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Heterotic/F-theory duality and Narasimhan-Seshadri equivalence

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Finding the F-theory dual of a Heterotic model with Wilson-line symmetry breaking presents the challenge of achieving the dual $\mathbb{Z}_2$-action on the F-theory model in such a way that the $\mathbb{Z}_2$-quotient is Calabi-Yau with an Enriques GUT surface over which $SU(5)_{gauge}$ symmetry is maintained. We propose a new way to approach this problem by taking advantage of a little-noticed choice in the application of Narasimhan-Seshadri equivalence between real E_8-bundles with Yang-Mills connection and their associated complex holomorphic $E_8^{\mathbb{C}}$-bundles, namely the one given by the real outer automorphism of $E_8^{\mathbb{C}}$ by complex conjugation. The triviality of the restriction on the compact real form E_8 allows one to introduce it into the $\mathbb{Z}_2$-action, thereby restoring E_8- and hence $SU(5)_{gauge}$ -symmetry on which the Wilson line can be wrapped.

1. Introduction

Duality between Heterotic models and F-theory models in String Theory begins with the compact real form E_8 of the simple complex algebraic group $E_8^{\mathbb{C}}$. On the Heterotic side one begins with a Calabi-Yau threefold V_3 elliptically fibered over a smooth del Pezzo surface B_2. V_3 comes equipped with two bundles

$$F_a \oplus F_b$$

each endowed with a Yang-Mills connection with structure group the compact real group E_8. The connection determines and is determined by its restriction to each elliptic fiber E_{b_2} for $b_2 \in B_2$.

F-theory begins with an elliptically fibered Calabi-Yau fourfold W_4 elliptically fibered over a Fano threefold B_3 with origins in the idealized Calabi-Yau fourfold with equation

$$y^2 = x^3 + a_0 z^5 \tag{1.1}$$

where z and a_0 are sections of the anti-canonical bundle of B_3. The exceptional fibers of a crepant resolution of W_4 over points of $S_{\mathrm{GUT}} := \{z = 0\} \subseteq B_3$ correspond to the positive simple roots of E_8 intersecting as dictated by the E_8-Dynkin diagram, i.e. they map precisely to the exceptional fibers of the crepant resolution of the E_8 rational double point surface singularity

$$\left\{y^2 = x^3 + z^5\right\} \subseteq \mathbb{C}^3. \tag{1.2}$$

As we will see in the following, this resolution is constructed entirely within the product of the Lie algebra of the complex algebraic group $E_8^{\mathbb{C}}$ and its set of Borel subalgebras, the latter being a smooth complex projective manifold [11].

1.1. Duality in the presence of an order-2 element in $\pi_1\left(V_3^{\vee}\right)$

It is often convenient that the Heterotic model, that we will henceforth denote as $V_3^{\vee}$ have an unbranched (Calabi-Yau) double cover V_3, or, said otherwise, that a Heterotic model V_3 admit a freely acting involution with

Calabi-Yau quotient. In particular, after E_8-symmetry is broken to $SU(5)$-symmetry on V_3/B_2, the last step in symmetry-breaking to what physicists call the 'Standard Model' MSSM is accomplished by wrapping what is referred to in string theory literature as a 'Wilson line' on the $\mathbb{Z}_2$-quotient $V_3^\vee/B_2^\vee$. The $\mathbb{Z}_2$ must also act trivially on the Yang-Mills connections on $F_a \oplus F_b$ so that the Heterotic quotient $V_3^\vee/B_2^\vee$ is equipped with two E_8-bundles

$$F_a^\vee \oplus F_b^\vee, \tag{1.3}$$

each with an inherited Yang-Mills connection. Said otherwise, the quotient preserves the E_8-symmetry of the initial bundles, as well as its breaking to $SU(5)$-symmetry on which the Wilson line is wrapped.

On the F-theory side, the $\mathbb{Z}_2$-action may have orbifold singularities but must be free on the smooth surface $S_{\mathrm{GUT}} \subseteq B_3$. Again the $\mathbb{Z}_2$-action must preserve the initial E_8-symmetry, as well as the breaking to $SU(5)$.[1]

However, in order that the F-theory quotient be Calabi-Yau, any $\mathbb{Z}_2$-action on the F-theory side will have to incorporate the involution

$$\frac{dx}{y} \mapsto -\frac{dx}{y}$$

on the relative one-form on the fibers of the elliptically fibered F-theory model W_4/B_3. As is easily seen in (1.1), a consequence is that each E_8-root, as represented by an exceptional curve over a point of S_{GUT}, is sent to its negative. On the other hand, one must start with E_8-symmetry on the F-theory quotient quotient as well.

Our conclusion will be that, in fact, whatever the $\mathbb{Z}_2$-action on W_4/B_3 turns out to be, it will have to somehow incorporate the symmetry

$$-I_4 : \mathfrak{h}^{\mathbb{C}}_{SU(5)} \to \mathfrak{h}^{\mathbb{C}}_{SU(5)}$$

on the complexified Cartan subalgebra of $SU(5)$, the involution that sends each root to minus itself, an involution that is not an element of the Weyl group $W(SU(5))$ but one that does preserve the symmetry with respect to the compact real group $SU(5)$.

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[1]The implications of this last issue seem not to have been fully appreciated in the literature. Another issue connected with Wilson line breaking in F-theory is the existence of vector-like exotics. We will deal with that issue separately in a forthcoming paper.

1.2. The example of $SU(2)$

To illustrate how this can work, we illustrate in the simplest case. We consider the A_1 rational double point surface singularity

$$y^2 = xz.$$

This is a quotient singularity via the map from the (u, v)-plane given by

$$\begin{aligned} y &= uv \\ x &= u^2 \\ z &= v^2. \end{aligned}$$

Assigning u and v weight $1/2$ the versal deformation space is the weighted homogeneous space

$$y^2 = xz + a_2 \tag{1.4}$$

of weight two where the Casimir polynomial algebra on the complexified Cartan $\mathfrak{h}^{\mathbb{C}}_{SU(2)}$ has generator

$$a_2 : \mathfrak{h}^{\mathbb{C}}_{SU(2)} \to \frac{\mathfrak{h}^{\mathbb{C}}_{SU(2)}}{W(SU(2))}.$$

The base extension

$$\left| \begin{pmatrix} x & y + \sqrt{a_2} \\ -y + \sqrt{a_2} & z \end{pmatrix} \right| = 0$$

has equivariant crepant resolution parametrized by $\mathfrak{h}^{\mathbb{C}}_{SU(2)}$ and with exceptional fiber parametrized by either the ratio of the rows or the ratio of the columns of the above 2×2 matrix. Interchanging rows and columns by transposition is accomplished by replacing y with $-y$, that is, by acting on the singularity (1.4) by the automorphism

$$(x, y, z) \mapsto (x, -y, z).$$

Via the faithful spin representation given by the quaternions, we may consider $SU(2)$ as a real matrix group with complexification $SL(2; \mathbb{C})$. What will be relevant for us in this paper is the following commutative diagram

$$\begin{tikzcd} & SU(2) \arrow[dl] \arrow[dr] & \\ SL(2;\mathbb{C}) \arrow[rr, "\iota"] & & SL(2;\mathbb{C}) \end{tikzcd}$$

where ι is complex conjugation. The equivariant crepant resolution of (1.4) involves a choice of Weyl chamber identifying the exceptional curve of the resolution with the positive root. The 'flop' $y \mapsto -y$ interchanges the two possible choices of positive simple root corresponding to the two possible equivariant crepant resolutions. The necessity of introducing the base extension in order to equivariantly resolve the family (1.4) and the 'flop' $y \mapsto -y$ interchanging the two possible equivariant crepant resolutions lies at the heart of the problem of finding a canonical crepant resolution of the Tate form (4.5) defining an F-theory model W_4.

The action of ι on the $\mathfrak{h}^{\mathbb{C}}_{SU(2)}$ is therefore $h \mapsto -h$. Notice that this action should *not* be thought of as the action of $W(SU(2))$ since the action of ι is trivial on the compact real group $SU(2)$. Rather it should be thought of as the transformation that interchanges each root ϱ with its negative $-\varrho$. In fact for $n > 2$, ι does not act on roots as an element of the Weyl group, the difference of the actions being given by the non-trivial symmetry of the Dynkin diagram.

1.3. Heterotic E_8 versus F-theory $E_8^{\mathbb{C}}$

The fact that the Heterotic model relies on the properties of the real group E_8 while the F-theory model relies on properties of the complex algebraic group $E_8^{\mathbb{C}}$ is a central theme of this paper. More specifically, as explained below, the passage from the real to the complex group in constructing Heterotic/F-theory duality rests on the choice between the two possible complexifications of the same real bundle via Narasimhan-Seshadri equivalence. The $\mathbb{Z}_2$-action must reverse that choice if symmetry of the real group and the Calabi-Yau property are to be simultaneously preserved on the quotient.

1.4. Narasimhan-Seshadri equivalence

Let $G_{\mathbb{R}}$ denote a compact simple real Lie group and let $G_{\mathbb{C}}$ denote the simple complex algebraic group for which $G_{\mathbb{R}}$ is the compact real form. Narasimhan-Seshadri equivalence on a compact Riemann surface C equates homomorphisms

$$\pi_1(C) \to G_{\mathbb{R}}$$

and semi-stable $G_{\mathbb{C}}$ vector bundles on C. The relevant remark is that, since $G_{\mathbb{R}}$ has a faithful real linear representation, therefore $G_{\mathbb{C}}$ can be defined by extension of scalars, that is, replacing real matrix entries with complex ones. Therefore complex conjugation induces a real involution

$$\iota : G_{\mathbb{C}} \to G_{\mathbb{C}}$$

that leaves $G_{\mathbb{R}}$ pointwise fixed. ι is of course not complex analytic. Thus the Narasimhan-Seshadri equivalence implicitly makes a choice of one of the two possible complex structures on the flat $G_{\mathbb{C}}$-bundle. The purpose of this paper is to point out a situation in which $G_{\mathbb{C}} = E_8^{\mathbb{C}}$ and the choice matters. This happens when introducing $\mathbb{Z}_2$-actions on F-theory/Heterotic dual manifolds with the property that the respective quotient manifolds continue to be dual.

1.5. Outline of the paper

In F-theory the exceptional components of the fibers of a crepant resolution $\tilde{W}_4/B_3$ of W_4/B_3 are identified with a system of positive simple roots of $SU(5)$. What is often less attended to in the presence of a $\mathbb{Z}_2$-action is the trajectory of those roots as initial E_8-symmetry is broken. In particular, on the Heterotic side the initial symmetry on $V_3^{\vee}/B_2^{\vee}$ is E_8-symmetry. Therefore a Heterotic dual $\tilde{W}_4^{\vee}/B_3^{\vee}$ should also manifest initial E_8-symmetry.

Section 2 is devoted to establishing the fact that, in order that $\tilde{W}_4^{\vee}/B_3^{\vee}$ be Calabi-Yau, the $\mathbb{Z}_2$-action must incorporate the standard involution

$$(x, y) \mapsto (x, -y) \tag{1.5}$$

on the Weierstrass form of the elliptic fibers of W_4/B_3. In addition it is shown how this last is compatible with the fact that the $\mathbb{Z}_2$-action on the Heterotic side that, as it must, incorporates the trivial involution

$$(x, y) \mapsto (x, y)$$

on the Weierstrass form of the elliptic fibers of the Heterotic model V_3/B_2. This Section also reviews the construction of the semi-stable limit in F-theory and the critical role that Narasimhan-Seshadri equivalence plays there.

Section 3 is devoted to showing that the one (real) involution on $E_8^{\mathbb{C}}$ that leaves E_8 pointwise fixed, namely complex conjugation, exactly reverses the sign of each E_8-root. That is, complex conjugation acts as minus the identity $(-I_8)$ on the Cartan subalgebra of $E_8^{\mathbb{C}}$.

Section 4 employs the Tate form for W_4/B_3 to imbed it in a family of rational double-point surface singularities that are in turn mapped into the semi-universal deformation of the E_8-rational double-point surface singularity

$$y^2 = x^3 + z^5.$$

Section 5 examines Brieskorn-Grothendieck equivariant crepant resolution of the semi-universal deformation of the E_8-rational double-point surface singularity. We show that the involution (1.5) is derived from the central involution $-I_8$ on the complex Cartan subalgebra $\mathfrak{h}_{E_8}^{\mathbb{C}}$ so that, by Section 3 it can be built into the $\mathbb{Z}_2$-action without breaking E_8-symmetry.

Section 6 tracks the $SU(5)$-roots, as manifest in the exceptional components of a general fiber of a crepant resolution $\tilde{W}_4/B_3$ of W_4/B_3 over S_{GUT}, back to their origins as E_8-roots exploiting the commutativity of the three-dimensional commutative diagram obtained by mapping the top row of

$$\begin{array}{ccc} SL(5;\mathbb{C}) & \rightarrow & E_8^{\mathbb{C}} \\ \uparrow & & \uparrow \\ SU(5) & \rightarrow & E_8 \\ \downarrow & & \downarrow \\ SL(5;\mathbb{C}) & \rightarrow & E_8^{\mathbb{C}} \end{array}$$

to the bottom row by the complex conjugate involution ι. It is exactly the commutativity of this diagram that allows us to claim that initial E_8-symmetry and subsequent $SU(5)$-symmetry are preserved on the F-theory quotient $W_4^{\vee}/B_3^{\vee}$.

Finally in Section 7 we state a conjecture that, if true, would derive from the Brieskorn-Grothendieck equivariant crepant resolution and the choice of positive Weyl chamber the construction of a 'canonical' crepant resolution of W_4/B_3.

2. *F*-theory/Heterotic Duality

2.1. Smooth elliptically-fibered Heterotic theory

The starting point in the construction of smooth Heterotic theory is an elliptically-fibered Calabi-Yau threefold V_3/B_2 over a smooth del Pezzo surface B_2 such that V_3/B_2 comes equipped with two two bundles

$$F_a \oplus F_b$$

with structure group the compact real group E_8 . Each bundle is endowed with a Yang-Mills connection, a connection determining and determined by its restriction to each elliptic fiber E_{b_2}. Each restriction is flat and therefore given by a homomorphism

$$\pi_1\left(E_{b_2}\right) \rightarrow E_8. \tag{2.1}$$

Since $\pi_1\left(E_{b_2}\right)$ is abelian, the image of the homomorphism can be conjugated into a maximal torus of E_8. So any semi-stable E_8-bundle with flat connection on E_{b_2} reduces to a unique homomorphism

$$\left\{\pi_1\left(E_{b_2}\right) \rightarrow \mathfrak{t}_{E_8} = \left(S^1\right)^8 \subseteq \left(\mathbb{C}^*\right)^8\right\}_{b_2 \in B_2}. \tag{2.2}$$

The exact sequence

$$0 \rightarrow \pi_{B_2}^* T_{B_2}^* \rightarrow T_{V_3}^* \rightarrow T_{V_3/B_2}^* \rightarrow 0$$

yields the equality

$$\det T_{V_3}^* = \pi_{B_2}^*\left(\det T_{B_2}^*\right) \otimes K_{V_3/B_2}.$$

Since V_3 is Calabi-Yau, $\det T_{V_3}^*$ is the trivial line bundle. Thus the $\mathbb{Z}_2$-action is either trivial on both of the right-hand factors or non-trivial on both. Castelnuovo's Rationality Criterion implies that there are no freely acting involutions on the del Pezzo surface B_2. Linearizing the action of B_2 around fixpoints yields the conclusion that either β_2 acts with finite fixpoint set and the action on relative one-forms in K_{V_3/B_2} is

$$\frac{dx}{y} \mapsto \frac{dx}{y}$$

or has a fixed curve along which the action on relative one-forms in K_{V_3/B_2} is

$$\frac{dx}{y} \mapsto \frac{-dx}{y}.$$

We will next see that the existence of an F-theory dual implies that the action of β_2 has only finite fixpoint set.

2.2. *F*-theory model

The starting point in the construction of F-theory is an elliptically fibered Calabi-Yau fourfold W_4/B_3 over a smooth Fano threefold B_3, itself fibered over the Heterotic B_2 with rational fibers. The F-theory model must be endowed with equivariant involutions

$$\begin{array}{ccc} W_4 & \xrightarrow{\tilde{\beta}_4} & W_4 \\ \downarrow & & \downarrow \\ B_3 & \xrightarrow{\beta_3} & B_3 \\ \downarrow & & \downarrow \\ B_2 & \xrightarrow{\beta_2} & B_2. \end{array}$$

Duality then requires that β_3 acts freely on the smooth anti-canonical divisor $S_{\mathrm{GUT}} \subseteq B_3$. Therefore S_{GUT} is a $K3$-surface and the quotient under the free $\mathbb{Z}_2$-action is an Enriques surface. Since S_{GUT} is ample, the involution β_3 can have only finite fixpoint set, a fact that in turn implies that the Heterotic β_2 can have only finite fixpoint set. As we have seen above, this implies that the involution $\tilde{\beta}_3$ on the Heterotic V_3 will have to act as

$$\frac{dx}{y} \mapsto \frac{dx}{y} \tag{2.3}$$

on relative one-forms in K_{V_3/B_2}.

The short exact sequence

$$0 \to \pi_{B_3}^* T_{B_3}^* \to T_{\tilde{W}_4}^* \to T_{\tilde{W}_4/B_3}^* \to 0 \tag{2.4}$$

of cotangent spaces to a crepant resolution $\tilde{W}_4/B_3$ of W_4/B_3 yields an equation

$$\det T_{\tilde{W}_4}^* = \pi_{B_3}^* \left(\det T_{B_3}^*\right) \otimes K_{\tilde{W}_4/B_3} \tag{2.5}$$

where $\det T^*_{B_3}$ has a meromorphic section ω_{GUT} with no zeros and simple pole along the $K3$-surface $S_{\mathrm{GUT}} \subseteq B_3$. Again the $\mathbb{Z}_2$-action is either trivial on both of the right-hand factors or non-trivial on both. The residue of ω_{GUT} is a nowhere vanishing holomorphic two-form on S_{GUT}. Since β_3 must act freely one concludes that

$$\beta_3^* (\omega_{\mathrm{GUT}}) = -\omega_{\mathrm{GUT}}. \tag{2.6}$$

So, in order that quotient $W_4^\vee$ be Calabi-Yau, (2.5) implies that the involution $\tilde{\beta}_4$ must act as

$$\frac{dx}{y} \mapsto -\frac{dx}{y} \tag{2.7}$$

on relative one-forms in $K_{\tilde{W}_4/B_3}$, that is, the involution $\tilde{\beta}_4$ will have to act as

$$(x, y) \mapsto (x, -y) \tag{2.8}$$

on the Weierstrass form on the fibers of W_4/B_3.

2.3. Preserving duality of the $\mathbb{Z}_2$-quotients from Heterotic to *F*-theory

The duality is realized by the canonical replacement of the restriction of the two bundles

$$F_a \oplus F_b$$

to each elliptic fiber E_{b_2} of V_3/B_2 by a union of elliptically fibered rational surfaces

$$(dP_a(b_2) \cup dP_b(b_2)) \to \mathbb{P}_{[a',a'']}(b_2) \cup \mathbb{P}_{[b',b'']}(b_2) \tag{2.9}$$

such that

$$dP_a(b_2) \cap dP_b(b_2) = E_{b_2}.$$

(2.9) is then a normal-crossing $K3$-surface elliptically fibered over the union of two $\mathbb{P}^1$'s meeting at a point. Taken together these normal-crossing $K3$-surfaces are the fibers of a fibration sequence

$$W_{4,0} \to B_{3,a} \cup B_{3,b} \to B_2$$

with total space a normal-crossing Calabi-Yau fourfold.

The above canonical replacement is permitted by three facts:

1) The Yang-Mills connections determine and are determined by their restrictions to flat connections on each elliptic fiber E_{b_2}.

2) The Narasimhan-Seshadri theorem allows replacement of the flat E_8-bundles

$$F_a(b_2) \oplus F_b(b_2)$$

on E_{b_2} with flat holomorphic $E_8^{\mathbb{C}}$-bundles

$$F_a^{\mathbb{C}}(b_2) \oplus F_b^{\mathbb{C}}(b_2)$$

where $E_8^{\mathbb{C}}$ is the algebraic group whose compact real form is E_8. (This is the point at which one of the two complexifications of the E_8-bundles is chosen. As we shall show below, the the $\mathbb{Z}_2$-action must incorporate a reversal of that choice in order that the F-theory quotient retain E_8-symmetry.)

3) In Section 4.5 of [6] Friedman-Morgan-Witten give us a classifying space for imbeddings of E_{b_2} into a rational elliptic surface $dP_9(b_2)$, each such corresponding canonically by a theorem of E. Looijenga [7] to an isomorphism class of flat $E_8^{\mathbb{C}}$-bundles F over E_{b_2}.

Namely one considers the family of 'dP_9-hypersurfaces'

$$\begin{aligned} y^2 = 4x^3 &- \left(g_2t^4 - \beta_1 st^3 - \ldots - \beta_4 s^4\right)x \\ &- \left(g_3t^6 - \alpha_2 s^2t^4 - \ldots - \alpha_6 s^6\right) \end{aligned} \tag{2.10}$$

in $\mathbb{P}^3_{1,1,2,3}$ parametrized by homogeneous forms α_j and β_j of weight j in a weighted projective space $\mathbb{P}^8_{1,2,2,3,3,4,4,5,6}$. Fixing the values of α_j and β_j we think of the solution set of (2.10) as a rational hypersurface in $\mathbb{P}^3_{1,1,2,3}$ with distinguished pencil

$$\gamma s + \delta t = 0. \tag{2.11}$$

The given elliptic curve E_{b_2} sits in each $dP_9(b_2)$ in (2.10) as the solution set to the equation

$$s = 0.$$

The associated sum of eight flat line bundles on E_{b_2} is given by the morphism

$$H_0^2(dP_9(b_2);\mathbb{Z}) \to \mathrm{Pic}^0(E_{b_2})$$

where $H_0^2(dP_9(b_2);\mathbb{Z})$ is the space of algebraic cycles on $dP_9(b_2)$ whose intersection number with E_{b_2} is zero. The intersection pairing on $H_0^2(dP_9(b_2);\mathbb{Z})$ is that of the E_8-Dynkin diagram.

Now (2.10) should be thought of as defining a fiber of a bundle or 'stack' over the moduli stack $\mathfrak{M}_{1,1}$ of elliptic curves given by their Weierstrass form. $\mathfrak{M}_{1,1}$ has a covering involution

$$((x,y),[s,t]) \mapsto ((x,-y),[-s,t]) \tag{2.12}$$

that lifts to dP_9-hypersurface involution

$$\begin{array}{c} ([s,t],(x,y),[\beta_1,\alpha_2,\beta_2,\alpha_3,\beta_3,\alpha_4,\beta_4,\alpha_5,\alpha_6]) \\ \downarrow \\ ([-s,t],(x,-y),[-\beta_1,\alpha_2,\beta_2,-\alpha_3,-\beta_3,\alpha_4,\beta_4,-\alpha_5,\alpha_6]) \end{array} \tag{2.13}$$

since the parity of the coefficients α_i and β_j in the weighted projective space $\mathbb{P}^8_{1,2,2,3,3,4,4,5,6}$ is matched by their degree. On the other hand (2.8) induced on $W_{4,0}/B_3$ and (2.3) induced on V_3/B_2 taken together will imply that at $s=0$ a 'logarithmic transform' (2.12) must be incorporated into the quotienting action along the elliptic fiber E_{b_2}. Only in this way can the two components of the quotient of

$$dP_a(b_2) \cup dP_b(b_2)$$

by the action of the involution retain the structure of dP_9's without multiple fibers.

The final step in passing from the Heterotic model to the F-theory model W_4/B_2 is then obtained by smoothing each normal-crossing $K3$-surface $dP_a(b_2) \cup dP_b(b_2)$ to obtain a smooth $K3$-surface fibered over the smoothing of $\mathbb{P}_{[a',a'']}(b_2) \cup \mathbb{P}_{[b',b'']}(b_2)$ to the fiber of B_3/B_2 over b_2. Thus one obtains a fibration sequence

$$W_4 \to B_3 \to B_2$$

that consists over generic $b_2 \in B_2$ of a smooth $K3$-surface elliptically fibered over the $\mathbb{P}^1$-fiber of B_3/B_2.

2.4. Compatibility of $\mathbb{Z}_2$-actions

The involution $\tilde{\beta}_4/\beta_3$ on W_4/B_3 with Calabi-Yau quotient $W_4^\vee/B_3^\vee$ must specialize to an involution $\tilde{\beta}_{4,0}/\beta_{3,0}$ on the semi-stable limit $W_{4,0}/(B_{3,a} \cup B_{3,b})$.

Key to understanding the $\mathbb{Z}_2$-action on $W_{4,0}$ is attending to the $\mathbb{Z}_2$-actions on the normal crossing $K3$-surfaces

$$(dP_a\,(b_2) \cup dP_b\,(b_2)) \to \mathbb{P}_{[a',a'']}\,(b_2) \cup \mathbb{P}_{[b',b'']}\,(b_2)$$

over the set Orb of fixpoints b_2 of the involution β_2. Since the action of the involution $\tilde{\beta}$ on V_3 is free, it must restrict to translation by a non-zero half-period on the elliptic fiber E_{b_2} over the fixed b_2.

Over each point of Orb, the involution $\tilde{\beta}_{4,0}/\beta_{3,0}$ induces compatible involutions on each of the two components $dP_{9,a}$ and $dP_{9,b}$ of the fiber. The $K3$-surface over a point $b_2 \in Orb$ induces an involution on each of the two dP_9-surfaces into which it splits in the semi-stable limit. The involution must specialize to translation by a given half-period δ on the fiber E_{b_2}, the intersection of the two dP_9's.

Lemma 1. *i) On the F-theory fiber of the semi-stable limit over a fixpoint of β_2, the involution*

$$([s,t]\,,(x,y)) \mapsto ([-s,t]\,,(x,-y))$$

in (2.13) acts on each of the two dP_9's. Therefore a section of the canonical bundle of the normal-crossing $K3$-surface away from its singular locus is given by setting $t = 1$ and writing the holomorphic two form

$$ds \wedge \frac{dx}{y}.$$

Therefore this form is locally invariant under the action of $\tilde{\beta}_4$ on $W_{4,0}$ since both factors ds and $\frac{dx}{y}$ are anti-invariant.

ii) On a small analytic neighborhood of E_{b_2}, the involution is given on each dP_9 by the so-called 'logarithmic transformation'

$$([s,t]\,,(x,y)) \mapsto ([-s,t]\,,((x,y)+\delta))\,. \tag{2.14}$$

However the canonical bundle near the crossing locus of the two dP_9's on the F-theory side is represented by two-form

$$d\log s \wedge \frac{dx}{y}$$

on each local component[2] *whose residue is the holomorphic one-form on the Heterotic side. is invariant under the action of* $\tilde{\beta}_4$ *on* $W_{4,0}$ *and so the its residual one-form*

$$\frac{dx}{y}$$

is invariant under the induced action of $\tilde{\beta}_3$ *as required on the Heterotic side.*

Proof. Since flat bundles on elliptic curves are invariant under translation, one sees by (2.10) that there are only two possibilities:

1) The E_8-bundles are pasted to themselves according to the identity isomorphism induced by translation by the half-period δ.

2) The pasting of each E_8-bundle incorporates the automorphism corresponding to the involution

$$([s,t]\,,(x,y)) \mapsto ([-s,t]\,,(x,-y))$$

given in (2.13) on each of the two dP_9's.

Possibility 1) is impossible since it would imply that the fiber of the fibration B_3/B_2 over the fixpoint would be pointwise invariant under the $\mathbb{Z}_2$-action. That would in turn imply that the $\mathbb{Z}_2$-action on the smooth anticanonical divisor $S_{\mathrm{GUT}} \subseteq B_3$ would also have fixpoints. This last eliminates the possibility of an F-theory quotient.

Possibility 2) however implies that $\mathbb{Z}_2$-action on B_3 has finite fixpoint set thereby allowing a free action on S_{GUT}. The $\mathbb{Z}_2$-action on the two dP_9-fibers over $b_2 \in Orb$ is then given on (2.10) by translation by the distinguished half-period of the common fiber $\{s_a = s_b = 0\}$ composed with (2.13). For any half-period δ of E_{b_2}, there is defined a so-called 'logarithmic transform' on the dP_9-fiber, that is, an involution that produces in the quotient a fiber of multiplicity two over $\left\{s^2 = 0\right\}$. Setting $t = 1$, the involution (2.14) takes the two-form

$$ds \wedge \frac{dx}{y}$$

to minus itself. On the other hand, the involution (2.13) leaves this same two-form invariant.

However if one removes the multiple fiber $E_{b_2}/\{(x,y) \equiv (x,y)+\delta\}$ from the quotient of (2.14) and removes the fiber E_{b_2} from the quotient $\frac{dP_9}{\{[s,t]\equiv[-s,t]\}}$,

[2]When smooth surfaces specialize to normal crossing surfaces, a holomorphic section of their canonical bundle specializes to a meromorphic section with logarithmic pole with cancelling residues on each of the two local components.

the remaining open surfaces are isomorphic. Then $\frac{dP_9}{\{[s,t]\equiv[-s,t]\}}$ corresponds to a flat E_8-bundle on $E_{b_2}/\left\{(x,y)\equiv(x,y)+\delta\right\}$ that pulls back to a flat E_8-bundle on E_{b_2} that is invariant under translation by δ. Along $s=0$ the meromorphic two-form

$$d\ln s\wedge\frac{dx}{y} \tag{2.15}$$

is invariant under (2.14) so that it must be the one that extends the the invariant two-form (2.13). Therefore its residue, dx/y is also invariant under the V_3-involution $\tilde{\beta}_3$. Said otherwise the quotient of the $\mathbb{Z}_2$-action yields the order-2 logarthmic transform of each of the two components

$$\left(dP_9^{\vee}/\mathbb{P}_{a,[s_a^2,t_a^2]}\right)\cup\left(dP_9^{\vee}/\mathbb{P}_{b,[s_b^2,t_t^2]}\right)$$

yielding the $\mathbb{Z}_2$-action of

$$\frac{dx}{y}\mapsto\frac{dx}{y}$$

on V_3/B_2. This action is necessary so that the Heterotic quotient be a Calabi-Yau threefold. Simultaneously the action is consistent with the $\mathbb{Z}_2$-action of

$$\frac{dx}{y}\mapsto-\frac{dx}{y} \tag{2.16}$$

on W_4/B_3 that is necessary so that the F-theory quotient be a Calabi-Yau fourfold. □

Lemma 1 is somewhat remarkable in its implications for the F-theory dual. Since the involution β_2 on B_2 has only finite fixpoint set, it acts with eigenvalue $(+1)$ on the canonical bundle of B_2 . So Lemma 1 says that for an F-theory dual with orbifold $\mathbb{Z}_2$ fundamental group[3], $\tilde{\beta}_4$ must act on the Weierstrass form of fibers of the F-theory dual by

$$(x,y)\mapsto(x,-y)\,.$$

Only in that way does the $\mathbb{Z}_2$-quotient become a Calabi-Yau fourfold.

[3]This will be useful for Wilson-line symmetry breaking.

3. Retaining E_8-symmetry

As we have just shown, thanks to the nature of the logarithmic transform above the fixpoints of β_2, the quotient

$$\frac{W_{4,0}}{\tilde{\beta}_{4,0}}$$

by the involution $\tilde{\beta}_{4,0}$ (induced on $W_{4,0}$ by the involution $\tilde{\beta}_4$ on W_4) retains the structure of the union of two dP_9's (without multiple fibers). Therefore quotienting the Heterotic model V_3/B_2 by $\tilde{\beta}_3/\beta_2$ does not break E_8-symmetry.

However (2.8) seems to imply that the involution $\tilde{\beta}_4/\beta_3$ does break E_8-symmetry on any crepant resolution $\tilde{W}_4/B_3$ of W_4/B_3. For example, in the idealized 'limit' example where the fibers of W_4/B_3 over points of S_{GUT} have E_8-singular fibers

$$y^2 = x^3 + a_0 z^5$$

the involution (2.8) sends each E_8-root as represented by the exceptional fibers of the crepant resolution to its negative. So the question becomes "How can one endow the F-theory model with an involution that leaves E_8 untouched but interchanges the E_8-roots with their negatives?"

We propose that the answer lies with the interchange, over each point $b_2 \in B_2$, of the two complex structures

$$\left(F_a^{\mathbb{C}} \oplus F_b^{\mathbb{C}}\right)\big|_{b_2}$$

on the elliptic curve E_{b_2} associated to the same flat real E_8-bundles

$$\left(F_a \oplus F_b\right)|_{b_2}.$$

In support of this proposal we cite the following Lemma.

Lemma 2. *For each root ρ of the compact real form $G_{\mathbb{R}}$ of a simple algebraic group $G_{\mathbb{C}}$, the involution ι exchanges the root space $\mathfrak{l}_\rho \subseteq g_{\mathbb{C}}$ with the root space $\mathfrak{l}_{-\rho} \subseteq g_{\mathbb{C}}$ and so acts as minus the identity on $[\mathfrak{l}_\rho, \mathfrak{l}_{-\rho}]$.*

Proof. For any pair of a root and its negative, consider the associated immersions

$$\begin{array}{ccc} SU(2) & \rightarrow & G_{\mathbb{R}} \\ \downarrow & & \downarrow \\ SL(2;\mathbb{C}) & \rightarrow & G_{\mathbb{C}}, \end{array} \quad . \tag{3.1}$$

It will suffice to show the assertion for the realization of the compact real form $SU(2)$ as the unit quaternions, its Lie algebra as the real vector space corresponding to the imaginary quaternions and the Lie algebra $\mathfrak{sl}(2;\mathbb{C})$ of the complex algebraic group $SL(2;\mathbb{C})$.

(*For physicists only*) That is, it will suffice to show the assertion for the roots of the compact real form $SU(2)$ with real Lie algebra the trace-zero hermitian 2×2 matrices with basis

$$T_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, -T_2 := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, T_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.2}$$

considered as the real subspace of the complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ of trace-zero 2×2 matrices with basis given by adding respective imaginary parts

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

With respect to the Cartan subalgebra generated by T_3 the root spaces are given by the eigenvectors

$$T_1 \pm i \cdot T_2$$

with real eigenvalues. These are exchanged by the action of Hermitian conjugation.

(*For mathematicians only*) That is, it will suffice to show the assertion for the roots of the compact real form $SU(2)$ with real Lie algebra the trace-zero skew-hermitian 2×2 matrices with basis

$$\mathbf{i} := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \mathbf{j} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathbf{k} := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

considered as the real subspace of the complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ of trace-zero 2×2 matrices with basis given by adding respective imaginary parts

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

With respect to the Cartan subalgebra generated by $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ the root spaces are given by the eigenvectors

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \pm \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \mathbf{j} \pm i \cdot \mathbf{k}.$$

These are then exchanged by the action of complex conjugation, and the roots are purely imaginary and so go to minus themselves under complex conjugation. □

Said otherwise, we retain E_8-symmetry under the $\mathbb{Z}_2$-action on the F- theory side by incorporating the involution ι into the $\mathbb{Z}_2$-action. In this way, we can admit the action (2.7) that forces the reversal of choice of Weyl chamber without breaking the symmetry with respect to the real group E_8 or with respect to the real group $SU(5)_{gauge}$.

Our claim is therefore that, in order to construct the F-theory dual of a Heterotic theory in which E_8-symmetry is preserved under a $\mathbb{Z}_2$-action, the quotient F-theory model can only be endowed with initial E_8-symmetry if the $\mathbb{Z}_2$-action incorporates the reversal of the choice of Weyl chamber. The reason is that the choice of Weyl chamber is used to identify a system of positive simple roots with exceptional components of the crepant resolution $\tilde{W}_4/B_3$ of the F-theory model W_4/B_3. Otherwise at the outset the $\mathbb{Z}_2$-action will simultaneously break E_8-symmetry on the F-theory dual while maintaining E_8-symmetry on the Heterotic model.

4. Breaking E_8-symmetry to $SU(5)$ in F-theory

Tracking the symmetry-breaking in F-theory and the Heterotic dual begins by breaking symmetry of E_8 to that of the first factor of a maximal subgroup

$$\frac{SU(5)_{gauge} \times SU(5)_{Higgs}}{\mathbb{Z}_5} \hookrightarrow E_8. \tag{4.1}$$

The inclusion (4.1) of rank-8 real compact semi-simple Lie groups yields an identification of maximal abelian subalgebras

$$\mathfrak{h}_{SU(5)_{gauge}} \times \mathfrak{h}_{SU(5)_{Higgs}} \to \mathfrak{h}_{E_8}, \tag{4.2}$$

an inclusion of Weyl groups

$$W\left(SU(5)_{gauge}\right) \times W\left(SU(5)_{Higgs}\right) \hookrightarrow W(E_8), \tag{4.3}$$

and a morphism

$$\mathfrak{h}^*_{E_8} \to \mathfrak{h}^*_{SU(5)_{gauge}} \times \mathfrak{h}^*_{SU(5)_{Higgs}} \tag{4.4}$$

of roots and of the respective rings of Casimir polynomials.

The associated symmetry-breaking is effected in terms of the 'Tate form'

$$wy^2 = x^3 + a_5xyw + a_4zx^2w + a_3z^2yw^2 + a_2z^3xw^2 + a_0z^5w^3 \tag{4.5}$$

of the defining equation for the F-theory model W_4/B_3. (4.5) defines W_4 as a hypersurface in a $\mathbb{P}^2$-bundle

$$P := \mathbb{P}\left(\mathcal{O}_{B_3} \oplus \mathcal{O}_{B_3}(2N) \oplus \mathcal{O}_{B_3}(3N)\right) \tag{4.6}$$

with homogeneous fiber coordinates $[w, x, y]$ over the base B_3. B_3 is a Fano manifold that we will assume to have very ample anti-canonical linear system whose generic divisor we denote by N and the the Calabi-Yau hypersurface $W_4 \subseteq P$ is completely determined by the choice of

$$z, a_0, a_2, a_3, a_4, a_5, \frac{y}{x} =: t \in H^0\left(K_{B_3}^{-1}\right).$$

By (2.6) we will require that

$$z \circ \beta_3 = -z$$

so that by (2.8)

$$a_j \circ \beta_3 = -a_j$$

for all j and

$$t \circ \beta_3 = -t.$$

Thus

$$z, a_0, a_2, a_3, a_4, a_5, \frac{y}{x} =: t \in H^0\left(K_{B_3}^{-1}\right)^{[-1]}, \tag{4.7}$$

the (-1)-eigenspace with respect to the involution β_3 on B_3.[4] We also initially assume that the a_2, a_3, a_4, a_5 are chosen generically in $H^0\left(K_{B_3}^{-1}\right)^{[-1]}$.

---

[4]For purposes of avoiding vector-like exotics in the F-theory quotient, we will always assume that

$$a_0 = -\sum_{j=2}^{5} a_j.$$

In particular the map

$$\psi_3 = (a_2, a_3, a_4, a_5) : B_3 \to \mathbb{P}^3$$

is a finite morphism with the defining equation for S_{GUT} given by a (smooth) generic hyperplane section

$$z = \sum_{j=2}^{5} \kappa_j a_j.$$

4.1. Tracking roots via rational double point surface singularities

Our device for tracking the behavior of roots begins by rewriting the equation of the GUT-surface S_{GUT} as

$$z = a_0 \cdot \sum_{j=2}^{5} \kappa_j c_j \tag{4.8}$$

where $c_j = a_j/a_0$. Letting

$$\begin{array}{c} B_3' := B_3 - \{a_0 = 0\} \\ W_4' := W_4 \times_{B_3} B_3' \end{array} \tag{4.9}$$

we divide (4.5) by a_0^6 and rescale by

$$\begin{array}{c} \frac{x}{a_0^2} \mapsto x \\ \frac{y}{a_0^3} \mapsto y \\ \frac{z}{a_0} \mapsto z \end{array}$$

to obtain

$$wy^2 = x^3 + c_5 xyw + c_4 zx^2 w + c_3 z^2 yw^2 + c_2 z^3 xw^2 + z^5 w^3 \tag{4.10}$$

with all entries invariant under the involution β_3 restricted to B_3'. In particular y now goes to y under the $\mathbb{Z}_2$-action, reflecting the fact that the Weyl chamber is no longer reversed when tracking the roots and $SU(5)$-symmetry is preserved! It is only when wrapping a Wilson line on the non-contractible loop on the $\mathbb{Z}_2$-quotient that symmetry is broken to that of the Standard Model [MSSM].

However to make this last equation compatible with the crepant resolution of W_4'/B_3', as in [4] where one has to interpret the c_j as Casimir polynomials giving the mapping

$$(c_2, c_3, c_4, c_5) : \mathfrak{h}_{SU(5)_{Higgs}} \to \frac{\mathfrak{h}_{SU(5)_{Higgs}}}{W\,(SU\,(5))} = \mathbb{C}^4,$$

in order to equivariantly resolve the family (4.10), one has to interpret the c_j as Casimir polynomials giving the mapping

$$(c_2, c_3, c_4, c_5) : \mathfrak{h}_{SU(5)_{gauge}} \to \frac{\mathfrak{h}_{SU(5)_{gauge}}}{W\,(SU\,(5))} = \mathbb{C}^4.$$

Then in order to preserve $SU\,(5)$-symmetry on the quotient of the $\mathbb{Z}_2$-action β_3, we must

1) replace each function c_j on $\mathfrak{h}_{SU(5)_{gauge}}$ with the composed function

$$c_j \circ (-I_4)$$

where $-I_4$ takes each root to minus itself,

and

2) send y to $-y$ reflecting the action of $-I_8$ on $\mathfrak{h}_{E_8}$.

This will be explained in more detail in what follows.

By setting $w = 1$, we will make

$$y^2 = x^3 + c_5xy + c_4zx^2 + c_3z^2y + c_2z^3x + z^5 \tag{4.11}$$

a weighted homogeneous deformation of weight 30 of the E_8 rational double point singularity

$$y^2 = x^3 + z^5$$

and simultaneously make (4.11) invariant with respect to the involution induced by

$$\begin{array}{ccc} \mathfrak{h}^{\mathbb{C}}_{SU(5)_{gauge}} \times \mathbb{C}^3 & & \mathfrak{h}^{\mathbb{C}}_{SU(5)_{gauge}} \times \mathbb{C}^3 \\ (h, (x, y, z)) & \mapsto & (-h, (x, -y, z)) \end{array} \tag{4.12}$$

(that takes c_j to $(-1)^j\, c_j$).

4.2. Deformation of the E_8 rational double point surface singularity

To understand the implications of this last assertion, we begin by choosing a nilpotent subregular element $X \in \mathfrak{e}_8$ whose commutator contains $\mathfrak{h}^{\mathbb{C}}_{E_8} = \mathfrak{h}_{E^{\mathbb{C}}_8}$

and write elements of the the Lie subalgebra $\ker(ad(X))$ as

$$\left((u,v),h,h'\right)\in\left(\mathbb{C}^2\times\mathfrak{h}_{SU(5)_{gauge}}\times\mathfrak{h}_{SU(5)_{Higgs}}\right)$$

and restrict to the subspace

$$((u,v),h)\in\left(\mathbb{C}^2\times\mathfrak{h}_{SU(5)_{gauge}}\times\{0\}\right).$$

To fit the product decomposition we must choose

$$X=X_{gauge}^{sr}+X_{Higgs}^{r}$$

where $X_{gauge}^{sr}\in\mathfrak{sl}(5;\mathbb{C})_{gauge}$ is subregular and $X_{Higgs}^{r}\in\mathfrak{sl}(5;\mathbb{C})_{Higgs}$ is regular and remark that their sum must be chosen to act faithfully on the four non-principal summands of the adjoint action of $\mathfrak{sl}(5;\mathbb{C})_{gauge}\times\mathfrak{sl}(5;\mathbb{C})_{Higgs}$ on $\mathfrak{e}_8$.

The Jacobson-Morozov theorem [2] states that any nilpotent element of $\mathfrak{sl}(5;\mathbb{C})$, in particular our subregular X_{gauge}^{sr} in the commutant of our fixed $\mathfrak{h}_{SU(5)_{gauge}}^{\mathbb{C}}\subseteq\mathfrak{sl}(5;\mathbb{C})$, completes to an $\mathfrak{sl}(2;\mathbb{C})$-triple $\left(X^{sr},Y^{sr},H^{sr}=[X^{sr},Y^{sr}]\right)$. Via (4.2) that triple imbeds as a subalgebra of $\mathfrak{e}_8^{\mathbb{C}}$.

We consider the following table of homogeneous forms

Entry	Weight=degree
$x=f_{10}(u,v)$	10
$y=f_{15}(u,v)$	15
$z=f_{6}(u,v)$	6
$c_j(h)$	j

on $\mathbb{C}^2\times\mathfrak{h}_{SU(5)_{gauge}}^{\mathbb{C}}$ with the property that the $f_k(u,v)$ are the generators of the invariant polynomials of the finite subgroup of $SU(2)$ lying in E_8 via (4.1) and the c_j are the Casimirs that define the mapping

$$\begin{array}{ccc}\mathbb{C}^2\times\mathfrak{h}_{SU(5)_{gauge}}^{\mathbb{C}} & \to & \mathbb{C}^3\times\frac{\mathfrak{h}_{SU(5)_{gauge}}^{\mathbb{C}}}{W(SU(5))}\\ ((u,v),h) & \mapsto & ((x,y,z),(c_2,c_3,c_4,c_5)).\end{array}\tag{4.13}$$

As we will explain in more detail in the next Section, this gives (4.11) the structure of a deformation

$$\mathcal{V}_{Tate}/\frac{\mathfrak{h}_{SU(5)_{gauge}}^{\mathbb{C}}}{W(SU(5))}.$$

of weighted homogeneous polynomials of weight 30 of the E_8 rational double point surface singularity has image given by the equation (4.11). Then the involution

$$((u,v),h) \mapsto ((-u,-v),-h)$$

is equivariant with the involution induced by (4.12).

Under the inclusion

$$\mathfrak{sl}(5;\mathbb{C})_{gauge} \times \mathfrak{sl}(5;\mathbb{C})_{Higgs} \hookrightarrow \mathfrak{e}_8^{\mathbb{C}} \tag{4.14}$$

that identifies maximal tori and hence Cartan subalgebras, we therefore have the induced map

(4.15)
$$\begin{gathered}\frac{\mathfrak{h}^{\mathbb{C}}_{SU(5)_{gauge}}}{W\big(SU(5)_{gauge}\big)} \times \frac{\mathfrak{h}^{\mathbb{C}}_{SU(5)_{Higgs}}}{W\big(SU(5)_{Higgs}\big)} \to \frac{\mathfrak{h}^{\mathbb{C}}_{E_8}}{W(E_8)} \\ \Big((c_2,c_3,c_4,c_5)_{gauge},(c_2,c_3,c_4,c_5)_{Higgs}\Big) \mapsto (a_{30},b_{24},b_{18},b_{12},c_{20},c_{14},c_8,c_2)\end{gathered}$$

where the coordinates of the 8-dimensional vector space $\frac{\mathfrak{h}^{\mathbb{C}}_{E_8}}{W(E_8)}$ are indexed and weighted by the standard basis of Casimir polynomial algebra of E_8.

Now the semi-universal deformation space of the rational double point surface singularity

$$\left\{y^2 = x^3 + z^5\right\} \subseteq \mathbb{C} \tag{4.16}$$

is given by

$$\begin{aligned} y^2 = x^3 + z^5 + a_{30} &+ \left(b_{24}z + b_{18}z^2 + b_{12}z^3\right) \\ &+ \left(c_{20}x + c_{14}xz + c_8xz^2 + c_2xz^3\right) \end{aligned} \tag{4.17}$$

where, in the first instance, the eight parameters a_j, b_j, c_j are considered as free parameters of an eight-dimensional complex vector space [3, 10]. The semi-universal family (4.17) forms a hypersurface in $\mathbb{C}^3 \times \frac{\mathfrak{h}^{\mathbb{C}}_{E_8}}{W(E_8)}$ parametrized by the map

$$\begin{gathered}\mathbb{C}^2 \times \mathfrak{h}^{\mathbb{C}}_{E_8} \to \mathcal{V}_8 \subseteq \mathbb{C}^3 \times \frac{\mathfrak{h}^{\mathbb{C}}_{E_8}}{W(E_8)} \\ ((u,v),h) \mapsto ((x,y,z),(c_2,c_8,c_{14},c_{20},b_{12},b_{18},b_{24},a_{30})).\end{gathered} \tag{4.18}$$

where one considers $(c_2,c_8,c_{14},c_{20},b_{12},b_{18},b_{24},a_{30})$ as the standard generators of the Casimir polynomial algebra of E_8. Again assigning

Entry	Weight
$x = f_{10}$	10
$y = f_{15}$	15
$z = f_6$	6
a_j, b_j, c_j	j

(4.17) becomes a weighted homogeneous polynomial of weight 30 on

$$\mathbb{C}^2 \times \mathfrak{h}_{E_8^{\mathbb{C}}}. \tag{4.19}$$

We denote this semi-universal deformation as

$$\mathcal{V}_{E_8} / \frac{\mathfrak{h}_{E_8}^{\mathbb{C}}}{W(E_8)}. \tag{4.20}$$

The fact that (4.11) is a weighted homogeneous deformation of weight 30 of the E_8 rational double point singularity implies by semi-universality that it is induced by pullback

$$\begin{array}{ccc} \mathcal{V}_{Tate} \times_{\frac{\mathfrak{h}_{SU(5)_{gauge}}^{\mathbb{C}}}{W(SU(5)_{gauge})}} \mathfrak{h}_{SU(5)_{gauge}}^{\mathbb{C}} & \rightarrow & \mathcal{V}_{E_8} \\ \downarrow & & \downarrow^{\pi} \\ \mathfrak{h}_{SU(5)_{gauge}}^{\mathbb{C}} & \rightarrow & \frac{\mathfrak{h}_{E_8}^{\mathbb{C}}}{W(E_8)} \end{array}$$

via (4.15) from (4.17).

We next examine the semi-universal deformation (4.20) in some detail.

5. Equivariant crepant resolution for the E_8 rational double point

We again begin with the E_8-singularity (4.16) whose minimal resolution has the property that the intersection matrix of its exceptional curves can be equated with the E_8-Dynkin diagram. (4.16) is a quotient singularity via the forms

$$\begin{array}{c} x = f_{10}(u,v) \\ y = f_{15}(u,v) \\ z = f_6(u,v) \end{array} \tag{5.1}$$

where $f_j(u,v)$ is a homogeneous form of degree j in the complex (u,v)-plane. The exceptional curves themselves are matched with simple roots corresponding to a choice of positive Weyl chamber. The involution ι interchanges that choice of positive Weyl chamber with its negative.

Letting ε denote a primitive fifth root of unity, the forms (5.1)are a minimal set of generators of the sub-ring of the polynomial ring $\mathbb{C}[u,v]$ made up of the polynomials that are invariant under the action of the the binary icosahedral group, that is, the finite subgroup $B \subseteq SU(2)$ of order 120 generated by

$$\left(\begin{array}{cc} \varepsilon^3 & 0 \\ 0 & \varepsilon^2 \end{array}\right), \frac{1}{\sqrt{5}}\left(\begin{array}{cc} -\varepsilon+\varepsilon^4 & \varepsilon^2-\varepsilon^3 \\ \varepsilon^2-\varepsilon^3 & \varepsilon-\varepsilon^4 \end{array}\right).$$

In fact

$$\begin{array}{ccc} \mathbb{C}^2 & \rightarrow & \mathbb{C}^2 \\ (u,v) & \mapsto & (x,z)=(f_{10},f_6) \end{array}$$

has general fiber of cardinality 120 on which B acts transitively. By degree, f_{15} does not lie in the polynomial ring $\mathbb{C}[f_{10},f_6]$ however satisfies the second-degree integral equation

$$wy^2 = x^3 + a_0 z^5 \tag{5.2}$$

and the degree of the mapping

$$\begin{array}{ccc} \mathbb{C}^2 & \rightarrow & \mathbb{C}^3 \\ (u,v) & \mapsto & (x,y,z)=(f_{10},f_{15},f_6) \end{array} \tag{5.3}$$

is 240. This is equivalent to the fact that

$$f_{15}(-u,-v) = -f_{15}(u,v). \tag{5.4}$$

Thus the symmetry $(u,v) \mapsto (-u,-v)$ commutes with the symmetry $(x,y,z) \mapsto (x,-y,z)$.

The breaking of E_8-symmetry is tracked by an unfolding of (4.16) in the semi-universal deformation space

$$\begin{aligned} y^2 = x^3 + z^5 + a_{30} + \left(b_{24}z + b_{18}z^2 + b_{12}z^3\right) \\ + \left(c_{20}x + c_{14}xz + c_8xz^2 + c_2xz^3\right) \end{aligned} \tag{5.5}$$

where, in the first instance, the eight parameters a_j, b_j, c_j are considered as free parameters of an eight-dimensional complex vector space that we will denote as

$$U_8 := \frac{\mathfrak{h}^{\mathbb{C}}_{E_8}}{W(E_8)}.$$

Since the roots of the various subgroups of E_8 to which the E_8-symmetry is broken are represented by the exceptional curves of the crepant resolution of a rational double-point singularity over a point of U_8 in (4.18), we will not be able to 'follow the roots' without understanding the Brieskorn-Grothendieck equivariant crepant resolution of semi-universal deformation of the E_8-rational double-point singularity as given in [3] and [10].

Unfortunately the equivariant Brieskorn-Grothendieck resolution cannot be a resolution over U_8. Rather one considers U_8 as the quotient

$$(c_2, c_8, c_{14}, c_{20}, b_{12}, b_{18}, b_{24}, a_{30}) : \mathfrak{h}_{E_8^{\mathbb{C}}} \to \frac{\mathfrak{h}_{E_8^{\mathbb{C}}}}{W\left(E_8^{\mathbb{C}}\right)} = U_8$$

by considering the eight parameters a_j, b_j, c_j in (4.18) as a standard basis of the $E_8^{\mathbb{C}}$ Casimir polynomials, then assigning weights

Entry	Weight
$x = f_{10}$	10
$y = f_{15}$	15
$z \equiv z = f_6$	6
a_j, b_j, c_j	j

so that (5.5) becomes a weighted homogeneous polynomial of weight 30 on

$$\mathbb{C}^2 \times \mathfrak{h}_{E_8^{\mathbb{C}}}. \tag{5.6}$$

From (4.18) we then have

$$\begin{array}{ccc} \mathbb{C}^2 \times \mathfrak{h}_{E_8^{\mathbb{C}}} & \to & \mathcal{V}_8 \times_{U_8} \mathfrak{h}_{E_8^{\mathbb{C}}} \\ \downarrow & & \downarrow \\ \mathbb{C}^2 \times U_8 & \to & \mathcal{V}_8 \subseteq \mathbb{C}^3 \times U_8 \end{array}$$

Somewhat miraculously, the equivariant crepant resolution $\tilde{\mathcal{V}}_8$ over $\mathfrak{h}_{E_8^{\mathbb{C}}}$ is canonically given by subvarieties of the incidence variety

$$\mathcal{I}_{E_8} := \{(x, B) : x \in B\} \subseteq E_8^{\mathbb{C}} \times \left\{B \leq E_8^{\mathbb{C}} : B\, a\, Borel\, subgroup\right\}.$$

To understand this, we begin with the regular elements of $E_8^{\mathbb{C}}$, that is those lying in only a finite number of Borel subgroups. Each component of the commutant of a regular element is a maximal torus. Choosing a maximal torus $\mathfrak{T}_8^{\mathbb{C}}$ for $E_8^{\mathbb{C}}$, the product

$$\mathfrak{h}_{E_8^{\mathbb{C}}} \times \mathbb{A}_{(u,v)}$$

imbeds in the Lie algebra $\mathfrak{e}_8^{\mathbb{C}}$ as the Lie algebra of the commutant subgroup $C(x_{sr})$ of a so-called subregular element $x_{sr} \in \mathfrak{T}_8^{\mathbb{C}}$, that is, one whose commutant in $E_8^{\mathbb{C}}$ contains $\mathfrak{T}_8^{\mathbb{C}}$ as a codimension-two subgroup. The Lie $\mathbb{C}{\cdot}H + \mathbb{A}_{(u,v)}$ should be considered as the Lie algebra of

$$\frac{C(x_{sr})}{\mathfrak{T}_8^{\mathbb{C}}}$$

and, as such, one of the $\mathfrak{su}(2) \otimes \mathbb{C}$ Lie algebras in (3.1), the complexification of the real subalgebra generated by the two real matrices in (3.2). Since ι acts as multiplication by (-1) on roots, it also takes a Borel subalgebra containing $\mathfrak{h}_{E_8^{\mathbb{C}}}$ to its opposite Borel subalgebra. So if we let u be the coordinate for $\mathfrak{l}_{\varrho}$ and let v be the coordinate for $\mathfrak{l}_{-\varrho}$ then via (5.7) ι induces the involution

$$(h; u, v) \mapsto (-h; -u, -v)$$

on $\mathfrak{h}_{E_8^{\mathbb{C}}} \times \mathbb{A}_{(u,v)}$.

For the maximal torus $\mathfrak{T}_8^{\mathbb{C}} = \exp\left(\mathfrak{h}_8^{\mathbb{C}}\right)$ of $E_8^{\mathbb{C}}$ we next identify neighborhoods of the identity under the exponential map

$$\begin{array}{ccccccc} \mathfrak{h}_{E_8^{\mathbb{C}}} \times \mathbb{A}_{(u,v)} & \hookrightarrow & \mathfrak{e}_8^{\mathbb{C}} & & C(x_{sr}) & \hookrightarrow & E_8^{\mathbb{C}} \\ \downarrow & & \downarrow & \stackrel{\exp}{\Longrightarrow} & \downarrow & & \downarrow \\ \mathfrak{h}_{E_8^{\mathbb{C}}} & \to & \mathfrak{h}_8^{\mathbb{C}}/W(E_8) & & \mathfrak{T}_8^{\mathbb{C}} & \to & \mathfrak{T}_8^{\mathbb{C}}/W(E_8) \end{array} \tag{5.7}$$

where the complex analytic map

$$E_8^{\mathbb{C}} \to \mathfrak{T}_8^{\mathbb{C}}/W(E_8)$$

assigns to the conjugacy class of $x \in E_8^{\mathbb{C}}$ the well-defined element $x_s \in \mathfrak{T}_8^{\mathbb{C}}/W(E_8)$ of its Jordan decomposition $x = x_s x_u$ into commuting semisimple and unipotent factors.

In a small neighborhood of the identity the set of subregular elements in $\mathfrak{T}_8^{\mathbb{C}}$ corresponds exactly under (5.7) to the set of the singular points of the versal deformation (5.5) and the Brieskorn-Grothendieck equivariant crepant resolution

$$\begin{array}{ccc} \tilde{\mathcal{V}}_8 & \rightarrow & \mathcal{V}_8 \\ \downarrow & & \downarrow \\ \mathfrak{h}_{E_8^{\mathbb{C}}} & \rightarrow & U_8 \end{array} \tag{5.8}$$

has exception fibers over sub-regular element x_{sr} given by subspaces

$$\{x_{sr}\} \times_{E_8} \mathcal{I}_{E_8}$$

consisting of those Borels B that contain x_{sr}. (See [3] and [10].)

Lemma 3. *The action of the complex conjugate involution ι reverses a choice of positive Weyl chamber of E_8 with its negative and therefore reverse the choice of Weyl chamber with respect to which the Brieskorn-Grothendieck equivariant crepant resolution is defined. This reversal induces the involution (2.8) on on the semi-universal deformation (4.18) of the E_8 rational double point surface singularity.*

Proof. The Brieskorn-Grothendieck equivariant crepant resolution is built entirely inside the product of

1) the commutator of a sub-regular element of the complex algebraic group $E_8^{\mathbb{C}}$

and

2) the set of Borel subgroups of $E_8^{\mathbb{C}}$.

Since all of the E_8-Casimirs are of even degree, $(u,v) \mapsto (-u,-v)$ commutes with the symmetry $(x,y,z) \mapsto (x,-y,z)$ in (4.18). But this last symmetry only commutes with the Brieskorn-Grothendieck equivariant crepant resolution if it is induced by a symmetry of (5.6). Now the functions in Table 5 are functions in the variables $(h;u,v)$ where the complex eight-tuple h is the parameter for a neighborhood of the origin in the Lie algebra of $E_8^{\mathbb{C}}$. The only function of odd weight is

$$y = f_{15}(h;u,v).$$

Therefore the involution

$$(h;u,v) \mapsto (-h;-u,-v)$$

induced by ι commutes with the projection to U_8 and sends $(x,y,z) \mapsto (x,-y,z)$. □

So the Brieskorn-Grothendieck equivariant crepant resolution is actually a pair of crepant resolutions $\dot{\mathcal{V}}_8$ and $\ddot{\mathcal{V}}_8$ of the pullback

$$\mathcal{V}_8 \times_{U_8} \mathfrak{h}_{E_8^{\mathbb{C}}}$$

of the family (5.5) to a family over the Cartan subalgebra $\mathfrak{h}_{E_8^{\mathbb{C}}}$. The two correspond to whether the resolution was grounded in a given choice of positive Weyl chamber or its negative. The two resolutions are related by a real analytic isomorphism over $\mathcal{V}_8 \times_{U_8} \mathfrak{h}_{E_8^{\mathbb{C}}}$ induced by $\iota \in \mathit{Gal}\,(\mathbb{C}/\mathbb{R})$. That is, we have the commutative diagram

$$\begin{array}{ccc} \dot{\mathcal{V}}_8 & \overset{\iota}{\longleftrightarrow} & \ddot{\mathcal{V}}_8 \\ \downarrow & & \downarrow \\ \mathcal{V}_8 & \overset{(x,y,z)\mapsto(x,-y,z)}{\longleftrightarrow} & \mathcal{V}_8. \end{array}$$

This diagram is then incorporated into a commutative diagram

$$\begin{array}{ccc} \dot{\mathcal{V}}_8 \times_{\left(\mathcal{V}_8 \times_{U_8} \mathfrak{h}_{E_8^{\mathbb{C}}}\right)} \ddot{\mathcal{V}}_8 & \rightarrow & \mathcal{V}_8 \\ \downarrow & & \downarrow \\ \mathfrak{h}_{E_8^{\mathbb{C}}} & \rightarrow & U_8 = \frac{\mathfrak{h}_{E_8^{\mathbb{C}}}}{W\left(E_8^{\mathbb{C}}\right)} \end{array} \tag{5.9}$$

for which the left-hand vertical map is smooth on each factor of the fibered product and the top horizontal map factors through crepant resolutions $\dot{\mathcal{V}}_8 \rightarrow \left(\mathcal{V}_8 \times_{U_8} \mathfrak{h}_{E_8^{\mathbb{C}}}\right)$ and $\ddot{\mathcal{V}}_8 \rightarrow \left(\mathcal{V}_8 \times_{U_8} \mathfrak{h}_{E_8^{\mathbb{C}}}\right)$ respectively.

These two resolutions have the the following four properties:

1) Each is determined by a choice of positive Weyl chamber in $\mathfrak{h}_8^{\mathbb{C}}$.

2) The mappings

$$\dot{\mathcal{V}}_8,\ \ddot{\mathcal{V}}_8/\mathfrak{h}_8^{\mathbb{C}} \rightarrow \left(\mathcal{V}_8 \times_{U_8} \mathfrak{h}_8^{\mathbb{C}}\right)/\mathfrak{h}_8^{\mathbb{C}} \tag{5.10}$$

are isomorphisms except over the singular locus of $\mathcal{V}_8 \times_{U_8} \mathfrak{h}_8^{\mathbb{C}}$.

3) Over a singular point $(z = 0, x = 0, y = 0; h)$ the fiber is the Dynkin curve for the minimal resolution of the rational double-point singularity corresponding to $h \in \mathfrak{h}_8^{\mathbb{C}}$.

4) Since all weights in Table (5) except that of y are even and ι acts as minus the identity on the vector space $\mathfrak{h}_{E_8^{\mathbb{C}}} \times \mathbb{A}_{(u,v)}$, we conclude that ι acts fiberwise on the semi-universal family (5.5) as

$$(u, v, x, y, z) \mapsto (-u, -v, x, -y, z)\,. \tag{5.11}$$

We therefore are able to conclude the following.

Theorem 4. *Via the Brieskorn-Grothendieck equivariant crepant resolution of the E_8 rational double-point (4.16), the action of the generator $\iota \in Gal\,(\mathbb{C}/\mathbb{R})$ corresponds to the action*

$$(x, y) \mapsto (x, -y) \tag{5.12}$$

on the Weierstrass form on the fibers of the semi-universal deformation of the E_8 rational double point.

6. Tracking the equivariant resolution under symmetry-breaking

6.1. Immersing W_4/B_3 into the versal deformation of the E_8 rational double point surface singularity

Next notice that the above mapping

$$(c_2, c_3, c_4, c_5) : B_3' \to \mathbb{C}^4 = \frac{\mathfrak{h}^{\mathbb{C}}_{SU(5)_{gauge}}}{W\,(SU\,(5))} \tag{6.1}$$

is such that, by (4.11) coupled with (4.8), it defines a commutative diagram

$$\begin{array}{ccc} W_4' & \to & \mathcal{V}_{Tate} \\ \downarrow & & \downarrow \\ B_3' & \to & \mathbb{C}^4. \end{array} \tag{6.2}$$

We next return to the isomorphism

$$\mathfrak{h}_{SU(5)_{gauge}} \times \mathfrak{h}_{SU(5)_{Higgs}} \to \mathfrak{h}_{E_8}$$

given in (4.2) inducing the epimorphism (4.2). We pull back the family $\mathcal{V}_{Tate}/\frac{\mathfrak{h}^{\mathbb{C}}_{SU(5)_{gauge}}}{W\left(SU(5)_{gauge}\right)}$ given by (4.11) under the map

$$\mathfrak{h}^{\mathbb{C}}_{SU(5)_{gauge}} \times \frac{\mathfrak{h}^{\mathbb{C}}_{SU(5)_{Higgs}}}{W\left(SU\,(5)_{Higgs}\right)} \to \frac{\mathfrak{h}_{E_8^{\mathbb{C}}}}{W\left(E_8^{\mathbb{C}}\right)}$$

where it becomes a weighted homogeneous family of rational double points of weight 30. Therefore by the semi-universality of the family (4.17) we have the induced commutative diagram

$$
(6.3)\qquad
\begin{array}{ccc}
\mathcal{V}_{Tate} \times_{\frac{\mathfrak{h}^{\mathbb{C}}_{SU(5)_{gauge}}}{W\left(SU(5)_{gauge}\right)}} \mathfrak{h}^{\mathbb{C}}_{SU(5)_{gauge}} & \rightarrow & \mathcal{V}_{E_8} \\
\downarrow & & \downarrow^{\pi} \\
\mathfrak{h}^{\mathbb{C}}_{SU(5)_{gauge}} & \rightarrow & \frac{\mathfrak{h}^{\mathbb{C}}_{E_8}}{W(E_8)}
\end{array}
$$

and so by §8 of [10] equivariant crepant resolutions

$$
(6.4)\qquad
\begin{array}{ccc}
\left\{\dot{\mathcal{V}}_{Tate}, \ddot{\mathcal{V}}_{Tate}\right\} \times_{\frac{\mathfrak{h}^{\mathbb{C}}_{SU(5)_{gauge}}}{W\left(SU(5)_{gauge}\right)}} \mathfrak{h}^{\mathbb{C}}_{SU(5)_{gauge}} & \rightarrow & \dot{\mathcal{V}}_{E_8}, \ddot{\mathcal{V}}_{E_8} \\
\downarrow & & \downarrow^{\tilde{\pi}} \\
\mathfrak{h}^{\mathbb{C}}_{SU(5)_{gauge}} & \rightarrow & \mathfrak{h}^{\mathbb{C}}_{E_8}.
\end{array}
$$

Now referring to the composition

$$
(6.5)\qquad
\begin{array}{ccccc}
W'_4 & \hookrightarrow & \mathcal{V}_{Tate} & \rightarrow & \mathcal{V}_{E_8} \\
\downarrow & & \downarrow & & \downarrow^{\pi} \\
B'_3 & \hookrightarrow & \mathbb{C}^4 = \frac{\mathfrak{h}^{\mathbb{C}}_{SU(5)_{gauge}}}{W\left(SU(5)_{gauge}\right)} & \rightarrow \mathbb{C}^8 = & \frac{\mathfrak{h}^{\mathbb{C}}_{E_8}}{W(E_8)}
\end{array}
$$

of (6.2) and (6.3), (6.4) lets us track roots over the central column of (6.5). It remains to track those roots as given by (6.4) to the exceptional curves of a crepant resolution $\tilde{W}_4/B_3$ of W_4/B_3, at least over a general point $p \in S_{\mathrm{GUT}}$.

Before proceeding to accomplish this last task, notice that $\iota = -I_8$ acts as

$$
\begin{aligned}
&\left(\left(a_{30}, b_{24}, b_{18}, b_{12}, c_{20}, c_{14}, c_8, c_2\right), (x, y, z)\right) \\
&\quad \mapsto \left(\left(a_{30}, b_{24}, b_{18}, b_{12}, c_{20}, c_{14}, c_8, c_2\right), (x, -y, z)\right)
\end{aligned}
$$

on the right-hand vertical map in (6.5), as

$$
\left(\left(c_2, c_3, c_4, c_5\right), (x, y, z)\right) \mapsto \left(\left(c_2, -c_3, c_4, -c_5\right), (x, -y, z)\right)
$$

on the central vertical map in (6.5), and as

$$
\left(b_3, (x, y, z)\right) \mapsto \left(\beta_3\left(b_3\right), (x, -y, -z)\right)
$$

on the left-hand vertical map in (6.5).[5]

[5]The lack of a sign change of y between the central vertical map of (6.5) and the left-hand vertical map under the action of $-I_8$ reflects the fact that the action

Theorem 5. *(6.2) allows us to track a crepant resolution of* W_4'/B_3' *back to the crepant resolution of the* E_8 *rational double point singularity over a general point* $p \in S_{\mathrm{GUT}} \cap B_3'$.

Proof. For general $p \in S_{\mathrm{GUT}}$, we define a holomorphic disk $D_p \subseteq B_3$ meeting S_{GUT} transversely at p and form

$$D_p \times_{B_3} W_4,$$

a smooth open elliptic surface whose resolution has an I_5-fiber over p by the Kodaira classification. Now D_p determines a normal vector ν_p to $S_{\mathrm{GUT}} \subseteq B_3$ at p that by the map from the left-hand column to the central column of (6.5) lifts to a non-zero nilpotent subregular element $X^{sr}_{gauge} = \tilde{\nu}_p$ in the nilpotent cone of the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}_{SU(5)}$ and so into the nilpotent cone of the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}_{E_8}$. We complete $X^{sr}_{gauge} = \tilde{\nu}_p$ to an

$$\mathfrak{sl}^{\mathbb{C}}_2 \; triple \subseteq \mathfrak{sl}^{\mathbb{C}}_5 \subseteq \mathfrak{g}^{\mathbb{C}}_{E_8}$$

via the Jacobson-Morosov theorem. Now the I_5-fiber over p in the crepant resolution $\tilde{W}_4'/B_3'$ of W_4'/B_3' implies that the decomposition of $\mathfrak{g}^{\mathbb{C}}_{SU(5)_{gauge}}$ as an $\mathfrak{sl}^{\mathbb{C}}_2$-module induced by the triple must have a simple summand decomposition that coincides with a simple decomposition associated with the I_5-fiber of $\dot{\mathcal{V}}_{Tate}/\mathfrak{h}^{\mathbb{C}}_{SU(5)_{gauge}}$, respectively $\ddot{\mathcal{V}}_{Tate}/\mathfrak{h}^{\mathbb{C}}_{SU(5)_{gauge}}$, and so of $\dot{\mathcal{V}}_{E_8}/\mathfrak{h}^{\mathbb{C}}_{E_8}$, respectively $\ddot{\mathcal{V}}_{E_8}/\mathfrak{h}^{\mathbb{C}}_{E_8}$, over a lifting of the image of p in $\frac{\mathfrak{h}^{\mathbb{C}}_{E_8}}{W(E_8)}$. Thus the fiber over p of the crepant resolution $\tilde{W}_4'/B_3'$ of W_4'/B_3' is induced by the fiber of the equivariant crepant resolution $\tilde{\mathcal{V}}_{E_8}/\mathfrak{h}^{\mathbb{C}}_{E_8}$ over a lifting of the image of p in $\frac{\mathfrak{h}^{\mathbb{C}}_{E_8}}{W(E_8)}$. □

$y \mapsto -y$ of the involution β_3 on B_3' will incorporate the reversal of positive roots with their negatives induced by the complex conjugate involution $-I_8$. That 'flop' interchanges the equivarant crepant resolutions $\dot{\mathcal{V}}_{E_8}/\mathfrak{h}^{\mathbb{C}}_{E_8}$ and $\ddot{\mathcal{V}}_{E_8}/\mathfrak{h}^{\mathbb{C}}_{E_8}$. It is only in this way that (4.11) becomes invariant under the action of β_3.

7. Crepant resolution conjecture

The diagram (6.5) and (5.8) induce a commutative diagram

$$\begin{array}{ccccc} W_4' \times_{B_3'} \mathfrak{h}^{\mathbb{C}}_{SU(5)_{gauge}} & \overset{\tilde{\vartheta}}{\dashrightarrow} & \tilde{\mathcal{V}}_{E_8} & \rightarrow & \mathcal{V}_{E_8} \\ \downarrow & & \downarrow & & \downarrow^{\pi} \\ B_3' \times_{\frac{\mathfrak{h}^{\mathbb{C}}_{SU(5)_{gauge}}}{W\left(SU(5)_{gauge}\right)}} \mathfrak{h}^{\mathbb{C}}_{SU(5)_{gauge}} & \rightarrow & \mathfrak{h}^{\mathbb{C}}_{E_8} & \rightarrow & \frac{\mathfrak{h}^{\mathbb{C}}_{E_8}}{W(E_8)} \end{array}$$

Conjecture 6. *The closure of the graph of the rational map $\tilde{\vartheta}$ in the above diagram is a crepant resolution of $W_4' \times_{B_3'} \mathfrak{h}^{\mathbb{C}}_{SU(5)_{gauge}}$.*

If true, this conjecture would allow is to track a crepant resolution of W_4'/B_3' back to the crepant resolution of the E_8 rational double point singularity over *every* $p \in S_{\mathrm{GUT}} \cap B_3'$, for example over the points of the matter and Higgs curves.

8. Conclusion

In this paper we have confronted a problem proposed but not fully resolved in [1], [5] and [8]. Our solution rests on the introduction of the complex conjugation operator into the $\mathbb{Z}_2$-action on W_4/B_3 to produce a Calabi-Yau quotient on which we still retain $SU(5)_{gauge}$-symmetry. This last assertion is proved by tracing the exceptional fibers of a crepant resolution $\tilde{W}_4/B_3$ of W_4/B_3 back to the E_8-roots from which they evolved using the Brieskorn-Grothendieck equivariant resolution of the semi-universal deformation of the E_8 rational double-point surface singularity.

One is still left with the task of explicitly constructing the B_3, the resolution $\tilde{W}_4/B_3$, and checking that the $\mathbb{Z}_2$-quotients have the phenomenologically correct invariant. Our strategy will be to first construct B_3 canonically from the geometry of A_4-roots space in such a way that it is both symmetric with respect to the action of complex conjugation and also has the desired numerical invariants. That done, the construction of $\tilde{W}_4/B_3$ and verification that it too has the correct numerical invariants will be relatively straightforward. The authors carried this program out in two related papers, "F-theory over a Fano threefold built from A_4-roots" [`arXiv:hep-th/1912.06902`] and "Heterotic-F-theory duality with Wilson line symmetry-breaking" [`arXiv:hep-th/1908.01913`].

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