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Unbroken $E_7 \times E_7$ nongeometric heterotic strings, stable degenerations and enhanced gauge groups in F-theory duals

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Eight-dimensional non-geometric heterotic strings with gauge algebra $\mathfrak{e}_8\mathfrak{e}_7$ were constructed by Malmendier and Morrison as heterotic duals of F-theory on K3 surfaces with $\Lambda^{1,1} \oplus E_8 \oplus E_7$ lattice polarization. Clingher, Malmendier and Shaska extended these constructions to eight-dimensional non-geometric heterotic strings with gauge algebra $\mathfrak{e}_7\mathfrak{e}_7$ as heterotic duals of F-theory on $\Lambda^{1,1} \oplus E_7 \oplus E_7$ lattice polarized K3 surfaces. In this study, we analyze the points in the moduli of non-geometric heterotic strings with gauge algebra $\mathfrak{e}_7\mathfrak{e}_7$, at which the non-Abelian gauge groups on the F-theory side are maximally enhanced. The gauge groups on the heterotic side do not allow for the perturbative interpretation at these points. We show that these theories can be described as deformations of the stable degenerations, as a result of coincident 7-branes on the F-theory side. From the heterotic viewpoint, this effect corresponds to the insertion of 5-branes. These effects can be used to understand nonperturbative aspects of nongeometric heterotic strings.

Additionally, we build a family of elliptic Calabi–Yau 3-folds by fibering elliptic K3 surfaces, which belong to the F-theory side of the moduli of non-geometric heterotic strings with gauge algebra $\mathfrak{e}_7\mathfrak{e}_7$, over $\mathbb{P}^1$. We find that highly enhanced gauge symmetries arise on F-theory on the built elliptic Calabi–Yau 3-folds.

1. Introduction

F-theory/heterotic duality [1–5] states that F-theory [1–3] compactification on an elliptic K3 fibered Calabi–Yau $(n+1)$-fold describes a theory physically equivalent to heterotic compactification[1] on an elliptic Calabi–Yau n-fold. Non-perturbative aspects of heterotic theory can be studied by utilizing this duality. F-theory/heterotic duality is strictly formulated when the stable degeneration limit[2] [5, 17] is taken on the F-theory side in which K3 fibers split into pairs of half K3 surfaces.

Recently, eight-dimensional non-geometric heterotic strings with unbroken $\mathfrak{e}_8\mathfrak{e}_7$ algebra were constructed by Malmendier and Morrison [24] by utilizing the F-theory/heterotic duality. The Narain space [25]

$$D_{2,18}/O(\Lambda^{2,18}) \tag{1}$$

gives the moduli space of eight-dimensional heterotic strings, and the double cover of this space,

$$D_{2,18}/O^{+}(\Lambda^{2,18}), \tag{2}$$

is equivalent to the moduli space of F-theory on elliptic K3 surfaces with a section. This is the statement of F-theory/heterotic duality. Malmendier and Morrison considered F-theory compactifications on elliptic K3 surfaces with $H \oplus E_8 \oplus E_7$ lattice polarization, namely elliptically fibered K3 surfaces with a type II^* fiber and a type III^* fiber with a global section, and

[1]Recent progress of heterotic strings can be found, for example, in [6–16].

[2]Stable degenerations in F-theory/heterotic duality have been studied recently, for example, in [18–23].

they constructed the moduli of heterotic strings with unbroken $\mathfrak{e}_8\mathfrak{e}_7$ algebra as the heterotic duals of them on the 2-torus. The moduli space of the non-geometric heterotic strings with unbroken $\mathfrak{e}_8\mathfrak{e}_7$ algebra constructed in [24] is given by

$$D_{2,3}/O^+(L^{2,3}). \tag{3}$$

Here $L^{2,3}$ denotes the orthogonal complement of $H \oplus E_8 \oplus E_7$ inside the K3 lattice Λ_{K3}, and the non-geometric heterotic strings constructed in [24] possess $O^+(L^{2,3})$-symmetry. Here $O^+(L^{2,3})$[3] mixes the complex structure moduli, the Kähler moduli and the moduli of Wilson line values. Therefore, the resulting heterotic strings do not have a geometric interpretation[4]; for this reason, the resulting heterotic strings are called *non-geometric* heterotic strings. A single Wilson line expectation value is non-zero for the non-geometric heterotic strings with unbroken $\mathfrak{e}_8\mathfrak{e}_7$ gauge algebra as constructed in [24]. The mathematical results of Kumar [28] and Clingher and Doran [29, 30], which gave the Weierstrass equations of elliptic K3 surfaces with a global section with E_8E_7 singularity, the coefficients of which are expressed as Siegel modular forms of even weight, were used in their construction.

Clingher, Malmendier, and Shaska [31] extended the construction of non-geometric heterotic strings by Malmendier and Morrison to non-geometric heterotic strings with unbroken $\mathfrak{e}_7\mathfrak{e}_7$ algebra. F-theory compactifications on elliptic K3 surfaces with $H \oplus E_7 \oplus E_7$ lattice polarization, namely K3 surfaces with a global section with two type III^* fibers, were considered, and eight-dimensional non-geometric heterotic strings on T^2 were obtained as the heterotic duals in their construction. The moduli of the resulting heterotic strings is parametrized by the space

$$D_{2,4}/O^+(L^{2,4}). \tag{4}$$

Here $D_{2,4}$ is the symmetric space of $O(2,4)$, namely, $D_{2,4}$ is defined as $O(2) \times O(4)\backslash O(2,4)$. The symmetric space $D_{2,4}$ is also referred to as the *bounded symmetric domain of type IV*. Here $L^{2,4}$ denotes the orthogonal complement of $H \oplus E_7 \oplus E_7$ in the K3 lattice Λ_{K3}. The complex structure moduli, Kähler moduli, and the moduli of Wilson line expectation values are mixed under the symmetry $O^+(L^{2,4})$, thus the heterotic strings constructed in [31] also do not have a geometric interpretation. Two Wilson line expectation values are

[3]The authors of [26] discussed connections of K3 surfaces with lattice polarizations, non-geometric heterotic strings, and $O^+(\Lambda^{2,2})$-modular forms.

[4]The authors of [27] discussed non-geometric type II theories.

non-trivial in non-geometric heterotic strings with unbroken $\mathfrak{e}_7\mathfrak{e}_7$ algebra. See also [32–40] for recent progress on non-geometric heterotic strings.

In this note, we analyze theories that correspond to the points in the moduli of eight-dimensional non-geometric heterotic strings on the 2-torus T^2 constructed in [31], at which the ranks of the non-Abelian gauge groups are enhanced to 18 on the F-theory side. These are the maximal enhancements of the non-Abelian gauge groups on the F-theory side. We mainly consider $E_8 \times E_8$ heterotic strings, rather than $SO(32)$ heterotic strings. (However, we do consider some applications to $SO(32)$ heterotic strings.) As only up to an $E_8 \times E_8 \times U(1)^4$ gauge group can arise in eight-dimensional $E_8 \times E_8$ heterotic strings compactified on the 2-torus [39], a consideration of the ranks of the non-Abelian gauge groups reveals that the gauge groups of the dual heterotic theories of these F-theory models do not allow for a perturbative interpretation. These heterotic strings include the non-perturbative effects of 5-branes. We find that these theories can be described as the deformations of the heterotic strings from the stable degeneration limit, in which the F-theory/heterotic duality strictly holds, and that these deformations result from the coincident 7-branes on the F-theory side. In the heterotic language, the effect of coincident 7-branes corresponds to the presence of 5-branes.

When the non-Abelian gauge groups on F-theory on an elliptic K3 surface are enhanced to rank 18, K3 surfaces become *extremal* K3 surfaces. A K3 surface is called attractive, when it has the Picard number $\rho = 20$, which is the highest value for a complex K3 surface. A complex elliptic K3 surface $f : X \to \mathbb{P}^1$ with a section is said to be extremal if the Picard number of X is 20 and the Mordell–Weil group $\mathrm{MW}(X, f)$ is finite. Owing to the classification result in [41], the complex structures of extremal K3 surfaces on which non-Abelian gauge groups on F-theory compactifications are enhanced to rank 18 in the moduli can be determined, and this enables us to deduce the Weierstrass equations of extremal K3 surfaces. By analyzing the deduced Weierstrass equations, we study F-theory compactifications and the non-geometric heterotic duals at these special points in the moduli.

We also discuss applications to $SO(32)$ heterotic strings in this study. We deduce the Weierstrass equations of elliptic K3 surfaces appearing as the compactification spaces of the F-theory duals of some $SO(32)$ heterotic strings, which are obtained as the transformations of $\mathfrak{e}_7\mathfrak{e}_7$ non-geometric heterotic strings.

In addition, we consider fibering elliptic K3 surfaces that belong to the F-theory side of the moduli of eight-dimensional $\mathfrak{e}_7\mathfrak{e}_7$ non-geometric heterotic strings, over $\mathbb{P}^1$, to build elliptically fibered Calabi–Yau 3-folds with a global section. We study F-theory compactifications on the resulting elliptic Calabi–Yau 3-folds[5]. We find that highly enhanced gauge groups arise in these compactifications. It is mainly local F-theory model buildings that have been discussed in recent studies [61–64]. However, the global aspects of the geometry need to be considered to discuss the issues of gravity. We investigate F-theory on elliptically fibered Calabi–Yau 3-folds from the global perspective in this study.

A similar organization can be found in [39].

This note is structured as follows. In Section 2, we briefly review F-theory compactifications, and we also review attractive K3 surfaces and extremal K3 surfaces that are technically necessary to analyze special points in the moduli of eight-dimensional non-geometric heterotic strings and F-theory duals. We also review the construction of non-geometric heterotic strings with unbroken $\mathfrak{e}_7\mathfrak{e}_7$ algebra in [31].

In Section 3, we discuss the special points in the eight-dimensional non-geometric heterotic moduli with unbroken $\mathfrak{e}_7\mathfrak{e}_7$ at which the ranks of the non-Abelian gauge symmetries on the F-theory side are enhanced to 18. The gauge groups in the heterotic strings which correspond to these points do not allow for the perturbative interpretations. We demonstrate that these theories can be seen as deformations of the stable degenerations as a result of the coincident 7-branes on the F-theory side. We also discuss applications to $SO(32)$ heterotic strings. We derive the Weierstrass equations of K3 elliptic fibrations appearing as the compactification spaces of the F-theory duals of some $SO(32)$ heterotic strings. We determine the gauge groups that arise on F-theory compactifications, including the global structures of the gauge groups. Some of the cases of elliptically fibered K3 surfaces with extended gauge groups in Section 3.3 can also be found in [57].

We build elliptically fibered Calabi–Yau 3-folds in Section 4 by fibering examples of elliptic K3 surfaces, which belong to the F-theory side of the moduli of eight-dimensional unbroken $\mathfrak{e}_7\mathfrak{e}_7$ non-geometric heterotic strings, over $\mathbb{P}^1$. We analyze F-theory compactifications on the resulting elliptic Calabi–Yau 3-folds. First, we consider the higher-dimensional analog of the construction of genus-one fibered K3 surfaces without a global section[6] to build genus-one fibered Calabi–Yau 3-folds without a section. This

[5]Recent discussions of F-theory compactifications on elliptic Calabi–Yau 3-folds can be found, e.g., in [39, 42–56]. The authors of [58–60] discussed F-theory on Calabi–Yau 3-folds with terminal singularities.

[6]Recent studies of F-theory compactifications on genus-one fibered spaces lacking a global section can be found in, for example, [58, 65–85].

construction ensures that the resulting 3-folds in fact satisfy the Calabi–Yau condition. Similar constructions of genus-one fibered Calabi–Yau 4-folds without a section using double covers can be found in [76]. Taking the Jacobian fibration[7] of the resulting genus-one fibered Calabi–Yau 3-folds yields elliptically fibered Calabi–Yau 3-folds with a global section. K3 fibers of these elliptic Calabi–Yau 3-folds belong to the F-theory side of the moduli of eight-dimensional non-geometric heterotic strings with unbroken $\mathfrak{e}_7\mathfrak{e}_7$ algebra. Therefore, the obtained elliptic Calabi–Yau 3-folds can be seen as the fibering of such K3 surfaces over the base curve $\mathbb{P}^1$. We deduce the gauge groups on F-theory compactifications on the elliptic Calabi–Yau 3-folds, and we find that some specific models do not have a $U(1)$ gauge field. We determine the Mordell–Weil groups of some models, and we obtain the global structures of the gauge groups of these models. We also deduce candidate matter spectra on F-theory on the constructed elliptically fibered Calabi–Yau 3-folds that satisfy the six-dimensional anomaly cancellation condition. We determine these candidate matter spectra directly from the global defining equations of the elliptically fibered Calabi–Yau 3-folds. We state our concluding remarks in Section 5.

2. Review of non-geometric heterotic strings with unbroken $\mathfrak{e}_7\mathfrak{e}_7$ algebra, F-theory, and extremal K3 surfaces

2.1. Review of F-theory compactifications

We briefly review F-theory compactifications on elliptic K3 surfaces. A similar review can be found in [39]. F-theory is compactified on spaces that admit a genus-one fibration. The complex structure of the genus-one fiber is identified with the axio-dilaton in F-theory compactification. This formulation allows the axio-dilaton to have $SL(2,\mathbb{Z})$ monodromy. Genus-one fibrations do not necessarily admit a global section; there are situations in which they have a global section, and those in which they do not. F-theory compactifications on elliptic fibrations with a global section have been investigated in recent studies, for example, in [21, 87–112]. Although Calabi–Yau genus-one fibration lacking a global section cannot be expressed in the Weierstrass form, when the Jacobian fibration of it exists, the Jacobian fibration yields an elliptic fibration with a global section. Calabi–Yau genus-one fibration Y and the Jacobian fibration $J(Y)$ have the identical types of the singular fibers, and they have the same discriminant loci.

[7] [86] discussed the Jacobians of elliptic curves.

Genus-one fibers degenerate over the codimension 1 locus in the base space, and this locus is referred to as the discriminant locus. Such degenerate fibers are called the singular fibers. When genus-one fiber degenerates, it becomes either $\mathbb{P}^1$ with a single singularity, or a sum of smooth $\mathbb{P}^1$'s meeting in specific ways. The types of the singular fibers of genus-one fibered surfaces were classified by Kodaira [113, 114]. Methods to determine the singular fibers of elliptic surfaces can be found in [115, 116].

In F-theory compactifications on genus-one fibrations, the non-Abelian gauge groups that form on the 7-branes correspond to the singular fibers of genus-one fibrations [3, 117]. The correspondences of the singular fibers and the singularity types of the compactification spaces are shown in Table 1 below. The corresponding monodromies and j-invariants of the singular fibers are also presented in the table.

Fiber type	J-invariant	Monodromy	Order of Monodromy	Singularity Type
I_0^*	regular	$-\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	2	D_4
I_m	∞	$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$	infinite	A_{m-1}
I_m^*	∞	$-\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$	infinite	D_{m+4}
II	0	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	6	none.
II^*	0	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	6	E_8
III	1728	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	4	A_1
III^*	1728	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	4	E_7
IV	0	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	3	A_2
IV^*	0	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	3	E_6

Table 1: Monodromies, j-invariants and the corresponding types of the singularities of singular fibers. “Regular” for j-invariant of I_0^* fiber means that j-invariant can take any finite value in $\mathbb{C}$ for I_0^* fiber.

The types of singular fibers of elliptic surfaces can be determined from the vanishing orders of the coefficients of the Weierstrass equations. The correspondences of the fiber types and the vanishing orders of the Weierstrass coefficients are shown in Table 2.

Fiber Type	Ord(f)	Ord(g)	Ord(Δ)
I_0	≥ 0	≥ 0	0
I_n $(n \geq 1)$	0	0	n
II	≥ 1	1	2
III	1	≥ 2	3
IV	≥ 2	2	4
I_0^*	≥ 2	3	6
	2	≥ 3	
I_m^* $(m \geq 1)$	2	3	$m+6$
IV^*	≥ 3	4	8
III^*	3	≥ 5	9
II^*	≥ 4	5	10

Table 2: List of the types of the singular fibers, and the corresponding vanishing orders of the coefficients, f, g, of the Weierstrass equation $y^2 = x^3 + f\,x + g$, and the orders of the discriminant, Δ.

When an elliptic fibration has a global section, the set of sections form a group, known as the Mordell–Weil group. The rank of the Mordell–Weil group gives the number of the $U(1)$ gauge fields in F-theory compactification on the elliptic fibration [3].

The second integral cohomology group $H^2(S, \mathbb{Z})$ of K3 surface S includes the information of the geometry of the K3 surface. This group has the lattice structure, and it is called the K3 lattice, Λ_{K3}. The K3 lattice is unimodular, even lattice of signature (3,19), and it is isometric to the direct sum of two E_8's and three hyperbolic planes [118]

$$\Lambda_{K3} \cong E_8^2 \oplus H^3. \tag{5}$$

The group of divisors (modulo algebraic equivalence) constitutes a sublattice inside the K3 lattice, called the Néron-Severi lattice $NS(S)$. When a K3 surface has an elliptic fibration with a global section, an elliptic fiber and a global section generate the hyperbolic plane H inside the Néron-Severi lattice. K3 surface S admitting an elliptic fibration with a section is equivalent to the condition that the Néron-Severi lattice $NS(S)$ contains the hyperbolic plane H [119]. When an elliptic K3 surface has the singular fibers, the Néron-Severi lattice $NS(S)$ contains the ADE lattices that correspond to the types of the singular fibers. For example, that a K3 surface S is $H \oplus E_7 \oplus E_7$-lattice polarized means that the Néron-Severi lattice $NS(S)$ includes the lattice $H \oplus E_7 \oplus E_7$ where the hyperbolic plane H is generated by the fiber class and the class of the section of the elliptic fibration. When an elliptic K3 surface has an elliptic fibration with a section the singular fibers of which include two type III^* fibers (or worse), then its Néron-Severi lattice contains the lattice $H \oplus E_7 \oplus E_7$ where the fiber class and the class of the section of the elliptic fibration generate H. One must require that the image of the lattice H embedded inside the Néron-Severi lattice includes a pseudo-ample class to ensure that it corresponds to an elliptic fibration with a section as mentioned in [24]. K3 surfaces with $H \oplus E_7 \oplus E_7$ lattice polarization are parametrized by the bounded symmetric domain of type IV, $D_{2,4}$, modded out by the symmetry of the orthogonal complement of the lattice $H \oplus E_7 \oplus E_7$ inside the K3 lattice Λ_{K3}:

$$D_{2,4}/O^+(L^{2,4}). \tag{6}$$

$L^{2,4}$ denotes the orthogonal complement of $H \oplus E_7 \oplus E_7$ inside the K3 lattice Λ_{K3}. Because the K3 lattice Λ_{K3} is isometric to $E_8^2 \oplus H^3$, $L^{2,4}$ can also be defined as the orthogonal complement of $E_7 \oplus E_7$ in the lattice $E_8^2 \oplus H^2$.

By utilizing the F-theory/heterotic duality, eight-dimensional non-geometric heterotic strings with unbroken $\mathfrak{e}_7\mathfrak{e}_7$, the moduli space of which is equivalent to (6) were constructed in [31]. In this note, we study the points in the moduli of such non-geometric heterotic strings at which the non-Abelian gauge symmetries are enhanced to rank 18 on the F-theory side.

2.2. Construction of $\mathfrak{e}_7\mathfrak{e}_7$ non-geometric heterotic strings by Clingher, Malmendier, and Shaska

We briefly review the construction of eight-dimensional non-geometric heterotic strings with unbroken $\mathfrak{e}_7\mathfrak{e}_7$ by Clingher, Malmendier and Shaska [31].

As stated previously, the moduli of elliptic K3 surfaces with a global section with two E_7 singularities, namely the K3 surfaces with $H \oplus E_7 \oplus E_7$ lattice polarization, are parameterized by the following space:

$$D_{2,4}/O^+(L^{2,4}). \tag{7}$$

The bounded symmetric domain of type IV, $D_{2,4}$, is known to be isomorphic to $\mathbf{H}_2$ [120]:

$$\mathbf{H_2} \cong D_{2,4}. \tag{8}$$

$\mathbf{H_2}$ is defined as
(9)

$$\mathbf{H_2} := \left\{ \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \in M_2(\mathbb{C}) \;\middle|\; 4\mathrm{Im}\, z_1 \mathrm{Im}\, z_4 > |z_2 - \overline{z_3}|^2 \text{ and } \mathrm{Im}\, z_4 > 0 \right\}.$$

As mentioned in [31], $\mathbf{H_2}$ is a generalization of the Siegel upper-half space $\mathbb{H}_2$ in the following sense:

$$\mathbb{H}_2 = \left\{ \omega \in \mathbf{H_2} \;\middle|\; \omega^t = \omega \right\}. \tag{10}$$

The modular group Γ acting on $\mathbf{H_2}$ is defined as

$$\Gamma = \left\{ G \in GL_4(\mathbb{Z}[i]) \;\middle|\; G^\dagger \begin{pmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{pmatrix} G = \begin{pmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{pmatrix} \right\}. \tag{11}$$

$\mathbf{1}_2$ denotes the 2×2 identity matrix. $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in the modular group Γ acts on $\omega \in \mathbf{H}_2$ as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \omega = (A\,\omega + B)(C\,\omega + D)^{-1}. \tag{12}$$

There is an involution, $\mathcal{T}$, that acts on $\mathbf{H}_2$ as

$$\mathcal{T} \cdot \omega = \omega^t. \tag{13}$$

The group $\Gamma_{\mathcal{T}}$ is defined to be the semi-direct product of the modular group Γ and $< \mathcal{T} >$:

$$\Gamma_{\mathcal{T}} := \Gamma \rtimes < \mathcal{T} > . \tag{14}$$

There is an isomorphism $\Gamma_{\mathcal{T}} \cong O^+(L^{2,4})$, and this induces the isomorphism (8) [120].

Under the isomorphism $\Gamma_{\mathcal{T}} \cong O^+(L^{2,4})$, the ring of $O^+(L^{2,4})$-modular forms corresponds to the ring of $\Gamma_{\mathcal{T}}$-modular forms of even characteristic [121], generated by the five modular forms J_k of weights $2k$, $k = 2, \cdots, 6$ [31]. See [31] for definitions of the modular forms J_k. In a special situation, the modular forms J_2, J_3, J_5, J_6 restrict to Igusa's generators [122], $\psi_4, \psi_6, \chi_{10}, \chi_{12}$ (and J_4 vanishes in this situation) [31].

The periods of $H \oplus E_7 \oplus E_7$ lattice polarized K3 surfaces S in $H^2(S, \mathbb{Z})$ determine points in $\mathbf{H}_2$. The Weierstrass coefficients of such elliptically fibered K3 surfaces were given in terms of $\Gamma_{\mathcal{T}}$-modular forms of even characteristic [31].

The Weierstrass equation of a K3 surface with $H \oplus E_7 \oplus E_7$ lattice polarization is given by [31]:

$$y^2 = x^3 + (e\,t^4 + c\,t^3 + a\,t^2)x + t^7 + g\,t^6 + (d\,e + f)\,t^5 + c\,d\,t^4 + b\,t^3. \tag{15}$$

Up to some scale factors, the coefficients are given in terms of the modular forms J_2, J_3, J_4, J_5, J_6 [31]:

$$\begin{aligned} c &= -J_5(\omega), \quad d = -\tfrac{1}{3}J_4(\omega), & e &= -3J_2(\omega) \\ f &= J_6(\omega), & g &= -2J_3(\omega) \\ a = -3d^2 &= -\frac{1}{3}J_4(\omega)^2, & b &= -2d^3 = \frac{2}{27}J_4(\omega)^3. \end{aligned} \tag{16}$$

The elliptically fibered K3 surface determines a point in $D_{2,4}$, and this also determines a point in $\mathbf{H}_2$ under the isomorphism (8), which we denote by ω.

Now, consider a manifold M and a line bundle Λ on M, and choose sections a, b, c, d, e, f, g of the line bundles $\Lambda^{\otimes 16}$, $\Lambda^{\otimes 24}$, $\Lambda^{\otimes 10}$, $\Lambda^{\otimes 8}$, $\Lambda^{\otimes 4}$, $\Lambda^{\otimes 12}$ and $\Lambda^{\otimes 6}$, respectively. When the sections a, b, c, d, e, f, g are identified as (16), because $\Gamma_{\mathcal{T}}$ is isomorphic to $O^+(L^{2,4})$, the compactification on M (which is the 2-torus when we consider 8D heterotic strings) gives a heterotic string theory with $O^+(L^{2,4})$-symmetry. The moduli space of eight-dimensional heterotic strings on the 2-torus T^2 decomposes into the product of the complex structure moduli, the Wilson line expectation values and Kähler moduli, in a suitable limit [123]. The complex structure moduli, the Wilson line expectation values and Kähler moduli are mixed under the $O^+(L^{2,4})$-symmetry. This represents the construction of non-geometric heterotic strings with $\mathfrak{e}_7\mathfrak{e}_7$ gauge algebra in [31].

The locus in the moduli in which the singularity ranks of elliptic K3 surfaces are enhanced satisfies 5-brane solutions on the heterotic side. The generic 5-brane solutions of non-geometric heterotic strings with $\mathfrak{e}_7\mathfrak{e}_7$ gauge algebra are discussed in [31].

Elliptic K3 surfaces with the lattice polarization $H \oplus E_7 \oplus E_7$ were described in [29] as the minimal resolution of the quartic hypersurfaces in $\mathbb{P}^3$ given by the following equations:

$$Y^2ZW - 4X^3Z + 3\alpha XZW^2 + \beta ZW^3 + \gamma XZ^2W - \frac{1}{2}(\zeta W^4 + \delta Z^2W^2) + \varepsilon XW^3 = 0, \tag{17}$$

where $[X : Y : Z : W]$ are homogeneous coordinates on $\mathbb{P}^3$. $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ are parameters, and $(\gamma, \delta) \neq (0, 0)$, and $(\varepsilon, \zeta) \neq (0, 0)$.

Making the following substitutions

$$\begin{aligned} X &= tx \\ Y &= y \\ W &= 4t^3 \\ Z &= 4t^4 \end{aligned} \tag{18}$$

yields the Weierstrass equation with two type III^* fibers as follows [31] :

$$y^2 = x^3 + 4t^3\,(\gamma t^2 - 3\alpha t + \varepsilon)\,x - 8t^5\,(\delta t^2 + 2\beta t + \zeta). \tag{19}$$

Type III^* fibers are at $t = 0$ and at $t = \infty$.

K3 surface with the lattice polarization $H \oplus E_7 \oplus E_7$ given by (17) always admits another fibration with a type II^* fiber and a type I_2^* fiber, as shown in [31], and the Weierstrass equation of this fibration is:

(20)
$$\begin{aligned} y^2 =& x^3 - \frac{1}{3}\big[9\alpha t^4 + 3(\gamma\zeta + \delta\varepsilon)\,t^3 + (\gamma\varepsilon)^2\,t^2\big]x \\ &+ \frac{1}{27}\big[27t^7 - 54\beta t^6 + 27(\alpha\gamma\varepsilon + \delta\zeta)\,t^5 + 9\gamma\varepsilon(\delta\varepsilon + \gamma\zeta)\,t^4 + 2(\gamma\varepsilon)^3\,t^3\big]. \end{aligned}$$

The Weierstrass equation (20) was used in [31] to construct eight-dimensional non-geometric heterotic strings with unbroken $\mathfrak{e}_7\mathfrak{e}_7$. (Compare the equation (20) with the equation (15).) Although the presence of two E_7 singularities in the Weierstrass equation is explicit in the equation (19), as stated in [31], the Weierstrass equation (19) does not necessarily extend over the entire parameter space. For this reason, the Weierstrass equation (20) was instead used to construct non-geometric heterotic strings with unbroken $\mathfrak{e}_7\mathfrak{e}_7$ in [31].

The K3 surface with the lattice polarization $H \oplus E_7 \oplus E_7$ (17) also admits another elliptic fibration further, the singular fibers of which include a type I_8^* fiber (or worse) [31]. This alternate fibration relates to $SO(32)$ heterotic string. The Weierstrass equation of this fibration is obtained by making the following substitutions into the equation (17) [31]:

$$\begin{aligned} X &= tx^3 \\ Y &= \sqrt{2}x^2y \\ W &= 2x^3 \\ Z &= 2x^2(-\varepsilon t + \zeta). \end{aligned} \tag{21}$$

The Weierstrass equation is [31] :

$$y^2 = x^3 + Ax^2 + Bx, \tag{22}$$

where

$$\begin{aligned} A &= t^3 - 3\alpha t - 2\beta \\ B &= (\gamma t - \delta)(\varepsilon t - \zeta). \end{aligned} \tag{23}$$

The discriminant is given by

$$\Delta = B^2\,(A^2 - 4B). \tag{24}$$

2.3. Extremal K3 surfaces

By the Shioda–Tate formula [124–126], the following equality holds for an elliptic surface S with a global section:

$$\mathrm{rk}\,L + \mathrm{rk}MW + 2 = \rho(S). \tag{25}$$

We have used rk L to denote the rank of the root lattice L generated by the fiber components of the reducible singular fibers of an elliptic surface S not meeting the zero section. The intersection matrix of root lattice L corresponds to a sum of ADE Dynkin diagrams. The Picard number $\rho(S)$ ranges from 2 to 20 for an elliptic K3 surface with a section. Thus, the rank of the singularity of an elliptic K3 surface S with a section is bounded by:

$$\mathrm{rk}\,L = \rho(S) - 2 - \mathrm{rk}MW \leq 18 - \mathrm{rk}MW. \tag{26}$$

Therefore, the rank of the singularity of an elliptic K3 surface S with a section can be 18 at the highest, and this value is achieved precisely when the Picard number attains the highest value 20, and the Mordell–Weil rank is 0. Physically, this means that the rank of the gauge group on F-theory compactification on an elliptic K3 surface is at most 18, and when the non-Abelian gauge group has the rank 18, it does not have a $U(1)$ gauge field.

K3 surfaces with Picard number 20 are called *attractive* K3 surfaces[8]. The complex structure moduli of the attractive K3 surfaces is known to be parametrized by three integers. The transcendental lattice $T(S)$ of a K3 surface S is the orthogonal complement of the Néron–Severi lattice $NS(S)$ inside the K3 lattice Λ_{K3}, and the transcendental lattices $T(S)$ of attractive K3 surfaces are positive-definite, even 2×2 lattices. The complex structure of an attractive K3 surface is determined by the transcendental lattice [128, 129]. The intersection form of the transcendental lattice of an attractive K3 surface can be transformed into the following form under the $GL_2(\mathbb{Z})$ action:

$$\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}. \tag{27}$$

Here a, b, c are integers, $a, b, c \in \mathbb{Z}$, and satisfy the relations:

$$a \geq c \geq b \geq 0. \tag{28}$$

Thus, the triplet of integers, a, b, c, parameterizes the complex structure moduli of the attractive K3 surfaces. We denote an attractive K3 surface, whose transcendental lattice has the intersection form (27) as $S_{[2a\ b\ 2c]}$ in this note.

An elliptic attractive K3 surfaces with a section is said to be *extremal* when it has Mordell–Weil rank 0. This condition is equivalent to an elliptic K3 surface with a section having singularity rank 18. Thus, the non-Abelian gauge group forming in F-theory compactification on an elliptic K3 surface has rank 18 precisely when the K3 surface is extremal. In Section 3, we study the points in the moduli of eight-dimensional non-geometric heterotic strings with unbroken $\mathfrak{e}_7\mathfrak{e}_7$ algebra at which the non-Abelian gauge groups are enhanced to rank 18 on the F-theory side. Elliptic K3 surfaces on the F-theory side become extremal at these points.

Elliptically fibered K3 surfaces generally admit distinct elliptic fibrations[9], and distinct elliptic fibrations have different singularity types and different Mordell–Weil groups. Physically, this means that the gauge groups

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[8]We refer to complex K3 surfaces with the highest Picard number 20 as attractive K3 surfaces, following the convention for the term used in [127].

[9]Genus-one fibered K3 surfaces in general admit both genus-one fibrations without a section, as well as elliptic fibrations with a section. However, as shown in

and $U(1)$ gauge fields that arise in F-theory compactification on an elliptic K3 surface with the fixed complex structure vary, because there still remains freedom to choose a fibration structure among the distinct choices of elliptic fibrations of that elliptic K3 surface[10].

The attractive K3 surface whose transcendental lattice has the intersection form

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \tag{29}$$

is particularly relevant to the contents of this study. The elliptic fibrations of the attractive K3 surface $S_{[2\ 0\ 2]}$ were classified in [131] and there are 13 types. We list these 13 types of elliptic fibration of the attractive K3 surface $S_{[2\ 0\ 2]}$ in Appendix A.

3. Special points in the moduli of eight-dimensional non-geometric heterotic strings and F-theory duals with enhanced gauge groups

3.1. Summary

There are finitely many points in the moduli of eight-dimensional non-geometric heterotic strings with unbroken $\mathfrak{e}_7\mathfrak{e}_7$ algebra, at which the non-Abelian gauge groups on the F-theory side are enhanced to rank 18.

In Section 3.2, we show that the heterotic strings at these special points in the moduli can be described as deformations of the stable degenerations, as a result of the coincident 7-branes on the F-theory side. This effect can be seen as the insertion of 5-branes in the heterotic language. We also discuss applications to $SO(32)$ heterotic strings.

As stated in Section 2.3, K3 surfaces become extremal on the F-theory side at these points in the moduli. The complex structures of the extremal K3 surfaces were classified in [41], and using this result, the complex structures of the extremal K3 surfaces at these points in the moduli can be determined. This enables us to determine the Weierstrass equations of the extremal K3 surfaces that appear as compactification spaces on the F-theory side in the

[130], the attractive K3 surface with discriminant four, $S_{[2\ 0\ 2]}$, only admits elliptic fibrations with a global section. The authors of [105] discussed F-theory compactification on the surface $S_{[2\ 0\ 2]}$, in relation to the appearances of the $U(1)$ factor.

[10]This point is discussed in [19].

moduli. By using this approach, we study the physics of the theories at the enhanced special points in the moduli.

In eight-dimensional $E_8 \times E_8$ heterotic strings on the 2-torus T^2, only the gauge groups up to $E_8 \times E_8 \times U(1)^4$ can arise in the perturbative description [39]. This implies that the heterotic dual of F-theory on an extremal elliptic K3 surface with the non-Abelian gauge group of rank 18 does not allow for the perturbative interpretation of the gauge group. This can reflect some non-perturbative aspects of the non-geometric heterotic strings.

3.2. F-theory on extremal K3 surfaces and non-geometric heterotic duals in the moduli

We discuss the points in the moduli of non-geometric heterotic strings with unbroken $\mathfrak{e}_7\mathfrak{e}_7$ algebra, at which the non-Abelian gauge symmetries on the F-theory side are enhanced to rank 18. K3 surfaces as compactification spaces on the F-theory side become extremal at these points. There are finitely many such points in the moduli, and the complex structures and the singularity types of the extremal K3 surfaces that appear in the moduli can be determined from Table 2 of [41]. Among these, those of the singularity types, which include E_8E_7, are studied in [39]. We do not discuss these extremal K3 surfaces in this note. Instead, we discuss the extremal K3 surfaces that belong to the moduli, the singularity types of which include E_7^2 [11].

The singularity types of the extremal K3 surfaces in the moduli of K3 surfaces with $H \oplus E_7 \oplus E_7$ lattice polarization, which do not include E_8, are as follows [41]:

$$E_7^2A_3A_1, \quad E_7^2D_4, E_7^2A_4, \quad E_7^2A_2^2. \tag{30}$$

We study F-theory on the extremal K3 surfaces possessing the first two singularity types in this note.

Because the perturbative eight-dimensional heterotic strings on T^2 can have up to $E_8 \times E_8 \times U(1)^4$ gauge group, the heterotic duals of F-theory on these extremal K3 surfaces do not allow for the perturbative interpretation of these gauge groups [39]. As we demonstrate in Sections 3.2.1 and 3.2.2, these theories can be seen as deformations of the stable degenerations as a result of the coincident 7-branes on the F-theory side. These theories satisfy multiple 5-brane solutions on the heterotic side.

[11] The singularity types of extremal K3 surfaces can also be enhanced to E_8D_7, as discussed in [31].

3.2.1. Extremal K3 surface with $E_7^2 A_3 A_1$ singularity. The complex structure of the extremal K3 surface with $E_7^2 A_3 A_1$ singularity is uniquely determined, and its transcendental lattice has the following intersection form [41]:

$$\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}. \tag{31}$$

Therefore, the attractive K3 surface $S_{[4\ 0\ 2]}$[12] admits an extremal fibration with the singularity type $E_7^2 A_3 A_1$, and F-theory on this extremal fibration has non-geometric heterotic dual with unbroken $\mathfrak{e}_7\mathfrak{e}_7$. The Weierstrass form of this extremal fibration can be found in [23] as

$$y^2 = x^3 - \frac{9}{16}(t^2 + s^2 + \frac{10}{3}ts)\, t^3 s^3\, x + \frac{9}{4} t^5 s^5\, (\frac{1}{4}t^2 + \frac{1}{4}s^2 + \frac{7}{18}ts), \tag{32}$$

the singular fibers of which consist of two type III^* fibers, a type I_4 fiber, and a type I_2 fiber. The above Weierstrass equation was obtained in [23] as the quadratic base change of an extremal rational elliptic surface. Geometrically, the quadratic base change of a rational elliptic surface is to glue a pair of identical rational elliptic surfaces. Extremal rational elliptic surfaces are the rational elliptic surfaces with a global section, the singularity types of which have rank 8. The types of singular fibers of the extremal rational elliptic surfaces were classified in [133]. The fiber types of the extremal rational elliptic surfaces are listed in Appendix B.

The complex structures of extremal rational elliptic surfaces are uniquely specified by the fiber types, except those with two fibers of type I_0^* (see [133]). The complex structures of extremal rational elliptic surfaces with two fibers of type I_0^* depend on the j-invariants of the fibers. The j-invariant j of an extremal rational elliptic surface with two type I_0^* fibers is constant over the base, and the fixed j specifies the complex structure [133]. In this study, we denote, for example, the extremal rational elliptic surface with a type III^* fiber and a type III fiber as $X_{[III,\ III^*]}$. We simply use n to denote a singular fiber of type I_n and m^* to represent a fiber of type I_m^*. The extremal rational elliptic surface with a type III^* fiber, a type I_2 fiber and a type I_1 fiber is denote as $X_{[III^*,\ 2,1]}$. Because the complex structure of an extremal rational elliptic surface with two type I_0^* fibers depends on the j-invariant of the elliptic fibers, we use $X_{[0^*,\ 0^*]}(j)$ to denote this extremal rational elliptic surface.

[12] The elliptic fibrations and the Weierstrass equations of the attractive K3 surface $S_{[4\ 0\ 2]}$ were obtained in [132].

Now we demonstrate that F-theory on an extremal K3 surface (32) can be seen as a deformation of stable degeneration, owing to an effect of coincident 7-branes. As deduced in [23], the K3 extremal fibration (32) is obtained as the quadratic base change of the extremal rational elliptic surface $X_{[III^*,\,2,1]}$ in which two type I_2 fibers and two type I_1 fibers collide. Whereas the quadratic base change of a rational elliptic surface generally yields an elliptic K3 surface, with twice as many singular fibers as the original rational elliptic surface, at the special limits at which singular fibers collide, the singularity type of the resulting K3 surface is enhanced. As discussed in [23], two identical extremal rational elliptic surfaces $X_{[III^*,\,2,1]}$ are glued together to yield an elliptic K3 surface, which we denote as S_1, the singular fibers of which consist of two type III^* fibers, two type I_2 fibers, and two type I_1 fibers. In the special limit at which 7-branes over which type I_2 fiber lies coincide with those over which type I_2 fiber lies, and 7-brane over which type I_1 fiber lies coincides with 7-brane over which type I_1 fiber lies, two type I_2 fibers are enhanced to type I_4 fiber, and two type I_1 fibers are enhanced to type I_2 fiber. Because a K3 surface with two type III^* fibers, a type I_4 fiber, and a type I_2 fiber has the singularity type $E_7^2A_3A_1$, the K3 surface S_1 deforms and it becomes an extremal K3 surface (32) in this limit. In short, F-theory on the extremal K3 surface (32) can be seen as deformation of the stable degeneration because of the coincident 7-branes.

As the singularity rank of a rational elliptic surface is up to 8, the non-Abelian gauge group that arises on F-theory on a generic K3 surface obtained as the reverse of the stable degeneration has rank up to 16. Here, by generic we mean a situation in which singular fibers of rational elliptic surfaces do not collide when they are glued together to yield an elliptic K3 surface. When the large radius limit is taken, the heterotic dual of this compactification admits a geometric interpretation. In special situations in which singular fibers collide, 7-branes become coincident and the singularity ranks of the resulting K3 surfaces enhance to become greater than 16. The gauge groups of the heterotic duals of F-theory compactifications on these K3 surfaces do not allow for geometric interpretation.

When the BPS solitons constitute a faithful representation of the gauge group G, the fundamental group of the gauge group G, $\pi_1(G)$, is isomorphic to the torsion part of the Mordell–Weil group of the elliptic fibration on which F-theory is compactified [134]. The Mordell–Weil group of the K3 extremal fibration (32) is isomorphic to $\mathbb{Z}_2$ [41, 132]. Thus, the global structure [134–136] of the gauge group that arises in F-theory compactification on this extremal K3 surface is [23]

$$E_7 \times E_7 \times SU(4) \times SU(2)/\mathbb{Z}_2. \tag{33}$$

Comparing the Weierstrass equation (32) with equation (19), we find that the following substitutions:

$$\begin{aligned} \alpha &= \frac{5}{32} \\ \beta &= -\frac{7}{128} \\ \gamma = \varepsilon &= -\frac{9}{64} \\ \delta = \zeta &= -\frac{9}{128} \end{aligned} \tag{34}$$

into equation (19) yield the Weierstrass equation (32). Plugging the substitutions (34) into the equation of the alternate fibration (20), we obtain the following equation:

$$y^2 = x^3 - \frac{1}{3}\,(10t^4 + 8t^3 + t^2)\,x + t^7 + \frac{56}{27}t^6 + \frac{26}{9}t^5 + \frac{8}{9}t^4 + \frac{2}{27}t^3, \tag{35}$$

with the discriminant

$$\Delta \sim t^{11}(t+2)^2(27t+4). \tag{36}$$

From equations (35) and (36), we find that the fibration (35) has a type II^* fiber at $t = \infty$, a type I_5^* fiber at $t = 0$, a type I_2 fiber at $t = -2$, and a type I_1 fiber at $t = -4/27$. Thus, the fibration (35) has singularity type $E_8D_9A_1$, and because the singularity type has rank 18, we deduce that this fibration is also extremal. Therefore, we find that the attractive K3 surface $S_{[4\ 0\ 2]}$ also admits an extremal fibration (35) with singularity type $E_8D_9A_1$. This agrees with the results in [41, 132].

3.2.2. Extremal K3 surface with $E_7^2D_4$ singularity. The complex structure of the extremal K3 surface with the singularity type $E_7^2D_4$ is uniquely determined, and the intersection form of the transcendental lattice is [41]:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}. \tag{37}$$

The Weierstrass equation of this extremal fibration of the attractive K3 surface $S_{[2\ 0\ 2]}$ is given as follows:

$$y^2 = x^3 + 4t^3(t-s)^2\, s^3\, x, \tag{38}$$

with the discriminant

$$\Delta \sim t^9 s^9\, (t-s)^6. \tag{39}$$

$[t:s]$ in the equation (38) denotes the homogeneous coordinate of the base $\mathbb{P}^1$. Two type III^* are at $[t:s]=[0:1]$ and $[1:0]$, and a type I_0^* fiber is at $[t:s]=[1:1]$.

As shown in [23], the extremal K3 fibration (38) can be seen as deformation of the stable degeneration. Gluing two identical extremal rational elliptic surfaces $X_{[III,\ III^*]}$ yields an elliptic K3 surface, S_2, the singular fibers of which have two type III^* fibers and two type III fibers. This is technically given by a generic quadratic base change of the extremal rational elliptic surface $X_{[III,\ III^*]}$, and this is the reverse of the stable degeneration. In a special limit at which two type III fibers collide, the elliptic K3 surface S_2 deforms to yield the extremal fibration (38) of the attractive K3 surface $S_{[2\ 0\ 2]}$ [23]. 7-branes over which type III fiber lies coincide with those over which type III fiber lies in this limit, at which colliding two type III fibers are enhanced to a type I_0^* fiber. Therefore, F-theory on the extremal K3 surface (38) can be seen as deformation of the stable degeneration as the consequence of the coincident 7-branes.

The Mordell–Weil group of the extremal elliptic fibration (38) is isomorphic to $\mathbb{Z}_2$ [41, 131]. Therefore, the gauge group on F-theory compactification on the extremal fibration (38) is [23]

$$E_7 \times E_7 \times SO(8)/\mathbb{Z}_2. \tag{40}$$

Comparing the equation (38) with the equation (19), we find that the following substitutions

$$\begin{aligned} \alpha &= \frac{2}{3} \\ \beta = \delta = \zeta &= 0 \\ \gamma = \varepsilon &= 1 \end{aligned} \tag{41}$$

into (19) yield the Weierstrass equation (38).

By plugging the substitutions (41) into the equation (20), we obtain the following Weierstrass equation:

$$y^2 = x^3 - \frac{1}{3}t^2(6t^2+1)x + \frac{1}{27}t^3(27t^4+18t^2+2), \tag{42}$$

with the discriminant

$$\Delta \sim t^{12}(27t^2+4). \tag{43}$$

We can confirm from the equations (42) and (43) that this alternate fibration in fact has a type II^* fiber at $t=\infty$, a type I_6^* fiber at $t=0$, and two type I_1 fibers at the roots of $27t^2+4=0$. Thus, the singularity type of the alternate fibration is E_8D_{10}, and we find that this fibration is also extremal. This gives the Weierstrass equation of the fibration no. 2 in Table A1 of the attractive K3 surface $S_{[2\ 0\ 2]}$ in Appendix A.

Double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ ramified along a bidegree (4,4) curve, given by the following equation:

$$\tau^2 = (t-\alpha_1)^3(t-\alpha_2)\,x^4 + (t-\alpha_3)^3(t-\alpha_2) \tag{44}$$

yields a genus-one fibered K3 surface lacking a global section, but admitting a bisection, and this K3 surface was considered in [75] in the context of F-theory compactifications on genus-one fibrations without a global section. x denotes the inhomogeneous coordinate of the first $\mathbb{P}^1$, and t denotes the inhomogeneous coordinate of the second $\mathbb{P}^1$, in the product $\mathbb{P}^1 \times \mathbb{P}^1$, respectively. α_1, α_2, α_3 are distinct points in $\mathbb{P}^1$. α's are superfluous parameters, and these can be mapped to:

$$\alpha_1 = 0, \;\; \alpha_2 = 1, \;\; \alpha_3 = \infty \tag{45}$$

under some appropriate automorphism of the base $\mathbb{P}^1$. The K3 genus-one fibration (44) has two type III^* fibers at $t=\alpha_1, \alpha_3$, and a type I_0^* fiber at $t=\alpha_2$ [75].

The Jacobian fibration of the K3 genus-one fibration (44) gives the extremal K3 elliptic fibration (38), as demonstrated in [75]. Utilizing this fact, in Section 4 we build an elliptically fibered Calabi–Yau 3-fold, by fibering the K3 genus-one fibration (44) over the base $\mathbb{P}^1$, then taking the Jacobian fibration of it. It turns out that the resulting Calabi–Yau 3-fold is a fibration of the extremal K3 surface (38) over the base $\mathbb{P}^1$. We also build a family of elliptic Calabi–Yau 3-folds which includes this Calabi–Yau 3-fold. These constructions will be discussed in detail in Section 4.

3.3. Applications to $SO(32)$ heterotic strings

As reviewed in Section 2.2, the K3 surface (17) with $H \oplus E_7 \oplus E_7$ lattice polarization admits another elliptic fibration, the singular fibers of which include a type I_8^* fiber, given by the Weierstrass equation:

$$y^2 = x^3 + (t^3 - 3\alpha t - 2\beta)\, x^2 + (\gamma t - \delta)(\varepsilon t - \zeta)\, x \tag{46}$$

with the discriminant

$$\Delta \sim (\gamma t - \delta)^2 (\varepsilon t - \zeta)^2 \big[(t^3 - 3\alpha t - 2\beta)^2 - 4(\gamma t - \delta)(\varepsilon t - \zeta)\big]. \tag{47}$$

Therefore, there is a birational map that transforms the elliptic fibration with two type III^* fibers (19) into an alternate fibration (46) with a type I_8^* fiber (or worse). Using this birational map, we send the extremal K3 elliptic fibrations with two E_7 singularities studied in Section 3.2 to another fibration with a type I_8^* fiber (or worse). This relates to $SO(32)$ heterotic strings.

As we saw previously in Section 3.2.2, the Weierstrass equation of the extremal fibration of the attractive K3 surface $S_{[2\ 0\ 2]}$ with singularity type $E_7^2 D_4$ is given by (38), with

$$\begin{aligned} \alpha &= \frac{2}{3} \\ \beta = \delta = \zeta &= 0 \\ \gamma = \varepsilon &= 1. \end{aligned} \tag{48}$$

By plugging these values (48) into the alternate fibration (46), we obtain the Weierstrass equation as

$$y^2 = x^3 + t(t^2 - 2)\, x^2 + t^2\, x, \tag{49}$$

with the discriminant

$$\begin{aligned} \Delta &\sim t^4 \left(t^2 (t^2 - 2)^2 - 4t^2\right) \\ &= t^8\, (t^2 - 4). \end{aligned} \tag{50}$$

In the homogeneous form, the discriminant is

$$\Delta \sim t^8 s^{14} (t^2 - 4s^2). \tag{51}$$

From equations (49) and (51), we deduce that the alternate fibration (49) has a type I_8^* fiber at $[t:s] = [1:0]$, a type I_2^* fiber at $[t:s] = [0:1]$, and two type I_1 fibers at $[t:s] = [2:1], [-2:1]$. Thus, the alternate fibration (49) has singularity type $D_{12}D_6$, and we find that this fibration is also extremal. We conclude that the alternate fibration (49) yields fibration no. 8 in Table A1 in Appendix A of the attractive K3 surface $S_{[2\ 0\ 2]}$.

The Mordell–Weil group of the fibration (49) is isomorphic to $\mathbb{Z}_2$ (see [41, 131]); therefore, the gauge group on F-theory compactification on the fibration (49) is

$$SO(24) \times SO(12)/\mathbb{Z}_2. \tag{52}$$

The attractive K3 surface $S_{[2\ 0\ 2]}$ has another extremal fibration with the singularity type $D_8^2 A_1^2$ [131]. (This is fibration no.9 in Table A1 in Appendix A.) As deduced in [23], the Weierstrass equation of this extremal fibration is given as follows:

$$y^2 = x^3 - 3t^2s^2\,(t^4 + s^4 - t^2s^2)\,x + (t^2 + s^2)\,t^3s^3\,(2t^4 - 5t^2s^2 + 2s^4), \tag{53}$$

with the discriminant

$$\Delta \sim t^{10}s^{10}\,(t-s)^2\,(t+s)^2. \tag{54}$$

Type I_4^* fibers are at $[t:s] = [1:0], [0:1]$, and type I_2 fibers are at $[t:s] = [1:1], [1:-1]$.

As shown in [23], extremal fibration (53) can be seen as deformation of the stable degeneration in which two extremal rational elliptic surfaces $X_{[4^*,\ 1,1]}$ are glued together. Gluing of two extremal rational elliptic surfaces $X_{[4^*,\ 1,1]}$ yields an elliptic K3 surface, which we denote by S_3, the singular fibers of which are two type I_4^* fibers and four type I_1 fibers. In a limit at which two pairs of type I_1 fibers collide, type I_1 fibers collide and they are enhanced to a type I_2 fiber. In this limit, K3 surface S_3 deforms to yield the attractive K3 surface with the extremal fibration (53) [23]. Therefore, extremal fibration (53) can be seen as deformation of the stable degeneration, as a result of coincident 7-branes over which type I_1 fibers lie.

The Mordell–Weil group of the fibration (53) is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ [41, 131]; therefore, the gauge group on F-theory compactification on the fibration (53) is [23]

$$SO(16) \times SO(16) \times SU(2)^2/\mathbb{Z}_2 \times \mathbb{Z}_2. \tag{55}$$

We saw previously in Section 3.2.1 that the attractive K3 surface $S_{[4\ 0\ 2]}$ admits an extremal fibration with the singularity type $E_7^2 A_3 A_1$, and the Weierstrass equation of this fibration is given by (32), with

$$\begin{aligned} \alpha &= \frac{5}{32} \\ \beta &= -\frac{7}{128} \\ \gamma = \varepsilon &= -\frac{9}{64} \\ \delta = \zeta &= -\frac{9}{128}. \end{aligned} \tag{56}$$

By plugging these into the equation (46), we obtain an alternate fibration given by:

$$y^2 = x^3 + (t^3 - \frac{15}{32}t + \frac{7}{64})\, x^2 + (\frac{9}{64})^2 (t - \frac{1}{2})^2\, x, \tag{57}$$

with the discriminant

$$\Delta \sim (t - \frac{1}{2})^7 (4t + 1)^2 (t + 1). \tag{58}$$

From the equations (57) and (58), we find that the alternate fibration has a type I_8^* fiber at $t = \infty$, a type I_1^* fiber at $t = \frac{1}{2}$, a type I_2 fiber at $t = -\frac{1}{4}$, and a type I_1 fiber at $t = -1$. Thus, the alternate fibration (57) has the singularity type $D_{12} D_5 A_1$, and this is also extremal. This result agrees with the elliptic fibrations with a section of the attractive K3 surface $S_{[4\ 0\ 2]}$ obtained in [132].

The Mordell–Weil group of the alternate fibration (57) is isomorphic to $\mathbb{Z}_2$ [41, 132]. Therefore, the gauge group on F-theory compactification on the fibration (57) is

$$SO(24) \times SO(10) \times SU(2)/\mathbb{Z}_2. \tag{59}$$

4. Jacobian Calabi–Yau 3-folds and F-theory compactifications

In this section, we fiber elliptic K3 surfaces over $\mathbb{P}^1$ to yield elliptically fibered Calabi–Yau 3-folds[13] with a global section, and we study six-dimensional F-theory compactifications with $N = 1$ supersymmetry on the constructed Calabi–Yau 3-folds. K3 fibers in this construction include an elliptic K3 surface that belongs to the F-theory side of the moduli of eight-dimensional non-geometric heterotic strings with unbroken $\mathfrak{e}_7\mathfrak{e}_7$ algebra, discussed in Section 3.2.2.

To be clear, we first consider genus-one fibered K3 surfaces lacking a global section, built as double covers of $\mathbb{P}^1 \times \mathbb{P}^1$ ramified along a (4,4) curve. The built genus-one fibered K3 surfaces do not have a global section, but they have a bisection [75]. We consider higher-dimensional analogs of these K3 surfaces to yield genus-one fibered Calabi–Yau 3-folds, built as double covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ ramified along a tridegree (4,4,4) surface. As we show in Section 4.1, the constructed Calabi–Yau 3-folds are genus-one fibered, but they lack a global section. These Calabi–Yau 3-folds still have a bisection [65]. The pullback of $\mathcal{O}(1)$ class in $\mathbb{P}^1$ yields a bisection [78].

Taking the Jacobian fibrations of these Calabi–Yau genus-one fibrations yields elliptically fibered Calabi–Yau 3-folds with a global section. The resulting elliptic Calabi–Yau 3-folds are also K3 fibered, and when we tune the coefficients of the defining equation, we obtain Calabi–Yau 3-folds, K3 fibers of which are the attractive K3 surface $S_{[2\ 0\ 2]}$ that belongs to the F-theory side of the moduli of non-geometric heterotic strings with unbroken $\mathfrak{e}_7\mathfrak{e}_7$ algebra.

We deduce the non-Abelian gauge groups on F-theory compactifications on the Jacobian Calabi–Yau 3-folds. We also perform a consistency check of the obtained gauge groups, by considering the symmetry that the elliptic fibers possess. Highly enhanced gauge groups arise when we choose specific coefficients of the defining equations of the Jacobian Calabi–Yau 3-folds. We determine the Mordell–Weil groups of some specific Calabi–Yau 3-folds, and we deduce the global structures of the gauge groups for F-theory on these spaces. We obtain some models without a $U(1)$ gauge field. Furthermore, we deduce viable candidate matter spectra on F-theory on the constructed elliptically fibered Calabi–Yau 3-folds that satisfy the six-dimensional anomaly cancellation condition.

[13]In [137–139] the elliptic fibrations of 3-folds were discussed.

4.1. Calabi–Yau 3-folds as double covers, Jacobian fibrations, and the discriminant locus

Double covers of the product of projective lines, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, ramified over a $(4,4,4)$ surface have the trivial canonical bundles, $K = 0$; therefore, they describe Calabi–Yau 3-folds. Fiber of projection onto $\mathbb{P}^1 \times \mathbb{P}^1$ is a double cover of $\mathbb{P}^1$ branched over four points, namely, it is an elliptic curve. Thus, projection onto $\mathbb{P}^1 \times \mathbb{P}^1$ gives a genus-one fibration. Fiber of projection onto $\mathbb{P}^1$ is a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ ramified along a $(4,4)$ curve, which yields a genus-one fibered K3 surface. Therefore, projection onto $\mathbb{P}^1$ yields a K3 fibration. These K3 surfaces do not have a section, but they have a bisection [75].

In this note, we consider in particular the double covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ given by the following equations:

$$\tau^2 = f_1(t)g_1(u)\,x^4 + f_2(t)g_2(u), \tag{60}$$

where x is the inhomogeneous coordinate on the first $\mathbb{P}^1$, and t and u are the inhomogeneous coordinates on the second and the third $\mathbb{P}^1$ in the product $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Here f_1, f_2 are polynomials in the variable t of degree four, and g_1, g_2 are polynomials of degree four in the variable u. By splitting the polynomials f_1, f_2, g_1, g_2 into linear factors, equation (60) can be rewritten as follows:

$$\tau^2 = \Pi_{i=1}^4 (t-\alpha_i) \cdot \Pi_{j=1}^4 (u-\beta_j) \cdot x^4 + \Pi_{k=5}^8 (t-\alpha_k) \cdot \Pi_{l=5}^8 (u-\beta_l). \tag{61}$$

K3 fiber of this genus-one fibered Calabi–Yau 3-fold is given by

$$\tau^2 = \Pi_{i=1}^4 (t-\alpha_i) \cdot x^4 + \Pi_{k=5}^8 (t-\alpha_k). \tag{62}$$

As shown in [75], K3 fiber (62) is genus-one fibered, but it does not have a global section. K3 fiber (62) has a bisection [75].

Using an argument similar to that in [76], we can show that the Calabi–Yau 3-fold (61) indeed lacks a rational section. Suppose it admits a rational section. Then, it restricts to a K3 fiber, and this gives a global section to the K3 fiber, leading to a contradiction. By an argument similar to those in [75, 76, 78], the genus-one fibered Calabi–Yau 3-fold (61) has a bisection[14].

[14]Thus, a discrete $\mathbb{Z}_2$ symmetry [65] arises in six-dimensional F-theory compactifications on the genus-one fibered Calabi–Yau 3-folds (61).

We can consider a special situation in which

$$\alpha_1 = \alpha_2 = \alpha_3, \quad \alpha_4 = \alpha_8, \quad \alpha_5 = \alpha_6 = \alpha_7. \tag{63}$$

This yields a genus-one fibered Calabi–Yau 3-fold

$$\tau^2 = (t-\alpha_1)^3(t-\alpha_4)\cdot \Pi_{j=1}^4(u-\beta_j)\cdot x^4 + (t-\alpha_5)^3(t-\alpha_4)\cdot \Pi_{l=5}^8(u-\beta_l), \tag{64}$$

and K3 fiber given by

$$\tau^2 = (t-\alpha_1)^3(t-\alpha_4)\, x^4 + (t-\alpha_5)^3(t-\alpha_4). \tag{65}$$

This is the genus-one fibered K3 surface (44) lacking a section, which we discussed in Section 3.2.2[15].

The Jacobian fibrations of the genus-one fibered Calabi–Yau 3-folds (61) yield elliptically fibered Calabi–Yau 3-folds with a section. The Jacobian fibrations are given by [140]

$$\tau^2 = \frac{1}{4}x^3 - \Pi_{i=1}^8(t-\alpha_i)\cdot \Pi_{j=1}^8(u-\beta_j)\cdot x. \tag{66}$$

Projection onto $\mathbb{P}^1 \times \mathbb{P}^1$ gives an elliptic fibration, and projection onto $\mathbb{P}^1$ yields a K3 fibration. K3 fiber of the projection onto $\mathbb{P}^1$ is given by

$$\tau^2 = \frac{1}{4}x^3 - \Pi_{i=1}^8(t-\alpha_i)\cdot x. \tag{67}$$

When parameters α are tuned as in (63), the Weierstrass equation of the Jacobian fibration becomes

$$\tau^2 = \frac{1}{4}x^3 - (t-\alpha_1)^3(t-\alpha_5)^3(t-\alpha_4)^2\cdot \Pi_{j=1}^8(u-\beta_j)\, x, \tag{68}$$

and K3 fiber (67) becomes

$$\tau^2 = \frac{1}{4}x^3 - (t-\alpha_1)^3(t-\alpha_5)^3(t-\alpha_4)^2\, x. \tag{69}$$

This is the Jacobian fibration of the K3 fiber (65), and this is the extremal fibration (38) of the attractive K3 surface $S_{[2\ 0\ 2]}$[16], which belongs to the

[15] As we stated previously in Section 3.2.2, $\alpha_1, \alpha_4, \alpha_5$ in equation (65) are superfluous parameters, and they can be mapped to $0, 1, \infty$ under certain automorphism of the base $\mathbb{P}^1$.

[16] Using a coordinate transformation, equation (69) can be replaced with

$$\tau^2 = x^3 + 4(t-\alpha_1)^3(t-\alpha_4)^2(t-\alpha_5)^3\, x.$$

F-theory side of the moduli of the non-geometric heterotic strings with unbroken $\mathfrak{e}_7\mathfrak{e}_7$ algebra.

The obtained Jacobian Calabi–Yau 3-folds (66) yield fibrations of K3 surfaces (67) over $\mathbb{P}^1$, and this family includes fibrations of the extremal K3 surface $S_{[2\ 0\ 2]}$, which we discussed in Section 3.2.2, over $\mathbb{P}^1$.

The discriminant of the Calabi–Yau Jacobian fibration (66) is given by the following equation:

$$\Delta \sim \Pi_{i=1}^{8}(t-\alpha_i)^3 \cdot \Pi_{j=1}^{8}(u-\beta_j)^3. \tag{70}$$

The discriminant locus of the Jacobian Calabi–Yau 3-fold (66) is given by the vanishing of the discriminant (70). Genus-one fibered Calabi–Yau 3-fold (61) and the Jacobian fibration (66) have the identical discriminant loci, and the identical configurations of the singular fibers.

From the equation (70), we find that the discriminant components of the Jacobian Calabi–Yau 3-fold (66) are given as follows:

$$\begin{aligned} A_i &= \{t=\alpha_i\} \quad (i=1,\cdots,8) \\ B_j &= \{u=\beta_j\} \quad (j=1,\cdots,8). \end{aligned} \tag{71}$$

In F-theory on the Jacobian Calabi–Yau 3-fold (66), 7-branes are wrapped on these components. Components A_i are isomorphic to $\{\mathrm{pt}\}\times\mathbb{P}^1$, and components B_j are isomorphic to $\mathbb{P}^1\times\{\mathrm{pt}\}$. Therefore, these are isomorphic to $\mathbb{P}^1$. The types of the singular fibers and the non-Abelian gauge groups on the 7-branes will be discusses in Section 4.2.

4.2. Non-Abelian gauge groups

We determine the non-Abelian gauge groups on F-theory on the Jacobian Calabi–Yau 3-folds constructed in Section 4.1. We also check the consistency of the obtained gauge groups.

4.2.1. Singular fibers of the Jacobian Calabi–Yau 3-folds, and non-Abelian gauge groups. From the Weierstrass equation (66) of the Jacobian Calabi–Yau 3-fold and the discriminant (70), we find that for a generic situation in which the coefficients α's and β's are mutually distinct, the types of the singular fibers on the components A_i, $i=1,\cdots,8$, and B_j, $j=1,\cdots,8$, are III. In this case, the gauge algebra that arises on F-theory compactification on the Jacobian Calabi–Yau 3-fold (66) is $\mathfrak{su}(2)^{16}$.

As stated previously, we can send $\alpha_1,\alpha_4,\alpha_5$ to $0,1,\infty$, and this yields (38).

When two of the coefficients, α_i and α_k, become coincident, a pair of type III fibers on the components A_i and A_k collides, and it is enhanced to a type I_0^* fiber. Because the polynomial

$$x^3 - \Pi_{j=1}^{8}(u-\beta_j)\cdot x \tag{72}$$

splits into the linear factor and the quadratic factor as:

$$x\left(x^2 - \Pi_{j=1}^{8}(u-\beta_j)\right), \tag{73}$$

type I_0^* fiber is semi-split [117]. The non-Abelian gauge group (as a local factor) that arises on the 7-branes wrapped on the component A_i is thus enhanced to $SO(7)$ in this situation [17] .

When three of the coefficients, α_i, α_k and α_l, become coincident, type III fibers on the components A_i, A_k, A_l collide, and they are enhanced to a type III^* fiber. Further enhancement breaks the Calabi–Yau condition, as stated in [75, 76]. An argument similar to that stated previously equally applies to β's and the components B_j. The results are presented in Table 3 below.

As discussed in Section 4.1, K3 fiber becomes most enhanced when parameters α are tuned as:

$$\alpha_1 = \alpha_2 = \alpha_3, \qquad \alpha_4 = \alpha_8, \qquad \alpha_5 = \alpha_6 = \alpha_7. \tag{75}$$

In this case, the gauge algebra that arises on F-theory compactification on the Jacobian Calabi–Yau 3-fold (68) is $\mathfrak{e}_7^2 \oplus \mathfrak{so}(7) \oplus \mathfrak{su}(2)^8$. K3 fiber becomes the attractive K3 surface $S_{[2\ 0\ 2]}$ with the singularity type $E_7^2 D_4$ (69) in this situation, and this attractive K3 surface was discussed in Section 3.2.2. The singularity type of the Jacobian Calabi–Yau 3-fold (66) is most enhanced,

[17] In a special situation in which there are four pairs of identical β's, e.g. $\beta_1 = \beta_5$, $\beta_2 = \beta_6$, $\beta_3 = \beta_7$, $\beta_4 = \beta_8$, the polynomial splits into three linear factors:

$$x\left(x - \Pi_{j=1}^{4}(u-\beta_j)\right)\left(x + \Pi_{j=1}^{4}(u-\beta_j).\right) \tag{74}$$

In this special situation, type I_0^* fiber over the component A_i becomes split, and the gauge group that forms on the 7-branes wrapped on the component A_i is enhanced to $SO(8)$.

Component	Fiber type	non-Abel. Gauge Group
$A_{1,\cdots,8}$	III	$SU(2)$
$A_{1,\cdots,8}$	I_0^*	$SO(7)$
$A_{1,\cdots,8}$	III^*	E_7
$B_{1,\cdots,8}$	III	$SU(2)$
$B_{1,\cdots,8}$	I_0^*	$SO(7)$
$B_{1,\cdots,8}$	III^*	E_7

Table 3: Fiber types and the gauge groups on the discriminant components.

when the following equalities hold among coefficients β's further:

(76) $$\beta_1 = \beta_2 = \beta_3, \quad \beta_4 = \beta_8, \quad \beta_5 = \beta_6 = \beta_7.$$

In this case, the Weierstrass equation of the Jacobian Calabi–Yau 3-fold becomes

(77)
$$\tau^2 = \frac{1}{4}x^3 - (t-\alpha_1)^3(t-\alpha_5)^3(t-\alpha_4)^2 \cdot (u-\beta_1)^3(u-\beta_5)^3(u-\beta_4)^2\, x.$$

The types of the singular fibers over the components, A_1, A_5, B_1, B_5, are enhanced to type III^*, and the types of the singular fibers over the components A_4 and B_4 are enhanced to type I_0^*. In this situation, the gauge algebra on the F-theory compactification of the Jacobian Calabi–Yau 3-fold (77) is enhanced to: $\mathfrak{e}_7^4 \oplus \mathfrak{so}(7)^2$.

We determine the Mordell–Weil groups of F-theory models on the Jacobian Calabi–Yau 3-folds (68). We deduce that they do not have a $U(1)$ gauge field, and we also deduce the global structures of the gauge groups formed in the models.

We saw previously that the Weierstrass equation of the Jacobian Calabi–Yau 3-fold becomes (68):

(78) $$\tau^2 = \frac{1}{4}x^3 - (t-\alpha_1)^3(t-\alpha_5)^3(t-\alpha_4)^2 \cdot \Pi_{j=1}^8(u-\beta_j)\, x,$$

when the K3 fibers are most enhanced, namely when K3 fibers become the attractive K3 surface $S_{[2\ 0\ 2]}$ with the singularity type $E_7^2 D_4$. The Mordell–Weil group of this extremal K3 surface is known [41, 131], and it is isomorphic to $\mathbb{Z}_2$. (See fibration no.4 in Table A1 in Appendix A.) Using an argument similar to those given in [76, 78], we consider the specialization to the K3 fiber to deduce that the Mordell–Weil group of the Jacobian Calabi–Yau 3-fold (78) is isomorphic to that of the K3 fiber. Therefore, we find that the Mordell–Weil group of the Jacobian Calabi–Yau 3-fold $J(Y)$ (78) is isomorphic to $\mathbb{Z}_2$:

$$MW(J(Y)) \cong \mathbb{Z}_2. \tag{79}$$

Thus, the global structure of the gauge group forming on the 7-branes is given as follows:

$$E_7^2 \times SO(7) \times SU(2)^8/\mathbb{Z}_2. \tag{80}$$

The F-theory on the Jacobian Calabi–Yau 3-folds (78) does not have a $U(1)$ gauge field.

4.2.2. Consistency check of the gauge groups. By an argument similar to those given in [75, 76], smooth genus-one fibers of the Calabi–Yau double covers (60) are invariant under the following transformation:

$$x \to e^{\frac{2\pi i}{4}} \cdot x. \tag{81}$$

We find from this that genus-one fibers of the Calabi–Yau double covers (60) possess complex multiplication of order 4; therefore, the generic genus-one fiber of the Calabi–Yau double cover (60) has j-invariant 1728. This requires the singular fibers to have j-invariant 1728 [75, 76]. Because the types of the singular fibers of Calabi–Yau genus-one fibration and those of the Jacobian fibration are identical, this means that the singular fibers of the Jacobian Calabi–Yau 3-fold (66) also have j-invariant 1728.

According to Table 1 in Section 2.1, the types of the singular fibers with j-invariant 1728 are: III, III^*, and I_0^*. (j-invariant of type I_0^* fiber can take the value 1728.) Thus, the Jacobian Calabi–Yau 3-fold (66) can have the singular fibers, only of types III, I_0^* and III^*. This agrees with the results obtained in Section 4.2.1. The monodromies of orders 2 and 4 characterize the non-Abelian gauge groups that form on F-theory compactifications of the Jacobian Calabi–Yau 3-folds (66).

4.3. Matter spectra

In this section, we deduce the candidate matter spectra on six-dimensional F-theory on the Jacobian Calabi–Yau 3-folds constructed in Section 4.1. As we stated previously, 7-branes are wrapped on the discriminant components given by

$$\begin{aligned} t &= \alpha_i \quad (i = 1, \ldots, 8) \\ u &= \beta_j \quad (j = 1, \ldots, 8). \end{aligned} \tag{82}$$

7-branes wrapping the components are isomorphic to $\mathbb{P}^1$. 7-branes intersect at the points $(t, u) = (\alpha_i, \beta_j)$, $i, j = 1, \ldots, 8$, in the base surface, and matter arises at these points.

The base surface B of the Jacobian Calabi–Yau 3-folds constructed in Section 4.1 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, $B \cong \mathbb{P}^1 \times \mathbb{P}^1$, thus the number of tensor multiplets that arise in F-theory compactification on the Jacobian Calabi–Yau 3-folds is [3]

$$\begin{aligned} T &= h^{1,1}(B = \mathbb{P}^1 \times \mathbb{P}^1) - 1 = 2 - 1 \\ &= 1. \end{aligned} \tag{83}$$

The six-dimensional anomaly cancellation condition [141–144] then requires that

$$H - V = 273 - 29 = 244. \tag{84}$$

In the equation (84), V and H represent the numbers of vector multiplets and hypermultiplets, respectively, in the 6D model.

For simplicity, we only consider the case in which K3 fibers are most enhanced, namely K3 fibers become the extremal K3 surface with the singularity type $E_7^2 D_4$. This corresponds to the case where the coefficients α satisfy the relations (63)

$$\alpha_1 = \alpha_2 = \alpha_3, \qquad \alpha_4 = \alpha_8, \qquad \alpha_5 = \alpha_6 = \alpha_7. \tag{85}$$

As we saw in Section 4.2.1, for this situation the Mordell–Weil rank of the Jacobian Calabi–Yau 3-folds is 0, and the Mordell–Weil group is isomorphic to $\mathbb{Z}_2$.

We will derive candidate matter representations in 6D F-theory on the Jacobian Calabi–Yau 3-folds for several situations where some pairs or triplets of β's coincide in Sections 4.3.1 - 4.3.6. A situation where multiple β's coincide corresponds to that multiple 7-branes wrapped on $\{u = \beta\}$ coincide. In this situation, the singularity type of the Jacobian Calabi–Yau 3-fold becomes enhanced.

We make a remark here that, while we will derive candidate matter spectra arising in 6D F-theory in Sections 4.3.1 - 4.3.6, they are *only* candidates of the matter representations of the 6D theory arising from the intersection of 7-branes on the Jacobian Calabi–Yau 3-folds. They do not necessarily yield the actual matter representations, which can only be obtained by analyzing the resolution of the singularities. We do not determine whether the derived candidate matter representations in this note yield the actual matter representations.

We yield several possibilities for matter representations through the collision of the singularities in Sections 4.3.1 - 4.3.6. They are all consistent with the anomaly cancellation condition, and they all fall within the structure of matter representations arising from codimension two singularities in F-theory discussed by the authors in [46].

The collision of singularities of the following types appear in Sections 4.3.1 - 4.3.6: (D_4, A_1), (E_7, A_1), (D_4, D_4), (E_7, D_4), and (E_7, E_7). Matter transforming in the 56-dimensional quaternionic fundamental representation of E_7 is denoted as $\frac{1}{2}\mathbf{56}$ in this note. Matter transforming in $\rho \oplus \overline{\rho}$ in the quaternionic representation, where ρ is an irreducible complex representation, yields a hypermultiplet; matter transforming in an quaternionic irreducible representation is referred to as a $\frac{1}{2}$-hypermultiplet [42]. $\frac{1}{2}$ in $\frac{1}{2}\mathbf{56}$ indicates to count 1/2 of the quaternionic dimension of the representation. We find that a $\frac{1}{2}$-hypermultiplet $\frac{1}{2}\mathbf{56}$ arises [18] through incomplete resolution [46] of the E_8 at the collision of type (E_7, A_1) as we will discuss in Section 4.3.1. We expect that matter representation arising at the collision of singularities of type (D_4, A_1) includes $\mathbf{7}$ owing to an analysis in [145]. Because $\mathbf{7} \otimes \mathbf{7}$ decomposes as [146] $\mathbf{27} \oplus \mathbf{21} \oplus \mathbf{1}$, we expect that matter representation arising at the collision of singularities of type (D_4, D_4) includes some combination of $\mathbf{27}$, $\mathbf{21}$ (and $\mathbf{1}$).

We yield the Weierstrass equations of the Jacobian Calabi–Yau 3-folds in Sections 4.3.1 - 4.3.6. Utilizing these equations, one obtains descriptions of the local geometries around the collisions of singularities. The associated matter representations can be obtained via the local analysis of codimension

[18] A pair of $\frac{1}{2}$-hypermultiplets associated with a quadratic parameter for an E_7 singularity was discussed in [145].

two singularities as discussed in [46] through the simultaneous resolution of singularities [147].

For example, the local geometry of the Jacobian Calabi–Yau 3-fold around the codimension two singularity at the collision of singularities of type (E_7, A_1) encountered in Sections 4.3.1 - 4.3.5 is locally given by the Weierstrass equation of the following form:

$$y^2 = x^3 - t^3 u\, x. \tag{86}$$

A codimension one E_7 singularity arises along the locus $t = 0$, and a codimension two singularity appears at $u = 0$ with singularity enhancement $E_7 \to E_8$. Matter representation arising from the codimension two singularity at the collision of singularities of type (E_7, A_1) can be obtained through a singularity resolution, and a $\frac{1}{2}$-hypermultiplet arises through the incomplete resolution of singularity as we will mention in Section 4.3.1.

4.3.1. Case where a pair of β coincide. First, we consider the case where a pair of β coincide:

$$\beta_1 = \beta_2. \tag{87}$$

Then the equation of the Jacobian Calabi–Yau 3-fold (66) becomes

$$\tau^2 = \frac{1}{4}x^3 - (t-\alpha_1)^3(t-\alpha_5)^3(t-\alpha_4)^2 \cdot (u-\beta_1)^2 \cdot \Pi_{j=3}^{8}(u-\beta_j) \cdot x \tag{88}$$

and the discriminant is given as follows

$$\Delta \sim (t-\alpha_1)^9(t-\alpha_5)^9(t-\alpha_4)^6 \cdot (u-\beta_1)^6 \cdot \Pi_{j=3}^{8}(u-\beta_j)^3. \tag{89}$$

The gauge algebra forming in F-theory compactification is $\mathfrak{e}_7^2 \oplus \mathfrak{so}(7)^2 \oplus \mathfrak{su}(2)^6$ [19]. Therefore, we have

$$V = 133 \times 2 + 21 \times 2 + 6 \times 3 = 326. \tag{90}$$

[19]Because the Mordell–Weil group of the Calabi–Yau elliptic fibration (68) is isomorphic to $\mathbb{Z}_2$, the global structure of the gauge group forming in F-theory compactification on the Calabi–Yau 3-fold (88) is $E_7^2 \times SO(7)^2 \times SU(2)^6/\mathbb{Z}_2$. Using an argument similar to that given here, the global structures of the gauge groups formed in F-theory compactifications in Sections 4.3.2 - 4.3.6 are found to be global $\mathbb{Z}_2$-quotients.

The anomaly cancellation condition (84) requires that

$$H = V + 244 = 326 + 244 = 570. \tag{91}$$

Matter arises at the 21 intersections

$$(\alpha_i, \beta_j) \qquad (i = 1, 4, 5, \;\; j = 1, 3, 4, 5, 6, 7, 8). \tag{92}$$

Parameters $\alpha_1, \alpha_4, \alpha_5$ can be sent to fixed values, e.g., 0, 1, ∞, under an automorphism of the first $\mathbb{P}^1$ in the base $\mathbb{P}^1 \times \mathbb{P}^1$. Therefore, these are superfluous and are not actual parameters of the complex structure deformation. Among parameters of the complex structure deformation $\beta_1, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8$, three of these can be fixed to specific values under an automorphism of the second $\mathbb{P}^1$ in the base $\mathbb{P}^1 \times \mathbb{P}^1$. Thus, the number of the effective parameters of the complex structure deformation is four. The number of the neutral hypermultiplets arising from the complex structure deformations is therefore given by

$$H^0 = 1 + 4 = 5. \tag{93}$$

Here, H^0 is used to denote the number of the neutral hypermultiplets. It follows that the sum of the dimensions of the representations of matter arising at the 21 intersections (α_i, β_j), $i = 1, 4, 5$, $j = 1, 3, 4, 5, 6, 7, 8$, must be

$$570 - 5 = 565 \tag{94}$$

to cancel the anomaly.

E_7 angularity and D_4 singularity collide at two intersections $(t, u) = (\alpha_1, \beta_1), (\alpha_5, \beta_1)$, and D_4 singularities collide at the intersection $(t, u) = (\alpha_4, \beta_1)$. E_7 singularity and A_1 singularity collide at the 12 intersections $(t, u) = (\alpha_i, \beta_j)$, $i = 1, 5$, $j = 3, \ldots, 8$. D_4 singularity and A_1 singularity collide at the six intersections $(t, u) = (\alpha_4, \beta_j)$, $j = 3, \ldots, 8$. When the types of colliding singularities are fixed, from a symmetry argument, the representations of matter arising at the intersections at which the fixed types of singularities collide should be identical.

If we assume that $\mathbf{56} \oplus \mathbf{7} \oplus \mathbf{1}$ arises at the two intersections where E_7 angularity and D_4 singularity collide, $\mathbf{27} \oplus \mathbf{1} \oplus \mathbf{1}$ arises at the intersection where D_4 singularities collide, $\frac{1}{2}\mathbf{56} \oplus \mathbf{2}$ arises at the 12 intersections where E_7 singularity and A_1 singularity collide and $\mathbf{7} \oplus \mathbf{1}$ arises at the six intersections where D_4 singularity and A_1 singularity collide, then the net dimension of the matter representations arising at the intersections of the 7-branes is

$$(56+7+1)\times 2+(27+1+1)+(28+2)\times 12+(7+1)\times 6=565. \quad (95)$$

Therefore, these matter representations yield a candidate of consistent matter spectrum on F-theory compactification on the Calabi–Yau 3-fold (88) that satisfies the anomaly cancellation condition. Here $\frac{1}{2}\mathbf{56}$ denotes the $\frac{1}{2}$-hypermultiplet of $\mathbf{56}$ of E_7[20]. Having $\frac{1}{2}\mathbf{56}\oplus\mathbf{1}$ instead of $\frac{1}{2}\mathbf{56}\oplus\mathbf{2}$ at the 12 intersections $(t,u)=(\alpha_i,\beta_j)$, $i=1,5$, $j=3,\ldots,8$, and $\mathbf{7}\oplus\mathbf{2}\oplus\mathbf{1}$ instead of $\mathbf{7}\oplus\mathbf{1}$ at the six intersections $(t,u)=(\alpha_4,\beta_j)$, $j=3,\ldots,8$ also yields another viable candidate of matter spectrum. Additionally, having matter representation $\mathbf{56}\oplus\mathbf{7}\oplus\mathbf{1}\oplus\mathbf{1}$, instead of $\mathbf{56}\oplus\mathbf{7}\oplus\mathbf{1}$, at the intersections where E_7 angularity and D_4 singularity collide, matter representation $\mathbf{21}$, instead of $\mathbf{27}\oplus\mathbf{1}\oplus\mathbf{1}$, at the intersection where two D_4 singularities collide, and matter representation $\mathbf{7}\oplus\mathbf{2}$, instead of $\mathbf{7}\oplus\mathbf{1}$, at the intersections where D_4 singularity and A_1 singularity collide also yields another viable candidate matter spectrum.

It is known that there are codimension two singularities where the apparent Kodaira singularity type does not need a complete resolution to yield a smooth Calabi–Yau manifold [46]. This actually happens for the collision of singularities of type (E_7,A_1) that we described here. We saw that a $\frac{1}{2}$-hypermultiplet $\frac{1}{2}\mathbf{56}$ yields a candidate matter arising at the collision of singularities of type (E_7,A_1). Analyzing the resolution of the singularity at the collision, one learns that $\frac{1}{2}$-hypermultiplet $\frac{1}{2}\mathbf{56}$ is an actual matter arising at the collision of type (E_7,A_1), and the $\frac{1}{2}$-hypermultiplet arises when the E_8 at the collision is incompletely resolved owing to the mechanism similar to that discussed in [46].

[20]The appearance of $\frac{1}{2}$-hypermultiplets of $\mathbf{56}$ of E_7 in F-theory compactification was discussed in [2, 3, 117]. The base of elliptically fibered Jacobian Calabi–Yau 3-folds (88), $\mathbb{P}^1\times\mathbb{P}^1$, is isomorphic to Hirzebruch surface $\mathbb{F}_0$. Weierstrass coefficient f of the Weierstrass equation $y^2=x^3+f\,x+g$ of Jacobian Calabi–Yau 3-fold (88) is given by

$$f=(t-\alpha_1)^3(t-\alpha_5)^3(t-\alpha_4)^2\cdot(u-\beta_1)^2\cdot\Pi_{j=3}^{8}(u-\beta_j).$$

Candidate $\frac{1}{2}$-hypermultiplets $\frac{1}{2}\mathbf{56}$ localized at the twelve intersections have an interpretation as localized at the intersections of E_7 loci $t=\alpha_{1,5}$ and the zeroes of $\Pi_{j=3}^{8}(u-\beta_j)$ [117].

4.3.2. Case where three pairs of the parameters $\boldsymbol{\beta}$ coincide. Next, we discuss the case where three pairs of the parameters β coincide:

$$\beta_1 = \beta_2 \quad (96)$$
$$\beta_3 = \beta_4$$
$$\beta_5 = \beta_6.$$

The equation of the Calabi–Yau 3-fold (68) becomes

$$\tau^2 = \frac{1}{4}x^3 - (t-\alpha_1)^3(t-\alpha_5)^3(t-\alpha_4)^2 \cdot (u-\beta_1)^2(u-\beta_3)^2(u-\beta_5)^2 \cdot \Pi_{j=7}^{8}(u-\beta_j)\cdot x \quad (97)$$

and the discriminant is given as follows

$$\Delta \sim (t-\alpha_1)^9(t-\alpha_5)^9(t-\alpha_4)^6 \cdot (u-\beta_1)^6(u-\beta_3)^6(u-\beta_5)^6 \cdot \Pi_{j=7}^{8}(u-\beta_j)^3. \quad (98)$$

The gauge algebra forming in F-theory compactification is: $\mathfrak{e}_7^2 \oplus \mathfrak{so}(7)^4 \oplus \mathfrak{su}(2)^2$. Thus, anomaly cancellation condition (84) requires that $H = 133 \times 2 + 21 \times 4 + 3 \times 2 + 244 = 600$. Using an argument similar to that given in Section 4.3.1, it turns out that the actual number of the parameters of the complex structure deformation is two, and we obtain the number of the neutral hypermultiplets arising from the complex structure deformations as $H^0 = 1 + 2 = 3$. The net representation dimensions of matter arising from the intersections of 7-branes must be $597(= 600 - 3)$ owing to the anomaly cancellation condition.

E_7 angularity and D_4 singularity collide at six intersections $(t,u) = (\alpha_i, \beta_j)$, $i = 1,5$, $j = 1,3,5$, and D_4 singularities collide at the three intersections $(t,u) = (\alpha_4, \beta_j)$, $j = 1,3,5$. E_7 singularity and A_1 singularity collide at the four intersections $(t,u) = (\alpha_i, \beta_j)$, $i = 1,5$, $j = 7,8$. D_4 singularity and A_1 singularity collide at the two intersections $(t,u) = (\alpha_4, \beta_j)$, $j = 7,8$.

We assume that $\mathbf{56} \oplus \mathbf{7} \oplus \mathbf{1}$ arises at the six intersections where E_7 angularity and D_4 singularity collide, $\mathbf{27} \oplus \mathbf{1} \oplus \mathbf{1}$ arises at the three intersection where D_4 singularities collide, $\frac{1}{2}\mathbf{56}$ arises at the four intersections where E_7 singularity and A_1 singularity collide and $\mathbf{7}$ arises at the two intersections where D_4 singularity and A_1 singularity collide, then a computation shows that this yields a consistent matter candidate on F-theory on the Jacobian Calabi–Yau 3-fold (97). Having $\mathbf{56} \oplus \mathbf{7}$, instead of $\mathbf{56} \oplus \mathbf{7} \oplus \mathbf{1}$, at the six intersections where E_7 angularity and D_4 singularity collide, and

$\mathbf{7} \oplus \mathbf{2} \oplus \mathbf{1}$, instead of $\mathbf{7}$, at the two intersections where D_4 singularity and A_1 singularity collide, also yields another viable candidate matter spectrum. Furthermore, having $\mathbf{56} \oplus \mathbf{7}$, instead of $\mathbf{56} \oplus \mathbf{7} \oplus \mathbf{1}$, at the six intersections where E_7 angularity and D_4 singularity collide, $\frac{1}{2}\mathbf{56} \oplus \mathbf{1}$, instead of $\frac{1}{2}\mathbf{56}$, at the four intersections where E_7 singularity and A_1 singularity collide and $\mathbf{7} \oplus \mathbf{1}$, instead of $\mathbf{7}$, at the two intersections where D_4 singularity and A_1 singularity collide, yields another viable candidate matter spectrum. In addition to these, having $\mathbf{56} \oplus \mathbf{7} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$, instead of $\mathbf{56} \oplus \mathbf{7} \oplus \mathbf{1}$, at the six intersections where E_7 angularity and D_4 singularity collide, $\mathbf{21}$, instead of $\mathbf{27} \oplus \mathbf{1} \oplus \mathbf{1}$, at the three intersections where two D_4 singularities collide, $\frac{1}{2}\mathbf{56} \oplus \mathbf{2}$, instead of $\frac{1}{2}\mathbf{56}$, at the four intersections where E_7 singularity and A_1 singularity collide and $\mathbf{7} \oplus \mathbf{2}$, instead of $\mathbf{7}$, at the two intersections where D_4 singularity and A_1 singularity collide, yields another viable candidate matter spectrum.

4.3.3. Case a triplet of parameters $\boldsymbol{\beta}$ coincide. We then consider the case a triplet of parameters β coincide:

$$\beta_1 = \beta_2 = \beta_3. \tag{99}$$

In this situation, the equation of the Calabi–Yau 3-fold (68) becomes

$$\tau^2 = \frac{1}{4}x^3 - (t-\alpha_1)^3(t-\alpha_5)^3(t-\alpha_4)^2 \cdot (u-\beta_1)^3 \cdot \Pi_{j=4}^{8}(u-\beta_j) \cdot x \tag{100}$$

and the discriminant is given as follows

$$\Delta \sim (t-\alpha_1)^9(t-\alpha_5)^9(t-\alpha_4)^6 \cdot (u-\beta_1)^9 \cdot \Pi_{j=4}^{8}(u-\beta_j)^3. \tag{101}$$

The gauge algebra forming in F-theory compactification is: $\mathfrak{e}_7^3 \oplus \mathfrak{so}(7) \oplus \mathfrak{su}(2)^5$. 7-branes intersect in eighteen points at $(t,u) = (\alpha_i, \beta_j)$, $i = 1,4,5$, $j = 1,4,5,6,7,8$. Using an argument similar to that given in Section 4.3.1, we obtain that the net representation dimensions of matter arising from these intersections must be 675 owing to the anomaly cancellation condition.

E_7 angularities collide at the two intersections $(t,u) = (\alpha_i, \beta_1)$, $i = 1,5$, and D_4 singularity and E_7 singularity collide at the intersection $(t,u) = (\alpha_4, \beta_1)$. E_7 singularity and A_1 singularity collide at the ten intersections $(t,u) = (\alpha_i, \beta_j)$, $i = 1,5$, $j = 4, \cdots, 8$. D_4 singularity and A_1 singularity collide at the five intersections $(t,u) = (\alpha_4, \beta_j)$, $j = 4, \cdots, 8$.

We assume that $\mathbf{133}$ arises at the two intersections where two E_7 angularities collide, $\mathbf{56} \oplus \mathbf{7} \oplus \mathbf{1}$ arises at the intersection where D_4 and E_7 singularities collide, $\frac{1}{2}\mathbf{56} \oplus \mathbf{2}$ arises at the ten intersections where E_7 singularity and A_1 singularity collide and $\mathbf{7} \oplus \mathbf{2}$ arises at the five intersections where D_4 singularity and A_1 singularity collide. Then, one can confirm that the anomaly cancels and this yields a consistent matter candidate on F-theory on the Jacobian Calabi–Yau 3-fold (100).

4.3.4. Case a triplet and a pair of parameters β coincide. We now consider the case a triplet and a pair of parameters β coincide:

$$\beta_1 = \beta_2 = \beta_3 \qquad \beta_4 = \beta_8. \tag{102}$$

In this situation, the equation of the Jacobian Calabi–Yau 3-fold (68) becomes:

$$\tau^2 = \frac{1}{4}x^3 - (t-\alpha_1)^3(t-\alpha_5)^3(t-\alpha_4)^2 \cdot (u-\beta_1)^3(u-\beta_4)^2 \cdot \Pi_{j=5}^{7}(u-\beta_j) \cdot x. \tag{103}$$

The discriminant is given as follows

$$\Delta \sim (t-\alpha_1)^9(t-\alpha_5)^9(t-\alpha_4)^6 \cdot (u-\beta_1)^9(u-\beta_4)^6 \cdot \Pi_{j=5}^{7}(u-\beta_j)^3. \tag{104}$$

The gauge algebra forming in F-theory compactification is: $\mathfrak{e}_7^3 \oplus \mathfrak{so}(7)^2 \oplus \mathfrak{su}(2)^3$. 7-branes intersect in fifteen points at $(t,u) = (\alpha_i, \beta_j)$, $i = 1,4,5$, $j = 1,4,5,6,7$. We find that the net representation dimensions of matter arising from these intersections must be 691 owing to the anomaly cancellation condition.

E_7 angularities collide at the two intersections $(t,u) = (\alpha_i, \beta_1)$, $i = 1,5$, and D_4 singularity and E_7 singularity collide at the three intersection points $(t,u) = (\alpha_4, \beta_1), (\alpha_i, \beta_4)$, $i = 1,5$. D_4 singularities collide at the intersection point $(t,u) = (\alpha_4, \beta_4)$. E_7 singularity and A_1 singularity collide at the six intersections $(t,u) = (\alpha_i, \beta_j)$, $i = 1,5$, $j = 5,6,7$. D_4 singularity and A_1 singularity collide at the three intersections $(t,u) = (\alpha_4, \beta_j)$, $j = 5,6,7$.

We assume that $\mathbf{133}$ arises at the two intersections where two E_7 angularities collide, $\mathbf{56} \oplus \mathbf{7} \oplus \mathbf{1}$ arises at the three intersection points where D_4 and E_7 singularities collide, $\mathbf{27} \oplus \mathbf{1} \oplus \mathbf{1}$ arises at the intersection point where two D_4 singularities collide, $\frac{1}{2}\mathbf{56} \oplus \mathbf{2}$ arises at the six intersections where E_7 singularity and A_1 singularity collide and $\mathbf{7} \oplus \mathbf{1}$ arises at the three intersections where D_4 singularity and A_1 singularity collide. Then, the anomaly cancels and this yields a consistent matter candidate on F-theory on the Jacobian Calabi–Yau 3-fold (103). There are a few possibilities whether to include **1** for matter representations at the intersections where two E_7 angularities collide and at the intersections where E_7 angularity and D_4 singularity collide, and whether to include **2** or **1** for matter representation at the intersections where D_4 and A_1 singularities collide, and whether matter arising at the intersection where two D_4 singularities collide includes **27** or **21** for other viable candidate matter spectra.

4.3.5. Case a triplet and two pairs of parameters $\boldsymbol{\beta}$ coincide. We consider the case a triplet and two pairs of parameters β coincide:

$$\begin{aligned} \beta_1 = \beta_2 &= \beta_3 \\ \beta_4 &= \beta_8 \\ \beta_5 &= \beta_6. \end{aligned} \tag{105}$$

In this situation, the equation of the Jacobian Calabi–Yau 3-fold (68) becomes:

$$\begin{aligned} \tau^2 = \frac{1}{4}x^3 - (t-\alpha_1)^3(t-\alpha_5)^3(t-\alpha_4)^2 \\ \cdot (u-\beta_1)^3(u-\beta_4)^2(u-\beta_5)^2(u-\beta_7)\cdot x. \end{aligned} \tag{106}$$

The discriminant is given as follows

$$\Delta \sim (t-\alpha_1)^9(t-\alpha_5)^9(t-\alpha_4)^6\cdot(u-\beta_1)^9(u-\beta_4)^6(u-\beta_5)^6(u-\beta_7)^3. \tag{107}$$

The gauge algebra forming in F-theory compactification is: $\mathfrak{e}_7^3 \oplus \mathfrak{so}(7)^3 \oplus \mathfrak{su}(2)$. 7-branes intersect in twelve points at $(t,u) = (\alpha_i, \beta_j)$, $i = 1,4,5$, $j = 1,4,5,7$. We find that the net representation dimensions of matter arising from these intersections must be 707 owing to the anomaly cancellation condition.

E_7 angularities collide at the two intersections $(t,u) = (\alpha_i, \beta_1)$, $i = 1,5$, and E_7 singularity and D_4 singularity collide at the five intersection points $(t,u) = (\alpha_i, \beta_j)$, $i = 1,5, j = 4,5$, (α_4, β_1). D_4 singularities collide at the two intersection points $(t,u) = (\alpha_4, \beta_j)$, $j = 4,5$. E_7 singularity and A_1 singularity collide at the two intersections $(t,u) = (\alpha_i, \beta_7)$, $i = 1,5$. D_4 singularity and A_1 singularity collide at the intersection $(t,u) = (\alpha_4, \beta_7)$.

We assume that **133** arises at the two intersections where two E_7 angularities collide, $\mathbf{56} \oplus \mathbf{7}$ arises at the five intersection points where E_7 and D_4 singularities collide, $\mathbf{27} \oplus \mathbf{1} \oplus \mathbf{1}$ arises at the two intersection points where two D_4 singularities collide, $\frac{1}{2}\mathbf{56} \oplus \mathbf{2}$ arises at the two intersections where E_7 singularity and A_1 singularity collide, and $\mathbf{7} \oplus \mathbf{1}$ arises at the intersection where D_4 singularity and A_1 singularity collide. Then, the anomaly cancels and this yields a consistent matter candidate on F-theory on the Jacobian Calabi–Yau 3-fold (106). There are a few possibilities whether to include **1** for matter representations at the intersections where two E_7 angularities collide, at the intersections where E_7 angularity and D_4 singularity collide, whether to include **1** or **2**, or not to include these, for matter representation at the intersections where E_7 and A_1 singularities collide and at the intersection where D_4 and A_1 singularities collide, and whether matter arising at the intersection where two D_4 singularities collide includes **27** or **21**, for other viable candidate matter spectra.

4.3.6. Case two triplets and a pair of parameters β coincide. We finally discuss the case two triplets and a pair of parameters β coincide:

$$\begin{aligned} \beta_1 = \beta_2 &= \beta_3 \\ \beta_4 &= \beta_8 \\ \beta_5 = \beta_6 &= \beta_7. \end{aligned} \tag{108}$$

In this situation, the equation of the Jacobian Calabi–Yau 3-fold (68) becomes:

(109)

$$\tau^2 = \frac{1}{4}x^3 - (t-\alpha_1)^3(t-\alpha_5)^3(t-\alpha_4)^2 \cdot (u-\beta_1)^3(u-\beta_5)^3(u-\beta_4)^2 \cdot x.$$

The discriminant is given as follows

$$\Delta \sim (t-\alpha_1)^9(t-\alpha_5)^9(t-\alpha_4)^6 \cdot (u-\beta_1)^9(u-\beta_5)^9(u-\beta_4)^6. \tag{110}$$

The gauge algebra forming in F-theory compactification is: $\mathfrak{e}_7^4 \oplus \mathfrak{so}(7)^2$. The parameters of the complex structure deformation, $\beta_1, \beta_4, \beta_5$, can be sent to fixed values under an automorphism of $\mathbb{P}^1$. Therefore, the number of the effective parameters of the complex structure deformation is zero, and the complex structure is fixed for this situation.

7-branes intersect in nine points at $(t,u) = (\alpha_i, \beta_j)$, $i = 1,4,5$, $j = 1,4,5$. We find that the net representation dimensions of matter arising from these intersections must be 817 owing to the anomaly cancellation condition.

E_7 angularities collide at the four intersections $(t,u) = (\alpha_i, \beta_j)$, $i = 1,5$, $j = 1,5$, and E_7 singularity and D_4 singularity collide at the four intersection points $(t,u) = (\alpha_4, \beta_j)$, $j = 1,5$, (α_i, β_4), $i = 1,5$. D_4 singularities collide at the intersection point $(t,u) = (\alpha_4, \beta_4)$.

We assume that **133** arises at the four intersections where two E_7 angularities collide, $\mathbf{56} \oplus \mathbf{7} \oplus \mathbf{1}$ arises at the four intersection points where E_7

and D_4 singularities collide, and $\mathbf{27} \oplus \mathbf{1} \oplus \mathbf{1}$ arises at the intersection point where two D_4 singularities collide. Then, the anomaly cancels and this yields a consistent matter candidate on F-theory on the Jacobian Calabi–Yau 3-fold (109). Analogous to other cases that we discussed previously, there are a few possibilities whether to include $\mathbf{1}$ for matter representation at the intersections where two E_7 angularities collide, and for matter representation at the intersections where E_7 and D_4 singularities collide, and whether matter arising at the intersection where two D_4 singularities collide includes $\mathbf{27}$ or $\mathbf{21}$, to yield other viable candidate matter spectra.

4.3.7. Summary and discussion of the obtained candidate matter representations. We have deduced candidate matter spectra on F-theory on the constructed elliptically fibered Calabi–Yau 3-folds. We have observed that it is natural to expect that $\mathbf{133}$ (or $\mathbf{133} \oplus \mathbf{1}$) arises at the collision of two E_7 singularities [21], to cancel the anomaly[22]. Under this assumption, we observed that matter representation arising at the collision of E_7 and A_1 singularities should include the $\frac{1}{2}$-hypermultiplet $\frac{1}{2}\mathbf{56}$ of E_7. If matter representation at this intersection includes $\mathbf{56}$, the net dimension of matter representations exceeds the number required by the anomaly cancellation condition by a large amount. We learned that $\frac{1}{2}$-hypermultiplet $\frac{1}{2}\mathbf{56}$ is an actual matter arising at the singularity at the collision of type (E_7, A_1), and $\frac{1}{2}\mathbf{56}$ arises when the E_8 at the collision is incompletely resolved. There appear a few possibilities for matter representation at the collision of E_7 and D_4 singularities, such as whether to include $\mathbf{1}$. We observed a few possibilities for matter representation at the collision of two D_4 singularities, such as $\mathbf{27} \oplus \mathbf{1} \oplus \mathbf{1}$ or $\mathbf{21}$. There are also a few possibilities for matter at the collision of E_7 and A_1 singularities, such as $\frac{1}{2}\mathbf{56}$, $\frac{1}{2}\mathbf{56} \oplus \mathbf{1}$, or $\frac{1}{2}\mathbf{56} \oplus \mathbf{2}$, and similarly for the collision of D_4 and A_1 singularities.

5. Conclusions

In this study, we have investigated the points in the eight-dimensional moduli of non-geometric heterotic strings with unbroken $\mathfrak{e}_7\mathfrak{e}_7$ algebra, at which the ranks of the non-Abelian gauge groups on the F-theory side are enhanced to 18. The gauge groups at these points do not allow for the perturbative

[21] The authors of [148] discussed the collision of two E_7 singularities in the context of four-dimensional conformal matter in F-theory.

[22] A similar observation was made for the collision of two E_6 singularities for elliptically fibered Calabi–Yau 3-folds of "Fermat-types" in [39], where it was argued that $\mathbf{78} \oplus \mathbf{1}$ is expected to arise at this intersection.

interpretation on the heterotic side. We demonstrated in this study that these theories can be seen as deformations of the stable degenerations owing to an effect of coincident 7-branes. This effect corresponds to the insertion of 5-branes from the heterotic viewpoint. We also discussed the application to $SO(32)$ heterotic strings.

K3 surfaces on the F-theory side of the moduli become extremal, when the non-Abelian gauge groups are enhanced to rank 18. We studied the Weierstrass equations of the extremal K3 surfaces that appear on the F-theory side of the eight-dimensional moduli of non-geometric heterotic strings with unbroken $\mathfrak{e}_7\mathfrak{e}_7$. The points in the moduli at which the ranks of the non-Abelian gauge groups are enhanced to 17 on the F-theory side also do not allow for the perturbative interpretations of the gauge groups on the heterotic side. It can be interesting to study these points in the moduli, and this can be a direction of future study.

We have also built elliptically fibered Calabi–Yau 3-folds, by fibering an elliptic K3 surface, which belongs to the F-theory side of the eight-dimensional moduli of non-geometric heterotic strings with unbroken $\mathfrak{e}_7\mathfrak{e}_7$ algebra, over $\mathbb{P}^1$. We analyzed six-dimensional F-theory compactifications on the built elliptic Calabi–Yau 3-folds. When we tune the parameters for the defining equations of these elliptic Calabi–Yau 3-folds, highly enhanced gauge groups form on the 7-branes. Eight-dimensional F-theory compactified on the extremal K3 fibers $S_{[2\ 0\ 2]}$ of these specific tuned Calabi–Yau spaces has non-geometric heterotic duals. Determining whether this duality extends to six-dimensional theories, namely whether F-theory compactifications on the total Calabi–Yau 3-folds have dual non-geometric heterotic strings, is a likely direction of future study.

We have also deduced viable candidate matter spectra on F-theory on the constructed elliptically fibered Calabi–Yau 3-folds, for the case when K3 fibers are most enhanced (63). There are certain ambiguities such as whether matter arising at intersections of 7-branes includes **1**, or **2**, or does not include these. There is also an ambiguity of whether matter arising at the collision of D_4 singularities includes **27** or **21**. Except for these ambiguities, the possibility of candidate matter spectra appears unique. We have observed that either **133** or $\mathbf{133} \oplus \mathbf{1}$ arises where two E_7 singularities collide. We have also observed that matter arising where E_7 and A_1 singularities collide should include the $\frac{1}{2}$-hypermultiplet, $\frac{1}{2}\mathbf{56}$ of E_7, to cancel the anomaly. We found that the $\frac{1}{2}$-hypermultiplet $\frac{1}{2}\mathbf{56}$ arises as an actual matter from the collision of singularities of type (E_7, A_1) through incomplete resolution of the singularity in our construction of elliptically fibered Calabi–Yau 3-folds. Confirming the actual matter spectra by analyzing the resolution of the singularities of the Jacobian Calabi–Yau 3-folds is also a likely direction of future study.

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Appendix A. Elliptic fibrations of attractive K3 $S_{[2\ 0\ 2]}$

[131] classified the types of the elliptic fibrations of the attractive K3 surface with the discriminant four, $S_{[2\ 0\ 2]}$, and computed the Mordell–Weil groups of the fibrations. We present in Table A1 the types of the elliptic fibrations and the Mordell–Weil groups of the attractive K3 surface $S_{[2\ 0\ 2]}$ determined in [131].

Elliptic fibrations of $S_{[2\ 0\ 2]}$	type of singularity	MW group
No.1	$E_8^2 A_1^2$	0
No.2	$E_8 D_{10}$	0
No.3	$D_{16} A_1^2$	$\mathbb{Z}_2$
No.4	$E_7^2 D_4$	$\mathbb{Z}_2$
No.5	$E_7 D_{10} A_1$	$\mathbb{Z}_2$
No.6	$A_{17} A_1$	$\mathbb{Z}_3$
No.7	D_{18}	0
No.8	$D_{12} D_6$	$\mathbb{Z}_2$
No.9	$D_8^2 A_1^2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
No.10	$A_{15} A_3$	$\mathbb{Z}_4$
No.11	$E_6 A_{11}$	$\mathbb{Z} \oplus \mathbb{Z}_3$
No.12	D_6^3	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
No.13	A_9^2	$\mathbb{Z}_5$

Table A1: List of the singularity types of the elliptic fibrations of K3 surface $S_{[2\ 0\ 2]}$, and the Mordell–Weil groups of the fibrations.

Appendix B. Types of the singular fibers of extremal rational elliptic surfaces

The types of the singular fibers of the extremal rational elliptic surfaces [133] are presented in Table B1. The complex structures of the extremal rational elliptic surfaces are uniquely specified by the types of the singular fibers, except rational elliptic surfaces with two type I_0^* fibers, $X_{[0^*,\, 0^*]}(j)$ [133].

Extremal rational elliptic surface	Type of singular fiber	Type of singularity
$X_{[II,\, II^*]}$	$II,\, II^*$	E_8
$X_{[III,\, III^*]}$	$III,\, III^*$	E_7A_1
$X_{[IV,\, IV^*]}$	$IV,\, IV^*$	E_6A_2
$X_{[0^*,\, 0^*]}(j)$	$I_0^*,\, I_0^*$	D_4^2
$X_{[II^*,\, 1,1]}$	$II^*\ I_1\ I_1$	E_8
$X_{[III^*,\, 2,1]}$	$III^*\ I_2\ I_1$	E_7A_1
$X_{[IV^*,\, 3,1]}$	$IV^*\ I_3\ I_1$	E_6A_2
$X_{[4^*,\, 1,1]}$	$I_4^*\ I_1\ I_1$	D_8
$X_{[2^*,\, 2,2]}$	$I_2^*\ I_2\ I_2$	$D_6A_1^2$
$X_{[1^*,\, 4,1]}$	$I_1^*\ I_4\ I_1$	D_5A_3
$X_{[9,1,1,1]}$	$I_9\ I_1\ I_1\ I_1$	A_8
$X_{[8,2,1,1]}$	$I_8\ I_2\ I_1\ I_1$	A_7A_1
$X_{[6,3,2,1]}$	$I_6\ I_3\ I_2\ I_1$	$A_5A_2A_1$
$X_{[5,5,1,1]}$	$I_5\ I_5\ I_1\ I_1$	A_4^2
$X_{[4,4,2,2]}$	$I_4\ I_4\ I_2\ I_2$	$A_3^2A_1^2$
$X_{[3,3,3,3]}$	$I_3\ I_3\ I_3\ I_3$	A_2^4

Table B1: List of the types of the singular fibers of extremal rational elliptic surfaces.

References

[1] C. Vafa, "Evidence for F-theory", *Nucl. Phys.* **B 469** (1996) 403 `arXiv:hep-th/9602022`.

[2] D. R. Morrison and C. Vafa, "Compactifications of F-theory on Calabi-Yau threefolds. 1", *Nucl. Phys.* **B 473** (1996) 74 `arXiv:hep-th/9602114`.

[3] D. R. Morrison and C. Vafa, "Compactifications of F-theory on Calabi-Yau threefolds. 2", *Nucl. Phys.* **B 476** (1996) 437 `arXiv:hep-th/9603161`.

[4] A. Sen, "F theory and orientifolds", *Nucl. Phys.* **B475** (1996) 562–578 `arXiv:hep-th/9605150`.

[5] R. Friedman, J. Morgan and E. Witten, "Vector bundles and F theory", *Commun. Math. Phys.* **187** (1997) 679–743 `arXiv:hep-th/9701162`.

[6] R. Blumenhagen, G. Honecker and T. Weigand, "Loop-corrected compactifications of the heterotic string with line bundles", *JHEP* **06** (2005) 020 `arXiv:hep-th/0504232`.

[7] R. Blumenhagen, G. Honecker and T. Weigand, "Supersymmetric (non-)Abelian bundles in the Type I and SO(32) heterotic string", *JHEP* **08** (2005) 009 `arXiv:hep-th/0507041`.

[8] T. Kimura and S. Mizoguchi, "Chiral generations on intersecting 5-branes in heterotic string theory", *JHEP* **04** (2010) 028 `arXiv:0912.1334 [hep-th]`.

[9] L. B. Anderson, A. Constantin, J. Gray, A. Lukas and E. Palti, "A comprehensive scan for heterotic SU(5) GUT models", *JHEP* **01** (2014) 047 `arXiv:1307.4787 [hep-th]`.

[10] L. B. Anderson, J. Gray and E. Sharpe, "Algebroids, heterotic moduli spaces and the Strominger system", *JHEP* **07** (2014) 037 `arXiv:1402.1532 [hep-th]`.

[11] X. de la Ossa and E. E. Svanes, "Holomorphic bundles and the moduli space of N=1 supersymmetric heterotic compactifications", *JHEP* **10** (2014) 123 `arXiv:1402.1725 [hep-th]`.

[12] L. B. Anderson and W. Taylor, "Geometric constraints in dual F-theory and heterotic string compactifications", *JHEP* **08** (2014) 025 `arXiv:1405.2074 [hep-th]`.

[13] X. de la Ossa, E. Hardy and E. E. Svanes, "The heterotic superpotential and moduli", *JHEP* **01** (2016) 049 arXiv:1509.08724 [hep-th].

[14] P. Candelas, X. de la Ossa and J. McOrist, "A metric for heterotic moduli", *Commun. Math. Phys.* **356** (2017) 567–612 arXiv: 1605.05256 [hep-th].

[15] A. Ashmore, X. de la Ossa, R. Minasian, C. Strickland-Constable and E. E. Svanes, "Finite deformations from a heterotic superpotential: holomorphic Chern-Simons and an L_∞ algebra", *JHEP* **10** (2018) 179 arXiv:1806.08367 [hep-th].

[16] P. Candelas, X. de la Ossa, J. McOrist and R. Sisca, "The universal geometry of heterotic vacua", *JHEP* **02** (2019) 038 arXiv: 1810.00879 [hep-th].

[17] P. S. Aspinwall and D. R. Morrison, "Point - like instantons on K3 orbifolds", *Nucl. Phys.* **B503** (1997) 533–564 arXiv:hep-th/9705104.

[18] L. B. Anderson, J. J. Heckman and S. Katz, "T-Branes and geometry", *JHEP* **05** (2014) 080 arXiv:1310.1931 [hep-th].

[19] A. P. Braun, Y. Kimura and T. Watari, "The Noether-Lefschetz problem and gauge-group-resolved landscapes: F-theory on K3 × K3 as a test case", *JHEP* **04** (2014) 050 arXiv:1401.5908 [hep-th].

[20] N. Cabo Bizet, A. Klemm and D. Vieira Lopes, " Landscaping with fluxes and the E8 Yukawa Point in F-theory", arXiv:1404.7645 [hep-th].

[21] M. Cvetič, A. Grassi, D. Klevers, M. Poretschkin and P. Song, "Origin of Abelian gauge symmetries in heterotic/F-theory duality," *JHEP* **1604** (2016) 041 arXiv:1511.08208 [hep-th].

[22] S. Mizoguchi and T. Tani, "Looijenga's weighted projective space, Tate's algorithm and Mordell-Weil Lattice in F-theory and heterotic string theory", *JHEP* **11** (2016) 053 arXiv:1607.07280 [hep-th].

[23] Y. Kimura, "Structure of stable degeneration of K3 surfaces into pairs of rational elliptic surfaces", *JHEP* **03** (2018) 045 arXiv:1710.04984 [hep-th].

[24] A. Malmendier and D. R. Morrison, "K3 surfaces, modular forms, and non-geometric heterotic compactifications", *Lett. Math. Phys.* **105** (2015) no.8, 1085–1118 arXiv:1406.4873 [hep-th].

[25] K. S. Narain, "New heterotic string theories in uncompactified dimensions <10", *Phys. Lett.* **169B** (1986) 41–46.

[26] J. McOrist, D. R. Morrison and S. Sethi, "Geometries, non-geometries, and fluxes", *Adv. Theor. Math. Phys.* **14** (2010) no.5 1515–1583 `arXiv:1004.5447 [hep-th]`.

[27] S. Hellerman, J. McGreevy and B. Williams, "Geometric constructions of nongeometric string theories", *JHEP* **01** (2004) 024 `arXiv: hep-th/0208174`.

[28] A. Kumar, "K3 surfaces associated with curves of genus two", *Int. Math. Res. Not.* **2008** (2008).

[29] A. Clingher and C. F. Doran, "Note on a geometric isogeny of K3 surfaces", *Int. Math. Res. Not.* **2011** (2011) 3657–3687.

[30] A. Clingher and C. F. Doran, "Lattice polarized K3 surfaces and Siegel modular forms", *Adv. Math.* **231** (2012) 172–212.

[31] A. Clingher, A. Malmendier and T. Shaska, "Six line configurations and string dualities" `arXiv:1806.07460 [math.AG]`.

[32] J. Gu and H. Jockers, "Nongeometric F-theory-heterotic duality", *Phys. Rev.* **D91** (2015) 086007 `arXiv:1412.5739 [hep-th]`.

[33] D. Lüst, S. Massai and V. Vall Camell, "The monodromy of T-folds and T-fects", *JHEP* **09** (2016) 127 `arXiv:1508.01193 [hep-th]`.

[34] A. Font, I. García-Etxebarria, D. Lust, S. Massai and C. Mayrhofer, "Heterotic T-fects, 6D SCFTs, and F-Theory", *JHEP* **08** (2016) 175 `arXiv:1603.09361 [hep-th]`.

[35] A. Malmendier and T. Shaska, "The Satake sextic in F-theory", *J. Geom. Phys.* **120** (2017) 290–305 `arXiv:1609.04341 [math.AG]`.

[36] I. García-Etxebarria, D. Lust, S. Massai and C. Mayrhofer, "Ubiquity of non-geometry in heterotic compactifications", *JHEP* **03** (2017) 046 `arXiv:1611.10291 [hep-th]`.

[37] A. Font and C. Mayrhofer, "Non-geometric vacua of the Spin($\mathbf{32}$)/$\mathbb{Z}_\mathbf{2}$ heterotic string and little string theories", *JHEP* **11** (2017) 064 `arXiv:1708.05428 [hep-th]`.

[38] A. Font, I. García-Etxebarria, D. Lüst, S. Massai and C. Mayrhofer, "Non-geometric heterotic backgrounds and 6D SCFTs/LSTs", PoS CORFU2016 (2017) 123 `arXiv:1712.07083 [hep-th]`.

[39] Y. Kimura, "Nongeometric heterotic strings and dual F-theory with enhanced gauge groups", *JHEP* **02** (2019) 036 `arXiv:1810.07657 [hep-th]`.

[40] E. Plauschinn, "Non-geometric backgrounds in string theory", *Phys.Rept.* **798** (2019) 1–122 `arXiv:1811.11203 [hep-th]`.

[41] I. Shimada and D.-Q. Zhang, "Classification of extremal elliptic K3 surfaces and fundamental groups of open K3 surfaces", *Nagoya Math. J.* **161** (2001), 23–54, `arXiv:math/0007171`.

[42] A. Grassi and D. R. Morrison, "Group representations and the Euler characteristic of elliptically fibered Calabi-Yau threefolds", *Jour. Alg. Geom.* **12** (2003) 321–356 `arXiv:math/0005196`.

[43] V. Kumar and W. Taylor, "String universality in six dimensions", *Adv. Theor. Math. Phys.* **15** (2011) 325–353 `arXiv:0906.0987 [hep-th]`.

[44] V. Kumar, D. R. Morrison and W. Taylor, "Mapping 6D N = 1 supergravities to F-theory", *JHEP* **02** (2010) 099 `arXiv:0911.3393 [hep-th]`.

[45] V. Kumar, D. R. Morrison and W. Taylor, "Global aspects of the space of 6D N = 1 supergravities", *JHEP* **11** (2010) 118 `arXiv:1008.1062 [hep-th]`.

[46] D. R. Morrison and W. Taylor, "Matter and singularities", *JHEP* **01** (2012) 022 `arXiv:1106.3563 [hep-th]`.

[47] A. Grassi and D. R. Morrison, "Anomalies and the Euler characteristic of elliptic Calabi-Yau threefolds", *Commun. Num. Theor. Phys.* **6** (2012) 51–127 `arXiv:1109.0042 [hep-th]`.

[48] F. Bonetti and T. W. Grimm, "Six-dimensional (1,0) effective action of F-theory via M-theory on Calabi-Yau threefolds", *JHEP* **05** (2012) 019 `arXiv:1112.1082 [hep-th]`.

[49] D. R. Morrison and W. Taylor, "Classifying bases for 6D F-theory models", *Central Eur. J. Phys.* **10** (2012) 1072–1088 `arXiv:1201.1943 [hep-th]`.

[50] D. R. Morrison and W. Taylor, "Toric bases for 6D F-theory models", *Fortsch. Phys.* **60** (2012) 1187–1216 `arXiv:1204.0283 [hep-th]`.

[51] W. Taylor, "On the Hodge structure of elliptically fibered Calabi-Yau threefolds", *JHEP* **08** (2012) 032 `arXiv:1205.0952 [hep-th]`.

[52] S. B. Johnson and W. Taylor, "Enhanced gauge symmetry in 6D F-theory models and tuned elliptic Calabi-Yau threefolds", *Fortsch. Phys.* **64** (2016) 581–644 arXiv:1605.08052 [hep-th].

[53] D. R. Morrison, D. S. Park and W. Taylor, "Non-Higgsable abelian gauge symmetry and F-theory on fiber products of rational elliptic surfaces", *Adv. Theor. Math. Phys.* **22** (2018) 177–245 arXiv:1610.06929 [hep-th].

[54] S. Monnier, G. W. Moore and D. S. Park, "Quantization of anomaly coefficients in 6D $\mathcal{N} = (1, 0)$ supergravity", *JHEP* **02** (2018) 020 arXiv:1711.04777 [hep-th].

[55] Y.-C. Huang and W. Taylor, "Comparing elliptic and toric hypersurface Calabi-Yau threefolds at large Hodge numbers", *JHEP* **02** (2019) 087 arXiv:1805.05907 [hep-th].

[56] S.-J. Lee, W. Lerche and T. Weigand, "A stringy test of the scalar weak gravity conjecture", *Nucl.Phys.* **B938** (2019) 321–350 arXiv:1810.05169 [hep-th].

[57] A. Clingher, T. Hill, and A. Malmendier, "The duality between F-theory and the heterotic string in $D = 8$ with two Wilson lines," *Lett. Math. Phys.* **110**, 3081–3104 (2020).

[58] V. Braun and D. R. Morrison, "F-theory on genus-one fibrations", *JHEP* **08** (2014) 132 arXiv:1401.7844 [hep-th].

[59] P. Arras, A. Grassi and T. Weigand, "Terminal singularities, Milnor numbers, and matter in F-theory", *J. Geom. Phys.* **123** (2018) 71–97 arXiv:1612.05646 [hep-th].

[60] A. Grassi and T. Weigand, "On topological invariants of algebraic threefolds with ($\mathbb{Q}$-factorial) singularities", arXiv:1804.02424 [math.AG].

[61] R. Donagi and M. Wijnholt, "Model building with F-theory," *Adv. Theor. Math. Phys.* **15**, 1237 (2011) arXiv:0802.2969 [hep-th].

[62] C. Beasley, J. J. Heckman and C. Vafa, "GUTs and exceptional branes in F-theory - I," *JHEP* **0901**, 058 (2009) arXiv:0802.3391 [hep-th].

[63] C. Beasley, J. J. Heckman and C. Vafa, "GUTs and exceptional branes in F-theory - II: Experimental Predictions," *JHEP* **0901**, 059 (2009) arXiv:0806.0102 [hep-th].

[64] R. Donagi and M. Wijnholt, "Breaking GUT groups in F-theory," *Adv. Theor. Math. Phys.* **15**, 1523 (2011) `arXiv:0808.2223 [hep-th]`.

[65] D. R. Morrison and W. Taylor, "Sections, multisections, and $U(1)$ fields in F-theory", *J. Singularities* **15** (2016) 126–149 `arXiv:1404.1527 [hep-th]`.

[66] L. B. Anderson, I. Garcia-Etxebarria, T. W. Grimm and J. Keitel, "Physics of F-theory compactifications without section", *JHEP* **12** (2014) 156 `arXiv:1406.5180 [hep-th]`.

[67] D. Klevers, D. K. Mayorga Pena, P. K. Oehlmann, H. Piragua and J. Reuter, "F-Theory on all Toric Hypersurface Fibrations and its Higgs Branches", *JHEP* **01** (2015) 142 `arXiv:1408.4808 [hep-th]`.

[68] I. Garcia-Etxebarria, T. W. Grimm and J. Keitel, "Yukawas and discrete symmetries in F-theory compactifications without section", *JHEP* **11** (2014) 125 `arXiv:1408.6448 [hep-th]`.

[69] C. Mayrhofer, E. Palti, O. Till and T. Weigand, "Discrete gauge symmetries by Higgsing in four-dimensional F-theory compactifications", *JHEP* **12** (2014) 068 `arXiv:1408.6831 [hep-th]`.

[70] C. Mayrhofer, E. Palti, O. Till and T. Weigand, "On discrete symmetries and Torsion homology in F-theory", *JHEP* **06** (2015) 029 `arXiv:1410.7814 [hep-th]`.

[71] V. Braun, T. W. Grimm and J. Keitel, "Complete intersection fibers in F-theory", *JHEP* **03** (2015) 125 `arXiv:1411.2615 [hep-th]`.

[72] M. Cvetič, R. Donagi, D. Klevers, H. Piragua and M. Poretschkin, "F-theory vacua with $\mathbb{Z}_3$ gauge symmetry", *Nucl. Phys.* **B898** (2015) 736–750 `arXiv:1502.06953 [hep-th]`.

[73] L. Lin, C. Mayrhofer, O. Till and T. Weigand, "Fluxes in F-theory compactifications on genus-one fibrations", *JHEP* **01** (2016) 098 `arXiv:1508.00162 [hep-th]`.

[74] Y. Kimura, "Gauge groups and matter fields on some models of F-theory without section", *JHEP* **03** (2016) 042 `arXiv:1511.06912 [hep-th]`.

[75] Y. Kimura, "Gauge symmetries and matter fields in F-theory models without section- compactifications on double cover and Fermat quartic K3 constructions times K3", *Adv. Theor. Math. Phys.* **21** (2017) no.8, 2087–2114 `arXiv:1603.03212 [hep-th]`.

[76] Y. Kimura, "Gauge groups and matter spectra in F-theory compactifications on genus-one fibered Calabi-Yau 4-folds without section - hypersurface and double cover constructions", *Adv. Theor. Math. Phys.* **22** (2018) no.6, 1489–1533 arXiv:1607.02978 [hep-th].

[77] M. Cvetič, A. Grassi and M. Poretschkin, "Discrete symmetries in heterotic/F-theory duality and mirror symmetry,", *JHEP* **06** (2017) 156 arXiv:1607.03176 [hep-th].

[78] Y. Kimura, "Discrete gauge groups in F-theory models on genus-one fibered Calabi-Yau 4-folds without section", *JHEP* **04** (2017) 168 arXiv:1608.07219 [hep-th].

[79] Y. Kimura, "K3 surfaces without section as double covers of Halphen surfaces, and F-theory compactifications", *PTEP* **2018** (2018) 043B06 arXiv:1801.06525 [hep-th].

[80] L. B. Anderson, A. Grassi, J. Gray and P.-K. Oehlmann, "F-theory on quotient threefolds with (2,0) discrete superconformal matter", *JHEP* **06** (2018) 098 arXiv:1801.08658 [hep-th].

[81] Y. Kimura, "$SU(n) \times \mathbb{Z}_2$ in F-theory on K3 surfaces without section as double covers of Halphen surfaces," *Adv. Theor. Math. Phys.* **24**, no.2, 459–490 (2020) arXiv:1806.01727 [hep-th].

[82] T. Weigand, "TASI lectures on F-theory", *PoS* **TASI2017** (2018) 016 arXiv:1806.01854 [hep-th].

[83] M. Cvetič, L. Lin, M. Liu and P.-K. Oehlmann, "An F-theory realization of the chiral MSSM with $\mathbb{Z}_2$-Parity", *JHEP* **09** (2018) 089 arXiv:1807.01320 [hep-th].

[84] M. Cvetič and L. Lin, "TASI lectures on Abelian and discrete symmetries in F-theory", *PoS* **TASI2017** (2018) 020 arXiv:1809.00012 [hep-th].

[85] Y.-C. Huang and W. Taylor, "On the prevalence of elliptic and genus one fibrations among toric hypersurface Calabi-Yau threefolds", *JHEP* **03** (2019) 014 arXiv:1809.05160 [hep-th].

[86] J. W. S. Cassels, *Lectures on Elliptic Curves*, London Math. Society Student Texts **24**, Cambridge University Press (1991).

[87] D. R. Morrison and D. S. Park, "F-theory and the Mordell-Weil group of elliptically-fibered Calabi-Yau threefolds", *JHEP* **10** (2012) 128 arXiv:1208.2695 [hep-th].

[88] C. Mayrhofer, E. Palti and T. Weigand, "U(1) symmetries in F-theory GUTs with multiple sections", *JHEP* **03** (2013) 098 `arXiv:1211.6742 [hep-th]`.

[89] V. Braun, T. W. Grimm and J. Keitel, "New Global F-theory GUTs with U(1) symmetries", *JHEP* **09** (2013) 154 `arXiv:1302.1854 [hep-th]`.

[90] J. Borchmann, C. Mayrhofer, E. Palti and T. Weigand, "Elliptic fibrations for $SU(5) \times U(1) \times U(1)$ F-theory vacua", *Phys. Rev.* **D88** (2013) no.4 046005 `arXiv:1303.5054 [hep-th]`.

[91] M. Cvetič, D. Klevers and H. Piragua, "F-theory compactifications with multiple U(1)-factors: Constructing elliptic fibrations with rational sections", *JHEP* **06** (2013) 067 `arXiv:1303.6970 [hep-th]`.

[92] V. Braun, T. W. Grimm and J. Keitel, "Geometric engineering in toric F-theory and GUTs with U(1) gauge factors," *JHEP* **12** (2013) 069 `arXiv:1306.0577 [hep-th]`.

[93] M. Cvetič, A. Grassi, D. Klevers and H. Piragua, "Chiral four-dimensional F-theory compactifications With SU(5) and multiple U(1)-factors", *JHEP* **04** (2014) 010 `arXiv:1306.3987 [hep-th]`.

[94] M. Cvetič, D. Klevers and H. Piragua, "F-Theory compactifications with multiple U(1)-factors: Addendum", *JHEP* **12** (2013) 056 `arXiv:1307.6425 [hep-th]`.

[95] M. Cvetič, D. Klevers, H. Piragua and P. Song, "Elliptic fibrations with rank three Mordell-Weil group: F-theory with U(1) x U(1) x U(1) gauge symmetry," *JHEP* **1403** (2014) 021 `arXiv:1310.0463 [hep-th]`.

[96] I. Antoniadis and G. K. Leontaris, "F-GUTs with Mordell-Weil U(1)'s," *Phys. Lett.* **B735** (2014) 226–230 `arXiv:1404.6720 [hep-th]`.

[97] M. Esole, M. J. Kang and S.-T. Yau, "A new model for elliptic fibrations with a rank one Mordell-Weil group: I. Singular fibers and semi-stable degenerations", `arXiv:1410.0003 [hep-th]`.

[98] C. Lawrie, S. Schäfer-Nameki and J.-M. Wong, "F-theory and all things rational: Surveying U(1) symmetries with rational sections", *JHEP* **09** (2015) 144 `arXiv:1504.05593 [hep-th]`.

[99] M. Cvetič, D. Klevers, H. Piragua and W. Taylor, "General U(1)×U(1) F-theory compactifications and beyond: geometry of unHiggsings and

novel matter structure," *JHEP* **1511** (2015) 204 arXiv:1507.05954 [hep-th].

[100] D. R. Morrison and D. S. Park, "Tall sections from non-minimal transformations", *JHEP* **10** (2016) 033 arXiv:1606.07444 [hep-th].

[101] M. Bies, C. Mayrhofer and T. Weigand, "Gauge backgrounds and zero-mode counting in F-theory", *JHEP* **11** (2017) 081 arXiv:1706.04616 [hep-th].

[102] M. Cvetič and L. Lin, "The global gauge group structure of F-theory compactification with U(1)s", *JHEP* **01** (2018) 157 arXiv:1706.08521 [hep-th].

[103] M. Bies, C. Mayrhofer and T. Weigand, "Algebraic cycles and local anomalies in F-theory", *JHEP* **11** (2017) 100 arXiv:1706.08528 [hep-th].

[104] M. Esole, M. J. Kang and S.-T. Yau, "Mordell-Weil torsion, anomalies, and phase transitions", arXiv:1712.02337 [hep-th].

[105] Y. Kimura and S. Mizoguchi, "Enhancements in F-theory models on moduli spaces of K3 surfaces with *ADE* rank 17", *PTEP* **2018** no. 4 (2018) 043B05 arXiv:1712.08539 [hep-th].

[106] M. Esole and M. J. Kang, "Flopping and slicing: SO(4) and Spin(4)-models", arXiv:1802.04802 [hep-th].

[107] Y. Kimura, "F-theory models on K3 surfaces with various Mordell-Weil ranks -constructions that use quadratic base change of rational elliptic surfaces", *JHEP* **05** (2018) 048 arXiv:1802.05195 [hep-th].

[108] S.-J. Lee, D. Regalado and T. Weigand, "6d SCFTs and U(1) flavour symmetries", *JHEP* **11** (2018) 147 arXiv:1803.07998 [hep-th].

[109] M. Esole and M. J. Kang, "Characteristic numbers of elliptic fibrations with non-trivial Mordell-Weil groups", arXiv:1808.07054 [hep-th].

[110] S. Mizoguchi and T. Tani, "Non-Cartan Mordell-Weil lattices of rational elliptic surfaces and heterotic/F-theory compactifications", *JHEP* **03** (2019) 121 arXiv:1808.08001 [hep-th].

[111] F. M. Cianci, D. K. Mayorga Pena and R. Valandro, "High U(1) charges in type IIB models and their F-theory lift", *JHEP* **04** (2019) 012 arXiv:1811.11777 [hep-th].

[112] W. Taylor and A. P. Turner, "Generic matter representations in 6D supergravity theories", *JHEP* **05** (2019) 081 `arXiv:1901.02012 [hep-th]`.

[113] K. Kodaira, "On compact analytic surfaces II", *Ann. of Math.* **77** (1963), 563–626.

[114] K. Kodaira, "On compact analytic surfaces III", *Ann. of Math.* **78** (1963), 1–40.

[115] A. Néron, "Modèles minimaux des variétés abéliennes sur les corps locaux et globaux", *Publications mathématiques de l'IHÉS* **21** (1964), 5–125.

[116] J. Tate, "Algorithm for determining the type of a singular fiber in an elliptic pencil", in *Modular Functions of One Variable IV*, Springer, Berlin (1975), 33–52.

[117] M. Bershadsky, K. A. Intriligator, S. Kachru, D. R. Morrison, V. Sadov and C. Vafa, "Geometric singularities and enhanced gauge symmetries", *Nucl. Phys.* **B 481** (1996) 215 `arXiv:hep-th/9605200`.

[118] J. Milnor, "On simply connected 4-manifolds", *Symposium Internacional de Topologia Algebraica (International Symposium on Algebraic Topology)*, Mexico City (1958) 122–128.

[119] S. Kondo, "Automorphisms of algebraic K3 susfaces which act trivially on Picard groups", *J. Math. Soc. Japan* **44** (1992) 75–98.

[120] K. Matsumoto, "Theta functions on the bounded symmetric domain of type $I_{2,2}$ and the period map of a 4-parameter family of K3 surfaces", *Math. Ann.* **295** (1993) no.3, 383–409.

[121] E. B. Vinberg, "On automorphic forms on symmetric domains of type *IV*", *Uspekhi Mat. Nauk* **65** (2010) no.3, 193–194.

[122] J. Igusa, "On Siegel modular forms of genus two", *Amer. J. Math.* **84** (1962) 175–200.

[123] K. S. Narain, M. H. Sarmadi and E. Witten, "A note on toroidal compactification of heterotic string theory", *Nucl. Phys.* **B279** (1987) 369–379.

[124] T. Shioda, "On elliptic modular surfaces", *J. Math. Soc. Japan* **24** (1972), 20–59.

[125] J. Tate, "Algebraic cycles and poles of zeta functions", in *Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963)*, Harper & Row (1965), 93–110.

[126] J. Tate, "On the conjectures of Birch and Swinnerton-Dyer and a geometric analog", *Séminaire Bourbaki* **9** (1964–1966), Exposé no. 306, 415–440.

[127] G. W. Moore, "Les Houches lectures on strings and arithmetic", `arXiv:hep-th/0401049`.

[128] I. I. Piatetski-Shapiro and I. R. Shafarevich, "A Torelli theorem for algebraic surfaces of type K3", *Izv. Akad. Nauk SSSR Ser. Mat.* **35** (1971), 530–572.

[129] T. Shioda and H. Inose, "On Singular K3 surfaces", in W. L. Jr. Baily and T. Shioda (eds.), *Complex analysis and algebraic geometry*, Iwanami Shoten, Tokyo (1977), 119–136.

[130] J.-H. Keum, "A note on elliptic K3 surfaces", *Trans. Amer. Math. Soc.* **352** (2000) 2077–2086.

[131] K.-I. Nishiyama, "The Jacobian fibrations on some K3 surfaces and their Mordell-Weil groups", *Japan. J. Math.* **22** (1996), 293–347.

[132] M. J. Bertin and O. Lecacheux, "Elliptic fibrations on the modular surface associated to $\Gamma_1(8)$", in *Arithmetic and geometry of K3 surfaces and Calabi-Yau threefolds*, 153–199, Fields Institute Commun. **67**, Springer (2013) `arXiv:1105.6312 [math.AG]`.

[133] R. Miranda and U. Persson, "On extremal rational elliptic surfaces", *Math. Z.* **193** (1986) 537–558.

[134] P. S. Aspinwall and D. R. Morrison, "Nonsimply connected gauge groups and rational points on elliptic curves", *JHEP* **9807** (1998) 012 `arXiv:hep-th/9805206`.

[135] P. S. Aspinwall and M. Gross, "The SO(32) heterotic string on a K3 surface", *Phys. Lett.* **B387** (1996) 735–742 `arXiv:hep-th/9605131`.

[136] C. Mayrhofer, D. R. Morrison, O. Till and T. Weigand, "Mordell-Weil torsion and the global structure of gauge groups in F-theory", *JHEP* **10** (2014) 16 `arXiv:1405.3656 [hep-th]`.

[137] N. Nakayama, "On Weierstrass models", *Algebraic Geometry and Commutative Algebra in Honor of Masayoshi Nagata*, (1988), 405–431.

[138] I. Dolgachev and M. Gross, "Elliptic three-folds I: Ogg-Shafarevich theory", *Journal of Algebraic Geometry* **3**, (1994), 39–80.

[139] M. Gross, "Elliptic three-folds II: Multiple fibres", *Trans. Amer. Math. Soc.* **349**, (1997), 3409–3468.

[140] S. Mukai, *An introduction to invariants and moduli*, Cambridge University Press (2003).

[141] M. B. Green, J. H. Schwarz and P. West, "Anomaly free chiral theories in six-dimensions", *Nucl. Phys.* **B254** (1985) 327–348.

[142] A. Sagnotti, "A note on the Green-Schwarz mechanism in open string theories", *Phys. Lett.* **B294** (1992) 196–203 arXiv:hep-th/9210127.

[143] J. Erler, "Anomaly cancellation in six-dimensions", *J. Math. Phys.* **35** (1994) 1819–1833 arXiv:hep-th/9304104.

[144] J. H. Schwarz, "Anomaly - free supersymmetric models in six-dimensions", *Phys. Lett.* **B371** (1996) 223–230 arXiv:hep-th/9512053.

[145] S. H. Katz and C. Vafa, "Matter from geometry," *Nucl. Phys.* **B497**, 146–154 (1997) arXiv:hep-th/9606086 [hep-th].

[146] R. Slansky, "Group theory for unified model building," *Physics Reports* **79** (1981) 1–128.

[147] S. H. Katz and D. R. Morrison, "Gorenstein threefold singularities with small resolutions via invariant theory for Weyl groups," *Jour. Alg. Geom.* **1** (1992) 449 arXiv:alg-geom/9202002.

[148] F. Apruzzi, J. J. Heckman, D. R. Morrison and L. Tizzano, "4D Gauge Theories with Conformal Matter", *JHEP* **09** (2018) 088 arXiv:1803.00582 [hep-th].

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