2), we have to choose a path Γ in P_{il} connecting $(\gamma_{34} \star \gamma_{23}) \star \gamma_{12}$ with $\gamma_{34} \star (\gamma_{23} \star \gamma_{12})$. In order to do so, we let $\varphi : [0,1] \to [0,1]$ be a smooth reparameterization such that $((\gamma_{34} \star \gamma_{23}) \star \gamma_{12})(t) = (\gamma_{34} \star (\gamma_{23} \star \gamma_{12}))(\varphi(t))$ for $t \in [0,1]$. We let Γ be induced from a fixed-ends homotopy h between $\mathrm{id}_{[0,1]}$ and φ. We let $\mathcal{T}_1$ be a trivialization of $(\gamma_{34} \star (\gamma_{23} \star \gamma_{12}))^*\mathcal{G}$, and then define a trivialization of $\mathcal{G}$ along Γ by $\mathcal{T} := h^*\mathcal{T}_1$. We set $\mathcal{T}_0 := \mathcal{T}|_{\{0\}\times[0,1]} = \varphi^*\mathcal{T}_1$. We note that Γ is a thin path with fixed endpoints, and that $\mathcal{T}$ is a trivialization with vanishing 2-form. Thus, by definition of the parallel transport $pt_{il}|_\Gamma$,

$$d_{(\gamma_{34}\star\gamma_{23})\star\gamma_{12},\gamma_{34}\star(\gamma_{23}\star\gamma_{12})} : \mathcal{R}_{il}|_{(\gamma_{34}\star\gamma_{23})\star\gamma_{12}}(\mathcal{T}_0) \to \mathcal{R}_{il}|_{\gamma_{34}\star(\gamma_{23}\star\gamma_{12})}(\mathcal{T}_1)$$

is the identity map. The pentagon diagram we have to prove thus reads as

$$\begin{tikzcd}
\mathcal{R}_{kl}|_{\gamma_{34}}(\iota_2^*\iota_2^*\mathcal{T}_0) \otimes \mathcal{R}_{jk}|_{\gamma_{23}}(\iota_1^*\iota_2^*\mathcal{T}_0) \otimes \mathcal{R}_{ij}|_{\gamma_{12}}(\iota_1^*\mathcal{T}_0) \arrow[r] \arrow[d, "\chi_{jkl}|_{\gamma_{23},\gamma_{34}}\otimes\mathrm{id}"'] & \mathcal{R}_{kl}|_{\gamma_{34}}(\iota_2^*\mathcal{T}_1) \otimes \mathcal{R}_{jk}|_{\gamma_{23}}(\iota_2^*\iota_1^*\mathcal{T}_1) \otimes \mathcal{R}_{ij}|_{\gamma_{12}}(\iota_1^*\iota_1^*\mathcal{T}_1) \arrow[d, "\mathrm{id}\otimes\chi_{ijk}|_{\gamma_{12},\gamma_{23}}"] \\
\mathcal{R}_{jl}|_{\gamma_{34}\star\gamma_{23}}(\iota_2^*\mathcal{T}_0) \otimes \mathcal{R}_{ij}|_{\gamma_{12}}(\iota_1^*\mathcal{T}_0) \arrow[d, "\chi_{ijl}|_{\gamma_{12},\gamma_{34}\star\gamma_{23}}"'] & \mathcal{R}_{kl}|_{\gamma_{34}}(\iota_2^*\mathcal{T}_1) \otimes \mathcal{R}_{ik}|_{\gamma_{23}\star\gamma_{12}}(\iota_1^*\mathcal{T}_1) \arrow[d, "\chi_{ikl}|_{\gamma_{23}\star\gamma_{12},\gamma_{34}}"] \\
\mathcal{R}_{il}|_{(\gamma_{34}\star\gamma_{23})\star\gamma_{12}}(\mathcal{T}_0) \arrow[r, equal] & \mathcal{R}_{il}|_{\gamma_{34}\star(\gamma_{23}\star\gamma_{12})}(\mathcal{T}_1).
\end{tikzcd}$$

The arrow on top consists of the canonical identifications r relating different choices of trivializations, and they are all identities according to the subsequent Remark 4.11. The remaining diagram then commutes due to the associativity of the composition of maps. □

Remark 4.11. Suppose $\gamma \in P_{ij}$ and $\varphi : [0,1] \to [0,1]$ is a smooth map with $\varphi(0) = 0$ and $\varphi(1) = 1$, such that $\gamma \circ \varphi = \gamma$. Let $\mathcal{T} : \gamma^*\mathcal{G} \to \mathcal{I}_0$ be a trivialization, and let $\mathcal{T}' := \varphi^*\mathcal{T}$. We have $\mathcal{R}_{ij}|_\gamma(\mathcal{T}) = \mathcal{R}_{ij}|_\gamma(\mathcal{T}')$, and we claim that $r_{\mathcal{T},\mathcal{T}'} = \mathrm{id}$. In order to see this, we may choose a smooth homotopy $h : [0,1]^2 \to [0,1]$ between the identity and φ, with fixed ends ($h(s,0) = 0$, $h(s,1) = 1$ for all $s \in [0,1]$) that fixes the path, $\gamma(h(s,t)) = \gamma(t)$ for all $s,t \in [0,1]$. Then, we consider the trivializations $h^*\mathcal{T}$ and $\mathrm{pr}_2^*\mathcal{T}$, where $\mathrm{pr}_2 : [0,1]^2 \to [0,1]$ is the projection. The hermitian line bundle $L := \Delta(h^*\mathcal{T}, \mathrm{pr}_2^*\mathcal{T})$ has a flat connection and thus admits a parallel unit-length section over $(\{0\} \times [0,1]) \cup ([0,1] \times \{0\}) \cup ([0,1] \times \{1\})$, where the two trivializations agree. With the connection, the section can be extended to all of $[0,1]^2$. In particular, we obtain a parallel section σ of $\Delta(\mathcal{T}', \mathcal{T})$ that is the identity over the end points. This shows $r_{\mathcal{T},\mathcal{T}'} = \mathrm{id}$.

Lemma 4.12. *Axiom (LBG3) is satisfied: ϕ_{ij} and χ_{ijk} are compatible with each other.*

Proof. We consider paths $\gamma_{12}, \gamma'_{12} \in P_{ij}$ and $\gamma_{23}, \gamma'_{23} \in P_{jk}$ as in (LBG3), and form the loops

$$\tau := (\gamma_{23} \star \gamma_{12}) \cup (\gamma'_{23} \star \gamma'_{12}) \quad , \quad \tau_{12} := \gamma_{12} \cup \gamma'_{12} \quad , \quad \tau_{23} := \gamma_{23} \cup \gamma'_{23}.$$

We consider corresponding trivializations of $\mathcal{G}$, namely $\mathcal{T}$ along τ, $\mathcal{T}_{12}$ along τ_{12}, and $\mathcal{T}_{23}$ along τ_{23}. The first objective is to compare these trivializations wherever two of them are defined. Over the common point y, we fix a 2-isomorphism $\sigma : \mathcal{T}_{12}|_{\frac{1}{2}} \Rightarrow \mathcal{T}_{23}|_0$. Since all paths have sitting instants there exist maps $\varphi, \varphi_{12}, \varphi_{23} : S^1 \to S^1$ such that

$$\begin{aligned}
\tau_{23} \circ \varphi_{23} \circ \iota_1 &= \tau_{23} = \tau \circ \varphi \circ \iota_1, \\
\tau_{23} \circ \varphi_{23} \circ \iota_1 &= \mathsf{c}_y = \tau_{12} \circ \varphi_{12} \circ \iota_1, \\
\tau_{12} \circ \varphi_{12} \circ \iota_2 &= \tau_{12} \circ \mathrm{rot}_\pi = \tau \circ \varphi \circ \iota_2,
\end{aligned}$$

where $\mathrm{rot}_\pi : S^1 \to S^1$ is the rotation by an angle of π. Due to the first and the third of these identities, we can find 2-isomorphisms $\phi_1 : \iota_1^*\varphi_{23}^*\mathcal{T}_{23} \Rightarrow \iota_1^*\varphi^*\mathcal{T}$ and $\phi_3 : \iota_2^*\varphi_{12}^*\mathcal{T}_{12} \Rightarrow \iota_2^*\varphi^*\mathcal{T}$. Because of the second identity, we have a 2-isomorphism $\phi_2 := \mathsf{c}_y^*\sigma : \iota_2^*\varphi_{23}^*\mathcal{T}_{23} \Rightarrow \iota_1^*\varphi_{12}^*\mathcal{T}_{12}$. We claim that we can choose these 2-isomorphism such that

$$\phi_1|_0 = \phi_3|_0 \bullet \phi_2|_0 \quad \text{and} \quad \phi_1|_1 = \phi_3|_1 \bullet \phi_2|_1. \tag{20}$$

Indeed, the first equation can be used to re-define σ such that this first equation is satisfied. Since $\phi_2 = \mathsf{c}_y^*\sigma$, we cannot repeat this for the second equation, so that we first obtain an error, a number $z \in \mathrm{U}(1)$. We consider the hermitian line bundle L_z over S^1 with connection of holonomy z. Then we replace $\mathcal{T}_{12}$ by $\mathcal{T}_{12} \otimes L_z$ and repeat the whole construction by fixing a new 2-isomorphism σ; then both equations in (20) are satisfied. Comparing with the definition of the fusion product given in Section 4.1, we have that

$$\begin{aligned}[\mathcal{T}_{23}, z_{23}] \otimes [\mathcal{T}_{12}, z_{12}] &\mapsto [\varphi_{23}^*\mathcal{T}_{23}, z_{23}] \otimes [\varphi_{12}^*\mathcal{T}_{12}, z_{12}] \\ &\mapsto [\varphi^*\mathcal{T}, z_{12}z_{23}] \mapsto [\mathcal{T}, z_{12}z_{23}]\end{aligned}$$

realizes the isomorphism λ' on top of the diagram of (LBG3). The remaining arrows of the diagram are labelled with the lifted path concatenation and the fusion representation, which, by our choices of trivializations, are just composition and scalar multiplication of linear maps, and the commutativity of the diagram reduces to the trivial fact that $z_{12}z_{23} \cdot (\varphi_{23} \circ \varphi_{12}) = (z_{23}\varphi_{23}) \circ (z_{12}\varphi_{12})$. □

4.6. Lifted constant paths

We equip the vector bundle $\mathcal{R}_{ii}$ with lifted constant paths. To that end, we consider $x \in Q_i$ and choose a trivialization $\mathcal{T} : \mathcal{G}|_x \to \mathcal{I}_0$. We have $\mathcal{R}_{ii}|_{\mathsf{c}_x}(\mathsf{c}_x^*\mathcal{T}) = \mathrm{End}(\Delta(\mathcal{E}_i|_x, \mathcal{T}))$, and readily define $\epsilon_i(x) := \mathrm{id}_{\Delta(\mathcal{E}_i|_x, \mathcal{T})}$.

Lemma 4.13. *The assignment $x \mapsto \epsilon_i(x)$ defines a smooth, parallel section along $\mathsf{c} : Q_i \to P_{ii}$.*

Proof. If $\mathcal{T}'$ is another trivialization of $\mathcal{G}|_x$, then there exists a 2-isomorphism $\psi : \mathcal{T} \Rightarrow \mathcal{T}'$ inducing the linear map $r_{\mathsf{c}_x^*\psi} : \mathcal{R}_{ii}|_{\mathsf{c}_x}(\mathsf{c}_x^*\mathcal{T}') \to \mathcal{R}_{ii}|_{\mathsf{c}_x}(\mathsf{c}_x^*\mathcal{T})$. We have $r_{\mathsf{c}_x^*\psi}(\mathrm{id}_{\Delta(\mathcal{E}_i|_x, \mathcal{T}')}) = \mathrm{id}_{\Delta(\mathcal{E}_i|_x, \mathcal{T})}$; this shows well-definedness. Next, we show directly that $\epsilon_i : Q_i \to \mathcal{R}_{ii}$ is smooth. Let $f : U \to Q_i$ be a plot of Q_i, i.e., a smooth map defined on an open subset $U \subseteq \mathbb{R}^n$. We have to show that $\tilde{c} := \epsilon_i \circ f : U \to \mathcal{R}_{ii}$ is a plot. Note that $c := \pi \circ \tilde{c} = \mathsf{c} \circ f$, where $\mathsf{c} : Q_i \to P_{ii}$ is the assignment of constant paths, and $c^\vee : U \times [0,1] \to P_{ii}$ is the map $(u,t) \mapsto \mathsf{c}_{f(u)}(t) = f(u)$. We choose an open subset $W \subseteq U$ such that there exists a trivialization $\mathcal{T} : f^*\mathcal{G}|_W \to \mathcal{I}_\rho$. Then we set $\mathcal{T}' := \mathrm{pr}_W^*\mathcal{T}$ for $\mathrm{pr}_W : [0,1] \times W \to W$, so that $\mathcal{T}'$ is a trivialization of $(c^\vee)^*\mathcal{G}$. The relevant vector bundle over W is

$$\mathrm{Hom}(\Delta((\mathrm{ev}_0 \circ c)^*\mathcal{E}_i, j_0^*\mathcal{T}'), \Delta((\mathrm{ev}_1 \circ c)^*\mathcal{E}_i, j_1^*\mathcal{T}')) = \mathrm{End}(\Delta(f^*\mathcal{E}_i, \mathcal{T})).$$

We note that $w \mapsto \tau(w) := \mathrm{id}_{\Delta(f^*\mathcal{E}_i,\mathcal{T})|_w}$ is a smooth section of this bundle over W. By definition of the diffeology on $\mathcal{R}_{ii}$, we have that $w \mapsto [i_w^*\mathcal{T}', \tau(w)] = [\mathsf{c}_w^*\mathcal{T}, \mathrm{id}_{\Delta(\mathcal{E}_i|_{f(w)},\mathcal{T}|_w)}] = \epsilon_i(f(w)) = \tilde{c}(w)$ is a plot, which was to show.

If γ is a path in Q_i, let $\mathcal{T}$ be a trivialization along γ, and consider the trivialization $\mathcal{S} := \mathrm{pr}_2^*\mathcal{T}$ over $[0,1]^2$. In particular, $\mathcal{S}$ is a trivialization along the path $t \mapsto \mathsf{c}_{\gamma(t)}$ connecting c_x with c_y in P_{ii}, and it has vanishing 2-form. We let $\mathcal{V} := \Delta(\gamma^*\mathcal{E}_i, \mathcal{T})$. Then, the parallel transport in $\mathcal{R}_{ii}$ along $\mathsf{c}_{\gamma(-)}$ is just the parallel transport in $\mathrm{End}(\mathcal{V})$:

$$pt_{ii}|_{\mathsf{c}_{\gamma(-)}} : \mathrm{End}(\mathcal{V})|_0 \to \mathrm{End}(\mathcal{V})|_1.$$

Since the section id into $\mathrm{End}(\mathcal{V}) = \mathcal{V}^* \otimes \mathcal{V}$ is parallel, we have the claim. □

Lemma 4.14. *Axiom (LBG4) is satisfied: ϵ_i provides units up to reparameterization for χ_{ijk}.*

Proof. Let $\mathcal{T}_0$ be a trivialization of $\mathcal{G}$ along a path γ from x to y, and let $\varphi : [0,1] \to [0,1]$ be a smooth map such that $\gamma \circ \varphi = \gamma \star \mathsf{c}_x$ (this uses the sitting instants of γ). Let h be a fixed-ends homotopy between $\mathrm{id}_{[0,1]}$ and φ, and let $\mathcal{T} := h^*\mathcal{T}$ as well as $\mathcal{T}_1 := \mathcal{T}|_{\{1\}\times[0,1]} = \varphi^*\mathcal{T}_0$. These are trivializations with vanishing 2-forms and they agree on $[0,1] \times \{0,1\}$, so that

$$d_{\gamma,\gamma\star\mathsf{c}_x} : \mathcal{R}_{ij}|_\gamma(\mathcal{T}_0) \to \mathcal{R}_{ij}|_{\gamma\star\mathsf{c}_x}(\mathcal{T}_1)$$

is the identity by definition of the connection pt_{ij}. Moreover,

$$\chi_{iij}|_{\mathsf{c}_x,\gamma} : \mathcal{R}_{ij}|_\gamma(\iota_2^*\mathcal{T}_1) \otimes \mathcal{R}_{ii}|_{\mathsf{c}_x}(\iota_1^*\mathcal{T}_1) \to \mathcal{R}_{ij}|_{\gamma\star\mathsf{c}_x}(\mathcal{T}_1)$$

is the composition. Due to Remark 4.11 we have identities $\mathcal{R}_{ii}|_{\mathsf{c}_x}(\iota_1^*\mathcal{T}_1) = \mathcal{R}_{ii}|_{\mathsf{c}_x}(\mathsf{c}_0^*(\mathcal{T}_0|_0))$ and $\mathcal{R}_{ij}|_\gamma(\iota_2^*\mathcal{T}_1) = \mathcal{R}_{ij}|_\gamma(\mathcal{T}_0)$. The first shows that $\epsilon_i(x) = \mathrm{id} \in \mathcal{R}_{ii}|_{\mathsf{c}_x}(\iota_1^*\mathcal{T}_1)$. The second shows that $\mathcal{R}_{ij}|_\gamma(\iota_2^*\varphi^*\mathcal{T}) = \mathcal{R}_{ij}|_\gamma(\mathcal{T}_0)$, and together with the first we have

$$\chi_{iij}|_{\mathsf{c}_x,\gamma}(v, \epsilon_i(x)) = d_{\gamma,\gamma\star\mathsf{c}_x}(v),$$

which is the first half of (LBG4). The second half is proved analogously. □

4.7. Lifted path reversal

We equip the vector bundle $\mathcal{R}_{ij}$ with a lifted path reversal. Let $\gamma \in P_{ij}$ with $x := \gamma(0)$ and $y := \gamma(1)$. We choose a trivialization $\mathcal{T}$ of $\gamma^*\mathcal{G}$, and let

$\overline{\mathcal{T}} := \mathsf{rev}^*\mathcal{T}$, where $\mathsf{rev} : [0,1] \to [0,1]$ is defined by $\mathsf{rev}(t) := 1 - t$. We have

$$\begin{aligned}\mathcal{R}_{ji}|_{\overline{\gamma}}(\overline{\mathcal{T}}) &= \mathrm{Hom}(\Delta(\mathcal{E}_j|_y, \overline{\mathcal{T}}|_0), \Delta(\mathcal{E}_i|_x, \overline{\mathcal{T}}|_1)) \\ &= \mathrm{Hom}(\Delta(\mathcal{E}_j|_y, \mathcal{T}|_1), \Delta(\mathcal{E}_i|_x, \mathcal{T}|_0)).\end{aligned}$$

We define the lifted path reversal by forming the adjoint linear map with respect to the given hermitian metrics,

$$\alpha_{ij}|_\gamma : \mathcal{R}_{ij}|_\gamma(\mathcal{T}) \to \overline{\mathcal{R}_{ji}}|_{\overline{\gamma}}(\overline{\mathcal{T}}) : \varphi \mapsto \varphi^*.$$

Lemma 4.15. *This defines a connection-preserving bundle morphism.*

Proof. If $\mathcal{T}'$ is another trivialization we consider a 2-isomorphism $\psi : \mathcal{T} \Rightarrow \mathcal{T}'$ and the corresponding canonical identification $r_\psi(\varphi) = \psi_1 \circ \varphi \circ \psi_0^{-1}$, for $\varphi \in \mathcal{R}_{ij}|_\gamma(\mathcal{T}')$. Since ψ_0 and ψ_1 are isometric isomorphisms, we have

$$\begin{aligned}(r_\psi(\varphi))^* &= (\psi_1 \circ \varphi \circ \psi_0^{-1})^* = \psi_0 \circ \varphi^* \circ \psi_1^{-1} \\ &= (\mathsf{rev}^*\psi)_1^* \circ \varphi^* \circ (\mathsf{rev}^*\psi)_0^{-1} = r_{\mathsf{rev}^*\psi}(\varphi^*),\end{aligned}$$

where $\mathsf{rev}^*\psi : \overline{\mathcal{T}} \to \overline{\mathcal{T}'}$. This shows the independence of the choice of the trivialization.

In the following we consider a smooth map $f : W \to P_{ij}$, where W is a smooth manifold, that admits a trivialization $\mathcal{T} : (f^\vee)^*\mathcal{G} \to \mathcal{I}_\rho$ defining the vector bundles $\mathcal{V}_\mathcal{T}$ and $\mathcal{W}_\mathcal{T}$ over W. We consider the pointwise path reversal, $\mathsf{rev}_W : W \times [0,1] \to W \times [0,1] : (w,t) \mapsto (w, 1-t)$. Then, $\overline{\mathcal{T}} := \mathsf{rev}_W^*\mathcal{T}$ is a trivialization of $\mathcal{G}$ along $f^\vee \circ \mathsf{rev}_W$. We have

$$\mathcal{V}_\mathcal{T} = \Delta((\mathrm{ev}_0 \circ f^\vee)^*\mathcal{E}_i, j_0^*\mathcal{T}) = \Delta((\mathrm{ev}_1 \circ f^\vee \circ R)^*\mathcal{E}_i, j_1^*\overline{\mathcal{T}}) = \mathcal{W}_{\overline{\mathcal{T}}},$$

and similarly $\mathcal{W}_\mathcal{T} = \mathcal{V}_{\overline{\mathcal{T}}}$. Now suppose $f := c|_W$ is the restriction of a plot $c : U \to P_{ij}$ to an open subset $W \subseteq U$, and we have a trivialization $\tau : W \times \mathbb{C}^k \to \mathrm{Hom}(\mathcal{V}_\mathcal{T}, \mathcal{W}_\mathcal{T})$, inducing a local trivialization of $\mathcal{R}_{ij}$. Taking the pointwise adjoint defines a local trivialization $\overline{\tau} : W \times \mathbb{C}^k \to \overline{\mathrm{Hom}(\mathcal{W}_\mathcal{T}, \mathcal{V}_\mathcal{T})} = \overline{\mathrm{Hom}(\mathcal{V}_{\overline{\mathcal{T}}}, \mathcal{W}_{\overline{\mathcal{T}}})}$, inducing a local trivialization of $\overline{\mathcal{R}_{ji}}$. Under these local trivializations, α_{ij} is the identity, and hence smooth.

In order to see that α_{ij} is connection-preserving, we consider a path $\Gamma \in PP_{ij}$ and compare the parallel transport $pt_{ij}|_\Gamma$ with $pt_{ji}|_{\widetilde{\mathsf{rev}}\circ\Gamma}$, where $\widetilde{\mathsf{rev}} : P_{ij} \to P_{ji} : \gamma \mapsto \bar{\gamma}$. We put $f := \Gamma$ in the situation described above. It is elementary to see that taking adjoints in the bundle of homomorphisms between hermitian vector bundles preserves induced connections. In the present

case, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(\mathcal{V}_{\mathcal{T}},\mathcal{W}_{\mathcal{T}})|_0 \arrow[r, "()^*"] \arrow[d, "pt"'] & \overline{\mathrm{Hom}(\mathcal{W}_{\mathcal{T}},\mathcal{V}_{\mathcal{T}})}|_0 \arrow[r, equal] \arrow[d, "pt"] & \overline{\mathrm{Hom}(\mathcal{V}_{\overline{\mathcal{T}}},\mathcal{W}_{\overline{\mathcal{T}}})}|_0 \arrow[d, "pt"] \\
\mathrm{Hom}(\mathcal{V}_{\mathcal{T}},\mathcal{W}_{\mathcal{T}})|_1 \arrow[r, "()^*"'] & \overline{\mathrm{Hom}(\mathcal{W}_{\mathcal{T}},\mathcal{V}_{\mathcal{T}})}|_1 \arrow[r, equal] & \overline{\mathrm{Hom}(\mathcal{V}_{\overline{\mathcal{T}}},\mathcal{W}_{\overline{\mathcal{T}}})}|_1
\end{tikzcd}$$

It remains to compare the integral of the 2-form ρ over $\Gamma^\vee$ with the integral of $\mathsf{rev}_W^*\rho$ over $(\widetilde{\mathsf{rev}}\circ\Gamma)^\vee = \mathsf{rev}_W\circ\Gamma^\vee$. Since rev_W is orientation-reversing, we get the opposite sign. In the definition of $pt_{ji}|_{\widetilde{\mathsf{rev}}\circ\Gamma}$, the exponential of this integral is considered as a scalar, which is multiplied with the parallel transport in the Hom-bundle. Since we are concerned with the complex conjugate vector bundle $\overline{\mathcal{R}_{ji}}$, the product has again the correct sign. This shows that the diagram

$$\begin{tikzcd}
\mathcal{R}_{ij}|_{\Gamma(0)}(\mathcal{T}_0) \arrow[r, "\alpha_{ij}"] \arrow[d, "pt_{ij}|_\Gamma"'] & \overline{\mathcal{R}_{ji}}|_{\overline{\Gamma(0)}}(\overline{\mathcal{T}}_0) \arrow[d, "pt_{ji}|_{\widetilde{\mathsf{rev}}\circ\Gamma}"] \\
\mathcal{R}_{ij}|_{\Gamma(1)}(\mathcal{T}_1) \arrow[r, "\alpha_{ij}"'] & \overline{\mathcal{R}_{ji}}|_{\overline{\Gamma(1)}}(\overline{\mathcal{T}}_1)
\end{tikzcd}$$

is commutative; hence, α_{ij} is connection-preserving. □

Lemma 4.16. *Axiom (LBG5) is satisfied: lifted path reversal is compatible with the metrics.*

Proof. As in the proofs of Lemmas 4.10 and 4.14, the endomorphisms $d_{\mathsf{c}_x,\overline{\gamma}\star\gamma}(\epsilon_i(x))$ and $d_{\mathsf{c}_x,\gamma\star\overline{\gamma}}(\epsilon_j(x))$ are identities, and the remaining equality is the obvious identity

$$\mathrm{tr}((\psi^*\circ\kappa)^*\circ\mathrm{id}) = \mathrm{tr}(\kappa^*\circ\psi) = \mathrm{tr}(\mathrm{id}\circ\psi\circ\kappa^*)$$

for traces and adjoints of endomorphisms ψ,κ of a finite-dimensional complex inner product space. □

Axioms (LBG6), (LBG7) and (LBG8) are obvious identities for adjoints of linear operators. The last two axioms are again more involved.

Lemma 4.17. *Axiom (LBG9) is satisfied: lifted path reversal is compatible with the fusion representation.*

Proof. We consider $(\gamma_1,\gamma_2)\in P_{ij}\times_{Q_i\times Q_j}P_{ij}$, the corresponding loop $\tau := \gamma_1\cup\gamma_2$, and a trivialization $\mathcal{T}:\tau^*\mathcal{G}\to\mathcal{I}_0$. In the diagram of (LBG9), the

clockwise composition

$$\mathcal{L}|_\tau \otimes \mathcal{R}_{ij}|_{\gamma_2}(\iota_2^*\mathcal{T}) \xrightarrow{\phi_{ij}} \mathcal{R}_{ij}|_{\gamma_1}(\iota_1^*\mathcal{T}) \xrightarrow{\alpha_{ij}} \overline{\mathcal{R}_{ji}}|_{\overline{\gamma_1}}(\overline{\iota_1^*\mathcal{T}})$$

sends $[\mathcal{T}, z] \otimes \varphi$ to $\overline{z} \cdot \varphi^*$. For the counter-clockwise direction, we let $\mathsf{rot}_\pi : S^1 \to S^1$ denote the rotation by an angle of π. Then, the map $d : \mathcal{L}|_{\gamma_2 \cup \gamma_1} \to \mathcal{L}|_{\overline{\gamma_1} \cup \overline{\gamma_2}}$ is given by $[\mathcal{T}, z] \mapsto [\mathsf{rot}_\pi^*\mathcal{T}, z]$, see [Wal16, Lemma 4.3.6]. We have to compose this with the isomorphism $\tilde{\lambda}$ of (2), which was computed in Section 4.1, see Remark 4.1. Thus, the map

$$\begin{aligned} \mathcal{L}|_\tau \otimes \mathcal{R}_{ij}|_{\gamma_2}(\iota_2^*\mathcal{T}) &\xrightarrow{\tilde{\lambda}\otimes\alpha_{ij}} \overline{\mathcal{L}}|_{\gamma_2\cup\gamma_1} \otimes \overline{\mathcal{R}}_{ji}|_{\overline{\gamma_2}}(\overline{\iota_2^*\mathcal{T}}) \\ &\xrightarrow{d\otimes\mathrm{id}} \overline{\mathcal{L}}|_{\overline{\gamma_1}\cup\overline{\gamma_2}} \otimes \overline{\mathcal{R}}_{ji}|_{\overline{\gamma_2}}(\overline{\iota_2^*\mathcal{T}}) \end{aligned}$$

is given by $[\mathcal{T}, z] \otimes \varphi \mapsto [\mathsf{rev}^*\mathcal{T}, \overline{z}] \otimes \varphi^* \mapsto [\mathcal{T}', \overline{z}] \otimes \varphi^*$, where $\mathcal{T}' := \mathsf{rot}_\pi^*\mathsf{rev}^*\mathcal{T}$. In order to obtain the counter-clockwise composition, it remains to compose the with fusion representation. We observe that $\iota_2^*\mathcal{T}' = \overline{\iota_2^*\mathcal{T}}$ and $\iota_1^*\mathcal{T}' = \overline{\iota_1^*\mathcal{T}}$; hence, the fusion representation results in $\phi_{ji}([\mathcal{T}', \overline{z}] \otimes \varphi^*) = \overline{z} \cdot \varphi^* \in \overline{\mathcal{R}_{ji}}|_{\overline{\gamma_1}}(\overline{\iota_1^*\mathcal{T}})$. This shows that the diagram of (LBG9) is commutative. □

Lemma 4.18. *The Cardy condition (LBG10) is satisfied.*

Proof. As in the proofs of Lemmas 4.10 and 4.14, the endomorphism $d_{\mathsf{c}_y, \gamma\star\mathsf{c}_x\star\overline{\gamma}}(\epsilon_j(y))$ is the identity. The remaining equality is the identity

$$\sum_{k=1}^{n} \varphi_k \circ \varphi \circ \varphi_k^* = \mathrm{tr}(\varphi) \cdot \mathrm{id},$$

where $(\varphi_1, ..., \varphi_n)$ is an orthonormal basis of $\mathcal{R}_{ij}|_\gamma(\mathcal{T})$ with respect to the metric defined in Section 4.2. It is straightforward to check this identity, for instance using elementary matrices. □

4.8. Functoriality of transgression

In Sections 4.1 to 4.7 we have defined our transgression functor on the level of objects. Now we provide its definition on the level of morphisms: we associate to a TBG 1-morphism $(\mathcal{A}, \psi) : (\mathcal{G}, \mathcal{E}) \to (\mathcal{G}', \mathcal{E}')$ a morphism (φ, ξ) between the transgressed LBG objects.

To start with, the transgression of U(1)-bundle gerbes is functorial [Wal10]: from the isomorphism $\mathcal{A} : \mathcal{G} \to \mathcal{G}'$ we obtain an isomorphism $\varphi : \mathcal{L} \to \mathcal{L}'$, which over a loop $\tau \in LM$ is given by $[\mathcal{T}, z] \mapsto [\mathcal{T} \circ \tau^* \mathcal{A}^{-1}, z]$, for $\mathcal{T}$ a trivialization of $\tau^*\mathcal{G}$. The isomorphism φ is connection-preserving and fusion-preserving.

Next, we define a vector bundle isomorphism $\xi_{ij} : \mathcal{R}_{ij} \to \mathcal{R}'_{ij}$ for all $i, j \in I$, using the 2-isomorphisms $\psi_i : \mathcal{E}_i \Rightarrow \mathcal{E}'_i \circ \mathcal{A}|_{Q_i}$. Let $\gamma \in P_{ij}$ be a path with $x := \gamma(0)$ and $y := \gamma(1)$, let $\mathcal{T} : \gamma^*\mathcal{G} \to \mathcal{I}_0$ be a trivialization, and let $\mathcal{T}' := \mathcal{T} \circ \gamma^* \mathcal{A}^{-1}$. We consider vector bundle isomorphisms

$$\begin{aligned} &\psi_{i,0} : \Delta(\mathcal{E}_i|_x, \mathcal{T}|_0) \to \Delta(\mathcal{E}'_i|_x, \mathcal{T}'|_0) \quad \text{and} \\ &\psi_{i,1} : \Delta(\mathcal{E}_j|_y, \mathcal{T}|_1) \to \Delta(\mathcal{E}'_j|_y, \mathcal{T}'|_1) \end{aligned}$$

defined as follows. The bicategory $\mathcal{G}rb^\nabla(M)$ provides a 2-isomorphism $\delta : \mathcal{A}^{-1} \circ \mathcal{A} \Rightarrow \mathrm{id}$, which induces a 2-isomorphism $\mathrm{id} \circ \delta^{-1} : \mathcal{T} \Rightarrow \mathcal{T}' \circ \gamma^* \mathcal{A}$. Now, $\psi_{i,0}$ is the composite

$$\begin{aligned} \Delta(\mathcal{E}_i|_x, \mathcal{T}|_0) &\xrightarrow{\Delta(\psi_i, \mathrm{id})} \Delta(\mathcal{E}'_i|_x \circ \mathcal{A}|_x, \mathcal{T}|_0) \\ &\xrightarrow{\Delta(\mathrm{id}, \mathrm{id} \circ \delta^{-1})} \Delta(\mathcal{E}'_i|_x \circ \mathcal{A}|_x, \mathcal{T}|'_0 \circ \mathcal{A}|_x) \xrightarrow{\Delta_{\mathcal{A}|_x}} \Delta(\mathcal{E}'_i|_x, \mathcal{T}'|_0), \end{aligned}$$

where $\Delta_{\mathcal{A}|_x}$ is defined in Remark 2.1 (c). $\psi_{i,1}$ is defined analogously. We have

$$\begin{aligned} \mathcal{R}_{ij}|_\gamma(\mathcal{T}) &= \mathrm{Hom}(\Delta(\mathcal{E}_i|_x, \mathcal{T}|_0), \Delta(\mathcal{E}_j|_y, \mathcal{T}|_1)) \\ \mathcal{R}'_{ij}|_\gamma(\mathcal{T}') &= \mathrm{Hom}(\Delta(\mathcal{E}'_i|_x, \mathcal{T}'|_0), \Delta(\mathcal{E}'_j|_y, \mathcal{T}'|_1)), \end{aligned}$$

and we finally define

$$\xi_{ij} : \mathcal{R}_{ij}|_\gamma(\mathcal{T}) \to \mathcal{R}'_{ij}|_\gamma(\mathcal{T}') : \varphi \mapsto \psi_{i,1} \circ \varphi \circ \psi_{i,0}^{-1}.$$

Lemma 4.19. *This defines a connection-preserving bundle isomorphism, which depends only on the 2-isomorphism class of* $(\mathcal{A}, \psi)$.

Proof. Independence of the choice of $\mathcal{T}$ is routine. For the smoothness, we consider a plot $c : U \to P_{ij}$, an open subset $W \subseteq U$, a trivialization $\mathcal{T} : (c|_W^\vee)^*\mathcal{G} \to \mathcal{I}_\rho$, and a bundle isomorphism $\tau : W \times \mathbb{C}^n \to \mathrm{Hom}(\mathcal{V}_\mathcal{T}, \mathcal{W}_\mathcal{T})$ over W, so that we have a local trivialization ϕ of $\mathcal{R}_{ij}$. We set $\mathcal{T}' := \mathcal{T} \circ (c|_W^\vee)^* \mathcal{A}^{-1}$ and obtain bundle isomorphisms $\psi_{i,0} : \mathcal{V}_\mathcal{T} \to \mathcal{V}_{\mathcal{T}'}$ and $\psi_{i,1} : \mathcal{W}_\mathcal{T} \to \mathcal{W}_{\mathcal{T}'}$ over W from exactly the same definition as above. Then we obtain a local trivialization of $\mathcal{R}'_{ij}$ using $\mathcal{T}'$ and $\tau'(w, v) := \psi_{i,1} \circ \tau(w, v) \circ \psi_{i,0}^{-1}$

for $w \in W$ and $v \in \mathbb{C}^n$; in these local trivializations the map ξ_{ij} is the identity, and hence smooth.

If $\Gamma : [0,1] \to P_{ij}$ is a path, we choose a trivialization $\mathcal{T} : (\Gamma^\vee)^*\mathcal{G} \to \mathcal{I}_\rho$. Then, the trivialization $\mathcal{T}' := \mathcal{T} \circ (\Gamma^\vee)^*\mathcal{A}^{-1}$ has the same 2-form ρ. Since the isomorphisms $\psi_{i,0}$ and $\psi_{i,1}$ over $[0,1]$ are connection-preserving, the induced isomorphism between Hom-bundles is connection-preserving, too. This shows that ξ_{ij} preserves connections.

We consider a 2-isomorphism φ between $(\mathcal{A}, \psi)$ and $(\mathcal{A}', \psi')$ and the corresponding vector bundle isomorphisms ξ_{ij} and ξ'_{ij}. For a path $\gamma \in P_{ij}$, let $\mathcal{T} : (\gamma^\vee)^*\mathcal{G} \to \mathcal{I}_0$ be a trivialization, $\mathcal{T}' := \mathcal{T} \circ \gamma^*\mathcal{A}^{-1}$ and $\mathcal{T}'' := \mathcal{T} \circ \gamma^*\mathcal{A}'^{-1}$. From the 2-isomorphism $\varphi : \mathcal{A} \Rightarrow \mathcal{A}'$ we construct another 2-isomorphism $\eta := \mathrm{id} \circ \gamma^*\varphi^\# : \mathcal{T}' \Rightarrow \mathcal{T}''$, where $\varphi^\# : \mathcal{A}^{-1} \Rightarrow \mathcal{A}'^{-1}$ denotes the inverse with respect to horizontal composition. We consider the following diagram of linear maps:

$$\begin{tikzcd}
 & \Delta(\mathcal{E}'_i|_x \circ \mathcal{A}|_x, \mathcal{T}|_0) \arrow[r, "\Delta(\mathrm{id}, \mathrm{id}\circ\delta_{\mathcal{A}}^{-1})"] \arrow[d, "\Delta(\mathrm{id}\circ\varphi, \mathrm{id})"] & \Delta(\mathcal{E}'_i|_x \circ \mathcal{A}|_x, \mathcal{T}'|_0 \circ \mathcal{A}|_x) \arrow[r, "\Delta_{\mathcal{A}|_x}"] \arrow[d, "\Delta(\mathrm{id}\circ\varphi, \eta|_0\circ\varphi)"] & \Delta(\mathcal{E}'_i|_x, \mathcal{T}'|_0) \arrow[d, "\Delta(\mathrm{id}, \eta|_0)"] \\
\Delta(\mathcal{E}_i|_x, \mathcal{T}|_0) \arrow[ur, "\Delta(\psi_i, \mathrm{id})"] \arrow[dr, "\Delta(\psi'_i, \mathrm{id})"'] & & & \\
 & \Delta(\mathcal{E}'_i|_x \circ \mathcal{A}'|_x, \mathcal{T}|_0) \arrow[r, "\Delta(\mathrm{id}, \mathrm{id}\circ\delta_{\mathcal{A}'}^{-1})"'] & \Delta(\mathcal{E}'_i|_x \circ \mathcal{A}|'_x, \mathcal{T}''|_0 \circ \mathcal{A}'|_x) \arrow[r, "\Delta_{\mathcal{A}|_y}"'] & \Delta(\mathcal{E}'_i|_x, \mathcal{T}''|_0)
\end{tikzcd}$$

The diagram is commutative: the triangular diagram on the left commutes due to the condition for TBG 2-isomorphisms, $\psi'_i = (\mathrm{id} \circ \varphi) \bullet \psi_i$. The diagram in the middle commutes because the 2-isomorphism $\delta_{\mathcal{A}} : \mathcal{A}^{-1} \circ \mathcal{A} \Rightarrow \mathrm{id}$ is natural with respect to 2-morphisms, and the diagram on the right-hand side commutes by definition of $\Delta_{\mathcal{A}|_x}$. The diagram gives the equation $\psi'_{i,0} = \Delta(\mathrm{id}, \eta|_0) \circ \psi_{i,0}$. Analogously, we have $\psi'_{i,1} = \Delta(\mathrm{id}, \eta|_1) \circ \psi_{i,1}$. Thus, for $\mathcal{R}_{ij}|_\gamma(\mathcal{T})$ we have:

$$\begin{aligned}
\xi'_{ij}(\varphi) = \psi'_{i,1} \circ \varphi \circ \psi'^{-1}_{i,0} &= \Delta(\mathrm{id}, \eta|_0) \circ \psi_{i,0} \circ \varphi \circ \psi_{i,1}^{-1} \circ \Delta(\mathrm{id}, \eta|_1)^{-1} \\
&= \Delta(\mathrm{id}, \eta|_0) \circ \xi_{ij}(\varphi) \circ \Delta(\mathrm{id}, \eta|_1)^{-1} = r_\eta(\xi_{ij}(\varphi)).
\end{aligned}$$

This shows that $\xi_{ij} = \xi'_{ij}$. □

The following lemma is a straightforward check, only using the definitions; its proof is left out for brevity.

Lemma 4.20. *The bundle isomorphism ξ_{ij} intertwines the fusion representations, and respects the lifted path composition, lifted constant paths, and lifted path reversal.* □

Thus, the pair (φ, ξ), with $\xi = \{\xi_{ij}\}_{i,j\in I}$ is a morphism in $\mathrm{LBG}(M, Q)$, and we call it the *transgression of the 1-morphism* $(\mathcal{A}, \psi)$. It is again straightforward to show that the assignment $(\mathcal{A}, \psi) \mapsto (\varphi, \xi)$ preserves identities and the composition, so that we have the following:

Proposition 4.21. *Transgression is a functor* $\mathscr{T} : \mathrm{TBG}(M, Q) \to \mathrm{LBG}(M, Q)$. □

4.9. Transgression of a trivial bundle gerbe, and reduction to the point

We analyze the transgression of TBG with a trivial bundle gerbe, i.e. $\mathcal{G} = \mathcal{I}_\rho$. We note that the bundle gerbe modules $\mathcal{E}_i$ are nothing but ordinary hermitian vector bundles E_i with connections over Q_i, and the 2-forms ω_i are determined by

$$\omega_i = \rho + \frac{1}{\mathrm{rk}(E_i)}\mathrm{tr}(\mathrm{curv}(E_i)).$$

Under transgression, we obtain the following. The line bundle is $\mathcal{L} = \mathbb{C}_{\rho_{LM}}$, the trivial line bundle equipped with the connection 1-form $\rho_{LM} := \int_{S^1} \mathrm{ev}^*\rho$, and the fusion product is just multiplication in the fibers [Wal11, Lemma 3.6]. Concerning the vector bundles $\mathcal{R}_{ij}$, we claim that we have a canonical, connection-preserving, unitary bundle isomorphism

$$\mathcal{R}_{ij} \cong \mathrm{Hom}(\mathrm{ev}_0^* E_i, \mathrm{ev}_1^* E_j) \otimes \mathbb{C}_{\rho_{PM}} \tag{21}$$

over P_{ij}, where $\rho_{PM} := \int_{[0,1]} \mathrm{ev}^*\rho$. Thus, $\mathcal{R}_{ij}$ is globally a bundle of homomorphisms between two vector bundles.

The isomorphism (21) is defined in the following way. We consider a path $\gamma \in P_{ij}$ from $x \in Q_i$ to $y \in Q_j$ and the trivialization $\mathcal{T} := \mathrm{id}_{\mathcal{I}_0}$ of $\gamma^*\mathcal{G} = \gamma^*\mathcal{I}_\rho = \mathcal{I}_0$. Then, $\Delta(\mathcal{E}_i|_x, \mathcal{T}|_0) = E_i|_x$ and $\Delta(\mathcal{E}_j|_y, \mathcal{T}|_1) = E_j|_y$, so that $\mathcal{R}_{ij}|_\gamma(\mathrm{id}_{\mathcal{I}_0}) = \mathrm{Hom}(E_i|_x, E_j|_y)$. It is straightforward to see that this defines a smooth bundle morphism (21). Concerning the connection, let $\Gamma : [0,1] \to P_{ij}$ be a path. Considering the trivialization $\mathcal{T} := \mathrm{id}_{\mathcal{I}_{(\Gamma^\vee)^*\rho}}$ of $(\Gamma^\vee)^*\mathcal{I}_\rho$, we obtain $\mathcal{V}_\mathcal{T} = (\mathrm{ev}_0 \circ \Gamma)^* E_i$ and $\mathcal{W}_\mathcal{T} = (\mathrm{ev}_1 \circ \Gamma)^* E_j$. Now, the parallel transport in $\mathcal{R}_{ij}$ is

$$pt_{ij}|_\Gamma : \mathcal{R}_{ij}|_{\Gamma(0)}(\mathrm{id}_{\mathcal{I}_0}) \to \mathcal{R}_{ij}|_{\Gamma(1)}(\mathrm{id}_{\mathcal{I}_0}) : \varphi \mapsto \exp\left(\int_{[0,1]^2} (\Gamma^\vee)^*\rho\right) \cdot pt(\varphi).$$

This is precisely the parallel transport in the vector bundle $\mathrm{Hom}(\mathrm{ev}_0^* E_i, \mathrm{ev}_1^* E_j) \otimes \mathbb{C}_{\rho_{PM}}$.

The fusion representation is trivial. The lifted path concatenation is the composition, under the isomorphisms (21), which is connection-preserving because $\star^*\rho_{PM} = \mathrm{pr}_{jk}^*\rho_{PM} + \mathrm{pr}_{ij}^*\rho_{PM}$ on $P_{jk} \times_{Q_j} P_{ij}$. The lifted constant path ϵ_i is the identity section, and the lifted path reversal is given by taking adjoint homomorphisms.

Finally, we consider the yet more trivial case of a target space (M, Q) with $M = \{*\}$ and $Q = \{*\}_{i\in I}$. The bundle gerbe $\mathcal{G}$ is trivial, as before, while the twisted vector bundles $\mathcal{E}_i$ give just a family $\{E_i\}_{i\in I}$ of finite-dimensional hermitian vector spaces. In other words,

$$\mathrm{TBG}^I := \mathrm{h}_1\mathrm{TBG}(*, \{*\}_{i\in I}) = \coprod_{i\in I} \mathcal{HV}ect_{fin} - .$$

Transgression to the category $\mathrm{LBG}^{(I)}$ described in Lemma 3.14 takes a family $\{E_i\}_{i\in I}$ to the object $(\mathcal{L}, \lambda, \mathcal{R}, \phi, \chi, \epsilon, \alpha)$ with $\mathcal{L} := \mathbb{C}$, $\lambda := \mathrm{id}$, $\mathcal{R}_{ij} := \mathrm{Hom}(E_i, E_j)$, $\phi := \mathrm{id}$, $\chi_{ijk} := \circ$, $\epsilon_i := \mathrm{id}_{E_i}$, $\alpha_{ij} = ()^*$.

5. Regression

In this section we construct our regression functor $\mathscr{R}_{i_0,x_0,\mathcal{F}_0} : \mathrm{LBG}(M, Q) \to \mathrm{TBG}(M, Q)$, depending on three parameters as explained below. We consider a LBG object $(\mathcal{L}, \lambda, \mathcal{R}, \phi, \chi, \epsilon, \alpha)$.

We first treat the line bundle $\mathcal{L}$ with its superficial connection and the fusion product λ. The regression of $(\mathcal{L}, \lambda)$ is discussed in [Wal16]. It requires the choice of a base point $x_0 \in M$. The regressed bundle gerbe $\mathcal{G} := \mathscr{R}_{x_0}(\mathcal{L}, \lambda)$ has the subduction (the diffeological analogue of a surjective submersion) $\mathrm{ev}_1 : P_{x_0}M \to M$, where $P_{x_0}M := \{\gamma \in PM \mid \gamma(0) = x_0\}$. The 2-fold fiber product is equipped with the smooth map $\cup : P_{x_0}M^{[2]} \to LM$, and the hermitian line bundle of $\mathcal{G}$ is $\cup^*\mathcal{L}$. The bundle gerbe product is the restriction of the fusion product λ to $P_{x_0}M^{[3]}$.

In the presence of branes, we need to choose a brane index $i_0 \in I$ such that $x_0 \in Q_{i_0}$. Over x_0 we have the simple algebra $\mathcal{A}_0 := \mathcal{A}_{i_0}|_{x_0}$ (Corollary 3.19), which is isomorphic to a Matrix algebra $\mathbb{C}^{n_0\times n_0}$, where $n_0 = \sqrt{\mathrm{rk}(\mathcal{R}_{i_0i_0})} = \|\epsilon_{i_0}(x_0)\|^2$ by Remark 2.6 (a). Thus, there exists an irreducible n_0-dimensional left $\mathcal{A}_0$-module $\mathcal{F}_0$, which we fix, too. In the following we consider $\mathcal{A}_0$ and $\mathcal{F}_0$ as bundles of algebras and left $\mathcal{A}_0$-modules over the point x_0, respectively.

For $i \in I$ we consider the subspace $P_i := \{\gamma \in P_{i_0i} \mid \gamma(0) = x_0\} \subseteq P_{x_0}M$, over which we find the bundle $\mathcal{R}_{i_0i}$ of $\mathrm{ev}_1^*\mathcal{A}_i$-$\mathrm{ev}_0^*\mathcal{A}_0$-bimodules (Proposition 3.8). Tensoring over $\mathcal{A}_0$ with the bundle $\mathrm{ev}_0^*\mathcal{F}_0$ of left $\mathrm{ev}_0^*\mathcal{A}_0$-modules, we

obtain a bundle $E_i := \mathcal{R}_{i_0 i} \otimes_{\mathrm{ev}_0^* \mathcal{A}_0} \mathrm{ev}_0^* \mathcal{F}_0$ of left $\mathrm{ev}_1^* \mathcal{A}_i$-modules over P_i. We forget the left $\mathrm{ev}_1^* \mathcal{A}_i$-action and consider E_i as a hermitian vector bundle. For later purpose, we note the following result.

Lemma 5.1. *We have* $\mathrm{rk}(E_i) = \sqrt{\mathrm{rk}(\mathcal{R}_{ii})}$ *for all* $i \in I$.

Proof. By Corollary 3.11, $\mathcal{R}_{i_0 i}|_\gamma$ establishes, for any $\gamma \in P_{ij}$ with $\gamma(1) =: x$, a Morita equivalence between $\mathcal{A}_0$ and $\mathcal{A}_i|_x$. The module $E_i|_\gamma$ is the image of $\mathcal{F}_0$ under the corresponding functor between representation categories. Irreducibility is preserved under Morita equivalence; hence, $E_i|_\gamma$ is an irreducible module of the algebra $\mathcal{A}_i|_x$, which is a simple algebra (Corollary 3.19) and has dimension $\mathrm{rk}(\mathcal{R}_{ii})$ (Remark 2.6 (a)). Matrix algebras have only one irreducible module up to isomorphism, namely the standard one; thus it has dimension $\sqrt{\mathrm{rk}(\mathcal{R}_{ii})}$. $\square$

As bundles over a point, $\mathcal{F}_0$ and $\mathcal{A}_0$ are equipped with trivial connections, for which the module action $\mathcal{F}_0 \otimes \mathcal{A}_0 \to \mathcal{F}_0$ is connection-preserving. Since the right module action $\mathcal{R}_{i_0 i} \otimes \mathrm{ev}_0^* \mathcal{A}_0 \to \mathcal{R}_{i_0 i}$ is connection-preserving (Lemma 3.7), it follows that E_i comes equipped with a connection, see Remark A.18.

The fusion representation ϕ induces a connection-preserving, unitary bundle isomorphism

$$\zeta_i : \cup^* \mathcal{L} \otimes \mathrm{pr}_2^* E_i \to \mathrm{pr}_1^* E_i.$$

over $P_i^{[2]}$; explicitly, it is induced by

$$\cup^* \mathcal{L} \otimes \mathrm{pr}_2^* \mathcal{R}_{i_0 i} \otimes \mathrm{ev}_0^* \mathcal{F}_0 \xrightarrow{\rho_{i_0 i} \otimes \mathrm{id}} \mathrm{pr}_1^* \mathcal{R}_{i_0 i} \otimes \mathrm{ev}_0^* \mathcal{F}_0.$$

Since the fusion representation intertwines the algebra actions (Lemma 3.13), this is well-defined under taking the quotient by the $\mathrm{ev}_0^* \mathcal{A}_0$-action. The compatibility between fusion representation and the fusion product (LBG1) ensures that the bundle isomorphism ζ_i satisfies the condition for making $\mathcal{E}_i := (E_i, \zeta_i)$ a $\mathcal{G}|_{Q_i}$-module. This completes the definition of our regression functor on the level of objects.

Next, we consider an LBG morphism (φ, ξ) between LBG objects $(\mathcal{L}, \lambda, \mathcal{R}, \phi, \chi, \epsilon, \alpha)$ and $(\mathcal{L}', \lambda', \mathcal{R}', \phi', \chi', \epsilon', \alpha')$. The regression of $\varphi : \mathcal{L} \to \mathcal{L}'$ to a 1-morphism $\mathcal{A} : \mathcal{G} \to \mathcal{G}'$ in $\mathcal{G}rb^\nabla(M)$ is discussed in [Wal16]; it is a "refinement" consisting simply of the bundle morphism $\cup^* \varphi : \cup^* \mathcal{L} \to \cup^* \mathcal{L}'$. The bundle isomorphisms $\xi_{ij} : \mathcal{R}_{ij} \to \mathcal{R}'_{ij}$ induce in the obvious way isomorphisms of the corresponding bundles of algebras and bimodules. In particular, we

obtain an algebra isomorphism $\xi_0 : \mathcal{A}_0 \to \mathcal{A}'_0$. Again, since the modules $\mathcal{F}_0$ and $\mathcal{F}'_0$ are irreducible modules over isomorphic matrix algebras, they must be isomorphic, and we can choose an intertwiner $f_0 : \mathcal{F}_0 \to \mathcal{F}'_0$. Then, the bundle isomorphism

$$\xi_{i_0 i} \otimes \mathrm{ev}_0^* f_0 : \mathcal{R}_{i_0 i} \otimes \mathrm{ev}_0^* \mathcal{F}_0 \to \mathcal{R}'_{i_0 i} \otimes \mathrm{ev}_0^* \mathcal{F}_0$$

induces a connection-preserving, unitary bundle isomorphism $\tilde{\psi}_i : E_i \to E'_i$, since $\xi_{i_0 i}$ and f_0 intertwine the $\mathcal{A}_0$ and $\mathcal{A}'_0$ actions. The diagram

$$\begin{tikzcd} \cup^* \mathcal{L} \otimes \mathrm{pr}_2^* E_i \arrow[r, "\epsilon_i"] \arrow[d, "\cup^* \varphi \otimes \mathrm{pr}_2^* \tilde{\psi}_i"'] & \mathrm{pr}_1^* E_i \arrow[d, "\mathrm{pr}_1^* \tilde{\psi}_i"] \\ \cup^* \mathcal{L}' \otimes \mathrm{pr}_2^* E'_i \arrow[r, "\epsilon'_i"'] & \mathrm{pr}_1^* E'_i \end{tikzcd} \tag{22}$$

is commutative, because $\xi_{i_0 i}$ commutes with the fusion representation. This means that $\tilde{\psi}_i$ gives rise to a 2-isomorphism $\psi_i : \mathcal{E}_i \Rightarrow \mathcal{E}'_i \circ \mathcal{A}$. Thus, forming the collection $\psi = \{\psi_i\}_{i \in I}$, the pair $(\mathcal{A}, \psi)$ is a TBG 1-morphism. It depends on the (non-canonical) choice of the intertwiner f_0; however, we have the following.

Lemma 5.2. *The 2-isomorphism class of $(\mathcal{A}, \psi)$ is independent of the choice of f_0.*

Proof. If f_0 and f'_0 are two choices, then, by Schur's lemma and because both respect hermitian metrics, they differ by a number $z \in \mathrm{U}(1)$, say $f'_0 = f_0 \cdot z$. This number determines a 2-isomorphism $\tilde{\varphi} : \mathcal{A} \Rightarrow \mathcal{A}$, by inducing it via the automorphism $z \cdot \mathrm{id}$ of $\cup^* \mathcal{L}'$. Let $\tilde{\psi}_i$ and $\tilde{\psi}'_i$ be the vector bundle morphisms determined by f_0 and f'_0, respectively, so that $\tilde{\psi}'_i = \tilde{\psi}_i \cdot z$. Then, we have $\psi'_i = (\mathrm{id}_{\mathcal{E}'_i} \circ \tilde{\varphi}) \bullet \psi_i$; this shows that $\tilde{\varphi}$ is the required 2-isomorphism. $\square$

We have thus provided the data for our regression functor $\mathscr{R}_{i_0, x_0, \mathcal{F}_0}$. The fact that it preserves the composition is easy to see by choosing, for the composed morphism, the composition of the intertwiners f_0 of the separate morphisms.

Remark 5.3. Up to canonical natural isomorphism, the regression functor is independent of all choices. This can be seen either manually, or it can be deduced from Theorem 1, which says that it is inverse to one fixed functor, the transgression functor.

6. Equivalence of target space and loop space perspectives

In this section we prove our main results: Theorem 2 (b) in Section 6.1, and Theorem 1 in Sections 6.2 and 6.3. Throughout this section, we fix a target space (M, Q), with $Q = \{Q_i\}_{i \in I}$.

6.1. Coincidence of the algebra bundles

We recall that TBG and LBG independently induce algebra bundles over the branes (Remark 2.2 and Section 3.1). We show that they coincide under transgression and regression.

We start with a TBG object $(\mathcal{G}, \mathcal{E})$. Let $\mathcal{R}_{ij}$ be the transgressed vector bundle over P_{ij}, and let $\mathcal{A}_i = \mathsf{c}^*\mathcal{R}_{ii}$ be the induced algebra bundle over Q_i. We let $\pi : Y \to M$ be the surjective submersion of the bundle gerbe $\mathcal{G}$. For a point $y \in Y$ with $x := \pi(y) \in Q_i$, we denote by $\mathcal{T}_y$ the corresponding trivialization of $\mathcal{G}|_x$, see Remark 2.4. Applying that remark to $s = \mathsf{c}_y$ as a section of $\pi : Y \to M$ along $\{x\} \to M$, we obtain a canonical vector bundle isomorphism $\varphi_y : \Delta(\mathcal{E}_i|_x, \mathcal{T}_y) \to E_i|_y$. It is straightforward to see that

$$\psi_y : \mathrm{End}(E_i)|_y \to \mathcal{A}_i|_x : \varphi \mapsto (\mathsf{c}_x^*\mathcal{T}_y, \varphi_y^{-1} \circ \varphi \circ \varphi_y)$$

induces a smooth, connection-preserving vector bundle isomorphism $\psi : \mathrm{End}(E_i) \to \pi^*\mathcal{A}_i$ over $\pi^{-1}(Q_i)$ that preserves the fiber-wise algebra structures. If $y' \in Y$ is another point with $\pi(y') = x$, we choose $\ell \in L|_{y,y'}$ and obtain a 2-isomorphism $\psi : \mathcal{T}_y \Rightarrow \mathcal{T}_{y'}$ in such a way that the diagram

$$\begin{tikzcd} \Delta(\mathcal{E}_i|_x, \mathcal{T}_{y'}) \arrow[r, "\varphi_{y'}"] \arrow[d, "\Delta(\psi,\mathrm{id})"'] & E_i|_{y'} \arrow[d, "\zeta_i(\ell\otimes -)"] \\ \Delta(\mathcal{E}_i|_x, \mathcal{T}_y) \arrow[r, "\varphi_y"'] & E_i|_y. \end{tikzcd} \tag{23}$$

is commutative (see again Remark 2.4). By (23), the bundle morphism

$$\mathrm{pr}_1^*\psi^{-1} \circ \mathrm{pr}_2^*\psi : \mathrm{pr}_2^*\mathrm{End}(E_i) \to \mathrm{pr}_1^*\mathrm{End}(E_i) \tag{24}$$

over $Y^{[2]}$ is given at (y, y') by $\varphi \mapsto \zeta_i(\ell \otimes -) \circ \varphi \circ \zeta_i(\ell \otimes -)^{-1}$. One can check that this is independent of ℓ; in fact, under the isomorphism $\mathrm{End}(E_i) = E_i^* \otimes E_i$, it is precisely the descent isomorphism $\tilde{\zeta}_i$ that defines the bundle $\mathrm{End}(\mathcal{E}_i)$. This proves that ψ descends to an isomorphism $\mathrm{End}(\mathcal{E}_i) \cong \mathcal{A}_i$ of algebra bundles over Q_i.

Now, consider an LBG object $(\mathcal{L}, \lambda, \mathcal{R}, \phi, \chi, \epsilon, \alpha)$, and the corresponding regression $(\mathcal{G}, \mathcal{E})$. We recall from Section 5 that $E_i = \mathcal{R}_{i_0 i} \otimes_{\mathrm{ev}_0^* \mathcal{A}_0} \mathrm{ev}_0^* \mathcal{F}_0$, which is a bundle of left $\mathrm{ev}_1^* \mathcal{A}_i$-modules over P_i. In particular, we have a homomorphism of algebra bundles

$$\mathrm{ev}_1^* \mathcal{A}_i \to \mathrm{End}(E_i).$$

We remark that E_i is the composition of two Morita equivalences: $\mathcal{R}_{i_0 i}$ is a Morita equivalence by Corollary 3.11, and, since it is an irreducible module, $\mathcal{F}_0$ is a Morita equivalence between $\mathcal{A}_0$ and $\mathbb{C}$. Hence, E_i is a Morita equivalence; in particular it is faithfully balanced, which implies that the above map is an isomorphism. The bundle of algebras $\mathrm{End}(\mathcal{E}_i)$ of TBG is now obtained via descent of $\mathrm{End}(E_i)$ along the path evaluation $\mathrm{ev}_1 : P_i \to Q_i$. Using that the bundle morphism ζ_i is defined from the fusion representation (see Section 5) and that the fusion representation commutes with the action of $\mathcal{A}_i$ (Lemma 3.13) one can show that the diagram

$$\begin{tikzcd}
 & \mathrm{ev}_1^* \mathcal{A}_i \arrow[dl] \arrow[dr] & \\
\mathrm{pr}_2^* \mathrm{End}(E_i) \arrow[rr, "\tilde{\zeta}_i"'] & & \mathrm{pr}_1^* \mathrm{End}(E_i)
\end{tikzcd}$$

is commutative. Thus, we obtain an algebra bundle isomorphism $\mathcal{A}_i \cong \mathrm{End}(\mathcal{E}_i)$.

6.2. Regression after Transgression

In this section we provide a natural isomorphism $\mathscr{R}_{i_0, x_0, \mathcal{F}_0} \circ \mathscr{T} \cong \mathrm{id}_{\mathrm{TBG}(M,Q)}$, contributing one half of the equivalence $\mathrm{TBG}(M, Q) \cong \mathrm{LBG}(M, Q)$. Thus, we construct for each TBG object $(\mathcal{G}, \mathcal{E})$ a 1-morphism $(\mathcal{A}, \psi) : (\mathscr{R}_{i_0, x_0, \mathcal{F}_0} \circ \mathscr{T})(\mathcal{G}, \mathcal{E}) \to (\mathcal{G}, \mathcal{E})$, and show that this is natural with respect to TBG morphisms.

We consider a TBG object $(\mathcal{G}, \mathcal{E})$, form the transgressed LBG object $(\mathcal{L}, \lambda, \mathcal{R}, \phi, \chi, \epsilon, \alpha)$, and consider the regressed bundle gerbe $\mathscr{R}_{x_0}(\mathcal{L}, \lambda)$. The bundle gerbe $\mathcal{G}$ consists of a surjective submersion $\pi : Y \to M$, a hermitian line bundle P over $Y^{[2]}$, and a bundle gerbe product μ. A 1-isomorphism $\mathcal{A} : \mathscr{R}_{x_0}(\mathcal{L}, \lambda) \to \mathcal{G}$ was constructed in [Wal16, Lemma 6.1.1] over the fiber product $Z := P_{x_0} M \times_M Y$. We review and slightly reformulate its construction in the following. The 1-isomorphism $\mathcal{A}$ consists of a hermitian line bundle K with connection over Z. Its construction involves the choice of a point $y_0 \in Y$ with $\pi(y_0) = x_0$. Let $\mathcal{T} : \gamma^* \mathcal{G} \to \mathcal{I}_0$ be a trivialization of $\mathcal{G}$ along

$\gamma \in P_{x_0}M$, consisting of a hermitian line bundle T over $Y_\gamma := [0,1]_\gamma \times_\pi Y$ and a bundle isomorphism τ over $Y_\gamma^{[2]}$. We define the complex inner product space

$$K|_{\gamma,y}(\mathcal{T}) := T|_{(0,y_0)} \otimes T|^*_{(1,y)}. \tag{25}$$

If $\psi : \mathcal{T} \Rightarrow \mathcal{T}'$ is a 2-isomorphism in $\mathcal{G}rb^\nabla([0,1])$, then it is given by a bundle isomorphism $\psi : T \to T'$ over Y_γ, and hence induces an obvious isomorphism $K|_{\gamma,y}(\mathcal{T}) \cong K|_{\gamma,y}(\mathcal{T}')$. We let $K|_{\gamma,y}$ be the set of equivalence classes of pairs $(\mathcal{T}, s \otimes \sigma)$ with a trivialization $\mathcal{T}$ and $s \otimes \sigma \in K|_{\gamma,y}(\mathcal{T})$.

The disjoint union of all these fibers define the total space of K. It can be equipped with a diffeology such that it becomes a hermitian line bundle. Further, one can define a connection on K in a canonical way. It remains to provide the bundle isomorphism over $Z^{[2]}$ which is part of the 1-isomorphism $\mathcal{A}$. Here,

$$\kappa|_{(\gamma_1,y_1),(\gamma_2,y_2)} : \mathcal{L}_{\gamma_1\cup\gamma_2} \otimes K|_{\gamma_2,y_2} \to K|_{\gamma_1,y_1} \otimes P_{y_1,y_2}$$

is defined as follows. We choose a trivialization $\mathcal{T} : (\gamma_1 \cup \gamma_2)^*\mathcal{G} \to \mathcal{I}_0$ and consider the induced trivializations $\mathcal{T}_1 : \gamma_1^*\mathcal{G} \to \mathcal{I}_0$ and $\mathcal{T}_2 : \gamma_2^*\mathcal{G} \to \mathcal{I}_0$. Let τ be the bundle isomorphism of the trivialization $\mathcal{T}$. We define for $s_0 \in T|_{(0,y_0)}$ and $\sigma \in T|^*_{(1,y_2)}$:

$$\begin{aligned}\kappa|_{(\gamma_1,y_1),(\gamma_2,y_2)}&((\mathcal{T}, z) \otimes (\mathcal{T}_2, s_0 \otimes \sigma)) \\ &:= z \cdot (\mathcal{T}_1, s_0 \otimes \tau^{tr-1}|_{(1,y_1),(1,y_2)}(\sigma)) \in K|_{\gamma_1,y_1}(\mathcal{T}_1) \otimes P_{y_1,y_2}.\end{aligned}$$

It is proved in [Wal16, Lemmata 6.1.1 & 6.1.3] that κ defines a connection-preserving unitary bundle isomorphism. This finishes the definition of the 1-isomorphism $\mathcal{A} : \mathscr{R}_{x_0}(\mathcal{L}, \lambda) \to \mathcal{G}$.

Next, we construct, for each brane index $i \in I$, a 2-isomorphism $\varphi_i : \mathcal{E}_i \circ \mathcal{A}|_{Q_i} \Rightarrow \mathcal{E}'_i$, where $\mathcal{E}'_i$ is the regressed bundle gerbe module. The inverses $\varphi := \{\varphi_i^{-1}\}_{i\in I}$ will then complete $\mathcal{A}$ into a TBG morphism $(\mathcal{A}, \varphi)$. We set $Z_i := Z|_{Q_i}$ and $K_i := K|_{Z_i}$. The 2-isomorphism φ_i consists of a bundle isomorphism $\varphi_i : K_i \otimes \mathrm{pr}_1^* E_i \to \mathrm{pr}_2^* E'_i$ over Z_i, where E'_i is the vector bundle of the regressed module $\mathcal{E}'_i$. Over a point $(y, \gamma) \in Z_i$ with $\pi(y) = \gamma(1) \in Q_i$ and $\gamma(0) = x_0$, this is an isomorphism

$$\varphi_i|_{y,\gamma} : K_i|_{y,\gamma} \otimes E_i|_y \to \mathcal{R}_{i_0 i}|_\gamma \otimes_{\mathcal{A}_0} \mathcal{F}_0. \tag{26}$$

Here, $\mathcal{A}_0 = \mathcal{R}_{i_0 i_0}|_{\mathsf{c}_{x_0}}$ is the fiber of the algebra bundle $\mathcal{A}_{i_0} = \mathsf{c}^*\mathcal{R}_{i_0 i_0}$ over Q_{i_0} over the point x_0. In order to define $\varphi_i|_{y,\gamma}$, we make two observations.

The first is the algebra isomorphism $\varphi_{y_0} : \mathrm{End}(E_{i_0})|_{y_0} \to \mathcal{A}_0$ constructed in Section 6.1, that allows us to choose $\mathcal{F}_0 = E_{i_0}|_{y_0}$ in the definition of the regression functor. The second observation is the following: the chosen lifts y_0 of $x_0 = \gamma(0)$ and y of $\gamma(1)$ yield a vector space isomorphism

$$\begin{aligned}(27)\quad \mathcal{R}_{i_0 i}|_\gamma(\mathcal{T}) &= \mathrm{Hom}(\Delta(\mathcal{E}_{i_0}|_{x_0}, \mathcal{T}|_0), \Delta(\mathcal{E}_i|_{\gamma(1)}, \mathcal{T}|_1)) \\ &\xrightarrow{\delta_{y_0,y}} \mathrm{Hom}(T|^*_{(0,y_0)} \otimes E_{i_0}|_{y_0}, T|^*_{(1,y)} \otimes E_i|_y) \\ &\cong E_{i_0}|^*_{y_0} \otimes T|_{(0,y_0)} \otimes T|^*_{(1,y)} \otimes E_i|_y.\end{aligned}$$

for any trivialization $\mathcal{T} : \gamma^*\mathcal{G} \to \mathcal{I}_0$, where $\delta_{y_0,y}$ is obtained from the isomorphisms $\Delta(\mathcal{E}_{i_0}|_{x_0}, \mathcal{T}|_0) \cong T|^*_{(0,y_0)} \otimes E_{i_0}|_{y_0}$ and $\Delta(\mathcal{E}_i|_{\gamma(1)}, \mathcal{T}|_1) \cong T|^*_{(1,y)} \otimes E_i|_y$ that result from the definition of Δ via descent. For another point $y' \in Y$ with $\pi(y) = \gamma(1)$, the two isomorphisms $\delta_{y_0,y}$ and $\delta_{y_0,y'}$ are related by the descent isomorphism

(28)

$$T^*|_{(1,y')} \otimes E_i|_{y'} \cong T^*|_{(1,y')} \otimes P^*_{y,y'} \otimes P_{y,y'} \otimes E_i|_{y'} \xrightarrow{\tau^{tr-1}\otimes\zeta_i} T^*|_{(1,y)} \otimes E_i|_y$$

that produces $\Delta(\mathcal{E}_i|_{\gamma(1)}, \mathcal{T}|_1)$. Now we are ready to give the definition of the isomorphism $\varphi_i|_{y,\gamma}$ of (26). We consider the linear map

$$T|_{(0,y_0)} \otimes T|^*_{(1,y)} \otimes E_i|_y \to E_{i_0}|^*_{y_0} \otimes T|_{(0,y_0)} \otimes T|^*_{(1,y)} \otimes E_i|_y \otimes E_{i_0}|_{y_0}$$

induced by the counit $\mathbb{C} \to E^* \otimes E$ of the vector space $E = E_{i_0}|_{y_0}$. The vector space on the left-hand side is isomorphic to $K_i|_{y,\gamma}(\mathcal{T}) \otimes E_i|_y$, and the vector space on the right-hand side is isomorphic to $\mathcal{R}_{i_0 i}|_\gamma(\mathcal{T}) \otimes E_{i_0}|_{y_0}$ via $\delta^{-1}_{y_0,y} \otimes \mathrm{id}$. Composed with the projection to the quotient $\mathcal{R}_{i_0 i}|_\gamma(\mathcal{T}) \otimes E_{i_0}|_{y_0} \to \mathcal{R}_{i_0 i}|_\gamma(\mathcal{T}) \otimes_{\mathcal{A}_0} E_{i_0}|_{y_0}$, it becomes an isomorphism, and this is the definition of $\varphi_i|_{y,\gamma}$. It is straightforward to check that this definition is compatible with a change of the trivialization, smooth, and connection-preserving.

Lemma 6.1. *For all $i \in I$, the bundle isomorphism φ_i induces a 2-isomorphism $\varphi_i : \mathcal{E}_i \circ \mathcal{A}|_{Q_i} \Rightarrow \mathcal{E}'_i$, i.e., the diagram*

$$\begin{tikzcd}
\mathcal{L}|_{\gamma_1\cup\gamma_2} \otimes K_i|_{y_2,\gamma_2} \otimes E_i|_{y_2} \arrow[r, "\kappa\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\varphi_i"'] & K_i|_{\gamma_1,y_1} \otimes P_{y_1,y_2} \otimes E_i|_{y_2} \arrow[r, "\mathrm{id}\otimes\zeta_i"] & K_i|_{y_1,\gamma_1} \otimes E_i|_{y_1} \arrow[d, "\varphi_i"] \\
\mathcal{L}|_{\gamma_1\cup\gamma_2} \otimes E'_i|_{\gamma_2} \arrow[rr, "\zeta'_i"'] & & E'_i|_{\gamma_1}
\end{tikzcd}$$

is commutative for any point $((y_1,\gamma_1),(y_2,\gamma_2)) \in Z^{[2]}_i$.

Proof. Choosing a trivialization $\mathcal{T}$ of $\mathcal{G}$ along $\gamma_1 \cup \gamma_2$, we obtain an isomorphism $\mathcal{L}_{\gamma_1\cup\gamma_2} \cong \mathbb{C}$. Substituting the definitions ζ_i' (via the fusion representation), of K_i, and of and φ_i (on the left using δ_{y_0,y_2} and on the right using δ_{y_0,y_1}) and using (28), the diagram can be reduced to

$$\begin{tikzcd}
 & T|_{(0,y_0)} \otimes T|^*_{(1,y_1)} \otimes P_{y_1,y_2} \otimes E_i|_{y_2} \arrow[dr, "\mathrm{id}\otimes\zeta_i"] & \\
T|_{(0,y_0)} \otimes T|^*_{(1,y_2)} \otimes E_i|_{y_2} \arrow[ur, "\kappa|_{\mathcal{L}=\mathbb{C}}\otimes\mathrm{id}"] \arrow[dr] & & T|_{(0,y_0)} \otimes T|^*_{(1,y_1)} \otimes E_i|_{y_1} \\
 & T|_{(0,y_0)} \otimes T^*|_{(1,y_2)} \otimes P^*_{y_1,y_2} \otimes P_{y_1,y_2} \otimes E_i|_{y_2} \arrow[ur, "\mathrm{id}\otimes\tau^{tr-1}\otimes\zeta_i"'] &
\end{tikzcd}$$

Consulting the definitions, we observe that this diagram is commutative. □

So far we have constructed a TBG 1-isomorphism $(\mathcal{A},\varphi) : (\mathcal{G}',\mathcal{E}_i') \to (\mathcal{G},\mathcal{E})$ for every TBG object $(\mathcal{G},\mathcal{E})$. With the following lemma we show that this assignment is natural with respect to TBG morphisms, and hence results in a natural isomorphism.

Lemma 6.2. *Let $(\mathcal{A},\psi) : (\mathcal{G}_1,\mathcal{E}_1) \to (\mathcal{G}_2,\mathcal{E}_2)$ be a TBG 1-morphism. For $a = 1, 2$, let $(\mathcal{G}_a',\mathcal{E}_a')$ be the TBG objects obtained by transgressing and then regressing $(\mathcal{G}_a,\mathcal{E}_a)$, and let $(\mathcal{A}_a,\varphi_a)$ be the corresponding 1-isomorphisms constructed above. Further, let $(\mathcal{A}',\psi')$ be obtained by transgressing and regressing $(\mathcal{A},\psi)$. Then, there exists a 2-isomorphism*

$$\begin{tikzcd}
(\mathcal{G}_1',\mathcal{E}_1') \arrow[r, "(\mathcal{A}',\psi')"] \arrow[d, "(\mathcal{A}_1,\varphi_1)"'] \arrow[dr, phantom, "\Swarrow\phi"] & (\mathcal{G}_2',\mathcal{E}_2') \arrow[d, "(\mathcal{A}_2,\varphi_2)"] \arrow[dl, phantom, "\Nearrow"] \\
(\mathcal{G}_1,\mathcal{E}_1) \arrow[r, "(\mathcal{A},\psi)"'] & (\mathcal{G}_2,\mathcal{E}_2)
\end{tikzcd}$$

in the bicategory $\mathrm{TBG}(M,Q)$*. In other words, the diagram*

$$\begin{tikzcd}
(\mathcal{G}_1',\mathcal{E}_1') \arrow[r, "(\mathcal{A}',\psi')"] \arrow[d, "(\mathcal{A}_1,\varphi_1)"'] & (\mathcal{G}_2',\mathcal{E}_2') \arrow[d, "(\mathcal{A}_2,\varphi_2)"] \\
(\mathcal{G}_1,\mathcal{E}_1) \arrow[r, "(\mathcal{A},\psi)"'] & (\mathcal{G}_2,\mathcal{E}_2).
\end{tikzcd}$$

is commutative in $\mathrm{h}_1(\mathrm{TBG}(M,Q))$*.*

Proof. The 2-isomorphism ϕ itself is already part of the naturality of ordinary bundle gerbe transgression and has been constructed in [Wal16, Lemma

6.1.4]. It remains to show that it is compatible with the 2-isomorphisms of the TBG morphisms. This is equivalent to the commutativity of the following pentagon diagram:

$$\begin{tikzcd}
 & \mathcal{E}_{2,i} \circ \mathcal{A}_2 \circ \mathcal{A}' \arrow[dl, Rightarrow, "\mathrm{id}\circ\phi"'] & \\
\mathcal{E}_{2,i} \circ \mathcal{A} \circ \mathcal{A}_1 & & \mathcal{E}'_{2,i} \circ \mathcal{A}' \arrow[ul, Rightarrow, "\varphi_{2,i}\circ\mathrm{id}"'] \\
\mathcal{E}_{1,i} \circ \mathcal{A}_1 \arrow[u, Rightarrow, "\psi_i\circ\mathrm{id}"] & & \mathcal{E}'_{1,i} \arrow[ll, Rightarrow, "\varphi_{1,i}"] \arrow[u, Rightarrow, "\psi'_i"']
\end{tikzcd}$$

We translate this further into a diagram of the bundle isomorphisms underlying these 2-isomorphisms. Let Q be the vector bundle of the 1-isomorphism $\mathcal{A} : \mathcal{G}_1 \to \mathcal{G}_2$, defined over the fibre product $Y_1 \times_M Y_2$ of the surjective submersions of the two bundle gerbes. Further, we recall that $\mathcal{A}' : \mathcal{G}'_1 \to \mathcal{G}'_2$ is the 1-isomorphism induced by the identity map on $P_{x_0}M$ and the bundle isomorphism $\varphi : \mathcal{L}_1 \to \mathcal{L}_2 : [\mathcal{T}, z] \mapsto [\mathcal{T} \circ \mathcal{A}^{-1}, z]$. Thus, its hermitian line bundle is $\mathcal{L}_2$. In the following we work over a point $(y_1, y_2, \gamma_1, \gamma_2) \in Y_1 \times_M Y_2 \times_M P_{x_0}M^{[2]}$, over which our diagram becomes the following diagram of linear maps:

$$\begin{tikzcd}
 & \mathcal{L}_2|_{\gamma_1\cup\gamma_2} \otimes K_2|_{y_2,\gamma_2} \otimes E_{2,i}|_{y_2} \arrow[dl, "\phi\otimes\mathrm{id}"'] & \\
K_1|_{y_1,\gamma_1} \otimes Q_{y_1,y_2} \otimes E_{2,i}|_{y_2} & & \mathcal{L}_2|_{\gamma_1\cup\gamma_2} \otimes E'_{2,i}|_{\gamma_2} \arrow[ul, "\mathrm{id}\otimes\varphi_{2,i}"'] \\
K_1|_{y_1,\gamma_1} \otimes E_{1,i}|_{y_1} \arrow[u, "\mathrm{id}\otimes\psi_i"] & & E'_{1,i}|_{\gamma_1} \arrow[ll, "\varphi_{1,i}"] \arrow[u, "\psi'_i"']
\end{tikzcd}$$

For further elaboration, we choose a trivialization $\mathcal{T}_1$ of $\mathcal{G}_1$ along $\gamma_1 \cup \gamma_2$, and consider its two halves $\iota_1^*\mathcal{T}_1$ and $\iota_2^*\mathcal{T}_1$, as well as the composite $\mathcal{T}_2 := \mathcal{T}_1 \circ \mathcal{A}^{-1}$ and its two halves $\iota_1^*\mathcal{T}_2$ and $\iota_2^*\mathcal{T}_2$. Then, using (25), we are able to express the line bundles K_1 and K_2 in terms of the line bundles T_1 of $\mathcal{T}_1$ and the line bundle Q, and we can express using (27) and the points y_1, y_2 the vector bundle $E'_{1,i}$ and $E_{2,i}$ in terms of $E_{1,i}$, $E_{2,i}$, T_1, and Q. The main point now is to compute the 2-isomorphism $\psi'_i : \mathcal{E}'_{1,i} \Rightarrow \mathcal{E}'_{2,i} \circ \mathcal{A}'$, defined as the regression of the isomorphism (φ, ξ), which in turn is the transgression of $(\mathcal{A}, \psi)$. For the regression we only need

$$\xi_{i_0 i} : \mathcal{R}_{1,i_0 i}|_{\gamma_1}(\iota_1^*\mathcal{T}_1) \to \mathcal{R}_{2,i_0 i}|_{\gamma_1}(\iota_1^*\mathcal{T}_2),$$

which is – under the above isomorphisms – determined by ψ_i via

$$\begin{aligned} &E_{1,i_0}|^*_{y_{1,0}} \otimes T_1|_{(0,y_{1,0})} \otimes T_1|^*_{(1,y_1)} \otimes E_{1,i}|_{y_1} \xrightarrow{\psi_{i_0}^{tr-1}\otimes\mathrm{id}\otimes\mathrm{id}\otimes\psi_i} \\ &\quad E_{2,i_0}|^*_{y_{2,0}} \otimes Q^*_{y_{1,0},y_{2,0}} \otimes T_1|_{(0,y_{1,0})} \otimes T_1|^*_{(1,y_1)} \otimes Q_{y_1,y_2} \otimes E_{2,i}|_{y_2}. \end{aligned}$$

For the subsequent regression of $\xi_{i_0 i}$ we need to fix an intertwiner $f_0 : E_{1,i_0}|_{y_{1,0}} \to E_{2,i_0}|_{y_{2,0}}$, see Section 5. Then – under our isomorphisms – the bundle isomorphism $\psi'_i : E'_{1,i}|_{\gamma_1} \to \mathcal{L}_2|_{\gamma_1\cup\gamma_2} \otimes E'_{2,i}|_{\gamma_2}$ becomes a map

$$\begin{aligned} &E_{1,i_0}|^*_{y_{1,0}} \otimes T_1|_{(0,y_{1,0})} \otimes T_1|^*_{(1,y_1)} \otimes E_{1,i}|_{y_1} \otimes E_{1,i_0}|_{y_{1,0}} \\ &\quad \to \mathcal{L}_2|_{\gamma_1\cup\gamma_2} \otimes E_{2,i_0}|^*_{y_{2,0}} \otimes Q^*_{y_{1,0},y_{2,0}} \\ &\qquad \otimes T_1|_{(0,y_{1,0})} \otimes T_1|^*_{(1,y_1)} \otimes Q_{y_1,y_2} \otimes E_{2,i}|_{y_2} \otimes E_{2,i_0}|_{y_{2,0}} \end{aligned}$$

given by

$$\omega \otimes s \otimes \sigma \otimes v \otimes v_0 \mapsto [\mathcal{T}_2, 1] \otimes \psi_{i_0}^{tr-1}(\omega) \otimes s \otimes \sigma \otimes \psi_i(v) \otimes f_0(v_0).$$

In the following we will use a specific intertwiner f_0, obtained by choosing an element $\chi_0 \in Q^*|_{y_{1,0},y_{2,0}}$ and setting $f_0 := (\chi_0 \otimes \mathrm{id}) \circ \psi_{i_0}|_{y_{1,0},y_{2,0}}$.

Finally, we recall the definition of the 2-isomorphism ϕ from [Wal16, Lemma 6.1.4]. Its bundle morphism

$$\begin{aligned} \phi : &\mathcal{L}_2|_{\gamma_1\cup\gamma_2} \otimes Q^*_{y_{1,0},y_{2,0}} \otimes T_1|_{(0,y_{1,0})} \otimes T_1|^*_{(1,y_1)} \otimes Q_{y_1,y_2} \\ &\to T_1|_{(0,y_{1,0})} \otimes T_1|_{(1,y_1)} \otimes Q_{y_1,y_2} \end{aligned}$$

is determined so that it sends $[\mathcal{T}_2, 1] \otimes \chi_0 \otimes s \otimes \sigma \otimes q$ to $s \otimes \sigma \otimes q$, where χ_0 is the element fixed above. We can now prove the commutativity of the pentagon diagram. The counter-clockwise composition acts as

$$\omega \otimes s \otimes \sigma \otimes v \otimes v_0 \mapsto \omega_0(f_0(v_0)) \cdot s \otimes \sigma \otimes q \otimes v',$$

where ω_0 is determined so that $\psi_{i_0}^{tr-1}(\omega) = \omega_0 \otimes \chi_0$, and $q \otimes v' := \psi_i(v)$. On the other hand, the clock-wise composition yields

$$\omega \otimes s \otimes \sigma \otimes v \otimes v_0 \mapsto \omega(v_0) \cdot s \otimes \sigma \otimes q \otimes v',$$

and it is straightforward to deduce the coincidence $\omega(v_0) = \omega_0(f_0(v_0))$ from the given definitions. Thus, the pentagon diagram is commutative. $\square$

6.3. Transgression after Regression

In this section we provide a natural equivalence $\mathscr{T} \circ \mathscr{R}_{i_0,x_0,\mathcal{F}_0} \cong \mathrm{id}_{\mathrm{LBG}(M,Q)}$, thus establishing the second half of the equivalence $\mathrm{TBG}(M,Q) \cong \mathrm{LBG}(M,Q)$. To that end, we construct for each LBG object $(\mathcal{L},\lambda,\mathcal{R},\phi,\chi,\epsilon,\alpha)$ a 1-morphism

$$(\varphi,\xi) : (\mathscr{T} \circ \mathscr{R}_{i_0,x_0,\mathcal{F}_0})(\mathcal{L},\lambda,\mathcal{R},\phi,\chi,\epsilon,\alpha) \to (\mathcal{L},\lambda,\mathcal{R},\phi,\chi,\epsilon,\alpha),$$

and show that it is natural with respect to LBG morphisms.

We let $(\mathcal{G},\mathcal{E}) := \mathscr{R}_{i_0,x_0,\mathcal{F}_0}(\mathcal{L},\lambda,\mathcal{R},\phi,\chi,\epsilon,\alpha)$ be the regressed TBG, and denote its transgressed LBG by $(\mathcal{L}',\lambda',\mathcal{R}',\phi',\chi',\epsilon',\alpha') := \mathscr{T}(\mathcal{G},\mathcal{E})$. In [Wal16, Section 6.2] we have constructed an isomorphism $\varphi : \mathcal{L}' \to \mathcal{L}$ of hermitian line bundles over LM. We recall this construction in a slightly adapted version. We consider a loop $\tau \in LM$ of the form $\tau = (\gamma_1 \star \gamma_0) \cup (\gamma_2 \star \gamma_0)$, where $\gamma_0 \in P_{x_0}M$ and $(\gamma_1,\gamma_2) \in PM^{[2]}$. Up to thin homotopy, every loop in M is of this form. Now let $\mathcal{T} : \tau^*\mathcal{G} \to \mathcal{I}_0$ be a trivialization, including a hermitian line bundle T with connection over $Z := S^1 {}_\tau\!\times_{\mathrm{ev}_1} P_{x_0}M$ and a connection preserving bundle isomorphism

$$\delta|_{(t,\beta_1),(t,\beta_2)} : \mathcal{L}|_{\beta_1\cup\beta_2} \otimes T_{(t,\beta_2)} \to T_{(t,\beta_1)}$$

over $Z^{[2]}$. We consider the obvious paths α_i in Z connecting $(0,\mathrm{id}\star\gamma_0)$ with $(\frac{1}{2},\gamma_i\star\gamma_0)$. Then, there exists a unique element $p \in \mathcal{L}|_\tau = \mathcal{L}|_{(\gamma_1\star\gamma_0)\cup(\gamma_2\star\gamma_0)}$ such that

$$\delta(p \otimes pt_{\alpha_2}(q)) = pt_{\alpha_1}(q)$$

for all $q \in T_{(0,\mathrm{id}\star\gamma_0)}$. We define $\varphi([\mathcal{T},z]) := p \cdot z$. Using the superficial connection on $\mathcal{L}$ and its parallel transport along thin homotopies, this definition can be extended to all loops $\tau \in LM$. This results in a fusion-preserving bundle morphism $\varphi : \mathcal{L}' \to \mathcal{L}$ that is independent of all involved choices, see [Wal16, Lemma 6.2.1 & 6.2.2].

Next, we construct, for $i,j \in I$, a vector bundle isomorphism

$$\xi_{ij} : \mathcal{R}'_{ij} \to \mathcal{R}_{ij}$$

over P_{ij}. First, we compute $\mathcal{R}'_{ij}$ over a path $\gamma \in P_{ij}$ connecting $x \in Q_i$ with $y \in Q_j$. Let $x_0 \in Q_{i_0}$ be the point and $\mathcal{F}_0$ be the $\mathcal{A}_{x_0}$-module chosen for regression, and recall that we constructed the bundle $E_i := \mathcal{R}_{i_0 i} \otimes_{\mathrm{ev}_0^*\mathcal{A}_0} \mathcal{F}_0$ over P_i. We choose a path $\gamma_0 \in P_{i_0 i}$ with $\gamma_0(0) = x_0$ and $\gamma_0(1) = x$, and

obtain a smooth map

$$s : [0,1] \to P_{x_0}M : t \mapsto \gamma^t \star \gamma_0, \tag{29}$$

where γ^t denotes the restriction of a path to $[0,t]$, reparameterized to $[0,1]$ (with sitting instants). In fact, s is a section along γ into the surjective submersion of $\gamma^*\mathcal{G}$. By Remark 2.4 this induces a trivialization $\mathcal{T} : \gamma^*\mathcal{G} \to \mathcal{I}_0$, together with an isomorphism

$$\begin{aligned} \mathcal{R}'_{ij}|_\gamma(\mathcal{T}) &= \mathrm{Hom}(\Delta(\mathcal{E}_i|_x, \mathcal{T}|_0), \Delta(\mathcal{E}_j|_y), \mathcal{T}|_1) \\ &\cong \mathrm{Hom}((s^*E_i)|_0, (s^*E_j)|_1) = \mathrm{Hom}(E_i|_{\mathsf{c}_x\star\gamma_0}, E_j|_{\gamma\star\gamma_0}). \end{aligned} \tag{30}$$

Using the definition of E_i and E_j via regression, as well as the lifted path concatenation, we obtain another isomorphism

$$\begin{aligned} \mathrm{Hom}(E_i|_{\mathsf{c}_x\star\gamma_0}, E_j|_{\gamma\star\gamma_0}) &= \mathrm{Hom}(\mathcal{R}_{i_0i}|_{\mathsf{c}_x\star\gamma_0} \otimes_{\mathcal{A}_{x_0}} \mathcal{F}_0, \mathcal{R}_{i_0j}|_{\gamma\star\gamma_0} \otimes_{\mathcal{A}_{x_0}} \mathcal{F}_0) \\ &\cong \mathrm{Hom}(\mathcal{R}_{i_0i}|_{\gamma_0} \otimes_{\mathcal{A}_{x_0}} \mathcal{F}_0, \mathcal{R}_{ij}|_\gamma \otimes_{\mathcal{A}_i|_x} \mathcal{R}_{i_0i}|_{\gamma_0} \otimes_{\mathcal{A}_{x_0}} \mathcal{F}_0) \\ &= \mathrm{Hom}(E_i|_{\gamma_0}, \mathcal{R}_{ij}|_\gamma \otimes_{\mathcal{A}_i|_x} E_i|_{\gamma_0}). \end{aligned} \tag{31}$$

Next, we consider the linear map

$$\mathcal{R}_{ij}|_\gamma \to \mathrm{Hom}(E_i|_{\gamma_0}, \mathcal{R}_{ij}|_\gamma \otimes_{\mathcal{A}_i|_x} E_i|_{\gamma_0}) \tag{32}$$

that sends a vector $v \in \mathcal{R}_{ij}|_\gamma$ to the homomorphism $w \mapsto v \otimes_{\mathcal{A}_i|_x} w$. Using that $\mathcal{A}_i|_x$ is simple and that $E_i|_{\gamma_0}$ is irreducible, it is straightforward to see that (32) is injective. Moreover, the dimensions on both sides coincide, so that it is in fact an isomorphism,

$$\begin{aligned} \dim(\mathcal{R}_{ij}|_\gamma) &\overset{\text{Remark 2.6 (b)}}{=} \sqrt{\mathrm{rk}(\mathcal{R}_{ii}) \cdot \mathrm{rk}(\mathcal{R}_{jj})} \overset{\text{Lemma 5.1}}{=} \mathrm{rk}(E_i) \cdot \mathrm{rk}(E_j) \\ &= \dim(\mathrm{Hom}(E_i|_{\mathsf{c}_x\star\gamma_0}, E_j|_{\gamma\star\gamma_0})) \overset{(31)}{=} \dim(\mathrm{Hom}(E_i|_{\gamma_0}, \mathcal{R}_{ij}|_\gamma \otimes_{\mathcal{A}_i|_x} E_i|_{\gamma_0})). \end{aligned}$$

Combining (30) and (31) with the inverse of (32), we obtain the isomorphism

$$\xi_{ij}|_\gamma : \mathcal{R}'_{ij}|_\gamma \to \mathcal{R}_{ij}|_\gamma.$$

Lemma 6.3. *$\xi_{ij}|_\gamma$ is independent of the choice of the path γ_0.*

Proof. If γ'_0 is another path connecting x_0 with $x = \gamma(0)$, then we have the two sections s, s' into the submersion of $\gamma^*\mathcal{G}$, inducing trivializations $\mathcal{T}$ and

$\mathcal{T}'$, respectively. Any element $p \in \mathcal{L}_{\gamma_0 \cup \gamma_0'}$ determines (via a thin homotopy) a section of $\mathcal{L}$ along (s, s'), and hence a hence a 2-isomorphism $\psi_p : \mathcal{T} \Rightarrow \mathcal{T}'$, see Remark 2.4. Our aim is to show that the following diagram is commutative, whose top row and bottom row are the isomorphisms ξ_{ij}^{-1}, defined using γ_0' and γ_0, respectively.

$$\begin{tikzcd}
\mathcal{R}_{ij}|_\gamma \arrow[r, "(32)"] \arrow[d, equal] & \mathrm{Hom}(E_i|_{\gamma_0'}, \mathcal{R}_{ij}|_\gamma \otimes_{\mathcal{A}_i|_x} E_i|_{\gamma_0'}) \arrow[r, "(31)"] \arrow[d, "((\mathrm{id}\otimes\rho_p\otimes\mathrm{id}_{\mathcal{T}_0})\circ - \circ(\rho_p\otimes\mathrm{id}_{\mathcal{T}_0})^{-1})"] & \mathrm{Hom}(E_i|_{\mathsf{c}_x\star\gamma_0'}, E_j|_{\gamma\star\gamma_0'}) \arrow[r, "(30)"] \arrow[d, "\zeta_j'\circ - \circ\zeta_i'^{-1}"] & \mathcal{R}_{ij}'|_\gamma(\mathcal{T}') \arrow[d, "r_{\psi_p}"] \\
\mathcal{R}_{ij}|_\gamma \arrow[r, "(32)"'] & \mathrm{Hom}(E_i|_{\gamma_0}, \mathcal{R}_{ij}|_\gamma \otimes_{\mathcal{A}_i|_x} E_i|_{\gamma_0}) \arrow[r, "(31)"'] & \mathrm{Hom}(E_i|_{\mathsf{c}_x\star\gamma_0}, E_j|_{\gamma\star\gamma_0}) \arrow[r, "(30)"'] & \mathcal{R}_{ij}'|_\gamma(\mathcal{T}).
\end{tikzcd}$$

The vertical maps in this diagram all depend on the choice of the point $p \in \mathcal{L}_{\gamma_0 \cup \gamma_0'}$: first, we have an isomorphism $\rho_p : \mathcal{R}_{i_0 i}|_{\gamma_0'} \to \mathcal{R}_{i_0 i}|_{\gamma_0}$ of right $\mathcal{A}_{x_0}$-modules, obtained by fusion with p. Second, we have linear isomorphisms $\zeta_i' : E_i|_{\mathsf{c}_x\star\gamma_0'} \to E_i|_{\mathsf{c}_x\star\gamma_0}$ and $\zeta_j' : E_j|_{\gamma\star\gamma_0'} \to E_j|_{\gamma\star\gamma_0}$ obtained using the gerbe module isomorphisms ζ_i and ζ_j, with the first argument fixed by the elements in $\mathcal{L}_{(\mathsf{c}_x\star\gamma_0)\cup(\mathsf{c}_x\star\gamma_0')}$ and $\mathcal{L}_{(\gamma\star\gamma_0)\cup(\gamma\star\gamma_0')}$ determined by p under thin homotopies. Now, all three subdiagrams are commutative: the one on the left commutes obviously by inspection, the one in the middle commutes due to the fact that ζ_i is defined (in the process of regression) by the fusion representation, and the one on the right-hand side commutes by definition of the isomorphism r_{ψ_p} (see Section 4.2) and Remark 2.4. $\square$

By the previous lemma, the isomorphisms $\xi_{ij}|_\gamma$ assemble into a well-defined map $\xi_{ij} : \mathcal{R}_{ij}' \to \mathcal{R}_{ij}$.

Lemma 6.4. *For all $i, j \in I$, the map ξ_{ij} is a connection-preserving bundle morphism.*

Proof. We first show the smoothness of ξ_{ij}. Let $c : U \to P_{ij}$ be a plot and let $W \subseteq U$ be a contractible open subset such that we can find a local trivialization $\phi : W \times \mathbb{C}^k \to W \times_{P_{ij}} \mathcal{R}_{ij}$. Due to the contractibility of W, there exists a point $w_0 \in W$ and a smooth map $W \to PW : w \mapsto \gamma_w$ with $\gamma_w(0) = w_0$ and $\gamma_w(1) = w$ for all $w \in W$. We let $c_0 := \mathrm{ev}_0 \circ c$ be the plot of initial points, set $x := c_0(w_0)$ and choose a path γ_0 in M connecting x_0 with x. For $w \in W$, we have the path $\gamma_w^c := c_0 \circ \gamma_w \in P_{ii}$ connecting x with $c_0(w)$. Finally (upon choosing a smoothing function which we suppress from the notation), for every $w \in W$ and $t \in [0, 1]$ we have a path $\beta_{w,t} \in PM$ defined by $\beta_{w,t}(s) := c(w)(ts)$; it connects $\beta_{w,t}(0) = c_0(w)$ with $\beta_{w,t}(1) = c(w)(t) =$

$c^\vee(w,t)$. The concatenation of these paths defines a smooth map

$$\sigma : [0,1] \times W \to P_{x_0}M : (t,w) \mapsto \beta_{w,t} \star (\gamma_w^c \star \gamma_0) \tag{33}$$

such that $\mathrm{ev}_1 \circ \sigma = c^\vee$. In other words, σ is a lift of $c^\vee$ to the surjective submersion of the regressed bundle gerbe $\mathcal{G}$ and hence defines a trivialization $\mathcal{T} : (c^\vee)^*\mathcal{G} \to \mathcal{I}_{\sigma^* B}$. We want to compute the corresponding vector bundles $\mathcal{V}_\mathcal{T}$ and $\mathcal{W}_\mathcal{T}$ over W, as defined in Section 4.2. For this purpose, we consider the map $\sigma' : W \to P_i$ defined by $\sigma'(w) := \gamma_w^c \star \gamma_0$, so that $(\sigma \circ j_0)(w) = \mathsf{c}_{c_0(w)} \star \sigma'(w)$. We obtain an isomorphism of vector bundles over W:

$$\begin{aligned} \mathcal{V}_\mathcal{T} &\cong (\sigma \circ j_0)^* E_i = (\sigma \circ j_0)^* \mathcal{R}_{i_0 i} \otimes_{\mathcal{A}_{x_0}} \mathcal{F}_0 \\ &\cong (\mathsf{c} \circ c_0)^* \mathcal{R}_{ii} \otimes_{c_0^* \mathcal{A}_i} \sigma'^* \mathcal{R}_{i_0 i} \otimes_{\mathcal{A}_{x_0}} \mathcal{F}_0 \cong \sigma'^* \mathcal{R}_{i_0 i} \otimes_{\mathcal{A}_{x_0}} \mathcal{F}_0 \end{aligned} \tag{34}$$

using, respectively, Remark 2.4, the definition of E_i under regression, Proposition 3.10, and the definition $\mathcal{A}_i|_{c_0(w)} = \mathcal{R}_{ii}|_{\mathsf{c}_{c_0(w)}}$. Analogously, we obtain an isomorphism

$$\mathcal{W}_\mathcal{T} \cong c^* \mathcal{R}_{ij} \otimes_{c_0^* \mathcal{A}_i} \sigma'^* \mathcal{R}_{i_0 i} \otimes_{\mathcal{A}_{x_0}} \mathcal{F}_0. \tag{35}$$

Combining these two isomorphisms, we have an isomorphism

$$\psi_{ij} : \mathrm{Hom}(\mathcal{V}_\mathcal{T}, \mathcal{W}_\mathcal{T}) \to c^* \mathcal{R}_{ij} \otimes_{c_0^* \mathcal{A}_i} \mathrm{End}(\mathcal{V}_\mathcal{T}),$$

of vector bundles over W, and by construction it restricts over each point $w \in W$ to the isomorphism $\xi_{ij}|_{c(w)}$. Using the given local trivialization ϕ of $\mathcal{R}_{ij}$, we consider the trivialization $\tau' : W \times \mathbb{C}^k \to \mathrm{Hom}(\mathcal{V}_\mathcal{T}, \mathcal{W}_\mathcal{T})$ defined by $\tau'(w,v) := \psi_{ij}^{-1}(\phi(w,v) \otimes \mathrm{id})$, and consider the associated local trivialization ϕ' of $\mathcal{R}'_{ij}$, which reads as

$$\phi'(w,v) := (w, [i_w^* \mathcal{T}, \tau'(w,v)]).$$

Under the two local trivializations ϕ and ϕ', the map ξ_{ij} corresponds to the identity, and hence it is smooth.

In order to show that ξ_{ij} is connection-preserving, we compute the local connection 1-form $\omega_{\phi'}$ associated to the local trivialization ϕ', and prove

that $\omega_{\phi'} = \omega_\phi$. According to the proof of Proposition 4.5 (see (16)), we have

$$\omega_{\phi'} = \omega_{\tau'} + \int_{[0,1]} \sigma^* B' \in \Omega^1(W, \mathfrak{gl}(\mathbb{C}^k)),$$

where B' is the curving of $\mathcal{G}'$. We observe from the constructions that the bundle isomorphisms (34) and (35), and hence the bundle isomorphism ψ_{ij} are connection-preserving. The definition of τ', together with the fact that the identity section in $\mathrm{End}(\mathcal{V}_\mathcal{T})$ is parallel, show that $\omega_{\tau'} = \omega_\phi$. It remains to prove that the 1-form

$$\int_{[0,1]} \sigma^* B' \in \Omega^1(W) \tag{36}$$

vanishes. This is complicated by the fact that the regressed curving B' is defined using a correspondence between 2-forms and smooth functions on a space of bigons; see [SW11] for the general theory of this correspondence. In our case, the 2-form $B' \in \Omega^2(P_{x_0}M)$ corresponds to a certain map $G_\mathcal{L}$ whose definition we will recall below. In Appendix B we show that the 1-form (36) corresponds to the map

$$PW \to \mathrm{U}(1) : \gamma \mapsto G_\mathcal{L}(\sigma_*(\Sigma_\gamma)),$$

where the bigon Σ_γ is defined by $\Sigma_\gamma := (\mathrm{id} \times \gamma)_*(\Sigma_{1,1})$, where $\Sigma_{1,1} : [0,1]^2 \to [0,1]^2$ is the so-called standard bigon. Using the definition of σ from (33), the bigon $\sigma_*(\Sigma_\gamma)$ is

$$(s,t) \mapsto \beta_{\gamma(\xi_2(s,t)),\xi_1(s,t)} \star (\gamma^c_{\gamma(\xi_2(s,t))} \star \gamma_0),$$

where ξ_1, ξ_2 are the two components of $\Sigma_{1,1}$, i.e. $\Sigma_{1,1} = (\xi_1, \xi_2)$. In the following we prove that $G_\mathcal{L}(\sigma_*(\Sigma_\gamma)) = 1$ for all $\gamma \in PW$; this shows that (36) is zero. In order to do this, we recall the definition of $G_\mathcal{L}(\Sigma)$ given in [Wal16, Section 5.2], which is essentially by parallel transport in $\mathcal{L}$ along a path γ_Σ in LM (see Figure 1 in Section 5.2 of [Wal16] for a picture of this path). We write $\chi_\gamma := \gamma_{\sigma_*(\Sigma_\gamma)}$ for simplicity; this is a path in LM given by

$$\begin{aligned} \chi_\gamma(t) = {} & (\beta_{\gamma(\xi_2(-,t)),\xi_1(-,t)}(1) \star \beta_{\gamma(\xi_2(0,t)),\xi_1(0,t)} \star \gamma^c_{\gamma(\xi_2(0,t))} \star \gamma_0) \\ & \cup (\mathsf{c} \star \beta_{\gamma(\xi_2(1,t)),\xi_1(1,t)} \star \gamma^c_{\gamma(\xi_2(1,t))} \star \gamma_0) \in LM. \end{aligned} \tag{37}$$

The end loops $\chi_\gamma(0)$ and $\chi_\gamma(1)$ are "flat", i.e. in the image of the map $\gamma \mapsto \gamma \cup \gamma$, along which $\mathcal{L}$ has the canonical flat section ν. In particular,

this defines elements $\nu_0, \nu_1 \in \mathcal{L}$ projecting to $\chi_\gamma(0)$ and $\chi_\gamma(1)$, respectively. Then, $G_{\mathcal{L}}(\sigma_*(\Sigma_\gamma))$ is defined by

$$pt_{\chi_\gamma}(\nu_0) = \nu_1 \cdot G_{\mathcal{L}}(\sigma_*(\Sigma_\gamma)). \tag{38}$$

We consider for $r \in [0,1]$ maps $\xi_1^r, \xi_2^r : [0,1]^2 \to [0,1]$ that constitute fixed-ends homotopies between ξ_1, ξ_2 and the map $(s,t) \mapsto t$. That is, we have $\xi_1^1 = \xi_1$ and $\xi_2^1 = \xi_2$, and $\xi_1^0(s,t) = \xi_2^0(s,t) = t$, as well as $\xi_1^r(s,0) = \xi_2^r(s,0) = 0$ and $\xi_1^r(s,1) = \xi_2^r(s,1) = 1$, for all $s,t \in [0,1]$. We consider the path χ_γ^r defined as in (37) but using ξ_1^r and ξ_2^r instead of ξ_1 and ξ_2. We regard $h : [0,1]^2 \to LM : (r,t) \mapsto \chi_\gamma^r(t)$ as a homotopy between the paths χ_γ and χ_γ^0 in LM. Calculating the latter paths explicitly, we notice that χ_γ^0 is a path trough flat loops. The homotopy h fixes the end-loops. We claim that the adjoint map $h^\vee : [0,1]^2 \times S^1 \to M$ has rank two; which can be checked explicitly using (37). Since the connection on $\mathcal{L}$ is superficial (see [Wal16, Lemma 2.2.3]) and ν is parallel, we have $pt_{\chi_\gamma}(\nu_0) = pt_{\chi_\gamma^0}(\nu_0) = \nu_1$. Comparing with (38), we have the claim. □

So far we have provided the data (φ, ξ) for a LBG morphism. It remains to show that it respects the fusion representation and the lifted path concatenation, see Remarks 3.2 and 3.12 (b). This is done in the following two lemmas.

Lemma 6.5. *The bundle isomorphism ξ_{ij} respects the fusion representations, i.e. the diagram*

$$\begin{tikzcd} \mathcal{L}'|_\tau \otimes \mathcal{R}'_{ij}|_{\gamma_2} \arrow[r, "\phi'_{ij}|_{\gamma_1,\gamma_2}"] \arrow[d, "\varphi\otimes\xi_{ij}"'] & \mathcal{R}'_{ij}|_{\gamma_1} \arrow[d, "\xi_{ij}"] \\ \mathcal{L}|_\tau \otimes \mathcal{R}_{ij}|_{\gamma_2} \arrow[r, "\phi_{ij}|_{\gamma_1,\gamma_2}"'] & \mathcal{R}_{ij}|_{\gamma_1} \end{tikzcd} \tag{39}$$

is commutative for all $\gamma_1, \gamma_2 \in P_{ij}$ with $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$, and $\tau := \gamma_1 \cup \gamma_2$.

Proof. We set $x := \gamma_1(0)$ and $y := \gamma_1(1)$. Let $\mathcal{T}$ be a trivialization of $\tau^*\mathcal{G}$, and let $p := \varphi(\mathcal{T}) \in \mathcal{L}|_\tau$. We choose a path γ_0 connecting x_0 with x and consider the corresponding sections

$$s_a : [0,1] \to P_{x_0}M : t \mapsto \gamma_a^t \star \gamma_0$$

along γ_a into the surjective submersions of $\mathcal{G}$, for $a = 1,2$. We have to study the induced trivializations $\mathcal{T}_a : \gamma_a^*\mathcal{G} \to \mathcal{I}_0$: their line bundles are $T_a|_{(t,\beta)} :=$

$$
\begin{tikzcd}
\Delta(\mathcal{E}_j|_y, \mathcal{T}_2|_1) \arrow[rrr, "\sigma"] \arrow[dr] \arrow[ddd] & & & \Delta(\mathcal{E}_j|_y, \mathcal{T}_1|_1) \arrow[dl] \arrow[ddd] \\
& \mathcal{L}|^*_{(\gamma_2\star\gamma_0)\cup(\gamma_2\star\gamma_0)} \otimes E_j|_{\gamma_2\star\gamma_0} \arrow[r, "\lambda(-\otimes p)^{tr}\otimes\mathrm{id}"] \arrow[d, "\tilde{\lambda}\otimes\mathrm{id}"'] & \mathcal{L}|^*_{(\gamma_2\star\gamma_0)\cup(\gamma_1\star\gamma_0)} \otimes E_j|_{\gamma_2\star\gamma_0} \arrow[d, "\tilde{\lambda}\otimes\mathrm{id}"] & \\
& \mathcal{L}|_{(\gamma_2\star\gamma_0)\cup(\gamma_2\star\gamma_0)} \otimes E_j|_{\gamma_2\star\gamma_0} \arrow[r, "\lambda(p\otimes-)\otimes\mathrm{id}"'] \arrow[dl, "\zeta_j"'] & \mathcal{L}|_{(\gamma_1\star\gamma_0)\cup(\gamma_2\star\gamma_0)} \otimes E_j|_{\gamma_2\star\gamma_0} \arrow[dr, "\zeta_j"] & \\
E_j|_{\gamma_2\star\gamma_0} \arrow[rrr, "\zeta_j(p\otimes-)"'] & & & E_j|_{\gamma_1\star\gamma_0}
\end{tikzcd}
$$

Figure 1.

$\mathcal{L}_{\beta\cup s_a(t)}$ and their isomorphisms are $\sigma_a|_{(t,\beta_1,\beta_2)} := \lambda_{\beta_1,\beta_2,s_a(t)}$, for $t \in [0,1]$ and paths $\beta, \beta_1, \beta_2 \in P_{x_0}M$ ending at $\gamma_a^t(1)$; see Remark 2.4. Let $\iota_a^*\mathcal{T}$ be the restrictions of $\mathcal{T}$ to trivializations along γ_a. Then, there exist parallel unit-length sections σ_a in $\Delta(\mathcal{T}_a, \iota_a^*\mathcal{T})$. They induce unit-length sections into $\Delta(\mathcal{T}_2|_0, \mathcal{T}_1|_0)$ and into $\Delta(\mathcal{T}_2|_1, \mathcal{T}_1|_1)$. Since $s_1(0) = s_2(0)$, we have $\mathcal{T}_1|_0 = \mathcal{T}_2|_0$ and can thus assume that the induced section into $\Delta(\mathcal{T}_2|_0, \mathcal{T}_1|_0)$ is the trivial one. The induced section into $\Delta(\mathcal{T}_2|_1, \mathcal{T}_1|_1)$ defines a 2-isomorphism $\sigma : \Delta(\mathcal{E}_j|_y, \mathcal{T}_2|_1) \Rightarrow \Delta(\mathcal{E}_j|_y, \mathcal{T}_1|_1)$. From the fact that the sections σ_k are parallel together with the definition of p, we conclude that σ is induced by

$$
T_1|_{(1,\beta)} = \mathcal{L}_{\beta,\gamma_1\star\gamma_0} \xrightarrow{\lambda(-\otimes p)} \mathcal{L}_{\beta,\gamma_2\star\gamma_0} = T_2|_{(1,\beta)}. \tag{40}
$$

We apply this to $\beta = \gamma_2 \star \gamma_0$ and obtain the diagram shown in Figure 1. Its outer vertical arrows are the maps of Remark 2.4 that have been used in order to define the isomorphisms of (30) over γ_1 and γ_2, respectively. The diagram in Figure 1 is commutative: the subdiagrams on the sides give the construction of Remark 2.4, the subdiagram on the bottom is the relation between ζ_j and λ from the definition of the 1-morphism $\mathcal{E}_j$, the subdiagram in the middle is obviously commutative, and the one on top is (40). We embed the diagram of Figure 1 twice as subdiagram B into a new diagram shown in Figure 2. In that diagram, subdiagram A commutes by construction of σ, and C commutes by definition of ϵ_j under regression. All other subdiagrams are obviously commutative. Since the equality $\mathcal{R}'_{ij}|_{\gamma_2}(\iota_2^*\mathcal{T}) = \mathcal{R}'_{ij}|_{\gamma_1}(\iota_1^*\mathcal{T})$ on top of this diagram is realized by $\phi'_{ij}|_{\gamma_1\cup\gamma_2}([\mathcal{T}, 1] \otimes -)$, it is now straightforward to conclude the commutativity of (39). $\square$

$$
\begin{tikzcd}
\mathcal{R}'_{ij}|_{\gamma_2}(\mathcal{T}_2) \arrow[dd, "\xi_{ij}|_{\gamma_2}"'] \arrow[ddr, equal] & \mathcal{R}'_{ij}|_{\gamma_2}(\iota_2^*\mathcal{T}) \arrow[l, "r_{\psi_2}"'] \arrow[rr, equal] \arrow[dr, equal] & & \mathcal{R}'_{ij}|_{\gamma_1}(\iota_1^*\mathcal{T}) \arrow[r, "r_{\psi_1}"] \arrow[dl, equal] & \mathcal{R}'_{ij}|_{\gamma_1}(\mathcal{T}_1) \arrow[dd, "\xi_{ij}|_{\gamma_1}"] \arrow[dl, equal] \\
& & \mathrm{Hom}(\Delta(\mathcal{E}_i|_x, \mathcal{T}|_0), \Delta(\mathcal{E}_j|_y), \mathcal{T}|_{\frac{1}{2}}) \arrow[dl, "\tilde{\sigma}_2"'] \arrow[dr, "\tilde{\sigma}_1"] & & \\
& \mathrm{Hom}(\Delta(\mathcal{E}_i|_x, \mathcal{T}_2|_0), \Delta(\mathcal{E}_j|_y, \mathcal{T}_2|_1)) \arrow[d, "(30)"] \arrow[rr, "\sigma\circ-"] & A & \mathrm{Hom}(\Delta(\mathcal{E}_i|_x, \mathcal{T}_1|_0), \Delta(\mathcal{E}_j|_y, \mathcal{T}_1|_1)) \arrow[d, "(30)"] & \\
& \mathrm{Hom}(E_i|_{\mathsf{c}_x\star\gamma_0}, E_j|_{\gamma_2\star\gamma_0}) \arrow[d, "(31)"] \arrow[rr, "\zeta_j(p\otimes-)\circ-"] & B \quad C & \mathrm{Hom}(E_i|_{\mathsf{c}_x\star\gamma_0}, E_j|_{\gamma_1\star\gamma_0}) \arrow[d, "(31)"] & \\
& \mathrm{Hom}(E_i|_{\gamma_0}, \mathcal{R}_{ij}|_{\gamma_2} \otimes_{\mathcal{A}_i|_x} E_i|_{\gamma_0}) \arrow[rr, "(\phi_{ij}(p\otimes-)\otimes\mathrm{id})\circ-"] & & \mathrm{Hom}(E_i|_{\gamma_0}, \mathcal{R}_{ij}|_{\gamma_1} \otimes_{\mathcal{A}_i|_x} E_i|_{\gamma_0}) & \\
\mathcal{R}_{ij}|_{\gamma_2} \arrow[ur, "(32)"] \arrow[rrr, "\phi_{ij}(p\otimes-)"'] & & & \mathcal{R}_{ij}|_{\gamma_1} \arrow[ul, "(32)"'] &
\end{tikzcd}
$$

Figure 2.

Lemma 6.6. *The bundle isomorphism ξ_{ij} respects the lifted path concatenations, i.e., the diagram*

$$
\begin{tikzcd}
\mathcal{R}'_{jk}|_{\gamma_{23}} \otimes \mathcal{R}'_{ij}|_{\gamma_{12}} \arrow[r, "\chi'_{ijk}"] \arrow[d, "\xi_{jk}\otimes\xi_{ij}"'] & \mathcal{R}'_{ik}|_{\gamma_{23}\star\gamma_{12}} \arrow[d, "\xi_{ik}"] \\
\mathcal{R}_{jk}|_{\gamma_{23}} \otimes \mathcal{R}_{ij}|_{\gamma_{12}} \arrow[r, "\chi_{ijk}"'] & \mathcal{R}_{ik}|_{\gamma_{23}\star\gamma_{12}}
\end{tikzcd}
$$

is commutative for all composable paths $\gamma_{12} \in P_{ij}$ and $\gamma_{23} \in P_{jk}$.

Proof. We choose a path γ_0 connecting x_0 with $\gamma_{12}(0)$, induce the section s of (29) along $\gamma_{23} \star \gamma_{12}$, and obtain a trivialization $\mathcal{T}_{13}$ of $(\gamma_{23} \star \gamma_{12})^*\mathcal{G}$. We have its restrictions to trivializations $\mathcal{T}_{12}$ and $\mathcal{T}_{23}$ over γ_{12} and γ_{23}, respectively; then we can use the definition of χ'_{ijk} under transgression, see Section 4.5. It is straightforward to see that $\mathcal{T}_{12}$ is induced by the section

$s_{12} := s \circ \iota_1$, and $\mathcal{T}_{23}$ is induced by the section $s_{23} := s \circ \iota_2$. While s_{12} is again of the form (29), and thus can be used to define ξ_{ij}, the section s_{23} is not of this form. Instead, we have $s_{23}(t) = (\gamma_{23}^t \star \gamma_{12}) \star \gamma_{12}$. In order to describe ξ_{jk}, we use the trivialization $\mathcal{T}'_{23}$ induced by the section s'_{23} defined by $s'_{23}(t) := \gamma_{23}^t \star (\gamma_{12} \star \gamma_0)$. In order to compare $\mathcal{T}_{23}$ and $\mathcal{T}'_{23}$, we consider the section $\tilde{s} : [0,1] \to \mathcal{L}$ along (s_{23}, s'_{23}), defined by

$$\tilde{s}(t) := d_{s_{23}(t) \cup s_{23}(t), s_{23}(t) \cup s'_{23}(t)}(\nu_{s_{23}(t)}).$$

Then, by Remark 2.4, we obtain a 2-isomorphism $\psi : \mathcal{T}_{23} \Rightarrow \mathcal{T}'_{23}$ and a commutative diagram

(41)

$$\begin{array}{ccccc}
\mathcal{R}'_{jk}|_{\gamma_{23}}(\mathcal{T}'_{23}) & = & \mathrm{Hom}(\Delta(\mathcal{E}_j|_y, \mathcal{T}'_{23}|_0), \Delta(\mathcal{E}_k|_z, \mathcal{T}'_{23}|_1)) & \longrightarrow & \mathrm{Hom}(E_j|_{\mathsf{c}_y \star (\gamma_{23} \star \gamma_{12})}, E_k|_{\gamma_{23} \star (\gamma_{12} \star \gamma_0)}) \\
\downarrow{\scriptstyle r_\psi} & & \downarrow{\scriptstyle \mathrm{Hom}(\psi_0, \psi_1)} & & \downarrow{\scriptstyle \mathrm{Hom}(\zeta_j(\tilde{s}(0) \otimes -), \zeta_k(\tilde{s}(1) \otimes -))} \\
\mathcal{R}'_{jk}|_{\gamma_{23}}(\mathcal{T}_{23}) & = & \mathrm{Hom}(\Delta(\mathcal{E}_j|_y, \mathcal{T}_{23}|_0), \Delta(\mathcal{E}_k|_z, \mathcal{T}_{23}|_1)) & \longrightarrow & \mathrm{Hom}(E_j|_{\gamma_{23} \star \gamma_{12}}, E_k|_{(\gamma_{23} \star \gamma_{12}) \star \gamma_0}).
\end{array}$$

Using a thin homotopy Γ_t between $s'_{23}(t) := \gamma_{23}^t \star (\gamma_{12} \star \gamma_0)$ and $s_{23}(t) = (\gamma_{23}^t \star \gamma_{12}) \star \gamma_0$, the fact that ζ_i is connection-preserving, and the fact that the canonical elements $\nu_{s_{23}(t)}$ are neutral under under ζ_i, one can show that $\zeta_j(\tilde{s}(0) \otimes -) = pt_{\Gamma_0}$ and $\zeta_k(\tilde{s}(1) \otimes -) = pt_{\Gamma_1}$, so that the morphism on the right hand side of the previous diagram is just $\mathrm{Hom}(pt_{\Gamma_0}, pt_{\Gamma_1})$. We now replace E_i by its explicit form as obtained from regression. In the first component, pt_{Γ_0} then becomes

$$d_{\mathsf{c}_y \star (\gamma_{12} \star \gamma_0), \gamma_{12} \star \gamma_0} \otimes_{\mathcal{A}_{x_0}} \mathrm{id}_{\mathcal{F}_0} : \mathcal{R}_{i_0 j}|_{\mathsf{c}_y \star (\gamma_{12} \star \gamma_0)} \otimes_{\mathcal{A}_{x_0}} \mathcal{F}_0 \to \mathcal{R}_{i_0 j}|_{\gamma_{12} \star \gamma_0} \otimes_{\mathcal{A}_{x_0}} \mathcal{F}_0,$$

which can be identified via (LBG4) with $\chi_{i_0 jj}|_{\gamma_{12} \star \gamma_0, \mathsf{c}_y}(\epsilon_j(y) \otimes -)^{-1}$. In the second component, pt_{Γ_1} becomes

$$\begin{aligned} d_{\gamma_{23} \star (\gamma_{12} \star \gamma_0), (\gamma_{23} \star \gamma_{12}) \star \gamma_0} \otimes_{\mathcal{A}_{x_0}} \mathrm{id}_{\mathcal{F}_0} : \mathcal{R}_{i_0 k}|_{\gamma_{23} \star (\gamma_{12} \star \gamma_0)} \otimes_{\mathcal{A}_{x_0}} \mathcal{F}_0 \\ \to \mathcal{R}_{i_0 k}|_{(\gamma_{23} \star \gamma_{12}) \star \gamma_0} \otimes_{\mathcal{A}_{x_0}} \mathcal{F}_0. \end{aligned}$$

In combination with the commutativity of diagram (41), this shows the commutativity of a diagram, which we find as a subdiagram in the upper left corner of the diagram shown in Figure 3. The commutativity of this diagram is what we want to show. The triangular subdiagrams commute by definition of one of their arrows. The four-sided diagram at the top is the definition of χ'_{ijk} under regression. The four-sided diagram below is just linear algebra, given that $s(0) = s_{12}(0)$, $s_{12}(1) = s_{23}(0)$, and

$$
\begin{tikzcd}
\mathcal{R}'_{jk}|_{\gamma_{23}}(\mathcal{T}'_{23}) \otimes \mathcal{R}'_{ij}|_{\gamma_{12}}(\mathcal{T}_{12}) \arrow[dd, "(30)"'] & \mathcal{R}'_{jk}|_{\gamma_{23}}(\mathcal{T}_{23}) \otimes \mathcal{R}'_{ij}|_{\gamma_{12}}(\mathcal{T}_{12}) \arrow[l, "r_{\mathcal{T}_{23},\mathcal{T}'_{23}}\otimes\mathrm{id}"'] \arrow[r, "\chi'_{ijk}"] \arrow[d, equal] & \mathcal{R}'_{ik}|_{\gamma_{23}\star\gamma_{12}}(\mathcal{T}) \arrow[dd, "(30)"] \arrow[d, equal] \\
 & \begin{array}{c}\mathrm{Hom}(\Delta(\mathcal{E}_j|_y, \mathcal{T}_{23}|_0), \Delta(\mathcal{E}_k|_z, \mathcal{T}_{23}|_1)) \\ \otimes\mathrm{Hom}(\Delta(\mathcal{E}_i|_x, \mathcal{T}_{12}|_0), \Delta(\mathcal{E}_j|_y, \mathcal{T}_{12}|_1))\end{array} \arrow[r, "\circ"'] \arrow[d, "\mathrm{Hom}(\varphi_{s_{23}(0)},\varphi_{s_{23}(1)})\otimes\mathrm{Hom}(\varphi_{s_{12}(0)},\varphi_{s_{12}(1)})"] & \mathrm{Hom}(\Delta(\mathcal{E}_i|_x, \mathcal{T}|_0), \Delta(\mathcal{E}_k|_z, \mathcal{T}|_1)) \arrow[dr, "\mathrm{Hom}(\varphi_{s(0)},\varphi_{s(1)})"'] \\
 & \begin{array}{c}\mathrm{Hom}(E_j|_{\gamma_{12}\star\gamma_0}, E_k|_{(\gamma_{23}\star\gamma_{12})\star\gamma_0}) \\ \otimes\mathrm{Hom}(E_i|_{\mathsf{c}_x\star\gamma_0}, E_j|_{\gamma_{12}\star\gamma_0})\end{array} \arrow[r, "\circ"'] & \mathrm{Hom}(E_i|_{\mathsf{c}_x\star\gamma_0}, E_k|_{(\gamma_{23}\star\gamma_{12})\star\gamma_0}) \arrow[dd, "(32)^{-1}\circ(31)"] \\
\begin{array}{c}\mathrm{Hom}(E_j|_{\mathsf{c}_y\star(\gamma_{12}\star\gamma_0)}, E_k|_{\gamma_{23}\star(\gamma_{12}\star\gamma_0)}) \\ \otimes\mathrm{Hom}(E_i|_{\mathsf{c}_x\star\gamma_0}, E_j|_{\gamma_{12}\star\gamma_0})\end{array} \arrow[dd, "(32)^{-1}\circ(31)"'] \arrow[ur, "\mathrm{Hom}(\chi^{-1}_{i_0jj},d)\otimes\mathrm{id}"] & \begin{array}{c}\mathrm{Hom}(E_j|_{\gamma_{12}\star\gamma_0}, \mathcal{R}_{ik}|_{\gamma_{23}\star\gamma_{12}} \otimes_{\mathcal{A}_i|_x} E_i|_{\gamma_0}) \\ \otimes\mathrm{Hom}(E_i|_{\gamma_0}, E_j|_{\gamma_{12}\star\gamma_0})\end{array} \arrow[ul, "\mathrm{Hom}(\mathrm{id},\chi_{i_0ik})\otimes\mathrm{Hom}(\chi_{i_0ii},\mathrm{id})"'] & \\
 & \begin{array}{c}\mathrm{Hom}(E_j|_{\gamma_{12}\star\gamma_0}, \mathcal{R}_{jk}|_{\gamma_{23}} \otimes_{\mathcal{A}_j|_y} \mathcal{R}_{ij}|_{\gamma_{12}} \otimes_{\mathcal{A}_i|_x} E_i|_{\gamma_0}) \\ \otimes\mathrm{Hom}(E_i|_{\gamma_0}, \mathcal{R}_{ij}|_{\gamma_{12}} \otimes_{\mathcal{A}_i|_x} E_i|_{\gamma_0})\end{array} \arrow[ul, "\mathrm{Hom}(\chi_{i_0jj},\chi_{i_0jk})\otimes\mathrm{Hom}(\chi_{i_0ii},\chi_{i_0ij})"'] \arrow[u, "\mathrm{Hom}(\mathrm{id},\chi_{ijk}\otimes\mathrm{id})\otimes\mathrm{Hom}(\mathrm{id},\chi_{i_0ij})"'] \arrow[d, "\mathrm{Hom}(\mathrm{id},\mathrm{id}\otimes\chi_{i_0ij})\otimes\mathrm{id}"'] & \\
 & \begin{array}{c}\mathrm{Hom}(E_j|_{\gamma_{12}\star\gamma_0}, \mathcal{R}_{jk}|_{\gamma_{23}} \otimes_{\mathcal{A}_j|_y} E_j|_{\gamma_{12}\star\gamma_0}) \\ \otimes\mathrm{Hom}(E_i|_{\gamma_0}, \mathcal{R}_{ij}|_{\gamma_{12}} \otimes_{\mathcal{A}_i|_x} E_i|_{\gamma_0})\end{array} & \\
\mathcal{R}_{jk}|_{\gamma_{23}} \otimes \mathcal{R}_{ij}|_{\gamma_{12}} \arrow[ur, "(32)"] \arrow[rr, "\chi_{ijk}"'] & & \mathcal{R}_{ik}|_{\gamma_{23}\star\gamma_{12}}
\end{tikzcd}
$$

Figure 3.

$s_{23}(1) = s(1)$. The Pentagon diagram in the middle has four tensor factors, split as $\mathrm{Hom}(-,-) \otimes \mathrm{Hom}(-,-)$, which commute separately. Indeed, commutativity in the first, third, and fourth factor is obvious, and in the second factor it is precisely the Pentagon diagram of (LBG2). Finally, there is a strangely shaped diagram at the lower right corner; this diagram commutes again by pure linear algebra. □

So far we have completed the definition of a LBG morphism (φ, ξ) between the transgression of the regression of a LBG object and this LBG object. It remains to prove that this construction depends naturally on the given LBG object; this is the content of the following lemma.

Lemma 6.7. *Let*

$$(\varphi, \xi) : (\mathcal{L}_1, \lambda_1, \mathcal{R}_1, \phi_1, \chi_1, \epsilon_1, \alpha_1) \to (\mathcal{L}_2, \lambda_2, \mathcal{R}_2, \phi_2, \chi_2, \epsilon_2, \alpha_2)$$

be a LBG morphism. We label the transgression of regressions of LBG objects and morphisms with primes, and denote by

$$\begin{aligned}(\varphi_1, \xi_1) &: (\mathcal{L}_1', \lambda_1', \mathcal{R}_1', \phi_1', \chi_1', \epsilon_1', \alpha_1') \to (\mathcal{L}_1, \lambda_1, \mathcal{R}_1, \phi_1, \chi_1, \epsilon_1, \alpha_1)\\ (\varphi_2, \xi_2) &: (\mathcal{L}_2', \lambda_2', \mathcal{R}_2', \phi_2', \chi_2', \epsilon_2', \alpha_2') \to (\mathcal{L}_2, \lambda_2, \mathcal{R}_2, \phi_2, \chi_2, \epsilon_2, \alpha_2)\end{aligned}$$

the LBG morphisms associated to the two LBG objects. Then, we have

$$(\varphi_2, \xi_2) \circ (\varphi', \xi') = (\varphi, \xi) \circ (\varphi_1, \xi_1).$$

Proof. Commutativity of the line bundle isomorphisms, $\varphi_2 \circ \varphi' = \varphi \circ \varphi_1$, has been shown in [Wal16, Section 6.2]. For the vector bundle isomorphisms, the relevant statement is the commutativity of the diagram

$$\begin{tikzcd} \mathcal{R}_{1,ij}'|_\gamma \arrow[r, "\xi_{ij}'"] \arrow[d, "\xi_{1,ij}"'] & \mathcal{R}_{2,ij}'|_\gamma \arrow[d, "\xi_{2,ij}"] \\ \mathcal{R}_{1,ij}|_\gamma \arrow[r, "\xi_{ij}"'] & \mathcal{R}_{2,ij}|_\gamma, \end{tikzcd} \tag{42}$$

for all paths $\gamma \in P_{ij}$. We pick an arbitrary $\gamma \in P_{ij}$ and set $x := \gamma(0)$ and $y := \gamma(1)$. For the construction of $\xi_{1,ij}$ and $\xi_{2,ij}$ we need to choose a path $\gamma_0 \in P_{i_0 i}$ with $\gamma_0(1) = x$, inducing sections s_1 and s_2 into the surjective submersions of $\gamma^*\mathcal{G}_1$ and $\gamma^*\mathcal{G}_2$, respectively, where $\mathcal{G}_1, \mathcal{G}_2$ are the regressed bundle gerbes, see (29). These sections, in turn, induce trivializations $\mathcal{T}_1$ and $\mathcal{T}_2$ of $\gamma^*\mathcal{G}_1$ and $\gamma^*\mathcal{G}_2$, respectively.

$$\begin{tikzcd}
\mathcal{R}'_{1,ij}|_\gamma(\mathcal{T}_1) \arrow[r, "\xi'_{ij}"] \arrow[d, "(30)"'] & \mathcal{R}'_{2,ij}|_\gamma(\mathcal{T}_2) \arrow[d, "(30)"] \\
\mathrm{Hom}(E_{1,i}|_{\mathsf{c}_x\star\gamma_0}, E_{1,j}|_{\gamma\star\gamma_0}) \arrow[r, "\mathrm{Hom}(\tilde\psi_i,\tilde\psi_j)"] \arrow[d, "(31)"'] & \mathrm{Hom}(E_{1,i}|_{\mathsf{c}_x\star\gamma_0}, E_{1,j}|_{\gamma\star\gamma_0}) \arrow[d, "(31)"] \\
\mathrm{Hom}(E_{1,i}|_{\gamma_0}, \mathcal{R}_{1,ij}|_\gamma \otimes_{\mathcal{A}_{1,i}|_x} E_{1,i}|_{\gamma_0}) \arrow[r, "\mathrm{Hom}(\xi_{i_0i}\otimes\mathrm{ev}_0^*f_0,\xi_{ij}\otimes\xi_{i_0i}\otimes\mathrm{ev}_0^*f_0)"] \arrow[d, "(32)"'] & \mathrm{Hom}(E_{2,i}|_{\gamma_0}, \mathcal{R}_{2,ij}|_\gamma \otimes_{\mathcal{A}_{2,i}|_x} E_{2,i}|_{\gamma_0}) \arrow[d, "(32)"] \\
\mathcal{R}_{1,ij}|_\gamma \arrow[r, "\xi_{ij}"'] & \mathcal{R}_{2,ij}|_\gamma.
\end{tikzcd}$$

Figure 4.

Let $(\mathcal{A},\psi)$ be the regression of (φ,ξ), i.e. $\mathcal{A}:\mathcal{G}_1\to\mathcal{G}_2$ is a 1-isomorphism induced from the line bundle morphism $\cup^*\varphi$, and $\psi=\{\psi_i\}_{i\in I}$ consists of 2-isomorphisms $\psi_i:\mathcal{E}_{1,i}\Rightarrow\mathcal{E}_{2,i}\circ\mathcal{A}$ induced from the vector bundle homomorphism $\tilde\psi_i:=\xi_{i_0i}\otimes\mathrm{ev}_0^*f_0:E_{1,i}\to E_{2,i}$, see Section 5. The transgression of $(\mathcal{A},\varphi)$ is defined by

$$\xi'_{ij}:\mathcal{R}_{1,ij}|_\gamma(\mathcal{T}_1)\to\mathcal{R}_{2,ij}|_\gamma(\mathcal{T}_2):\varphi\mapsto\psi_{i,1}\circ\varphi\circ\psi_{i,0}^{-1},$$

see Section 4.8. Evaluating the construction of the isomorphisms $\psi_{i,1}$ and $\psi_{i,0}$ in the present situation, using that the trivializations $\mathcal{T}_1$ and $\mathcal{T}_2$ are induced from sections, we obtain commutative diagrams

$$\begin{tikzcd}
\Delta(\mathcal{E}_{1,i}|_x,\mathcal{T}_1|_0) \arrow[r, "\psi_{i,0}"] \arrow[d, "\varphi_s"'] & \Delta(\mathcal{E}_{2,i}|_x,\mathcal{T}_2|_0) \arrow[d, "\varphi_s"] \\
E_{1,i}|_{\mathsf{c}_x\star\gamma_0} \arrow[r, "\tilde\psi_i"'] & E_{2,i}|_{\mathsf{c}_x\star\gamma_0}
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
\Delta(\mathcal{E}_{1,j}|_y,\mathcal{T}_1|_1) \arrow[r, "\psi_{i,1}"] \arrow[d, "\varphi_s"'] & \Delta(\mathcal{E}_{2,j}|_y,\mathcal{T}_2|_1) \arrow[d, "\varphi_s"] \\
E_{1,j}|_{\gamma\star\gamma_0} \arrow[r, "\tilde\psi_j"'] & E_{2,j}|_{\gamma\star\gamma_0}
\end{tikzcd}$$

where vertical arrows are the bundle morphisms of Remark 2.4. This shows the commutativity of the first subdiagram of the diagram shown in Figure 4. The diagram in the middle is commutative because ξ_{ij} is compatible with the lifted path concatenation, and the diagram at the bottom is obviously commutative. The outer shape of the diagram of Figure 4 is (42). $\square$

Appendix A. Vector bundles and algebra bundles over diffeological spaces

A.1. Diffeological vector bundles

In this section we provide the basics about vector bundles over diffeological spaces. For the definition we follow [IZ13, Art. 8.9]. Let X be a diffeological space.

Definition A.1. A *complex vector bundle* of rank k over X consists of the following structure:

(a) a diffeological space E, the total space,

(b) a smooth map $\pi : E \to X$, the projection,

(c) a complex vector space structure on each fiber $E|_x := \pi^{-1}(\{x\})$, for all $x \in X$.

The following condition has to be satisfied: for each plot $c : U \to X$ and each point $u \in U$ there exists an open neighborhood $u \in W \subseteq U$ and a diffeomorphism

$$\phi : W \times \mathbb{C}^k \to W \times_X E$$

that covers the identity map on W, and restricts to a linear map $\phi|_w : \mathbb{C}^k \to E|_{c(w)}$ over each point $w \in W$.

A *morphism* between vector bundles E and E' over X is a smooth map $\varphi : E \to E'$ that commutes with the projections and restricts to a linear map $\varphi|_x : E|_x \to E|'_x$ over each fiber. Hermitian vector bundles and unitary bundle morphisms are defined in the obvious analogous way.

As usual, vector bundles can be associated to principal bundles via representations. For principal bundles over diffeological spaces we use the definition of [Wal12].

Lemma A.2. *Suppose G is a Lie group and $\rho : G \to \mathrm{GL}(\mathbb{C}^k)$ is a Lie group homomorphism. Suppose further that P is a principal G-bundle over X. Let $E := P \times_G \mathbb{C}^k$ be equipped with the quotient diffeology, the projection induced from P, and the fibrewise vector space structure of $\mathbb{C}^k$. Then, E is a vector bundle over X.*

Proof. Since the projection $\pi : P \to X$ of a principal G-bundle is a subduction, every plot $c : U \to X$ and every point $u \in U$ admit an open neighborhood $u \in W \subseteq U$ with a lift: a plot $\tilde{c} : W \to P$ such that $\pi \circ \tilde{c} = c$. Then we define a local trivialization of E by

$$\phi(w, v) := (w, [\tilde{c}(w), v]).$$

This is obviously smooth and fiber-wise linear. An inverse is defined in the following way. Suppose $(w, [p, v]) \in W \times_X E$. Since P is a principal G-bundle, there exists a unique $g_{p,w} \in G$ such that $p = \tilde{c}(w) g_{p,w}$. We set $\phi^{-1}(w, [p, v]) := (w, \rho(g_{p,w})(v))$. It is easy to check that this is inverse to ϕ. In order to check the smoothness of ϕ^{-1}, let $d : W' \to W \times_X E$ be a plot, i.e. $d = (d_1, d_2)$ where $d_1 : W' \to W$ is a smooth map and $d_2 : W' \to E$ is a plot of E, such that $\pi \circ d_2 = c \circ d_1$. We have to show that $\phi^{-1} \circ d : W' \to W \times \mathbb{C}^k$ is smooth, which can be done locally. By definition of the quotient diffeology of E, W' can be covered by smaller open sets W'' such that $d_2|_{W''} = [p, v]$, where $p : W'' \to P$ is a plot of P and $v : W'' \to \mathbb{C}^k$ is smooth. Now we have to check that

$$\phi^{-1} \circ d|_{W''} : W'' \to W \times \mathbb{C}^k : w \mapsto (d_1(w), \rho(g_{p(w), d_1(w)})(v(w)))$$

is smooth. This follows from the definition of principal G-bundles, according to which the map $\delta : P \times_X P \to G$ that induces $g_{p,v} := \delta(p, v)$ is smooth. □

Remark A.3.

(a) If the image of ρ in Lemma A.2 is contained in $U(\mathbb{C}^k)$, then E is a hermitian vector bundle.

(b) It is easy to check that all familiar operations with vector bundles can be performed: pullback, tensor product, dual bundles, Hom-bundles etc.

(c) If X is a smooth manifold, considered as a diffeological space, and E is a vector bundle over X in the sense of Definition A.1, then there exists a unique smooth manifold structure on E that induces the given diffeology and gives E the structure of a smooth vector bundle over X. Indeed, this smooth manifold structure is defined via local trivializations of E, whose transition functions are smooth.

We continue with connections on vector bundles over diffeological spaces, for which no established definition exists. Since tangent vectors in diffeological spaces are notoriously difficult to handle, we define connections via their

parallel transport. First we recall the following prerequisite. By a *path* in X we understand a smooth map $\gamma : [0,1] \to X$ with sitting instants, and we denote by $PX \subseteq C^\infty([0,1], X)$ the space of paths, equipped with its natural diffeology. A smooth map f between smooth manifolds is said to have *rank* k if $\mathrm{rk}(\mathrm{d}f_x) \leq k$ for all points x in its domain. A smooth map has rank k if and only if the pullback of every $(k+1)$-forms vanishes [SW11, Lemma 4.2]. That condition makes sense for smooth maps between diffeological spaces, and we define the rank of maps between diffeological spaces in this way.

Definition A.4. Two paths $\gamma_1, \gamma_2 \in PX$ in a diffeological space X are called *thin homotopic*, if there exists a path $h \in PPX$ such that

(a) it is a homotopy: $h(0) = \gamma_1$ and $h(1) = \gamma_2$

(b) it fixes end-points: $h(s)(0) = \gamma_1(0) = \gamma_2(0)$ and $h(s)(1) = \gamma_1(1) = \gamma_2(1)$ for all $s \in [0,1]$.

(c) it is thin: the map $h^\vee : [0,1]^2 \to X : (s,t) \mapsto h(s)(t)$ has rank one.

Now we are in position to define a connection on a vector bundle E over a diffeological space X.

Definition A.5. A *connection* on E is a family of linear maps $pt_\gamma : E|_{\gamma(0)} \to E|_{\gamma(1)}$, for each path γ in X, such that the following conditions are satisfied:

(a) pt_γ depends only on the thin homotopy class of the path γ.

(b) $pt_{\gamma_2 \star \gamma_1} = pt_{\gamma_2} \circ pt_{\gamma_1}$ for all composable paths in X.

(c) for each local trivialization $\phi : W \times \mathbb{C}^k \to W \times_X E$ of E there exists a 1-form $\omega_\phi \in \Omega^1(W, \mathfrak{gl}(\mathbb{C}^k))$ such that for every $\gamma \in PW$ the diagram

$$\begin{tikzcd}
\mathbb{C}^k \arrow[r, "\exp(\omega_\phi)(\gamma)"] \arrow[d, "\phi|_{\gamma(0)}"'] & \mathbb{C}^k \arrow[d, "\phi|_{\gamma(1)}"] \\
E_{c(\gamma(0))} \arrow[r, "pt_{c\circ\gamma}"'] & E_{c(\gamma(1))}
\end{tikzcd}$$

is commutative, where $\exp(\omega_\phi)(\gamma) \in \mathrm{GL}(\mathbb{C}^k)$ is the path-ordered exponential of ω_ϕ along γ.

Remark A.6. Under the assumption that conditions (a) and (b) hold, (c) is equivalent to the following condition.

(c') for each local trivialization $\phi : W \times \mathbb{C}^k \to W \times_X E$ of E the map $p_\phi : PW \to \mathrm{GL}(\mathbb{C}^k)$, where $p_\phi(\gamma) \in \mathrm{GL}(\mathbb{C}^k)$ is the linear isomorphism

$$\mathbb{C}^k \xrightarrow{\phi|_{\gamma(0)}} E_{c(\gamma(0))} \xrightarrow{pt_{c\circ\gamma}} E_{c(\gamma(1))} \xrightarrow{\phi|_{\gamma(1)}^{-1}} \mathbb{C}^k,$$

is smooth.

The equivalence uses the theory of smooth functors [SW09]. Indeed, if ω_ϕ exists, then the map p_ϕ is $\exp(\omega_\phi) : PW \to \mathrm{GL}(\mathbb{C}^k)$, and hence smooth. Conversely, if p_ϕ is smooth, then it follows from (a) and (b) that it defines a smooth functor $p_\phi : \mathcal{P}_1(W) \to B\mathrm{GL}(\mathbb{C}^k)$, corresponding to a 1-form ω_ϕ such that $p_\phi = \exp(\omega_\phi)$.

If E is hermitian, then a connection is called *unitary* if pt_γ is unitary and the 1-forms ω_ϕ of all local trivializations ϕ take values in $\mathfrak{u}(n)$. A connection is called *flat* if pt_γ depends only on the homotopy class of γ.

Let E and E' be vector bundles over X equipped with connections pt and pt', respectively. A bundle morphism $\varphi : E \to E'$ is called *connection-preserving* if it commutes with the parallel transport, i.e. the diagram

$$\begin{tikzcd} E|_x \arrow[r, "pt_\gamma"] \arrow[d, "\varphi"'] & E|_y \arrow[d, "\varphi"] \\ E'|_x \arrow[r, "pt'_\gamma"'] & E'|_y \end{tikzcd}$$

is commutative for all paths $\gamma \in PX$, with $x := \gamma(0)$ and $y := \gamma(1)$.

We shall verify that our notion of a connection is compatible with the existing notion of a connection on a principal bundle (defined in [Wal12] as a Lie algebra-valued 1-form) under the associated bundle construction of Lemma A.2.

Lemma A.7. *Suppose P is a principal G-bundle over X and $\omega \in \Omega^1(P, \mathfrak{g})$ is a connection on P. We denote by $\tau^\omega_\gamma : P_{\gamma(0)} \to P_{\gamma(1)}$ the parallel transport along a path $\gamma \in PX$. Let $\rho : G \to \mathrm{GL}(\mathbb{C}^k)$ be a Lie group homomorphism. Then, the formula*

$$pt_\gamma([p, v]) := [\tau^\omega_\gamma(p), v]$$

defines a connection on the associated vector bundle $P \times_G \mathbb{C}^k$.

Proof. We use [Wal12, Prop. 3.2.10] for the properties of the parallel transport τ^ω. It is G-equivariant by item (b); hence pt_γ is well-defined. Further,

pt_γ is linear by construction. It is compatible with path composition due to item (a), and it only depends on the thin homotopy class due to item (b). It remains to check the compatibility with a local trivialization ϕ, obtained as described in Lemma A.2 as $\phi(w,v) := (w, [\tilde{c}(w), v])$. We set $\omega_\phi := \rho_*(\tilde{c}^*\omega)$. Suppose $\gamma \in PW$. Set $\tilde{\gamma} := \tilde{c} \circ \gamma$; this is a lift of $c \circ \gamma \in PX$. By [Wal12, Def. 3.2.9] we have $\tau^{\omega}_{c\circ\gamma}(\tilde{\gamma}(0)) = \tilde{\gamma}(1) \cdot \exp(\omega)(\tilde{\gamma})$. The calculus for path ordered exponentials implies that

$$\rho(\exp(\omega)(\tilde{\gamma})) = \exp(\rho_*(\tilde{c}^*\omega))(\gamma) = \exp(\omega_\phi)(\gamma). \tag{A.1}$$

Now, the commutativity of the diagram in Definition A.5 is straightforward to check. □

Remark A.8. If X is a smooth manifold, then a connection in the sense of Definition A.5 furnishes a transport functor [SW09]. These are equivalent to ordinary connections on ordinary vector bundles. Thus, our approach to vector bundles and connections over diffeological spaces reduces consistently to the classical theory over smooth manifolds.

Remark A.9. The treatment of curvature in terms of parallel transport involves transport 2-functors, and drops a bit out of the context of this article, see [SW09, Sec. 7.2] and [SW13, Sec. 3.4]. The content of [SW13, Lemma 3.4.3] is that the curvature of a connection on a vector bundle E is given locally by the endomorphism valued 2-form

$$\Omega_\phi := \mathrm{d}\omega_\phi + \frac{1}{2}[\omega_\phi \wedge \omega_\phi] \in \Omega^2(W, \mathfrak{gl}(\mathbb{C}^k)),$$

where $\omega_\phi \in \Omega^1(W, \mathfrak{gl}(\mathbb{C}^k))$ is the 1-form of a local trivialization. Globally, one can consider its trace, which is a globally-defined 2-form $\mathrm{tr}(\Omega_\phi) = \mathrm{tr}(\mathrm{d}\omega_\phi)$.

A.2. Superficial connections on path spaces

Let M be a smooth manifold, and let PM denote the diffeological space of paths in M with sitting instants. A path $\Gamma \in PPM$ is called *thin* if $\Gamma^\vee : [0,1]^2 \to M$ has rank one, and it is called *fixed-ends*, if the end-paths $s \mapsto \Gamma(s)(0)$ and $s \mapsto \Gamma(s)(1)$ are constant. A fixed-ends thin path Γ makes the paths $\Gamma(0), \Gamma(1) \in PM$ thin homotopic in the sense of Definition A.4.

Definition A.10. Two paths $\Gamma_1, \Gamma_2 \in PPM$ are called *rank-two-homotopic*, if there exists $h \in PPPM$ satisfying the following conditions:

(a) It is a homotopy, i.e. $h(0) = \Gamma_1$ and $h(1) = \Gamma_2$.

(b) It fixes the paths of end-points: for all $r, s \in [0,1]$ we have

$$h(r)(s)(0) = \Gamma_1(s)(0) = \Gamma_2(s)(0) \quad \text{and} \quad h(r)(s)(1) = \Gamma_1(s)(1) = \Gamma_2(s)(1).$$

(c) $h^\vee : [0,1]^3 \to M$ has rank two.

For such a homotopy we write $h_0 \in PPM$ for the path $h_0(r) := h(r)(0)$ connecting $\Gamma_1(0)$ with $\Gamma_2(0)$, and $h_1 \in PPM$ for the path $h_1(r) := h(r)(1)$ connecting $\Gamma_1(1)$ with $\Gamma_2(1)$. Note that h_0 and h_1 are fixed-ends paths by (b).

Definition A.11. Let E be a vector bundle over PM. A connection pt on E is called *superficial*, if the following two conditions are satisfied:

(i) Parallel transport along a fixed-ends thin path is independent of that path. More precisely, if $\Gamma_1, \Gamma_2 \in PPM$ are fixed-ends thin paths with $\Gamma_1(0) = \Gamma_2(0)$ and $\Gamma_1(1) = \Gamma_2(1)$, then $pt_{\Gamma_1} = pt_{\Gamma_2}$.

(ii) If $\Gamma_1, \Gamma_2 \in PPM$ are rank-two-homotopic via $h \in PPPM$, then the following diagram is commutative:

$$\begin{tikzcd}
E|_{\Gamma_1(0)} \arrow[r, "pt_{\Gamma_1}"] \arrow[d, "pt_{h_0}"'] & E|_{\Gamma_1(1)} \arrow[d, "pt_{h_1}"] \\
E|_{\Gamma_2(0)} \arrow[r, "pt_{\Gamma_2}"'] & E|_{\Gamma_2(1)}
\end{tikzcd}$$

Via Definition A.11 (i), a superficial connection on E determines for each pair (γ, γ') of thin homotopic paths a canonical map $d_{\gamma,\gamma'} : E|_\gamma \to E|_{\gamma'}$ by putting $d_{\gamma,\gamma'} := pt_\Gamma$ for some fixed-ends thin path Γ connecting γ with γ'. We also note that Definition A.11 (ii) implies that $pt_{\Gamma_1} = pt_{\Gamma_2}$ if the rank two homotopy h fixes the end-points.

Remark A.12. Two paths $\Gamma_1, \Gamma_2 \in PPM$ can be rank-two-homotopic in the sense of Definition A.10 or thin homotopic as paths in $X = PM$ in the sense of Definition A.4. In general, both conditions are different and none implies the other.

A.3. Bundles of algebras and bimodules

By an algebra we will always mean a unital, associative algebra over $\mathbb{C}$, and all algebra homomorphisms and representations will be unital. We first fix some terminology. Let X be a diffeological space.

(a) An *algebra structure* on a vector bundle E over X is a bundle morphism $\mu : E \otimes E \to E$ over X such that over each point $x \in X$ the map $\mu|_x : E|_x \otimes E|_x \to E|_x$ equips $E|_x$ with the structure of an algebra, and the section $x \mapsto 1_x$ of unit elements is smooth.

(b) An algebra structure on a vector bundle E is called *local*, if for each plot $c : U \to X$ and each point $x \in U$ there exist an algebra $A_{c,x}$, an open neighborhood $x \in V \subseteq U$ and a diffeomorphism $\phi : V \times A_{c,x} \to V \times_X E$ that induces the identity on V and its restriction $\phi|_v : A_{c,x} \to E|_{c(v)}$ to the fiber over each $v \in V$ is an algebra isomorphism.

(c) A vector bundle E with local algebra structure is called *algebra bundle* or *bundle of algebras*, if the algebras $A_{c,x}$ can be chosen independently of the plot c and the point x.

Analogous terminology will be used for various types of algebras, for instance, involutive algebras and Frobenius algebras. We remark that the necessity of carefully distinguishing between these types of bundles is *not* caused by the fact that we work over diffeological spaces; the same types exist over smooth manifolds and have to be distinguished.

The following results explain in a nice way the role of connections in relation to algebra bundles.

Lemma A.13. *Suppose a vector bundle E with algebra structure μ admits a connection pt for which μ is connection-preserving. Then, μ is local. Moreover, the restriction of E to each path-connected component of X is an algebra bundle.*

Proof. That μ is connection-preserving means that the isomorphisms $pt_\gamma : E|_x \to E|_y$ are algebra isomorphisms. Consider a local trivialization $\phi : V \times \mathbb{C}^k \to V \times_X E$ of the vector bundle E, for a plot $c : U \to X$ and $V \subseteq U$ a contractible open set. For a fixed smooth contraction of V to a point $x_0 \in V$, we obtain a smooth map $\gamma : V \to PV$ assigning to a point $x \in V$ a path γ_x from x_0 to x. We induce an algebra structure on $\mathbb{C}^k$ such that $\phi|_{x_0} : \mathbb{C}^k \to E|_{c(x_0)}$ is an algebra isomorphism, and denote that algebra by A_{c,x_0}. In general, $\phi|_x : A_{c,x_0} \to E|_{c(x)}$ is not an algebra homomorphism for

$x \neq x_0$. However, consider the new trivialization $\phi' : V \times A_{c,x_0} \to V \times_X E$ defined by $\phi'(x,v) := \phi(x, \exp(\omega_\phi)(\gamma_x)v)$, where ω_ϕ is a local connection 1-form for ϕ. We claim that $\phi'|_x : A_{c,x_0} \to E|_{c(x)}$ is an algebra homomorphism for all $x \in V$. Indeed, we get from Definition A.5 (c)

$$\phi'|_x = \phi|_x \circ \exp(\omega_\phi)(\gamma_x) = pt_{c\circ\gamma_x} \circ \phi|_{x_0},$$

and this is a composition of algebra homomorphisms. This shows that μ is local.

Now, fix an arbitrary plot $c_0 : U_0 \to X$, a point $x_0 \in U_0$, an algebra A_{c_0,x_0} with a local algebra trivialization ϕ_0 around x_0. For any other plot $c : U \to X$, $x \in U$ and local algebra trivialization $\phi : V \times A_{c,x} \to V \times_X E$ defined in $x \in V \subseteq U$, choose a path $\gamma \in PX$ connecting x_0 with x. Parallel transport and the algebra isomorphisms $\phi_0|_{x_0}$ and $\phi|_x$ determine an algebra isomorphism $A_{c_0,x_0} \cong A_{c,x}$. Pre-composing with ϕ produces a new local trivialization ϕ' defined over V with typical fiber A_{c_0,x_0}. Thus, E is an algebra bundle over the path-connected component of x_0. □

If X is not path-connected, then local algebra structures have non-isomorphic typical fibers over the different connected components, in general. However, since the underlying vector bundle has the same rank everywhere, all these algebras have the same dimension. Since simple algebras of the same dimension are necessarily isomorphic, we obtain the following.

Lemma A.14. *If E carries a local algebra structure such that for all $x \in X$ the algebra $E|_x$ is simple, then E is an algebra bundle.* □

Next, we introduce terminology for bundles of bimodules. Let A and B be vector bundles over X with algebra structures.

(a) An *A-B-bimodule structure* on a vector bundle M over X is a pair (λ, ρ) of vector bundle morphisms $\lambda : A \otimes M \to M$ and $\rho : M \otimes B \to M$ such that over each point $x \in X$ the linear maps $\lambda|_x : A|_x \otimes M|_x \to M|_x$ and $\rho|_x : M|_x \otimes B|_x \to M|_x$ define commuting left and right algebra actions.

(b) An A-B-bimodule structure is called *local*, if for each plot $c : U \to X$ and each $x \in U$ there exist an open neighborhood $x \in V \subseteq U$, algebras A_0 and B_0, an A_0-B_0-bimodule M_0 and diffeomorphisms

$$\phi_A : V \times A_0 \to V \times_X A, \quad \phi_B : V \times B_0 \to V \times_X B,$$
$$\phi_M : V \times M_0 \to V \times_X M$$

covering the identity on V, such that for each $v \in V$ the restrictions $\phi_A|_v : A_0 \to A|_{c(v)}$ and $\phi_B|_v : B_0 \to B|_{c(v)}$ are algebra isomorphisms, and the restriction $\phi_M|_v : M_0 \to M|_{c(v)}$ is a bimodule intertwiner (along $\phi_A|_v$ and $\phi_B|_v$).

(c) We say that M is a *bundle of A-B-bimodules*, or *A-B-bimodule bundle*, if A and B are bundles of algebras, with typical fibers A_0 and B_0, respectively, and M_0 can be chosen independently of c and x as an A_0-B_0-bimodule.

Lemma A.15. *Suppose A, B are vector bundles over X with connections and connection-preserving algebra structures, and suppose M is a vector bundle over X with connection and a connection-preserving A-B-bimodule structure (λ, ρ). Then, (λ, ρ) is local. Moreover, the restriction of M to each path-connected component is a bundle of A-B-bimodules.*

Proof. The proof is analogous to Lemma A.13 and left out for brevity. □

Lemma A.16. *Suppose A and B are bundles of simple algebras over X, and M is a vector bundle with a local A-B-bimodule structure (λ, ρ). Suppose further that M is faithfully balanced, i.e., the induced maps $\tilde{\lambda} : A \to \mathrm{End}_B(M)$ and $\tilde{\rho} : B \to \mathrm{End}_A(M)^{op}$ are fiber-wise isomorphisms. Then, M is a bundle of A-B-bimodules.*

Proof. Faithfully balanced bimodules establish Morita equivalences. The claim then follows from the statement that Morita equivalent simple algebras are Morita equivalent in a unique way (up to isomorphism). Indeed, every Morita equivalence has to be irreducible (as an $A \otimes B^{op}$-module), because if it was a direct sum of two bimodules, it would not be invertible. However, if A and B are simple, then $A \otimes B^{op}$ is again simple, so it has a unique irreducible module. □

Finally, we discuss the composition of bimodule bundles, i.e. their tensor product over an algebra bundle. This tends to be difficult and is treated in [KLW]. Let A, B, C be vector bundles over X with algebra structures, let M be a vector bundle with A-B-bimodule structure (λ_M, ρ_M), and let N be a vector bundle with B-C-bimodule structure (λ_N, ρ_N). Over each point $x \in X$ we consider the subspace $K_x \subseteq M|_x \otimes N|_x$ generated by elements of the form

$$\rho_M|_x(m \otimes b) \otimes n - m \otimes \lambda_N|_x(b \otimes n)$$

for all $m \in M|_x$, $n \in N|_x$ and $b \in B|_x$. We consider the disjoint union of the quotient spaces $(M|_x \otimes N|_x)/K_x$ for all $x \in X$ and denote it by $M \otimes_B N$. It will be equipped with the obvious left A-action λ_M and the right C-action ρ_N, and be equipped with the unique diffeology making the projection $M \otimes N \to M \otimes_B N$ a subduction. In general, $M \otimes_B N$ will not even be a vector bundle. Under assumptions of locality and semisimplicity, however, we have the following result [KLW, Thm. 4.2.6 & Cor. 3.1.12]:

Lemma A.17. *If the algebra structures on A, B, C are local and semisimple, and the bimodule structures on M and N are local, then $M \otimes_B N$ is a vector bundle, and (λ_M, ρ_N) is a local A-C-bimodule structure.*

Remark A.18. Consider again bundles of algebras A, B, C and bundles M of A-B-bimodules and N of B-C-bimodules. We assume that all bundles are equipped with connections, in such a way that the algebra structures and the bimodule structures are connection-preserving. Then, there is a naturally defined connection on $M \otimes_B N$, for which the A-C-bimodule structure is connection-preserving. In order to see this, we only have to observe that the sub-vector bundle U is invariant under parallel transport.

Appendix B. Fiber integration for smooth 2-functors

In this section we provide a result for the theory of smooth functors of [SW09, SW11]. It is used in Lemma 6.4 as one step to establish our main theorem, but also might be interesting in other contexts. Let X be a smooth manifold. We recall that there is a bijection

$$\mathcal{F}un^\infty(\mathcal{P}_1(X), B\mathrm{U}(1)) \cong \Omega^1(X), \tag{B.2}$$

where the left hand side consists of smooth functors defined on the smooth path groupoid of X with values in the Lie groupoid $B\mathrm{U}(1)$ (it has a single object and $\mathrm{U}(1)$ as the manifold of morphisms) [SW09, Prop. 4.7]. Analogously, there is a bijection

$$\mathcal{F}un^\infty(\mathcal{P}_2(X), BB\mathrm{U}(1)) \cong \Omega^2(X), \tag{B.3}$$

where the left hand side consists of smooth 2-functors defined on the smooth path 2-groupoid of X with values in the Lie 2-groupoid $BB\mathrm{U}(1)$ (one object, one 1-morphism, and $\mathrm{U}(1)$ as the manifold of 2-morphisms) [SW11, Theorem

2.21]. We define a ,,fiber integration map“ for smooth 2-functors, i.e. a map

$$\int_{[0,1]} : \mathcal{F}un^\infty(\mathcal{P}_2([0,1]\times X), BB\mathrm{U}(1)) \to \mathcal{F}un^\infty(\mathcal{P}_1(X), B\mathrm{U}(1)), \tag{B.4}$$

and prove the following result.

Proposition B.1. *The fiber integration map* (B.4) *corresponds under above bijections to the ordinary fiber integration of differential forms, i.e. the diagram*

$$\begin{tikzcd} \mathcal{F}un^\infty(\mathcal{P}_2([0,1]\times X), BB\mathrm{U}(1)) \arrow[r, "\int_{[0,1]}"] \arrow[d] & \mathcal{F}un^\infty(\mathcal{P}_1(X), B\mathrm{U}(1)) \arrow[d] \\ \Omega^2([0,1]\times X) \arrow[r, "\int_{[0,1]}"'] & \Omega^1(X) \end{tikzcd} \tag{B.5}$$

is commutative.

The fiber integration map (B.4) is defined as follows. Suppose F is a smooth 2-functor on $[0,1]\times X$, and $\gamma \in PX$ is a path in X. Then,

$$\left(\int_{[0,1]} F\right)(\gamma) := F(\Sigma_\gamma) \quad \text{with} \quad \Sigma_\gamma := (\mathrm{id}\times\gamma)_*(\Sigma_{1,1}). \tag{B.6}$$

Here, $\Sigma_{s,t}$ is the *standard bigon* in $[0,1]^2$, and $\mathrm{id}\times\gamma : [0,1]^2 \to [0,1]\times X$ pushes it to a bigon in $[0,1]\times X$. The standard bigon $\Sigma_{s,t}$ is the uniquely defined bigon that fills the rectangle spanned by $(0,0)$ and (s,t), see [SW11, Sec. 2.2.1]. We note that Σ_γ is a bigon between the path $(\gamma_1, \mathsf{c}_{\gamma(1)}) \star (\mathsf{c}_0, \gamma)$ and the path $(\mathsf{c}_1, \gamma) \star (\gamma_1, \mathsf{c}_{\gamma(0)})$, where γ_1 is the standard path in $[0,1]$.

Lemma B.2. (B.6) *defines a smooth functor* $\int_{[0,1]} F : \mathcal{P}_1(X) \to B\mathrm{U}(1)$.

Proof. We have to check that $\int_{[0,1]} F$ respects the composition, that it only depends on the thin homotopy class of γ, and that it is smooth. The latter follows directly from the smoothness of F. If $h : [0,1]^2 \to X$ is a thin homotopy between γ and γ', then it induces a homotopy

$$[0,1]^3 \to [0,1]\times X : (r,s,t) \mapsto (\mathrm{id}\times h(r,-))(\Sigma_{1,1}(s,t))$$

between $(\mathrm{id} \times \gamma)_*(\Sigma_{1,1})$ and $(\mathrm{id} \times \gamma')_*(\Sigma_{1,1})$. This homotopy has rank two, because h has rank one, and thus the product $\mathrm{id} \times h$ has rank two. By inspection one checks that it restricts to a thin homotopy between the boundary paths. This shows that Σ_γ and $\Sigma_{\gamma'}$ are rank-two-homotopic, which implies $F(\Sigma_\gamma) = F(\Sigma_{\gamma'})$. Finally, if γ and γ' are composable paths in X, then

$$\Sigma_{\gamma' \star \gamma} = (\Sigma_{\gamma'} \circ \mathsf{c}_\gamma) \bullet (\mathsf{c}_{\gamma'} \circ \Sigma_\gamma),$$

where c_γ is the constant bigon for γ, and $\bullet$ and $\circ$ denote the vertical and horizontal composition of 2-morphisms in $\mathcal{P}_2(X)$, respectively. Since F is a 2-functor and $BB\mathrm{U}(1)$-valued, this gives $F(\Sigma_{\gamma' \star \gamma}) = F(\Sigma_{\gamma'}) \cdot F(\Sigma_\gamma)$. $\square$

Now we prove the commutativity of the diagram (B.5). Consider a point $x \in X$ and a tangent vector $v \in T_x X$ represented by a curve $\gamma : \mathbb{R} \to X$ with $\gamma(0) = x$. Using the description of the bijections (B.2) and (B.3), and starting with a 2-functor F, following diagram (B.5) clockwise yields

$$-\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 \left(\int_{[0,1]} F \right) (\gamma_*(\gamma_t)) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 F(\Sigma_{\gamma_*(\gamma_t)}),$$

where γ_t is the standard path in $\mathbb{R}$ from 0 to t. Counter-clockwise, we have

$$-\int_0^1 \mathrm{d}r \; \frac{\mathrm{d}^2}{\mathrm{d}s\mathrm{d}t}\bigg|_{0,0} F(\Gamma_r(\Sigma_{s,t})),$$

where $\Gamma_r : \mathbb{R}^2 \to [0,1] \times X$ represents the tangent vectors $\partial_r \in T_r[0,1]$ and $v \in T_x X$; for instance, we can put $\Gamma_r(s,t) := (r+s, \gamma(t))$. In order to compare these two expressions, we consider the map

$$\tilde{\Gamma}_r : \mathbb{R}^2 \to \mathbb{R}^2 : (s,t) \to (r+s, t).$$

Then, we have

$$(\tilde{\Gamma}_r(\Sigma_{s,t}) \circ \mathsf{c}) \bullet (\mathsf{c} \circ \Sigma_{r,t}) = \Sigma_{s+r,t}$$

as bigons in $\mathbb{R}^2$. Further, $\Gamma_r = (\mathrm{id} \times \gamma) \circ \tilde{\Gamma}_r$, and thus

$$(\Gamma_r(\Sigma_{s,t}) \circ \mathsf{c}) \bullet (\mathrm{id} \times \gamma)(\mathsf{c} \circ \Sigma_{r,t}) = (\mathrm{id} \times \gamma)(\Sigma_{s+r,t}).$$

We apply F and obtain

$$F(\Gamma_r(\Sigma_{s,t})) \cdot F((\mathrm{id} \times \gamma)(\Sigma_{r,t})) = F((\mathrm{id} \times \gamma)(\Sigma_{s+r,t})).$$

Differentiating, we get

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} F(\Gamma_r(\Sigma_{s,t})) = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=r} F((\mathrm{id}\times\gamma)(\Sigma_{s,t}))\cdot F((\mathrm{id}\times\gamma)(\Sigma_{r,t}))^{-1}.$$

This is the pullback of the Maurer-Cartan form on U(1) along $r \mapsto F((\mathrm{id}\times\gamma)(\Sigma_{r,t}))$; hence we have

$$\exp\left(\int_0^1 \mathrm{d}r\, \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_0 F(\Gamma_r(\Sigma_{s,t}))\right) = F((\mathrm{id}\times\gamma)(\Sigma_{1,t})).$$

Taking the derivative with respect to t and evaluating at $t=0$ gives

$$\int_0^1 \mathrm{d}r\, \frac{\mathrm{d}^2}{\mathrm{d}s\mathrm{d}t}\bigg|_{0,0} F(\Gamma_r(\Sigma_{s,t})) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 F((\mathrm{id}\times\gamma)(\Sigma_{1,t})).$$

It remains to observe that the bigons $\Sigma_{\gamma_*(\gamma_t)} = (\mathrm{id}\times\gamma_*(\gamma_t))(\Sigma_{1,1})$ and $(\mathrm{id}\times\gamma)(\Sigma_{1,t})$ are thin homotopic, which is straightforward to see.

Appendix C. Pullback of bundle gerbes along rank-one maps

We provide the following, general result about bundle gerbes and bundle gerbe morphisms. We need it in the proof of Proposition 4.5.

Theorem C.1. *Let M be a smooth manifold, X be a compact smooth manifold, and $\phi : X \to M$ be a smooth map of rank at most one.*

(a) If $\mathcal{G}$ is a bundle gerbe with connection over M, then its pullback along ϕ admits a parallel trivialization, i.e. a 1-isomorphism $\mathcal{T} : \phi^*\mathcal{G} \to \mathcal{I}_0$ in $\mathcal{G}rb^\nabla(X)$.

(b) If $\mathcal{A} : \mathcal{G} \to \mathcal{G}'$ is a 1-morphism in $\mathcal{G}rb^\nabla(M)$, and $\mathcal{T} : \phi^*\mathcal{G} \to \mathcal{I}_0$ and $\mathcal{T}' : \phi^*\mathcal{G}' \to \mathcal{I}_0$ are parallel trivializations, then there exists a hermitian vector bundle E with flat connection, and a 2-isomorphism in $\mathcal{G}rb^\nabla(X)$:

$$\begin{tikzcd}
\phi^*\mathcal{G} \arrow[r, "\phi^*\mathcal{A}"] \arrow[d, "\mathcal{T}"'] & \phi^*\mathcal{G}' \arrow[d, "\mathcal{T}'"] \arrow[dl, Rightarrow] \\
\mathcal{I}_0 \arrow[r, "E"'] & \mathcal{I}_0
\end{tikzcd}$$

Before proving Theorem C.1, we shall point out the following corollary, which appeared already as [Wal16, Prop. 3.3.1]. It follows from Theorem C.1

(a) and the definition of surface holonomy as the exponential of the integral over X of the 2-form ρ of any trivialization $\mathcal{T} : \phi^*\mathcal{G} \to \mathcal{I}_\rho$.

Corollary C.2. *If X is an oriented closed surface and $\mathcal{G}$ is a bundle gerbe with connection over M, then the surface holonomy of $\mathcal{G}$ around ϕ is trivial.* □

In the remainder of this section we proof Theorem C.1. We work on the level of cocycle data, with respect to an open cover $\{U_\alpha\}_{\alpha\in A}$ of M. A bundle gerbe $\mathcal{G}$ with connection is given by a triple (B, A, g), where $g_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \to \mathrm{U}(1)$, $A_{\alpha\beta} \in \Omega^1(U_\alpha \cap U_\beta)$, and $B_\alpha \in \Omega^2(U_\alpha)$ satisfy the cocycle conditions

$$\begin{aligned} &B_\beta - B_\alpha = \mathrm{d}A_{\alpha\beta}, \quad A_{\beta\gamma} - A_{\alpha\gamma} + A_{\alpha\beta} = \mathrm{dlog}(g_{\alpha\beta\gamma}), \\ &\text{and} \quad g_{\beta\gamma\delta}\, g_{\alpha\beta\delta} = g_{\alpha\gamma\delta}\, g_{\alpha\beta\gamma}. \end{aligned}$$

We can assume that each cocycle is normalized in the sense that $A_{\alpha\alpha} = 0$ and $g_{\alpha\beta\gamma} = 1$ whenever $|\{\alpha, \beta, \gamma\}| < 3$. With respect to cocycles (B, A, g) and (B', A', g') for bundle gerbes $\mathcal{G}$ and $\mathcal{G}'$, respectively, a 1-morphism $\mathcal{A} : \mathcal{G} \to \mathcal{G}'$ is given by a pair (Π, G), where $G_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{U}(n)$ and $\Pi_\alpha \in \Omega^1(U_\alpha, \mathfrak{u}(n))$ satisfy

$$\begin{aligned} B'_\alpha &= B_\alpha + \tfrac{1}{n}\mathrm{tr}(\mathrm{d}\Pi_\alpha) \\ \Pi_\beta + A_{\alpha\beta} &= A'_{\alpha\beta} + \mathrm{Ad}^{-1}_{G_{\alpha\beta}}(\Pi_\alpha) + \mathrm{dlog}(G_{\alpha\beta}) \\ G_{\alpha\gamma} \cdot g_{\alpha\beta\gamma} &= g'_{\alpha\beta\gamma} \cdot G_{\alpha\beta} \cdot G_{\beta\gamma}; \end{aligned}$$

see, e.g. [Gaw05]. Here, n is the rank of the vector bundle of the 1-morphism, and we regard $\mathrm{U}(1) \subseteq \mathrm{U}(n)$ as the central subgroup, and similarly we view $\mathbb{R} \subseteq \mathfrak{u}(n)$ as diagonal matrices.

We construct a refinement of the open cover $\{U_\alpha\}_{\alpha\in A}$ with properties adapted to the map $\phi : X \to M$. Let $K := \phi(X) \subseteq M$ be equipped with the subspace topology, so that $\{U_\alpha \cap K\}_{\alpha\in A}$ is an open cover of K. By [Sar65, Theorem 2] and [Chu63, Proposition 1.3] the covering dimension of K is $\dim(K) \leq 2$. Thus, there exists a refinement $\{V_\alpha\}_{\alpha\in B}$ consisting of open sets $V_\alpha \subseteq K$ and a refinement map $r : B \to A$ with $V_\alpha \subseteq U_{r(\alpha)} \cap K$, such that all non-trivial 3-fold intersections are empty. Since V_α is open in K, there exists an open set $\tilde{V}_\alpha \subseteq M$ such that $V_\alpha = \tilde{V}_\alpha \cap K$. Now we collect all open sets $\tilde{V}_\alpha$ and all sets $U_\alpha \setminus K$, which are open since $K \subseteq M$ is closed, as it is a compact subset of a Hausdorff space. This results in a new open cover

$\{W_\alpha\}_{\alpha\in C}$ of M that is a refinement of the original one, and no point $x \in K$ is contained in an intersection $W_\alpha \cap W_\beta \cap W_\gamma$ with α, β, γ distinct.

Since we have a refinement, we can assume that our cocycles (B, A, g), (B', A', g') and (Π, G) are defined with respect to $\{W_\alpha\}_{\alpha\in C}$. Now we prove (a). Let $\{\psi_\alpha\}_{\alpha\in C}$ be a smooth partition of unity subordinated to the open cover $\{W_\alpha\}_{\alpha\in C}$. We define

$$\rho_\alpha := \sum_{\beta\in C} \psi_\beta A_{\alpha\beta} \in \Omega^1(W_\alpha)$$

and check that

$$\rho_\alpha - \rho_\beta = A_{\alpha\beta} + \sum_{\gamma\in C} \psi_\gamma \mathrm{dlog}(g_{\alpha\beta\gamma}).$$

We change our cocycle (B, A, g) by the 1-forms ρ_α, and obtain an equivalent cocycle $(\tilde{B}, \tilde{A}, g)$ with $\tilde{B}_\alpha = B_\alpha + \mathrm{d}\rho_\alpha$ and $\tilde{A}_{\alpha\beta} = A_{\alpha\beta} - \rho_\alpha + \rho_\beta$. Then we perform the pullback along ϕ. The fact that ϕ is of rank one implies $\phi^*\tilde{B}_\alpha = 0$. Since there are no non-trivial 3-fold intersections, we have $\phi^* g_{\alpha\beta\gamma} = 1$, and the above calculation implies $\phi^*\tilde{A}_{\alpha\beta} = 0$. Thus, the pullback results in $(0, 0, 1)$, which is a cocycle for $\mathcal{I}_0$. Translating between cocycle data and geometric objects, this implies the existence of the parallel trivialization $\mathcal{T}$.

It remains to prove (b). Under the change of local data from (B, A, g) to $(\tilde{B}, \tilde{A}, g)$, and similarly for (B', A', g'), we obtain new local data $(\tilde{\Phi}, \tilde{G})$ for the 1-morphism. Pulling back along ϕ, we obtain local data $(\phi^*\tilde{\Pi}, \phi^*\tilde{G})$ for a 1-endomorphism of $(0, 0, 1)$. Since ϕ is of rank one, we have $\mathrm{d}(\phi^*\tilde{\Pi}) = \phi^*(\mathrm{d}\tilde{\Pi}) = 0$, and the absence of non-trivial 3-fold intersections implies the usual cocycle condition for $\phi^*\tilde{G}$. Thus, $(\phi^*\tilde{\Pi}, \phi^*\tilde{G})$ is local data for a vector bundle E with flat connection over X.

Table of Notation

LBG	loop space brane geometry, Section 2.1
TBG	target space brane geometry, Section 2.2
PX	the space of smooth paths in X with sitting instants
$\gamma_1 \star \gamma_2$	denotes the concatenation of paths
$\overline{\gamma}$	denotes the reversed path
c_x	denotes the constant path at a point x
$\gamma_1 \cup \gamma_2$	the loop $\overline{\gamma_2} \star \gamma_1$, when γ_1 and γ_2 have a common initial point and a common end point
d_{γ_1,γ_2}	the parallel transport of a superficial connection along a (arbitrary) thin path connecting γ_1 with γ_2
$\Delta(\mathcal{E},\mathcal{F})$	a vector bundle obtained from two twisted vector bundles, see (4)
$\overline{V}$	the complex conjugate vector space
V^*	the dual vector space, $V^* := \mathrm{Hom}(V,\mathbb{C})$
φ^*	the adjoint of a linear map between complex inner product spaces
S^1	the circle, $S^1 = \mathbb{R}/\mathbb{Z}$.
ι_1	the map $[0,1] \to S^1 : t \mapsto \frac{1}{2}t$
ι_2	the map $[0,1] \to S^1 : t \mapsto 1 - \frac{1}{2}t$
i_x	for $x \in X$, is the map $[0,1] \to X \times [0,1] : t \mapsto (x,t)$
j_t	for $t \in [0,1]$ and some space X, is $X \to X \times [0,1] : x \mapsto (x,t)$
$f^\vee$	for $f : X \to C^\infty(Y,Z)$, denotes the adjoint map $X \times Y \to Z : (x,y) \mapsto f(x)(y)$.

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