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Twisted gauge fields

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We propose a generalisation of the notion of associated bundles to a principal bundle constructed via group action cocycles rather than representations of the structure group. We devise a notion of connection generalising Ehresmann connection on principal bundles, giving rise to the appropriate covariant derivative on sections of these twisted associated bundles (and on twisted tensorial forms). We study the action of the group of vertical automorphisms on the objects introduced (active gauge transformations). We also provide the gluing properties of the local representatives (passive gauge transformations). The latter are generalised gauge fields: They satisfy the gauge principle of physics, but are of a different geometric nature than standard Yang-Mills fields. We also examine the conditions under which this new geometry coexists and mixes with the standard one. We show that (standard) conformal tractors and Penrose's twistors can be seen as simple instances of this general picture. We also indicate that the twisted geometry arises naturally in the definition and study of anomalies in quantum gauge field theory.

1. Introduction

Classical gauge field theory is founded on the differential geometry of connections on fibered spaces: Ehresmann (principal) connections underlies Yang-Mills type theories, relevant for particle physics, while Cartan connections are the foundation for gauge theories of gravitation. Since Wigner, it is admitted that given a symmetry (either spatio-temporal of internal) identified as a Lie group H, different kinds of (fundamental) particles correspond to different (irreducible) representations (ρ, V) of H. But particles are actually manifestations of (quantized) fields. So, one considers a principal bundle $\mathcal{P}(\mathcal{M}, H)$ over spacetime $\mathcal{M}$, to which are associated - for each representation - bundles E whose sections $s : \mathcal{M} \to E \in \Gamma(E)$ describe matter fields of different kinds (to be further quantized). Ehresmann connections on $\mathcal{P}$ give Yang-Mills potentials on $\mathcal{M}$ and induce covariant differentiation on $\Gamma(E)$ that represents the minimal coupling of matter fields to gauge interactions. Similarly, Cartan connections on $\mathcal{P}$ give gravitational potentials on $\mathcal{M}$, and in many interesting cases also induce covariant differentiation representing the coupling of matter fields to gravity.

A gauge field theory is specified by choosing a Lagrangian. This choice is constrained by a list of desiderata that might be long and in part guided by empirical data. But at the top of the list is that the Lagrangian should satisfy the gauge principle: it must be invariant (or quasi-invariant) under local (point dependent) transformations of the field variables. These gauge transformations are induced by the group of vertical automorphisms of the underlying principal bundle $\text{Aut}_v(\mathcal{P})$, which is a normal subgroup of its group of automorphisms $\text{Aut}(\mathcal{P})$, itself the subgroup of $\text{Diff}(\mathcal{P})$ that preserve the fibration structure and projects as diffeomorphisms of spacetime. We have the short exact sequence, $\text{Aut}_v(\mathcal{P}) \xrightarrow{\iota} \text{Aut}(\mathcal{P}) \xrightarrow{\pi} \text{Diff}(\mathcal{M})$. The gauge principle should then be understood as a direct, though abstract, extension of the principle of general covariance at the heart of General Relativity (GR), and the requirement of gauge invariance as a rather natural generalisation of the requirement of diffeomorphism invariance. Our most successful theories of fundamental physics, from GR to the Standard Model (SM), are gauge field theories of this kind.

Various generalisations, some far reaching, of the above differential geometric framework with their associated notions of connection have been proposed, sometimes with physical motivations, often with relevant physical applications. The rise to popularity of supersymmetry and supergravity e.g. inspired the study of differential super-geometry, super-bundles and super-connections [1]. The development of derivation-based noncommutative geometry (NCG) opened the possibility to give a geometrico-algebraic interpretation to the potential of the the scalar field in the electroweak model when the latter is unified with the gauge potential in a noncommutative connection [2, 3]. The same feat is made possible by defining a special class of connections on Lie algebroids [4, 5], a framework presenting the advantage of being closer to standard differential geometry now familiar to most. Famously, NCG à la Connes - using spectral triplets/actions - has ambitions matching its abstraction since it was advocated as having the ressources to explain in a unified way various features of the SM as well as naturally incorporating GR with it [6–8]. See [9] for a short review on formulations of gauge theories, and [10] for application of NGC in physics.

Here we put forward what we believe is an original generalisation of connections on principal bundles. The main new ingredient is a cocycle for the action of the structure group H on $\mathcal{P}$, from which a new notion of *twisted* associated bundles is defined. A corresponding notion of connection on $\mathcal{P}$ is needed, that generalises Ehresmann's and induces a good notion of covariant differentiation on sections of these twisted bundles, and more generally on the space of twisted tensorial forms - whose subspace of degree 0 is isomorphic with the space of twisted sections. We develop this picture in extensive computational details. The expert differential geometer might find this repetitive, but we believe it benefits the broader potential readership.

We also take care to verify that the construction is well-behaved under bundle morphisms (the construction is functorial), which means in particular that there is a well-behaved right action of $\mathrm{Aut}_v(\mathcal{P})$ on the new objects, defining their active gauge transformations. The local picture on $\mathcal{M}$ is also detailed, where the local representatives of the above global objects are seen as generalised - or twisted - gauge fields. Indeed, they provide the means to naturally implement the gauge principle, while being of a different geometric nature than Yang-Mills fields.

The conditions necessary for the above twisted geometry to coexist with the standard one are examined, and we provide the complete explicit treatment of this *mixed* geometry. In particular, mixed gauge fields appear in the local picture. The building of twisted/mixed gauge theories is then briefly sketched. We also attempt to identify a subclass of twisted/mixed connections that seems to be a reasonable generalisation of Cartan connections.

Our proposal is conservative, but it has relevant contacts with the literature both in mathematics and physics. We indeed indicate how conformal tractors and Penrose' local twistors can be seen as simple - and in a precise sense, degenerate - examples of the general framework to be developed here. As an aside, conformal gravity is shown to be an instance of mixed gauge theory hiding in plain sight. Finally, we point out that the twisted geometry we advocate appears naturally in the study of anomalies in quantum gauge field theory.

2. Twisted functions and twisted associated vector bundles

Given $\mathcal{P}$ a H-principal fiber bundle and G a Lie group, consider the smooth map:

$$\begin{aligned} C : \mathcal{P} \times H &\to G \\ (p, h) &\mapsto C_p(h), \qquad \text{s.t} \quad C_p(hh') = C_p(h)\, C_{ph}(h'). \end{aligned} \tag{1}$$

This is known as a *group action cocycle* [11] (see also the mathematical literature concerned with abstract ergodic theory [12, 13]). From the very definition follows:

$$\begin{aligned} C_p(h') = C_p(e)\, C_p(h') \quad &\to \quad C_p(e) = e'. \\ C_p(h) = C_p(h)\, C_{ph}(e) \quad &\to \quad C_{ph}(e) = e'. \\ C_p\big(hh^{-1}\big) = C_p(e) = e' = C_p(h)\, C_{ph}\big(h^{-1}\big) \quad &\to \quad C_p(h)^{-1} = C_{ph}\big(h^{-1}\big). \\ C_p\big(h^{-1}h\big) = e' = C_p(h^{-1})\, C_{ph^{-1}}(h) \quad &\to \quad C_p\big(h^{-1}\big) = C_{ph^{-1}}(h)^{-1}. \end{aligned} \tag{2}$$

Notice that the defining relation 1 can be seen as an equivariance relation on $\mathcal{P}$,

$$C_{ph}(h') = C_p(h)^{-1} C_p(hh'), \quad \text{or} \quad R_h^* C(h') = C(h)^{-1} C(hh'). \tag{3}$$

The differential of this map is, $dC_{|(p,h)} = dC(h)_{|p} + dC_{p|h} : T_p\mathcal{P} \oplus T_hH \to T_{C_p(h)}G$, where $\ker dC(h)_{|p} = T_hH$ and $\ker dC_{p|h} = T_p\mathcal{P}$, with by definition:

$$\begin{aligned} dC(h)_{|p}(X_p) &= \tfrac{d}{dt} C_{\phi_t}(h)|_{t=0}, \qquad && \phi_t \text{ the flow of } X \in \Gamma(T\mathcal{P}) \text{ and } \phi_{t=0} = p, \\ dC_{p|h}(Y_h) &= \tfrac{d}{dt} C_p(\varphi_t)|_{t=0}, \qquad && \varphi_t \text{ the flow of } Y \in \Gamma(TH) \text{ and } \varphi_{t=0} = h. \end{aligned}$$

Notice that $C_p(h)^{-1} dC_{|(p,h)} : T_p\mathcal{P} \oplus T_hH \to T_{e'}G = \text{Lie}G$.

2.1. Twisted functions

Given a representation (ρ, V) of G, define the space $\Omega^0_{\text{eq}}\big(\mathcal{P}, C(H)\big)$ of V-valued C-twisted equivariant smooth functions on $\mathcal{P}$ as:

$$\begin{aligned} \varphi : \mathcal{P} &\to V \qquad \text{s.t} \quad R^*_h \varphi = \rho\left[C(h)^{-1}\right]\varphi. \\ p &\mapsto \varphi(p). \end{aligned} \tag{4}$$

This is a well behaved space under the right action of H on $\mathcal{P}$ since on the one hand $\varphi(phh') = \rho\left[C_p(hh')^{-1}\right]\varphi(p)$, and on the other hand:

$$\begin{aligned} \varphi(phh') &= \rho\left[C_{ph}(h')^{-1}\right]\varphi(ph) \\ &= \rho\left[C_{ph}(h')^{-1}\right]\rho\left[C_p(h)^{-1}\right]\varphi(p) \\ &= \rho\left[C_{ph}(h')^{-1}C_p(h)^{-1}\right]\varphi(p) \\ &= \rho\left[\big(C_p(h)C_{ph}(h')\big)^{-1}\right]\varphi(p) = \rho\left[C_p(hh')^{-1}\right]\varphi(p). \end{aligned}$$

2.2. Twisted associated vector bundles

We generalise the usual construction of associated bundles to $\mathcal{P}(\mathcal{M}, H)$ via representations of the structure group H. Define a right H-action on $\mathcal{P} \times V$ by,

$$\begin{aligned} (\mathcal{P} \times V) \times H &\to \mathcal{P} \times V \\ \big((p,v), h\big) &\mapsto \big(ph, \rho\big[C_p(h)^{-1}\big]v\big) \end{aligned}$$

This is well defined since we have on the one hand

$$\big((p,v), hh'\big) \mapsto \big(phh', \rho\left[C_p(hh')^{-1}\right]v\big),$$

and on the other hand we find

$$\begin{aligned}\big((ph, \rho\left[C_p(h)^{-1}\right] v),\, h'\big) &\mapsto \big((ph)h', \rho\left[C_{ph}(h')^{-1}\right]\left(\rho\left[C_p(h)^{-1}\right] v\right)\big)\\ &= \left(phh', \rho\left[\left(C_p(h)C_{ph}(h')\right)^{-1}\right] v\right).\end{aligned}$$

Declare equivalent the paires in $\mathcal{P} \times H$ that are related by this right action, note $\sim$ the equivalence relation, and $[p, v]$ the equivalence class of (p, v). We have then the C-twisted associated bundle $E^C = \mathcal{P} \times_{\rho(C)} V := \mathcal{P} \times V / \sim$. Denoting $\Gamma\big(E^C\big)$ the space of sections of the C-twisted associated bundle, as in the usual case we have the isomorphism $\Gamma\big(E^C\big) \simeq \Omega^0_{\text{eq}}\big(\mathcal{P}, C(H)\big)$.

The question then arise as to find natural differential operators on $\Gamma\big(E^C\big) \simeq \Omega^0_{\text{eq}}\big(\mathcal{P}, C(H)\big)$, and in particular a good notion of covariant differentiation. This requires the appropriate notion of connection.

3. Twisted connection and covariant differentiation

3.1. Vertical vector fields

The space $\Gamma(T\mathcal{P})$ of vector fields on $\mathcal{P}$ is a subalgebra of the derivations of smooth functions on $\mathcal{P}$, $\Omega^0(\mathcal{P})$. The right action of H on $\mathcal{P}$ gives the canonical space of vertical vector fields $\Gamma(V\mathcal{P})$, that naturally act on any equivariant function of $\mathcal{P}$. In particular, for $X^v \in \Gamma(V\mathcal{P})$, $X \in \text{Lie}H$, and $\varphi \in \Omega^0_{\text{eq}}\big(\mathcal{P}, C(H)\big)$ we have:

$$\begin{aligned}\left(X^v\varphi\right)(p) &= \tfrac{d}{d\tau}\varphi\left(pe^{\tau X}\right)\big|_{\tau=0} = \tfrac{d}{d\tau}\,\rho\left[C_p\left(e^{\tau X}\right)^{-1}\right]\varphi(p)\Big|_{\tau=0}\\ &= -\tfrac{d}{d\tau}\,\rho\left[C_p\left(e^{\tau X}\right)\right]\varphi(p)\big|_{\tau=0}\\ &= -\rho_*\left[\tfrac{d}{d\tau}C_p\left(e^{\tau X}\right)\big|_{\tau=0}\right]\varphi(p).\end{aligned}$$

Where ρ_* is the induced representation of Lie$G$. The relation $X^v(\varphi) = -\rho_*\left[\tfrac{d}{d\tau}C\left(e^{\tau X}\right)\big|_{\tau=0}\right]\varphi$, is nothing but the infinitesimal C-equivariance of φ, eq (4).

For later use, we also derive an identity flowing from the action of $[X^v, Y^v]$ and $[X, Y]^v$ on φ. First, we simply have

$$[X, Y]^v\varphi = -\rho_*\left[\tfrac{d}{dt}C_p\left(e^{t[X,Y]}\right)\big|_{t=0}\right]\varphi.$$

Then,

$$
\begin{aligned}
&(X^v(Y^v\varphi))\,(p) = \left(-X^v \tfrac{d}{d\tau}\,\rho\left[C\left(e^{\tau Y}\right)\right]\big|_{\tau=0}\varphi\right)(p)\\
&\quad= -\tfrac{d}{d\sigma}\tfrac{d}{d\tau}\,\rho\left[C_{pe^{\sigma X}}\left(e^{\tau Y}\right)\right]\varphi\left(pe^{\sigma X}\right)\big|_{\tau=0,\,\sigma=0}\,,\\
&\quad= -\tfrac{d}{d\sigma}\tfrac{d}{d\tau}\,\rho\left[C_p\left(e^{\sigma X}\right)^{-1}C_p\left(e^{\sigma X}e^{\tau Y}\right)\right]\rho\left[C_p\left(e^{\sigma X}\right)^{-1}\right]\big|_{\tau=0,\,\sigma=0}\varphi(p)\,,\\
&\quad= -\tfrac{d}{d\sigma}\,\rho\left[C_p\left(e^{\sigma X}\right)^{-1}\right]\tfrac{d}{d\tau}\,\rho\left[C_p\left(e^{\sigma X}e^{\tau Y}\right)\right]\big|_{\tau=0}\,\rho\left[C_p\left(e^{\sigma X}\right)^{-1}\right]\big|_{\sigma=0}\varphi(p)\,,\\
&\quad= \Big(-\tfrac{d}{d\sigma}\,\rho\left[C_p\left(e^{\sigma X}\right)^{-1}\right]\big|_{\sigma=0}\tfrac{d}{d\tau}\,\rho\left[C_p\left(e^{\tau Y}\right)\right]\big|_{\tau=0}\\
&\qquad-\tfrac{d}{d\sigma}\tfrac{d}{d\tau}\,\rho\left[C\left(e^{\sigma X}e^{\tau Y}\right)\right]\big|_{\sigma=0\,\tau=0}\\
&\qquad-\tfrac{d}{d\tau}\,\rho\left[C_p\left(e^{\tau Y}\right)\right]\big|_{\tau=0}\tfrac{d}{d\sigma}\,\rho\left[C_p\left(e^{\sigma X}\right)^{-1}\right]\big|_{\sigma=0}\Big)\varphi(p).
\end{aligned}
$$

By the same process

$$
\begin{aligned}
(Y^v(X^v\varphi))\,(p) = \Big(&-\tfrac{d}{d\tau}\,\rho\left[C_p\left(e^{\tau Y}\right)^{-1}\right]\big|_{\tau=0}\tfrac{d}{d\sigma}\,\rho\left[C_p\left(e^{\sigma X}\right)\right]\big|_{\sigma=0}\\
&-\tfrac{d}{d\tau}\tfrac{d}{d\sigma}\,\rho\left[C\left(e^{\tau Y}e^{\sigma X}\right)\right]\big|_{\sigma=0\,\tau=0}\\
&-\tfrac{d}{d\sigma}\,\rho\left[C_p\left(e^{\sigma X}\right)\right]\big|_{\sigma=0}\tfrac{d}{d\tau}\,\rho\left[C_p\left(e^{\tau Y}\right)^{-1}\right]\big|_{\tau=0}\Big)\varphi(p).
\end{aligned}
$$

Combining the two results, and since $[X,Y]^v = [X^v, Y^v]$, we obtain the identity,

$$
\begin{aligned}
(5)\quad &[X,Y]^v\,\varphi(p) = [X^v, Y^v]\,\varphi(p) \quad \Rightarrow\\
&\tfrac{d}{dt}C_p\left(e^{t[X,Y]}\right)\big|_{t=0} = \tfrac{d}{d\sigma}\tfrac{d}{d\tau}\left(C_p\left(e^{\sigma X}e^{\tau Y}\right) - C_p\left(e^{\tau Y}e^{\sigma X}\right)\right)\big|_{\sigma=0,\,\tau=0}.
\end{aligned}
$$

3.2. Connection

In the usual case, a covariant derivative on sections $\Gamma(E) \simeq \Omega^0_{\text{eq}}(P,V)$ of a bundle E associated to $\mathcal{P}$ is obtained through the definition of an horizontal distribution $H\mathcal{P}$ that complements the canonical vertical distribution $V\mathcal{P}$ on $\mathcal{P}$, so that $\forall p \in \mathcal{P}$, $T_p\mathcal{P} = V_p\mathcal{P} \oplus H_p\mathcal{P}$ and $R_{h*}H_p\mathcal{P} = H_{ph}\mathcal{P}$. This is Ehresmann's far reaching notion of a connection. Most often, such an horizontal distribution is defined as the kernel of a connection 1-form $\omega \in \Omega^1(\mathcal{P}, \text{Lie}H)$ ($\forall p \in \mathcal{P}, H_p\mathcal{P} := \ker\omega_p$) satisfying the algebraic axioms - equivalent to the previous geometric axioms: $\omega_p(X^v_p) = X$, for $X \in \text{Lie}H$, and $R^*_h\omega_{ph} = \text{Ad}_{h^{-1}}\omega_p$. The covariant derivative of a section $\varphi \in \Omega^0_{\text{eq}}(P,V)$ along an horizontal vector field $X^h \in \Gamma(H\mathcal{P})$ is then $X^h\varphi$, and satisfies $R^*_hX^h\varphi = \rho(h^{-1})X^h\varphi$, so that $X^h\varphi \in \Omega^0_{\text{eq}}(P,V)$.

The curvature of the connection form is defined as $\Omega(X,Y) := d\omega(X^h, Y^h)$, for $X, Y \in \Gamma(T\mathcal{P})$. It is clearly a tensorial 2-form (Ad-equivariant and horizontal): $\Omega \in \Omega^2_{\text{tens}}(\mathcal{P}, \text{Ad})$. It also happens to satisfy Cartan's structure equation: $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$. The curvature is the motivating example for defining the exterior covariant derivative on equivariant forms, $D : \Omega^\bullet_{\text{eq}}(\mathcal{P}, \rho) \to \Omega^\bullet_{\text{tens}}(\mathcal{P}, \rho)$. Denoting the horizontal projection by $|^h : \Gamma(T\mathcal{P}) \to \Gamma(H\mathcal{P})$, one indeed defines $D := d \circ |^h$, so that on the one hand $\Omega = D\omega$, and on the other hand $X^h\varphi = D\varphi(X)$. The exterior covariant derivative thus generalises the covariant derivative of sections. It happens that on tensorial forms, it can be expressed in terms of the connection 1-form : $D\alpha = d\alpha + \rho_*(\omega)\alpha$, for $\alpha \in \Omega^\bullet_{\text{tens}}(P, \rho)$.

It turns out that to define a notion of covariant differentiation on sections of C-twisted associated bundles - or C-twisted equivariant functions - it is unnecessary to define an horizontal distribution. All that is required is the appropriate notion of connection 1-form.

Let us propose a Lie$G$-valued 1-form $\omega \in \Omega^1(\mathcal{P}, LieG)$ satisfying:

$$\text{(I)} \qquad \omega_p(X^v_p) = \tfrac{d}{d\tau} C_p\left(e^{\tau X}\right)\big|_{\tau=0} = dC_{p|e}(X), \quad \text{for } X^v_p \in V_p\mathcal{P} \text{ generated by } X \in \text{Lie}H.$$

Given that

$$\begin{aligned} R_{*h}X^v_p &= \tfrac{d}{d\tau} p e^{\tau X} h\big|_{\tau=0} = \tfrac{d}{d\tau} p h h^{-1} e^{\tau X} h\big|_{\tau=0} \\ &= \tfrac{d}{d\tau} p h e^{\tau h^{-1} X h}\big|_{\tau=0} =: (\text{Ad}_{h^{-1}}X)^v_{ph}, \end{aligned}$$

we deduce the equivariance of ω on $V\mathcal{P}$:

$$\begin{aligned} \left(R^*_h\omega_{ph}\right)\left(X^v_p\right) &= \omega_{ph}\left(R_{h*}X^v_p\right) = \omega_{ph}\left((\text{Ad}_{h^{-1}}X)^v_{ph}\right) \\ &= \tfrac{d}{d\tau} C_{ph}\left(e^{\tau \text{Ad}_{h^{-1}}X}\right)\big|_{\tau=0} = \tfrac{d}{d\tau} C_{ph}\left(h^{-1}e^{\tau X}h\right)\big|_{\tau=0}, \\ &= \tfrac{d}{d\tau} C_{ph}\left(h^{-1}\right) C_p\left(e^{\tau X}h\right)\big|_{\tau=0} \\ &= C_p\left(h\right)^{-1} \tfrac{d}{d\tau} C_p\left(e^{\tau X}\right) C_{pe^{\tau X}}\left(h\right)\big|_{\tau=0}, \\ &= C_p\left(h\right)^{-1}\left(\underbrace{\tfrac{d}{d\tau} C_p\left(e^{\tau X}\right)\big|_{\tau=0}}_{\omega_p\left(X^v_p\right)} C_p\left(h\right) + \underbrace{\tfrac{d}{d\tau} C_{pe^{\tau X}}\left(h\right)}_{(X^v C(h))(p)}\right), \\ &= \left(C_p\left(h\right)^{-1}\omega_p\, C_p(h) + C_p\left(h\right)^{-1} dC(h)_{|p}\right)\left(X^v_p\right), \end{aligned}$$

We extend this result by requiring that it holds on the full tangent bundle of $\mathcal{P}$, i.e. ω has prescribed equivariance:

$$\text{(II)} \qquad R_h^*\omega_{ph} = C_p\,(h)^{-1}\,\omega_p\,C_p(h) + C_p\,(h)^{-1}\,dC(h)_{|p}.$$

This prescription is well-behaved with respect to (w.r.t) the right action of H on $\mathcal{P}$. Indeed, for $h, h' \in H$ we have the composition of pullbacks $R_{h'}^* \circ R_h^* = (R_h \circ R_{h'})^* = R_{h'h}^*$. So that on the one hand $R_{h'}^*\,(R_h^*\omega) = R_{h'h}^*\omega = C(h'h)^{-1}\omega\,C(h'h) + C(h'h)^{-1}dC(h'h)$. On the other hand, a direct computation using (3) gives:

$$\begin{aligned}
R_{h'}^*\,(R_h^*\omega) &= R_{h'}^*\left(C(h)^{-1}\omega\,C(h) + C(h)^{-1}dC(h)\right),\\
&= C(h'h)^{-1}C(h')\left(C(h')^{-1}\omega\,C(h') + C(h')^{-1}dC(h')\right)C(h')^{-1}C(h'h)\\
&\quad + C(h'h)^{-1}C(h')\,d\left(C(h')^{-1}C(h'h)\right),\\
&= C(h'h)^{-1}\omega\,C(h'h) + C(h'h)^{-1}dC(h')\cdot C(h')^{-1}C(h'h)\\
&\quad + C(h'h)^{-1}C(h')\,dC(h')^{-1}\cdot C(h'h) + C(h'h)^{-1}dC(h'h),\\
&= C(h'h)^{-1}\omega\,C(h'h) + C(h'h)^{-1}dC(h'h) = R_{h'h}^*\omega.
\end{aligned}$$

The axioms (I) and (II) define our notion of *twisted connection.* Let us denote $\mathcal{C}(\mathcal{P})^T$ the space of these connections.

For later use, and because it is a result in its own right, we here give the infinitesimal version of the equivariance law (II) of ω. Given $X^v \in \Gamma(V\mathcal{P})$, we have:

$$\begin{aligned}
\text{(6)}\quad (L_{X^v}\omega)_p &:= \tfrac{d}{d\tau} R^*_{e^{\tau X}}\,\omega_{pe^{\tau X}}\,\big|_{\tau=0}\\
&= \tfrac{d}{d\tau}\left(C_p\left(e^{\tau X}\right)^{-1}\omega_p\,C_p\left(e^{\tau X}\right) + C_p\left(e^{\tau X}\right)^{-1} dC\left(e^{\tau X}\right)_p\right)\big|_{\tau=0}\,,\\
&= -\underbrace{\tfrac{d}{d\tau}C_p\left(e^{\tau X}\right)\big|_{\tau=0}}_{dC_{p|e}(X))}\omega_p + \omega_p\,\tfrac{d}{d\tau}C_p\left(e^{\tau X}\right)\big|_{\tau=0}\\
&\quad - \tfrac{d}{d\tau}C_p\left(e^{\tau X}\right)\big|_{\tau=0}\underbrace{dC(e)_p}_{=0} + d\left(\tfrac{d}{d\tau}C_p\left(e^{\tau X}\right)\big|_{\tau=0}\right),
\end{aligned}$$

$$\text{(II.b)}\quad L_{X^v}\omega = d\big(dC_{|e}(X)\big) + \left[\omega, dC_{|e}(X)\right].$$

3.3. Covariant differentiation

Consider the space of C-equivariant differential forms

$$\Omega^\bullet_{\text{eq}}\big(\mathcal{P}, C(H)\big) := \left\{\alpha \in \Omega^\bullet(\mathcal{P}, V)\,|\,R_h^*\alpha_{ph} = \rho\left[C_p\,(h)^{-1}\right]\alpha_p\right\}.$$

The infinitesimal version of this equivariance property is given by the Lie derivative along a vertical vector field:

$$
\begin{aligned}
L_{X^v}\alpha &= \tfrac{d}{d\tau} R^*_{e^{\tau X}}\alpha\big|_{\tau=0} = \tfrac{d}{d\tau}\rho\left[C\left(e^{\tau X}\right)^{-1}\right]\alpha\big|_{\tau=0} \\
&= -\rho_*\left[\tfrac{d}{d\tau}C\left(e^{\tau X}\right)\big|_{\tau=0}\right]\alpha.
\end{aligned} \tag{7}
$$

Elements of the subspace of tensorial C-equivariant forms $\Omega^\bullet_{\text{tens}}\big(\mathcal{P}, C(H)\big)$ further satisfies $\alpha(X^v, Y, \ldots) = 0$ for $X^v \in \Gamma(V\mathcal{P})$. Sections of a C-twisted associated bundles are also C-tensorial 0-forms, $\varphi \in \Omega^0_{\text{tens}}\big(\mathcal{P}, C(H)\big) = \Omega^0_{\text{eq}}\big(\mathcal{P}, C(H)\big)$.

Proposition 1. *The exterior covariant derivative defined as $D := d + \rho_*(\omega)$ preserves both $\Omega_{eq}\big(\mathcal{P}, C(H)\big)$ and $\Omega_{tens}\big(\mathcal{P}, C(H)\big)$.*

Proof. First, we show that $D : \Omega^\bullet_{\text{eq}}\big(\mathcal{P}, C(H)\big) \to \Omega^\bullet_{\text{eq}}\big(\mathcal{P}, C(H)\big)$. For $\alpha \in \Omega^\bullet_{\text{eq}}\big(\mathcal{P}, C(H)\big)$:

$$
\begin{aligned}
R^*_h D\alpha &= dR^*_h\alpha + \rho_*\left(R^*_h\omega\right)R^*_h\alpha, \\
&= d\rho\left[C_p(h)^{-1}\right]\cdot\alpha + \rho\left[C_p(h)^{-1}\right]d\alpha \\
&\quad + \rho_*\left(C(h)^{-1}\omega\, C(h) + C(h)^{-1}dC(h)\right)\rho\left[C(h)^{-1}\right]\alpha, \\
&= \rho\left[C(h)^{-1}\right]\big(d\alpha + \rho_*(\omega)\alpha\big) = \rho\left[C(h)^{-1}\right]D\alpha.
\end{aligned}
$$

So indeed $D\alpha \in \Omega^\bullet_{\text{eq}}\big(\mathcal{P}, C(H)\big)$. Here we used the second defining property (II) of ω.

Then we show that $D : \Omega^\bullet_{\text{tens}}\big(\mathcal{P}, C(H)\big) \to \Omega^\bullet_{\text{tens}}\big(\mathcal{P}, C(H)\big)$. It is enough to prove it for $\alpha \in \Omega^1_{\text{tens}}\big(\mathcal{P}, C(H)\big)$:

$$
\begin{aligned}
D\alpha(X^v, Y) &= \left(d\alpha + \rho_*(\omega)\alpha\right)(X^v, Y), \\
&= \underbrace{d\alpha(X^v, Y)}_{(L_{X^v}\alpha)(Y)} + \rho_*(\omega(X^v))\alpha(Y) - \rho_*(\omega(Y))\,\underbrace{\alpha(X^v)}_{=0}, \\
&= -\rho_*\left[\tfrac{d}{d\tau}C\left(e^{\tau X}\right)\big|_{\tau=0}\right]\alpha(Y) + \rho_*\left[\tfrac{d}{d\tau}C\left(e^{\tau X}\right)\big|_{\tau=0}\right]\alpha(Y) = 0.
\end{aligned}
$$

We used Cartan's magic formula $L_X = i_X d + di_X$ and the first defining property (I) of ω. □

This operator provide the adequate notion of covariant differentiation on $\Gamma\big(E^C\big) \simeq \Omega^0_{\text{tens}}\big(\mathcal{P}, C(H)\big)$. Indeed for any section φ, its covariant derivative is $D\varphi = d\varphi + \rho_*(\omega)\varphi \in \Omega^1_{\text{tens}}\big(\mathcal{P}, C(H)\big)$.

3.4. Curvature

The curvature 2-form of the twisted connection is defined via Cartan's structure equation: $\Omega := d\omega + \frac{1}{2}[\omega,\omega]$. It then identically satisfies the Bianchi identity: $d\Omega + [\omega,\Omega] = 0$.

Proposition 2. *The curvature is a C-tensorial 2-form, $\Omega \in \Omega^2_{tens}\big(\mathcal{P}, C(H)\big)$.*

Proof. We begin by proving, using (II), that $\Omega \in \Omega^2_{\text{eq}}\big(\mathcal{P}, C(H)\big)$:

$$\begin{aligned}
R_h^*\Omega &= dR_h^*\omega + \tfrac{1}{2}[R_h^*\omega, R_h^*\omega],\\
&= d\left(C(h)^{-1}\omega\, C(h) + C(h)^{-1}dC(h)\right)\\
&\quad + \tfrac{1}{2}\left[C(h)^{-1}\omega\, C(h) + C(h)^{-1}dC(h), C(h)^{-1}\omega\, C(h) + C(h)^{-1}dC(h)\right],\\
&= dC(h)^{-1}\omega\, C(h) + C(h)^{-1}d\omega\, C(h) - C(h)^{-1}\omega dC(h) + dC(h)^{-1}dC(h)\\
&\quad + \tfrac{1}{2}C(h)^{-1}[\omega,\omega]C(h) + [C(h)^{-1}\omega\, C(h), C(h)^{-1}dC(h)]\\
&\quad + \tfrac{1}{2}[C(h)^{-1}dC(h), C(h)^{-1}dC(h)],\\
&= C(h)^{-1}\left(d\omega + \tfrac{1}{2}[\omega,\omega]\right)C(h),\\
&= C(h)^{-1}\Omega\, C(h).
\end{aligned}$$

The curvature Ω of the twisted connection ω is thus an $\mathrm{Ad}_{C(H)}$-equivariant 2-form. Then we prove that Ω is tensorial. Firstly:

$$\begin{aligned}
\Omega(X^v, Y^v) &= d\omega(X^v, Y^v) + [\omega(X^v), \omega(Y^v)],\\
&= X^v \cdot \omega(Y^v) - Y^v \cdot \omega(X^v) - \omega([X^v, Y^v]) + [\omega(X^v), \omega(Y^v)].
\end{aligned}$$

Now, using (I):

$$\begin{aligned}
X^v \cdot \omega(Y^v) &= X^v\left(\tfrac{d}{d\tau}C_p\left(e^{\tau Y}\right)\big|_{\tau=0}\right) = \tfrac{d}{d\sigma}\tfrac{d}{d\tau}C_{pe^{\sigma X}}\left(e^{\tau Y}\right)\big|_{\tau=0,\,\sigma=0}\\
&= \tfrac{d}{d\sigma}\tfrac{d}{d\tau}C_p\left(e^{\sigma X}\right)^{-1}C_p\left(e^{\sigma X}e^{\tau Y}\right)\big|_{\tau=0,\,\sigma=0},\\
&= \tfrac{d}{d\sigma}\left(C_p\left(e^{\sigma X}\right)^{-1}\tfrac{d}{d\tau}C_p\left(e^{\sigma X}e^{\tau Y}\right)\big|_{\tau=0}\right)\big|_{\sigma=0},\\
&= -\tfrac{d}{d\sigma}C_p\left(e^{\sigma X}\right)\big|_{\sigma=0}\tfrac{d}{d\tau}C_p\left(e^{\tau Y}\right)\big|_{\tau=0} + \tfrac{d}{d\sigma}\tfrac{d}{d\tau}C_p\left(e^{\sigma X}e^{\tau Y}\right)\big|_{\tau=0,\,\sigma=0}
\end{aligned}$$

Idem:

$$-Y^v \cdot \omega(X^v) = \tfrac{d}{d\tau}C_p\left(e^{\tau Y}\right)\big|_{\tau=0}\tfrac{d}{d\sigma}C_p\left(e^{\sigma X}\right)\big|_{\sigma=0} - \tfrac{d}{d\tau}\tfrac{d}{d\sigma}C_p\left(e^{\tau Y}e^{\sigma X}\right)\big|_{\tau=0,\,\sigma=0}.$$

But then we have,

$$\begin{aligned} -\omega([X^v, Y^v]) &= -\omega([X,Y]^v) = -\tfrac{d}{dt} C_p\left(e^{t[X,Y]}\right)\big|_{t=0} \\ \text{and} \quad [\omega(X^v), \omega(Y^v)] &= \left[\tfrac{d}{d\sigma} C_p\left(e^{\sigma X}\right)\big|_{\sigma=0}, \tfrac{d}{d\tau} C_p\left(e^{\tau Y}\right)\big|_{\tau=0}\right]. \end{aligned}$$

Finally, by using the identity (5) we see by inspection that $\Omega(X^v, Y^v) = 0$. Secondly:

$$\begin{aligned} \Omega(X^v, Y) &= X^v \cdot \omega(Y) - Y \cdot \omega(X^v) - \omega([X^v, Y]) + [\omega(X^v), \omega(Y)], \\ &= X^v \cdot \omega(Y) - \omega([X^v, Y]) - \Big(d\omega(X^v) + [\omega, \omega(X^v)] \Big)(Y). \end{aligned}$$

Now,

$$\begin{aligned} \left(L_{X^v}\omega\right)(Y) &= \big((i_{X^v} d + d i_{X^v})\omega\big)(Y) = d\omega(X^v, Y) + d(\omega(X^v))(Y), \\ &= X^v \cdot \omega(Y) - Y \cdot \omega(X^v) - \omega([X^v, Y]) + Y \cdot \omega(X^v) \\ &= X^v \cdot \omega(Y) - \omega([X^v, Y]). \end{aligned}$$

But also, by (6) $L_{X^v}\omega = d\omega(X^v) + [\omega, \omega(X^v)]$. So, by inspection we have that $\Omega(X^v, Y) = 0$. Which finishes to demonstrate that $\Omega \in \Omega^2_{\text{tens}}\big(\mathcal{P}, C(H)\big)$. □

This fact then allows to see that the Bianchi identity can be written as $D\Omega = 0$. It is by the way easy to show that, as usual, $D^2\alpha = DD\alpha = \rho_*(\Omega)\alpha$.

4. Functoriality

In this section we check that the twisted objects defined above enjoy the same functoriality as standard constructions, i.e. that principal bundle morphisms induce morphisms of associated twisted bundles, and of spaces of twisted connections and twisted equivariant forms. Of particular interest is the special case of vertical automorphisms of a principal bundle, giving rise to (active) gauge transformations.

4.1. Naturality under bundle morphisms

Consider a H-bundle (iso)morphism $\phi : \mathcal{P} \to \mathcal{P}'$, with $\phi(ph) = \phi(p)h$, which induces a smooth map (diffeomorphism) $\bar{\phi} : \mathcal{M} \to \mathcal{M}'$. Given $C' : \mathcal{P}' \times H \to G$, a H-cocycle on $\mathcal{P}'$, $C := \phi^* C' : \mathcal{P} \times H \to G$ is a H-cocycle on $\mathcal{P}$. Indeed,

$C_p(h) := C'_{\phi(p)}(h)$, so

$$\begin{aligned} C_p(hh') = C'_{\phi(p)}(hh') &= C'_{\phi(p)}(h)C'_{\phi(p)h}(h') \\ &= C'_{\phi(p)}(h)C'_{\phi(ph)}(h') =: C_p(h)C_{ph}(h'). \end{aligned}$$

So, H-bundle morphisms induce morphisms of twisted bundles in the following way (we omit the representation ρ for simplicity):

$$\begin{aligned} \tilde{\phi} : E^C &\to E'^{C'}, \\ [p, v] &\mapsto \tilde{\phi}([p, v]) := [\phi(p), v], \\ [ph, C_p(h)^{-1}v] &\mapsto \tilde{\phi}([ph, C_p(h)^{-1}v]) \\ &\quad := [\phi(ph), C_p(h)^{-1}v] =: [\phi(p)h, C'_{\phi(p)}(h)^{-1}v]. \end{aligned}$$

Naturally this implies the existence of a morphism of spaces of sections. Indeed, given $\varphi' \in \Omega^0_{\text{eq}}(\mathcal{P}', C'(H))$, a twisted function on $\mathcal{P}'$ such that $R^*_h\varphi' = C'(h)^{-1}\varphi'$, the map $\varphi := \phi^*\varphi'$ is such that $\varphi(ph) := \varphi'(\phi(ph)) = \varphi'(\phi(p)h) = C'_{\phi(p)}(h)^{-1}\varphi'(\phi(p)) =: C_p(h)^{-1}\varphi(p)$. We thus have the morphism $\phi^* : \Omega^0_{\text{eq}}(\mathcal{P}', C'(H)) \to \Omega^0_{\text{eq}}(\mathcal{P}, C(H))$, which, by the isomorphism mentioned in section 2.2, induces the morphism $\tilde{\phi}^* : \Gamma(E'^{C'}) \to \Gamma(E^C)$.

More generally, ϕ induces a morphism of spaces of C-equivariant forms, $\phi^* : \Omega^\bullet_{\text{eq}}(\mathcal{P}', C'(H)) \to \Omega^\bullet_{\text{eq}}(\mathcal{P}, C(H))$. Indeed, given $\alpha' \in \Omega^\bullet_{\text{eq}}(\mathcal{P}', C'(H))$ and $R_h \circ \phi = \phi \circ R_h$, the form $\alpha := \phi^*\alpha'$ is such that (still omitting ρ) $R^*_h\alpha := R^*_h\phi^*\alpha' = \phi^*R^*_h\alpha' = \phi^*\left(C'(h)^{-1}\alpha'\right) = \phi^*C'(h)^{-1}\phi^*\alpha' =: C(h)^{-1}\alpha$. So, $\alpha \in \Omega^\bullet_{\text{eq}}(\mathcal{P}, C(H))$. Also, since for $X^v_p \in V_p\mathcal{P}$ we have $\phi_*X^v_p = X^v_{\phi(p)} \in V_{\phi(p)}\mathcal{P}'$, pullback by ϕ preserves horizontality and the above morphism restricts to the spaces of C-tensorial forms, $\phi^* : \Omega^\bullet_{\text{tens}}(\mathcal{P}', C'(H)) \to \Omega^\bullet_{\text{tens}}(\mathcal{P}, C(H))$.

One can further show that there are induced morphisms of spaces of twisted connections $\phi^* : \mathcal{C}(\mathcal{P}')^T \to \mathcal{C}(\mathcal{P})^T$. If $\mathcal{P}'$ is endowed with a twisted connection $\omega' \in \mathcal{C}(\mathcal{P}')^T$, which therefore satisfies, for $q \in \mathcal{P}'$, $\omega'_q(X^v_q) = dC_{q|e}(X)$, $X \in \text{Lie}H$, and $R^*_h\omega'_{qh} = C'_q(h)^{-1}\omega'_qC'_q(h) + C'_q(h)^{-1}dC'(h)_{|q}$, then $\omega := \phi^*\omega'$ satisfies on the one hand,

$$\begin{aligned} \omega_p(X^v_p) := \phi^*\omega'_{\phi(p)}(X^v_p) = \omega'_{\phi(p)}(\phi_*X^v_p) &= \omega'_{\phi(p)}(X^v_{\phi(p)}) \\ &= dC'_{\phi(p)|e}(X) =: dC_{p|e}(X), \end{aligned}$$

and on the other hand, for $X_p \in T_p\mathcal{P}$,

$$\begin{aligned}
R_h^*\omega_{ph}(X_p) &= \omega_{ph}(R_{h*}X_p) := \phi^*\omega'_{\phi(ph)}(R_{h*}X_p) \\
&= \omega'_{\phi(p)h}(\phi_* R_{h*}X_p) = \omega'_{\phi(p)h}(R_{h*}\phi_* X_p) = R_h^*\omega'_{\phi(p)h}(\phi_* X_p), \\
&= \left(C'_{\phi(p)}(h)^{-1}\omega'_{\phi(p)}C'_{\phi(p)}(h) + C'_{\phi(p)}(h)^{-1}dC'(h)_{|\phi(p)}\right)(\phi_* X_p), \\
&=: C_p(h)^{-1}\phi^*\omega'_{\phi(p)}(X_p)\, C_p(h) + C_p(h)^{-1}d(C'(h)\circ\phi)_{|p}(X_p), \\
&= \left(C_p(h)^{-1}\omega_p\, C_p(h) + C_p(h)^{-1}dC(h)_{|p}\right)(X_p).
\end{aligned}$$

It is then a twisted connection on $\mathcal{P}$, $\omega \in \mathcal{C}(\mathcal{P})^T$.

From the above we obtain readily that a covariant derivative $D' = d + \rho_*(\omega')$ on $\mathcal{P}'$ pulls-back as a covariant derivative $D := \phi^* D'$ on $\mathcal{P}$. In particular it means that H-bundle morphisms ϕ induce morphisms of twisted associated bundles equiped with covariant derivatives, $\tilde{\phi} : (E^C, D) \to (E'^{C'}, D')$.

If $\mathcal{M} = \mathcal{M}'$ and $\bar{\phi} = \mathrm{id}_{\mathcal{M}}$, then ϕ is a bundle equivalence, and we have established equivalence of the associated twisted structures. The above functoriality holds also in the special case $\mathcal{P} = \mathcal{P}'$ with $\phi \in \mathrm{Aut}(\mathcal{P})$ covering $\bar{\phi} \in \mathrm{Diff}(\mathcal{M})$, and therefore in the case $\phi \in \mathrm{Aut}_v(\mathcal{P})$ covering $\bar{\phi} = \mathrm{id}_{\mathcal{M}}$. The latter is of particular interest for physical application to gauge theories.

4.2. Action of vertical automorphisms and gauge transformations

The group of vertical automorphisms of the principal bundle $\mathcal{P}$ is a subgroup of its group of diffeomorphisms, $\mathrm{Aut}_v(\mathcal{P}) := \Big\{\Phi \in \mathrm{Diff}(\mathcal{P})\,|\,\Phi(ph) = \Phi(p)h \text{ for } h \in H, \text{ and } \pi\circ\Phi = \pi\Big\}$. It acts on itself by composition of maps. The gauge group is defined as $\mathcal{H} := \left\{\gamma : \mathcal{P} \to H\,|\, R_h^*\gamma = h^{-1}\gamma h\right\}$. Both group are isomorphic by the identification $\Phi(p) = p\gamma(p)$. The group of vertical automorphisms acts by pullback on $\Omega^\bullet(\mathcal{P})$. A pullback by $\Phi \in \mathrm{Aut}_v(\mathcal{P})$ will then equivalently be called an active gauge transformation by $\gamma \in \mathcal{H}$. Now, for $\Psi \in \mathrm{Aut}_v(\mathcal{P})$ associated to the elements $\eta \in \mathcal{H}$, we have: $\Psi^*\gamma(p) = \gamma(\Psi(p)) = \gamma(p\eta(p)) = \eta(p)^{-1}\gamma(p)\eta(p)$. So the action of the gauge group $\mathcal{H}$ on itself, noted γ^η, is defined as: $\gamma^\eta := \Psi^*\gamma = \eta^{-1}\gamma\eta$. It reflects the defining equivariance of its elements.

To give the gauge transformations of the objects defined in the previous sections, we need to first consider the smooth map,

$$\begin{aligned} C(\gamma) : \mathcal{P} &\to G, \\ p &\mapsto C_p\left(\gamma(p)\right). \end{aligned}$$

Its equivariance is,

$$\begin{aligned} R_h^* C(\gamma)(p) &= C_{ph}\left(\gamma(ph)\right) = C_{ph}\left(h^{-1}\gamma(p)h\right) \\ &= C_{ph}\big(h^{-1}\big) C_p\left(\gamma(p)h\right) = C_p(h)^{-1} C_p(\gamma(p)) C_{p\gamma(p)}(h), \end{aligned}$$

$$R_h^* C(\gamma) = C(h)^{-1} C(\gamma h) = C(h)^{-1} C(\gamma)\, \Phi^* C(h). \tag{8}$$

Correspondingly we have,

$$\begin{aligned} \Psi^* C(\gamma)(p) &= C_{\Psi(p)}\big(\gamma\left(\Psi(p)\right)\big) = C_{p\eta(p)}\left(\eta(p)^{-1}\gamma(p)\eta(p)\right) \\ &= C_p\left(\eta(p)\right)^{-1} C_p\big(\gamma(p)\eta(p)\big). \end{aligned}$$

$$C(\gamma)^\eta := \Psi^* C(\gamma) = C(\eta)^{-1} C(\gamma\eta). \tag{9}$$

This map is given by the composition,

$$\begin{aligned} &\mathcal{P} \xrightarrow{\Delta} \mathcal{P} \times \mathcal{P} \xrightarrow{\mathrm{id}\times\gamma} \mathcal{P} \times H \xrightarrow{\quad C \quad} G, \\ &p \longmapsto (p,p) \longmapsto (p,\gamma(p)) \longmapsto C_p(\gamma(p)). \end{aligned}$$

So its differential $dC(\gamma)_{|p} : T_p\mathcal{P} \to T_{C_p(\gamma(p))}G$ is,

$$\begin{aligned} &T_p\mathcal{P} \xrightarrow{d\Delta} T_p\mathcal{P} \times T_p\mathcal{P} \xrightarrow{\mathrm{id}\times d\gamma} T_p\mathcal{P} \times T_{\gamma(p)}G \\ &\qquad \xrightarrow{dC(\gamma(p))_{|p} + dC_{p|\gamma(p)}} T_{C_p(\gamma(p))}G, \\ &X_p \longmapsto (X_p, X_p) \longmapsto \Big(X_p, \underbrace{d\gamma_{|p}(X_p)}_{[X(\gamma)](p)}\Big) \\ &\qquad \longmapsto dC(\gamma(p))_{|p}(X_p) \,+\, \underbrace{dC_{p|\gamma(p)}\left(d\gamma_p(X_p)\right)}_{dC_p(\gamma)_{|p}(X_p)}. \end{aligned}$$

So, if ϕ_τ is the flow of X with $\phi_{\tau=0} = p$, we have:

$$\begin{aligned} dC(\gamma)_{|p}(X_p) &= dC(\gamma(p))_{|p}(X_p) + dC_p(\gamma)_{|p}(X_p) \\ &= \tfrac{d}{d\tau}\left\{C_{\phi_\tau}(\gamma(p)) + C_p\left(\gamma(\phi_\tau)\right)\right\}\Big|_{\tau=0}. \end{aligned} \tag{10}$$

Notice then that $C_p(\gamma(p))^{-1}dC(\gamma)_{|p} : T_p\mathcal{P} \to T_{e'}G = \mathrm{Lie}G$. We are then ready to state and prove the following two propositions.

Proposition 3. *The active gauge transformation of a twisted connection, noted ω^γ, is*

$$\omega^\gamma := \Phi^*\omega = C(\gamma)^{-1}\omega\, C(\gamma) + C(\gamma)^{-1}dC(\gamma) \ \in \mathcal{C}(\mathcal{P})^T. \tag{11}$$

Proof. Given the standard result $\Phi_* X_p = R_{\gamma(p)*}X_p + [\gamma^{-1}d\gamma]_{|p}(X_p)\big|^v_{\Phi(p)}$ for $X \in \Gamma(T\mathcal{P})$, we have:

$$\left(\Phi^*\omega\right)_p(X_p) = \omega_{\Phi(p)}\left(\Phi_* X_p\right) = \omega_{\Phi(p)}\left(R_{\gamma(p)*}X_p + \gamma(p)^{-1}d\gamma_{|p}(X_p)\big|^v_{\Phi(p)}\right).$$

The first term is easily worked out, by (II):

$$R^*_{\gamma(p)}\omega_{p\gamma(p)}(X_p) = \left(C_p\left(\gamma(p)\right)^{-1}\omega_p\, C_p\left(\gamma(p)\right) + C_p\left(\gamma(p)\right)^{-1} dC\left(\gamma(p)\right)_{|p}\right)(X_p)$$

The second term needs special attention. By (I) we have:

$$\begin{aligned}
\omega_{\Phi(p)}\left(\gamma(p)^{-1}d\gamma_{|p}(X_p)\big|^v_{\Phi(p)}\right) &= \tfrac{d}{d\tau}C_{p\gamma(p)}\left(e^{\tau\,\gamma(p)^{-1}d\gamma_{|p}(X_p)}\right)\big|_{\tau=0} \\
&= C_p\left(\gamma(p)\right)^{-1}\tfrac{d}{d\tau}C_p\left(\gamma(p)e^{\tau\,\gamma(p)^{-1}d\gamma_{|p}(X_p)}\right)\big|_{\tau=0}, \\
&= C_p\left(\gamma(p)\right)^{-1}dC_{p|\gamma(p)}\left(\gamma(p)\tfrac{d}{d\tau}e^{\tau\,\gamma(p)^{-1}d\gamma_{|p}(X_p)}\big|_{\tau=0}\right) \\
&= C_p\left(\gamma(p)\right)^{-1}dC_{p|\gamma(p)}\left(d\gamma_{|p}(X_p)\right), \\
&= C_p\left(\gamma(p)\right)^{-1}dC_p(\gamma)_{|p}(X_p).
\end{aligned}$$

Finally, we then obtain,

$$\begin{aligned}
\left(\Phi^*\omega\right)_p(X_p) &= \Big(C_p\left(\gamma(p)\right)^{-1}\omega_p\, C_p\left(\gamma(p)\right) \\
&\qquad + C_p\left(\gamma(p)\right)^{-1}\left(dC\left(\gamma(p)\right)_{|p} + dC_p(\gamma)_{|p}\right)\Big)(X_p), \\
&= \left(C_p\left(\gamma(p)\right)^{-1}\omega_p\, C_p\left(\gamma(p)\right) + C_p\left(\gamma(p)\right)^{-1}dC(\gamma)_{|p}\right)(X_p).
\end{aligned}$$

By the way, from section 4.1 above, we have $\Phi^* : \mathcal{C}(\mathcal{P})^T \to \mathcal{C}(\mathcal{P})^T$. So indeed $\omega^\gamma \in \mathcal{C}(\mathcal{P})^T$. □

Proposition 4. *The active gauge transformation of a C-tensorial form, noted α^γ, is*

$$\alpha^\gamma := \Phi^*\alpha = \rho\left[C(\gamma)^{-1}\right]\alpha \ \in \Omega^\bullet_{tens}\big(\mathcal{P}, C(H)\big). \tag{12}$$

Proof. Proceeding as above we get,

$$\begin{aligned}(\Phi^*\alpha)_p\,(X_p,\ldots) &= \alpha_{\Phi(p)}\left(R_{\gamma(p)*}X_p + [\gamma^{-1}d\gamma]_{|p}(X_p)\big|^v_{\Phi(p)},\ldots\right),\\ &= \alpha_{\Phi(p)}\left(R_{\gamma(p)*}X_p,\ldots\right),\\ &= R^*_{\gamma(p)}\alpha_{\Phi(p)}(X_p,\ldots) = \rho\left[C_p\left(\gamma(p)\right)^{-1}\right]\alpha_p(X_p,\ldots).\end{aligned}$$

Also, from section 4.1 we have $\Phi^*\colon \Omega^\bullet_{\text{tens}}\big(\mathcal{P},C(H)\big) \to \Omega^\bullet_{\text{tens}}\big(\mathcal{P},C(H)\big)$. So indeed $\alpha^\gamma \in \Omega^\bullet_{\text{tens}}\big(\mathcal{P},C(H)\big)$. □

From this follows the gauge transformations of the curvature, of sections and their covariant derivatives:

$$\Omega^\gamma := \Phi^*\Omega = C(\gamma)^{-1}\Omega\, C(\gamma), \tag{13}$$

$$\varphi^\gamma := \Phi^*\varphi = \rho\left[C(\gamma)^{-1}\right]\varphi, \tag{14}$$

$$(D\varphi)^\gamma := \Phi^* D\varphi = \rho\left[C(\gamma)^{-1}\right]D\varphi. \tag{15}$$

Using (11), equation (13) is alternatively found from the Cartan structure equation by having $\Omega^\gamma = d\omega^\gamma + \frac{1}{2}[\omega^\gamma,\omega^\gamma]$, and equation (15) by having $(D\varphi)^\gamma = D^\gamma\varphi^\gamma = d\varphi^\gamma + \rho_*(\omega^\gamma)\varphi^\gamma$.

Finally, we explicitly verify the following.

Proposition 5. *The action of* $\mathrm{Aut}_v(\mathcal{P}) \simeq \mathcal{H}$ *on* $\mathcal{C}(\mathcal{P})^T$ *and* $\Omega^\bullet_{tens}\big(\mathcal{P},C(H)\big)$ *is a right action.*

Proof. Given $\Phi,\Psi \in \mathrm{Aut}_v(\mathcal{P})$ associated respectively to the elements $\gamma,\eta \in \mathcal{H}$, and using (9) we have

$$\begin{aligned}(\omega^\gamma)^\eta &:= \Psi^*\left(\Phi^*\omega\right) = \Psi^*\left(C(\gamma)^{-1}\omega\, C(\gamma) + C(\gamma)^{-1}dC(\gamma)\right),\\ &= C(\gamma\eta)^{-1}C(\eta)\left(C(\eta)^{-1}\omega\, C(\eta) + C(\eta)^{-1}dC(\eta)\right)C(\eta)^{-1}C(\gamma\eta)\\ &\quad + C(\gamma\eta)^{-1}C(\eta)\, d\left(C(\eta)^{-1}C(\gamma\eta)\right),\\ &= C(\gamma\eta)^{-1}\omega\, C(\gamma\eta) + C(\gamma\eta)^{-1}dC(\gamma\eta).\end{aligned}$$

This shows the consistency of the notation for active gauge transformations, in terms of which the above result is simply $(\omega^\gamma)^\eta = \omega^{\gamma\eta}$. This extends easily to tensorial forms. For $\alpha \in \Omega^\bullet_{\text{tens}}\big(\mathcal{P}, C(H)\big)$:

$$\begin{aligned}(\alpha^\gamma)^\eta := \Psi^*\left(\Phi^*\alpha\right) = \Psi^*\left(\rho\left[C(\gamma)^{-1}\right]\alpha\right)\\ = \rho\left[C(\gamma\eta)^{-1}C(\eta)\right]\rho\left[C(\eta)^{-1}\right]\alpha = \rho\left[C(\gamma\eta)^{-1}\right]\alpha.\end{aligned}$$

Which is simply $(\alpha^\gamma)^\eta = \alpha^{\gamma\eta}$. Further gauge transformations of Ω, φ and $D\varphi$ are special cases of this result. □

We thus have well-defined spaces $\mathcal{C}(\mathcal{P})^T$, and $\Omega^\bullet_{\text{tens}}\big(\mathcal{P}, C(H)\big)$ endowed with an exterior covariant derivative D, with a consistent right action of the gauge group $\mathcal{H} \simeq \text{Aut}_v(\mathcal{P})$ of the underlying principal bundle $\mathcal{P}$.

5. Local description

We now turn to the local description of the global objects just described. As a principal bundle, $\mathcal{P}$ is locally trivialisable, meaning that for any open subset $\mathcal{U}$ of the base manifold $\mathcal{M}$: $\mathcal{P}_{|\mathcal{U}} \simeq \mathcal{U} \times H$. Given a local trivialising section $\sigma : \mathcal{U} \to \mathcal{P}_{|\mathcal{U}}$, any form $\beta \in \Omega^\bullet(\mathcal{P})$ can be pulled-back as a form $b := \sigma^*\beta \in \Omega(\mathcal{U})$. Also, any vector $X \in \Gamma(T\mathcal{U})$ can be pushed-forward as a vector $\sigma_* X \in \Gamma(T\mathcal{P}_{|\mathcal{U}})$.

The pullbacks of the connection and its curvature are respectively Lie$G$-valued 1-form and 2-form on $\mathcal{U}$. We denote $A := \sigma^*\omega \in \Omega^1(\mathcal{U}, \text{Lie}G)$, and $F := \sigma^*\Omega \in \Omega^2(\mathcal{U}, \text{Lie}G)$. By the naturality of the pullback, Cartan's structure equation still holds: $F = dA + {}^1\!/\!_2[A, A]$. The pullback of a section $\varphi \in \Omega^0_{\text{tens}}\big(\mathcal{P}, C(H)\big) \simeq \Gamma\big(E^C\big)$ is a V-valued map on $\mathcal{U}$ that we denote $\phi := \sigma^*\varphi \in \Omega^0(\mathcal{U}, V)$. Still by naturality of the pullback we have that: $D\phi := \sigma^* D\varphi \in \Omega^1(\mathcal{U}, V)$, with $D\phi = d\phi + \rho_*(A)\phi$. In general, let us denote the pullbacks of a C-tensorial form and its exterior covariant derivative $\alpha, D\alpha \in \Omega^\bullet_{\text{tens}}(\mathcal{P}, C(H))$ by $a := \sigma^*\alpha, Da := \sigma^* D\alpha \in \Omega^\bullet(\mathcal{U}, V)$, with $Da = da + \rho_*(A)a$.

If this framework would apply to (particle) physics - which happens on $\mathcal{M}$, describing space-time - A would be a generalised/twisted gauge potential and F would be its field strength, while ϕ would be a generalised/twisted matter field and $D\phi$ would describe its minimal coupling to A.

As forms on $\mathcal{U} \subset \mathcal{M}$, the local variables A, F, a and Da do not have equivariance w.r.t the structure group H of $\mathcal{P}$. Nevertheless, the fact that they are shadows of global objects shows both in their gluing properties from one open subset of $\mathcal{M}$ to another, and in the local version of their gauge transformations. These are discussed in the next two sections.

5.1. Gluing properties: passive gauge transformations

Consider $\mathcal{U},\mathcal{U}' \subset \mathcal{M}$ such that $\mathcal{U} \cap \mathcal{U}' \neq \emptyset$, endowed with local sections $\sigma : \mathcal{U} \to \mathcal{P}_{|\mathcal{U}}$ and $\sigma' : \mathcal{U}' \to \mathcal{P}_{|\mathcal{U}'}$. On the overlap, both sections are related as

$$\sigma' = \sigma g, \qquad \text{with} \qquad \begin{aligned} g : \mathcal{U} \cap \mathcal{U}' &\to H, \\ x &\mapsto g(x). \end{aligned}$$

It is a standard result that for $X_x \in T_x\mathcal{M}$, $x \in \mathcal{U} \cap \mathcal{U}'$, the pushforwards by σ' and σ are related by

$$\sigma'_* X_x = R_{g(x)*}\left(\sigma_* X_x\right) + [g^{-1}dg]_{|x}(X_x)\big|^{v}_{\sigma'(x)}, \tag{16}$$

with $\sigma'_* X_x \in T_{\sigma'(x)}\mathcal{P}_{|\mathcal{U}'}$ and $\sigma_* X_x \in T_{\sigma(x)}\mathcal{P}_{|\mathcal{U}}$.

Furthermore, let us introduce the maps $C_\sigma(h) := \sigma^* C(h) : \mathcal{U} \to G$, with $h \in H$, as well as $C_\sigma(g) : \mathcal{U} \to G$. Notice that, just like $C(\gamma)$ has a double dependence on $p \in \mathcal{P}$, $C_\sigma(g)$ has a double dependence on $x \in \mathcal{U} \cap \mathcal{U}' \subset \mathcal{M}$. Their counterparts on $\mathcal{U}'$ are $C_{\sigma'}(h) := \sigma'^* C(h) : \mathcal{U}' \to G$ and $C_{\sigma'}(g') : \mathcal{U}' \to G$, and we have

$$\begin{aligned} C_{\sigma'(x)}(h) &= C_{\sigma(x)g(x)}(h) = C_{\sigma(x)}(g(x))^{-1} C_{\sigma(x)}\big(g(x)h\big), \\ C_{\sigma'(x)}(g'(x)) &= C_{\sigma(x)g(x)}\big(g'(x)\big) = C_{\sigma(x)}(g(x))^{-1} C_{\sigma(x)}\left(g(x)g'(x)\right), \end{aligned} \tag{17}$$

We are ready to state the following,

Proposition 6. *The gluing properties of the local representatives of a twisted connection and a tensorial form are,*

$$A' = C_\sigma(g)^{-1} A\, C_\sigma(g) + C_\sigma(g)^{-1} dC_\sigma(g), \tag{18}$$

$$a' = \rho\left[C_\sigma(g)^{-1}\right] a. \tag{19}$$

Proof. For the connection, using (16) and (I)-(II), we have:

$$\begin{aligned} A'_x(X_x) &= \sigma'^* \omega_{\sigma'(x)}(X_x) = \omega_{\sigma'(x)}(\sigma'_* X_x) \\ &= \omega_{\sigma'(x)}\left(R_{g(x)*}\left(\sigma_* X_x\right) + [g^{-1}dg]_{|x}(X_x)\big|^{v}_{\sigma'(x)}\right), \\ &= R^*_{g(x)}\omega_{\sigma'(x)}(\sigma_* X_x) + \tfrac{d}{d\tau} C_{\sigma'(x)}\left(e^{\tau\, [g^{-1}dg]_{|x}(X_x)}\right)\big|_{\tau=0}, \end{aligned}$$

$$
\begin{aligned}
&= \left(C_{\sigma(x)}(g(x))^{-1}\omega_{\sigma(x)}C_{\sigma(x)}(g(x)) + C_{\sigma(x)}(g(x))^{-1}dC(g(x))_{|\sigma(x)}\right)(\sigma_*X_x) \\
&\quad + \tfrac{d}{d\tau}C_{\sigma(x)}(g(x))^{-1}C_{\sigma(x)}\left(g(x)\, e^{\tau\, g(x)^{-1}dg_{|x}(X_x)}\right)\Big|_{\tau=0}, \\
&= C_{\sigma(x)}(g(x))^{-1}\sigma^*\omega_{\sigma(x)}(X_x)\, C_{\sigma(x)}(g(x)) + C_{\sigma(x)}(g(x))^{-1}dC_\sigma(g(x))_{|x}(X_x) \\
&\quad + C_{\sigma(x)}(g(x))^{-1}dC_{\sigma(x)|\, g(x)}\Bigg(\underbrace{g(x)\tfrac{d}{d\tau}e^{\tau\, g(x)^{-1}dg_{|x}(X_x)}\Big|_{\tau=0}}_{dg_{|x}(X_x)}\Bigg), \\
&= \Bigg(C_{\sigma(x)}(g(x))^{-1}A_x\, C_{\sigma(x)}(g(x)) \\
&\qquad + C_{\sigma(x)}(g(x))^{-1}\big(\underbrace{dC_\sigma(g(x))_{|x} + dC_{\sigma(x)}(g)_{|x}}_{dC_\sigma(g)_{|x}}\big)\Bigg)(X_x).
\end{aligned}
$$

Likewise for a tensorial form,

$$
\begin{aligned}
a'_x(X_x,\ldots) &= \sigma'^*\alpha_{\sigma'(x)}(X_x,\ldots) = \alpha_{\sigma'(x)}(\sigma'_*X_x,\ldots), \\
&= \alpha_{\sigma'(x)}\left(R_{g(x)*}(\sigma_*X_x) + [g^{-1}dg]_{|x}(X_x)\big|^v_{\sigma'(x)},\ldots\right), \\
&= R^*_{g(x)}\alpha_{\sigma'(x)}(\sigma_*X_x,\ldots), = \rho\left[C_{\sigma(x)}(g(x))^{-1}\right]\alpha_{\sigma(x)}(\sigma_*X_x,\ldots), \\
&= \rho\left[C_{\sigma(x)}(g(x))^{-1}\right]\sigma^*\alpha_{\sigma(x)}(X_x,\ldots), \\
&= \rho\left[C_{\sigma(x)}(g(x))^{-1}\right]a_x(X_x,\ldots).
\end{aligned}
$$

□

The last result holds true for the exterior covariant derivative: $(Da)' = \rho\left[C_\sigma(g)^{-1}\right]Da$, which is also found from (18) by having $(Da)' = D'a' = da' + \rho_*(A')\,a'$. In the language of physics, this would be an implementation of the *gauge principle*. As a special case of (19), we obtain the gluing properties of the local representatives of the curvature, sections and their covariant derivative,

$$
\begin{aligned}
&F' = C_\sigma(g)^{-1}F\,C_\sigma(g), \quad \phi' = \rho\left[C_\sigma(g)^{-1}\right]\phi \\
&\text{and} \quad (D\phi)' = D'\phi' = \rho\left[C_\sigma(g)^{-1}\right]D\phi.
\end{aligned} \tag{20}
$$

The first result can be obtain from Cartan's structure equation and (18) by having, $F' = dA' + \frac{1}{2}[A', A']$.

Suppose now that we have a third open subset $\mathcal{U}''$ such that $\mathcal{U}''\cap\mathcal{U}'\cap\mathcal{U}\neq\emptyset$, and consider a section $\sigma'' : \mathcal{U}'' \to G$ such that on $\mathcal{U}''\cap\mathcal{U}'\cap\mathcal{U}$,

$$
\sigma'' = \sigma'g' = \sigma g g', \quad \text{where} \quad \begin{aligned} g' : \mathcal{U}''\cap\mathcal{U}'\cap\mathcal{U} &\to H, \\ x &\mapsto g'(x). \end{aligned}
$$

We check that the gluing properties are well-behaved across open subsets. Using (17) and (18) we find that,

(21)
$$\begin{aligned} A'' &= C_{\sigma'}(g')^{-1} A' C_{\sigma'}(g') + C_{\sigma'}(g')^{-1} dC_{\sigma'}(g'), \\ &= C_\sigma(gg')^{-1} C_\sigma(g) \left(C_\sigma(g)^{-1} A\, C_\sigma(g) + C_\sigma(g)^{-1} dC_\sigma(g) \right) C_\sigma(g)^{-1} C_\sigma(gg') \\ &\quad + C_\sigma(gg')^{-1} C_\sigma(g) d \left(C_\sigma(g)^{-1} C_\sigma(gg') \right), \\ &= C_\sigma(gg')^{-1} A\, C_\sigma(gg') + C_\sigma(gg')^{-1} dC_\sigma(gg'). \end{aligned}$$

In the same way, using (17) and (19),

(22)
$$a'' = \rho \left[C_{\sigma'}(g')^{-1} \right] a' = \rho \left[C_\sigma(gg')^{-1} C_\sigma(g) \right] \rho \left[C_\sigma(g)^{-1} \right] a = \rho \left[C_\sigma(gg')^{-1} \right] a.$$

So that, $(D\alpha)'' = \rho \left[C_\sigma(gg')^{-1} \right] Da$, which is also obtained from $(Da)'' = D''a'' = da'' + \rho_*(A'')\, a''$. As special cases of this result, we have:

(23)
$$F'' = C_\sigma(gg')^{-1} F\, C_\sigma(gg'), \quad \phi'' = \rho \left[C_\sigma(gg')^{-1} \right] \phi \quad \text{and} \quad (D\phi)'' = \rho \left[C_\sigma(gg')^{-1} \right] D\phi.$$

The gluing properties (18)-(19) in proposition 6 resemble the active gauge transformations of propositions 3 and 4. But while the latter describe the transformation of global objects (i.e. living on $\mathcal{P}$) into new global objects, the former merely describe how the same global objects are seen from different open subsets of $\mathcal{M}$ - or from the same subset but through different local sections. This justifies the terminology *passive gauge transformations* for the gluing properties, that is of common use in physics.

This is in close analogy with changes of coordinate representations of intrinsic geometric objects on $\mathcal{M}$ in (pseudo) Riemannian geometry and General Relativistic physics, which are dubbed *passive diffeomorphisms* due to their formal identity with the action of Diff$(\mathcal{M})$ which transforms intrinsic objects into new ones. Elements of Diff$(\mathcal{M})$ are therefore sometimes called *active diffeomorphisms*.

Yet, there is obviously also a local representation of the active gauge transformations discussed in section 4.2. This is the object of the next section.

5.2. Local active gauge transformations

Let us first denote the local representatives on $\mathcal{U} \subset \mathcal{M}$ of the gauge group elements $\gamma, \eta \in \mathcal{H}$ by upright greek letters, $\upgamma := \sigma^*\gamma : \mathcal{U} \to H$, and $\upeta := \sigma^*\eta : \mathcal{U} \to H$. The local gauge group on $\mathcal{U}$ is then simply defined as $\mathcal{H}_{\text{loc}} := \left\{\upgamma : \mathcal{U} \to H \,|\, \upgamma^{\upeta} = \upeta^{-1}\upgamma\upeta\right\}$, where the defining property is the pullback by σ of the action of $\mathcal{H}$ on itself. We then define the smooth map,

$$
\begin{aligned}
C_\sigma(\upgamma) := \sigma^* C(\gamma) : \mathcal{U} &\to G, \\
x &\mapsto C_{\sigma(x)}(\upgamma(x)),
\end{aligned}
$$

which resembles the map $C_\sigma(g)$ introduced above. Its local active gauge transformation is,

$$
C_\sigma(\upgamma)^{\upeta} := \sigma^*\left(C(\gamma)^\eta\right) = \sigma^*\left(C(\eta)^{-1}C(\gamma\eta)\right) = C_\sigma(\upeta)^{-1}C_\sigma(\upgamma\upeta) \tag{24}
$$

Notice the close formal analogy with (17). The local active gauge transformations of a connection and tensorial forms are then:

$$
\begin{aligned}
A^{\upgamma} &= \sigma^*\omega^\gamma = \sigma^*\left(C(\gamma)^{-1}\omega\, C(\gamma) + C(\gamma)^{-1}dC(\gamma)\right), \\
&= C_\sigma(\upgamma)^{-1}A\, C_\sigma(\upgamma) + C_\sigma(\upgamma)^{-1}dC_\sigma(\upgamma).
\end{aligned} \tag{25}
$$

and

$$
\begin{aligned}
a^{\upgamma} &= \sigma^*\alpha^\gamma = \sigma^*\left(\rho\left[C(\gamma)^{-1}\right]\alpha\right), \\
&= \rho\left[C_\sigma(\upgamma)^{-1}\right]a.
\end{aligned} \tag{26}
$$

This latter result holds true for the exterior covariant derivative, $(Da)^{\upgamma} = \rho\left[C_\sigma(\upgamma)^{-1}\right]Da$, which is also obtained from (25) via $(Da)^{\upgamma} = D^{\upgamma}a^{\upgamma} = da^{\upgamma} + \rho_*\left(A^{\upgamma}\right)a^{\upgamma}$. This is again an implementation of the *gauge principle.*

As a special case of (26), we obtain the local active gauge transformations of the curvature, sections and their covariant derivative,

$$
\begin{aligned}
&F^{\upgamma} = C_\sigma(\upgamma)^{-1}F\, C_\sigma(\upgamma), \quad \phi^{\upgamma} = \rho\left[C_\sigma(\upgamma)^{-1}\right]\phi \\
&\text{and} \quad (D\phi)^{\upgamma} = D^{\upgamma}\phi^{\upgamma} = \rho\left[C_\sigma(\upgamma)^{-1}\right]D\phi.
\end{aligned} \tag{27}
$$

The first result being also obtained from Cartan's structure equation and (25) via $F^{\upgamma} = dA^{\upgamma} + \frac{1}{2}[A^{\upgamma}, A^{\upgamma}]$.

Finally, from (24) is is easily seen that,

$$(A^{\gamma})^{\eta} = A^{\gamma\eta} = C_\sigma(\gamma\eta)^{-1} A\, C_\sigma(\gamma\eta) + C_\sigma(\gamma\eta)^{-1} dC_\sigma(\gamma\eta),$$
$$(a^{\gamma})^{\eta} = a^{\gamma\eta} = \rho \left[C_\sigma(\gamma\eta)^{-1}\right] a.$$

Further transformations of F, ϕ and $D\phi$ ensue. This shows that the action of $\mathcal{H}_{\rm loc}$ on local objects on $\mathcal{U} \subset \mathcal{M}$ is a well-behaved right action.

Let us reiterate a standard yet important gauge theoretic observation: The local active gauge transformations (25)-(26), relating local representatives seen through the same section σ (by the same observer) of different global objects, are formally indistinguishable from the passive gauge transformations (18)-(19), relating local representatives seen through different sections σ and σ' (by distinct observers) of the same global objects. This is clear by observing that $\sigma^*\Phi(p) = \Phi(\sigma(x)) = \sigma(x)\gamma(\sigma(x)) = \sigma(x)\gamma(x)$. So the pullback by σ of objects actively transformed by Φ/γ is formally equivalent to the pullback of the untransformed objects by a new local section $\sigma' = \sigma\gamma$. Nevertheless, in physics, symmetry under active gauge transformations is of much greater conceptual importance than the mere symmetry under passive ones. In the case of general relativistic physics for example, while symmetry of the theory under coordinate changes translates as a principle - the principle of general relativity - of democratic access to intrinsic objects of $\mathcal{M}$ which is thus at first seen as identical to the objective spacetime, symmetry under $\mathrm{Diff}(\mathcal{M})$ implies that the manifold $\mathcal{M}$ and its points are non-physical and that only relative field configurations over it - that can be diffeomorphically dragged - have physical meaning (this is the famous "hole argument"). In Yang-Mills gauge theories - and a fortiori here - it is still not entirely clear how one should interpret the two types of formally equivalent symmetries.[1]

6. Mixing with the standard situation

In this section we consider the minimal conditions under which the geometry described above mixes and coexists with the standard one. At the risk of some repetition, we thus slightly generalise the previous construction.

[1]This matter is distinct from another important discussion, mainly addressed by philosophers of physics, regarding the demarcation criterion between *substantial* and *artificial* gauge symmetries. A main takeaway is that substantial gauge symmetries (either passive or active) in Yang-Mills theories signal non-local physical properties or phenomena, while artificial symmetries do not. See [14] and references therein.

We will first suppose that the structure group of the bundle $\mathcal{P}$ is a direct product $H \times K$, whose elements are written $(h,k) = hk$ for $h \in H$ and $k \in K$. With the two subgroups commuting, the composition law is simply $hk \cdot h'k' = hkh'k' = hh'kk' \in H \times K$. The right action on $\mathcal{P}$ is thus $R_{hk} = R_{kh}$, and the right actions of the two subgroups commute: $R_h \circ R_k = R_k \circ R_h$.

We also consider the (inner) semi-direct product group $G \rtimes K$, whose elements are written gk for $g \in G$ and $k \in K$. The two subgroups do not commute, the composition law is $gk \cdot g'k' = gkg'k' = g\, kg'k^{-1} \cdot kk' \in G \rtimes K$, and the group morphism $K \to \text{Aut}(G)$ defining the semi-direct product is $k \mapsto \text{Conj}(k)$. Finally, we require that the representation (ρ, V) of G extends to a representation of $G \rtimes K$, and is therefore a representation for both subgroups.

The group of vertical automorphisms is also a direct product $\text{Aut}_v(\mathcal{P}) = \text{Aut}_v(\mathcal{P}, H) \times \text{Aut}_v(\mathcal{P}, K)$, with elements $\Psi = (\Phi, \Xi)$. Correspondingly, the gauge group is $\mathcal{H} \times \mathcal{K}$ with elements $(\gamma, \zeta) = \gamma\zeta$. The association is $\Psi(p) = p\gamma(p)\zeta(p)$. Because of the commutativity of the actions of H and K we have,

$$
\begin{aligned}
R_k^*\gamma &= \gamma, \qquad & \Xi^*\gamma &= \gamma, \\
R_h^*\zeta &= \zeta, \qquad & \Phi^*\zeta &= \zeta.
\end{aligned} \tag{28}
$$

From this we have indeed that $\Psi = \Phi \circ \Xi = \Xi \circ \Phi$.

Consider X^v and $Y^v \in \Gamma(V\mathcal{P})$ generated respectively by $X \in \text{Lie}H$ and $Y \in \text{Lie}K$. We have the infinitesimal versions of the above equivariance laws,

$$
L_{Y^v}\gamma = Y^v(\gamma) = 0 \qquad \text{and} \qquad L_{X^v}\zeta = X^v(\zeta) = 0. \tag{29}
$$

Still by commutativity of the action of H and K we get,

$$
\begin{aligned}
R_{k*}X_p^v &= X_{pk}^v, \qquad & R_{k*}Y_p^v &= \text{Ad}_{k^{-1}}Y\big|_{pk}^v, \\
R_{h*}Y_p^v &= Y_{ph}^v, \qquad & R_{h*}X_p^v &= \text{Ad}_{k^{-1}}X\big|_{ph}^v,
\end{aligned} \tag{30}
$$

Also, for $Z^v = \{X^v, Y^v\}$ it is easily shown that $\Phi_* Z_p^v = Z_{\Phi(p)}^v$ and $\Xi_* Z_p^v = Z_{\Xi(p)}$.

The definition of the cocycle map C prescribes its H-equivariance. We need to specify also its K-equivariance. It is easily found that the simplest choice compatible with its H-equivariance is, for $h' \in H$ and $k \in K$,

$$
\begin{aligned}
&R_k^* C(h') = k^{-1}C(h')k, \\
&\text{whose infinitesimal version is } \ L_{Y^v}C(h') = [C(h'), Y].
\end{aligned} \tag{31}
$$

Indeed, from (3) and (31), one has on the one hand

$$C_{pkh}(h') = C_{pk}(h)^{-1}C_{pk}(hh') = k^{-1}C_p(h)^{-1}k \cdot k^{-1}C_p(hh')k.$$

And on the other hand,

$$C_{phk}(h') = k^{-1}C_{ph}(h')k = k^{-1}C_p(h)^{-1}C_p(hh')k.$$

The infinitesimal equivariance if obtained from

$$\left(L_{Y^v}C(h')\right)(p) = \tfrac{d}{d\tau}C_{pe^{\tau Y}}(h')\Big|_{\tau=0}.$$

It follows that the $\mathcal{K}$-gauge transformation of this map is,

$$C(h')^{\zeta} := \Xi^* C(h') = \zeta^{-1}C(h')\zeta. \tag{32}$$

In the same way, the H-equivariance of the map $C(\gamma)$ is known from (8). From above, we get its K-equivariance

$$R_k^* C(\gamma) = k^{-1}C(\gamma)k, \text{ with infinitesimal version } L_{Y^v}C(\gamma) = [C(\gamma), Y]. \tag{33}$$

From which follows that,

$$C(\gamma)^{\zeta} := \Xi^* C(\gamma) = \zeta^{-1}C(\gamma)\zeta. \tag{34}$$

Now, let us see what we can do with these ingredients.

6.1. Mixed vector bundles and tensorial forms

Given a representation (ρ, V) of $G \rtimes K$, we define the mixed vector bundle $\mathcal{E}^C = \mathcal{P} \times_{C(H)\rtimes K} V := \mathcal{P} \times V/\sim$, with equivalence relation $(p, v) \sim \left(phk = pkh,\, \rho\left(k^{-1}C_p(h)^{-1}\right)v\right)$. It is well defined because on the one hand we have,

$$\begin{aligned}(p, v) &\sim \left(ph, \rho\left[C_p(h)^{-1}\right]v\right)\\ &\sim \left(phk, \rho\left(k^{-1}\right)\rho\left[C_p(h)^{-1}\right]v\right) = \left(phk, \rho\left[k^{-1}C_p(h)^{-1}\right]v\right).\end{aligned}$$

On the other hand, using (31),

$$\begin{aligned}(p, v) \sim \left(pk, \rho\left(k^{-1}\right)v\right) &\sim \left(pkh, \rho\left[C_{pk}(h)^{-1}\right]\rho\left(k^{-1}\right)v\right)\\ &= \left(pkh, \rho\left[k^{-1}C_p(h)^{-1}k\right]\rho\left(k^{-1}\right)v\right)\\ &= \left(pkh, \rho\left[k^{-1}C_p(h)^{-1}\right]v\right).\end{aligned}$$

It is clear that the twisted bundle $E^C = \mathcal{P} \times_{C(H)} V$ is a subbundle of $\mathcal{E}^C$, and so is the standard vector bundle $E = \mathcal{P} \times_K V := \mathcal{P} \times V/\sim$ for the equivalence relation $(p, v) \sim \big(pk, \rho(k^{-1})v\big)$. Hence the name for $\mathcal{E}^C$.

Define the space of $C(H) \rtimes K$-tensorial differential forms,

$$\Omega^\bullet_{\text{tens}}\big(\mathcal{P}, C(H)\rtimes K\big) = \Big\{\alpha \in \Omega^\bullet(\mathcal{P}, V) \,\big|\, \alpha_p(Z^v_p, \dots) = \ 0 \text{ for } Z^v = \{X^v, Y^v\}, \\ \text{and } R^*_{hk}\alpha = R^*_{kh}\alpha = \rho\big(C_p(h)k\big)^{-1}\alpha\Big\}.$$

Clearly, these are in particular both $C(H)$-tensorial and K-tensorial, and we indeed verify the compatibility relations:

$$\begin{aligned} R^*_k\left(R^*_h\alpha_{pkh}\right) &= R^*_k\left(\rho\left(C_p(h)^{-1}\right)\alpha_{pk}\right) = \rho\left(C_{pk}(h)^{-1}\right)R^*_k\alpha_{pk} \\ &= \rho\left(k^{-1}C_p(h)^{-1}k\right)\rho\left(k^{-1}\right)\alpha_p = \rho\left(k^{-1}C_p(h)^{-1}\right)\alpha_p, \\ R^*_h\left(R^*_k\alpha_{phk}\right) &= R^*_h\left(\rho\left(k^{-1}\right)\alpha_{ph}\right) = \rho\left(k^{-1}\right)\rho\left(C_p(h)^{-1}\right)\alpha_p. \end{aligned}$$

As per the usual argument, there is an isomorphism

$$\Gamma\big(\mathcal{E}^C\big) \simeq \Omega^0_{\text{tens}}\big(\mathcal{P}, C(H)\rtimes K\big).$$

Again, the question arises as to the adequate notion of connection on $\mathcal{P}$ that provides a good covariant derivative on $\Omega^\bullet_{\text{tens}}\big(\mathcal{P}, C(H)\rtimes K\big)$, and on sections of $\mathcal{E}^C$ in particular.

6.2. Mixed twisted connections

We endow the bundle $\mathcal{P}(\mathcal{M}, H\times K)$ with a connection $\omega \in \Omega^1_{\text{eq}}\big(\mathcal{P}, \text{Lie}\big(G \rtimes K\big)\big)$ satisfying,

(I*) $\omega_p\left(X^v_p + Y^v_p\right) = \frac{d}{d\tau}C_p\left(e^{\tau X}\right)\big|_{\tau=0} + Y = dC_{p|e}(X) + Y \ \in \text{Lie}G \oplus \text{Lie}K,$

(II*) $R^*_{hk}\omega_{phk} = R^*_{kh}\omega_{pkh} = [C_p(h)k]^{-1}\omega_p\,[C_p(h)k] + [C_p(h)k]^{-1}d[C(h)k]_{|p}.$

It is clear that on the H-subbundle, ω satisfies the properties (I)-(II) of a twisted connection, while on the K-subbundle it satisfies the definition of a standard K-principal connection: $\omega_p(Y^v_p) = Y$ and $R^*_k\omega_{pk} = \text{Ad}_{k^{-1}}\omega_p$. We therefore call a 1-form ω defined by (I*) and (II*), a *mixed* twisted connection.

6.2.1. Covariant derivative. The infinitesimal version of the equivariance property of tensorial forms is,

$$\begin{aligned} L_{X^v+Y^v}\alpha &= \tfrac{d}{d\tau}R^*_{e^{\tau(X+Y)}}\alpha\big|_{\tau=0} = \tfrac{d}{d\tau}\,\rho\left[e^{-\tau Y}C\left(e^{\tau X}\right)^{-1}\right]\alpha\big|_{\tau=0} \\ &= -\rho_*\left[Y + \tfrac{d}{d\tau}C\left(e^{\tau X}\right)\big|_{\tau=0}\right]\alpha = -\rho_*\big(Y + dC_e(X)\big)\,\alpha. \end{aligned}$$

With this in mind, we obtain the following.

Proposition 7. *The exterior covariant derivative defined as $D := d + \rho_*(\omega)$ preserves both $\Omega^\bullet_{eq}\big(\mathcal{P}, C(H)\rtimes K\big)$ and $\Omega^\bullet_{tens}\big(\mathcal{P}, C(H)\rtimes K\big)$.*

Proof. First, using (II*),we show that

$$D : \Omega^\bullet_{\text{eq}}\big(\mathcal{P}, C(H)\rtimes K\big) \to \Omega^\bullet_{\text{eq}}\big(\mathcal{P}, C(H)\rtimes K\big).$$

For $\alpha \in \Omega^\bullet_{\text{eq}}\big(\mathcal{P}, C(H)\rtimes K\big)$:

$$\begin{aligned} R^*_{hk}D\alpha &= dR^*_{hk}\alpha + \rho_*\left(R^*_{hk}\omega\right)R^*_{hk}\alpha, \\ &= d\rho\big(C(h)k\big)^{-1}\cdot\alpha + \rho\big(C(h)k\big)^{-1}d\alpha \\ &\quad + \rho_*\left([C(h)k]^{-1}\omega\,[C(h)k] + [C(h)k]^{-1}d[C(h)k]\right)\rho\big(C(h)k\big)^{-1}\alpha, \\ &= \rho\big(C(h)k\big)^{-1}\big(d\alpha + \rho_*(\omega)\alpha\big) = \rho\big(C(h)k\big)^{-1}D\alpha. \end{aligned}$$

Then, using (I*), we show that $D : \Omega^\bullet_{\text{tens}}\big(\mathcal{P}, C(H)\rtimes K\big) \to \Omega^\bullet_{\text{tens}}\big(\mathcal{P}, C(H)\rtimes K\big)$. It is enough to prove it for $\alpha \in \Omega^1_{\text{tens}}\big(\mathcal{P}, C(H)\rtimes K\big)$:

$$\begin{aligned} D\alpha(X^v+Y^v, Z) &= \big(d\alpha + \rho_*(\omega)\alpha\big)(X^v+Y^v, Z), \\ &= \underbrace{d\alpha(X^v+Y^v, Z)}_{(L_{X^v+Y^v}\alpha)(Z)} + \rho_*(\omega(X^v+Y^v))\,\alpha(Z) \\ &\quad - \rho_*(\omega(Z))\,\underbrace{\alpha(X^v+Y^v)}_{=0}, \\ &= -\rho_*\big(Y + dC_e(X)\big)\,\alpha(Z) + \rho_*\big(dC_e(X) + Y\big)\,\alpha(Z) = 0. \end{aligned}$$

$\square$

In particular, D provides a good notion of covariant differentiation of sections of the mixed vector bundle $\mathcal{E}^C$.

6.2.2. Curvature. The curvature of the mixed connection is defined in the usual way, so that we have the result:

Proposition 8. *The curvature $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$ is a mixed tensorial 2-form, $\Omega \in \Omega^2_{tens}\big(\mathcal{P}, C(H) \rtimes K\big)$. It satisfies a Bianchi identity $D\Omega = d\Omega + [\omega, \Omega] = 0$.*

Proof. The equivariance is proven the usual way via (II*),

$$
\begin{aligned}
R^*_{hk}\Omega &= dR^*_{hk}\omega + \tfrac{1}{2}[R^*_{hk}\omega, R^*_{hk}\omega], \\
&= d\left([C(h)k]^{-1}\omega\,[C(h)k] + [C(h)k]^{-1}d[C(h)k]\right) \\
&\quad + \tfrac{1}{2}\big[[C(h)k]^{-1}\omega\,[C(h)k] + [C(h)k]^{-1}d[C(h)k], \\
&\qquad [C(h)k]^{-1}\omega\,[C(h)k] + [C(h)k]^{-1}d[C(h)k]\big], \\
&= \dots \\
&= [C(h)k]^{-1}\left(d\omega + \tfrac{1}{2}[\omega,\omega]\right)[C(h)k] = [C(h)k]^{-1}\Omega\,[C(h)k].
\end{aligned}
$$

Now, taking $Y^v = 0$ in (I*), the horizontality of Ω w.r.t. H-vertical vector fields is proven as in Proposition 2. Taking $X^v = 0$ in (I*), ω is a standard Ehresmann K-connection, so the horizontality of Ω w.r.t. K-vertical vector fields is proven the usual way. By linearity, Ω is $H \times K$-horizontal. It is therefore $C(H) \rtimes K$-tensorial. Since here $\rho = \mathrm{Ad}$, the covariant derivative is $D\Omega = d\Omega + [\omega, \Omega]$ and vanishes by definition of Ω. □

It is easily seen that another standard result that extends to the mixed case is that for $\alpha \in \Omega^\bullet_{\text{tens}}\big(\mathcal{P}, C(H) \rtimes K\big)$, $D^2\alpha = DD\alpha = \rho_*(\Omega)\alpha$.

6.3. Mixed gauge transformations

The gauge transformations of the mixed connection and tensorial forms assume a simple form because the actions of $\mathrm{Aut}_v(\mathcal{P}, H) \simeq \mathcal{H}$ and $\mathrm{Aut}_v(\mathcal{P}, K) \simeq \mathcal{K}$ commute. Indeed we have the following.

Proposition 9. *The gauge transformations of ω and $\alpha \in \Omega^\bullet_{tens}\big(\mathcal{P}, C(H) \rtimes K\big)$ are,*

$$\omega^{\gamma\zeta} = [C(\gamma)\zeta]^{-1}\omega\,[C(\gamma)\zeta] + [C(\gamma)\zeta]^{-1}d[C(\gamma)\zeta], \tag{35}$$

$$\alpha^{\gamma\zeta} = \rho\big[C(\gamma)\zeta\big]^{-1}\alpha. \tag{36}$$

Proof. First, notice that the push forward of a vector $X_p \in T_p\mathcal{P}$ by a vertical automorphism $\Psi \in \mathrm{Aut}_v(\mathcal{P}, H \times K)$ is,

$$\begin{aligned}\Psi_* X_p &= R_{(\gamma\zeta)(p)*}X_p + [(\gamma\zeta)(p)]^{-1}d(\gamma\zeta)_p(X_p)\big|^v_{\Psi(p)},\\ &= R_{\gamma(p)\zeta(p)*}X_p + \gamma(p)^{-1}d\gamma_{|p}(X_p)\big|^v_{\Psi(p)} + \zeta(p)^{-1}d\zeta_{|p}(X_p)\big|^v_{\Psi(p)}\end{aligned}$$

Therefore, the full gauge transformation of the mixed connection is by definition,

$$\begin{aligned}\omega_p^{\gamma\zeta}(X_p) &:= (\Psi^*\omega)_p\,(X_p) = \omega_{\Psi(p)}\,(\Psi_* X_p)\\ &= R^*_{\gamma(p)\zeta(p)}\omega_{\Psi(p)}(X_p) + \tfrac{d}{d\tau}C_{\Psi(p)}\left(e^{\gamma(p)^{-1}d\gamma_p(X_p)}\right)\big|_{\tau=0}\\ &\quad + \zeta(p)^{-1}d\zeta_{|p}(X_p),\\ &= \big([C_p(\gamma(p))\zeta(p)]^{-1}\omega_p[C_p(\gamma(p))\zeta(p)]\\ &\qquad + [C_p(\gamma(p))\zeta(p)]^{-1}d[C(\gamma(p))\zeta(p)]_{|p}\big)(X_p)\\ &\quad + \zeta(p)^{-1}\tfrac{d}{d\tau}C_{p\gamma(p)}\left(e^{\gamma(p)^{-1}d\gamma_p(X_p)}\right)\big|_{\tau=0}\,\zeta(p) + \zeta(p)^{-1}d\zeta_{|p}(X_p),\\ &= [C_p(\gamma(p))\zeta(p)]^{-1}\omega_p(X_p)[C_p(\gamma(p))\zeta(p)]\\ &\quad + \zeta(p)^{-1}\,C_p(\gamma(p))^{-1}dC(\gamma(p))_{|p}(X_p)\,\zeta(p)\\ &\quad + \zeta(p)^{-1}\left(C_p(\gamma(p))^{-1}dC_p(\gamma)_{|p}\right)(X_p)\,\zeta(p) + \zeta(p)^{-1}d\zeta_{|p}(X_p),\\ &= [C_p(\gamma(p))\zeta(p)]^{-1}\omega_p(X_p)[C_p(\gamma(p))\zeta(p)]\\ &\quad + \zeta(p)^{-1}\,C_p(\gamma(p))^{-1}dC(\gamma)_{|p}(X_p)\,\zeta(p) + \zeta(p)^{-1}d\zeta_{|p}(X_p),\\ &= [C_p(\gamma(p))\zeta(p)]^{-1}\omega_p(X_p)[C_p(\gamma(p))\zeta(p)]\\ &\quad + \zeta(p)^{-1}\,C_p(\gamma(p))^{-1}d[C(\gamma)\zeta]_{|p}(X_p),\\ &= \big([C(\gamma)\zeta]^{-1}\omega\,[C(\gamma)\zeta] + [C(\gamma)\zeta]^{-1}d[C(\gamma)\zeta]\big)_{|p}\,(X_p).\end{aligned}$$

In the same way, for a mixed tensorial form,

$$\begin{aligned}\alpha_p^{\gamma\zeta}(X_p,\ldots) &:= (\Psi^*\alpha)_p\,(X_p,\ldots) = \alpha_{\Psi(p)}\,(\Psi_* X_p,\ldots)\\ &= \alpha_{\Psi(p)}\left(R_{\gamma(p)\zeta(p)*}X_p,\ldots\right) = R^*_{\gamma(p)\zeta(p)}\alpha_{\Psi(p)}(X_p,\ldots),\\ &= \rho\big[C_p(\gamma(p))\zeta(p)\big]^{-1}\alpha_p(X_p,\ldots) = \left(\rho\big[C(\gamma)\zeta\big]^{-1}\alpha\right)_{|p}(X_p,\ldots).\end{aligned}$$

□

As special cases of (36), we obtain the gauge transformations of the curvature, of mixed sections and their covariant derivative,

$$\begin{aligned}&\Omega^{\gamma\zeta} = [C(\gamma)\zeta]^{-1}\Omega\,[C(\gamma)\zeta], \quad \varphi^{\gamma\zeta} = \rho\big[C(\gamma)\zeta\big]^{-1}\varphi,\\ &\text{and}\quad (D\varphi)^{\gamma\zeta} = \rho\big[C(\gamma)\zeta\big]^{-1}D\varphi.\end{aligned}$$

It is clear from (35) that ω transforms in particular as a twisted connection under $\mathcal{H}$ and a standard connection under $\mathcal{K}$. So, on the one hand, by (28),

$$\begin{aligned}(\omega^\zeta)^\gamma &= \Phi^*(\Xi^*\omega) = \Phi^*\left(\zeta^{-1}\omega\,\zeta + \zeta^{-1}d\zeta\right) = \zeta^{-1}\Phi^*\omega\,\zeta + \zeta^{-1}d\zeta,\\ &= \zeta^{-1}\left(C(\gamma)^{-1}\omega\,C(\gamma) + C(\gamma)^{-1}dC(\gamma)\right)\zeta + \zeta^{-1}d\zeta,\\ &= \zeta^{-1}C(\gamma)^{-1}\omega\,C(\gamma)\zeta + \zeta^{-1}C(\gamma)^{-1}d\left(C(\gamma)\zeta\right).\end{aligned}$$

On the other hand, by (34) ,

$$\begin{aligned}(\omega^\gamma)^\zeta &= \Xi^*(\Phi^*\omega) = \Xi^*\left(C(\gamma)^{-1}\omega\,C(\gamma) + C(\gamma)^{-1}dC(\gamma)\right),\\ &= \zeta^{-1}C(\gamma)^{-1}\zeta\left(\zeta^{-1}\omega\,\zeta + \zeta^{-1}d\zeta\right)\zeta^{-1}C(\gamma)\zeta\\ &\quad + \zeta^{-1}C(\gamma)^{-1}\zeta d\left(\zeta^{-1}C(\gamma)\zeta\right),\\ &= \zeta^{-1}C(\gamma)^{-1}\omega\,C(\gamma)\zeta + \zeta^{-1}C(\gamma)^{-1}d\left(C(\gamma)\zeta\right).\end{aligned}$$

In the same way, from (36) it is clear that α transforms in particular as a twisted tensorial form under $\mathcal{H}$ and as a standard tensorial form under $\mathcal{K}$. So that,

$$\begin{aligned}(\alpha^\zeta)^\gamma &= \Phi^*(\Xi^*\alpha) = \Phi^*\left(\rho\left(\zeta^{-1}\right)\alpha\right)\\ &= \rho\left(\zeta^{-1}\right)\rho\left[C(\gamma)^{-1}\right]\alpha = \rho\left[C(\gamma)\zeta\right]^{-1}\alpha,\\ (\alpha^\gamma)^\zeta &= \Xi^*(\Phi^*\alpha) = \Xi^*\left(\rho\left[C(\gamma)^{-1}\right]\alpha\right)\\ &= \rho\left[\zeta^{-1}C(\gamma)^{-1}\zeta\right]\rho\left(\zeta^{-1}\right)\alpha = \rho\left[C(\gamma)\zeta\right]^{-1}\alpha.\end{aligned} \tag{37}$$

The results (35)-(36) indeed express the commutativity of the action of $\mathcal{H}$ and $\mathcal{K}$.

We can use a notational game that allows to perform symbolically the computation of gauge transformations we have seen so far, and may be used as a memory trick. If ω^ζ denotes the result of a gauge transformation by $\mathcal{K}$, a further such transformation is noted $(\omega^\zeta)^\xi = (\omega^\xi)^{\zeta^\xi} = (\omega^\xi)^{\xi^{-1}\zeta\xi} = \omega^{\zeta\xi}$. Now, denote $\omega^\gamma = \omega^{C(\gamma)}$ the result of a gauge transformation by $\mathcal{H}$. A further such transformation if then $(\omega^\gamma)^\eta = (\omega^{C(\gamma)})^\eta = (\omega^\eta)^{C(\gamma)^\eta} = (\omega^{C(\eta)})^{C(\eta)^{-1}C(\gamma\eta)} = \omega^{C(\gamma\eta)}$. Then, $(\omega^\gamma)^\zeta = (\omega^{C(\gamma)})^\zeta = (\omega^\zeta)^{C(\gamma)^\zeta} = (\omega^\zeta)^{\zeta^{-1}C(\gamma)\zeta} = \omega^{C(\gamma)\zeta}$. Also, $(\omega^\zeta)^\gamma = (\omega^\gamma)^{\zeta^\gamma} = (\omega^{C(\gamma)})^\zeta = \omega^{C(\gamma)\zeta}$. So we could note $\omega^{\gamma\zeta}$ as $\omega^{C(\gamma)\zeta}$ instead. This would have the advantage of making the mixed structure clear. Idem for twisted or mixed tensorial forms.

6.4. Local version

To be complete, and at the risk of some redundancy, we provide in the next two subsections the local description of the above mixed global geometry.

6.4.1. Passive mixed gauge transformations. Consider again $\mathcal{U}, \mathcal{U}' \subset \mathcal{M}$ such that $\mathcal{U} \cap \mathcal{U}' \neq \emptyset$, with the local sections $\sigma : \mathcal{U} \to \mathcal{P}_{|\mathcal{U}}$ and $\sigma' : \mathcal{U}' \to \mathcal{P}_{|\mathcal{U}'}$ related by,

$$
\begin{aligned}
\sigma' &= \sigma g \ell, \qquad \text{where } g : \mathcal{U} \cap \mathcal{U}' \to H, \text{ and } \quad \ell : \mathcal{U} \cap \mathcal{U}' \to K. \\
&= \sigma \ell g \qquad\qquad\qquad\quad x \mapsto g(x) \qquad\qquad\qquad\quad x \mapsto \ell(x)
\end{aligned}
$$

Then, for $X_x \in T_x\mathcal{M}$, $x \in \mathcal{U} \cap \mathcal{U}'$, the pushforwards by σ' and σ are related by

$$
\begin{aligned}
\sigma'_* X_x &= R_{g(x)\ell(x)*}\left(\sigma_* X_x\right) + [(g\ell)^{-1} d(g\ell)]_{|x}(X_x)\big|^v_{\sigma'(x)}, \\
&= R_{g(x)\ell(x)*}\left(\sigma_* X_x\right) + [g^{-1}dg]_{|x}(X_x)\big|^v_{\sigma'(x)} + [\ell^{-1}d\ell]_{|x}(X_x)\big|^v_{\sigma'(x)}.
\end{aligned}
$$

From this we find the following gluing properties.

Proposition 10. *The gluing properties of the local representatives on $\mathcal{U}$ and $\mathcal{U}'$ of a mixed connection and a mixed tensorial forms are,*

$$A' = [C_\sigma(g)\ell]^{-1} A\, [C_\sigma(g)\ell] + [C_\sigma(g)\ell]^{-1} d[C_\sigma(g)\ell], \tag{38}$$

$$a' = \rho\, [C_\sigma(g)\ell]^{-1} a. \tag{39}$$

Proof. The proof is as for active gauge transformations in Proposition 9. Using (I*)-(II*) as well as the proof of Proposition 6, we find:

$$
\begin{aligned}
A'_x(X_x) &= (\sigma'^*\omega)_x(X_x) = \omega_{\sigma'(x)}\left(\sigma'_* X_x\right), \\
&= \omega_{\sigma'(x)}\Big(R_{g(x)\ell(x)*}\left(\sigma_* X_x\right) + [g^{-1}dg]_{|x}(X_x)\big|^v_{\sigma'(x)} \\
&\qquad + [\ell^{-1}d\ell]_{|x}(X_x)\big|^v_{\sigma'(x)}\Big), \\
&= R^*_{g(x)\ell(x)}\omega_{\sigma'(x)}\left(\sigma_* X_x\right) + \tfrac{d}{d\tau} C_{\sigma'(x)}\left(e^{\tau\,[g^{-1}dg]_{|x}(X_x)}\right)\big|_{\tau=0} \\
&\quad + [\ell^{-1}d\ell]_{|x}(X_x), \\
&= \big([C_{\sigma(x)}\left(g(x)\right)\ell(x)]^{-1}\omega_{\sigma(x)}\,[C_{\sigma(x)}\left(g(x)\right)\ell(x)] \\
&\quad + [C_{\sigma(x)}\left(g(x)\right)\ell(x)]^{-1} d[C(g(x))\ell(x)]_{|\sigma(x)}\big)(\sigma_* X_x) \\
&\quad + \ell(x)^{-1}\left(\tfrac{d}{d\tau} C_{\sigma(x)g(x)}\left(e^{\tau\,[g^{-1}dg]_{|x}(X_x)}\right)\big|_{\tau=0}\right)\ell(x) + [\ell^{-1}d\ell]_{|x}(X_x),
\end{aligned}
$$

$$
\begin{aligned}
&= \big([C_{\sigma(x)}\,(g(x))\,\ell(x)]^{-1} A_x\,[C_{\sigma(x)}\,(g(x))\,\ell(x)] \\
&\quad + [C_{\sigma(x)}\,(g(x))\,\ell(x)]^{-1} d[C_\sigma(g(x))\ell(x)]_{|x}\big)(X_x) \\
&\quad + \ell(x)^{-1} C_{\sigma(x)}(g(x))^{-1} dC_{\sigma(x)}(g)_{|x}(X_x)\,\ell(x) + [\ell^{-1} d\ell]_{|x}(X_x), \\
&= \big([C_{\sigma(x)}\,(g(x))\,\ell(x)]^{-1} A_x\,[C_{\sigma(x)}\,(g(x))\,\ell(x)] \\
&\quad + [C_{\sigma(x)}\,(g(x))\,\ell(x)]^{-1} d[C_\sigma(g)\ell(x)]_{|x}\big)(X_x) + [\ell^{-1} d\ell]_{|x}(X_x), \\
&= \big([C_{\sigma(x)}\,(g(x))\,\ell(x)]^{-1} A_x\,[C_{\sigma(x)}\,(g(x))\,\ell(x)] \\
&\quad + [C_{\sigma(x)}\,(g(x))\,\ell(x)]^{-1} d[C_\sigma\,(g)\,\ell]_{|x}\big)(X_x).
\end{aligned}
$$

For the local representatives of a tensorial form $\alpha \in \Omega^\bullet_{\text{tens}}\big(\mathcal{P}, C(H) \rtimes K\big)$ we get:

$$
\begin{aligned}
a'_x(X_x, \ldots) &= \big(\sigma'^* \alpha\big)_x\,(X_x, \ldots) = \alpha_{\sigma'(x)}\,\big(\sigma'_* X_x, \ldots\big) \\
&= \alpha_{\sigma'(x)}\,\big(R_{g(x)\ell(x)*}\,(\sigma_* X_x)\,, \ldots\big) = R^*_{g(x)\ell(x)} \alpha_{\sigma(x)}\,(\sigma_* X_x, \ldots)\,, \\
&= \rho\,\big[C_{\sigma(x)}(g(x))\ell(x)\big]^{-1} \alpha_{\sigma(x)}\,(\sigma_* X_x, \ldots) \\
&= \rho\,\big[C_{\sigma(x)}(g(x))\ell(x)\big]^{-1} a_x\,(X_x, \ldots)\,.
\end{aligned}
$$

□

This last result is also valid for the exterior covariant derivative, which is also found by having $(Da)' = D'a' = da' + \rho_*\,(A')\,a'$. As special cases of (39), we have the gluings of the local representatives of the curvature, sections and their covariant derivative:

$$
\begin{aligned}
&F' = [C_\sigma(g)\ell]^{-1} F\,[C_\sigma(g)\ell], \quad \phi' = \rho\,[C_\sigma(g)\ell]^{-1}\phi \\
&\text{and} \quad (D\phi)' = D'\phi' = \rho\,[C_\sigma(g)\ell]^{-1} D\phi.
\end{aligned} \tag{40}
$$

And as usual the first result is also obtained from Cartan structure equation and (38).

6.4.2. Local active mixed gauge transformations. On a single open set $\mathcal{U} \subset \mathcal{M}$ with local section $\sigma : \mathcal{U} \to \mathcal{P}_{|\mathcal{U}}$, the local gauge group is $\mathcal{H}_{\text{loc}} \times \mathcal{K}_{\text{loc}}$, with $\mathcal{H}_{\text{loc}} := \big\{\gamma : \mathcal{U} \to H \,|\, \gamma^\eta = \eta^{-1}\gamma\eta\big\}$ and $\mathcal{K}_{\text{loc}} := \{\zeta : \mathcal{U} \to K \,|\, \zeta^\xi = \xi^{-1}\zeta\xi\}$, but also - as local version of (28) - $\gamma^\zeta = \gamma$ and $\zeta^\gamma = \zeta$. The $\mathcal{H}_{\text{loc}}$-transformation of the map $C_\sigma(\gamma) : \mathcal{U} \to G$ is given by (24). Its $\mathcal{K}_{\text{loc}}$-transformation, the counterpart of (34), is

$$
C_\sigma(\gamma)^\zeta = \zeta^{-1} C_\sigma(\gamma)\zeta. \tag{41}
$$

The local mixed active gauge transformations of A and a are simply,

$$A^{\gamma\zeta} = \sigma^*\omega^{\gamma\zeta} = [C_\sigma(\gamma)\zeta]^{-1}A\,[C_\sigma(\gamma)\zeta] + [C_\sigma(\gamma)\zeta]^{-1}d[C_\sigma(\gamma)\zeta],. \tag{42}$$

$$a^{\gamma\zeta} = \sigma^*\alpha^{\gamma\zeta} = \rho[C_\sigma(\gamma)\zeta]^{-1}a. \tag{43}$$

We prove as in section 6.3 that $\mathcal{H}_{\text{loc}}$ and $\mathcal{K}_{\text{loc}}$ commute. Using (41), one easily shows that $\left(A^\gamma\right)^\zeta = \left(A^\zeta\right)^\gamma$ and $\left(a^\gamma\right)^\zeta = \left(a^\zeta\right)^\gamma$. Here also we could write (42) as $A^{C(\gamma)\zeta}$ and (43) as $a^{C(\gamma)\zeta}$.

Again, (43) hold for (Da) and is also found via $(Da)^{\gamma\zeta} = D^{\gamma\zeta}a^{\gamma\zeta}$. The mixed local gauge transformations for F, ϕ and $D\phi$ are found as special cases,

$$\begin{aligned} &F^{\gamma\zeta} = [C_\sigma(\gamma)\zeta]^{-1}F\,[C_\sigma(\gamma)\zeta], \qquad \phi^{\gamma\zeta} = \rho\,[C_\sigma(\gamma)\zeta]^{-1}\,\phi \\ &\text{and} \quad (D\phi)^{\gamma\zeta} = \rho\,[C_\sigma(\gamma)\zeta]^{-1}\,D\phi. \end{aligned} \tag{44}$$

As always, the transformation of F is also obtained from Cartan's structure equation and (42).

Finally, we notice the exact formal resemblance between (38)-(39) and (42)-(43), and refer back to our comments at the end of section 5.2.

7. Action of the Lie algebra of vertical automorphisms

Let us denote the Lie algebras of $\text{Aut}_v(\mathcal{P}, H)$ and $\mathcal{H}$ respectively by $\mathfrak{aut}_v(\mathcal{P}, H)$ and $\text{Lie}\mathcal{H}$. Given $\Phi_\tau \in \text{Aut}_v(\mathcal{P}, H)$, we have $\Phi_\tau(p) = p\gamma_\tau(p) = pe^{\tau\,\chi(p)}$ with $\chi : \mathcal{P} \to \text{Lie}H$. Since $R_h^*\gamma_\tau = h^{-1}\gamma_\tau h$, it comes that $R_h^*\,\chi = \text{Ad}_{h^{-1}}\chi$. We have then $\text{Lie}\mathcal{H} := \{\,\chi : \mathcal{P} \to \text{Lie}H \mid R_h^*\,\chi = \text{Ad}_{h^{-1}}\chi\}$. In other terms $\text{Lie}\mathcal{H} = C^\infty_{\text{Ad}}(\mathcal{P}, \text{Lie}H) \simeq \Gamma\,(\mathcal{P} \times_{\text{Ad}} \text{Lie}H)$.

Now, Φ_τ is the flow of the vector field

$$\chi^v_p := \tfrac{d}{d\tau}\Phi_\tau(p)\big|_{\tau=0} = \tfrac{d}{d\tau}pe^{\tau\,\chi(p)}\big|_{\tau=0}. \tag{45}$$

By the way, $\chi^v_{ph} := \frac{d}{d\tau}phe^{\tau\,\chi(ph)} = \frac{d}{d\tau}phe^{\tau\,h^{-1}\chi(p)h} = \frac{d}{d\tau}ph\,h^{-1}e^{\tau\,\chi(p)}h = \frac{d}{d\tau}R_h\left(pe^{\tau\,\chi(p)}\right) =: R_{h*}\,\chi^v_p$. That is χ^v is a right-invariant vector field, $\chi^v \in \Gamma^H(T\mathcal{P})$. Conversely it is easily shown that the flow $\phi_\tau : \mathcal{P} \to \mathcal{P}$ of a right-invariant vertical vector field is such that $\phi_\tau(ph) = \phi_\tau(p)h$ and $\pi \circ \phi_\tau = \pi$, i.e. it is a vertical automorphism. So $\mathfrak{aut}_v(\mathcal{P}, H) = \Gamma^H(T\mathcal{P}) \cap \Gamma(V\mathcal{P})$. The definition (45) describes explicitly the isomorphism $\mathfrak{aut}_v(\mathcal{P}, H) \simeq \text{Lie}\mathcal{H}$.

7.1. Infinitesimal gauge transformations

To begin with, we want the infinitesimal counterpart of propositions (3) and (4). For this, let us first give the following result,

$$\frac{d}{d\tau}C_p\left(e^{\tau\,\chi(p)}\right)\Big|_{\tau=0}=dC_{p|e}\left(\frac{d}{d\tau}e^{\tau\,\chi(p)}\Big|_{\tau=0}\right)=dC_{p|e}(\chi(p)). \tag{46}$$

Remember that in $dC_{p|e}$, d is not de Rham derivative on $\mathcal{P}$ but signifies the pushforward of the map $C_p : H \to G$. Given this, it is easy to prove the following,

Proposition 11. *The infinitesimal active gauge transformations of the connection and of C-tensorial forms are,*

$$\begin{aligned}&L_{\chi^v}\omega = d\left(dC_{|e}(\chi)\right)+\left[\omega, dC_{|e}(\chi)\right],\\ &L_{\chi^v}\alpha = -\rho_*\left(dC_{|e}(\chi)\right)\alpha.\end{aligned} \tag{47}$$

Proof. The Lie derivative of ω w.r.t χ^v is,

$$\begin{aligned}L_{\chi^v}\omega :=&\frac{d}{d\tau}\,\Phi^*_\tau\omega\,\Big|_{\tau=0}=\frac{d}{d\tau}\,C\left(e^{\tau\,\chi}\right)^{-1}\omega\, C\left(e^{\tau\,\chi}\right)+C\left(e^{\tau\,\chi}\right)^{-1}dC\left(e^{\tau\,\chi}\right)\Big|_{\tau=0},\\ =&-\frac{d}{d\tau}C\left(e^{\tau\,\chi}\right)\Big|_{\tau=0}\omega+\omega\,\frac{d}{d\tau}C\left(e^{\tau\,\chi}\right)\Big|_{\tau=0}+d\,\frac{d}{d\tau}C\left(e^{\tau\,\chi}\right)\Big|_{\tau=0}.\end{aligned}$$

Using (46) the result follows. As for $\alpha \in \Omega^\bullet_{\text{tens}}\big(\mathcal{P}, C(H)\big)$,

$$L_{\chi^v}\alpha :=\frac{d}{d\tau}\,\Phi^*_\tau\alpha\,\Big|_{\tau=0}=\frac{d}{d\tau}\,\rho\left[C\left(e^{\tau\,\chi}\right)^{-1}\right]\alpha\,\Big|_{\tau=0}=-\rho_*\left[\frac{d}{d\tau}C\left(e^{\tau\,\chi}\right)\Big|_{\tau=0}\right]\alpha.$$

□

From this is immediately deduced that,

$$\begin{aligned}&L_{\chi^v}\Omega=\left[\Omega, dC_{|e}(\chi)\right],\quad L_{\chi^v}\varphi=-\rho_*\left(dC_{|e}(\chi)\right)\varphi,\\ &\text{and}\quad L_{\chi^v}D\varphi=-\rho_*\left(dC_{|e}(\chi)\right)D\varphi.\end{aligned}$$

It suffices to pullback (47) on $\mathcal{U}\subset\mathcal{M}$ to obtain the infinitesimal active gauge transformations of the local representatives, counterparts of (25) and (26). Given $\chi := \sigma^*\chi : \mathcal{U} \to \text{Lie}H \in \text{Lie}\mathcal{H}_{\text{loc}}$, this gives

$$\begin{aligned}\delta_\chi A &= \sigma^*\left(L_{\chi^v}\omega\right) = d\left(dC_{\sigma|e}(\chi)\right)+\left[A, dC_{\sigma|e}(\chi)\right],\\ \delta_\chi a &= \sigma^*\left(L_{\chi^v}\alpha\right) = -\rho_*\left(dC_{\sigma|e}(\chi)\right)a.\end{aligned} \tag{48}$$

Alternatively, these are obtained in a way analogous to the above proof, by defining $\delta_\chi A := \frac{d}{d\tau}\,A^{\gamma_\tau}\big|_{\tau=0}$ and $\delta_\chi a := \frac{d}{d\tau}\,a^{\gamma_\tau}\big|_{\tau=0}$.

The infinitesimal versions of the gluing properties, proposition 6, are not related to the action of $\mathfrak{aut}_v(\mathcal{P}, H) \simeq \mathrm{Lie}\mathcal{H}$ but are obtained by a completely analogous computation. On poses $g_\tau(x) = e^{\tau\lambda(x)}$, with $\lambda : \mathcal{U} \to \mathrm{Lie}H$, and defines:

$$(49)\qquad \begin{aligned} \delta_\lambda A &:= \tfrac{d}{d\tau} A'\big|_{\tau=0} = d\left(dC_{\sigma|e}(\lambda)\right) + \left[A, dC_{\sigma|e}(\lambda)\right], \\ \delta_\lambda a &:= \tfrac{d}{d\tau} a'\big|_{\tau=0} = -\rho_*\left(dC_{\sigma|e}(\lambda)\right) a. \end{aligned}$$

And as usual, these passive infinitesimal gauge transformations are formally identical to the active ones, Eq.(48).

7.2. Mixed case

In the same spirit as in the above considerations, under the assumptions of section 6 we are interested in the infinitesimal counterpart of the mixed gauge transformations of the connection (35) and of C-tensorial forms (36). Let us denote $\mathrm{Lie}\mathcal{K} := \{\upsilon : \mathcal{P} \to K \mid R_k^*\upsilon = \mathrm{Ad}_{k^{-1}}\upsilon\}$. It is isomorphic to $\mathfrak{aut}_v(\mathcal{P}, K)$ via the definition $\upsilon_p^v := \frac{d}{d\tau} p e^{\tau\,\upsilon(p)}\big|_{\tau=0}$. Elements of $\mathrm{Lie}\mathcal{H}$ and $\mathrm{Lie}\mathcal{K}$ also satisfy: $R_k^*\chi = \chi$ and $R_h^*\upsilon = \upsilon$. We have then

$$\mathfrak{aut}_v(P) = \mathfrak{aut}_v(\mathcal{P}, H) \oplus \mathfrak{aut}_v(\mathcal{P}, K) \simeq \mathrm{Lie}\mathcal{H} \oplus \mathrm{Lie}\mathcal{K},$$

and $\xi^v \in \mathfrak{aut}_v(\mathcal{P})$ s.t $\xi^v = \chi^v + \upsilon^v$ is generated by $\xi = \chi + \upsilon$.

In preparation for the next proposition, consider that:

$$(50)\qquad \begin{aligned} \tfrac{d}{d\tau} C\left(e^{\tau\chi}\right) e^{\tau\upsilon}\big|_{\tau=0} &= \tfrac{d}{d\tau} C\left(e^{\tau\chi}\right) e\,\big|_{\tau=0}\, e_K + e_G \tfrac{d}{d\tau} e^{\tau\upsilon}\big|_{\tau=0} \\ &= dC_{|e}(\chi) + \upsilon. \end{aligned}$$

Proposition 12. *The infinitesimal mixed gauge transformations of the connection and C-tensorial forms are,*

$$(51)\qquad \begin{aligned} L_{\xi^v}\omega &= d\left(dC_{|e}(\chi) + \upsilon\right) + \left[\omega, dC_{|e}(\chi) + \upsilon\right] = L_{\chi^v}\omega + L_{\upsilon^v}\omega, \\ L_{\xi^v}\alpha &= -\rho_*\left(dC_{|e}(\chi) + \upsilon\right)\alpha = L_{\chi^v}\alpha + L_{\upsilon^v}\alpha. \end{aligned}$$

Proof. It goes exactly as in the proof of proposition 11, using proposition 9:

$$\begin{aligned} L_{\xi^v}\omega &:= \tfrac{d}{d\tau}\left(\Phi \circ \Xi\right)_\tau^* \omega\,\big|_{\tau=0} \\ &= \tfrac{d}{d\tau}\left[C\left(e^{\tau\chi}\right)e^{\tau\upsilon}\right]^{-1}\omega\left[C\left(e^{\tau\chi}\right)e^{\tau\upsilon}\right] \\ &\quad + \left[C\left(e^{\tau\chi}\right)e^{\tau\upsilon}\right]^{-1} d\left[C\left(e^{\tau\chi}\right)e^{\tau\upsilon}\right]\big|_{\tau=0}, \\ &= -\tfrac{d}{d\tau}C\left(e^{\tau\chi}\right)e^{\tau\upsilon}\big|_{\tau=0}\,\omega + \omega\,\tfrac{d}{d\tau}C\left(e^{\tau\chi}\right)e^{\tau\upsilon}\big|_{\tau=0} + d\,\tfrac{d}{d\tau}C\left(e^{\tau\chi}\right)e^{\tau\upsilon}\big|_{\tau=0}. \end{aligned}$$

The result follows from (50). Idem for $\alpha \in \Omega^\bullet_{\text{tens}}\big(\mathcal{P}, C(H)\rtimes K\big)$:

$$L_{\xi^\upsilon}\alpha := \tfrac{d}{d\tau}\,(\Phi\circ\Xi)^*_\tau\,\alpha\,\big|_{\tau=0} = \tfrac{d}{d\tau}\,\rho\,[C\,(e^{\tau\,\chi})\,e^{\tau\upsilon}]^{-1}\,\alpha\,\big|_{\tau=0} \\ = -\rho_*\left[\tfrac{d}{d\tau}C\,(e^{\tau\,\chi})\,e^{\tau\upsilon}\big|_{\tau=0}\right]\alpha.$$

□

The local version on $\mathcal{U}\subset\mathcal{M}$ of this, the linear couterpart of (42)-(43), is obtained either by pullback or by $\delta A := \tfrac{d}{d\tau}\,A^{\gamma_\tau\zeta_\tau}\big|_{\tau=0}$ and $\delta a := \tfrac{d}{d\tau}\,a^{\gamma_\tau\zeta_\tau}\big|_{\tau=0}$. Either way, given $\upsilon := \sigma^*\upsilon : \mathcal{U}\to \text{Lie}K \in \text{Lie}\mathcal{K}_{\text{loc}}$, we get

$$\begin{aligned} \delta A &= d\left(dC_{\sigma|e}(\chi)+\upsilon\right) + \left[A, dC_{\sigma|e}(\chi)+\upsilon\right] = \delta_\chi A + \delta_\upsilon A \\ \delta a &= -\rho_*\left(dC_{\sigma|e}(\chi)+\upsilon\right) a = \delta_\chi a + \delta_\upsilon a. \end{aligned} \tag{52}$$

The linearised gluing properties, proposition 10, are obtained analogously. One poses $\ell_\tau(x) = e^{\tau\nu(x)}$, with $\nu : \mathcal{U}\to LieK$, and defines:

$$\begin{aligned} \delta A &:= \tfrac{d}{d\tau}\,A'\big|_{\tau=0} = d\left(dC_{\sigma|e}(\lambda)+\nu\right) + \left[A, dC_{\sigma|e}(\lambda)+\nu\right] = \delta_\lambda A + \delta_\nu A, \\ \delta a &:= \tfrac{d}{d\tau}\,a'\big|_{\tau=0} = -\rho_*\left(dC_{\sigma|e}(\lambda)+\nu\right) a = \delta_\lambda a + \delta_\nu a. \end{aligned} \tag{53}$$

7.3. A word on BRST

The BRST formalism is widely used in physics as an efficient algebraic way to handle infinitesimal gauge transformations. The BRST cohomology is rich, and allows notably to classify viable Lagrangian and gauge anomalies (stemming from gauge symmetry breaking), [15, 16] . The geometric origin of its heuristic rules has been explored by several authors, and the most satisfying view holds that it is grounded in the infinite dimensional analogue of the Chevalley-Eilenberg construction [17] associated to the gauge group. The so-called ghost being essentially the Maurer-Cartan form on $\mathcal{H}$, while the BRST operator is the de Rham differential on the infinite dimensional group manifold. See [18, 19]. Here we give the minimal BRST presentation reproducing the above linearised local active gauge transformations.

One considers that all objects have a new grading in addition to the de Rham form degree, the so-called ghost degree. The forms A, F, ϕ are attributed ghost degree 0, while χ and υ now stands for ghost fields[2] and have ghost degree 1. Let us denote for simplicity $dC_{\sigma|e}(\chi) =: c(\chi)$. The BRST

[2]Which are generic place holders for specific elements of Lie$\mathcal{H}$ and Lie$\mathcal{K}$, i.e. they are the Maurer-Cartan forms on $\mathcal{H}$ and $\mathcal{K}$ respectively.

differential, noted s, increases by one unit the ghost number and is such that $sd + ds = 0$ and $s^2 = 0$. So that $(d + s)^2 = 0$. It acts as

$$sA = -dc(\chi) - [A, c(\chi)]\,, \qquad sF = [F, c(\chi)] \qquad s\phi = -\rho_*(c(\chi))\phi,$$
$$\text{and} \qquad sc(\chi) = -\tfrac{1}{2}[c(\chi), c(\chi)].$$

The last equality stems from requiring $s^2 = 0$ on A, F or ϕ. In the mixed case, just replace $c(\chi)$ above by the total ghost $c(\chi) + \upsilon$. It is clear by linearity that the total BRST operator splits accordingly as $s = s_\chi + s_\upsilon$. The last equality above in particular nicely encapsulates the relations $s_\upsilon \upsilon = -\frac{1}{2}[\upsilon, \upsilon]$ and $s_\chi c(\chi) = -\frac{1}{2}[c(\chi), c(\chi)]$ - ensuring that we have BRST subalgebras for both sectors - as well as $s_\chi \upsilon = 0$ and $s_\upsilon c(\chi) = -[c(\chi), \upsilon]$ which are the BRST versions of the fourth equality (28) and of (34).

8. Twisted Cartan connection

Cartan connections are ancestors to Ehresmann connections. Ehresmann gave them their first recognisably modern definition while proposing his own generalisation (see [20] also [21] and reference therein). We refer to [22] and [23] for modern in depth treatments of the subject.

Given a principal bundle $\mathcal{P}(\mathcal{M}, H)$, and Lie$G' \supset$ Lie$H$, a Cartan connection is a differential 1-form $\varpi \in \Omega^1(\mathcal{P}, \text{Lie}G')$ enjoying the same two defining properties of an Ehresmann connection, $\varpi_p(X^v_p) = X \in \text{Lie}H$ and $R^*_h \varpi_{ph} = \text{Ad}_{h^{-1}}\varpi_p$, but it is also required that $\varpi_p : T_p\mathcal{P} \to \text{Lie}G'$ be a linear isomorphism $\forall p \in \mathcal{P}$. That is, the Cartan connection defines an absolute parallelism on $\mathcal{P}$. This means that $\dim\mathcal{M} = \dim \text{Lie}G'/\text{Lie}H$, and more precisely that given the projection $\tau : \text{Lie}G' \to \text{Lie}G'/\text{Lie}H$, $\tau(\varpi)$ is a soldering form (whose local version induces a metric structure on $\mathcal{M}$ if a bilinear form $\eta : \text{Lie}G'/\text{Lie}H \times \text{Lie}G'/\text{Lie}H \to \mathbb{R}$ is given). This is as well expressed by the fact that the Cartan connection induces a soldering, i.e. an isomorphism of vector bundles : $T\mathcal{M} \simeq \mathcal{P} \times_{\text{Ad}(H)} \text{Lie}G'/\text{Lie}H$. These facts make Cartan connections especially well suited to gauge theories of gravity.

The curvature is still given by Cartan's structure equation $\Omega = d\varpi + \frac{1}{2}[\varpi, \varpi] \in \Omega^2_{\text{tens}}(\mathcal{P}, \text{Lie}G')$, and $\Theta := \tau(\Omega)$ is the torsion. Gauge transformations are given by the action of $\text{Aut}_v(\mathcal{P}, H) \simeq \mathcal{H}$, as it would for an Ehresmann connection: $\varpi^\gamma := \Phi^*\varpi = \gamma^{-1}\varpi\gamma + \gamma^{-1}d\gamma$ and $\Omega^\gamma := \Phi^*\Omega = \gamma^{-1}\Omega\gamma$.

Since Cartan connections can be seen as special cases of Ehresmann connections, and since twisted connections are a particular generalisation of

Ehresmann connections, we would like to define a special class of twisted connections that would give an acceptable generalisation of Cartan connections. We might call these twisted Cartan connections. There might be more than one clever way to define such a class. To emulate the desirable properties of Cartan connections, especially regarding the gauge treatment of gravity, the following desiderata seem sufficient.

We consider again a principal bundle $\mathcal{P}(\mathcal{M}, H)$ and a cocycle for the group action of H on $\mathcal{P}$, $C : \mathcal{P} \times H \to G$. We demand that the twisted Cartan connection be $\varpi \in \Omega^1(\mathcal{P}, \text{Lie}G')$, where the Lie algebra $\text{Lie}G' \supset \text{Lie}G$ is such that $\dim \text{Lie}G'/\text{Lie}G = \dim\mathcal{M}$, and satisfies:

(I) $\quad \varpi_p(X^v_p) = \frac{d}{d\tau} C_p\left(e^{\tau X}\right)\big|_{\tau=0} = dC_{p|e}(X) \in \text{Lie}G, \quad$ for $X \in \text{Lie}H$,

(II) $\quad R^*_h \varpi_{ph} = C_p(h)^{-1} \varpi_p\, C_p(h) + C_p(h)^{-1} dC(h)_{|p}$,

(III) $\quad \ker \varpi_p = \emptyset, \quad \forall p \in \mathcal{P}$, and $\quad \text{Lie}G'/\text{Lie}G \subset \text{Im}\, \varpi_p$.

Thus, the first two axioms are those of a twisted connection as originally defined, but the third is a weakened version of the one satisfied by a Cartan connection: We only require that ϖ be an injection, not a linear isomorphism. Still, it is sufficient to define a soldering.

Proposition 13. *A twisted Cartan connection on $\mathcal{P}(\mathcal{M}, H)$ induces a soldering, i.e. a vector bundle isomorphism $T\mathcal{M} \simeq \mathcal{P} \times_{Ad_{C(H)}} LieG'/LieG$.*

Proof. Consider the following diagram

$$
\begin{tikzcd}
V_p\mathcal{P} \arrow[r, "\varpi_p", "\simeq"'] \arrow[d] & \text{Im}\,\varpi_p \cap \text{Lie}G \arrow[d] \\
T_p\mathcal{P} \arrow[r, "\varpi_p", "\simeq"'] \arrow[d, "\pi_*"] & \text{Im}\,\varpi_p \cap \text{Lie}G' \arrow[d, "\tau"] \\
T_x\mathcal{M} \arrow[r, dashed] & \begin{array}{l}\text{Im}\,\varpi_p \,\cap\, \text{Lie}G'/\text{Lie}G \\ =\, \text{Lie}G'/\text{Lie}G\end{array}
\end{tikzcd}
$$

The first columns is clearly a short exact sequences, and the second column is one also because of (III). The maps in the two upper rows are isomorphisms due to (I) and (III), so there must be an isomorphism in in the bottom row that makes the diagram commute. Such an isomorphism depends on $p \in \mathcal{P}$ and is given only up to the C-twisted adjoint action of H.

Indeed, denote $\beta_p : T_x\mathcal{M} \to \text{Lie}G'/\text{Lie}G$, and consider $X_p \in T_p\mathcal{P}$ projecting as $\pi_* X_p = X_x \in T_x\mathcal{M}$. The commutativity of the bottom square means $\beta_p(X_x) = \beta_p(\pi_* X_p) = \tau \circ \varpi_p(X_p)$. But also,

$$\begin{aligned}\beta_{ph}(X_x) &= \beta_{ph}(\pi_* R_{h*} X_p) = \tau \circ \varpi_{ph}(R_{h*}X_p) = \tau \circ \text{Ad}_{C_p(h)^{-1}} \varpi_p(X_p) \\ &= \text{Ad}_{C_p(h)^{-1}} \tau \circ \varpi_p(X_p) = \text{Ad}_{C_p(h)^{-1}} \beta_p(X_x),\end{aligned}$$

where (II) is used. Now, if one defines the map,

$$\begin{aligned}\iota :\ \mathcal{P} \times \text{Lie}G'/\text{Lie}G \quad &\to \quad T\mathcal{M} \\ (p, t) \quad &\mapsto \ \left(\pi(p),\, \beta_p^{-1}(t)\right),\end{aligned}$$

it appears that:

$$\begin{aligned}\iota\left(ph, \text{Ad}_{C_p(h)^{-1}} t\right) &= \left(\pi(ph),\, \beta_{ph}^{-1}\left(\text{Ad}_{C_p(h)^{-1}} t\right)\right) \\ &= \left(\pi(p),\, \beta_p^{-1}(t)\right) = \iota(p, t).\end{aligned}$$

Thus, we have actually a bundle map $\iota : \mathcal{P} \times_{\text{Ad}_{C(H)}} \text{Lie}G'/\text{Lie}G \to T\mathcal{M}$, which is a bundle equivalence because it covers the identity on $\mathcal{M}$ and is an isomorphism of fibers. $\square$

A corollary is that there is a bijective correspondance between vector fields $X \in \Gamma(T\mathcal{M})$ and $\text{Ad}_{C(H)}$-equivariant maps $\varphi : \mathcal{P} \to \text{Lie}G'/\text{Lie}G$.

Relatedly, it is clear that a twisted Cartan connection defines a soldering form $\theta := \tau \circ \varpi \in \Omega^1(\mathcal{P}, \text{Lie}G'/\text{Lie}G)$ which is both horizontal and $\text{Ad}_{C(H)}$-equivariant: $\theta(X^v) = 0$ and $R_h^*\theta = \text{Ad}_{C(h)^{-1}}\theta$. That is, the soldering form is tensorial, $\theta \in \Omega^1_{\text{tens}}\big(\mathcal{P}, C(H)\big)$. Locally, on $\mathcal{U} \subset \mathcal{M}$ endowed with a section σ, its pullback is $e := \sigma^*\theta \in \Omega^1(\mathcal{U}, \text{Lie}G'/\text{Lie}G)$. Given a coordinate system $\{x^\mu\}$ on $\mathcal{U}$ and a basis $\{u_a\}$ of $\text{Lie}G'/\text{Lie}G$, $e = e^a{}_\mu\, dx^\mu \otimes u_a$ where $e^a{}_\mu$ is a vielbein field. If there is a symmetric non-degenerate bilinear form $\eta : \text{Lie}G'/\text{Lie}G \times \text{Lie}G'/\text{Lie}G \to \mathbb{R}$, a metric $g : \Gamma(T\mathcal{U}) \times \Gamma(T\mathcal{U}) \to \mathbb{R}$ is induced via $g := \eta \circ e$. If η is Ad_G-invariant (possibly up to a non-vanishing multiplicative factor), the local metric g extends to a (conformal) metric on $\mathcal{M}$.

Let V be a $(\text{Lie}G', G)$-module, i.e. it supports an action of $\text{Lie}G'$ via ρ'_*, and an action of G via ρ which is such that $\rho_* = \rho'_{*|\text{Lie}G}$. The space $\Omega^\bullet_{\text{tens}}\big(\mathcal{P}, C(H)\big)$ of V-valued twisted tensorial forms is defined as in section 3.3, in particular their equivariance is $R_h^*\alpha = \rho\,[C(h)]^{-1}\,\alpha$. The twisted Cartan connection, on account of (I) and (II), induces an exterior covariant derivative on this space defined as usual by $D := d + \rho'_*(\varpi)$.

The curvature of the twisted Cartan connection is again given by $\Omega = d\varpi + \frac{1}{2}[\varpi, \varpi] \in \Omega^2(\mathcal{P}, \text{Lie}G')$, and the torsion is $\Theta := \tau(\Omega)$. The curvature is a $\text{Ad}_{C(H)}$-tensorial form so D acts on it, but trivially so, which gives a Bianchi identity: $D\Omega = d\Omega + [\varpi, \Omega] = 0$.

Gauge transformations are of course given by the action of $\text{Aut}_v(\mathcal{P}, H) \simeq \mathcal{H}$. On account of (I) and (II), the twisted Cartan connection transforms as: $\varpi^\gamma := \Phi^*\varpi = C(\gamma)^{-1}\varpi\, C(\gamma) + C(\gamma)^{-1}dC(\gamma)$. We already know that for twisted tensorial forms $\alpha^\gamma := \Phi^*\alpha = \rho\,[C(\gamma)]^{-1}\,\alpha$. In particular, this gives the gauge transformations of the curvature, the torsion and the soldering form: $\Omega^\gamma = C(\gamma)^{-1}\Omega\, C(\gamma)$, $\Theta^\gamma = C(\gamma)^{-1}\Theta\, C(\gamma)$ and $\theta^\gamma = C(\gamma)^{-1}\theta\, C(\gamma)$.

8.1. Mixed Cartan connection

The previous construction easily extends to the mixed case. Consider the principal bundle $\mathcal{P}(\mathcal{M}, H \times K)$ with a cocycle $C : \mathcal{P} \times H \to G$. A mixed Cartan connection is $\varpi \in \Omega^1(\mathcal{P}, \text{Lie}G')$, with $\text{Lie}G' \supset \text{Lie}(G \rtimes K)$ such that $\dim \text{Lie}G'/\text{Lie}(G \rtimes K) = \dim\mathcal{M}$, satisfying:

$$\text{(I}^\star\text{)} \qquad \begin{aligned} \varpi_p(X^v_p + Y^v_p) &= \tfrac{d}{d\tau}C_p\left(e^{\tau X}\right)\big|_{\tau=0} + Y \\ &= dC_{p|e}(X) + Y \in \text{Lie}(G \rtimes K), \end{aligned}$$

$$\text{(II}^\star\text{)} \qquad \begin{aligned} R^*_{hk}\varpi_p phk = R^*_{kh}\varpi_p pkh &= [C_p(h)k]^{-1}\varpi_p\,[C_p(h)k] \\ &\quad + [C_p(h)k]^{-1}d[C(h)k]_{|p}, \end{aligned}$$

$$\text{(III}^\star\text{)} \qquad \ker\varpi_p = \emptyset, \quad \forall p \in \mathcal{P}, \text{and} \quad \text{Lie}G'/\text{Lie}(G \rtimes K) \subset \text{Im}\,\varpi_p.$$

The first two axioms are those of a mixed connection, the third is a slightly extended version of the one satisfied by a twisted Cartan connection and ensures that, mutadis mutandis, the result of Proposition 13 generalises as $T\mathcal{M} \simeq \mathcal{P} \times_{\text{Ad}_{C(H)\rtimes K}} \text{Lie}G'/\text{Lie}(G \rtimes K)$. As above, the projection $\tau : \text{Lie}G' \to \text{Lie}G'/\text{Lie}(G \rtimes K)$, allows to define the soldering form $\theta \in \Omega^1\big(\mathcal{P}, C(H) \rtimes K\big)$. Given a $(\text{Lie}G', G)$-module V, the space $\Omega^\bullet_{\text{tens}}\big(\mathcal{P}, C(H) \rtimes K\big)$ of V-valued mixed tensorial forms is defined as in section 6.1. A mixed Cartan connection, on account of (I*) and (II*), induces an exterior covariant derivative $D := d + \rho'_*(\varpi)$ on this space. The curvature $\Omega \in \Omega^2_{\text{tens}}\big(\mathcal{P}, C(H) \rtimes K\big)$ satisfies Bianchi, $D\Omega = 0$, and the torsion is still $\Theta := \tau(\Omega)$. On account of (I*) and (II*), the gauge transformation of the mixed Cartan connection under $\text{Aut}_v(\mathcal{P}) \simeq \mathcal{H} \times \mathcal{K}$ is, $\varpi^{\gamma\zeta} := \Psi^*\varpi = [C(\gamma)\zeta]^{-1}\varpi\,[C(\gamma)\zeta] + [C(\gamma)\zeta]^{-1}d[C(\gamma)\zeta]$. For mixed tensorial forms it is given by (36) of Proposition 9, which gives the gauge transformations of the curvature, the torsion and the soldering form as special cases.

8.2. Reductive and parabolic mixed Cartan geometries

Following a well established nomenclature for Cartan geometries [22, 23], we signal and briefly characterise two notable classes of mixed Cartan geometries.

If there is an Ad_G- invariant decomposition $\mathrm{Lie}G' = \mathrm{Lie}G + V^n$, we call the mixed Cartan geometry *reductive*. In this case, there is clean splitting of the mixed Cartan connection $\varpi = \omega + \theta$ into a mixed connection ω and a mixed soldering form θ. The curvature splits in the same way as the sum of the curvature of ω and the torsion, given by $\Theta = d\theta + [\omega, \theta]$.

If G' is semi-simple and G is a parabolic subgroup corresponding to a $|k|$-grading of $\mathrm{Lie}G' = \bigoplus\limits_{-k \leq i \leq k} \mathfrak{g}'_i$, s.t. $[\mathfrak{g}'_i, \mathfrak{g}'_j] \subset \mathfrak{g}'_{i+j}$, the associated mixed Cartan geometry will be called *parabolic*. Both the mixed Cartan connection and its curvature split along the $|k|$-grading.

9. Twisted gauge theories

Concerning applications of this twisted/mixed geometry to physics, the local representatives are to be regarded as generalised gauge fields: They satisfy the gauge principle while being of a different geometric nature than standard gauge fields. Given the usual ingredients, it is easy to come up with Lagrangian functionals that specify twisted gauge theories in a way that exactly parallels the construction of Yang-Mills or gravity gauge theories.

To give prototypical examples, consider the Killing form on $\mathrm{Lie}G$ or $\mathrm{Lie}G'$ - which, for special linear groups, reduces to the trace operator, $B(XY) = \mathrm{Tr}(XY)$ for $X, Y \in \mathrm{Lie}G^{(\prime)}$. Suppose also that the representation maps ρ'_*, ρ of the $(\mathrm{Lie}G', G)$-module V are unitary for some bilinear form $\langle\, ,\, \rangle$ on V. Finally, suppose $\mathcal{M}$ endowed with a metric g, giving a Hodge operator $* : \Omega^\bullet(\mathcal{M}) \to \Omega^{m-\bullet}(\mathcal{M})$. Then, one can write the gauge-invariant Lagrangian,

$$L(A, \phi) = \tfrac{1}{2}\,\mathrm{Tr}\,(F \wedge *F) + \langle D\phi, *D\phi\rangle + U\left(\langle \phi, \phi\rangle\right) *\mathbb{1},$$

where U is some potential. Or, if ϕ is a spinor (fermions), so that we note it ψ instead, we can write:

$$L(A, \psi) = \tfrac{1}{2}\,\mathrm{Tr}\,(F \wedge *F) + \langle \psi, \not{D}\psi\rangle - m\langle \psi, *\psi\rangle.$$

with $\not{D} = \gamma \wedge *D$ where $\gamma = \gamma_a\, e^a{}_\mu\, dx^\mu = \gamma_\mu\, dx^\mu$ is the Dirac gamma matrices-valued 1-form (with $e^a{}_\mu$ the vielbein associated to the metric g). It describes the dynamics of the twisted gauge potential A coupled to a twisted Dirac field of mass m. Quite obviously, the field equations for A and ψ obtained from the action $S(A, \phi) = \int_{\mathcal{M}} L(A, \phi)$, are respectively Yang-Mills' equation sourced by ψ - $D * F = J(\psi)$ - and Dirac's equation - $(\not{D} - m)\psi = 0$.

10. Applications: Conformal tractors, twistors, and anomalies in QFT

Tractors are sections of a $n+2$-dim real vector bundle over a conformal n-manifold $(\mathcal{M}, [g])$, the tractor bundle $\mathcal{T}$. These sections have a special covariance under Weyl rescaling of the metric, that is under change of metric within the conformal class. This bundle is endowed with a covariant derivative often called the tractor connection $\nabla^{\mathcal{T}}$. As a matter of fact, $\left(\mathcal{T}, \nabla^{\mathcal{T}}\right)$ is the basis of a tractor calculus which is the analogue for conformal manifolds of the tensorial calculus on (pseudo) Riemannian manifolds [24]. Recently, it has found some applications in physics, especially General Relativity [25].

Twistors are sections of a 4-dim complex vector bundle over a conformal 4-manifold, the (local) twistor bundle $\mathbb{T}$. They have special covariance under Weyl rescaling, so $\mathbb{T}$ is endowed with a covariant derivative $\nabla^{\mathbb{T}}$ - the twistor connection - and $\left(\mathbb{T}, \nabla^{\mathbb{T}}\right)$ is the basis of a twistorial calculus which is the analogue of spinorial calculus on (pseudo) Riemannian manifolds [26, 27]. Penrose originally devised twistors with physics purpose in mind, and nowadays they found renewed relevance in string theory and loop quantum gravity [28].

Twistors are the spin version of $n = 4$ tractors. Initially, and still in many standard presentations, both tractors and twistors are constructed in a similar way: bottom-up, by prolongation of differential equations. Starting on a conformal manifold, one defines the so-called Almost Einstein (AE) equation and twistor equation, which are prolonged into closed linear systems. These are rewritten as linear operators acting on multiplets of variables (the parallel tractors and global twistors respectively). The special covariance of these multiplets under Weyl rescaling is given by definition and commute with the action of their respective associated linear operators, which are thus called tractor and twistor connections. The multiplets are interpreted as parallel sections of vector bundles, the tractor and (local) twistor bundles. One can consult [24, 25] for the details of this procedure in the tractor case, and the reference text [27] for the twistor case (or [29] where the case of paraconformal (PCF) manifolds is treated in a very similar fashion).

The fact that $\mathcal{T}$ and $\mathbb{T}$ had a close relationship with the principal bundle of the conformal Cartan geometry [22] did not go unnoticed [24, 30]. In the modern treatment of Cartan geometries, the term tractor assume a broader meaning: Bundles associated to a Cartan geometry via a restriction of a representation of $G' \supset H$ are termed tractor bundles, and for parabolic Cartan geometries the covariant derivatives on these bundles induced by a Cartan connection are called tractor connections (see [23] sections 1.5.7 and 3.1.22). But then the original meaning is only recovered via special sections of the underlying Cartan principal bundle known as Weyl structures ([23], section 5.1). In the cases under consideration, the bottom-up procedure via prolongation described above has been occasionally deemed more explicit [29], or more intuitive and direct [24] than the viewpoint in terms of vector bundles associated to the conformal Cartan geometry. But this constructive approach requires a rather significant amount of computation.

An alternative top-down approach is proposed in [31, 32] where $\mathcal{T}$ and $\mathbb{T}$ are explicitly constructed via gauge symmetry reduction of the real and complex vector bundles naturally associated to the conformal Cartan geometry, while $\nabla^{\mathcal{T}}$ and $\nabla^{\mathbb{T}}$ are shown to be induced by the reduced Cartan connection. This alternative constructive procedure relies on a general scheme of gauge symmetry reduction known as the *dressing field method* [33]. It is computationally more economical than the bottom-up approach via prolongation, and allows to generalise tractors and twistors to manifolds with torsion. It preserves the insight of the more abstract modern treatment - by articulating how $\mathcal{T}$ and $\mathbb{T}$ are associated to the conformal Cartan geometry - yet it is more user friendly.[3] Finally, most relevant for the present work, this method hints at the fact that tractors and twistors can be seen as simple, and somewhat degenerated, examples of the twisted/mixed geometry elaborated above. This last point is elaborated in the next two subsections.

After a third section where we comment briefly on conformal gravity, in a fourth and final subsection we show how the twisted geometry arises naturally in the very definition of anomalies in quantum gauge field theory, and therefore ought be relevant to their study.

[3]For an explanation of the relation between the dressing fiel method and the notion of Weyl structure, one can consult appendix A in [34].

10.1. The case of conformal tractors

Consider a conformal 4-manifold $(\mathcal{M},[g])$. A tractor is a map $\varphi:\mathcal{U}\subset\mathcal{M}\to\mathbb{R}^6$, $x\mapsto\varphi(x)=\left(\begin{smallmatrix}\rho\\ \ell\\ \sigma\end{smallmatrix}\right)$ with $\ell=\ell^a\in\mathbb{R}^4$, $\rho\in\mathbb{R}$ and $\sigma\in\mathbb{R}_*$. The (generalised) tractor connection is $\varpi=\left(\begin{smallmatrix}0 & P & 0\\ e & A & P^t\\ 0 & e^t & 0\end{smallmatrix}\right)$, where $A\in\Omega^1\big(\mathcal{U},\mathfrak{so}(1,3)\big)$, $e=e^a{}_\mu dx^\mu\in\Omega^1\big(\mathcal{U},\mathbb{R}^4\big)$ is the soldering form, and $P\in\Omega^1\big(\mathcal{U},\mathbb{R}^{4*}\big)$. The operation $|^t$ is the transposition via the Minkowski metric η. The tractor connection enters the definition of the tractor derivative $D\varphi=d\varphi+\varpi\varphi$. The tractor curvature is $\bar\Omega=\left(\begin{smallmatrix}f & \mathsf{C} & 0\\ \mathsf{T} & \mathsf{W} & \mathsf{C}^t\\ 0 & \mathsf{T}^t & -f\end{smallmatrix}\right)$, where $\mathsf{W}\in\Omega^2\big(\mathcal{U},\mathfrak{so}(1,3)\big)$, $\mathsf{T}\in\Omega^2\big(\mathcal{U},\mathbb{R}^4\big)$ is the torsion, $\mathsf{C}\in\Omega^2\big(\mathcal{U},\mathbb{R}^{4*}\big)$, and $f\in\Omega^2\big(\mathcal{U},\mathbb{R}\big)$ is such that $f_{ab}=P_{[ab]}$.

If one imposes the conditions $\text{Ricc}(\mathsf{W}):=\mathsf{W}^a{}_{bac}=0$ and $\mathsf{T}=0$, several consequences follow. First A is the Lorentz/spin connection (expressed as a function of the components of e). Then $f=0$ (by the Bianchi identify) and P is the Schouten 1-form (with components the symmetric Schouten tensor). Finally, W and C are the Weyl and Cotton 2-forms. In this case, ϖ is the standard tractor connection.

The (local) gauge group of Weyl rescalings is $\mathcal{W}:=\big\{z:\mathcal{U}\in\mathcal{M}\to\mathbb{R}^+_*\mid z^{z'}=z'^{-1}zz'=z\big\}$. The Weyl gauge transformations of the above variables are obtained via elements of the form

$$C(z)=\begin{pmatrix}z & \Upsilon(z) & {}^{z^{-1}}\!/2\,\Upsilon(z)\Upsilon(z)^t\\ 0 & \mathbb{1} & z^{-1}\Upsilon(z)^t\\ 0 & 0 & z^{-1}\end{pmatrix},\qquad\text{where}\quad \Upsilon(z)=\Upsilon(z)_a:=z^{-1}\partial_\mu z\;e^\mu{}_a\in\mathbb{R}^{4*}.\tag{54}$$

We have indeed,

$$\begin{aligned}&\varphi^z=C(z)^{-1}\varphi, &&\varpi^z=C(z)^{-1}\varpi\,C(z)+C(z)^{-1}dC(z),\\ &(D\varphi)^z=D^z\varphi^z=C(z)^{-1}D\varphi, &&\bar\Omega^z=C(z)^{-1}\bar\Omega\,C(z).\end{aligned}\tag{55}$$

It is easily verified that for $z,z'\in\mathcal{W}$ one has $C(z)^{z'}=C(z')^{-1}C(zz')$ (because $e^z=ze$), which is an instance of (24), local version of the gauge transformation law of a cocycle (9). So that (55) is indeed a special case of (25)-(27), local version of the gauge transformations of twisted fields as described in Proposition 3 and 4. We conclude that tractor variables are indeed twisted gauge fields w.r.t. the Weyl gauge group $\mathcal{W}$.

The (local) Lorentz gauge group is $\mathcal{SO}:=\Big\{\mathsf{S}=\left(\begin{smallmatrix}1&0&0\\0&S&0\\0&0&1\end{smallmatrix}\right),\ S:\mathcal{U}\to SO(1,3)\mid \mathsf{S}^{\mathsf{S}'}=\mathsf{S}'^{-1}\mathsf{S}\mathsf{S}'\Big\}$. The Lorentz gauge transformations of the tractor variables are,

$$\varphi^{\mathsf{S}} = \mathsf{S}^{-1}\varphi, \qquad \varpi^{\mathsf{S}} = \mathsf{S}^{-1}\varpi\,\mathsf{S} + \mathsf{S}^{-1}d\mathsf{S},$$
$$(D\varphi)^{\mathsf{S}} = D^{\mathsf{S}}\varphi^{\mathsf{S}} = \mathsf{S}^{-1}D\varphi, \qquad \bar{\Omega}^{\mathsf{S}} = \mathsf{S}^{-1}\bar{\Omega}\,\mathsf{S}.$$

Which means that they are standard gauge fields w.r.t. $\mathcal{SO}$. By the way, it is easily verified that $C(z)^{\mathsf{S}} = \mathsf{S}^{-1}C(z)\,\mathsf{S}$ (because $e^S = S^{-1}e$), which is a special case of (41), local version of (34). This ensures that the actions of $\mathcal{W}$ and $\mathcal{SO}$ on the tractor variables commute and that we have,

$$\varphi^{\,z\mathsf{S}} = [C(z)\mathsf{S}]^{-1}\varphi, \qquad \varpi^{\,z\mathsf{S}} = [C(z)\mathsf{S}]^{-1}\varpi\,[C(z)\mathsf{S}] + [C(z)\mathsf{S}]^{-1}d[C(z)\mathsf{S}],$$
$$(D\varphi)^{\,z\mathsf{S}} = D^{\,z\mathsf{S}}\varphi^{\,z\mathsf{S}} = [C(z)\mathsf{S}]^{-1}D\varphi, \qquad \bar{\Omega}^{\,z\mathsf{S}} = [C(z)\mathsf{S}]^{-1}\bar{\Omega}\,[C(z)\mathsf{S}],$$

that is a special case of (42)-(44), local version of the gauge transformations for mixed gauge fields as given in Proposition 9. We conclude that tractor variables are mixed gauge fields w.r.t. the gauge group $\mathcal{W} \times \mathcal{SO}$.

From the above local construct we can attempt to recover the global twisted geometry that it stems from. Consider the bundle $\mathcal{P}\big(\mathcal{M}, W \times SO(1,3)\big)$, with $W := \mathbb{R}^+_*$, as well as a cocycle map $C : \mathcal{P} \times W \to G$ where the target group is defined as,

$$G = \left\{ \begin{pmatrix} 1 & r & 1/2\, rr^t \\ 0 & \mathbb{1} & r^t \\ 0 & 0 & 1 \end{pmatrix} \rtimes \begin{pmatrix} z & 0 & 0 \\ 0 & \mathbb{1} & 0 \\ 0 & 0 & z^{-1} \end{pmatrix} = \begin{pmatrix} z & r & z^{-1}/2\, rr^t \\ 0 & \mathbb{1} & z^{-1}r^t \\ 0 & 0 & z^{-1} \end{pmatrix} \,\middle|\, r \in \mathbb{R}^{4*}, z \in W \right\}.$$

For $z \in \mathcal{W}$, the local cocycle (54) is a map $C(z) : \mathcal{U} \to G$. Consider also the (inner) semidirect product group

$$G \rtimes SO := \left\{ \begin{pmatrix} z & r & z^{-1}/2\, rr^t \\ 0 & \mathbb{1} & z^{-1}r^t \\ 0 & 0 & z^{-1} \end{pmatrix} \rtimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} z & rS & z^{-1}/2\, rr^t \\ 0 & S & z^{-1}r^t \\ 0 & 0 & z^{-1} \end{pmatrix} \,\middle|\, S \in SO(1,3) \right\},$$

where the group morphism $SO \to \mathrm{Aut}(G)$ defining the semidirect structure is $S \mapsto \mathrm{Conj}(S)$. Its Lie algebra is

$$\mathrm{Lie}(G \rtimes SO) = \left\{ \begin{pmatrix} \varepsilon & \iota & 0 \\ 0 & s & \iota^t \\ 0 & 0 & -\varepsilon \end{pmatrix} \,\middle|\, \varepsilon \in \mathbb{R}^+_*, s \in \mathfrak{so}(1,3), \iota \in \mathbb{R}^{4*} \right\}.$$

Clearly, the tractor connection takes value in the bigger Lie algebra $\mathrm{Lie}G' = \mathrm{Lie}(G \rtimes SO) \oplus \mathbb{R}^4 = \left\{ \begin{pmatrix} \varepsilon & \iota & 0 \\ \tau & s & \iota^t \\ 0 & \tau^t & -\varepsilon \end{pmatrix} \,\middle|\, \tau \in \mathbb{R}^4 \right\}$, and $\dim\ \mathrm{Lie}G'/\mathrm{Lie}(G \rtimes SO) = \dim\mathcal{M}$. Therefore, it appears that the tractor connection ϖ is a local mixed

Cartan connection as defined in section 8.1. By the way, there is a decomposition of LieG' that shows it is graded:

$$\begin{aligned}
&\text{Lie}G' = \mathfrak{g}'_{-1} + \mathfrak{g}'_0 + \mathfrak{g}'_1 = \left\{ \left(\begin{smallmatrix} 0 & 0 & 0 \\ \tau & 0 & 0 \\ 0 & \tau^t & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} \varepsilon & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & -\varepsilon \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & \iota & 0 \\ 0 & 0 & \iota^t \\ 0 & 0 & 0 \end{smallmatrix}\right) \mid \dots \right\}, \\
&[\mathfrak{g}'_i, \mathfrak{g}'_j] \in \mathfrak{g}'_{i+j},
\end{aligned}$$

and $\text{Lie}(G \rtimes SO) = \mathfrak{g}'_0 + \mathfrak{g}'_1$ is a parabolic subalgebra. So tractor geometry is an instance of parabolic mixed Cartan geometry. In particular the tractor bundle is a mixed vector bundle associated to $\mathcal{P}$, $\mathcal{T} = \mathcal{P} \times_{C(W) \rtimes SO} \mathbb{R}^6$, as defined in section 6.1.

Notice however that, in view of (54), the $\mathcal{P}$-dependance of the cocycle map $C : \mathcal{P} \times W \to G$ is localised in the coefficients of the soldering form $({e^a}_\mu)$ in the definition of $\Upsilon(z)$, which also contains a derivative of z. So, for constant $z \in W$ the $\mathcal{P}$-dependance vanishes, and the cocycle reduces to a group morphism $C : W \to G$, $z \mapsto C(z) = \left(\begin{smallmatrix} z & 0 & 0 \\ 0 & \mathbb{1} & 0 \\ 0 & 0 & z^{-1} \end{smallmatrix}\right)$. Then, at the level of $W \times SO$-equivariance, the W-twisted side of tractor geometry seems to degenerate into a standard non-twisted case. Only at the level of $\mathcal{W} \times \mathcal{SO}$-action/gauge transformations does the cocycle structure, and the $\mathcal{W}$-twisted geometry, is manifest. Because of this, the fact that tractors are mixed gauge fields and provide an instance of a new type of geometry might go unnoticed.[4] The same goes for twistors, as we now show.

10.2. The case of local twistors

We follow closely the treatment given in the previous section. But first, let us start by reminding some basic results and fix our notations.

Denote Minkowski space by $\mathsf{M} := (\mathbb{R}^4, \eta)$, and consider $\left\{\sigma_a^{AA'}\right\}_{a=0,\dots,3}$, a basis of 2×2 hermitian matrices $\text{Herm}(2,\mathbb{C}) = \{M \in M_2(\mathbb{C}) \mid M^* = M\}$, where $*$ denote Hermitian transposition. There is a vector space isomorphism $\mathsf{M} \to \text{Herm}(2,\mathbb{C})$, $x = x^a \mapsto \bar{x} = \bar{x}^{AA'} := x^a \sigma_a^{AA'} = \frac{1}{2}\left(\begin{smallmatrix} x^0+x^3 & x^1-ix^2 \\ x^1+ix^2 & x^0-x^3 \end{smallmatrix}\right)$. Upper case Latin letters are Weyl spinor indices, with values 0 and 1. The spacetime interval is then given by $x^T\eta x = 4\det(\bar{x})$. The isomorphism for the dual is $\mathbb{R}^{4*} \to \text{Herm}(2,\mathbb{C})$, $r = x^t = x^T\eta \mapsto \bar{r} := x^0\sigma_0 - x^i\sigma_i = \frac{1}{2}\left(\begin{smallmatrix} x^0-x^3 & -x^1+ix^2 \\ -x^1-ix^2 & x^0+x^3 \end{smallmatrix}\right)$.

[4]From a physicist's point of view it is clear that something unusual is going on, because gauge elements $C(z)$ of type (54) clearly cannot come from the mere gauging of a Lie group.

Correspondingly, we have the double cover group morphism $SL(2,\mathbb{C}) \to SO(1,3)$, $\pm\bar{S} \mapsto S$. It is a spin representation of the Lorentz group. So the action $SO(1,3) \times \mathsf{M} \to \mathsf{M}$, $(S,x) \mapsto Sx$ preserving η, is represented by the action $SL(2,\mathbb{C}) \times \mathrm{Herm}(2,\mathbb{C}) \to \mathrm{Herm}(2,\mathbb{C})$, $(\bar{S},\bar{x}) \mapsto \bar{S}\bar{x}\bar{S}^*$ preserving det. The associated Lie algebra isomorphism $\mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{so}(1,3)$, $\bar{s} \mapsto s$, implies that the action $\mathfrak{so}(1,3) \times \mathsf{M} \to \mathsf{M}$, $(s,x) \mapsto sx$, is represented by the action $\mathfrak{sl}(2,\mathbb{C}) \times \mathrm{Herm}(2,\mathbb{C}) \to \mathrm{Herm}(2,\mathbb{C})$, $(\bar{s},\bar{x}) \mapsto \bar{s}\bar{x} + \bar{x}\bar{s}^*$.

On a conformal 4-manifold $\mathcal{M}$, a (local) twistor is a map $\psi : \mathcal{U} \subset \mathcal{M} \to \mathbb{C}^4$, $x \mapsto \psi(x) = \left(\begin{smallmatrix} \pi \\ \omega \end{smallmatrix}\right)$ with $\pi, \omega \in \mathbb{C}^2$ dual Weyl spinors. The (generalised) twistor connection is $\bar{\varpi} = \left(\begin{smallmatrix} -\bar{A}^* & -i\,\bar{P} \\ i\,\bar{e} & \bar{A} \end{smallmatrix}\right)$, with $\bar{A} \in \Omega^1\big(\mathcal{U}, \mathfrak{sl}(2,\mathbb{C})\big)$ and $\bar{e}, \bar{P} \in \Omega^1\big(\mathcal{U}, \mathrm{Herm}(2,\mathbb{C})\big)$. It enters the definition of the twistor derivative (or twistor transport), $\bar{D} := d + \bar{\varpi}$. The twistor curvature is

$$\bar{\Omega} = \left(\begin{smallmatrix} -\bar{\mathsf{W}}^* + f/2\mathbb{1}_2 & -i\,\bar{\mathsf{C}} \\ i\,\bar{\mathsf{T}} & \bar{\mathsf{W}} - f/2\mathbb{1}_2 \end{smallmatrix}\right),$$

with $\bar{\mathsf{W}} \in \Omega^2\big(\mathcal{U}, \mathfrak{sl}(2,\mathbb{C})\big)$, $\bar{\mathsf{T}}, \bar{\mathsf{C}} \in \Omega^2\big(\mathcal{U}, \mathrm{Herm}(2,\mathbb{C})\big)$, and $f \in \Omega^2\big(\mathcal{U}, \mathbb{R}\big)$. Here again, imposing $\mathrm{Ricc}(\bar{\mathsf{W}}) = 0$ and $\bar{\mathsf{T}} = 0$ implies that $\bar{A}$ is the spin connection, $f = 0$, and $\bar{P}, \bar{\mathsf{C}}, \bar{\mathsf{W}}$ are the Schouten, Cotton and Weyl tensors. So that $\bar{\varpi}$ is the standard twistor connection.

The (local) Weyl gauge transformations of these twistor variables are obtained via elements of type

(56)
$$\bar{C}(z) = \begin{pmatrix} z^{1/2} & -i\,z^{-1/2}\,\bar{\Upsilon}(z) \\ 0 & z^{-1/2} \end{pmatrix}, \quad \text{where} \quad \bar{\Upsilon}(z) = \Upsilon(z)_{AA'} \in \mathrm{Herm}(2,\mathbb{C}),$$

so that,

$$\begin{aligned} &\psi^z = \bar{C}(z)^{-1}\psi, \qquad \bar{\varpi}^z = \bar{C}(z)^{-1}\bar{\varpi}\,\bar{C}(z) + \bar{C}(z)^{-1}d\bar{C}(z), \\ &(\bar{D}\psi)^z = \bar{D}^z\psi^z = \bar{C}(z)^{-1}\bar{D}\psi, \qquad \bar{\Omega}^z = \bar{C}(z)^{-1}\bar{\Omega}\,\bar{C}(z). \end{aligned} \tag{57}$$

One verifies again that given $z, z' \in \mathcal{W}$, one has $\bar{C}(z)^{z'} = \bar{C}(z')^{-1}\bar{C}(zz')$, as instance of (24) and a local version of (9). It follows that (57) is a special case of (25) -(27), local versions of Propositions 3 and 4. Twistor variables are therefore twisted gauge fields w.r.t. the Weyl gauge group $\mathcal{W}$.

The (local) spin gauge group is

$$\mathcal{SL} := \left\{ \bar{\mathsf{S}} = \left(\begin{smallmatrix} \bar{S}^{-1*} & 0 \\ 0 & \bar{S} \end{smallmatrix}\right),\ \bar{S} : \mathcal{U} \to SL(2,\mathbb{C}) \,\middle|\, \bar{\mathsf{S}}^{\bar{\mathsf{S}}'} = \bar{\mathsf{S}}'^{-1}\bar{\mathsf{S}}\bar{\mathsf{S}}' \right\}.$$

The spin gauge transformations of the twistor variables are,

$$\begin{aligned} &\psi^{\bar{\mathsf{S}}} = \bar{\mathsf{S}}^{-1}\psi, \qquad \bar{\varpi}^{\bar{\mathsf{S}}} = \bar{\mathsf{S}}^{-1}\bar{\varpi}\,\bar{\mathsf{S}} + \bar{\mathsf{S}}^{-1}d\bar{\mathsf{S}}, \\ &(\bar{D}\psi)^{\bar{\mathsf{S}}} = D^{\bar{\mathsf{S}}}\psi^{\bar{\mathsf{S}}} = \bar{\mathsf{S}}^{-1}\bar{D}\psi, \qquad \bar{\Omega}^{\bar{\mathsf{S}}} = \bar{\mathsf{S}}^{-1}\bar{\Omega}\,\bar{\mathsf{S}}. \end{aligned}$$

so they are standard gauge fields w.r.t. $\mathcal{SL}$. Also, one verifies that $\bar{C}(z)^{\bar{\mathsf{S}}} = \bar{\mathsf{S}}^{-1}\bar{C}(z)\,\bar{\mathsf{S}}$, a special case of (41) and a local version of (34). So the actions of $\mathcal{W}$ and $\mathcal{SO}$ on the twistor variables commute and we have,

$$\begin{aligned} &\psi^{\,z\bar{\mathsf{S}}} = [\bar{C}(z)\bar{\mathsf{S}}]^{-1}\psi, \qquad \bar{\varpi}^{\,z\bar{\mathsf{S}}} = [\bar{C}(z)\bar{\mathsf{S}}]^{-1}\bar{\varpi}\,[\bar{C}(z)\bar{\mathsf{S}}] + [\bar{C}(z)\bar{\mathsf{S}}]^{-1}d[\bar{C}(z)\bar{\mathsf{S}}], \\ &(\bar{D}\psi)^{\,z\bar{\mathsf{S}}} = \bar{D}^{\,z\bar{\mathsf{S}}}\psi^{\,z\bar{\mathsf{S}}} = [\bar{C}(z)\bar{\mathsf{S}}]^{-1}\bar{D}\psi, \qquad \bar{\Omega}^{\,z\bar{\mathsf{S}}} = [\bar{C}(z)\bar{\mathsf{S}}]^{-1}\bar{\Omega}\,[\bar{C}(z)\bar{\mathsf{S}}], \end{aligned}$$

as special case of (42)-(44) and local version of Proposition 9. Thus, like tractors, twistor variables are mixed gauge fields w.r.t. the gauge group $\mathcal{W} \times \mathcal{SL}$.

Again, we can guess the global twisted geometry from the local data. Consider the bundle $\mathcal{P}\big(\mathcal{M},\, W \times SL(2,\mathbb{C})\big)$, and a cocycle $C : \mathcal{P} \times W \to \bar{G}$ where $\bar{G} = \left\{ \left(\begin{smallmatrix} \mathbb{1} & -i\,\bar{r} \\ 0 & \mathbb{1} \end{smallmatrix}\right) \rtimes \left(\begin{smallmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{smallmatrix}\right) = \left(\begin{smallmatrix} z^{1/2} & -i\,z^{-1/2}\,\bar{r} \\ 0 & z^{-1/2} \end{smallmatrix}\right) \,\middle|\, \bar{r} \in \mathrm{Herm}(2,\mathbb{C}), z \in W \right\}$. For $z \in \mathcal{W}$, the local cocycle (56) is a map $\bar{C}(z) : \mathcal{U} \to \bar{G}$. Consider also the semidirect product group

$$\begin{aligned} \bar{G} \rtimes SL := \Big\{ &\left(\begin{smallmatrix} z^{1/2} & -i\,z^{-1/2}\,\bar{r} \\ 0 & z^{-1/2} \end{smallmatrix}\right) \rtimes \left(\begin{smallmatrix} \bar{S}^{-1*} & 0 \\ 0 & \bar{S} \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} z^{1/2}\bar{S}^{-1*} & -i\,z^{-1/2}\,\bar{r}\bar{S} \\ 0 & z^{-1/2}\bar{S} \end{smallmatrix}\right) \,\Big|\, \bar{S} \in SL(2,\mathbb{C}) \Big\}, \end{aligned}$$

where the group morphism $SL \to \mathrm{Aut}(\bar{G})$ of the semidirect structure is $\bar{S} \mapsto \mathrm{Conj}(\bar{S})$. Its Lie algebra is $\mathrm{Lie}(\bar{G} \rtimes SL) = \left\{ \left(\begin{smallmatrix} -(\bar{s}-\varepsilon/2\mathbb{1})^* & -i\,\bar{\iota} \\ 0 & \bar{s}-\varepsilon/2\mathbb{1} \end{smallmatrix}\right) \,\middle|\, \varepsilon \in \mathbb{R}^+_*, s \in \mathfrak{sl}(2,\mathbb{C}), \bar{\iota} \in \mathrm{Herm}(2,\mathbb{C}) \right\}$. The twistor connection manifestly takes value in a bigger Lie algebra,

$$\begin{aligned}\mathrm{Lie}\bar{G}' &= \mathrm{Lie}(\bar{G} \rtimes SL) \oplus \mathrm{Herm}(2,\mathbb{C}) \\ &= \left\{ \begin{pmatrix} -(\bar{s}-\varepsilon/2\mathbb{1})^* & -i\,\bar{\iota} \\ i\,\bar{\tau} & \bar{s}-\varepsilon/2\mathbb{1} \end{pmatrix} \,\middle|\, \bar{\tau} \in \mathrm{Herm}(2,\mathbb{C}) \right\},\end{aligned}$$

and dim $\mathrm{Lie}\bar{G}'/\mathrm{Lie}(\bar{G} \rtimes SL) = \dim\mathcal{M}$. Therefore, the twistor connection $\bar{\varpi}$ is a local mixed Cartan connection. Obviously, like its real counterpart, $\mathrm{Lie}\bar{G}'$ is graded:

$$\begin{aligned}&\mathrm{Lie}\bar{G}' = \bar{\mathfrak{g}}'_{-1} + \bar{\mathfrak{g}}'_0 + \bar{\mathfrak{g}}'_1 = \left\{ \begin{pmatrix} 0 & 0 \\ -i\,\bar{\tau} & 0 \end{pmatrix} + \begin{pmatrix} -(\bar{s}-\varepsilon/2\mathbb{1})^* & 0 \\ 0 & \bar{s}-\varepsilon/2\mathbb{1} \end{pmatrix} + \begin{pmatrix} 0 & -i\,\bar{\iota} \\ 0 & 0 \end{pmatrix} \,\middle|\, \dots \right\}, \\ &[\bar{\mathfrak{g}}'_i, \bar{\mathfrak{g}}'_j] \in \bar{\mathfrak{g}}'_{i+j},\end{aligned}$$

with $\mathrm{Lie}(\bar{G} \rtimes SL) = \bar{\mathfrak{g}}'_0 + \bar{\mathfrak{g}}'_1$ a parabolic subalgebra. So, twistor geometry is an instance of parabolic mixed Cartan geometry, and the twistor bundle is a mixed vector bundle $\mathbb{T} = \mathcal{P} \times_{\bar{C}(W) \rtimes SL} \mathbb{C}^4$.

Notice again that, as in the tractor case, given (54), the $\mathcal{P}$-dependance of the cocycle map $\bar{C} : \mathcal{P} \times W \to \bar{G}$ comes from the coefficients of the soldering form $({e^a}_\mu)$ entering the definition of $\bar{\Upsilon}(z)$, containing a derivative of z. For constant $z \in W$ the cocycle then reduces to a group morphism $\bar{C} : W \to \bar{G}$, $z \mapsto \bar{C}(z) = \begin{pmatrix} z^{1/2}\mathbb{1} & 0 \\ 0 & z^{-1/2}\mathbb{1} \end{pmatrix}$. Again, at the level of $W \times SO$-equivariance the W-twisted side of twistor geometry is degenerated, and only at the level of $\mathcal{W} \times \mathcal{SO}$-gauge transformations does the cocycle structure and the $\mathcal{W}$-twisted geometry are manifest.

10.3. Conformal gravity as a twisted gauge theory

An attempt to interpret local twistors and the twistor covariant derivative as gauge fields in the spirit of Yang-Mills theory, compatible with the gauge principle of field theory, was first proposed in [35]. This work is cited in the reference text of Penrose and Rindler [27].[5] From the above considerations, it appears clearly that actually twistors better fit in the generalised geometry developed in this paper, and are therefore not Yang-Mills gauge fields, but rather twisted/mixed gauge fields - that indeed provide a new satisfying instantiation of the gauge principle.

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[5]The footnote p.133 reads, "*Local twistors [...] can under certain circonstances be thought of as defining a kind of Yang-Mills theory, cf Merkulov (1984)(Bach tensor current)*.".

It was further shown in [35] that the Yang-Mills equation for the twistor connection $\bar{\varpi}$ reproduces the Bach equation of conformal (or Weyl) gravity, and that upon examination, the Yang-Mills type Lagrangian for $\bar{\varpi}$ is indeed the Weyl tensor-squared Lagrangian of conformal gravity.

This is most clearly understood in the context of Section 9. First, define the Killing forms $\bar{B}$ and B on Lie$\bar{G}'$ and LieG'. Given $\bar{M}, \bar{N} \in$ Lie$\bar{G}'$, $\bar{B}(\bar{M}, \bar{N}) := \frac{1}{2}\big(\mathrm{Tr}(\bar{M}\bar{N}) + \mathrm{Tr}(\bar{N}^*\bar{M}^*)\big)$. Given $M, N \in$ LieG', $B(M, N) :=$ $\mathrm{Tr}(MN)$. The same formulae hold for $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{so}(1,3)$ and define $\bar{B}_{\mathfrak{sl}(2,\mathbb{C})}$ and $B_{\mathfrak{so}(1,3)}$, which must coincide since $\mathfrak{sl}(2,\mathbb{C}) \simeq \mathfrak{so}(1,3)$. As a matter of fact, for $m, n \mapsto \bar{m}, \bar{n}$ one has $\bar{B}_{\mathfrak{sl}(2,\mathbb{C})}(\bar{m}, \bar{n}) = B_{\mathfrak{so}(1,3)}(m, n)$. The Yang-Mills Lagrangians associated to the standard tractor and twistor connections both reproduce conformal gravity,

$$\left.\begin{array}{l} L_{\mathrm{YM}}(\varpi) = \frac{1}{2}B(\Omega, *\Omega) = \frac{1}{2}B_{\mathfrak{so}(1,3)}(\mathsf{W}, *\mathsf{W}) \\ L_{\mathrm{YM}}(\bar{\varpi}) = \frac{1}{4}\bar{B}(\bar{\Omega}, *\bar{\Omega}) = \frac{1}{2}\bar{B}_{\mathfrak{sl}(2,\mathbb{C})}(\bar{\mathsf{W}}, *\bar{\mathsf{W}}) \end{array}\right\} = \frac{1}{2}\,\mathrm{Tr}(\mathsf{W} \wedge *\mathsf{W}) = L_{\mathrm{Weyl}}(e).$$

It is then no surprise that the field equations obtained by varying the action $S_{\mathrm{YM}}(\bar{\varpi})$ w.r.t. $\bar{\varpi}$, or $S_{\mathrm{YM}}(\varpi)$ w.r.t. ϖ, on the one hand, and by varying the action $S_{\mathrm{Weyl}}(e)$ w.r.t. e on the other hand, should coincide. In the first case we obtain the Yang-Mills equations for the standard tractor and twistor connections, and in the second case we obtain the Bach equation:

$$\begin{tikzcd}
\frac{\delta S_{\mathrm{YM}}(\bar{\varpi})}{\delta \bar{\varpi}} = 0 \ \rightarrow\ \bar{D} * \bar{\Omega} = 0 \arrow[rr, leftrightarrow] \arrow[dr, bend right, leftrightarrow] & & \frac{\delta S_{\mathrm{YM}}(\varpi)}{\delta \varpi} = 0 \ \rightarrow\ D * \Omega = 0 \arrow[dl, bend left, leftrightarrow] \\
& \frac{\delta S_{\mathrm{Weyl}}(e)}{\delta e} = 0 \ \rightarrow\ B_{ab} = 0 &
\end{tikzcd}$$

where B_{ab} is the Bach tensor. The equivalence on the right side of the diagram was first noticed in [36]. From this we conclude that conformal gravity is a mixed $\mathcal{W} \times \mathcal{SO}$ -gauge theory hiding in plain sight.

10.4. Application to anomalies in QFT

Anomalies arise in QFT when the quantization of a classical gauge theory fails to uphold the gauge invariance. First discovered through perturbative methods, (consistent) anomalies were found to be characterized by BRST cohomological methods and obtainable via Stora-Zumino descent equations. They finally came to be understood as degree 1 elements in the cohomology of Lie$\mathcal{H}$ [18], and soon after as $\mathcal{H}$-1-cocycles [37, 38], see in particular [39]. Interestingly, in [39] and [40, 41], an infinite dimensional *twisted line bundle* appears as a relevant object in the study of anomalies (see also [42, 43], or [44] more recently). It comes out as follows.

Consider a Yang-Mills gauge theory such that the relevant fields space is the space $\mathcal{A}$ of Ehresmann connections of a H-principal bundle $\mathcal{P}$ with gauge group $\mathcal{H}$. Under proper restrictions, $\mathcal{A}$ is itself a $\mathcal{H}$-principal bundle over the moduli space $\mathcal{A}/\mathcal{H}$, where one is here in the realm of infinite dimensional Hilbert manifolds [45, 46].

A quantum (vacuum) functional is a smooth map $W\colon \mathcal{A}\to \mathbb{C}^*$. For a gauge invariant quantized theory, W is s.t. $W(A^\gamma)=W(A)$, $\gamma \in \mathcal{H}$, it is therefore projectable and descends to a functional on the base $\mathcal{A}/\mathcal{H}$. But an anomalous functional is s.t. $W(A^\gamma) = C(A,\gamma)^{-1}W(A)$, where $C\colon \mathcal{A}\times \mathcal{H} \to U(1)$ is $C(A,\gamma) = \exp\{-i2\pi f(A,\gamma)\}$. The functional in the phase is the Wess-Zumino term, also called *integrated anomaly* since indeed $\lim\limits_{\tau\to 0} f(A,\gamma_\tau)/\tau = a(\chi, A)$ is the anomaly (linear in $\chi \in$ Lie$\mathcal{H}$).

Now, consistency of the right action of the gauge group $W(A^{\gamma\gamma'})= W((A^\gamma)^{\gamma'})$ implies the cocycle relation $C(A,\gamma\gamma') = C(A,\gamma)\, C(A^\gamma,\gamma')$, which is a form of the Wess-Zumino consistency condition for the gauge anomaly. An anomalous quantum functional W is then a *twisted* C-equivariant function on $\mathcal{A}$, i.e. a section of the twisted associated line bundle $\mathcal{L}^C := \mathcal{A}\times_{C(\mathcal{H})} \mathbb{C}^*$.

As far as I can tell, the peculiar geometrical nature of twisted line bundles like $\mathcal{L}^C$ was first stressed in [47, 48].[6] These references also introduce a

[6]In the introduction of the latter reference we read, "*[...] recently objects (called generalized associated bundles hereafter) have appeared in the physics literature, about whose general structure little seems to be known*". And after defining the twisted line bundle with fiber $\mathbb{C}$ we find, just below Eq.(2.3), the comment "*bundles of this kind have recently appeared in the physics literature (mainly in relation with anomalies). Their geometrical structure, however, was not further investigated.*" The present paper happens to contribute to this investigation.

connection for a twisted line bundle $L^C := \mathcal{P} \times_{C(H)} \mathbb{C}^*$ associated to a H-principal bundle $\mathcal{P}$, with $C : \mathcal{P} \times H \to \mathbb{C}^*$, that turns out to be a special case of twisted connection defined in section 3.2.

Working on a patch $\mathcal{U} \subset \mathcal{M}$, consider a map $u : \pi^{-1}(\mathcal{U}) \subset \mathcal{P} \to H$ defined by the equivariance property $R_h^* u = h^{-1}u$.[7] This we call a - locally defined - dressing field [33]. The map $C(u) : \pi^{-1}(\mathcal{U}) \to H$, is such that $C_{ph}(u(ph)) = C_{ph}(h^{-1}u(p)) = C_{ph}(h^{-1})\, C_p(u(p)) = C_p(h)^{-1}C_p(u(p))$. So its equivariance is $R_h^* C(u) = C(h)^{-1}C(u)$ (call it a twisted dressing field, see [31, 32, 34]). Build then the 1-form $\Gamma := C(u)dC(u)^{-1}$ satisfying, for $X_p^v \in V_p\mathcal{P}$ generated by $X \in \text{Lie}H$,

$$\begin{aligned}
\Gamma_p(X_p^v) &= C_p(u(p))dC(u)_p^{-1}(X_p^v) = C_p(u(p))[X^v(C(u))]^{-1}(p) \\
&= C_p(u(p))\tfrac{d}{d\tau}C_{pe^{\tau X}}\left(u(pe^{\tau X})\right)^{-1}\big|_{\tau=0}, \\
&= \tfrac{d}{d\tau}C_p\left(e^{\tau X}\right)\big|_{\tau=0} = dC_{p|e}(X),
\end{aligned}$$

and, for $h \in H$ and $X_p \in T_p\mathcal{P}$,

$$\begin{aligned}
R_h^*\Gamma_{ph} &= C_{ph}(u(ph))dR_h^*C(u)_{|p} = C_p(h)^{-1}C_p(u(p))\, d\left(C(u)^{-1}C(h)\right)_{|p}, \\
&= C_p(h)^{-1}\Gamma_p C_p(h) + C_p(h)^{-1}dC(h)_{|p}.
\end{aligned}$$

Thus, $\Gamma \in \mathcal{C}(\mathcal{P}_{|\mathcal{U}})^T$. Given a partition of unity $\{\delta_i\}$ subordinate to a covering $\{\mathcal{U}_i\}$ of $\mathcal{M}$, the Γ_i's can be glued into a non flat twisted connection $\Gamma = \sum(\pi^*\delta_i)\Gamma_i \in \mathcal{C}(\mathcal{P})^T$, with curvature $\Omega = d\Gamma \in \Omega^2_{\text{tens}}(\mathcal{P}, C)$. It defines a covariant derivative $D = d + \Gamma$ on $\Omega^\bullet_{\text{tens}}(\mathcal{P}, C)$, and on sections of L^C in particular.

Applied to the case $\mathcal{P} \Rightarrow \mathcal{A}$ and $L^C \Rightarrow \mathcal{L}^C$,we see that the twisted connection Γ is given essentially by the Wess-Zumino term $f(A, \gamma)$.

11. Conclusion

In this paper, we have constructed a geometry that generalises associated vector bundles E built via representations (ρ, V) of the structure group H of a principal bundle $\mathcal{P}(\mathcal{M}, H)$. These generalised associated bundles E^C are built from cocycles for the action of the structure group on the principal bundle, $C : \mathcal{P} \times H \to G$, and representations (ρ, V) of the target group G instead of H. We therefore call these - unimaginatively - *twisted* associated

[7] Such a map always exists on a trivial bundle, here the bundle $\mathcal{P}_{|\mathcal{U}} = \mathcal{U} \times H$.

bundles. We have also characterised the space of V-valued twisted tensorial forms $\Omega^\bullet_{\text{tens}}\big(\mathcal{P}, C(H)\big)$, whose subspace of degree 0 is isomorphic with the space $\Gamma(E^C)$ of sections of twisted bundles. We have then defined a notion of twisted connection form on $\mathcal{P}$ that generalises Ehresmann connection 1-forms in providing a good exterior covariant derivative on $\Omega^\bullet_{\text{tens}}\big(\mathcal{P}, C(H)\big)$ - thus allowing to define a parallel transport on $\Gamma(E^C)$ - and whose curvature belongs to $\Omega^2_{\text{tens}}\big(\mathcal{P}, C(H)\big)$. As usual, the gauge transformations of the twisted connections and tensorial forms are obtained from the action of vertical automorphisms, $\text{Aut}_v\,\big(\mathcal{P}, H\big)$, of the principal bundle. These geometrical objects provide a new way to implement the gauge principle of physics, so that the local representatives on $\mathcal{M}$ can be seen as twisted gauge fields generalising Yang-Mills type gauge fields. The possibility of building twisted gauge theories straightforwardly ensue.

As the most immediate extension of this new framework, we have considered the case of bundles associated to, and tensorial forms on, a principal bundle $\mathcal{P}(\mathcal{M}, H \times K)$, which behave as twisted objects w.r.t. the action of H, but as standard objects w.r.t. the action of K. For this reason we call them respectively *mixed* associated bundles $\mathcal{E}^C$ and *mixed* tensorial forms $\Omega^\bullet_{\text{tens}}\big(\mathcal{P}, C(H) \rtimes K\big)$. We then defined a corresponding notion of mixed connection, which is both a H-twisted connection and a K-standard (Ehresmann) connection. It induces a good covariant derivative on $\Omega^\bullet_{\text{tens}}\big(\mathcal{P}, C(H) \rtimes K\big)$ and $\Gamma(\mathcal{E}^C)$, and its curvature is mixed tensorial.

In the same way that Cartan connection 1-forms are a distinguished subclass of Ehresmann connection 1-forms, we have proposed a definition for a subclass of our twisted/mixed connections that may be a sensible generalisation of Cartan connections. We then call these twisted/mixed Cartan connections.

To convince ourselves that all this is not idle exploration, we have shown that conformal tractors and local twistors can be seen as simple and slightly degenerate instances of the general framework presented here. The tractor and twistor bundles are mixed vector bundles, while the tractor and twistor connections are mixed Cartan connections. This clarifies and puts on firmer mathematical ground the attempt [35] to interpret local twistors as gauge fields of a kind: they are indeed mixed gauge fields as defined here. We are then led to the surprising conclusion that conformal gravity is an unsuspected example of twisted/mixed gauge theory.

At least one other example could have been added to the list: projective tractors. Like conformal tractors, these can be constructed bottom-up via prolongation [24], but they can also be more economically obtained via the dressing field method [33]. In the latter case, the twisted structure is more readily and explicitly seen. We refrained from presenting this case because it would have added little to the discussion and we wanted to spare the reader an elaboration that might have felt repetitive.

Keen readers will perhaps have noted that twisted objects can be related to standard constructions by "hiding" the cocycle structure. One can indeed build the twisted associated bundle $\mathcal{Q} = \mathcal{P} \times_{C(H)} G$, which is a G-principal bundle under the right action of G on itself. One can show that the standard associated bundle $\mathcal{Q} \times_G V$ is isomorphic to the twisted bundle $\mathcal{P} \times_{C(H)} V$, and that a twisted connection on $\mathcal{P}$ induces an Ehresmann connection on $\mathcal{Q}$ (and vice-versa). In our view this interesting fact does not imply that the twisted geometry described here isn't worthy of further study. No more than the well-known fact that Lie$G$-valued Cartan connections on some H-principal bundle $\mathcal{P}$ induce Ehresmann connections on the G-principal associated bundle $\mathcal{Q} = \mathcal{P} \times_H G$ (and vice-versa[8], see e.g. [22] Appendix A, §3.) means that Cartan geometry isn't a worthy subject in its own right.

One might therefore be interested in further developing and understanding the twisted/mixed geometry, so as to e.g. see how standard notions such as holonomy, characteristic classes, Chern-Weyl theory, etc... export to this new context. And further still, one may want to examine how this framework extends to the super-differential geometric setup, eyeing possible applications to physics and supergravity.

The relevance of the twisted geometry is perhaps especially easy to argue for in view of the application to physics we have already hinted at: Cocycles in the form of Wess-Zumino terms appear most naturally in the study of anomalous quantum functionals on the $\mathcal{H}$-bundle $\mathcal{A}$ of Ehresmann connections of a H-principal bundle $\mathcal{P}$. These are C-equivariant functionals, i.e. section of a line twisted bundle associated to $\mathcal{A}$. The natural covariant differentiation of such functionals would require to endow $\mathcal{A}$ with a twisted connection, perhaps more general than the one defined by the cocycle/Wess-Zumino term introduced in [47, 48]. The twisted geometry may also prove useful in relation to works on boundaries in gauge theories. For example, in [49] so-called *field dependent gauge transformations* are introduced that could perhaps be better understood as $\mathcal{H}$-valued cocycles ($C\colon \mathcal{A} \times \mathcal{H} \to \mathcal{H}$), and $\mathcal{A}$ is endowed with a connection that might be interpreted as a twisted connection: if their "field dependent gauge transformation" g is indeed seen as a cocycle, compare equations (3.9a)-(3.10)/(3.9b) with (I)-(II)/(47).

[8]Provided that the Ehresmann connection ω satisfies $\ker \omega \cap \phi_* T\mathcal{P} = \emptyset$, with the bundle map $\phi : \mathcal{P} \to \mathcal{Q}$.

Finally, let us notice that a priori twisted/mixed gauge theories can be quantized following the same strategies used for standard gauge theories. There seems to be no objections to using path integration methods, and we have seen that the BRST framework is general enough to accommodate our new geometric objects. It is also conceivable to attempt canonical quantization, after all this is what Penrose proposed for twistors (see e.g. *twistor quantization* on p.142 of [27] and references therein).

However a new layer of complexity may appear here. Indeed, the gauge transformations of our new gauge fields are twisted by cocycles. Up until now, we did not articulate what is the equivalence relation on these cocycles and how is defined the associated cohomology. What, if anything, does this cohomology add to the gauge structure of mixed gauge fields? How does it interact with the usual gauge-BRST cohomology familiar in gauge theory? How does it relate to the quantisation problem, and are there e.g. new kind of anomalies associated to it? These intriguing questions seem worthy of further investigation.

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