

Symplectic duality and implosions

ANDREW DANCER, AMIHAY HANANY, AND FRANCES KIRWAN

We discuss symplectic and hyperkähler implosion and present candidates for the symplectic duals of the universal hyperkähler implosion for various groups.

1. Introduction

Implosion is an abelianisation construction that originated in symplectic geometry [14] and for which a hyperkähler analogue was developed in a series of papers [5–8]. In particular a complex-symplectic analogue of the universal symplectic implosion for a compact simple group was introduced, which in the A_n case (ie the group $SU(n+1)$) is in fact hyperkähler as a stratified space. The universal implosion for K carries a complex-symplectic action of $K_{\mathbb{C}} \times T_{\mathbb{C}}$ where $T_{\mathbb{C}}$ is the complexification of the maximal torus T . In the A_n case this is the complexification of an action of $K \times T$ which preserves the hyperkähler structure (that is, it is isometric and triholomorphic). There is also an action of $Sp(1)$ that rotates complex structures.

This data suggests that there should be a symplectic dual of the implosion. In this paper we present candidates for the symplectic duals in the A_n and D_n cases, including some computational evidence. We also include a discussion of implosions and their links to quiver varieties and the Moore-Tachikawa category, which we hope will be of interest to string theorists and algebraic geometers.

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2. Symplectic implosion

In this section we review the symplectic implosion construction of Guillemin, Jeffrey and Sjamaar [14]

The idea is that given a space M with Hamiltonian action of a compact group K , one can form the imploded space M_{impl} with a Hamiltonian action

of the maximal torus T of K , such that the symplectic reduction M of K agrees with the reduction of M_{impl} by T as long as we reduce at levels in the closed positive Weyl chamber. We can summarise this, using the usual notation for symplectic quotients, as:

$$M //_{\xi} K = M_{\text{impl}} //_{\xi} T \quad : \quad \xi \in \bar{\mathfrak{t}}_+^*$$

Fortunately the problem of constructing symplectic implosions can be reduced to the case $M = T^*K$, which in this sense plays a universal role for Hamiltonian spaces with K action. The key point here is that T^*K has a Hamiltonian $K \times K$ action so when we form the implosion $(T^*K)_{\text{impl}}$ with respect to, say, the right K action, the implosion has a $K \times T$ action, because the left K action survives the implosion process. Now the implosion of a general symplectic manifold X with Hamiltonian K -action can be obtained by reducing $X \times (T^*K)_{\text{impl}}$ by the diagonal K action, producing a space X_{impl} with T action. The reduction of X by K , at any element ξ of a chosen closed positive Weyl chamber in the dual \mathfrak{k}^* of the Lie algebra of K , coincides with the reduction of X_{impl} by T at ξ . In this sense the implosion abelianises the K action on X .

The space $(T^*K)_{\text{impl}}$ is referred to therefore as the *universal symplectic implosion* for K . It is explicitly constructed as a symplectic stratified space, by considering the product $K \times \bar{\mathfrak{t}}_+^*$ of the group and the closed positive Weyl chamber, and then performing certain collapsing operations as follows.

To motivate this, recall that the universal implosion should carry a Hamiltonian $K \times T$ action. The reductions by T at points in the closed positive Weyl chamber should coincide with the reductions of T^*K by the right K factor in the $K \times K$ action on T^*K . These reductions are exactly the coadjoint orbits of K : the K action on these coadjoint orbits is induced by the left K action on T^*K , or equivalently by the K factor in the $K \times T$ action on $(T^*K)_{\text{impl}}$.

Now, for $K \times \bar{\mathfrak{t}}_+^*$ the T moment map is projection onto the $\bar{\mathfrak{t}}_+^*$ factor so the reduction at level ξ is just $(K \times \{\xi\})/T \cong K/T$. This gives the correct picture for ξ in the open Weyl chamber, but not for ξ in the lower-dimensional faces of the chamber.

If we stratify the product $K \times \bar{\mathfrak{t}}_+^*$ by the faces of the Weyl chamber, then the choice of stratum corresponds to a choice of stabiliser C for ξ , and the coadjoint orbit of ξ is now K/C . Therefore to obtain the coadjoint orbits on reduction by T , we must quotient each stratum by the commutator $[C, C]$. Now the reduction by T at level ξ is $(K \times \{\xi\})/T.[C, C] = K/\text{Stab}_K(\xi)$ as required.

Hence the implosion is the symplectic stratified space obtained from $K \times \mathfrak{t}_+^*$ by stratifying by the faces of the Weyl chamber and quotienting by the commutator of the stabiliser associated to each stratum. In particular no collapsing occurs on the open Weyl chamber as C is then abelian. This yields the top stratum $K \times \mathfrak{t}_+^*$.

3. Nonreductive quotients

As often is the case with constructions in symplectic geometry, there is an alternative description of the universal symplectic implosion in terms of algebraic geometry.

We recall that geometric invariant theory (GIT) defines the quotient $X//G$ of an affine variety X over \mathbb{C} by the action of a complex reductive group G to be the affine variety $\text{Spec}(\mathcal{O}(X)^G)$ associated to the algebra $\mathcal{O}(X)^G$ of G -invariant regular functions on X . This is well-defined because in this situation the algebra $\mathcal{O}(X)^G$ is finitely generated.

Moreover the inclusion of $\mathcal{O}(X)^G$ in $\mathcal{O}(X)$ induces a natural G -invariant morphism from X to $X//G$. When G is reductive this morphism is always surjective, and points of X become identified in $X//G$ if and only if the closures of their G -orbits meet in the semistable locus of X .

If G is nonreductive then this picture can break down because the algebra of invariants is not necessarily finitely generated so $\text{Spec}(\mathcal{O}(X)^G)$ need not define an affine variety. Even if the algebra of invariants is finitely generated, so that the GIT quotient exists, the natural morphism $X \rightarrow X//G$ is not necessarily surjective, and its image is in general not a subvariety of the GIT quotient but only a constructible subset [10] (ie a finite union of intersections of open sets and closed sets).

It was shown in [14] that the universal symplectic implosion for a compact group K can be identified with the nonreductive GIT quotient $K_{\mathbb{C}}//N$. Here $K_{\mathbb{C}}$, the complexification of K , is a complex affine variety, and N denotes the maximal unipotent subgroup of $K_{\mathbb{C}}$. Although N is not reductive, the algebra of invariants $\mathcal{O}(K_{\mathbb{C}})^N$ is finitely generated so $K_{\mathbb{C}}//N$ exists as an affine variety. In fact $K_{\mathbb{C}}//N$ may be viewed as the canonical affine completion of the quasi-affine variety $K_{\mathbb{C}}/N$, which embeds naturally as an open subset of $K_{\mathbb{C}}//N$ with complement of codimension at least two. The restriction map from $\mathcal{O}(K_{\mathbb{C}}//N)$ to $\mathcal{O}(K_{\mathbb{C}}/N)$ is thus an isomorphism, and both algebras can be identified with the algebra of N -invariant regular functions on $K_{\mathbb{C}}$.

Moreover, there is a natural description of $K_{\mathbb{C}}//N$ as a stratified space, where the strata may be identified with $K_{\mathbb{C}}/[P, P]$ and P ranges over the

$2^{\text{rank}K}$ standard parabolics of $K_{\mathbb{C}}$. The top stratum, corresponding to choosing P to be the Borel subgroup B , is the quasi-affine variety $K_{\mathbb{C}}/N$. This stratification agrees with the symplectic stratification of section 2. In particular, using the Iwasawa decomposition $K_{\mathbb{C}} = KAN$, we may view the top stratum as KA , the open subset of the implosion corresponding to the interior of the positive Weyl chamber for K .

The simplest example as discussed in [14], is $K = SU(2)$. Now the N action on $K_{\mathbb{C}} = SL(2, \mathbb{C})$ is:

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \mapsto \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} + nx_{11} \\ x_{21} & x_{22} + nx_{21} \end{pmatrix}$$

with invariant ring freely generated by x_{11} and x_{21} , so $K_{\mathbb{C}}//N = \mathbb{C}^2$. There are two strata, the top one $SL(2, \mathbb{C})/N = \mathbb{C}^2 - \{0\}$ and the bottom one $\{0\}$. (As the closed Weyl chamber for $SU(2)$ is $[0, \infty)$, these coincide with the symplectic strata $SU(2) \times (0, \infty)$ and $(SU(2) \times \{0\})/SU(2)$).

So we see, as in the general case, that the implosion provides an affine completion of the quasi-affine top stratum. Notice that the canonical morphism $K_{\mathbb{C}} \rightarrow K_{\mathbb{C}}//N = \mathbb{C}^2$ defined by $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \mapsto \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}$ is not surjective, but instead has image the constructible set $\mathbb{C}^2 - \{0\}$.

In this case the strata actually fit together to form a smooth variety, but if K has a simple factor of rank greater than one, the implosion is always singular.

This picture has been generalised by Kirwan [16] to the case of quotients $K_{\mathbb{C}}//U_P$ where U_P is the unipotent radical of a parabolic subgroup P . This nonreductive quotient still exists as a variety and there is an interpretation in terms of a generalised version of the symplectic implosion construction of section 2. These spaces are referred to as partial symplectic implosions. They have an action of $K_{\mathbb{C}} \times L_P$ where L_P is the reductive Levi subgroup of P (recall P is the semidirect product $U_P \rtimes L_P$).

4. Hyperkähler implosion

In [5] we considered an analogue of the universal implosion for hyperkähler geometry. The starting point is the observation by Kronheimer [21] that $T^*K_{\mathbb{C}}$ carries a complete hyperkähler metric that is preserved by an action of $K \times K$. This action is not only isometric but also triholomorphic, that is, it preserves each individual complex structure I, J, K .

Kronheimer's construction proceeds by identifying $T^*K_{\mathbb{C}}$ with the moduli space of solutions to Nahm's equations

$$\frac{dT_i}{dt} + [T_0, T_i] = [T_j, T_k],$$

where (ijk) is a cyclic permutation of (123) , for smooth maps $T_i : [0, 1] \rightarrow \mathfrak{k}$. The moduli space is formed by quotienting by the gauge group of maps $g : [0, 1] \rightarrow K$ such that $g(0) = g(1) = Id$.

The residual gauge action by gauge transformations not necessarily equal to the identity at the endpoints $0, 1$ gives rise to the hyperkähler $K \times K$ action.

Note also that there is an isometric $SO(3)$ action given by rotating the triple (T_1, T_2, T_3) of Nahm matrices. This action is *not* triholomorphic but acts transitively on the 2-sphere of complex structures.

The identification of the Kronheimer moduli space with $T^*K_{\mathbb{C}}$ involves of course a choice of complex structure l . However all such complex structures are equivalent under the $SO(3)$ action. Note also that the l -holomorphic symplectic structure defined by the holomorphic parallel 2-form $\omega_J + i\omega_K$ is just the standard $K_{\mathbb{C}} \times K_{\mathbb{C}}$ -invariant holomorphic symplectic structure that $T^*K_{\mathbb{C}}$ has as the cotangent bundle of a complex manifold. (We shall usually use the term *complex-symplectic structure* for holomorphic symplectic structure in this paper).

$T^*K_{\mathbb{C}}$ is thus the hyperkähler analogue of the symplectic $K \times K$ -space $T^*K = K_{\mathbb{C}}$. As the universal symplectic implosion is the nonreductive quotient $K_{\mathbb{C}}//N$, it makes sense in the hyperkähler setting to consider a suitable reduction of $T^*K_{\mathbb{C}}$ by N , more specifically the complex-symplectic quotient (in the sense of geometric invariant theory) of $T^*K_{\mathbb{C}}$ by N .

As the complex-symplectic structure on $T^*K_{\mathbb{C}}$ is the standard one, its associated moment map is just projection onto the $\mathfrak{k}_{\mathbb{C}}^*$ factor of $T^*K_{\mathbb{C}} = K_{\mathbb{C}} \times \mathfrak{k}_{\mathbb{C}}^*$. The zero locus for this moment map is therefore $K_{\mathbb{C}} \times \mathfrak{n}^{\circ}$ where \mathfrak{n}° is the annihilator in $\mathfrak{k}_{\mathbb{C}}^*$ of the Lie algebra \mathfrak{n} of N .

We are therefore led to define the universal hyperkähler implosion for K to be the geometric invariant theory (GIT) quotient $(K_{\mathbb{C}} \times \mathfrak{n}^{\circ})//N$ where N is a maximal unipotent subgroup of the complexified group $K_{\mathbb{C}}$. It is sometimes convenient to choose an invariant inner product, and identify the annihilator \mathfrak{n}° with the opposite Borel subalgebra \mathfrak{b} .

As N is nonreductive, it is a nontrivial result that the algebra of N -invariants is finitely generated and hence the quotient exists as an affine variety. This was shown in the case $K = SU(n)$ in [5] and in general follows from results of Ginzburg-Riche [13] (see the discussion in [8] for example).

The universal hyperkähler implosion carries a complex-symplectic action of $K_{\mathbb{C}} \times T_{\mathbb{C}}$ where T is the standard maximal torus of K . The $K_{\mathbb{C}}$ action is just left translation on the $K_{\mathbb{C}}$ factor, while the $T_{\mathbb{C}}$ action is right translation on the $K_{\mathbb{C}}$ factor together with the adjoint action on the \mathfrak{n}° factor. Of course the fact we are restricting to \mathfrak{n}° means that the right $K_{\mathbb{C}}$ action on $K_{\mathbb{C}} \times \mathfrak{k}_{\mathbb{C}}^*$ is broken to a $T_{\mathbb{C}}$ action.

A naive guess might be that, by analogy with the symplectic case, the complex-symplectic torus reductions of the implosion will give us the coadjoint orbits for the complex Lie algebra $\mathfrak{k}_{\mathbb{C}}$. However this cannot be exactly right, as only semisimple coadjoint orbits in the complex Lie algebra are closed. The complex-symplectic quotients by the torus action are instead the Kostant varieties; that is, the varieties in $\mathfrak{k}_{\mathbb{C}}^*$ obtained by fixing the values of the invariant polynomials for this Lie algebra [4, 18]. The Kostant varieties are in general stratified spaces whose strata are distinct complex coadjoint orbits. The minimal stratum is the semisimple orbit and the top stratum is the regular orbit, which is open and dense in the Kostant variety with complement of codimension at least 2. (For $K_{\mathbb{C}} = SL(n, \mathbb{C})$ the elements of the regular orbit are characterised by the minimal polynomial being equal to the characteristic polynomial, the latter being fixed by the choice of Kostant variety).

Note that, just as the symplectic implosion has real dimension $\dim_{\mathbb{R}} K + \text{rank } K$, so the hyperkähler implosion has *complex* dimension equal to $\dim_{\mathbb{C}} K_{\mathbb{C}} + \text{rank } K_{\mathbb{C}}$, consistent with the fact that the Kostant varieties have complex dimension $\dim_{\mathbb{C}} K_{\mathbb{C}} - \text{rank } K_{\mathbb{C}}$.

5. Hyperkähler quiver diagrams

The description in the previous section is rather abstract and although it makes plain the complex-symplectic structure, it is less clear that this actually comes from a hyperkähler metric.

In [5] we considered the case when $K = SU(n)$. In this situation the universal hyperkähler implosion can be identified with a hyperkähler quotient using quiver diagrams, and thus can be seen to be genuinely a stratified hyperkähler space rather than just a complex-symplectic one.

We shall consider quivers $Q = (Q_0, Q_1)$ where Q_0 is the set of vertices and Q_1 the set of edges. For each edge $e \in Q_1$, we denote $o(e)$ and $i(e)$ the outgoing and incoming vertices of the edge. To each vertex j we associate a complex vector space V_j of dimension N_j .

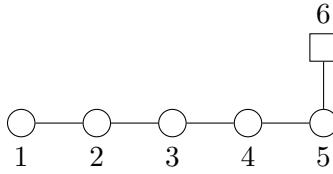


Figure 1: Quiver for the nilpotent cone of A_5 .

In the simplest case one can associate to the quiver the flat quaternionic space

$$M = \bigoplus_{e \in Q_1} \text{hom}(V_{i(e)}, V_{o(e)}) \oplus \text{hom}(V_{o(e)}, V_{i(e)})$$

and the group $K = \prod_{j \in Q_0} U(V_j)$, with its natural action on M :

$$\alpha_e \mapsto g_{o(e)} \alpha_e g_{i(e)}^{-1}, \quad \beta_e \mapsto g_{i(e)} \beta_e g_{o(e)}^{-1} \quad (e \in Q_1),$$

In more physical language, to each edge joining vertices labelled by dimensions N_i and N_j we associate the hypermultiplets $\mathbb{H}^{N_i N_j}$ transforming in the bifundamental representation of $U(N_i) \times U(N_j)$. Fixing a complex structure and identifying this with $\text{hom}(\mathbb{C}^{N_i}, \mathbb{C}^{N_j}) \oplus \text{hom}(\mathbb{C}^{N_j}, \mathbb{C}^{N_i})$ as above corresponds physically to decomposing the hypermultiplet into chiral and antichiral multiplets.

This action preserves the hyperkähler structure so one may form the hyperkähler reduction $M // K$. More generally, one may hyperkähler reduce by a subgroup K_1 of K , so that the quotient $M // K_1$ retains a residual hyperkähler action of $N_K(K_1)/K_1$ where $N_K(K_1)$ denotes the normaliser of K_1 in K . In particular, one may define a normal subgroup K_1 of K by choosing a subset $Q \subset Q_0$ and defining $K_1 = K_Q := \prod_{j \in Q} U(V_j)$. That is, we ‘turn off’ the action at the nodes in $Q_0 - Q$. The hyperkähler quotient now has a residual action of $K/K_Q \cong \prod_{j \notin Q} U(V_j)$.

The vertices $j \in Q$ where the group still acts are called *gauge nodes* and the vertices $j \in Q_0 - Q$ where the action has been turned off are the *flavour nodes*. The gauge nodes are denoted by circles and the flavour nodes by square boxes.

Example 5.1. Consider the A_n diagram with dimension vector $(1, 2, \dots, n)$ where the n -dimensional node is a flavour node. (The figure shows the $n = 6$ case)

So we hyperkähler reduce by $U(1) \times \dots \times U(n - 1)$ and leave the residual action $U(n)$. The hyperkähler quotient is known by the work of Kobayashi-Swann [17] (see also [19]) to be the nilpotent variety for $A_{n-1} = SL(n, \mathbb{C})$

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This motivated the quiver description of hyperkähler implosion for $K = SU(n)$ developed in [5]. The implosion is required to have a $SU(n) \times T$ action with hyperkähler reduction by T giving the Kostant varieties, in particular reduction at level zero giving the nilpotent variety. It is natural therefore to consider the same quiver as above, but with the action of $H = \prod_{j=1}^{n-1} SU(j)$, rather than $K = \prod_{j=1}^{n-1} U(j)$. The resulting hyperkähler quotient $M // H$ is a stratified hyperkähler space with a residual action of the torus $T = K/H$ as well as a commuting action of $SU(n)$.

We can also consider the implosion as a complex-symplectic quotient. It is the geometric invariant theory quotient, of the zero locus of the complex moment map $\mu_{\mathbb{C}}$ for the H action, by the complexification

$$H_{\mathbb{C}} = \prod_{j=1}^{n-1} SL(j, \mathbb{C})$$

of H ,

The complex moment map equation $\mu_{\mathbb{C}} = 0$ is equivalent to the equations

$$(5.2) \quad \beta_{i+1}\alpha_{i+1} - \alpha_i\beta_i = \lambda_{i+1}^{\mathbb{C}} I \quad (i = 0, \dots, n - 2),$$

for (free) complex scalars $\lambda_1^{\mathbb{C}}, \dots, \lambda_{n-1}^{\mathbb{C}}$. The complex numbers λ_i combine to give the complex-symplectic moment map for the residual action of $K_{\mathbb{C}}/H_{\mathbb{C}}$ which we can identify with the maximal torus $T_{\mathbb{C}}$ of $K_{\mathbb{C}}$.

Note that, as usual with linear hyperkähler quotients at level zero, we also have an $Sp(1)$ action on the implosion that *rotates* the complex structures. If we view the quaternionic summands $\text{hom}(V_i, V_{i+1}) \oplus \text{hom}(V_{i+1}, V_i)$ associated to each edge of the quiver as quaternionic space $\mathbb{H}^{N_i N_{i+1}}$ then the quiver group H may be viewed as acting on $\mathbb{H}^{N_i N_{i+1}}$ on the left while the quaternionic structure is acting on the right by $-i, -j, -k$ etc. Now multiplication by unit quaternions on the right gives an isometric action, rotating complex structures, and commuting with the action of H . It therefore acts on the hyperkähler moment map $\mu : M \rightarrow \mathfrak{h}^* \otimes \mathbb{R}^3$ by rotation on \mathbb{R}^3 and hence preserves the hyperkähler quotient at level zero. Moreover, as the level is zero and the moment map is homogeneous quadratic, we have a scaling action of the positive reals. We can summarise this as saying the

$SU(n)$ -implosion has a conical structure, and as such is expected to fit into the symplectic duality framework discussed in section 7.

For other classical groups we do not as yet have a quiver description of the implosion. This is because the analogues of the quiver description of the nilpotent varieties involve *orthosymplectic quivers*, that is, quivers where the groups attached to the vertices are alternately orthogonal and symplectic groups [17]. Unlike the unitary groups, we cannot write these groups as extensions of tori by subgroups, so we cannot mimic the above construction by considering quivers with just the subgroups acting.

6. Moore-Tachikawa category

In [22] Moore and Tachikawa proposed a category whose objects were complex semisimple or reductive groups and where morphisms between G_1 and G_2 are complex-symplectic manifolds with $G_1 \times G_2$ action. (Strictly speaking a morphism is a triple (X, G_1, G_2) where X is such a complex-symplectic manifold, ie the ordering of the objects is specified). There is also supposed to be a commuting circle action acting on the complex-symplectic form with weight 2. Composition of morphisms $X \in \text{Mor}(G_1, G_2)$ and $Y \in \text{Mor}(G_2, G_3)$ proceeds by forming the product $X \times Y$ with $G_1 \times (G_2 \times G_2) \times G_3$ action and then taking the complex-symplectic quotient by the diagonal G_2 action. The resulting quotient is complex-symplectic with residual $G_1 \times G_3$ action so lies in $\text{Mor}(G_1, G_3)$ as required. The Kronheimer space $T^*K_{\mathbb{C}}$ is complex-symplectic with $K_{\mathbb{C}} \times K_{\mathbb{C}}$ action and defines the identity element in $\text{Mor}(G, G)$ with $G = K_{\mathbb{C}}$.

In this picture the implosion for K may be viewed as an element of $\text{Mor}(K_{\mathbb{C}}, T_{\mathbb{C}})$. The process of imploding a complex-symplectic manifold with $K_{\mathbb{C}}$ action to obtain a manifold with $T_{\mathbb{C}}$ action, as described in section 2 but in the complex-symplectic case, is now exactly that of composition of morphisms with the implosion, to obtain a map:

$$\text{Mor}(1, K_{\mathbb{C}}) \rightarrow \text{Mor}(1, T_{\mathbb{C}})$$

Note that one could enrich the data of complex-symplectic manifolds to hyperkähler manifolds in these definitions, using the fact that the complex-symplectic quotient by G_2 coincides with the hyperkähler quotient by the maximal compact subgroup of G_2 . However now $T^*K_{\mathbb{C}}$ is no longer exactly the identity, as pointed out by Moore-Tachikawa. The metric is shifted by a factor representing the length of the interval on which the Nahm data is defined to produce the Kronheimer space.

7. Symplectic duality

It is conjectured that there is a duality between certain complex-symplectic (that is, holomorphic symplectic) varieties, that physically may be interpreted as duality (the notion of duality is explained below and is different than other forms of dualities in physics) between Higgs and Coulomb branches of a 3d $\mathcal{N} = 4$ theory.

The complex-symplectic varieties concerned usually in fact have a hyperkähler structure, and arise either as hyperkähler cones or as deformations thereof. In many cases the Higgs branch cone occurs as the zero level set of a hyperkähler quotient construction $M // G$ (the moduli space of vacua), while the deformations occur by moving the level set away from zero. In physics the resulting deformation parameters are called Fayet-Iliopoulos parameters.

For symplectic duality constructions we want the complex-symplectic varieties to have a circle action that acts on the complex symplectic form with weight 2 (in terms of the hyperkähler structure, the circle action fixes one complex structure I but rotates the J, K so the I -holomorphic form $\omega_J + i\omega_K$ is scaled rather than being invariant under the action).

As mentioned in §5, linear hyperkähler quotients at level zero have a $Sp(1)$ action rotating the complex structures. Making a deformation that breaks this $Sp(1)$ down to the circle action fixing the specific complex structure I corresponds to changing the level set to $(\lambda, 0, 0)$ where $\lambda \in \mathfrak{g}^*$. As the level set at which the hyperkähler reduction is performed must lie in the centre of G , the number of deformation parameters is the dimension of the center of G .

On the Coulomb side, the deformation parameters are the masses. The duality is supposed to interchange the rank of the hyperkähler isometry group of a space and the number of deformation parameters for its dual. More precisely, the Cartan algebra of the flavour group of the Higgs branch is identified with the space of mass parameters, and the Cartan algebra of the flavour group of the Coulomb branch with the space of Fayet-Iliopoulos parameters.

Nakajima (see [23] for example) has suggested that in the case when the Higgs branch is a hyperkähler quotient $M // G$ by a compact group G , the Coulomb branch should be birational to $T^*(T_{\mathbb{C}}^{\vee})/W$, the quotient by the Weyl group of the cotangent bundle of the complexified dual maximal torus of G . We therefore expect

$$\dim_{\mathbb{R}}(\text{Coulomb branch}) = 4 \text{ rank } G.$$

Physically, the birational equivalence represents quantum corrections to the classical description of the Coulomb branch.

One example where the theory is completely worked out is *hypertoric manifolds*, that is, hyperkähler quotients of flat quaternionic space by tori. (See [2] for example). As in [1] we consider quotients of \mathbb{H}^d by a subtorus N of T^d . The torus is defined by vectors $u_1, \dots, u_d \in \mathbb{R}^n$: explicitly we define $\mathfrak{n} = \text{Lie}N$ to be the kernel of the map $\beta : \mathbb{R}^d = \text{Lie}T^d \rightarrow \mathbb{R}^n$ defined by $\beta : e_i \mapsto u_i$, where e_1, \dots, e_d is the standard basis for \mathbb{R}^d . On the Lie algebra level, we have an exact sequence

$$0 \rightarrow \mathfrak{n} \rightarrow \mathbb{R}^d \xrightarrow{\beta} \mathbb{R}^n \rightarrow 0$$

On the Lie group level we have:

$$1 \rightarrow N \rightarrow T^d \rightarrow T^n \rightarrow 1$$

The hypertoric $M = \mathbb{H}^d // N$ has real dimension $4d - 4(d - n) = 4n$. and admits a residual action of the quotient torus $T^n = T^d/N$. The number of deformation parameters for M is $\text{rank } N = d - n$ and the rank of the isometry group is n .

Now the dual hypertoric variety is defined to be the hyperkähler quotient of \mathbb{H}^d by the dual torus \hat{T}^n

$$1 \rightarrow \hat{T}^n \rightarrow \hat{T}^d \rightarrow \hat{N} \rightarrow 1$$

Now the number of deformation parameters is n and the rank of the isometry group is $\text{rank } \hat{N} = \text{rank } N = d - n$, in accordance with the principle of symplectic duality. The dimension of the dual hypertoric is $4(d - n)$, illustrating how dimension can change under duality.

As usual in toric or hypertoric geometry, this duality can be viewed as a combinatorial phenomenon, in this case known as Gale duality. Given a vector space V of dimension n with spanning vectors u_1, \dots, u_d , we can form the space of linear dependency relations $\{(\alpha_1, \dots, \alpha_d) : \sum_{i=1}^d \alpha_i u_i = 0\}$. This is a $d - n$ dimensional vector space W with d distinguished elements w_1, \dots, w_d in the dual vector space W^* defined by $w_i : (\alpha_1, \dots, \alpha_d) \mapsto \alpha_i$. This duality, interchanging n and $d - n$, implements the above duality between the hypertorics of dimension $4n$ and $4(d - n)$.

In this case, both the Higgs and Coulomb branches are given by finite-dimensional hyperkähler quotients. However there are cases where one space

is given by such a construction but its dual is not—we call these *non-Lagrangian* theories.

Various relations between a quiver variety and its symplectic dual have been developed in the physics literature.

The crucial concept here is that of a *balanced* node. In the case of a unitary quiver with dimensions N_j at nodes j , the *balance* of a node j is

$$-2N_j + \sum_{k \text{ adjacent to } j} N_k$$

and we say the node is *balanced* if the balance is zero.

For a nice physical theory we would like all the gauge nodes to have balance greater than or equal to -1 . If this holds and there is a node with balance equal to -1 the quiver is called *minimally unbalanced*, while if all nodes have nonnegative balance with at least one of positive balance, we say it is *positively balanced*.

In the case of unitary quivers, the balanced gauge nodes should form the Dynkin diagram of the semisimple part of (a subgroup of) the hyperkähler isometry group of the dual space. (Unbalanced nodes give abelian symmetries). This refines the earlier idea that deformation parameters coming from the unitary gauge nodes should give an abelian algebra of symmetries in the dual—if the nodes are balanced then the associated abelian symmetry group is realised as the maximal torus of a larger semisimple group.

For example, in the nilpotent variety quiver of Example 5.1 all nodes except the final flavour node are balanced. This gives an A_{n-1} Dynkin diagram which should give $SU(n)$ symmetry group of the dual. In fact the dual is still the nilpotent variety.

Example 7.1. Consider the quiver diagram in Figure 2 corresponding to the hyperkähler quotient $\mathbb{H}^d // U(1)$ where we have 1 gauge node (with dimension 1) and 1 flavour node (with dimension d)

This is a hypertoric, with symplectic dual $\mathbb{H}^d // T^{d-1}$. The latter space gives the cyclic Kleinian singularity $\mathbb{C}^2 / \mathbb{Z}_d$ or its deformations, the A_{d-1} multi-instanton metrics whose topology is generated by a chain of $d-1$ rational curves with self-intersection -2 .

If $d \neq 2$ then we have no balanced nodes in the diagram, but if $d = 2$ then the gauge node is balanced. This reflects the fact that for $d = 2$ the dual space is Eguchi-Hanson which has a triholomorphic $SU(2)$ action, an enlargement of the triholomorphic $U(1)$ action that occurs for general d . \diamond

Figure 2: $U(1)$ with d flavors.

One can study the varieties occurring in symplectic duality by finding the Hilbert series of their coordinate ring (the chiral ring in physics terminology). This series counts the dimension m_d of the degree d parts of the ring

$$HS(t) = \sum_{d=0}^{\infty} m_d t^d.$$

The variable t is called the *fugacity*.

Cremonesi-Hanany-Zaffaroni [3] have derived a formula, the *monopole formula* to compute the Hilbert series of the Coulomb branch of a quiver variety obtained as a hyperkähler reduction of a flat quaternionic space by a group G . We are counting monopole operators whose gauge field has a Dirac monopole singularity, with associated magnetic charge living in the weight lattice $\Gamma_{\hat{G}}$ of the Langlands dual \hat{G} . Their formula involves contributions from the stabiliser groups of each element of the lattice:

$$(7.2) \quad HS(t) = \sum_{m \in \Gamma_{\hat{G}}/W_{\hat{G}}} t^{2\Delta(m)} P_G(m, t)$$

Here

$$P_G(m, t) = \prod_i \frac{1}{1 - t^{2d_i(m)}}$$

where the $d_i(m)$ are the exponents of the stabiliser group $G_m = \text{Stab}_G(m)$ —that is, the degrees of the generators (Casimirs) for the ring of invariants of G_m under the adjoint representation. We can also interpret $P_G(m, t)$ as the Poincaré polynomial of the classifying space BG_m .

The term $\Delta(m)$ is given by

$$\Delta(m) = - \sum_{\alpha \in R^+} |\alpha(m)| + \frac{1}{2} \sum_b |b(m)|$$

where R^+ denotes the set of positive roots in G , and the second sum is taken over the weights in the given representation M .

Plethystic techniques have been developed (eg [11]) to compute from the Hilbert series the generators, relations and higher-order syzygies of the chiral ring.

Note that the t^2 term of the Hilbert series is expected to give the dimension of the global symmetry group.

8. Duals of implosions

We now consider what kind of space would be dual to the $SU(n)$ implosion. The latter space has an action of $SU(n) \times T^{n-1}$, so this suggests we look at a quiver whose balanced nodes give the Dynkin diagram A_{n-1} and whose unbalanced nodes give the torus factor.

We consider the quiver from Example 5.1 that gives the nilpotent variety. Now replace the flavour node (box) with dimension n by a bouquet of n $U(1)$ nodes attached to the $(n-1)$ -dimensional gauge node. This ensures that the $(n-1)$ -dimensional node remains balanced, as well as the gauge nodes lower down the chain. So the balanced nodes do form the A_{n-1} Dynkin diagram as required, giving a $SU(n)$ action on the dual. The $U(1)$ nodes are unbalanced (for $n \neq 3$) and generate a T^{n-1} action on the dual (it is T^{n-1} , not T^n , as one $U(1)$ ‘decouples’, ie acts trivially. This is the diagonally embedded $U(1) \hookrightarrow U(1)^n \times \prod_{k=1}^{n-1} Z(U(k))$, where Z denotes the centre). Note that the balance of the $U(1)$ nodes is always at least -1 , and is positive for $n \geq 4$.

Example 8.1. If $n = 2$ this is just an A_3 diagram with dimension 1 at each node. As the diagonal $U(1)$ acts trivially this represents the trivial hyperbolic $\mathbb{H}^2 // T^2$ and its dual is $\mathbb{H}^2 // \{1\} = \mathbb{H}^2$. This is correct as the universal hyperkähler implosion for $SU(2)$ is indeed \mathbb{H}^2 . \diamond

Example 8.2. If $n = 3$ we have a star-shaped quiver (affine \tilde{D}_4 Dynkin diagram) with dimension 2 at the central node and dimension 1 at the four nodes radially connected to it (one from the tail of the truncated A_2 diagram and three from the bouquet). Uniquely in this case all nodes (even the bouquet ones) are balanced, so we expect, after decoupling, an $SO(8)$ symmetry in its dual.

This is correct, as the $SU(3)$ universal hyperkähler implosion may be identified with the Swann bundle of the quaternionic Kähler Grassmannian $\tilde{\text{Gr}}_4(\mathbb{R}^8) = SO(8)/S(O(4) \times O(4))$ of oriented 4-planes in \mathbb{R}^8 . The $SO(8)$

symmetry of the quaternionic Kähler base lifts to a symmetry of the hyperkähler Swann bundle (see Example 8.7 of [5] for a discussion). \diamond

As the $SU(n)$ implosion has been described as a reduction by a product of special unitary groups in $\mathfrak{5}$, we expect it has no deformation parameters. This checks with the fact that the proposed dual has no residual hyperkähler isometries, as all nodes are gauge and not flavour nodes.

In fact, we expect for general groups that the implosion has no deformation parameters, as we obtain it as the nonreductive quotient $(K_{\mathbb{C}} \times \mathfrak{n}^{\circ})//N$ and the maximal unipotent group N has trivial maximal torus so no characters.

For a global symmetry of $SU(n) \times U(1)^{n-1}$ we expect the coefficient of the t^2 term in the Hilbert series to be $n^2 - 1 + n - 1 = n^2 + n - 2$. In addition, due to the balance of $n - 3$ of each $U(1)$ node in the Bouquet, there are generators of the chiral ring which arise from the $U(1)$ nodes that contribute 2 per each $U(1)$ at order t^{n-1} . These correspond to one monopole operator of positive charge and one with negative charge under the corresponding $U(1)$ global symmetry. We expect the Hilbert series to get contributions

$$HS_n = (n^2 + n - 2)t^2 + 2nt^{n-1} + \dots$$

Let us see how this fits in examples. For $n = 2$ we get $4t$ that represent the 4 generators of \mathbb{H}^2 . They contribute 6 more quadratic bilinears that enhance the global symmetry from $SU(2) \times U(1)$ to $Sp(2)$. For $n = 3$ the affine \tilde{D}_4 quiver indeed confirms that the global symmetry is enhanced from $SU(3) \times U(1)^2$ to $SO(8)$. For $n > 3$ perturbative computations confirm the t^2 coefficient.

One can further refine the expression for the Hilbert Series in equation (7.2) by introducing a fugacity z_i for each magnetic charge m_i of $U(1)_i$ in the bouquet for $i = 1 \dots n$, resulting in a function of $n + 1$ variables $HS(t, z_i)$. This expression can be further integrated

$$(1 - t^2)^{n-1} \prod_i \oint_{|z_i|=1} \frac{dz_i}{z_i} HS(t, z_i)$$

resulting in the expression for the Hilbert series of the nilpotent cone of $SL(n)$ which takes a particularly simple form

$$\frac{\prod_{i=1}^n (1 - t^{2i})}{(1 - t^2)^{n^2}}$$

This constitutes a non trivial test of the proposed quiver for the $SU(n)$ implosion.

We can also check that this is consistent with Nakajima’s picture.

The rank of the group $U(1)^{n-1} \times \prod_{i=1}^{n-1} U(i)$ by which we quotient in the bouquet quiver is $\frac{1}{2}(n+2)(n-1) = \frac{1}{2}(n^2+n-2)$ and the real dimension of the implosion is

$$\dim_{\mathbb{R}} SL(n, \mathbb{C}) + \dim_{\mathbb{R}}(T_{\mathbb{C}}^{n-1}) = 2(n^2+n-2).$$

Going in the reverse direction, the implosion is produced as a hyperkähler quotient by $\prod_{i=1}^{n-1} SU(i)$ which has rank $\frac{1}{2}(n-1)(n-2)$. The quaternionic dimension of the bouquet quiver variety is

$$n(n-1) + \sum_{i=1}^{n-2} i(i+1) - (n-1 + \sum_{i=1}^{n-1} i^2)$$

which works out as $\frac{1}{2}(n-1)(n-2)$ as desired.

For example, if $n = 3$ then we have the affine \tilde{D}_4 Dynkin diagram, giving one of Kronheimer’s examples [20] of real dimension 4, ie quaternionic dimension 1. This corresponds to the fact that the $SU(3)$ implosion is a hyperkähler quotient of a linear space by $SU(2)$.

We also make some remarks on partial hyperkähler implosions, ie complex symplectic quotients of $T^*K_{\mathbb{C}}$ by the unipotent radical U_P of a parabolic P . (It is as yet a conjecture that these exist as algebraic varieties, that is, that the algebra of U_P -invariants in $K_{\mathbb{C}} \times \mathfrak{u}_P^{\circ}$ is finitely generated).

In the case $K = SU(n)$, of course, the parabolics are indexed by ordered partitions $n = n_1 + \dots + n_r$ and the corresponding Levi subgroup is $S(GL(n_1, \mathbb{C}) \times \dots \times GL(n_r, \mathbb{C}))$.

As $SL(n, \mathbb{C})/P = SU(n)/S(U(n_1) \times \dots \times U(n_r))$, we see that

$$\dim_{\mathbb{R}} P = n^2 - 2 + \sum_{i=1}^r n_i^2$$

and

$$\dim_{\mathbb{R}} U_P = n^2 - \sum_{i=1}^r n_i^2$$

so the dimension of the partial implosion should be

$$\dim_{\mathbb{R}}(SL(n, \mathbb{C}) \times \mathfrak{u}_P^{\circ}) // U_P = 2(n^2 - 2 + \sum_{i=1}^r n_i^2)$$

Note that as $\sum_{i=1}^r n_i = n$, the sum $\sum_{i=1}^r n_i^2$ has the same parity as n^2 so the expression inside the bracket above is even, as required.

If all $n_i = 1$ of course P is the Borel and we recover the dimension of the standard implosion as above.

A natural candidate for the dual would be the quiver diagram we obtain by taking the basic diagram for the nilpotent quiver, excising the dimension n flavour node, and then attaching r legs, each of them an A_{n_i} quiver with the dimension n_i node next to the dimension $n - 1$ node of the original diagram.

So the remaining nodes of the original diagram are all balanced, giving an $SU(n)$ symmetry in the implosion. Moreover on each leg, all nodes except the $\dim n_i$ ones are balanced, yielding $SU(n_i)$ symmetries for $i = 1, \dots, r$. Also, the r unbalanced nodes (ie the $\dim n_i$ ones of the attached legs) would yield, after decoupling, $r - 1$ Abelian symmetries. These nodes have balance $n - n_i - 2$ which is always at least -1 and is positive unless our partition is $n = (n - 2) + 2, (n - 2) + 1 + 1$ or $(n - 1) + 1$.

So overall, we would get $SU(n) \times S(U(n_1) \times \dots \times U(n_r))$ symmetry, as required.

The group by which we perform the hyperkähler quotient is

$$G = S(U(1) \times \dots \times U(n - 1)) \times \prod_{i=1}^r U(1) \times \dots \times U(n_i)$$

which has rank

$$\frac{1}{2}(n^2 - 2 + \sum_{i=1}^r n_i^2)$$

So the real dimension of the implosion is 4 times the rank of G , in accordance with Nakajima's picture. The dimensions and symmetry groups therefore work out correctly—we hope to further investigate this picture in a future work.

9. Orthosymplectic examples

For other classical groups we have to revisit the notion of balance, as well as the prescription for finding the symmetry group of the dual (see eg [15], [12]).

In the case of orthosymplectic quivers (where we use the physics notation $USp(n) = Sp(n/2)$), there are 2 cases to consider:

(i) that of an orthogonal node labelled by $SO(N)$, with neighbours $USp(N_j) = Sp(N_j/2)$. The balancing condition is

$$2N = 2 + \sum N_j$$

where the sum is taken over all nodes adjacent to the $SO(N)$ one.

(ii) a symplectic mode $USp(N)$ with neighbours $SO(N_j)$. Now the balancing condition is

$$2N = -2 + \sum N_j$$

Let us consider the D_n case. The quiver defining the nilpotent variety is a chain with $2n - 2$ gauge nodes $SO(2), USp(2), SO(4), \dots, USp(2n - 2)$ and then a flavour node $SO(2n)$. The gauge nodes are all balanced, yielding in the orthosymplectic situation a $SO(2n)$ symmetry in the dual space.

For the dual of the implosion, we can mimic the construction in the A_n case, removing the flavour node and replacing it with a bouquet of n $SO(2)$ nodes. This keeps the $USp(2n - 2)$ gauge node (and the preceding gauge nodes) balanced, so we still have an $SO(2n)$ -symmetry in the implosion as required. The unbalanced nodes now yield a T^n symmetry in the implosion, which again is correct. As in the A_n case, we have no flavour nodes, reflecting the fact we do not expect deformation parameters in the implosion.

We can carry out a check using the calculations of Zhenghao Zhong [26] of the Hilbert series for the Coulomb branch of these quivers for $n = 3, 4, 5, 6, 7$. The t^2 coefficient, which is expected to give the dimension of the global symmetry group, is 18, 32, 50, 72, 98 in these cases. So in each of these cases we obtain

$$2n^2 = n + 2n(2n - 1)/2 = \text{rank } SO(2n) + \dim SO(2n)$$

as expected for the complex dimension of the symmetry group of the $SO(2n)$ implosion.

The rank of the group by which we are performing the hyperkähler quotient is $n + 2 \sum_{i=1}^{n-1} i = n^2$, and the real dimension of the $SO(2n)$ -implosion is $4n^2$, in accordance with our expectation.

So for D_n although the original implosion does not appear to have a quiver description (ie is non-Lagrangian) the dual *does* arise as a quiver variety.

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JESUS COLLEGE, OXFORD, OX1 3DW, UK

E-mail address: `dancer@maths.ox.ac.uk`

THEORETICAL PHYSICS GROUP, THE BLACKETT LABORATORY

IMPERIAL COLLEGE, PRINCE CONSORT ROAD, LONDON SW7 2AZ, UK

E-mail address: `a.hanany@imperial.ac.uk`

NEW COLLEGE, OXFORD, OX1 3BN, UK

E-mail address: `kirwan@maths.ox.ac.uk`

