ADV. THEOR. MATH. PHYS.
Volume 25, Number 7, 1847–1952, 2021

Two-dimensional perturbative scalar QFT and Atiyah-Segal gluing

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We study the perturbative quantization of 2-dimensional massive scalar field theory with polynomial (or power series) potential on manifolds with boundary. We prove that it fits into the functorial quantum field theory framework of Atiyah-Segal. In particular, we prove that the perturbative partition function defined in terms of integrals over configuration spaces of points on the surface satisfies an Atiyah-Segal type gluing formula. Tadpoles (short loops) behave nontrivially under gluing and play a crucial role in the result.

1 Introduction **1848**

2 Preliminaries **1854**

3 Perturbative quantization **1859**

4 Heuristic analysis of path integrals and gluing formulae **1879**

5 Regularization of tadpoles **1885**

6 Formal Fubini Theorem and Atiyah-Segal gluing **1903**

7 Functoriality **1910**

S.K. and K.W. would like to thank the University of Zurich where a part of this work was written, and acknowledge partial support of NCCR SwissMAP, funded by the Swiss National Science Foundation, and by the COST Action MP1405 QSPACE, supported by COST (European Cooperation in Science and Technology), and the SNF grant No. 200020_172498/1 during their affilitation with the University of Zurich. K. W. acknowledges further support from a BMS Dirichlet postdoctoral fellowship and the SNF Postdoc Mobility grant P2ZHP2_184083, and would like to thank the Humboldt-Universität Berlin, in particular the group of Dirk Kreimer, and the University of Notre Dame for their hospitality.

8 Discussion and outlook **1920**

Appendix A Examples **1922**

Appendix B Trace of stress-energy tensor and tadpole. Trace anomaly **1939**

References **1947**

1. Introduction

In recent years, *Functorial Quantum Field Theories* (FQFTs), as proposed by Atiyah and Segal [2],[53], have been the subject of intense mathematical investigation, see e.g. [9], [49], [55] and references therein. The rough idea is that a quantum field theory corresponds to a functor

$$Z\colon \mathbf{Cob} \to \mathbf{Hilb}$$

from a cobordism category, possibly equipped with extra structure, to the category of Hilbert spaces. Examples of such functors from *topological* cobordism categories (called TQFTs, short for *Topological Quantum Field Theories*) abound, see e.g. [50], [58], [16]. On the other hand, there are very few examples known for geometric cobordism categories. The first examples (in dimension greater than one) are: 2-dimensional Yang-Mills theory (Migdal-Witten, [42], [61]) and 2-dimensional free fermion conformal field theory (Segal [53] and Tener [57]) – for the cobordism category endowed with area form or conformal structure, respectively. An example of an invertible FQFT for the Spin Riemannian cobordism category is constructed by Dai and Freed [12]. In [32] it was shown that free massive scalar field theory provides an example of such a FQFT in even dimensions, for the Riemannian cobordism category.

In this paper we give a new example of an interacting FQFT on the Riemannian cobordism category arising from the perturbative path integral.

1.1. Main results

We are considering the perturbative quantization of the scalar field theory defined classically by the action functional

$$S_\Sigma(\phi) = \int_\Sigma \frac{1}{2} d\phi \wedge * d\phi + \int_\Sigma \frac{m^2}{2}\phi^2 \,\mathrm{dVol}_\Sigma + \int_\Sigma p(\phi)\,\mathrm{dVol}_\Sigma$$

with Σ an oriented surface endowed with Riemannian metric, $\phi \in C^\infty(\Sigma)$ the field, $m > 0$ a parameter ("mass") and $p(\phi) = \sum_{k\geq 3} \frac{p_k}{k!}\phi^k$ a polynomial (or possibly power series) interaction potential.

The first main result of this paper is that there is an Atiyah-Segal gluing formula for the perturbative partition function (Definition 5.2). The result is set up as follows. First, we define a vector space

$$H_Y = \left\{ \Psi(\eta) = \sum_{n\geq 0} \int_{C_n^\circ(Y)} dx_1 \cdots dx_n \, \psi_n(x_1, \ldots, x_n)\eta(x_1)\cdots\eta(x_n) \right\}$$

associated to a 1-dimensional Riemannian manifold $Y = S^1 \sqcup \cdots \sqcup S^1$ (Definition 3.3). Vectors in H_Y are functionals on $C^\infty(Y) \ni \eta$ and are parameterized by the "n-particle wave functions" ψ_n – formal power series in $\hbar^{1/2}$ with coefficients given by smooth symmetric functions on the open configuration space of n points on Y with certain types of singularities on diagonals allowed (in particular, logarithmic singularities on codimension 1 diagonals), see Definition 3.2.

The perturbative partition function of a surface Σ is then given as[1]

(1.1)
$$Z_\Sigma(\eta) = e^{-\frac{1}{2}\int_{\partial\Sigma} \mathrm{dVol}_{\partial\Sigma}\, \eta D_\Sigma(\eta)} \det{}^{-\frac{1}{2}}(\Delta + m^2) \sum_\Gamma \frac{\hbar^{E-N-\frac{n}{2}} F_\Gamma(\eta)}{|\,\mathrm{Aut}(\Gamma)|} \quad \in H_{\partial\Sigma}$$

where:

- $\hbar$ is a formal parameter of quantization.
- D_Σ is the Dirichlet-to-Neumann operator.
- $\det(\Delta + m^2)$ is the zeta-regularized determinant over functions on Σ with Dirichlet boundary conditions.

[1]For the sake of preliminary exposition, we are being slightly imprecise writing that Z_Σ is an element of the space of boundary states – see Section 3.4 for details (in particular, Remarks 3.20, 3.4 and 3.19).

- The sum runs over graphs Γ with N bulk vertices and n boundary vertices (which are univalent), with no boundary-boundary edges.[2] Here E is the total number of edges and $|\operatorname{Aut}(\Gamma)|$ is the order of the automorphism group of the graph.
- The Feynman evaluation of a graph $F_\Gamma(\eta)$ is calculated as follows.
  - Boundary vertices of Γ are placed at points $x_i \in \partial\Sigma$ and are decorated with $\eta(x_i)$;
  - bulk vertices are placed at points $y_j \in \Sigma$ and are decorated by $-p_v$ (the coefficient of the interaction polynomial, with v the valence of the vertex);
  - edges between distinct vertices are decorated by the Green's function for $\Delta + m^2$ (or its normal derivative for bulk-boundary edges),
  - an edge connecting a vertex to itself is decorated by the zeta-regularized evaluation of the Green's function on the diagonal, see Definition 5.8.

  Then $F_\Gamma(\eta)$ is calculated as the product of all decorations integrated over positions of all points x_i, y_j.

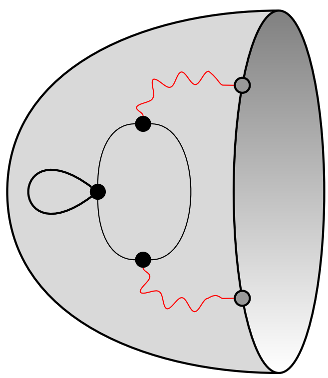


Figure 1. A typical Feynman graph on a hemisphere. Straight lines are decorated with the Green's function (for Dirichlet boundary conditions) while wavy lines are decorated with its normal derivative.

Finally, if Σ is a closed surface with a decomposition $\Sigma = \Sigma_L \cup_Y \Sigma_R$, where Y is the boundary of Σ_L and Σ_R, we define a pairing

$$\langle \cdot, \cdot \rangle_{\Sigma_L, Y, \Sigma_R} \colon H_Y \otimes H_Y \to \mathbb{R}[[\hbar^{1/2}]] \tag{1.2}$$

[2] In fact, the exponential prefactor in (1.1) can be seen as the contribution of boundary-boundary edges – see Remark 3.19.

as follows:

(1.3)

$$\langle \Psi^L(\eta), \Psi^R(\eta)\rangle = \det{}^{-\frac{1}{2}}(D_{\Sigma_L} + D_{\Sigma_R})\cdot$$
$$\cdot \sum_{\mathfrak{m}\in\mathfrak{M}_{m+n}} \int_{C^\circ_{m+n}(Y)} dx_1 \cdots dx_{m+n}\, \psi^L_m(x_1 \ldots, x_m)\psi^R_n(x_{m+1}, \ldots, x_{m+n})\cdot$$
$$\cdot \prod_{(i,j)\in\mathfrak{m}} K(x_i, x_j)$$

Here $\psi^{L,R}$ are the wavefunctions for the states $\Psi^{L,R}$ and K is the Green's function for the operator $D_{\Sigma_L} + D_{\Sigma_R}$. The sum runs over perfect matchings $\mathfrak{m}$ of $m + n$ elements.

The Atiyah-Segal gluing formula can then be formulated as follows.

Theorem 1.1. *Let $\Sigma = \Sigma_L \cup_Y \Sigma_R$. Then*

$$Z_\Sigma = \langle \widehat{Z}_{\Sigma_L}, \widehat{Z}_{\Sigma_R}\rangle_{\Sigma_L,Y,\Sigma_R}$$

where $\widehat{Z}_{\Sigma_L}$, $\widehat{Z}_{\Sigma_R}$ are given by (1.1) specialized to Σ_L, Σ_R with the exponential prefactor omitted.

The proof is based, roughly, on the idea that the value of a Feynman graph Γ on Σ can be presented in terms of values of its subgraphs Γ_L, Γ_R located on Σ_L and Σ_R, glued using the interface Green's functions K, see Figure 2.

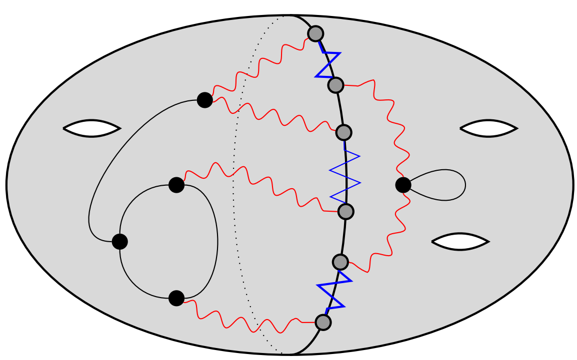


Figure 2. When gluing contributions from two Feynman graphs, their boundary vertices are connected by the Green's function of the Dirichlet-to-Neumann operator (zig-zag line).

This result has a generalization for Σ a non-closed surface, see Theorem 6.1.

The second result is that the construction above can be upgraded to a functor from the symmetric monoidal semi-category (i.e. without identity morphisms) of Riemannian cobordisms to Hilbert spaces. The main problem obstructing functoriality of the previous result is the fact that the pairing (1.2) depends not just on the gluing interface Y but also on the adjacent surfaces.

We define the adjusted partition function

$$\overline{Z}_\Sigma(\eta) = e^{\frac{1}{2}\int_{\partial\Sigma} \mathrm{dVol}_{\partial\Sigma}\, \eta\varkappa(\eta)} Z_\Sigma(\eta)$$

where $\varkappa = (\Delta + m^2)^{1/2}\big|_{\partial\Sigma}$ is the square root of the Helmholtz operator on the boundary. We also define a new adjusted (functorial) pairing $\langle,\rangle_{2\varkappa}$ on H_Y, given by a similar formula to (1.3) where the operator $D_{\Sigma_L} + D_{\Sigma_R}$ is replaced by $2\varkappa$. Furthermore, we replace H_Y with its L^2 completion.

Theorem 1.2. *The assignment of the Hilbert space H_Y to each closed Riemannian 1-manifold Y (endowed with a two-sided collar) and of the adjusted partition function $\overline{Z}_\Sigma$ to each Riemannian 2-cobordism constitutes a functor* $\mathbf{Riem}^2 \to \mathbf{Hilb}$.

1.2. Tadpoles

In various treatments of scalar theory, tadpole diagrams were set to zero (this corresponds to a particular renormalization scheme – in flat space, this is tantamount to normal ordering, see e.g. [22], [54]). However, in our framework this prescription contradicts locality in Atiyah-Segal sense, see Section 5.1. One good solution is to prescribe to the tadpole diagrams the zeta-regularized diagonal value of the Green's function. We prove that assigning to a surface its zeta-regularized tadpole is compatible with locality, see Proposition 5.18. However there are other consistent prescriptions (for instance, the tadpole regularized via point-splitting and subtracting the singular term, see Section 5.4). This turns out to be related to Wilson's idea of RG flow in the space of interaction potentials, see Section 5.5.

In the free theory, the zeta-regularized tapdole can be interpreted in terms of the trace of the (classical) stress-energy tensor. Generally, the trace of the quantum stress-energy tensor $\big\langle \operatorname{tr} T_q(x)\big\rangle$ (the reaction of the partition function to an infinitesimal Weyl transform of the metric) differs from the expectation value of the classical stress-energy tensor by a "trace anomaly"

– an effect well-known in conformal field theory. In Appendix B we obtain an expression for the trace anomaly in the interacting massive scalar theory (Proposition B.4):

$$\left\langle \operatorname{tr} T_q(x) \right\rangle - \left\langle \operatorname{tr} T_{\mathrm{cl}}(x) \right\rangle = \frac{\hbar}{4\pi} \left\langle \frac{K(x)}{6} - m^2 - \frac{\partial^2}{\partial \phi^2} p(\phi) \right\rangle$$

with $K(x)$ the scalar curvature of Σ at the point x. Brackets $\langle \cdots \rangle$ stand for the expectation value defined using the perturbative path integral, as a sum of connected graphs with a single marked vertex (except in $\left\langle \operatorname{tr} T_q(x) \right\rangle$ where brackets are a part of notation). We also comment on how to compare the $m \to 0$ limits of trace anomaly and partition function with conformal field theory, see Remark B.3.

1.3. Plan of the paper

Let us briefly outline the plan of the paper.

- In Section 2 we recall some facts on classical field theory and free massive scalar field theory.
- In Section 3, we define the perturbative quantization of scalar field theory on manifolds with boundary[3]. We use heuristics of path integrals to motivate our construction and then give rigorous definitions and proofs.
- In Section 4, we show how to heuristically derive gluing formulae for regularized determinants and Green's functions from formal Fubini theorems for path integrals. These gluing formulae have been proven in other contexts in the literature, and we briefly review these mathematical results.
- In Section 5 we study the regularization of tadpole diagrams and how it interacts with gluing.
- In Section 6, we state and prove Theorem 1.1 above (in the general form).

[3]Our approach is slightly different from the one in [9]: we use the second-order formalism whereas [9] works in the first order. Also, a technical point: we use a different extension of boundary fields into the bulk, for the splitting of fields as boundary fields plus fluctuations, – we use the extension as a solution of equations of motion whereas [9] uses a "discontinuous extention." We plan to discuss the relation between the two approaches in a future publication.

- In Section 7, we promote our results to the functorial framework and state and prove Theorem 1.2 above.
- In Appendix A, we present a collection of explicit examples of zeta-regularized determinants and tadpole functions (and gluing thereof) and Dirichlet-to-Neumann operators. We prove that the Dirichlet-to-Neumann operator differs from the square root of the Helmholtz operator on the boundary by a pseudodifferential operator of order ≤ -2 (a sharp bound) – Proposition A.3.
- In Appendix B we discuss the relation of the tadpole and the stress-energy tensor, and obtain the trace anomaly.

1.4. Related work

The cutting and gluing of perturbative partition functions in the context of first-order gauge theories is discussed in [9]. An example of a computation in the context of Chern-Simons theory was done in [60], [11]. An approach to perturbative Chern-Simons invariants of 3-manifolds via cutting and gluing in the context of algebraic topology was discussed in [35]. Another approach to gluing of functional integrals - with a view towards supersymmetry - is considered in [13],[14]. Gluing in the context of the perturbative path integral approach to quantum mechanics was considered in [30].

1.5. Acknowledgements

We would like to thank Alberto S. Cattaneo, Liviu Nicolaescu, Nicolai Reshetikhin, Stephan Stolz and Donald Youmans for helpful remarks and discussions.

2. Preliminaries

2.1. Classical field theory

A classical (Lagrangian) field theory on a compact oriented d-dimensional Riemannian manifold Σ, possibly with nonempty boundary, consists of:

(i) The space of fields $\mathcal{F}_\Sigma$, which is the space sections of a vector bundle[4] over Σ.

[4] Or, more generally, a sheaf. However, the case relevant for this paper is the one of the trivial $\mathbb{R}$-bundle.

(ii) A Lagrangian $\mathcal{L}_\Sigma$, which is a function on $\mathcal{F}_\Sigma$ with values in the space of d-forms on Σ, i.e. $\mathcal{L}_\Sigma : \mathcal{F}_\Sigma \to \Omega^d(\Sigma), \phi \mapsto \mathcal{L}_\Sigma(\phi)$ such that it is *local* in the sense that it depends only on the fields and finitely many derivatives thereof.

Using the Lagrangian, we define the action functional $S_\Sigma : \mathcal{F}_\Sigma \to \mathbb{R}, S_\Sigma(\phi) = \int_\Sigma \mathcal{L}_\Sigma(\phi)$.

The variation δS_Σ of the action functional S_Σ has the so-called Euler-Lagrange term and a boundary term. The Euler-Lagrange term gives rise to the equation of motion and its solutions are called the classical solutions. Let $EL_\Sigma \subset \mathcal{F}_\Sigma$ denote the space of the classical solutions. The boundary term induces a one-form on EL_Σ.

Let $\mathbf{Y}$ denote a $(d-1)$-dimensional manifold Y together with a one-sided d-dimensional collar around Y. Associated to Y we have a space Φ_Y which consists of the solutions to the equation of motion on $\mathbf{Y}$, and a one-form α_Y on Φ_Y that arises from the boundary contribution of the variation $\delta S_{\mathbf{Y}}$ on $\mathcal{F}_{\mathbf{Y}}$. We can identify Φ_Y, at least when $\mathcal{L}_\Sigma$ is nice, with the Cauchy data space C_Y which is the information on the fields and their derivatives along Y so that the equation of motion has a unique solution. Let $\omega_Y = \delta\alpha_Y$, then, ω_Y defines a presymplectic structure on C_Y and it is symplectic[5] in nice situations. For instance, for the free scalar field theory (C_Y, ω_Y) is a symplectic vector space.

Now assume that $\partial\Sigma = Y$, then we have a surjective submersion $\pi_\Sigma : \mathcal{F}_\Sigma \to C_Y$. Let $L_\Sigma = \pi_\Sigma(EL_\Sigma)$. Then, L_Σ is an isotropic submanifold of C_Y and it will be a Lagrangian submanifold when $\mathcal{L}_\Sigma$ is nice. We assume that L_Σ is Langangian for this discussion.

Hence, a classical field theory assigns to a compact oriented Riemannian $(d-1)$-dimensional manifold Y (more precisely, Y has a one-sided collar) a symplectic manifold (C_Y, ω_Y). Moreover, if $\partial\Sigma = Y$, then L_Σ is a Lagrangian submanifold of C_Y. More generally if $\partial\Sigma = \overline{\partial\Sigma^{\text{in}}} \sqcup \partial\Sigma^{\text{out}}$, then L_Σ is a Lagrangian submanifold of $\overline{C_{\partial\Sigma^{\text{in}}}} \times C_{\partial\Sigma^{\text{out}}}$ and it can be regarded as a relation which if often called a canonical relation [59]. Here $\overline{\partial\Sigma^{\text{in}}}$ is used to emphasize that the intrinsic orientation on $\partial\Sigma^{\text{in}}$ is the opposite of the induced orientation from Σ (whereas the bar in $\overline{C_{\partial\Sigma^{\text{in}}}}$ denotes the change of sign of the symplectic form). Furthermore, if $\Sigma = \Sigma_1 \cup_Y \Sigma_2$, then $L_\Sigma = L_{\Sigma_1} \circ L_{\Sigma_2}$

[5] C_Y is typically infinite-dimensional in the setting of field theory, and nondegeneracy of symplectic structure is understood in the weak sense, i.e. the map $TM \to T^*M$ induced by ω is injective, but not necessarily surjective.

where the $\circ$ means the composition of relations. We refer to [59] for a discussion of the symplectic category and [7] for a more elaborate discussion and axiomatization of classical field theory in a more general setting.

2.2. Free massive scalar theory

Here, we illustrate the discussion above using the free massive scalar field theory. Let Σ be a compact oriented Riemannian manifold of dimension d with $\partial\Sigma = Y$. Let m be a positive real number. For the massive free scalar field theory on Σ, the space of fields is $\mathcal{F}_\Sigma = C^\infty(\Sigma)$ and the Lagrangian is given by

$$\mathcal{L}(\phi) = \frac{1}{2}(d\phi \wedge * d\phi + m^2 \phi \wedge *\phi) \text{ and } S(\phi) = \frac{1}{2}\int_\Sigma d\phi \wedge * d\phi + m^2\phi\wedge *\phi.$$

Moreover,

$$\delta S = \int_\Sigma \delta\phi \wedge (d * d\phi + m^2 * \phi) + \int_{\partial\Sigma} \delta\phi \wedge * d\phi$$

and the equation of motion is

$$(\Delta_\Sigma + m^2)\phi = 0$$

which is also known as *Helmholtz* equation. Here, and throughout the paper, Δ_Σ is the Laplace-de Rham operator $\Delta = dd^* + d^*d$ (where d^* is the codifferential) restricted to 0-forms.[6] The Cauchy data space C_Y is given by $C^\infty(Y) \oplus C^\infty(Y)$, and $\pi_\Sigma \colon \mathcal{F}_\Sigma \to C_Y$ is given by $\phi \mapsto \left(\iota_Y^*(\phi), \iota_Y^*\left(\frac{\partial\phi}{\partial\nu}\right)\right)$, where ν is the outward pointing unit normal vector field along Y. Furthermore, $\alpha_Y = \int_Y \chi\delta\phi \,\mathrm{dVol}_Y$ and the symplectic form ω_Y is given by $\omega_Y = \delta\alpha_Y = \int_Y \delta\chi\delta\phi\,\mathrm{dVol}_Y$. Here δ is understood as de Rham differential on $C^\infty(Y)$, this means that when evaluated on two vectors (ϕ_1, χ_1) and (ϕ_2, χ_2) the result is

$$\omega_Y((\phi_1,\chi_1)(\phi_2,\chi_2)) = \int_Y (\chi_1\phi_2 - \phi_1\chi_2)\,\mathrm{dVol}_Y.$$

L_Σ is the graph of the Dirichlet-to-Neumann operator D_Σ on Y, which is defined as follows:

[6] In particular, it has nonnegative spectrum, so coincides with *minus* the usual Laplace operator on flat space.

Definition 2.1. *Let $\eta \in C^\infty(Y)$ and $\phi_\eta \in C^\infty(\Sigma)$ be the solution of the Helmholtz equation on Σ with $\iota_Y^*(\phi_\eta) = \eta$ given by Lemma 2.8. Then we define $D_\Sigma \colon C^\infty(Y) \to C^\infty(Y)$ by*

$$D_\Sigma(\eta) := \iota_Y^* \left(\frac{\partial \phi_\eta}{\partial \nu} \right).$$

It is known that D_Σ is symmetric from which it follows that L_Σ is Lagrangian. Furthermore, if $\Sigma = \Sigma_1 \cup_Y \Sigma_2$, one can verify that $L_\Sigma = L_{\Sigma_1} \circ L_{\Sigma_2}$ [8, 32].

Remark 2.2. *When $\Sigma = \Sigma_1 \cup_Y \Sigma_2$, the Dirichlet-to-Neumann operator D_{Σ_1,Σ_2} along Y is defined as the sum of normal derivatives, with respect to the induced orientations on Y, of solutions of Helmholtz equation on Σ_1 and Σ_2:*

$$D_{\Sigma_1,\Sigma_2} = D_{\Sigma_1} + D_{\Sigma_2}. \tag{2.1}$$

2.3. Green's functions

Let Σ be a closed oriented Riemannian manifold and P be an elliptic differential operator on Σ such that P is invertible, then P has a unique Green's function $G(x, y)$, see for example [56], Chapter 7:

The PDE

$$P_y G(x, y) = \delta_x(y)$$

has unique distributional solution $G(x, y)$ and it is the integral kernel of P^{-1}. Moreover,

$$G(x, y) \in C^\infty (\Sigma \times \Sigma \setminus \mathrm{diag}).$$

More generally, if Σ is a compact oriented manifold with boundary, then one can define Green's function by imposing boundary conditions. For example, for the Dirichlet boundary condition, we have the following definition.

Definition 2.3. *Let Σ be a compact manifold with $\partial\Sigma \neq \varnothing$ and P be an elliptic operator on Σ. Then the boundary value problem*

$$P_y G(x, y) = \delta_x(y)$$

with $G(x, y) = 0$ on $\partial\Sigma$ has a unique distributional solution $G(x, y)$. We call such a $G(x, y)$ Green's function with Dirichlet boundary condition and denote it by $G_\Sigma^D(x, y)$.

Remark 2.4. *Green's functions for other boundary conditions are defined similarly and Green's functions may not be unique for a general boundary condition.*

In this paper, we are mostly interested in the Green's function of $\Delta_\Sigma + m^2$ where Δ_Σ is the nonnegative Laplacian on Σ. One well-known technique to construct Green's functions of an elliptic operator P on a manifold with boundary is the method of images (see for example, [22]) which we describe below.

Let Σ be a smooth compact oriented Riemannian manifold with boundary and $\partial\Sigma = \partial_1\Sigma \sqcup \partial_2\Sigma$ We want to construct a Green's function that satisfies the Dirichlet boundary condition on $\partial_1\Sigma$ and the Neumann boundary condition on $\partial_2\Sigma$. The idea here is to use the "doubling twice trick" (we took this name from [9]): we first glue a copy of Σ along $\partial_1\Sigma$ and denote the resulting manifold by Σ'. Note that there is a canonical isomorphism S_1 which is the reflection about $\partial_1\Sigma$. Next, we glue Σ' with itself to get a closed Riemannian manifold Σ''. Let S_2 denote the reflection along the boundary of Σ'. Since Σ'' is a closed manifold we have the Green's function G'' for Σ''. We use G'' to define Green's function on Σ with the desired properties. For this purpose, we define

$$G(x,y) = G''(x,y) + G''(x, S_2(y)) - G''(x, S_1 y) - G''(x, S_2 \circ S_1(y)).$$

Let us verify that $G(x,y)$ is indeed a desired Green's function.

Lemma 2.5. *G(x,y) is a Green's function for P that satisfies Dirichlet boundary condition on $\partial_1\Sigma$ and the Neumann boundary condition on $\partial_2\Sigma$.*

Proof. Let $x \in \Sigma$. Then $P_y G(x,y) = \delta_x(y)$ as $\delta_x(S(y)) = 0$ for $y \in \Sigma$. By construction $G(x,y)$ satisfies the Dirichlet boundary condition on $\partial_1\Sigma$ and the Neumann boundary condition on $\partial_2\Sigma$. □

Example 2.6. *Consider Σ to be the first quadrant in the $\mathbb{R}^2$ and $P = \Delta$. We want a Green's function that satisfies the Dirichlet boundary condition on $y = 0$ and the Neumann boundary condition on $x = 0$. In this case Σ'' is the whole $\mathbb{R}^2$. We know that $-\frac{1}{2\pi}\log((d(x,y),(\alpha,\beta))$ is the Green's function on $\mathbb{R}^2$. Thus, in this case,*

$$\begin{aligned} G((x,y),(\alpha,\beta)) = &-\frac{1}{2\pi}\log((d((x,y),(\alpha,\beta))) - \frac{1}{2\pi}\log((d(x,y),(-\alpha,\beta)) \\ &+\frac{1}{2\pi}\log((d(x,y),(\alpha,-\beta)) + \frac{1}{2\pi}\log((d(x,y),(-\alpha,-\beta)) \end{aligned}$$

One can easily check G has the desired properties.

From now onward a Green's function always refers to a Green's function associated to $\Delta + m^2$. The following fact about Green's functions on a compact oriented Riemannian manifold Σ is well known, see for example [56], Chapter 7; the item (ii) follows from the expansion of the Green's function near the diagonal (see for example [39]).

Lemma 2.7. *(i) The Green's function $G_\Sigma^D(x,y)$, satisfying Dirichlet boundary condition, is symmetric and it defines a positive bounded operator on $L^2(\Sigma)$.*

(ii) In the case $\dim \Sigma = 2$, in a neighborhood of the diagonal $\Delta(\Sigma)$ in $\Sigma \times \Sigma$, away from the boundary, we have

$$G_\Sigma^D(x,y) = -\frac{1}{2\pi}\log(d(x,y)) + H(x,y),$$

where H is in C^1.

We can use Green's function on a manifold with boundary to construct solutions to boundary value problems [17]:

Lemma 2.8. *Let $\eta \in C^\infty(\partial\Sigma)$. Define ϕ_η^Σ on Σ by*

$$\phi_\eta^\Sigma(x) = -\int_{\partial\Sigma} \frac{\partial G_\Sigma^D(x,y)}{\partial \nu}\eta(y)\,dy,$$

then ϕ_η^Σ is the unique solution of the Dirichlet boundary value problem for $\Delta_\Sigma + m^2$ on Σ with boundary value η. Moreover, ϕ_η is smooth on Σ.

We will sometimes drop the subscript Σ if it is clear from the context. We will also drop D from G_Σ^D because, in this paper, we consider the Green's function either on a closed manifold Σ or with respect to the Dirichlet boundary condition when $\partial\Sigma$ is non-empty.

3. Perturbative quantization

In this section we consider the perturbative quantization of scalar field theory with a potential $p \in \mathbb{R}[[\phi]]$ - i.e. the evaluation of the partition function by formally applying the method of steepest descent. As usual, terms in the resulting power series are labeled by Feynman graphs that are evaluated according to Feynman rules. The result is a functional on the boundary fields (more precisely, on the leaf space of a polarization on C_Y).

3.1. Formal Gaussian integrals and moments

The path integrals appearing in this paper are all integrals over vector spaces, and can be reduced to expressions of the form

$$\int_{\phi\in C^\infty(\Sigma,\partial\Sigma)} e^{-\frac{1}{2\hbar}(\phi,A\phi)}\phi(x_1)\cdots\phi(x_n)D\phi,$$

where Σ is a Riemannian manifold, $C^\infty(\Sigma,\partial\Sigma)$ denotes smooth functions which vanish on the boundary, $A\colon C^\infty(\Sigma,\partial\Sigma)\to C^\infty(\Sigma,\partial\Sigma)$ is a linear operator, $(\phi,\psi)=\int_\Sigma \phi\psi\,\mathrm{dVol}_\Sigma$, and $\hbar$ is a formal parameter. One way to define these integrals is just to simply postulate the rules for finite-dimensional Gaussian moments in infinite dimensions. We will very briefly review this idea, as it is essential to the paper. Details can be found in many places, for instance [47],[49],[43]. For $n=0$, we want to define the "formal Gaussian integral"

$$\int_{\phi\in C^\infty(\Sigma,\partial\Sigma)} e^{-\frac{1}{2\hbar}(\phi,A\phi)}D\phi := \frac{1}{(\det A)^{\frac{1}{2}}} \tag{3.1}$$

Heuristically this defines a certain normalization of the path integral measure (absorbing an infinite power of $2\pi\hbar$). However, since A is an operator on an infinite-dimensional space, we need to be careful about the determinant. In this paper, following [26], we will use the zeta-regularization.
The values of Gaussian moments can be elegantly given using the notion of perfect matchings:

Definition 3.1. *If S is a set, a* perfect matching *on S is a collection $\mathfrak{m}$ of disjoint two-element subsets of S such that $\bigcup\mathfrak{m}=S$. The set of perfect matchings on $\{1,\ldots,n\}$ is denoted $\mathfrak{M}_n$.*

For instance, $\mathfrak{m}=\{\{1,3\},\{2,4\}\}$ is a perfect matching on $S=\{1,2,3,4\}$. Again, simply extending the finite-dimensional result to infinite dimensions yields the following definition:

$$\int_{\phi \in C^\infty(\Sigma,\partial\Sigma)} e^{-\frac{1}{2\hbar}(\phi,A\phi)}\phi(x_1)\cdots\phi(x_n)D\phi := \frac{\hbar^n}{(\det A)^{\frac{1}{2}}} \sum_{\mathfrak{m}\in\mathfrak{M}_n} \prod_{\{i,j\}\in\mathfrak{m}} A^{-1}(x_i,x_j) \tag{3.2}$$

where A^{-1} is the integral kernel of the inverse of A. This definition works fine as long as $x_i \neq x_j$ for all $i \neq j$, but if two x_i's coincide, we run into trouble because A^{-1} is typically singular on the diagonal. In this section, we will resolve this issue by *normal ordering*, which has the effect of neglecting any terms containing $A^{-1}(x_i, x_i)$. However, for the purpose of gluing, we will need to resort to another mechanism explained in Section 5.
A standard combinatorial argument, for which we again refer to the literature (e.g. the references above), then shows that one can conveniently label all terms in integrals such as

$$\int_{C^\infty(\Sigma,\partial\Sigma)} e^{-\frac{1}{2\hbar}(\phi,A\phi)-\frac{1}{\hbar}p(\phi)},$$

where p is a polynomial, by graphs. These graphs are called Feynman graphs and the rules to evaluate them are called Feynman rules. Below, we will define the path integrals in question through these graphs and rules.

3.2. The path integral on a manifold with boundary

3.2.1. The general picture. Let Σ be a compact oriented Riemannian manifold with $\partial\Sigma = Y$. We may have $Y = \varnothing$. Recall from Section 2 that we then have a presymplectic manifold (C_Y, ω_Y). Let us assume it is symplectic. For the quantization we need some extra data, namely a polarization P_Y of C_Y. We assume that P_Y is such that the space of leaves B_Y of the associated foliation is a smooth manifold. Let $\mathrm{q}_Y : C_Y \to B_Y$ be the quotient map. If $\pi_\Sigma^{-1}(\mathrm{q}_Y^{-1}(\eta)) \cap EL_\Sigma$ is a finite set, then the formal expression

$$Z_\Sigma(\eta) = \int_{\pi_\Sigma^{-1}(\mathrm{q}_Y^{-1}(\eta))} e^{-\frac{S(\phi)}{\hbar}}\, D\phi \tag{3.3}$$

can be defined using the formal version of the method of steepest descent.

3.2.2. Free scalar theory. Let us again consider our main example, the free massive scalar field theory defined by the action

$$S(\phi) = \frac{1}{2}\int_\Sigma d\phi \wedge *d\phi + m^2 \phi \wedge *\phi$$

We recall that

$$\begin{aligned} C_Y &= C^\infty(Y) \oplus C^\infty(Y) \ni (\eta, \psi), \\ \pi_\Sigma(\phi) &= \left(\iota_Y^*(\phi), \iota_Y^*\left(\frac{\partial\phi}{\partial\nu}\right)\right), \\ \omega_Y &= \int_Y \delta\eta\delta\psi. \end{aligned}$$

In a symplectic vector space, a nice class of polarizations is given by Lagrangian subspaces. In particular, the splitting $C_Y = C^\infty(Y) \oplus C^\infty(Y)$ is Lagrangian, so that there are two obvious polarizations on C_Y. We will choose the polarization for which q_Y is the projection on the first component. Thus, for $\eta \in C^\infty(Y)$, we have

$$\pi_\Sigma^{-1}(q_Y^{-1}(\eta)) = \{\phi \in C^\infty(\Sigma), \iota_Y^*(\phi) = \eta\}$$

and

$$\mathrm{q}_Y^{-1}(\pi_\Sigma^{-1}(\eta)) \cap EL_\Sigma = \{\phi_\eta\}$$

where ϕ_η is the unique solution of the Dirichlet problem for $\Delta_\Sigma + m^2$ with boundary value η. The assignment $s\colon \eta \mapsto \phi_\eta$, defines a section of the short exact sequence of vector spaces

$$\begin{tikzcd} C^\infty(\Sigma, Y) \arrow[r, hook] & C^\infty(\Sigma) \arrow[r, two heads, "q_Y \circ \pi_\Sigma"] & C^\infty(Y) \arrow[l, bend left, "s"] \end{tikzcd}$$

Hence, we can write $\phi = \hat{\phi} + \phi_\eta$ where $\hat{\phi}$ vanishes on Y. Moreover, $S(\phi) = S(\hat{\phi}) + S(\phi_\eta)$. We can then rewrite[7] the formal expression (3.3) as follows:

$$\int_{\pi_\Sigma^{-1}(\mathrm{q}_Y^{-1}(\eta))} e^{-\frac{S(\phi)}{\hbar}}\, D\phi = e^{-\frac{S(\phi_\eta)}{\hbar}} \int_{\pi_\Sigma^{-1}(\mathrm{q}_Y^{-1}(0))} e^{-\frac{S(\hat{\phi})}{\hbar}}\, D\hat{\phi}$$

The latter expression is, formally, a Gaussian integral over the vector space $C_0^\infty(\Sigma)$ functions which vanish on the boundary of Σ. Thus it makes sense

[7] Assuming translation invariance of the functional measure $D\phi$.

to define it, analogously to the finite-dimensional case, as

$$Z_\Sigma(\eta) = \int_{\pi_\Sigma^{-1}(\mathrm{q}_Y^{-1}(\eta))} e^{-\frac{S(\phi)}{\hbar}} D\phi := (\det(\Delta_\Sigma + m^2))^{-\frac{1}{2}} e^{-\frac{S(\phi_\eta)}{\hbar}} \tag{3.4}$$

where, $S(\phi_\eta)$ is given by

$$S(\phi_\eta) = \frac{1}{2}\int_{\partial\Sigma} \eta D_\Sigma \eta \,\mathrm{dVol}_{\partial\Sigma} \tag{3.5}$$

and the Dirichlet boundary condition is used for the zeta-regularized determinant. In this paper, unless stated otherwise, the zeta-regularized determinant will be taken with respect to the Dirichlet boundary condition when the boundary is present.

Later on, we will be interested in a decomposition of the boundary into two components, $\partial\Sigma = \partial_L\Sigma \sqcup \partial_R\Sigma$ (these components are allowed to be empty or disconnected). Then $\eta = \eta_L + \eta_R$, where η_i is supported on $\partial_i\Sigma$. Since the Dirichlet-to-Neumann operator is symmetric, we can rewrite

$$S(\phi_\eta) = S(\phi_{\eta_L}) + S(\phi_{\eta_R}) + \int_{\partial\Sigma_L} \eta_L D_\Sigma \eta_R \,\mathrm{dVol}_{\partial\Sigma_L},$$

and the latter term can be expanded[8] as

$$\begin{aligned}\int_{\partial\Sigma_L} \eta_L D_\Sigma \eta_R \,\mathrm{dVol}_{\partial\Sigma_L} &= -\int_{(y,y')\in\partial\Sigma_L\times\partial\Sigma_R} \frac{\partial}{\partial\nu(x)}\frac{\partial G_\Sigma(x,y)}{\partial\nu(y)}\eta_L(x)\eta_R(y)dydy' \\ &=: S_{L,R}(\eta_L,\eta_R)\end{aligned}$$

Thus, the partition function can be expanded as

$$\begin{aligned}Z_\Sigma(\eta_L,\eta_R) &= \int_{\pi_\Sigma^{-1}(\mathrm{q}_Y^{-1}(\eta))} e^{-\frac{S(\phi)}{\hbar}} D\phi \\ &:= (\det(\Delta_\Sigma + m^2))^{-\frac{1}{2}} e^{-\frac{S(\phi_{\eta_L})}{\hbar}} e^{-\frac{S(\phi_{\eta_R})}{\hbar}} e^{-\frac{S_{L,R}(\eta_L,\eta_R)}{\hbar}}.\end{aligned} \tag{3.6}$$

3.2.3. Interacting theory. From now on, unless stated otherwise, Σ is assumed to be a *two-dimensional* Riemannian manifold.

[8]To condense the notation, we let $\int_Y f(y)dy := \int_Y f(y)\,\mathrm{dVol}_Y(y)$.

Let

$$p(\phi) = \sum_{k\geq 0} \frac{p_k}{k!}\phi^k$$

be a formal power series. We are interested in the interacting massive scalar field theory where the Lagrangian has the form

$$L(\phi) = \frac{1}{2}(d\phi \wedge *d\phi + m^2\phi \wedge *\phi) + *p(\phi)$$

so that the action functional is $S = S_0 + S_{\mathrm{int}}$ with

$$S_0 = \frac{1}{2}\int_\Sigma d\phi \wedge *d\phi + m^2\phi \wedge *\phi \text{ and } S_{\mathrm{int}} = \int_\Sigma *p(\phi).$$

We will consider this theory as a perturbation of the free theory, in order to define the perturbative partition function

$$Z_\Sigma(\eta, \hbar) = \int_{\pi_\Sigma^{-1}(\mathrm{q}_Y^{-1}(\eta))} e^{-\frac{S_0(\phi)+S_{\mathrm{int}}(\phi)}{\hbar}}\, D\phi$$

Let $\eta \in C^\infty(Y)$. The assignment $\eta \mapsto \phi_\eta$, where ϕ_η is the unique solution to the Dirichlet boundary value problem with boundary value η, defines a section of $q_Y \circ \pi_\Sigma : \mathcal{F}_\Sigma \to B_Y$. Hence, we can write $\phi = \hat{\phi} + \phi_\eta$ where $\hat{\phi}$ vanishes on Y. Moreover, $S_0(\phi) = S_0(\hat{\phi}) + S_0(\phi_\eta)$. Now, we can write

$$Z_\Sigma(\eta, \hbar) = e^{-\frac{S_0(\phi_\eta)}{\hbar}} \int_{\pi_\Sigma^{-1}(\mathrm{q}_Y^{-1}(0))} e^{-\frac{1}{\hbar}(S_0(\hat{\phi})+S_{\mathrm{int}}(\hat{\phi}+\phi_\eta))}\, D\hat{\phi}. \tag{3.7}$$

The integral on the right hand side is again a formal path integral, and as such we would like to define as a formal perturbed Gaussian integral as explained above.

There is a subtlety here: one would prefer the partition function to be a formal power series in $\hbar$, i.e. it should not contain negative powers of $\hbar$.[9] In the closed case, it is enough to assume $p_0 = p_1 = 0$ to achieve this. In the presence of boundary however, it is necessary to express the partition

[9] For instance, because a product of two formal Laurent series with infinitely many negative powers is generally ill-defined.

function instead in terms of the rescaled boundary field

$$\tilde{\eta} = \hbar^{-1/2}\eta \quad \Leftrightarrow \quad \eta = \hbar^{1/2}\tilde{\eta}.$$

In terms of $\tilde{\eta}$, (3.7) reads

$$Z_\Sigma(\tilde{\eta}, \hbar) = e^{-S_0(\phi_{\tilde{\eta}})} \int_{\pi_\Sigma^{-1}(\mathrm{q}_Y^{-1}(0))} e^{-\frac{1}{\hbar}(S_0(\hat{\phi}) + S_{\mathrm{int}}(\hat{\phi} + \sqrt{\hbar}\phi_{\tilde{\eta}})} D\hat{\phi}. \tag{3.8}$$

For the rest of the paper, we will work with the rescaled boundary field $\tilde{\eta}$, and we will treat it as an element of $C^\infty(\partial\Sigma)$. Unless otherwise stated, we will assume $p_0 = p_1 = p_2 = 0$.[10]

3.2.4. The space of boundary states - a perturbative model. Heuristically, the space of boundary states should be the space of square integrable functions on the space of leaves of the polarization: $H_Y = L^2(B_Y) = L^2(C^\infty(Y))$. Of course, in the field theory setting, one has to be very careful about the measure used to define these L^2 spaces, and in many cases it is convenient to drop the measure theory altogether and work with a different model for the space of states. Here we will present (Definitions 3.3, 3.9) a natural model in the context of perturbative quantization, in which a gluing formula can be formulated. In Section 7, we will revisit it from a more measure-theoretic perspective, and will introduce another pairing of states leading to a functorial interpretation of the gluing formula.

Let $C_n^\circ(Y)$ denote the open configuration space of n points in Y:

$$C_n^\circ(Y) = \{(y_1, \ldots, y_n) \in Y^n | y_i \neq y_j, \forall i \neq j\}.$$

To introduce the model for the space of states, we will need the following auxiliary definition (which is motivated by the properties of Feynman graphs, see Proposition 3.17 below).

Definition 3.2. *We say that a smooth function $f(y_1, \ldots, y_n)$ on the open configuration space $C_n^\circ(Y)$ has* admissible singularities on diagonals *if:*

---

[10]This condition guarantees that there are finitely many Feynman diagrams contributing to each order in $\hbar$. We can also relax this assumption and allow nonzero p_0, p_1, p_2, see the discussion in Section 3.4.1.

(a) f has (at most) logarithmic singularity when two points collide:

$$f = \mathrm{O}(\log d(y_i, y_j))$$

as $y_i \to y_j$.

(b) Assume that a set of $k \geq 3$ points $y_{i_1}, \ldots, y_{i_k}$ coalesce, so that pairwise distances satisfy $\epsilon < d(y_{i_r}, y_{i_s}) < C\epsilon$ (with C a constant). Then

$$f = \mathrm{O}\Big(\frac{1}{\epsilon^{k-2}}\Big)$$

as $\epsilon \to 0$.

(c) Behavior near a general diagonal: assume that several disjoint subsets $S_1, \ldots, S_p \subset \{1, \ldots, n\}$ of points coalesce at p different points on Y, with each coalescing cloud of points at pairwise distances of order ϵ. Then

$$f = \mathrm{O}\Big(\prod_{j=1}^{p} g_{|S_j|}(\epsilon)\Big)$$

as $\epsilon \to 0$, where

$$g_k(\epsilon) = \begin{cases} \log \epsilon, & k = 2 \\ \frac{1}{\epsilon^{k-2}}, & k \geq 3 \end{cases}$$

We will denote the space of smooth functions on $C_n^\circ(Y)$ with admissible singularities on diagonals allowed as $C^\infty_{\mathrm{adm}}(C_n^\circ(Y))$.

We remark that all functions in $C^\infty_{\mathrm{adm}}(C_n^\circ(Y))$ are integrable. However, they are generally not in L^p for $p \geq 2$ due to the singularities arising at a collapse of ≥ 3 points.

The perturbative model for the space of boundary states is constructed in two steps: first we introduce the "pre-space of states" and then (Definition 3.9) we will introduce the space of states proper as its appropriate completion.

Definition 3.3. *Let Y be a closed 1-dimensional manifold. For $n \in \mathbb{N}, n \geq 1$, we define $H_Y^{(n)}$ to be the space of functionals $\Psi\colon C^\infty(Y) \to \mathbb{R}$ of the form*

$$\Psi(\tilde{\eta}) = \int_{C_n^\circ(Y)} \psi(y_1, \ldots, y_n)\tilde{\eta}(y_1) \cdots \tilde{\eta}(y_n)\, dy_1 \ldots dy_n, \tag{3.9}$$

where ψ is the "wave function" of the state Ψ and it is a smooth, symmetric function on $C_n^\circ(Y)$ with admissible singularities (in the sense of Definition

3.2) allowed on diagonals. We say that ψ represents Ψ. We are allowing the wave function ψ to take values in formal power series $\mathbb{R}[[\hbar^{1/2}]]$. Moreover, we define $H^{(0)} = \mathbb{R}[[\hbar^{1/2}]]$. Finally, we define the pre-space of states[11] *as*

$$H^{\mathrm{pre}} = \bigoplus_{n\geq 0} H^{(n)}.$$

In particular, the vector space associated to the empty manifold $Y = \varnothing$ is just $\mathbb{R}[[\hbar^{1/2}]]$.

Remark 3.4. *Notice that*[12]

$$\begin{aligned} S_0(\phi^\Sigma_{\tilde\eta}) &= -\left[\frac{1}{2}\int_{\partial\Sigma\times\partial\Sigma} dy\, dy'\, \frac{\partial^2 G_\Sigma(y,y')}{\partial\nu(y)\partial\nu(y')}\tilde\eta(y)\tilde\eta(y')\right]_{\mathrm{reg}} \\ &:= -\lim_{\epsilon\to 0}\left(\frac{1}{2}\int_{(y,y')\in\partial\Sigma\times\partial\Sigma,\, d(y,y')>\epsilon} dy\, dy'\, \frac{\partial^2 G_\Sigma(y,y')}{\partial\nu(y)\partial\nu(y')}\tilde\eta(y)\tilde\eta(y') \right. \\ &\qquad \left. - \frac{1}{\pi\epsilon}\int_{\partial\Sigma} dy\, \widetilde{\eta}(y)^2\right) \end{aligned} \tag{3.10}$$

Since the second normal derivative of the Green's function behaves as $\mathrm{O}\left(\frac{1}{d(y,y')^2}\right)$*, it is worse than a logarithmic singularity allowed for a 2-point collapse and thus* $e^{-S_0(\phi^\Sigma_{\tilde\eta})} \notin H^{\mathrm{pre}}_{\partial\Sigma}$*.*

In fact this singularity is strong enough to be non-integrable on the diagonal of the configuration space and the integral needs to be understood in the regularized sense, as in the second line above.

Remark 3.5. *If Y has several components $Y = Y_1 \sqcup \ldots \sqcup Y_n$, then the associated pre-space of states factorizes as a (projective) tensor product*

$$H^{\mathrm{pre}}_Y \cong H^{\mathrm{pre}}_{Y_1} \otimes \cdots \otimes H^{\mathrm{pre}}_{Y_n}. \tag{3.11}$$

---

[11]This model for the pre-space of states is twice larger than what we need, in the following sense. The space H^{pre}_Y is $\frac{1}{2}\mathbb{Z}\times\mathbb{Z}$-bigraded by the $\hbar$-degree and polynomial degree in $\tilde\eta$, d_1 and d_2 respectively. The perturbative state induced from a surface on the boundary always has only monomials satisfying the "selection rule" $d_1 - \frac{d_2}{2} \in \mathbb{Z}$.

[12]One way to prove Eq. (3.10) is as follows. In $S_0[\phi_{\tilde\eta}] = \frac{1}{2}\int_\Sigma d\phi_{\tilde\eta} * d\phi_{\tilde\eta} + m^2\phi_{\tilde\eta} * \phi_{\tilde\eta} = \frac{1}{2}\int_\Sigma d(\phi_{\tilde\eta} * d\phi_{\tilde\eta})$ use the expansion $\phi_\eta(x) = -\int_{\partial\Sigma}\frac{\partial G_\Sigma(x,y')}{\partial\nu(y)}\eta(y')\,dy'$ for the second factor and use Stokes' theorem for the complement of a small half-disk of radius ϵ around y'. The second term in the second line in (3.10) arises (asymptotically) as the contribution of the boundary of that half-disk.

3.2.5. The gluing pairing. In this subsection we define the pairing that will be used to formulate the gluing theorem. The notation is as follows. We consider a cobordism $(\Sigma, \partial_L\Sigma, \partial_R\Sigma)$. We then consider a decomposition of Σ along a hypersurface (curve) Y: $\Sigma = \Sigma_L \cup_Y \Sigma_R$, such that $\partial\Sigma_L = \partial_L\Sigma \cup Y$ and $\partial\Sigma_R = \partial_R\Sigma \cup Y$. The heuristic idea to define the pairing is as follows: if Ψ_1 is a functional of boundary fields of the left cobordism $\tilde{\eta}_L, \tilde{\eta}_Y$, and Ψ_2 is a functional of the boundary fields $\tilde{\eta}_Y, \tilde{\eta}_R$ on the right cobordism, then we want to define

$$\langle \Psi_1, \Psi_2\rangle(\tilde{\eta}_L, \tilde{\eta}_R) = \int_{\tilde{\eta}_Y} \Psi_1(\tilde{\eta}_L, \tilde{\eta}_Y)\Psi_2(\tilde{\eta}_Y, \tilde{\eta}_R)\, \mathcal{D}\tilde{\eta}_Y,$$

where $\mathcal{D}\tilde{\eta}_Y$ is the "Lebesgue measure" on $C^\infty(Y)$. To get to a mathematical definition, we notice that partition functions always include a factor of $e^{-S_0(\phi_{\tilde{\eta}_Y})}$. Thus, it makes sense to extract that factor and thus arrive at a formal *Gaussian* measure on $C^\infty(Y)$, for which we can use the ideas of Section 3.1:

$$\begin{aligned}&\int_{\tilde{\eta}_Y} \Psi_1(\tilde{\eta}_L, \tilde{\eta}_Y)\Psi_2(\tilde{\eta}_Y, \tilde{\eta}_R)\, \mathcal{D}\tilde{\eta}_Y \\ &\text{``}{=}\text{''} \int_{\tilde{\eta}_Y} \widehat{\Psi_1}(\tilde{\eta}_L, \tilde{\eta}_Y)\widehat{\Psi_2}(\tilde{\eta}_Y, \tilde{\eta}_R) e^{-S_0(\phi^{\Sigma_L}_{\tilde{\eta}_Y}) - S_0(\phi^{\Sigma_R}_{\tilde{\eta}_Y})}\, \mathcal{D}\tilde{\eta}_Y\end{aligned}$$

With this idea in mind, we now define a map describing the formal integral over $C^\infty(Y)$ with respect to $e^{-S_0(\phi^{\Sigma_L}_{\tilde{\eta}_Y}) - S_0(\phi^{\Sigma_R}_{\tilde{\eta}_Y})}\mathcal{D}\tilde{\eta}_Y$.

Definition 3.6. *Let $\Sigma = \Sigma_L \cup_Y \Sigma_R$ and D_{Σ_L,Σ_R} the Dirichlet-to-Neumann operator along Y defined in Remark 2.2. Let K be the integral kernel of the inverse of D_{Σ_L,Σ_R}. We define the map $\langle\cdot\rangle_{\Sigma_L,Y,\Sigma_R}\colon H_Y^{(n)} \to \mathbb{C}$, called the* expectation value map*, by*

$$\langle \Psi\rangle_{\Sigma_L,Y,\Sigma_R} = \frac{1}{\det(D_{\Sigma_L,\Sigma_R})^{\frac{1}{2}}} \sum_{\mathfrak{m}\in\mathfrak{M}_n} \int_{C_n^\circ(Y)} \psi(y_1,\ldots,y_n) \cdot \prod_{\{v_1,v_2\}\in\mathfrak{m}} K(y_{v_1}, y_{v_2})\, dy_1\cdots dy_n. \tag{3.12}$$

The extension of this map to H_Y^{pre} is also denoted by $\langle\cdot\rangle_{\Sigma_L,Y,\Sigma_R}$.

Since $K(y,y') = \mathrm{O}(\log d(y,y'))$ at $y \to y'$ and ψ has admissible singularities on diagonals, the integral is convergent.

Remark 3.7. *This map is of course nothing but a formal integration over the field $\tilde{\eta} \in C^\infty(Y)$. It has an interesting interpretation as the expectation value of the observable $\Psi \colon C^\infty(Y) \to \mathbb{C}$ with respect to the theory on Y with space of fields $C^\infty(Y)$ and (non-local) action functional*

$$S_Y = \int_Y \tilde{\eta} D_{\Sigma_L,Y,\Sigma_R} \tilde{\eta} \,\mathrm{dVol}_Y.$$

This explains the notation.

Since sets of odd cardinality do not have any perfect matchings, the map $\langle\cdot\rangle$ vanishes on $H^{(n)}$, for n odd.

For $\Psi \in H^{(n)}$, $\Psi' \in H^{(m)}$ two states with wave functions $\psi \in C^\infty_{\mathrm{adm}}(C^\circ_n(Y))$, $\psi' \in C^\infty_{\mathrm{adm}}(C^\circ_m(Y))$, we can form a new state $\Psi \odot \Psi' \in H^{(n+m)}$ whose wave function is the symmetrized tensor product $\psi \odot \psi' \in C^\infty_{\mathrm{adm}}(C^\circ_{n+m}(Y))$ given by

$$\begin{aligned}(\psi \odot \psi')&(y_1, \ldots, y_{n+m}) \\ &= \frac{1}{(n+m)!} \sum_\sigma \psi(y_{\sigma(1)}, \ldots, y_{\sigma(n)})\, \psi'(y_{\sigma(n+1)}, \ldots, y_{\sigma(n+m)})\end{aligned} \tag{3.13}$$

with σ running over permutations of $n+m$ elements.

The pairing is then simply the composition of the expectation value map with the multiplication $\odot$:

Definition 3.8. *Let $\Sigma = \Sigma_L \cup_Y \Sigma_R$. Then, we define a pairing*

$$\langle\cdot,\cdot\rangle_{\Sigma_L,Y,\Sigma_R} \colon H^{\mathrm{pre}}_Y \times H^{\mathrm{pre}}_Y \to \mathbb{R}[[\hbar^{1/2}]]$$

by

$$\left\langle \Psi, \Psi' \right\rangle_{\Sigma_L,Y,\Sigma_R} = \left\langle \Psi \odot \Psi' \right\rangle_{\Sigma_L,Y,\Sigma_R} \tag{3.14}$$

and extending bilinearly.

Definition 3.9. *We define the space of states H_Y associated to a Riemannian 1-manifold Y as the completion (order-by-order in $\hbar$) of H^{pre}_Y with respect to the pairing $\langle\cdot,\cdot\rangle_{\Sigma_L,Y,\Sigma_R}$.*

Remark 3.10. *If $\Sigma = \Sigma_L \cup_Y \Sigma_R$ with $\partial\Sigma_i = Y_i \sqcup Y$, $i \in \{L, R\}$, then $\partial\Sigma = Y_L \sqcup Y_R$. By using the isomorphisms $H_{\partial\Sigma_i} \cong H_{Y_i} \otimes H_Y$ and $H_{\partial\Sigma} \cong H_{Y_L} \otimes$*

H_{Y_R}, *the pairing extends to a map*

$$\langle \cdot, \cdot \rangle_{\Sigma_L, Y, \Sigma_R} \colon H_{\partial \Sigma_L} \otimes H_{\partial \Sigma_R} \to H_{\partial \Sigma}.$$

3.3. Feynman graphs

In this subsection we introduce the Feynman graphs relevant for this paper.

Definition 3.11. *A Feynman graph Γ is given by the following data:*

1) *Three disjoint finite sets (V_b, V_L, V_R), called the set of bulk and left resp. right boundary vertices. Their union, $V = V_b \sqcup V_L \sqcup V_R$ is called the set of vertices. $V_\partial = V_L \sqcup V_R$ is called the set of boundary vertices.*
2) *A finite set H with an incidence map $i \colon H \to V$*
3) *An involution $\tau \colon H \to H$ without fixed points (representing the edges)*

such that for all $v \in V_L \sqcup V_R$, we have $|i^{-1}(v)| = 1$ (boundary vertices are univalent).

The edge set $E(\Gamma)$ of the graph is by definition the set of orbits of τ. We denote by $E_i(\Gamma)$ the edges that contain i boundary vertices. Thus $E(\Gamma) = E_0(\Gamma) \sqcup E_1(\Gamma) \sqcup E_2(\Gamma)$. We give them different graphical representations (see Table 1). Some examples of graphs are shown in Figure 3.

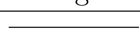
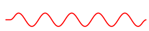
edge	set	name
	E_0	bulk edge
	E_1	bulk-boundary edge
	E_2	boundary-boundary edge

Table 1. Edges in Feynman diagrams.

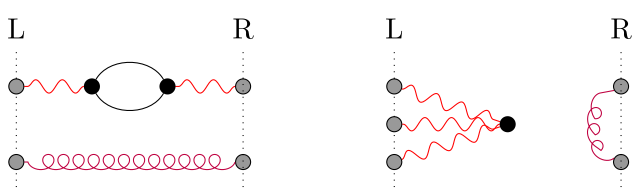


Figure 3. Some examples of Feynman graphs.

We shall also require the notion of automorphism of a graph.

Definition 3.12. *An automorphism φ of a graph Γ is given by a pair of bijections:*

$$V(\varphi)\colon V \to V$$

and

$$H(\varphi)\colon H \to H$$

which commute with the incidence map i and the involution τ, i.e.

$$\begin{array}{ccc} H & \xrightarrow{i} & V \\ \downarrow{\scriptstyle H(\varphi)} & & \downarrow{\scriptstyle V(\varphi)} \\ H & \xrightarrow{i} & V \end{array}$$

V is required to respect the decomposition $V = V_b \sqcup V_L \sqcup V_R$.

The automorphism also induces maps on the sets of edges, denoted by $E_i(\varphi)$ or $E(\varphi)$.

Below we will rely on the following simple observation:

Proposition 3.13. *Suppose all bulk vertices are at least trivalent. Then $\ell(\Gamma) := |E(\Gamma)| - |V_b(\Gamma)| - \frac{1}{2}|V_\partial(\Gamma)| \geq 0$, with equality if and only if there are no bulk vertices.*

Proof. The assumption implies that the number of half-edges in the graph is at least $3|V_b(\Gamma)| + |V_\partial(\Gamma)|$. This implies

$$|E(\Gamma)| - \frac{3}{2}|V_b(\Gamma)| - \frac{1}{2}|V_\partial(\Gamma)| \geq 0,$$

which in turn implies the statement. □

3.4. Feynman rules and the perturbative path integral

Associated to a graph Γ is a certain configuration space C_Γ:

Definition 3.14. *Given a Feynman graph Γ, we define the associated configuration space of Γ in a cobordism $(\Sigma, \partial_L\Sigma, \partial_R\Sigma)$ as*

$$C^\circ_\Gamma(\Sigma) \equiv C^\circ_\Gamma := \{f\colon V \to \Sigma, f \text{ injective}, f(V_L) \subset \partial_L\Sigma, f(V_R) \subset \partial_R\Sigma\}$$

If Γ has k_l resp k_r left resp. right boundary vertices and l bulk vertices, picking an enumeration of V_b, V_L, V_R identifies C_Γ° as the open subset of $\Sigma^l \times \partial_L\Sigma^{k_l} \times \partial_R\Sigma^{k_r}$ given by removing all diagonals. We now define the weight $F(\Gamma)$ as a functional of the boundary fields by associating a certain function (depending on the boundary fields) on C_Γ° to the graph and integrating it over C_Γ° against the measure $\mathrm{dVol}_{C_\Gamma^\circ(\Sigma)}$ induced by the embedding into $\Sigma^l \times \partial_L\Sigma^{k_l} \times \partial_R\Sigma^{k_r}$. Namely, $F(\Gamma)$ can be defined as follows:

Definition 3.15. *Let Γ be a Feynman graph with n boundary vertices $v_1, \ldots, v_n$ and no short loops. Let $\pi_i\colon C_\Gamma^\circ(\Sigma) \to Y$ denote the projection to the i-th boundary point. Then $F(\Gamma)$ is the map $F(\Gamma)\colon C^\infty(\Sigma) \to \mathbb{R}$ defined by*

$$F(\Gamma)[\tilde\eta] = \int_{C_\Gamma(\Sigma)} \omega_\Gamma \pi_1^*\tilde\eta \cdots \pi_n^*\tilde\eta \,\mathrm{dVol}_{C_\Gamma^\circ(\Sigma)} \tag{3.15}$$

where

$$\omega_\Gamma = \prod_{v\in V_b(\Gamma)} (-p_{val(v)}) \prod_{\{\alpha,\beta\}\in E_0} G_\Sigma(x_\alpha, x_\beta)\cdot \cdot \prod_{\{x_\alpha,y_i\}\in E_1} \Big(-\frac{\partial G_\Sigma(x_\alpha,y_i)}{\partial\nu(y_i)}\Big) \prod_{\{y_i,y_j\}\in E_2} \Big(-\frac{\partial^2 G_\Sigma(y_i,y_j)}{\partial\nu(y_i)\partial\nu(y_j)}\Big) \tag{3.16}$$

is the product of propagators and their normal derivatives according to the combinatorics of the graph.[13]

Remark 3.16. *The configuration space C_Γ° has a natural map p to $C_\Gamma^\partial(\Sigma) := C_{V_L}^\circ(\partial_L\Sigma) \times C_{V_R}^\circ(\partial_R\Sigma)$ given by forgetting the bulk points. The fiber of this map over a pair f_1, f_2 of configurations is the open configuration space of $\Sigma \setminus (f_1(V_L) \cup f_2(V_R))$. Thus, we can define $\psi_\Gamma := p_*\omega_\Gamma$, where p_* denotes integration (pushforward) along the fibers of p. ψ_Γ is a function on $C_\Gamma^\partial(\Sigma)$ whose regularity we will study below. We can then rewrite* (3.15) *as*

$$F(\Gamma)[\tilde\eta] = \int_{C_\Gamma^\partial(\Sigma)} \psi_\Gamma \pi_1^*\tilde\eta \ldots \pi_n^*\tilde\eta \,\mathrm{dVol}_{(\partial\Sigma)^n}. \tag{3.17}$$

We will call ψ_Γ the wave function *associated to Γ. Even though ψ_Γ does not need to be symmetric under permutation of the boundary points, only its symmetric part will contribute to the integral* (3.17)*.*

[13]In the case if Γ contains boundary-boundary edges connecting a boundary component to itself, the integral (3.15) needs to be regularized as in Remark 3.4.

We will now show that the coefficients of Feynman graphs have the nice regularity properties that we want.

Proposition 3.17. *Let Γ be a graph without short loops and without boundary-boundary edges connecting ∂_L to ∂_L or ∂_R to ∂_R. Then the corresponding wave function ψ_Γ is a smooth function on the open configuration space with admissible singularities on diagonals, as in Definition 3.2:*

$$\psi_\Gamma \in C^\infty_{\mathrm{adm}}(C^\partial_\Gamma(\Sigma))$$

Proof. Given a graph Γ, we are interested in the integral

$$\begin{aligned} f(y_1,\ldots,y_n) = \int_{\Sigma^N} d^2x_1\cdots d^2x_N \prod_{(\alpha,\beta)\in E_0} G^0(x_\alpha,x_\beta)\cdot \\ \cdot \prod_{(\alpha,i)\in E_1} G^1(x_\alpha,y_i)\cdot \prod_{(i,j)\in E_2} G^2(y_i,y_j) \end{aligned} \tag{3.18}$$

as a function of n pairwise distinct boundary points $y_1,\ldots,y_n\in Y$. Note that $\psi_\Gamma = f\cdot(-1)^{\#E_1+\#E_2}\prod_{v\in V_b}(-p_{val(v)})$ – a constant multiple of f. Here we denoted $N=|V_b|$ the number of bulk vertices, $n=|V_L|+|V_R|$ the number of boundary vertices; we denoted G^k the Green's function ($k=0$), its first ($k=1$) or second ($k=2$) normal derivative at the boundary appearing in (3.16). Note that we have the following asymptotics:

$$\begin{aligned} G^0(x_\alpha,x_\beta) \underset{x_\alpha\to x_\beta}{\sim} -\frac{1}{2\pi}\log d(x_\alpha,x_\beta), \\ G^1(x_\alpha,y_i) \underset{x_\alpha\to y_i}{=} \mathrm{O}\left(\frac{1}{d(x_\alpha,y_i)}\right), \quad G^2(y_i,y_j) \underset{y_i\to y_j}{=} \mathrm{O}\left(\frac{1}{d(y_i,y_j)^2}\right). \end{aligned} \tag{3.19}$$

In fact, the arguments of G^2 never approach each other in (3.18), since in Γ we didn't allow boundary-boundary edges connecting a boundary component to itself.

First note that, for $y_1,\ldots,y_n$ fixed pairwise distinct boundary points, (3.18) is a convergent integral: if $p\geq 2$ bulk points $x_{\alpha_1},\ldots,x_{\alpha_p}$ coalesce at pairwise distances $\epsilon < d(x_{\alpha_r},x_{\alpha_s}) < C\epsilon$, the integrand behaves as $\mathrm{O}(\log^a\epsilon)$ (with a the number of edges in the collapsing subgraph), which gives an integrable singularity. If $p\geq 1$ bulk points collapse at a boundary point y_i, the integrand behaves as

$$\mathrm{O}\Big(\frac{\log^a\epsilon}{\epsilon^b}\Big)$$

where $b = 1$ if in Γ the vertex y_i is connected to one of the collapsing bulk points and $b = 0$ otherwise. This again gives an integrable singularity.[14]

Smoothness on the open configuration space of the boundary follows from standard arguments.

Next, we turn to the analysis of the singularities. Consider the asymptotic regime for (3.18) when y_i approaches y_j. In the limit $y_i \to y_j$, the integral converges unless there is a bulk vertex x_α connected to both y_i and y_j. If there is such a vertex x_α, the integral is divergent as $y_i \to y_j$. Consider the integral over a half-disk D^+_δ of radius $\delta \gg d(y_i, y_j)$ centered at y_j, where D^+_δ is contained in a geodesic normal chart. The integral in x_α over the complement $\Sigma \setminus D^+_\delta$ does not create a singularity. The integral $\int_{D^+_\delta} \frac{d^2 x_\alpha}{d(x_\alpha,y_i)d(x_\alpha,y_j)}$ can be modeled by the corresponding integral for the flat case, since the leading singularities are the same in both cases and the metric in geodesic normal coordinates satisfies $g_{ij} = \delta_{ij} + \mathrm{O}(\delta^2)$. Thus, one obtains the estimate[15]

$$\int_{D^+_\delta} \frac{d^2 x_\alpha}{d(x_\alpha, y_i) d(x_\alpha, y_j)} \underset{y_i \to y_j}{=} \mathrm{O}\Big(\log \frac{\delta}{d(y_i, y_j)}\Big) \tag{3.20}$$

Thus, the worst possible singularity of (3.18) at a codimension one diagonal of the configuration space is the logarithmic one.

Consider a subset of boundary points $y_{i_1}, \ldots, y_{i_k}$, with $k \geq 3$ coalescing at pairwise distances $\epsilon < d(y_{i_r}, y_{i_s}) < C\epsilon$. In this asymptotic regime, the strongest singularity in (3.18) arises from the situation when a single bulk vertex x_α connected to each of the coalescing y's by an edge, is colliding onto them. By a similar argument to the above, this situation is modeled by an integral over a (flat) half-plane

$$\int_{\Pi_+} \frac{d^2 x_\alpha}{d(x_\alpha, y_{i_1}) \cdots d(x_\alpha, y_{i_k})} \underset{\epsilon \to 0}{=} \mathrm{O}\Big(\frac{1}{\epsilon^{k-2}}\Big) \tag{3.21}$$

[14] Note that here it is essential that the boundary vertices are univalent - otherwise we could have gotten $b > 1$ which would lead to a non-integrable singularity.

[15] Indeed, denote that the integral in the l.h.s. of (3.20) by J. It is a function of δ and the distance $d(y_i, y_j)$. Set $y_j = 0$ for convenience. Making a rescaling $x_\alpha \mapsto \Lambda x_\alpha, y_i \mapsto \Lambda y_i$ under the integral, we see that $J(\Lambda\delta, \Lambda d(y_i, y_j)) = J(\delta, d(y_i, y_j))$. Therefore, $J = J(\frac{\delta}{d(y_i,y_j)})$. Next, the integrand behaves in J behaves as $\sim \frac{1}{r^2}$, with $r = d(x_\alpha, y_j)$, when $r \gg d(y_i, y_j)$. Therefore, in the asymptotic regime $\delta \gg d(y_i, y_j)$, we have $J \sim \int_0^\pi d\theta \int_{Cd(y_i,y_j)}^\delta \frac{r dr}{r^2} \sim \pi \log \frac{\delta}{d(y_i,y_j)}$.

This estimate follows from a scaling argument: denoting the l.h.s. by $I(y_{i_1},\ldots,y_{i_k})$, we have $I(\Lambda y_{i_1},\ldots,\Lambda y_{i_k}) = \Lambda^{2-k} I(y_{i_1},\ldots,y_{i_k})$ for any $\Lambda > 0$, as follows from a scaling substitution in the integral $y_{i_r} \mapsto \Lambda y_{i_r}$, $x_\alpha \mapsto \Lambda x_\alpha$.

Finally, if we have a simultaneous collapse of several subsets of boundary points (at different points on the boundary), the respective worst-case-scenario asymptotics is given by a product of model integrals (3.20), (3.21) corresponding to the collapsing subsets. $\square$

It should be noted that naively extending the definition of the Feynman rules to diagrams with self-loops would yield ill-defined results, as the Green's function is singular on the diagonal. One way to overcome this divergence problem is to not apply the formal integral to the exponential of the action, but to apply *normal ordering* before applying the formal integral. Put simply, this has the effect of removing short loops[16]. This leads to the following definition[17].

Definition 3.18. *We define the* normal ordered perturbative partition function (3.7) *by*

$$Z^{\mathrm{no}}_{\Sigma}(\tilde\eta_L,\tilde\eta_R) := \frac{1}{\det(\Delta_\Sigma + m^2)^{\frac12}} \sum_{\Gamma} \frac{\hbar^{\ell(\Gamma)} F(\Gamma)[\tilde\eta_L,\tilde\eta_R]}{|\operatorname{Aut}(\Gamma)|} \tag{3.22}$$

where the sum is over all Feynman graphs without self-loops, $\ell(\Gamma) = |E(\Gamma)| - |V_b(\Gamma)| - |\frac12 V_\partial(\Gamma)| \in \frac12\mathbb{Z}_{\geq 0}$ and $F(\Gamma)$ is the Feynman weight of the Feynman graph Γ.

Remark 3.19. *Since boundary vertices are univalent, the contributions of the E_2 edges can be factored out. They yield precisely the exponential of $-S_0(\phi_{\tilde\eta})$. Hence, we can write*

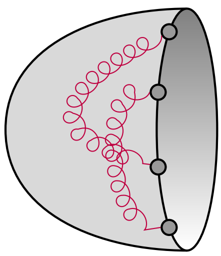
$$\begin{aligned} Z^{\mathrm{no}}_{\Sigma}(\tilde\eta_L,\tilde\eta_R) &= \frac{e^{-S_0(\phi_{\tilde\eta_L+\tilde\eta_R})}}{\det(\Delta_\Sigma + m^2)^{\frac12}} \sum_{\{\Gamma\colon E_2(\Gamma)=\emptyset\}} \frac{\hbar^{\ell(\Gamma)} F(\Gamma)[\tilde\eta_L,\tilde\eta_R]}{|\operatorname{Aut}(\Gamma)|}. \\ &= Z^{\mathrm{free}}_{\Sigma}(\widetilde\eta_L,\widetilde\eta_R) Z^{\mathrm{pert,no}}_{\Sigma}(\widetilde\eta_L,\widetilde\eta_R) \end{aligned}$$

---

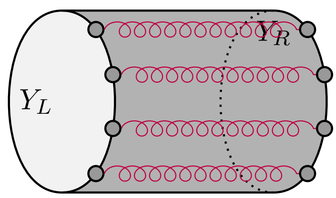
[16] We refer to the literature, e.g. [22] for an explanation of why this is the case.

[17] This definition is just a neat way to rewrite the result of a formal computation of the path integral (3.7) using the methods sketched in Section 3.1, for a deeper discussion, we refer again to the literature, e.g. [47],[49],[43].

Here Z_Σ^{free} *is the partition function of the free theory* (3.6) *and* $Z_\Sigma^{\text{pert,no}}(\widetilde{\eta}_L, \widetilde{\eta}_R)$ *is given by the sum of all diagrams containing no boundary edges. By construction, there are only finitely many diagrams at each order in* $\hbar$ *contributing to* $Z_\Sigma^{\text{pert,no}}$. *Thus,* $Z_\Sigma^{\text{pert,no}} \in H_{\partial\Sigma}^{\text{pre}}$. *Expanding* $S_0(\phi_{\tilde{\eta}_L+\tilde{\eta}_R}) = S_0(\phi_{\tilde{\eta}_L}) + S_0(\phi_{\tilde{\eta}_R}) + S_{L,R}(\tilde{\eta}_L, \tilde{\eta}_R)$, *we see that the first two terms generate Feynman diagrams connecting the left resp. right boundary to themselves, while the third term generates diagrams connecting the two, see Figure 4 below. This observation will be important in the proof of the gluing formula.*



(a) Graphs contributing to the exponential prefactor $e^{-\frac{1}{2}\int_Y \eta D_\Sigma \eta \mathrm{dVol}_Y}$. Curled red edges are decorated with normal derivatives of Green's functions at both boundary points.



(b) Graphs contributing to the exponential factor containing the "off-diagonal Y_L-Y_R" block of the Dirichlet-to-Neumann operator.

Figure 4. Expansion of different "blocks" of Dirichlet-to-Neumann operator

We further define the adjusted partition function as

$$\begin{aligned}\widehat{Z_\Sigma^{\text{no}}}(\widetilde{\eta}_L, \widetilde{\eta}_R) &= e^{S_0(\phi_{\tilde{\eta}_L})+S_0(\phi_{\tilde{\eta}_R})} Z_\Sigma^{\text{no}}(\widetilde{\eta}_L, \widetilde{\eta}_R) \\ &= \frac{e^{-S_{L,R}(\tilde{\eta}_L, \tilde{\eta}_R)}}{\det(\Delta_\Sigma + m^2)^{\frac{1}{2}}} \sum_{\{\Gamma :\, E_2(\Gamma)=\emptyset\}} \frac{\hbar^{\ell(\Gamma)} F(\Gamma)[\tilde{\eta}_L, \tilde{\eta}_R]}{|\operatorname{Aut}(\Gamma)|} \\ &\in H_{\partial_L\Sigma} \otimes H_{\partial_R\Sigma}\end{aligned} \tag{3.23}$$

Remark 3.20. *Note that (3.22) is not an element of the space of states (due to bad singularity in the exponential prefactors, cf. Remark 3.4), whereas (3.23) is in the space of states.*[18]

[18]Note however that (3.23) in general is not in the pre-space of states (which was was defined via direct sum, not a direct product) due to infinitely many diagrams of type 4b contributing in the order $O(\hbar^0)$. In the completion, (3.23) is a legitimate element.

3.4.1. Digression: Allowing coefficients p_0, p_1, p_2. The assumption that $p_0 = p_1 = p_2 = 0$ is important to have finitely many diagrams contributing in every order in $\hbar$ (excluding the boundary-boundary edges). In fact, we can reduce the general case to the case where these coefficients are absent by resumming the corresponding diagrams. Consider again the path integral

$$Z_\Sigma^{m,p}(\eta,\hbar) = \int_{\pi_\Sigma^{-1}(\mathrm{q}_Y^{-1}(\eta))} e^{-\frac{S_0(\phi)+S_{\mathrm{int}}(\phi)}{\hbar}}\, D\phi$$

for non-rescaled boundary field, with p_0, p_1, p_2 possibly non-zero.

- The linear term $p_1\phi$ in $p(\phi)$ shifts the critical point of $\frac{1}{2}m^2\phi^2 + p(\phi)$ away from zero to some value $\phi_{\mathrm{cr}} \in p_1\mathbb{R}[[p_1]]$. Let

  $$p(\phi_{\mathrm{cr}} + \psi) = \widetilde{p}(\psi) = \sum_{n\geq 0, n\neq 1} \frac{\widetilde{p}_n}{n!}\psi^n$$

  Note that $\widetilde{p}(\psi)$ is a power series in the shifted field ψ with vanishing linear term. Thus we obtain

  $$Z_\Sigma^{m,p}(\eta,\hbar) = Z^{m,\tilde{p}}(\eta - \phi_{\mathrm{cr}}, \hbar).$$

  This corresponds to resumming the tree diagrams generated by the term $p_1\phi$ in the action. In fact, one can show[19] that

  $$\phi_{\mathrm{cr}}(x) = \sum_{T \text{ rooted tree}} F(T)$$

  where the root of the tree is labeled by $x \in \Sigma$, see Figure 5.

- The constant term $\widetilde{p}_0$ in the new potential $\widetilde{p}(\psi)$ can be carried out of the path integral. This corresponds to resumming the disconnected vertices generated by $\widetilde{p}_0$. Thus, we obtain

  $$Z^{m,\tilde{p}}(\eta - \phi_{\mathrm{cr}}, \hbar) = e^{-\frac{\widetilde{p}_0}{\hbar}\mathrm{Area}(\Sigma)}\, Z_\Sigma^{m,\widetilde{p}_{\geq 2}}(\eta - \phi_{\mathrm{cr}}, \hbar).$$

[19] This follows from writing the critical point equation $m^2\phi + p'(\phi) = 0$ in the form $\phi = -\frac{p_1}{m^2} - \frac{1}{m^2}\sum_{n\geq 2}\frac{p_n}{(n-1)!}\phi^{n-1} =: \Xi(\phi)$. Its solution can be then written as the limit of iterations $\phi_{\mathrm{cr}} = \lim_{N\to\infty}\Xi^N(0)$ which can in turn be presented as a sum over rooted trees. Here we note that for ϕ a constant function, one has $\Xi(\phi)(x) = \int d^2y\, G(x,y)(-p_1) + \int d^2y\, G(x,y)\sum_{n\geq 2}\frac{-p_n}{(n-1)!}\phi(y)^{n-1}$ (which is also a constant function).

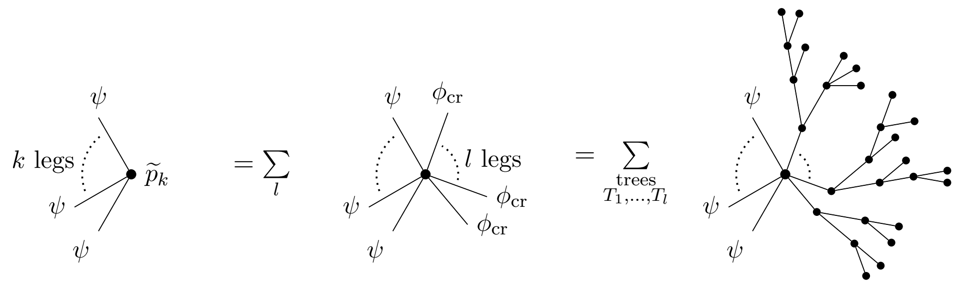


Figure 5. Resumming trees into $\widetilde{p}$.

- Finally, the quadratic term can be accounted for as a shift of mass, $\widetilde{m}^2 = m^2 + p_2$. Indeed, resumming diagrams containing a binary vertex (see Figure 6) we obtain a new propagator $G' = \sum_{k\geq 0}(-p_2)^k G^{k+1}$ which coincides with the Neumann series for $A + p_2$, where $A = \Delta + m^2$:

$$(A + p_2 I)^{-1} = A^{-1}\left(I - (-p_2 A^{-1})\right)^{-1} = A^{-1}\sum_{k\geq 0}(-p_2)^k A^{-k}.$$

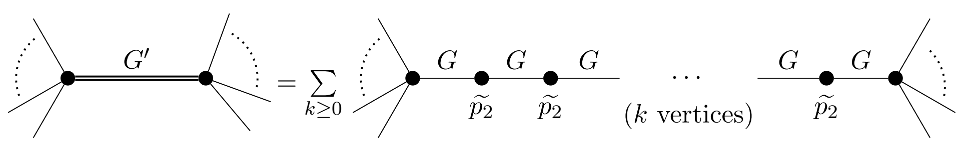


Figure 6. Resumming binary vertices

Thus, the path integral for the interaction potential p reduces, by these manipulations (shift of the integration variable by a constant $\phi \to \psi$, carrying a constant out and absorbing a quadratic term into the free part), to a path integral with interaction potential $\widetilde{p}_{\geq 3} = \sum_{n\geq 3}\frac{\widetilde{p}_n}{n!}\psi^n$ and mass $\widetilde{m}$:

$$Z_\Sigma^{m,p}(\eta,\hbar) = e^{-\frac{\widetilde{p}_0}{\hbar}\mathrm{Area}(\Sigma)}\ Z_\Sigma^{\widetilde{m},\widetilde{p}_{\geq 3}}(\eta - \phi_{\mathrm{cr}},\hbar).$$

Mathematically, the r.h.s. here should be regarded as the definition of the left hand side.

4. Heuristic analysis of path integrals and gluing formulae

In this section, we discuss the heuristic analysis of path integrals associated to the free massive scalar field theory on compact oriented Riemannian manifolds and explain how they lead to the gluing formula for zeta regularized determinants and Green's functions. Furthermore, we give a rigorous proof of the gluing formula (see Proposition (4.2) below) for the Green's functions.

In this section Σ is a compact oriented Riemannian manifold, not necessarily of dimension two.

4.1. BFK gluing formula for the zeta-regularized determinants

First, we consider the partition function of the free massive scalar field theory and explain how it leads to the Burghelea-Friedlander-Kappeler (BFK) gluing formula for the zeta-regularized determinants [5, 37].

We are interested in the path integrals of the form:[20]

$$Z_\Sigma = \int_{\mathcal{F}_\Sigma} e^{-S_0(\phi)}\, D\phi \tag{4.1}$$

When Σ is closed, we can rewrite

$$S_0(\phi) = \frac{1}{2}\int_\Sigma \phi(\Delta_\Sigma + m^2)\phi\, \mathrm{dVol}(\Sigma)$$

This means that the integral in (4.1) is a Gaussian integral. Hence, we have

$$Z_\Sigma = \det(\Delta_\Sigma + m^2)^{-\frac{1}{2}}.$$

More generally, if $\partial\Sigma = Y$ and $Y \neq \varnothing$, the partition function is defined by (3.4), which is:

$$Z_\Sigma(\eta) = (\det(\Delta_\Sigma + m^2))^{-\frac{1}{2}} e^{-S(\phi_\eta)}.$$

Let Σ be a closed oriented Riemannian manifold obtained by gluing two compact oriented Riemannian manifolds glued along a common boundary component Y: $\Sigma = \Sigma_L \cup_Y \Sigma_R$, $\partial\Sigma_L = Y$ and $\partial\Sigma_R = \overline{Y}$. Recall that for $\eta \in$

[20]For the purpose of this section we can set $\hbar = 1$.

$C^\infty(Y)$, we have for $i \in \{L, R\}$,

$$Z_{\Sigma_i}(\eta) = (\det(\Delta_{\Sigma_i} + m^2))^{-\frac{1}{2}} e^{-S_i(\phi_\eta)}$$

where S_i are defined as in (3.5). Let us assume that there is a formal Fubini's theorem (also known as locality of path integrals):

$$\int_{\mathcal{F}_\Sigma} e^{-S(\phi)}\, D\phi = \int_\eta \left(\int_{\pi^{-1}(\eta)} e^{-(S_L(\phi_L)+S_R(\phi_R))}\, D\Phi \right) D\eta. \tag{4.2}$$

Then, this suggests the following gluing relation for the zeta-regularized determinants:

$$\det(\Delta_\Sigma + m^2) = \det(\Delta_{\Sigma_L} + m^2) \det(\Delta_{\Sigma_R} + m^2) \det(D_{\Sigma_L,\Sigma_R}) \tag{4.3}$$

In summary, the locality of path integrals suggests a gluing formula for the zeta-regularized determinants. In fact, (4.3) is a theorem first proved by BFK in [5] when Σ is two-dimensional. It was later generalized to the case of arbitrary even dimension by Lee [37] under the assumption that the Riemannian metric is a product metric near the boundary.

Remark 4.1. *The gluing relation (4.3) admits a generalization to the case when Y is not necessarily a dividing hypersurface: for $Y \subset \Sigma$ any compact hypersurface, one has* $\det(\Delta_\Sigma + m^2) = \det(\Delta_{\Sigma\backslash Y} + m^2)\ \det(D_\Sigma)$. *Here $\Sigma\backslash Y$ is understood as Σ with two additional boundary components, Y and $\bar{Y}$; D_Σ is the sum of Dirichlet-to-Neumann operators at Y and at $\bar{Y}$.*

4.2. Path integral representation of Green's function and a gluing relation

In this subsection, we again consider the free massive scalar theory. Let Σ be a compact oriented Riemannian manifold with $\partial\Sigma = Y$ and $f \in C^\infty(\Sigma)$. Let us define an observable O_f, which is by definition a function on the space of fields, by

$$O_f(\phi) = \int_\Sigma f\phi\, \mathrm{dVol}_\Sigma$$

The expectation of value of O_f defines a function on the space of boundary fields:

$$\begin{aligned}\langle O_f\rangle(\eta) &= \int_{\pi^{-1}(\eta)} e^{-S(\phi)} O_f\, D\phi \\ &= \int_{\pi^{-1}(\eta)} e^{-S(\hat{\phi})-S(\phi_\eta)} \left(\hat{O}_f(\hat{\phi}) + O_f(\phi_\eta)\right) D\phi \\ &= \det\left(\Delta_\Sigma + m^2\right)^{-\frac{1}{2}} e^{-S(\phi_\eta)} O_f(\phi_\eta)\end{aligned}$$

where $\hat{O}_f$ is the observable on the space of fields which vanish on the boundary and it is defined by

$$\hat{O}_f(\hat{\phi}) = \int_\Sigma f\hat{\phi}\,\mathrm{dVol}_\Sigma.$$

Given $f, g \in C^\infty(\Sigma)$ and a boundary field η, one can show that:

$$\begin{aligned}\langle O_f O_g\rangle(\eta) = \det&\left(\Delta_\Sigma + m^2\right)^{-\frac{1}{2}} e^{-S(\phi_\eta)} \\ &\cdot \left(O_f(\phi_\eta)O_g(\phi_\eta) + \int_{\Sigma\times\Sigma} f(x)G_\Sigma(x,x')g(x')d^2x d^2x'\right)\end{aligned}$$

In particular,

$$\langle O_f O_g\rangle(0) = \det\left(\Delta_\Sigma + m^2\right)^{-\frac{1}{2}} \int_{\Sigma\times\Sigma} f(x)G_\Sigma(x,x')g(x')\, d^2x d^2x'$$

and taking $f = \delta_x$ and $g = \delta_{x'}$ we get the path integral represention of Green's function.

4.2.1. Gluing relation for Green's functions. The path integral representation of the Green's function and the formal Fubini type argument suggest that the Green's function with respect to the Dirichlet boundary condition satisfy a gluing relation and it can be proven rigorously (cf. Proposition 4.2). Let Σ be a compact oriented Riemannian manifold obtained by gluing two compact oriented Riemannian manifolds glued along a common boundary component Y: $\Sigma = \Sigma_L \cup_Y \Sigma_R$, $\partial\Sigma_L = \overline{Y_L} \sqcup Y$ and $\partial\Sigma_R = \overline{Y} \sqcup Y_R$. Let $i \in \{L, R\}$. Let G_{Σ_i} be Green's functions on Σ_i and G_Σ be the Green's function on Σ. For $f, g \in C^\infty(\Sigma)$ and $\eta_i \in C^\infty(Y_i)$, by definition we have

$$\langle O_f O_g \rangle (\eta_L, \eta_R) = \int_{\pi^{-1}(\eta_L,\eta_R)} e^{-S(\phi)} O_f O_g \, D\phi.$$

Let us assume that there is a formal Fubini's theorem:

$$\begin{aligned} &\int_{\pi^{-1}(\eta_L,\eta_R)} e^{-S(\phi)} O_f O_g \, D\phi \\ &\quad = \int_\eta \left(\int_{\pi^{-1}(\eta,\eta_L,\eta_R)} e^{-(S(\phi_L)+S(\phi_R))} O_f O_g \, D\Phi \right) D\eta, \end{aligned} \tag{4.4}$$

where $\Phi = (\phi_L, \phi_R)$, $\phi_i \in C^\infty(\Sigma_i)$ and $\eta \in C^\infty(Y)$.

Next, we want to analyze (4.4) in various situations. We first fix some notations. Given $\eta_i \in C^\infty(Y_i)$, we use $\phi_{(\eta_L,\eta_R)}$ to denote the unique solution to Dirichlet boundary value problem on Σ associated to $\Delta_\Sigma + m^2$ with boundary values η_L and η_R. Similarly, given $\eta \in C^\infty(Y)$, we will use $\phi^{(L)}_{(\eta_L,\eta)}$ and $\phi^{(R)}_{(\eta,\eta_R)}$ for the solutions to Dirichlet boundary value problems on Σ_L and Σ_R respectively.

Case (i): Assume both f and g are supported in Σ_L. Then,

$$\begin{aligned} \langle O_f O_g \rangle (\eta_L, \eta_R) &= \det(\Delta_{\Sigma_L} + m^2)^{-\frac{1}{2}} \det(\Delta_{\Sigma_R} + m^2)^{-\frac{1}{2}} \\ &\quad \cdot \int_\eta e^{-\left(S(\phi^{(R)}_{\eta,\eta_R}) + S(\phi^{(L)}_{\eta_L,\eta})\right)} \Bigg\{ \int_{\Sigma_L} f \phi^{(L)}_{\eta_L,\eta} \, \mathrm{dVol}_{\Sigma_L} \cdot \int_{\Sigma_L} g \phi^{(L)}_{\eta_L,\eta} \, \mathrm{dVol}_{\Sigma_L} \\ &\qquad + \int_{\Sigma_L \times \Sigma_L} f(x) g(x')) G_{\Sigma_L}(x, x') \, d^2x d^2x' \Bigg\} D\eta \\ &= \det(\Delta_{\Sigma_L} + m^2)^{-\frac{1}{2}} \det(\Delta_{\Sigma_R} + m^2)^{-\frac{1}{2}} \det D^{-\frac{1}{2}}_{\Sigma_L,\Sigma_R} e^{-S(\phi_{(\eta_L,\eta_R)})} \\ &\Bigg\{ \int_{\Sigma_L \times \Sigma_L} f(x) g(x') G_{\Sigma_L}(x, x') \, d^2x d^2x' + \\ &\int_{\Sigma_L \times \Sigma_L} \left(\int_{Y \times Y} \frac{\partial G_{\Sigma_L}(x, y)}{\partial \nu(y)} f(x) g(x') \frac{\partial G_{\Sigma_L}(y', x')}{\partial \nu(y')} K(y, y') \, dy dy' \right) d^2x d^2x' \Bigg\} \\ &= \det(\Delta_\Sigma + m^2)^{-\frac{1}{2}} e^{-S(\phi_{(\eta_L,\eta_R)})} \Bigg\{ \int_{\Sigma_L \times \Sigma_L} f(x) g(x') G_{\Sigma_L}(x, x') \, d^2x d^2x' + \\ &\int_{\Sigma_L \times \Sigma_L} \left(\int_{Y \times Y} \frac{\partial G_{\Sigma_L}(x, y)}{\partial \nu(y)} f(x) g(x') \frac{\partial G_{\Sigma_L}(y', x')}{\partial \nu(y')} K(y, y') \, dy dy' \right) d^2x d^2x' \Bigg\}, \end{aligned}$$

where K is the integral kernel of the inverse of the Dirichlet-to-Neumann operator D_{Σ_L,Σ_R}. We have used the gluing formula for the Dirichlet-to-Neumann operators [31] which amounts gluing of solutions of Helmholtz equations and the BFK gluing formula for the zeta-regularized determinants above.

Case (ii): Suppose f is supported in Σ_L and g is supported in Σ_R. Then as above,

$$
\begin{aligned}
&\langle O_f O_g \rangle (\eta_L, \eta_R) \\
&= \int_\eta \left(\int_{\pi^{-1}(\eta,\eta_L,\eta_R)} e^{-S(\phi_L)-S(\phi_R)} \int_{\Sigma_L} f\phi_L \, \mathrm{dVol}_{\Sigma_L} \int_{\Sigma_R} g\phi_R \, \mathrm{dVol}_{\Sigma_R} \, D\Phi \right) D\eta \\
&= \int_{\Sigma_L \times \Sigma_R} \left(\int_{Y\times Y} \frac{\partial G_{\Sigma_L}(x,y)}{\partial \nu(y)} f(x) g(x') \frac{\partial G_{\Sigma_R}(y',x')}{\partial \nu(y')} K(y,y') \, dydy' \right) d^2x d^2x'
\end{aligned}
$$

If we take $f = \delta_x$ and $g = \delta_{x'}$ in the relations above, they suggest a gluing relation for the Green's function. This gluing relation can be proven mathematically which is the content of the following proposition.

Proposition 4.2. *The Green's functions satisfy the following gluing relation:*

(i) For $i \in \{L, R\}$ and $x, x' \in \Sigma_i$:

$$G_\Sigma(x,x') - G_{\Sigma_i}(x,x') = \int_{Y\times Y} \frac{\partial G_{\Sigma_i}(x,y)}{\partial \nu(y)} K(y,y') \frac{\partial G_{\Sigma_i}(y',x')}{\partial \nu(y')} \, dydy'$$

(ii) For $x \in \Sigma_L$ and $x' \in \Sigma_R$:

$$G_\Sigma(x,x') = \int_{Y\times Y} \frac{\partial G_{\Sigma_L}(x,y)}{\partial \nu(y)} K(y,y') \frac{\partial G_{\Sigma_R}(y',x')}{\partial \nu(y')} \, dydy'$$

Proof. This proposition follows from the proof of Theorem 2.1 [6] and the Green's identity. Let us consider the case when $x, y \in \Sigma_L$, the other cases follow similarly. In Theorem 2.1 [6], it is shown that $K(y,y') = G_\Sigma(y,y')$ on Y. By the Green's identity, we have

$$G_\Sigma(x,x') - G_{\Sigma_L}(x,x') = -\int_Y G_\Sigma(x,y') \frac{\partial G_{\Sigma_L}(y',x')}{\partial \nu(y')} \, dy' \tag{4.5}$$

and

$$-\int_Y \frac{\partial G_{\Sigma_L}(x,y)}{\partial \nu(y)} G_\Sigma(y,y') \, dy = G_\Sigma(x,y'). \tag{4.6}$$

Now the proposition, when $x, x' \in \Sigma_L$, follows from combining (4.5) and (4.6). The other cases follow similarly. □

Remark 4.3. *The gluing relation for the Green's function can be pictorially represented as in Figure 7. We represent the kernel of the inverse Dirichlet-to-Neumann operator by a zig-zag:* 

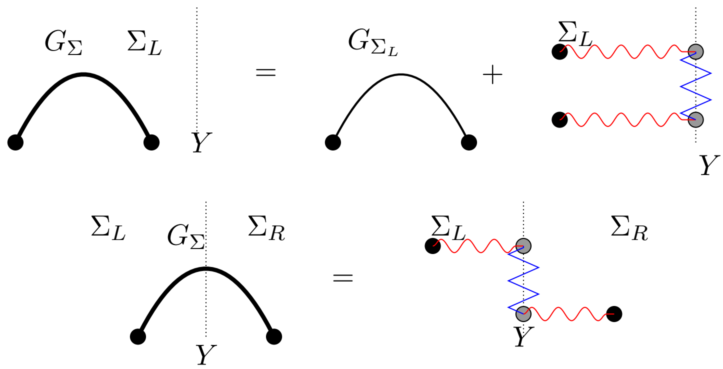


Figure 7. Gluing relation for the Green's function. Thick lines mean one should associate the function corresponding to Σ.

Remark 4.4. *The gluing formula implies similar formulae for the normal derivatives of the Green's function. These look schematically like the ones in Figure 8.*

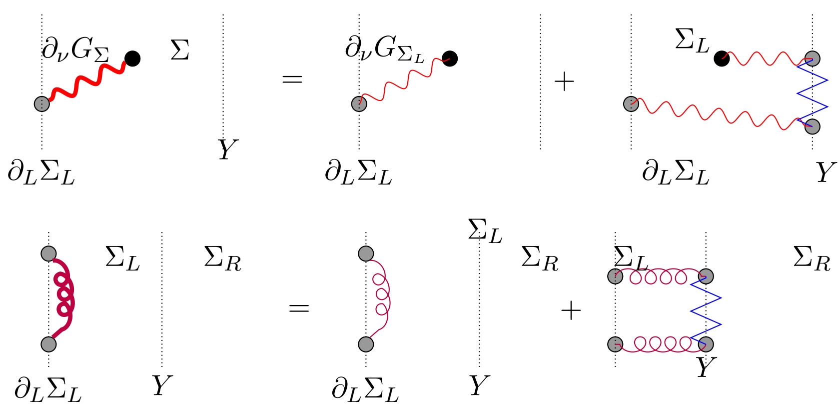


Figure 8. Gluing relation for normal derivatives of Green's functions. Thick lines mean one should associate the function corresponding to Σ.

The goal of this paper is to show that the gluing formulae for determinants and Green's functions imply a gluing formula - a formal Fubini's theorem - for the perturbative partition functions. However, as it turns out, the gluing formula for the Green's function is not compatible with normal ordering, i.e. considering only Feynman diagrams without tadpoles (short loops) as in Section 3. We discuss this issue in the next section.

5. Regularization of tadpoles

In principle, the formal application of Wick's theorem results in graphs with short loops. Under the usual Feynman rules, those would be assigned $G(x, x)$ - the (undefined) value of the Green's function on the diagonal. Normal ordering is tantamount to defining $G(x, x) = 0$, and with this assignment one obtains a well-defined perturbative partition function, as was shown in Section 3. Below, we will explain why this definition is not quite satisfactory. We will then show how to overcome those problems by introducing more sophisticated regularizations $\tau(x)$ for $G(x, x)$. Finally, we discuss the relation between this approach and the ultra-violet cutoffs oftentimes used in quantum field theory.

5.1. Why introduce tadpoles?

First, the normal-ordered partition function does not satisfy the gluing formula:

$$Z_\Sigma \neq \langle \widehat{Z_{\Sigma_L}}(\tilde{\eta}), \widehat{Z_{\Sigma_R}}(\tilde{\eta}) \rangle_{\Sigma_L, Y, \Sigma_R}$$

Indeed, from Equation (3.12) we see that the right hand side contains terms of the form in Figure 9

$$\int_{Y \times Y} \frac{\partial G_{\Sigma_i}(x, y)}{\partial \nu(y')} K(y, y') \frac{\partial G_{\Sigma_i}(y', x)}{\partial \nu(y')} \, dy dy',$$

which do not appear from the gluing formula in Remark 4.3 for the Green's

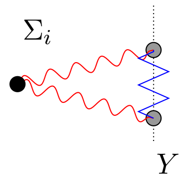


Figure 9. Diagrams which violate normal ordering when gluing

function if there are no tadpole diagrams. One can see in examples that they do not vanish.
Second, defining $G(x,x) = 0$ is inconsistent with zeta-regularization of the determinant already at the level of the free theory, in the following sense. Namely, we can consider a quadratic perturbation

$$S = \frac{1}{2}\int_\Sigma d\phi \wedge *d\phi + m^2\phi \wedge *\phi + \alpha\phi \wedge *\phi.$$

If we include α in the free action, the corresponding partition function is

$$Z = \det(\Delta_\Sigma + m^2 + \alpha)^{-\frac{1}{2}}.$$

On the other hand, treating α as a perturbation, by the convention that $G(x,x) = 0$ we obtain

$$Z = \det(\Delta_\Sigma + m^2)^{-\frac{1}{2}}.$$

We are thus led to look for another assignment $\tau(x) = G(x,x)$ which will resolve these issues. This motivates the definitions in the subsection below.

5.2. Tadpole functions

Given $\tau \in L^p(\Sigma)$ for arbitrarily large p, we can define the corresponding Feynman rules $F^\tau(\Gamma)$ where we evaluate short loops using τ. Since short loops are often called *tadpoles*, we will refer to τ as a *tadpole function.*

Lemma 5.1. *Let Γ be a Feynman diagram, possibly with short loops. Then $F^\tau(\Gamma) \in C^\infty_{\mathrm{adm}}(C^\circ_n(\partial\Sigma))$.*

Proof. By inspection of the proof of Proposition 3.17, we see that it adapts to this case. □

We can now define the partition function with tadpole τ.

Definition 5.2. *We define the partition function with respect to τ:*

$$Z^\tau_\Sigma(\tilde\eta_L, \tilde\eta_R) = \frac{1}{\det(\Delta_\Sigma + m^2)^{\frac{1}{2}}}\sum_\Gamma \frac{F^\tau(\Gamma)}{|\mathrm{Aut}(\Gamma)|}\hbar^{\ell(\Gamma)} \tag{5.1}$$

Suppose we have two manifolds Σ_L, Σ_R with a common boundary component Y and tadpole functions τ_i on Σ_i for $i \in \{L, R\}$. We can then define

a function $\tau_L * \tau_R$ on $\Sigma_L \cup_Y \Sigma_R$ by setting for $x \in \Sigma_i$
(5.2)
$$(\tau_L * \tau_R)(x) = \tau_i(x) + \int_{(y,y')\in Y\times Y} \frac{\partial G_{\Sigma_i}(x,y)}{\partial\nu(y)} K(y,y') \frac{\partial G_{\Sigma_i}(y',x)}{\partial\nu(y')} dydy'.$$

Lemma 5.3. *The following holds:*

$$\tau_L * \tau_R \in L^p(\Sigma_L \cup_Y \Sigma_R)$$

for any $p > 0$.

Proof. The second term on the r.h.s. in (5.2) is smooth away from Y and behaves as $\mathrm{O}(\log d(x,Y))$ near Y (as can be shown by considering a model integral on a half-space), which implies the statement, cf. Lemma 5.12 below. □

Definition 5.4. *We call an assignment of tadpole functions* τ_Σ *to surfaces* Σ *a* local *assignment if it satisfies the gluing formula*

$$\tau_{\Sigma_L \cup_Y \Sigma_R} = \tau_L * \tau_R \tag{5.3}$$

Pictorially, this gluing formula can be represented as in Figure 10.

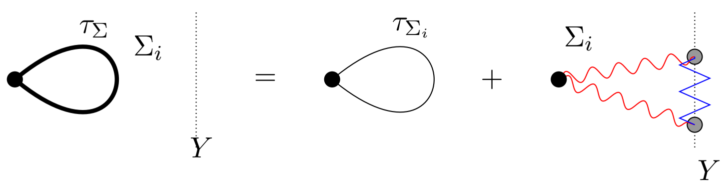


Figure 10. Gluing relation for local tadpole functions

This definition assures compatibility with the gluing formula. Another property we can ask for is the consistency with zeta-regularization:

Definition 5.5 (Compatibility with zeta-regularization). *Let* τ_Σ *be a tadpole function.*

i) We say that τ_Σ *is* weakly compatible *with zeta-regularization if*

$$\int_\Sigma \tau_\Sigma \, \mathrm{dVol}_\Sigma = \frac{d}{dm^2} \log\det(\Delta_\Sigma + m^2).$$

ii) Let $F\colon C^\infty_c(\Sigma\setminus\partial\Sigma)\supset U\to\mathbb{R}$ *be given by*

$$F(\alpha)=\log\det(\Delta_\Sigma+m^2+\alpha).$$

Then F *is Fréchet differentiable in a neighbourhood of* $0\in U$. *We say that* τ_Σ *is* strongly compatible *with zeta-regularization if*

$$DF(0)\alpha=\int_\Sigma \tau_\Sigma(x)\alpha(x)d^2x,$$

i.e. τ_Σ *is the distribution representing* $DF(0)$.

In the next two subsections, we will show that there exists a local assignment which is consistent with the zeta-regularization.

5.3. Zeta-regularized tadpole

Let Σ be a closed and oriented two dimensional Riemannian manifold. Let $\theta_{\mathrm{A}}(x,x',t)$ be the integral kernel of $e^{-t\mathrm{A}}$ i.e. the heat kernel. Let $\theta_{\mathrm{A}}(x,t)$ denote $\theta_{\mathrm{A}}(x,x,t)$. Then, we define the *local zeta function* associated to A as follows:

Definition 5.6. *The local zeta function associated to* A *is denoted by* $\zeta_{\mathrm{A}}(s,x)$ *and defined as*

$$\zeta_{\mathrm{A}}(s,x):=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\theta_{\mathrm{A}}(x,t)\,\mathrm{d}t. \tag{5.4}$$

The relation between the zeta function of A and local zeta function of A is given by

$$\zeta_{\mathrm{A}}(s)=\int_\Sigma \zeta_{\mathrm{A}}(s,x)\,d^2x.$$

We can use the small time asymptotics of the heat kernel to investigate the local zeta function. We first recall that [15, 21, 41]

$$\theta_{\mathrm{A}}(x,x',t)=e^{-m^2t}\frac{e^{-d(x,x')^2/4t}}{4\pi t}\left(a_0(x,x')+a_1(x,x')t+\mathrm{O}(t^2)\right)$$

for $t\to 0$. In particular, when $t\to 0$,

$$\theta_{\mathrm{A}}(x,t)=\frac{e^{-m^2t}}{4\pi t}\left(1+a_1(x)t+\mathrm{O}(t^2)\right) \tag{5.5}$$

where $a_1(x) = a_1(x,x)$. It is known that [41] that

$$a_1(x) = \frac{1}{6}\mathrm{R}(x)$$

where R is the scalar curvature of Σ. We can use these properties of the heat kernel together with its large time behavior to show that $\zeta_{\mathrm{A}}(s,x)$ is holomorphic for $\mathrm{Re}(s) > 1$, it has meromorphic extension to $\mathbb{C}$ and $\zeta_{\mathrm{A}}(s,x)$ is holomorphic at $s = 0$. This is the content of the following lemma.

Lemma 5.7. *For each* $x \in \Sigma$, $\zeta_{\mathrm{A}}(s,x)$ *is holomorphic for* $\mathrm{Re}(s) > 1$ *and* $\zeta_{\mathrm{A}}(s,x)$ *has meromorphic extension to* $\mathbb{C}$ *and it is holomorphic at* $s = 0$. *Moreover,*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\zeta_{\mathrm{A}}(s,x) = {} & \frac{1}{4\pi}m^2(\log m^2 - 1) - \frac{1}{4\pi}a_1(x)\,\log m^2 \\ & + \int_0^\infty \frac{e^{-m^2 t}\left(\theta_{\Delta_\Sigma}(x,t) - g(x,t)\right)}{t}\,\mathrm{d}t,\end{aligned}$$

where $g(x,t) = \dfrac{1}{4\pi t} + a_1(x)$. *Furthermore, the assignment* $x \mapsto \dfrac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\zeta_A(s,x)$ *is smooth.*

Proof. For large $\mathrm{Re}(s)$, a simple computation shows

$$\begin{aligned}\zeta_{\mathrm{A}}(s,x) = {} & \frac{1}{4\pi}\frac{1}{(s-1)m^{2s-2}} + \frac{a_1(x)}{4\pi m^{2s}} \\ & + \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-tm^2}\left(\theta_{\Delta_\Sigma}(x,t) - g(x,t)\right)\,\mathrm{d}t\end{aligned} \tag{5.6}$$

This representation of $\zeta_A(s,x)$ proves the first part of the lemma. Now, the expression for the derivative at $s = 0$ follows from using the fact that Γ has a pole of order one at $s = 0$ while differentiating $\zeta_A(s,x)$. Finally, smoothness of coefficients of the local heat kernel expansion in the interior of Σ implies the assignment $x \mapsto \dfrac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\zeta_{\mathrm{A}}(s,x)$ is smooth. $\square$

From the proof of Lemma 5.7, we see that $\zeta_{\mathrm{A}}(s,x)$ has a simple pole at $s = 1$. However, we can consider the finite part of the local zeta function $s = 1$. This motivates the following definition.

Definition 5.8. *We define the tadpole function via zeta regularization by*

$$\tau_\Sigma^{\mathrm{reg}}(x) = \boldsymbol{f.p.}_{s=1}\zeta_{\mathrm{A}}(s,x) := \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=1}(s-1)\zeta_{\mathrm{A}}(s,x). \tag{5.7}$$

Note that $\tau_\Sigma^{\mathrm{reg}}(x)$ is the constant term in the Laurent series expansion of $\zeta_{\mathrm{A}}(s,x)$ at $s=1$.

One can think of $\tau_\Sigma^{\mathrm{reg}}(x)$ as a regularization[21] of the value of the Green's function $\zeta_A(1,x) = G_\Sigma(x,x)$ on the diagonal.

In fact, we can write $\tau_\Sigma^{\mathrm{reg}}(x)$ more explicitly as shown in the following lemma.

Lemma 5.9. *The following holds:*

$$\tau_\Sigma^{\mathrm{reg}}(x) = -\frac{1}{4\pi}\log m^2 + \int_0^\infty e^{-m^2 t}\left(\theta_{\Delta_\Sigma}(x,t) - \frac{1}{4\pi t}\right)\mathrm{d}t.$$

Furthermore,

$$\tau_\Sigma^{\mathrm{reg}}(x) = -\frac{d}{dm^2}\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\zeta_{\mathrm{A}}(s,x).$$

Proof. Using the proof of Lemma 5.7, we observe that

$$\zeta_{\mathrm{A}}(1+\varepsilon,x) = \frac{1}{4\pi\varepsilon} - \frac{1}{4\pi}\log m^2 + \int_0^\infty e^{-m^2 t}\left(\theta_{\Delta_\Sigma}(x,t) - \frac{1}{4\pi t}\right)\mathrm{d}t + \mathrm{O}(\varepsilon)$$

as $\varepsilon \to 0$. Now, the first part of the lemma follows from the definition of the finite part. The second part of the lemma follows from differentiating in m^2 the explicit representation of $\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\zeta(s,x)$ in Lemma 5.7. □

Remark 5.10. *The zeta-regularized tadpole function $\tau_\Sigma^{\mathrm{reg}}$ is an invariant for the action of the group of isometries of the Riemannian metric on Σ. In particular, if the group of isometries acts transitively on Σ then $\tau_\Sigma^{\mathrm{reg}}$ is constant.*

More generally, we can define the local zeta function on a compact manifold with boundary. Let $\theta_{\mathrm{A}}(x,x',t)$ be the integral kernel of A with respect

[21] In fact, it would be more appropriate to call it *renormalization*: we first regularize by shifting s away from 1 and then we subtract the singular part $\frac{\mathrm{const}}{s-1}$.

to the Dirichlet boundary condition. Then, it is well known [41] for small t that

$$\theta_{\mathrm{A}}(x,x',t) = e^{-m^2 t}\frac{e^{-d(x,x')^2/4t}}{4\pi t}\left(a_0(x,x')+b_0(x,x')\sqrt{t}+a_1(x,x')t+\mathrm{O}(t^{3/2})\right).$$

Here b_0 appears as a contribution from the boundary and it is supported at the boundary[22]. Hence,

$$\theta_{\mathrm{A}}(x,t) = \frac{e^{-m^2 t}}{4\pi t}\left(1+b_0(x)\sqrt{t}+a_1(x)t+\mathrm{O}(t^{3/2})\right) \tag{5.8}$$

as $t\to 0$ where $b_0(x)=b_0(x,x)$ and $a_1(x)=a_1(x,x)$.

The local zeta function and the tadpole function $\tau_\Sigma^{\mathrm{reg}}(x)$ via zeta regularization are defined as above. The discussion above regarding the meromorphic extension of $\zeta_{\mathrm{A}}(s,x)$ does not change. In particular, $\zeta_{\mathrm{A}}(s,x)$ is holomorphic at $s=0$. Furthermore:

Lemma 5.11. *Let Σ be a two dimensional compact Riemannian manifold with smooth boundary. Then, the following holds:*

(i)

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\zeta_{\mathrm{A}}(s,x) = \frac{1}{4\pi}m^2(\log m^2-1) - \frac{m b_0(x)}{2\sqrt{\pi}} - \frac{a_1(x)\log m^2}{4\pi} + \int_0^\infty \frac{e^{-m^2 t}\left(\theta_\Delta(x,t)-\tilde{g}(x,t)\right)}{t}\,\mathrm{dt}$$

where $\tilde{g}(x,t) = \frac{1}{4\pi t}\left(1+b_0(x)\sqrt{t}+a_1(x)\,t\right)$.

(ii)

$$\tau_\Sigma^{\mathrm{reg}}(x) = -\frac{\mathrm{d}}{\mathrm{d}m^2}\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\zeta_{\mathrm{A}}(s,x)$$

[22] For any fixed x,x' away the boundary, b_0 will not appear in the asymptotic expansion. However, upon restricting to the diagonal and integrating, one will have a contribution coming from b_0. See e.g. [41], for a detailed statement see [24, Theorem 3.12].

Proof. One can show that

$$\begin{aligned}\zeta_{\mathrm{A}}(s,x) = \frac{1}{4\pi}\left(\frac{1}{(s-1)m^{2s-2}} + \frac{\Gamma(s-1/2)\,b_0(x)}{\Gamma(s)\,m^{2s-1}} + \frac{a_1(x)}{m^{2s}}\right) \\ + \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-tm^2}\left(\theta_{\Delta_\Sigma}(x,t) - \tilde{g}(x,t)\right)\,\mathrm{d}t\end{aligned} \tag{5.9}$$

Differentiating at $s = 0$ and using $\Gamma(-1/2) = -2\sqrt{\pi}$, we get the first part of the lemma. From this expression, it also follows that

$$\zeta_A(1+\varepsilon,x) = \frac{1}{4\pi\varepsilon} - \frac{1}{4\pi}\log m^2 + \int_0^\infty e^{-m^2t}\left(\theta_{\Delta_\Sigma}(x,t) - \frac{1}{4\pi t}\right)\,\mathrm{d}t + \mathrm{O}(\varepsilon)$$

as $\varepsilon \to 0$. This shows that

$$\tau_\Sigma^{\mathrm{reg}}(x) = -\frac{1}{4\pi}\log m^2 + \int_0^\infty e^{-m^2t}\left(\theta_{\Delta_\Sigma}(x,t) - \frac{1}{4\pi t}\right)\,\mathrm{d}t.$$

The second part of the lemma follows from a simple computation. □

The behavior of $\tau_\Sigma^{\mathrm{reg}}$ in the interior of Σ is similar to the case when there is no boundary. However, it is not clear how it behaves near the boundary. We will show that the behavior of $\tau_\Sigma^{\mathrm{reg}}$ is comparable to that of the function $x \mapsto \log(d(x,\partial\Sigma))$ up to a bounded function. We begin the analysis with the following lemma.

Lemma 5.12. *The function $x \mapsto \log d(x,\partial\Sigma)$ is in $L^p(\Sigma)$ for any $p \geq 1$.*

Proof. Let $f(x) = \log d(x,\partial\Sigma)$. Assume that the Riemannian metric is a product metric near the boundary. Then, the volume form on a collar neighborhood the boundary can be written as $dt \wedge \mathrm{dVol}_{\partial\Sigma}$ and consequently it is possible to find $C > 0$ and $a > 0$ such that

$$\int_\Sigma |f|^p \leq C\int_0^a |\log t|^p\, dt.$$

For the general case, let $T_r(\partial\Sigma) = \{x \in \Sigma : d(x,\partial\Sigma) \leq r\}$ be the tube around $\partial\Sigma$ constructed using the normal vector field [23, Chapter 3]. Then, for small $r > 0$, the volume form on T_r can be written, using Fermi coordinates, in the form $h(t)dt \wedge \mathrm{dVol}_{\partial\Sigma}$ [23, Theorem 9.22], where $h(t) = 1 + \mathrm{O}(t)$. In these coordinates, we have $d(x,\partial\Sigma) = t$, which implies $x \mapsto \log d(x,\partial\Sigma)$ is integrable in this coordinate neighborhood by the result for the product metric. □

Next, we compare the behavior of $\tau_\Sigma^{\mathrm{reg}}$ to that of the function $\kappa(x) = \mathbf{f.p.}_{s=1}\xi(s,x)$ where

$$\xi(s,x) = \frac{1}{\Gamma(s)} \int_0^1 t^{s-1}\Xi(x,t)\,\mathrm{d}t$$

and

$$\Xi(x,t) = \frac{1}{4\pi t}\left(1 - \exp\left(-\frac{d(x,\partial\Sigma)^2}{t}\right)\right).$$

First, we show:

Lemma 5.13. *The function* $\kappa - \dfrac{1}{2\pi}\log(d(\cdot,\partial\Sigma))$ *is bounded in* Σ.

Proof. Let $u = d(x,\partial\Sigma)^2$. Then, it follows that

$$\xi(s,x) = -\frac{u^{s-1}}{4\pi\Gamma(s)}\left(\Gamma(1-s) - \int_0^u (e^{-t}-1)t^{-s}\,dt\right).$$

Hence,

$$\xi(1+\varepsilon,x) = -\frac{u^{\varepsilon}}{4\pi\varepsilon\Gamma(\varepsilon)}\left(\Gamma(-\varepsilon) - \int_0^u (e^{-t}-1)t^{-(1+\varepsilon)}\,dt\right)$$

Now, the lemma follows from the definition of κ using the fact that $u^\varepsilon = 1 + \varepsilon\log u + \mathrm{O}(\varepsilon^2)$ as $\varepsilon \to 0$. $\square$

Following a strategy similar to the study of the heat kernel expansion on a manifold with boundary [41], we can check that $\tau_\Sigma^{\mathrm{reg}} - \kappa$ is bounded. Now, combining the discussion above, we have the following result.

Proposition 5.14. *The zeta-regularized tadpole behaves near* $\partial\Sigma$ *as*

$$\tau_\Sigma^{\mathrm{reg}} = \frac{1}{2\pi}\log d(\cdot,\partial\Sigma) + f$$

with f *a bounded function.*

As a consequence of this proposition and Lemma 5.12, we have:

Corollary 5.15. τ_Σ^{reg} *is in* L^p *for all* $p \geq 1$.

Also, the consistency of $\tau_\Sigma^{\mathrm{reg}}$ with the zeta-regularization is immediate.

Corollary 5.16. *The zeta-regularized tadpole is consistent with the zeta-regularization: we have*

$$\int_\Sigma \tau_\Sigma^{\mathrm{reg}}\,\mathrm{dVol}_\Sigma = \frac{\mathrm{d}}{\mathrm{d}m^2}\log\det(\Delta_\Sigma + m^2).$$

Furthermore, we also have compatibilty with the zeta-regularization in the strong sense.

Lemma 5.17. $\tau_\Sigma^{\mathrm{reg}}$ *is compatible with zeta-regularization in the strong sense.*

Proof. As usual, we write $A = \Delta_\Sigma + m^2$. We consider the function $F(\alpha) = \log\det(A+\alpha)$, which is defined for $\alpha \in U \subset C_c^\infty(\Sigma \setminus \partial\Sigma)$, where U is a neighborhood of 0. Then we have to show that

$$DF(0)\alpha = \int_\Sigma \tau_\Sigma^{\mathrm{reg}}(x)\alpha(x)d^2x$$

where D is the Fréchet derivative. Applying again the Mellin transform, we have that

$$\begin{aligned} F(\varepsilon\alpha) &= -\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\int_\Sigma \zeta_{A+\epsilon\alpha}(s,x) \\ &= -\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\int_\Sigma \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\theta_{A+\varepsilon\alpha}(x,t)dtd^2x \end{aligned}$$

The heat operator for $A+\epsilon\alpha$ can be represented by perturbation theory as

$$e^{-t(A+\epsilon\alpha)} = \sum_{k=0}^\infty \varepsilon^k W_k(t)$$

where $W_0 = e^{-tA}$. The relevant object for us is the trace of the first order perturbation $W_1(t)$ which can be computed as

$$\operatorname{tr} W_1(t) = -t\int_\Sigma d^2x\,\alpha(x)\,\theta_A(x,t).$$

We therefore conclude that

$$F(\varepsilon\alpha) = F(0) + \varepsilon\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\int_\Sigma \frac{1}{\Gamma(s)}\int_0^\infty t^s\alpha(x)\theta_{\mathrm{A}}(x,t)dtd^2x + O(\varepsilon^2)$$

Now, exploiting $s\Gamma(s) = \Gamma(s+1)$ we realize that the term of order ε is

$$DF(0)\alpha = \int_\Sigma \underbrace{\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \frac{s}{\Gamma(s+1)} \int_0^\infty t^s \theta_{\mathrm{A}}(x,t)dt}_{=\tau_\Sigma^{\mathrm{reg}}(x)} \alpha(x) d^2x$$
$$= \int_\Sigma \tau_\Sigma^{\mathrm{reg}}(x)\alpha(x)d^2x.$$

□

Proposition 5.18. *The assignment $\Sigma \mapsto \tau_{reg}^\Sigma$ is a local assignment.*

For the proof we investigate another tadpole function, given by the so-called point splitting.

5.4. Point-splitting tadpole

Let Σ be a compact oriented Riemannian manifold possibly with boundary. Let us consider the operator $A = \Delta_\Sigma + m^2$. We will always impose the Dirichlet boundary condition if Σ has boundary and we assume boundary to be closed. We recall that the Green's function on Σ associated to A can be written as

$$G_\Sigma(x,x') = -\frac{1}{2\pi}\log(d(x,x')) + \mathrm{H}(x,x')$$

near the diagonal of $\Sigma \times \Sigma$, where H is in C^1 in a neighborhood of the diagonal in $\Sigma \setminus \partial\Sigma \times \Sigma \setminus \partial\Sigma$. The singular support of the distribution G_Σ is the diagonal.

Definition 5.19. *We define* $\tau_\Sigma^{\mathrm{split}}(x) := \lim_{x' \to x} \left[G_\Sigma(x,x') + \frac{1}{2\pi}\log(d(x,x'))\right]$.

We can think of $\tau_\Sigma^{\mathrm{split}}$ as a way to regularize G_Σ on the diagonal via "point-splitting". In the following lemma, we state some properties of $\tau_\Sigma^{\mathrm{split}}$.

Lemma 5.20. *The point-splitting tadpole $\tau_\Sigma^{\mathrm{split}}$ is C^1 on $\Sigma \setminus \partial\Sigma$ and $\tau_\Sigma^{\mathrm{split}} \in \mathrm{L}^p(\Sigma)$ for any $p \geq 1$. Moreover,*

$$\tau_\Sigma^{\mathrm{split}}(x) = -\frac{\log m^2}{4\pi} + \frac{\log 2 - \gamma}{2\pi} + \int_0^\infty e^{-tm^2}\left(\theta_{\Delta_\Sigma}(x,t) - \frac{1}{4\pi t}\right) \mathrm{dt}$$

where γ is the Euler's constant.

Proof. The fact that the point-splitting tadpole $\tau_\Sigma^{\text{split}}$ is C^1 on $\Sigma \setminus \partial\Sigma$ follows from the definition. To show $\tau_\Sigma^{\text{split}} \in L^p(\Sigma)$, it suffices to show the function f on Σ defined by $x \mapsto \frac{1}{2\pi}\log(d(x,\partial\Sigma))$ is in $L^p(\Sigma)$ as $f - \tau_\Sigma^{\text{split}}$ is locally bounded in a collar neighborhood of the boundary. We have already shown in Lemma 5.12 that $f \in L^p(\Sigma)$. This completes the proof of $\tau_\Sigma^{\text{split}} \in L^p(\Sigma)$. Another proof of this assertion can be given by comparing $\tau_\Sigma^{\text{split}}$ with τ_Σ^{reg} using Corollary 5.21, which uses only the second part of this lemma, and applying Corollary 5.15.

For the last part, let us recall that

$$G_\Sigma(x,x') = \int_0^\infty e^{-tm^2}\theta_{\Delta_\Sigma}(x,x',t)\,\mathrm{dt}$$

and

$$\frac{1}{2\pi}K_0(md(x,x')) = \int_0^\infty \frac{1}{4\pi t}e^{-tm^2-d(x,x')^2/4t}\,\mathrm{dt}$$

where $K_0(z)$ is the modified Bessel's function. The $z \to 0$ asymptotics of $K_0(z)$ implies

$$\frac{1}{2\pi}K_0(md(x,x')) = -\frac{\log(md(x,x'))}{2\pi} + \frac{\log 2 - \gamma}{2\pi} + \mathrm{O}(m^2 d(x,x')^2)$$

as $d(x,x') \to 0$. We can rewrite this as:

$$\frac{\log(d(x,x'))}{2\pi} = -\frac{1}{2\pi}K_0(md(x,x')) - \frac{\log m^2}{4\pi} + \frac{\log 2 - \gamma}{2\pi} + \mathrm{O}(m^2 d(x,x')^2)$$

as $d(x,x') \to 0$. Using this, we see that

$$\begin{aligned}
&\lim_{x' \to x}\left(G_\Sigma(x,x') + \frac{1}{2\pi}\log(d(x,x'))\right) \\
&\qquad = -\frac{\log m^2}{4\pi} + \frac{\log 2 - \gamma}{2\pi} + \int_0^\infty e^{-tm^2}\left(\theta_{\Delta_\Sigma}(x,t) - \frac{1}{4\pi t}\right)\mathrm{dt}.
\end{aligned}$$

$\square$

Corollary 5.21. *For $x \in \Sigma \setminus \partial\Sigma$, we have* $\tau_\Sigma^{\text{reg}}(x) - \tau_\Sigma^{\text{split}}(x) = \dfrac{\gamma - \log 2}{2\pi}$.

We note that the point-splitting tadpole function is invariant under the action of the group of the isometries of the Riemannian metric. From this it follows that:

Proposition 5.22. *Let Σ be a closed oriented Riemannian manifold such that the group of isometries act transitively on Σ, then $\tau_\Sigma^{\mathrm{split}}$ is constant on Σ. In particular, if Σ is a closed Riemannian surface with constant scalar curvature, then $\tau_\Sigma^{\mathrm{split}}$ is constant.*

Finally, we can give two examples of local assignments.

Proposition 5.23. *The assignment $\Sigma \mapsto \tau_\Sigma^{\mathrm{split}}$ is a local assignment of tadpole functions. In particular, $\tau_\Sigma^{\mathrm{reg}}(x)$ is also a local assignment of tadpole functions.*

Proof. We can show $\tau_\Sigma^{\mathrm{split}}$ is a local assignment by a direct application of the gluing formula for G_Σ and we can complete the proof using Corollary 5.21. □

We end this subsection with the following remark concerning the appearance of tadpole functions in other context in the literature.

Remark 5.24. *The zeta-regularized tadpole and the point-splitting tadpole appear in the study of conformally covariant elliptic operators such as Yamabe operator and Paneitz operator on a closed Riemannian manifold Σ [1]. In this context, these functions are known as the mass function of the operators and they are used in the study of mass theorems, regularized traces, and conformal variation of regularized traces, we refer to [1, 39, 45, 46, 51, 52] for details.*

5.5. Some comments on the relation to RG flow in 2D scalar field theory

In this paper we regularize 2d partition functions by choosing a tadpole function. In the above we constructed two particular examples of tadpole functions, the "zeta-regularized" and the "point-splitting" one. The purpose of this subsection is to relate these to the setup of renormalization group and RG flow.

5.5.1. Renormalized action (action with counterterms). Consider the 2D scalar field theory with action

$$S(\phi) = \underbrace{\int_\Sigma \frac{1}{2} d\phi \wedge *d\phi + \frac{m^2}{2}\phi^2\, dvol}_{S_{free}} + p(\phi)\, dvol$$

with $p(\phi) = \sum_n \frac{p_n}{n!}\phi^n$ a polynomial interaction (or more generally a formal power series).

Say, we are interested in the normalized path integral

$$Z_{norm} = \frac{\int \mathcal{D}\phi \; e^{-\frac{1}{\hbar}S(\phi)}}{\int \mathcal{D}\phi \; e^{-\frac{1}{\hbar}S_{free}(\phi)}} \tag{5.10}$$

and we define it by a perturbative expansion with Green's function regularized via proper time cut-off:

$$G_\Lambda(x,x') = \int_{1/\Lambda^2}^{\infty} dt \; e^{-m^2 t}\theta_\Delta(x,x',t)$$

with θ_Δ the heat kernel for the Laplacian, $x, x' \in \Sigma$ and Λ a very large cut-off having the dimension of mass. We note that, for $x' \neq x$, $\lim_{\Lambda\to\infty} G_\Lambda(x,x') = G(x,x')$ exists, whereas on the diagonal we have the asymptotic behavior

$$G_\Lambda(x,x) \underset{\Lambda\to\infty}{\sim} \frac{\log\Lambda}{2\pi} + \underbrace{\widetilde{\tau}(x)}_{\text{finite}} \tag{5.11}$$

Lemma 5.25. *The finite part $\widetilde{\tau}(x)$ appearing in the r.h.s. of (5.11) differs from the zeta-regularized tadpole by a universal constant:*

$$\widetilde{\tau}(x) = \tau^{\text{reg}}(x) - \frac{\gamma}{4\pi} \tag{5.12}$$

with γ the Euler constant.

Proof. Indeed, we find

$$\begin{aligned} G_\Lambda(x,x) &= \int_{1/\Lambda^2}^{\infty} dt\, e^{-m^2 t}\theta_\Delta(x,t) \\ &= \int_{1/\Lambda^2}^{\infty} dt\, e^{-m^2 t}\left(\theta_\Delta(x,t) - \frac{1}{4\pi t}\right) + \underbrace{\int_{1/\Lambda^2}^{\infty} dt\, e^{-m^2 t}\frac{1}{4\pi t}}_{\frac{1}{4\pi}E_1\left(\frac{m^2}{\Lambda^2}\right)} \\ &\underset{\Lambda\to\infty}{\sim} \frac{\log\Lambda}{2\pi} \underbrace{-\frac{\gamma + \log m^2}{4\pi} + \int_0^{\infty} dt\, e^{-m^2 t}\left(\theta_\Delta(x,t) - \frac{1}{4\pi t}\right)}_{\widetilde{\tau}(x)} \\ &\quad + O\left(\frac{m^2}{\Lambda^2}\right). \end{aligned}$$

Here $E_1(u)=\int_u^\infty dt\,\frac{e^{-t}}{t}$ is the exponential integral and we used its asymptotic behavior $E_1(u)\sim -\log u-\gamma+O(u)$ at $u\to 0$. Comparing this formula for $\widetilde{\tau}$ with the result for τ^{reg} (Lemma 5.9), we obtain (5.12). □

For the normalized path integral (5.10) to be finite, we must assume that the coefficients of $p(\phi)=p_\Lambda(\phi)$ in the numerator depend on Λ in such a way that the limit $\lim_{\Lambda\to\infty} Z_{norm}$ exists. For that to happen, $p_\Lambda(\phi)$ must have the following form:

$$
\begin{aligned}
(5.13)\qquad p_\Lambda(\phi) &= \sum_{n\geq 0}\frac{p_n}{n!}\sum_{k=0}^{[\frac{n}{2}]}(-1)^k(2k-1)!!\begin{pmatrix} n\\ 2k\end{pmatrix}\cdot\left(\frac{\hbar}{2\pi}\log\Lambda\right)^k\phi^{n-2k}\\
&= \sum_{n\geq 0}p_n\sum_{k=0}^{[\frac{n}{2}]}\frac{1}{k!}\left(-\frac{\hbar}{4\pi}\log\Lambda\right)^k\cdot\frac{\phi^{n-2k}}{(n-2k)!}
\end{aligned}
$$

– here we are essentially subtracting from $p_{\mathrm{naive}}(\phi)=\sum\frac{p_n}{n!}\phi^n$ the "counterterms" compensating for the tadpole divergencies encountered when computing the path integral (5.10) using p_{naive}.

Note that (5.13) satisfies the differential equation

$$
(5.14)\qquad \frac{\partial}{\partial\log\Lambda}\,p_\Lambda(\phi)=-\frac{\hbar}{4\pi}\,\frac{\partial^2}{\partial\phi^2}\,p_\Lambda(\phi)
$$

- one can see at as a heat equation with "time" coordinate $\log\Lambda$ and "space" coordinate ϕ, and (5.13) is the general solution with initial condition given by $p_{\mathrm{naive}}(\phi)$ at "time" $\log\Lambda=0$.[23]

Some examples of solutions:

$\frac{\phi^2}{2}-\frac{\hbar}{4\pi}\log\Lambda$	shift of mass (gets additively renormalized),
$\Lambda^{-\frac{\hbar}{4\pi}\alpha^2}e^{\alpha\phi}$	potential of Liouville theory,
$\Lambda^{\frac{\hbar}{4\pi}\alpha^2}\cos(\alpha\phi)$	potential of sine-Gordon theory.

In the last two examples the potential is multiplicatively renormalized (attains an anomalous dimension).[24]

[23] One can call (5.14) the RG flow equation "at the ultraviolet end": it tells how the local counterterms change when the cut-off is changed infinitesimally.

[24] Cf. the fact that in free massless scalar field theory – a prototypical CFT – $:e^{\alpha\phi}:$ is a vertex operator of holomorphic/antiholomorphic dimension $h=\bar{h}=-\frac{\hbar}{8\pi}\alpha^2$ (in our normalization convention), and thus total scaling dimension $h+\bar{h}=-\frac{\hbar}{4\pi}\alpha^2$ and spin $h-\bar{h}=0$.

5.5.2. Tadpoles vs. RG flow ("petal diagram resummation"). Let $Z^{\tau,p}$ be the partition function on a surface Σ (possibly with boundary) for the massive scalar field with interaction potential

$$p(\phi) = \sum_n \frac{p_n}{n!}\phi^n, \tag{5.15}$$

defined using the tadpole function $\tau = \tau(x)$.

We denote $\mathrm{Fun}(\Sigma)$ the space of smooth functions in the interior of Σ which behave as $O(|\log d(x,\partial\Sigma)|^N)$ near the boundary, for some power N.

We will consider the setup where the coefficients p_n of the interaction potential themselves are allowed to be functions on Σ valued in power series in $\hbar$, i.e., $p = p(\phi, x, \hbar) \in \mathrm{Fun}(\Sigma)[[\phi,\hbar]]$.[25] We have the following.

Proposition 5.26 (Petal diagram resummation).

(i) We have the equality of partition functions

$$Z^{\tau,p} = Z^{0,\widetilde{p}} \tag{5.16}$$

Here the right hand side is defined with zero tadpole and

$$\widetilde{p}(\phi,x,\hbar) := \sum_{n\geq 0} p_n(x,\hbar) \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{1}{k!}\left(\frac{\hbar\,\tau(x)}{2}\right)^k \cdot \frac{\phi^{n-2k}}{(n-2k)!} \tag{5.17}$$

(ii) Denote the r.h.s. of (5.17) by $\mathcal{R}_\tau(p)$. We have $\mathcal{R}_{\tau_1+\tau_2}(p) = \mathcal{R}_{\tau_1}(\mathcal{R}_{\tau_2}(p))$.[26]

(iii) If $q = \mathcal{R}_{\tau_1-\tau_2}(p)$, then $Z^{\tau_1,p} = Z^{\tau_2,q}$.

(iv) The transformed potential (5.17) satisfies the "local RG flow equation":

$$\frac{\partial}{\partial\tau}\mathcal{R}_\tau(p) = \frac{\hbar}{2}\frac{\partial^2}{\partial\phi^2}\mathcal{R}_\tau(p) \tag{5.18}$$

which holds pointwise on Σ.

---

[25] The reason for introducing this extended setup is that the transformation (5.17) below generally (for a non-constant tadpole function τ) transforms a potential with constant coefficients to one with non-constant coefficients.

[26] Thus, $\mathcal{R}$ defines an action of the additive group $\mathrm{Fun}(\Sigma)$ on interaction potentials $p \in \mathrm{Fun}(\Sigma)[[\phi,\hbar]]$ – the "local RG flow." Note that, for a constant function τ, $\mathcal{R}_\tau$ transforms potentials with constant coefficients $p \in \mathbb{R}[[\phi,\hbar]]$ to potentials with constant coefficients.

(v) For a potential $p(\phi) = \sum_j c_j e^{\alpha_j \phi}$ given by a sum of exponents, with α_j, c_j independent on ϕ (but possibly depending on $x, \hbar$), the corresponding transformed potential (5.17) is:

$$\mathcal{R}_\tau(p) = \sum_j c_j e^{\frac{\hbar\tau}{2}\alpha_j^2} e^{\alpha_j \phi}$$

Proof. For (i), one shows (5.16) as a resummation of perturbation theory: summation of the "petal diagrams" for the theory with potential p yields the vertices of the theory with potential

$$\widetilde{p} = \sum_{n\geq 0}\sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{p_n}{n!}\phi^{n-2k}\begin{pmatrix} n \\ 2k \end{pmatrix}(2k-1)!!\,(\hbar\tau)^k \tag{5.19}$$

where the combinatorial coefficient $\begin{pmatrix} n \\ 2k \end{pmatrix}(2k-1)!!$ counts the number of ways to attach k edges to a vertex with n incident half-edges. Expression (5.19) simplifies to (5.17).

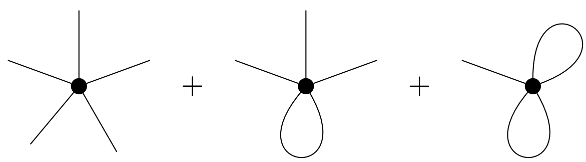


Figure 11. Petal diagram resummation

Item (ii) is straightforward: denoting the n-th coefficient p_n in $p(\phi)$ (normalized as in (5.15)) by $[p]_n$, we have from (5.17) that

$$[\mathcal{R}_T(p)]_n \sum_{k\geq 0}\left(\frac{\hbar\,\tau}{2}\right)^k p_{n+2k}$$

and therefore

$$[\mathcal{R}_{\tau_1}(\mathcal{R}_{\tau_2}(p))]_n = \sum_{k_1,k_2\geq 0}\frac{1}{k_1!}\left(\frac{\hbar\,\tau_1}{2}\right)^{k_1}\frac{1}{k_2!}\left(\frac{\hbar\,\tau_2}{2}\right)^{k_2} p_{n+2k_1+2k_2}$$
$$\underset{l=k_1+k_2}{=}\ \sum_{l\geq 0}\left(\frac{\hbar\,(\tau_1+\tau_2)}{2}\right)^l p_{n+2l} = [\mathcal{R}_{\tau_1+\tau_2}(p)]_n$$

Item (iii) is a generalization of (i) (since the case $q=0$ is (i)) and it follows from (i) and (ii):

$$Z^{\tau_2,q} \underset{(i)}{=} Z^{0,\mathcal{R}_{\tau_2}(q)} = Z^{0,\mathcal{R}_{\tau_2}(\mathcal{R}_{\tau_1-\tau_2}(p))} \underset{(ii)}{=} Z^{0,\mathcal{R}_{\tau_1}(p)} \underset{(i)}{=} Z^{\tau_1,p}$$

The equation (iv) follows immediately from (5.17) by applying the relevant derivatives to the r.h.s.

Item (v) is the observation that when applied to an exponential $p(\phi) = e^{\alpha\phi} = \sum_{n\geq 0} \frac{\alpha^n}{n!}\phi^n$, the transformation (5.17) yields

$$\sum_{n\geq 0}\sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{1}{k!}\left(\frac{\hbar\tau}{2}\right)^k \alpha^n \frac{\phi^{n-2k}}{(n-2k)!} = \sum_{n\geq 0}\sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{1}{k!}\left(\frac{\hbar\tau}{2}\alpha^2\right)^k \frac{(\alpha\phi)^{n-2k}}{(n-2k)!} = e^{\frac{\hbar\tau}{2}\alpha^2} e^{\alpha\phi}$$

Then (v) follows by $\mathrm{Fun}(\Sigma)[[\hbar]]$-linearity of the transformation $\mathcal{R}_T$. □

Proposition 5.26 implies the following.

Corollary 5.27. *One has the equality*

$$Z^{\tau^{\mathrm{reg}},p} = Z^{\tau_\Lambda,p_\Lambda} \tag{5.20}$$

Here the l.h.s. is the partition function for an interaction potential $p(\phi)$, calculated using the zeta-regularized tadpole $\tau^{\mathrm{reg}}(x)$. The r.h.s. is the partition function with the tadpole $\tau_\Lambda(x) := G_\Lambda(x,x)$ (with G_Λ defined via proper time cut-off, as in (5.11)) and with the "renormalized" interaction potential (or "potential with counterterms") given by

$$p_\Lambda = \mathcal{R}_{\tau^{\mathrm{reg}}-\tau_\Lambda}(p) \underset{\Lambda\to\infty}{\sim} \mathcal{R}_{-\frac{\log\Lambda}{2\pi}+\frac{\gamma}{4\pi}}(p)$$

Here in the last point we used the result (5.12).

The r.h.s. of (5.20) is *almost* the same as the computation of the perturbative path integral for the theory using the cut-off-regularized Green's function (5.11), with the action including counterterms, which are fine-tuned – see (5.20) – so that the path integral is finite. "Almost" – because in (5.20) only the Green's functions in the tadpoles are regularized while the Green's functions between distinct vertices are the exact ones. However, the distinction between these two regularizations for Feynman diagrams becomes negligible as $\Lambda\to\infty$.

Another way to present the result (5.20) is:

$$Z^{\widetilde{\tau},p'} \underset{\Lambda\to\infty}{\leftarrow} Z^{\tau_\Lambda,p_\Lambda} \tag{5.21}$$

where $\widetilde{\tau}$ is as in (5.11) – the cut-off-renormalized tadpole (i.e., with cut-off regularization imposed and with the singular term subtracted), with $p' = \mathcal{R}_{\frac{\gamma}{4\pi}}(p)$ a finite transformation of the potential (arising from the difference in zeta vs cut-off renormalization schemes). Here $p_\Lambda = \mathcal{R}_{-\frac{\log\Lambda}{2\pi}}(p')$. Note that this is the same as formula (5.13) from Section 5.5.1 if we identify $p' = p_{\mathrm{naive}}$. Thus, in the asymptotic equality (5.21) we either subtract a singular part from the tadpole (in the l.h.s.), or we add counterterms to the action (in the r.h.s.).

Cutting into tiny squares (a heuristic picture).[27] Consider a cellular subdivision X_ϵ of the surface Σ into small squares sq_i, of linear size of order $\epsilon = \frac{1}{\Lambda}$, with $\epsilon \to 0$. One can consider two different pictures (local assignments of tadpoles):

I Set the tadpole functions to zero for each small square, $\tau_{sq_i} = 0$. By the gluing formula for tadpoles, this leads to a glued tadpole $\tau_\Sigma \sim -\frac{\log\epsilon}{2\pi} +$ (finite part) on the surface; τ_Σ is a version of cut-off regularized tadpole $G_\Lambda(x,x)$, see (5.11).

II Set the tadpole functions for the small squares and for Σ to their zeta-regularized values. Then on a small square, we have $\tau_{sq_i} \sim \frac{\log\epsilon}{2\pi} +$ (finite part) (this is the $\epsilon \to 0$ asymptotics of an explicit answer for a flat square) and τ_Σ is finite (ϵ-independent).

In the first picture, we need to define the partition function using the renormalized potential p_Λ, in order to have a finite result; in the second picture, we are taking the non-renormalized potential p and have a finite result.

6. Formal Fubini Theorem and Atiyah-Segal gluing

In this section, we finally prove the gluing formula for the perturbative partition function. It comes in different flavors, according our choice of tadpole function. If τ is a local assignment of tadpole functions, we write $Z^\tau_\Sigma := Z^{\tau_\Sigma}_\Sigma$.

[27]We call it a heuristic picture because it relies on cutting with corners which is yet to be fully understood.

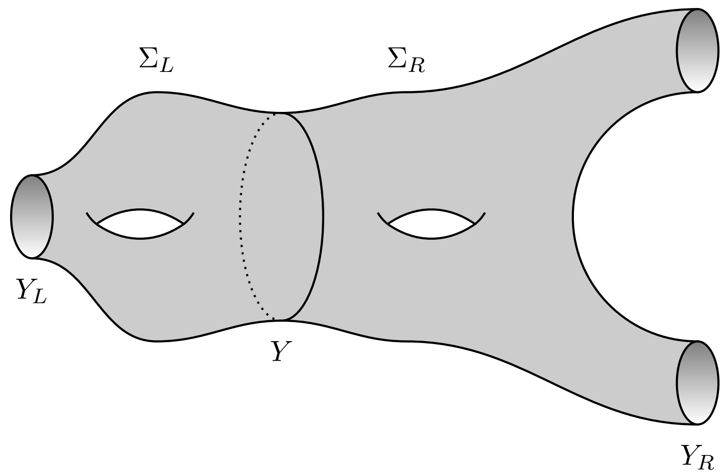


Figure 12. A cobordism Σ from Y_L to Y_R with a decomposition $\Sigma = \Sigma_L \cup_Y \Sigma_R$.

6.1. The gluing formula

Let Σ be a two-dimensional compact Riemannian cobordism with boundary $\partial\Sigma = Y_L \sqcup Y_R$. Let Y be a collection of circles in Σ such that $\Sigma = \Sigma_L \cup_Y \Sigma_R$, with $Y_L \subset \Sigma_L$ and $Y_R \subset \Sigma_R$. The main goal of this section is to prove the following theorem.

Theorem 6.1. *Let $\Sigma = \Sigma_L \cup_Y \Sigma_R$ as above and τ_L, τ_R be tadpole functions on Σ_L,Σ_R respectively. Then*[28]

$$e^{-S_0(\phi^{\Sigma_L}_{\tilde{\eta}_L})} e^{-S_0(\phi^{\Sigma_R}_{\tilde{\eta}_R})} \langle \widehat{Z^{\tau_L}_{\Sigma_L}}, \widehat{Z^{\tau_R}_{\Sigma_R}} \rangle_{\Sigma_L, Y, \Sigma_R} = Z^{\tau_L * \tau_R}_{\Sigma}. \tag{6.1}$$

The following is an immediate corollary.

[28]The slightly unwieldy notation is due to the fact that when gluing only over a part of the boundary, there are in general three different "blocks" of the Dirichlet-to-Neumann operator that play different roles in the gluing: The block supported on the gluing interface is formally absorbed in the Gaussian measure, and therefore deleted from the partition function, the off-diagonal block participates in the gluing as in Figure 8, and the block supported on the remaining boundary component does not interact with gluing gets corrected into the Dirichlet-to-Neumann operator of the glued bulk (as in the second part of Figure 8).

Corollary 6.2. *If τ is an assignment of local tadpole functions, we have*

$$e^{-S_0(\phi_{\tilde{\eta}_L}^{\Sigma_L})}e^{-S_0(\phi_{\tilde{\eta}_R}^{\Sigma_R})}\langle\widehat{Z^{\tau}_{\Sigma_L}},\widehat{Z^{\tau}_{\Sigma_R}}\rangle_{\Sigma_L,Y,\Sigma_R}=Z^{\tau}_{\Sigma}. \tag{6.2}$$

In particular, this holds for the perturbative partition function defined using the zeta-regularized tadpole.

To prove this theorem we introduce some auxiliary structures on Feynman diagrams.

6.2. More on Feynman diagrams

This strategy of the proof was inspired by Johnson-Freyd's paper [30].

Definition 6.3 (Decorated Feynman graphs).

i) A decoration *of a Feynman graph is a pair of functions*[29]

$$f=(dec_V\colon V(\Gamma)\to\{L,R\},dec_E\colon E(\Gamma)\to\{u,c\}).$$

ii) A decoration is admissible *if $f(V_X)\subset\{X\}, X=L,R$, and all edges between a vertex decorated L and a vertex decorated R are decorated by c.*

iii) A decorated Feynman graph *is a pair (Γ,f) of a Feynman graph Γ and a decoration f of Γ.*

The automorphism group $\mathrm{Aut}(\Gamma)$ acts on the set of decorations. Two decorations of Γ that are related by an automorphism of Γ are called *isomorphic.* The set of isomorphism classes of decorations is denoted $dec(\Gamma)$.

Definition 6.4. *An* automorphism *of a decorated Feynman graph is an automorphism $\varphi\in\mathrm{Aut}(\Gamma)$ that fixes the decoration: $dec_V(V_I(\varphi)(v))=dec_V(v)$, $dec_E(E(\varphi)(e))=dec_E(e)$, i.e. the decorated automorphism group is the stabilizer of the decoration under the action of $\mathrm{Aut}(\Gamma)$ on the set of decorations. We denote the set of automorphisms of a decorated graph by $\mathrm{Aut}^{dec}(\Gamma)$.*

Notice that all edge types - E_0,E_1,E_2 - can be cut. We introduce a set of Feynman rules for decorated graphs.

[29] "u" is for "uncut", "c" is for "cut".

Definition 6.5. *Let (Γ, f) be a decorated Feynman graph and let $\Sigma = \Sigma_L \cup_Y \Sigma_R$. Also, let τ_L, τ_R be tadpole functions on Σ_L, Σ_R. Then we define the weight $F_\Sigma^{dec,\tau_L,\tau_R}$ of the decorated Feynman graph as follows: For a bulk vertex labeled $X \in \{L, R\}$, we integrate over Σ_X. Edges decorated by u between different X vertices are assigned G_{Σ_X} or its appropriate derivatives, or the tadpole function τ_X. Edges labeled by c are assigned the second term in the gluing formula for the appropriate derivative of the Green's function. Tadpoles labeled by c are assigned the second term in gluing formula for tadpoles.*

Lemma 6.6. *For all graphs Γ, we have*

$$\frac{F_\Sigma^{\tau_L * \tau_R}(\Gamma)}{|\operatorname{Aut}(\Gamma)|} = \sum_{f \in dec(\Gamma)} \frac{F_\Sigma^{dec,\tau_L,\tau_R}(\Gamma^f)}{|\operatorname{Aut}^{dec}(\Gamma^f)|}. \tag{6.3}$$

Proof. Decompose $F_\Sigma(\Gamma)$ using $\int_\Sigma = \int_{\Sigma_L} + \int_{\Sigma_R}$ and the gluing formula for the propagator $G_\Sigma = G_u + G_c$ between L and L (resp. R and R) vertices and $G_\Sigma = G_c$ between L and R vertices. Every term in the resulting sum is labeled by a decoration f of Γ. Isomorphic decorations will evaluate to the same weight. Thus, we obtain

$$\frac{F_\Sigma^{\tau_L * \tau_R}(\Gamma)}{|\operatorname{Aut}(\Gamma)|} = \sum_{f \in dec(\Gamma)} F_\Sigma^{dec,\tau_L,\tau_R}(\Gamma^f) \frac{|\operatorname{Aut}(\Gamma) \cdot f|}{|\operatorname{Aut}(\Gamma)|}.$$

By the orbit-stabilizer theorem, we obtain

$$\frac{|\operatorname{Aut}(\Gamma) \cdot f|}{|\operatorname{Aut}(\Gamma)|} = \frac{1}{|\operatorname{Aut}^{dec}(\Gamma^f)|}$$

and the claim follows. $\square$

We define a gluing operation $*$ on Feynman graphs: Denote Feynman graphs with no $R - R$ edges by Gr_R and Feynman graphs with no $L - L$ edges by Gr_L. For $\Gamma_L \in \mathrm{Gr}_R, \Gamma_R \in \mathrm{Gr}_L$ we define

$$\Gamma_L * \Gamma_R := \sum_{\sigma \text{ perfect matching of } V_R(\Gamma_L) \sqcup V_L(\Gamma_R)} \Gamma^{dec}(\sigma, \Gamma_L, \Gamma_R),$$

where $\Gamma^{dec}(\sigma, \Gamma_L, \Gamma_R)$ is the decorated graph obtained by decorating vertices in Γ_X with X, edges in Γ_X by u (for "uncut") and connecting the boundary vertices specified by σ to an edge, decorated c (for "cut"), between the bulk vertices attached to these boundary vertices. In the language of Definition 3.11 we set $V_L(\Gamma^{dec}) = V_L(\Gamma_L)$,$V_R(\Gamma^{dec}) = V_R(\Gamma_R)$, $V_b(\Gamma^{dec}) = V_b(\Gamma_L) \sqcup V_b(\Gamma_R)$. The set of half-edges is the union of all half-edges incident to these vertices. The map τ specifying the edges is extended by the perfect matching σ. These new edges are decorated c, all other edges are decorated u, the vertices carry the obvious decorations. See Figure 13.

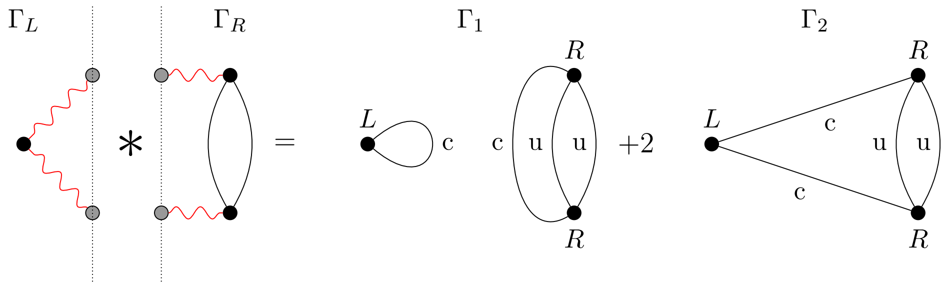


Figure 13. The gluing operation on graphs. The first term corresponds to the perfect matching which matches vertices on either side, the second term to the two matchings identifying the vertices on different sides.

Remark 6.7. *We can glue graphs with different amounts of boundary vertices. For instance, the graph Γ_L in Figure 13 could be glued to the empty graph on the right hand side, and these terms are important for the gluing formula for the Green's (or tadpole) function.*

The gluing operation lands in formal linear combinations of decorated graphs.

$$\Gamma_L * \Gamma_R = \sum_{\Gamma^{dec} \in \operatorname{supp}(\Gamma_L * \Gamma_R)} m^{\Gamma^{dec}}_{\Gamma_L, \Gamma_R} \Gamma^{dec}.$$

For instance, in the example of Figure 13, we have $m^{\Gamma_1}_{\Gamma_L,\Gamma_R} = 1$ and $m^{\Gamma_2}_{\Gamma_L,\Gamma_R} = 2$. Then:

Lemma 6.8. *Let Γ^{dec} be a decorated Feynman graph. Then there is a unique $\Gamma_L \in \mathrm{Gr}_R, \Gamma_R \in \mathrm{Gr}_L$ such that $\Gamma^{dec} \in \operatorname{supp}(\Gamma_L * \Gamma_R)$.*

Proof. Delete every c-decorated edge $e = \{v_1, v_2\}$ in Γ^{dec} and replace it with vertices v_1', v_2' and edges $e_1 = v_1, v_1', e_2 = \{v_2, v_2'\}$. This results in two disconnected graphs Γ_L and Γ_R containing vertices decorated L and R respectively (they are disconnected since edges between L and R vertices are decorated c). The newly added vertices are declared right resp. left boundary vertices if connected to a vertex decorated L resp. R. Forgetting the decorations, this is the unique combination of graphs that will contain Γ^{dec} in its support after gluing. □

Notice that the pairing $\left\langle F^{\tau_L}_{\Sigma_L}(\Gamma_L), F^{\tau_R}_{\Sigma_R}(\Gamma_R)\right\rangle_{\Sigma_L,Y,\Sigma_R}$ makes sense also if the graphs Γ_L and Γ_R have $L-L$ (resp. $R-R$ edges).

Lemma 6.9. *Using notation as above, for $\Gamma_L \in \mathrm{Gr}_R$ and $\Gamma_R \in \mathrm{Gr}_L$ we have*

$$\left\langle F^{\tau_L}_{\Sigma_L}(\Gamma_L), F^{\tau_R}_{\Sigma_R}(\Gamma_R)\right\rangle_{\Sigma_L,Y,\Sigma_R} = \frac{F^{dec,\tau_L,\tau_R}_{\Sigma}(\Gamma_L * \Gamma_R)}{\det(D_{\Sigma_L,\Sigma_R})^{\frac{1}{2}}} \tag{6.4}$$

Here we extend F^{dec}_{Σ} linearly to formal linear combinations of decorated graphs.

Proof. The only nontrivial point here is that the integral kernel we used to define the pairing is the same kernel as the one appearing in the gluing formula for the Green's function. Apart from this fact the proof is a matter of plugging in the definitions. □

Finally we require the following combinatorial lemma.

Lemma 6.10. *Using notation as above*

$$\frac{\Gamma_L}{|\operatorname{Aut}(\Gamma_L)|} * \frac{\Gamma_R}{|\operatorname{Aut}(\Gamma_R)|} = \sum_{\Gamma^{dec}\in \operatorname{supp}\Gamma_L * \Gamma_R} \frac{\Gamma^{dec}}{|\operatorname{Aut}^{dec}(\Gamma^{dec})|}. \tag{6.5}$$

Proof. This is essentially a consequence of the orbit-stabilizer theorem. Notice we can rewrite (6.5) as

$$m^{\Gamma^{dec}}_{\Gamma_L,\Gamma_R} \overset{!}{=} \frac{|\operatorname{Aut}(\Gamma_L)||\operatorname{Aut}(\Gamma_R)|}{|\operatorname{Aut}^{dec}(\Gamma^{dec})|} \tag{6.6}$$

which is already suggestive of the group action we want to consider. Namely, the group $\operatorname{Aut}\Gamma_L \times \operatorname{Aut}\Gamma_R$ acts on the set $\mathfrak{m}$ of perfect matchings of $V_R(\Gamma_L) \sqcup V_L(\Gamma_R)$. Two perfect matchings related by this group action will define the same decorated graph, so that for any $m^{\Gamma^{dec}_{\Gamma_L,\Gamma_R}} = |(\operatorname{Aut}\Gamma_L \times \operatorname{Aut}\Gamma_R)\cdot\sigma|$ for any $\sigma \in \mathfrak{m}$ that defines Γ^{dec}. The main realization is that the stabilizer group of σ is isomorphic to the automorphism group of the decorated graph Γ^{dec}: The condition that a pair (φ_L, φ_R) stabilizes σ is equivalent to asking that $\varphi_L \sqcup \varphi_R$ preserves incidence of cut edges. Equation (6.6) is then just the orbit-stabilizer theorem. □

6.3. Proof of the gluing formula

We now have all the necessary ingredients to prove Theorem 6.1 (we suppress dependence on arguments after the first line).

Proof of Theorem 6.1. Recall that we denote by Gr_L (resp. Gr_R) the set of all Feynman graphs containing *no* edges between left (resp. right) boundary vertices. Then, as noted in Remark 3.19, we have

$$\sum_{\Gamma} \frac{F^{\tau}_{\Sigma}(\Gamma)}{|\mathrm{Aut}(\Gamma)|} = \sum_{\Gamma \in \mathrm{Gr}_R} e^{-S_0(\phi^{\Sigma}_{\tilde{\eta}_R})} \frac{F^{\tau}_{\Sigma}(\Gamma)}{|\mathrm{Aut}(\Gamma)|}$$

In particular, for a surface Σ with $\partial\Sigma = \partial_L\Sigma \cup \partial_R\Sigma$, we have

$$e^{-S_0(\phi^{\Sigma}_{\tilde{\eta}_L})} \widehat{Z^{\tau}_{\Sigma}} = \frac{1}{\det(\Delta_{\Sigma_L} + m^2)^{\frac{1}{2}}} \sum_{\Gamma \in \mathrm{Gr}_R} \frac{F^{\tau}_{\Sigma}(\Gamma)}{|\mathrm{Aut}(\Gamma)|}. \tag{6.7}$$

The proof of the gluing formula is now a simple consequence of our previous work. Consider again a cobordism $(\Sigma, \partial_L\Sigma, \partial_R\Sigma)$ with a decomposition $\Sigma = \Sigma_L \cup_Y \Sigma_R$, where $(\Sigma_L, \partial_L\Sigma, Y), (\Sigma_R, Y, \partial_R\Sigma_R)$ are two cobordisms with common boundary component Y. Then, we have

$$\begin{aligned}
&e^{-S_0(\phi^{\Sigma_L}_{\tilde{\eta}_L})} e^{-S_0(\phi^{\Sigma_R}_{\tilde{\eta}_R})} \left\langle \widehat{Z^{\tau_L}_{\Sigma_L}}(\tilde{\eta}_L, \tilde{\eta}), \widehat{Z^{\tau_R}_{\Sigma_R}}(\tilde{\eta}, \tilde{\eta}_R) \right\rangle_{\Sigma_L, Y, \Sigma_R} = \\
&(\text{Eq. (6.7)}) = \frac{1}{\det(\Sigma_L + m^2)^{\frac{1}{2}} \det(\Sigma_R + m^2)^{\frac{1}{2}}} \\
&\qquad \cdot \sum_{\substack{\Gamma_L \in \mathrm{Gr}_R \\ \Gamma_R \in \mathrm{Gr}_L}} \left\langle \frac{F^{\tau_L}_{\Sigma_L}(\Gamma_L)}{|\mathrm{Aut}(\Gamma_L)|}, \frac{F^{\tau_R}_{\Sigma_R}(\Gamma_R)}{|\mathrm{Aut}(\Gamma_R)|} \right\rangle_{\Sigma_L, Y, \Sigma_R} \\
&(\text{Eq. (6.4)}) = \frac{1}{\det(\Sigma_L + m^2)^{\frac{1}{2}} \det(\Sigma_R + m^2)^{\frac{1}{2}} \det(D_{\Sigma_L, \Sigma_R})^{\frac{1}{2}}} \\
&\qquad \cdot \sum_{\substack{\Gamma_L \in \mathrm{Gr}_R \\ \Gamma_R \in \mathrm{Gr}_L}} \frac{F^{dec, \tau_L, \tau_R}_{\Sigma}(\Gamma_L * \Gamma_R)}{|\mathrm{Aut}(\Gamma_L)| \cdot |\mathrm{Aut}(\Gamma_R)|}
\end{aligned}$$

$$\text{(Eqs. (4.3),(6.5))} = \frac{1}{\det(\Delta_\Sigma + m^2)^{\frac{1}{2}}} \sum_{\substack{\Gamma_L \in \mathrm{Gr}_R \\ \Gamma_R \in \mathrm{Gr}_L}} \sum_{\Gamma^{dec} \in \Gamma_L * \Gamma_R} \frac{F_\Sigma^{dec,\tau_L,\tau_R}(\Gamma^{dec})}{|\operatorname{Aut}^{dec}(\Gamma^{dec})|}$$

$$\text{(Lemma 6.8)} = \frac{1}{\det(\Delta_\Sigma + m^2)^{\frac{1}{2}}} \sum_{\Gamma^{dec}} \frac{F_\Sigma^{dec,\tau_L,\tau_R}(\Gamma^{dec})}{|\operatorname{Aut}^{dec}(\Gamma^{dec})|}$$

$$\text{(Eq. (6.3))} = \frac{1}{\det(\Delta_\Sigma + m^2)^{\frac{1}{2}}} \sum_{\Gamma} \frac{F_\Sigma^{\tau_L * \tau_R}(\Gamma)}{|\operatorname{Aut}(\Gamma)|} = Z_\Sigma^{\tau_L * \tau_R}.$$

□

7. Functoriality

Returning to the discussion of the introduction, it is a natural question whether the assignment $Y \mapsto H_Y, \Sigma \mapsto Z_\Sigma$ has an interpretation as a functor. It turns out that the answer to this question is positive, provided source and target category are adequately defined, and one introduces the correct mathematical setup. The main problem is that in the treatment of Section 3 the pairing on the space of boundary states depends on the bulk. We'll briefly describe the idea how to remedy this. Remember that heuristically we want the pairing to be integration against the "Lebesgue measure" on the space of boundary fields:

$$\langle \Psi_1, \Psi_2 \rangle = \int_{\tilde{\eta}_Y} \Psi_1(\tilde{\eta}_L, \tilde{\eta}_Y)\Psi_2(\tilde{\eta}_Y, \tilde{\eta}_R).$$

Of course this formal Lebesgue measure does not depend on the bulk, but it is not mathematically well-defined, so another idea is needed. From the point of view of perturbation theory, it was natural to use the factor $e^{-S_0(\phi_{\tilde{\eta}_Y})}$ to define a formal measure with respect to which we were defining the pairing, but this factor depends on the bulk, since $S_0(\phi_{\tilde{\eta}_Y}) = \frac{1}{2}\int_Y \tilde{\eta} D_\Sigma \tilde{\eta}\, \mathrm{dVol}_Y$. The trick to obtain a functor is to realize that dependence on the bulk is "small" in the sense that $D_\Sigma = \sqrt{\Delta_Y + m^2} + S$, where S is a compact operator[30] that contains all the bulk dependence. We can thus use the operator $\sqrt{\Delta_Y + m^2}$ to define an *actual Gaussian measure* on a completion of $C^\infty(Y)$. This Gaussian measure will then be corrected to the one induced by the Dirichlet-to-Neumann operator by a "small" contribution from the

[30]In fact, $\delta = (\Delta_Y + m^2)^{-\frac{1}{2}} S$ is a trace-class operator, see Proposition A.3. Moreover, if the metric Σ in the neighborhood of Y is the product metric, then S is a smoothing operator. We refer to Appendix A.3.3 for examples.

bulk. Thus, we will multiply the partition function by a factor to obtain its correct normalization. But on a heuristic level, nothing happens at all: we are merely splitting the Gaussian factor in the partition function in a different way. In this section we will spell out the details of this idea and prove that this is enough to make the partition function functorial.

7.1. The source category

The source category is the semicategory (i.e. category without identity morphisms) $\mathbf{Riem}^2$ of 2-dimensional Riemannian cobordisms defined as follows:

- Objects are closed Riemannian 1-manifolds with two-sided collars $Y \times (-\epsilon, \epsilon)$ with an arbitrary metric restricting to the metric on Y on $Y \times \{0\}$.
- A morphism from $Y_L \times (-\epsilon_L, \epsilon_L)$ to $Y_R \times (-\epsilon_R, \epsilon_R)$ is a Riemannian cobordism Σ with $\partial_X \Sigma = Y_X$ such that $\partial_L \Sigma$ has a collar (tubular neighborhood) in Σ isometric to $Y_L \times [0, \epsilon_L)$ and $\partial_R \Sigma$ has a collar isometric to $Y_R \times (-\epsilon_R, 0]$.

Composition of morphisms is well-defined since the 2-sided collars ensure that metrics can be glued smoothly.

We refer the reader to [27, 55] for a detailed discussion of the Riemannian cobordism category.

7.2. The space of boundary states revisited and the target category

We turn to describing the space of states associated to a closed 1-manifold. It is constructed as as in Section 3, but with Dirichlet-to-Neumann operator replaced by the square root of the Helmholtz operator on the boundary.

In order to prove functoriality of the appropriately adjusted partition function, we also introduce the measure-theoretic formulation of the space of states and compare it with the conventional Fock space formulation.

7.2.1. Space of boundary states - perturbative picture. Consider again the vector spaces $H_Y^{(n)}$ of Definition 3.3: functionals Ψ on $C^\infty(Y)$ of the form

$$\Psi(\widetilde{\eta}) = \int_{C_n^\circ(Y)} \psi(y_1, \ldots, y_n) \tilde{\eta}(y_1) \cdots \widetilde{\eta}(y_n) \, dy_1 \ldots dy_n$$

where $\psi \in C^\infty_{\mathrm{adm}}(C^\circ_n(Y))[[\hbar^{1/2}]]$ is symmetric and has admissible singularities (Definition 3.2) on diagonals, and set $H^{\mathrm{pre}}_Y = \bigoplus_{n\geq 0} H^{(n)}_Y$. Then we define a new pairing on H^{pre}, similar in form to the one of Definition 3.8, but with the Dirichlet-to-Neumann operator replaced by (twice) the square root of the Helmholtz operator on Y. Notice that this pairing will be intrinsic to the metric on Y, in particular, it can be defined without reference to any bulk manifold. We define the space of states as the completion of H^{pre} with respect to that pairing.

Definition 7.1. *Let g be a Riemannian metric on Y and $m > 0$ and consider the Helmholtz operator $\Delta_g + m^2$ on Y. Denote $\varkappa$ the square root of this operator. We define the pairing*

$$\begin{aligned} \langle\cdot,\cdot\rangle_{2\varkappa} \colon H^{(n)}_Y \times H^{(m)}_Y &\to \mathbb{R}[[\hbar^{1/2}]] \\ \langle \Psi, \Psi'\rangle_{2\varkappa} &= \langle \Psi \odot \Psi'\rangle_{2\varkappa} \end{aligned} \tag{7.1}$$

where $\odot$ is the symmetric tensor product defined in (3.13) *and $\langle\cdot\rangle_{2\varkappa} \colon H^{(k)} \to \mathbb{R}[[\hbar^{1/2}]]$ the $2\varkappa$-expectation value map*

$$\begin{aligned} \langle \Psi\rangle_{2\varkappa} = \frac{1}{(\det 2\varkappa)^{\frac{1}{2}}} \sum_{\mathfrak{m}\in\mathfrak{M}_n} \int_{C^\circ_n(Y)} &\psi(y_1,\ldots,y_n) \\ \prod_{\{v_1,v_2\}\in\mathfrak{m}} &(2\varkappa)^{-1}(y_{v_1},y_{v_2})\, dy_1\cdots dy_n. \end{aligned} \tag{7.2}$$

We extend the pairing bilinearly to $H^{\mathrm{pre}}_Y = \bigoplus_{n\geq 0} H^{(n)}_Y$.

The completion of H^{pre}_Y with respect to the pairing $\langle\cdot,\cdot\rangle_{2\varkappa}$ coincides (as a topological vector space) with H_Y, see Definition 3.9, since operators $2\varkappa$ and D_{Σ_L,Σ_R} are sufficiently close.

Notice that H_Y has the decomposition

$$H_Y = \bigoplus \overline{H}^{(n)}_Y$$

where $\overline{H}^{(n)}_Y$ is understood as $H^{(n)}_Y$ completed with respect to the restriction of $\langle\cdot,\cdot\rangle_{2\kappa}$ degree-wise in $\hbar$. However, this is *not* a decomposition into orthogonal subspaces.

7.2.2. Space of boundary states - Fock space picture. We now want to compare our model of the space of states to the more classical notion of Fock space. This is a space naturally associated to a pseudodifferential operator on $C^\infty(Y)$ defined as follows.

Definition 7.2. *Denote $V_{2\varkappa}$ the completion of $C^\infty(Y)$ with respect to the pairing*[31]

$$\langle f,g\rangle'_{2\varkappa} = \frac{1}{2}\int_{Y\times Y} f(x)\,\varkappa^{-1}(x,y)\,g(y)dxdy. \tag{7.3}$$

The Fock space model *of the space of boundary states associated with Y is*

$$F_+(Y) := \widehat{\bigoplus_{k=0}^{\infty}} S^k V_{2\varkappa} \ \otimes \mathbb{R}[[\hbar^{1/2}]]$$

where S^kV is the k-th symmetric power of the Hilbert space V and $\widehat{\bigoplus}$ denotes the completed orthogonal sum of Hilbert spaces.

Remark 7.3. *The pairing (7.3) on $C^\infty(Y)$ is the covariance of a Gaussian probability measure $\mu_{2\varkappa}$ on $D'(Y)$–the space of distributions on Y. This means that $\widehat{\bigoplus_{k=0}^{\infty}} S^kV_{2\varkappa}$ is isomorphic to $L^2(D'(Y),\mu_{2\varkappa})$ via a canonical isomorphism [4, 25, 54]. The isomorphism is constructed by considering the Wiener chaos decomposition [29] of $L^2(D'(Y),\mu_{2\varkappa})$ obtained from the so-called normal odering procedure. From this, it follows that $F_+(Y)$ is isomorphic to $L^2(D'(Y),\mu_{2\varkappa})[[\hbar^{1/2}]]$.*

Notice that in the decomposition of $H_Y = \bigoplus H_Y^{(n)}$, the individual components $H_Y^{(n)}$ ("n-particle sectors") are not orthogonal to each other. On the other hand, in the Fock space we have $S^kV_{2\varkappa} \perp S^lV_{2\varkappa}$ for $l \neq k$. Nevertheless, the completion H_Y of H_Y^{pre} can also be decomposed into an orthogonal direct sum. This can be done by using the normal ordering, which for example, for[32] $\psi \in C^\infty_{\mathrm{adm}}(C_n^\circ(Y))^{S_n}[[\hbar^{1/2}]]$ is defined by:

$$\begin{aligned}
:\psi := \psi(y_1,\ldots,y_n) &- \binom{n}{2}\int_{Y\times Y}\psi(y_1,\ldots,y_n)(2\kappa)^{-1}(y_1,y_2)\,dy_1dy_2\\
&+\binom{n}{4}\int_{Y^4}\psi(y_1,\ldots,y_n)(2\kappa)^{-1}(y_1,y_2)(2\kappa)^{-1}(y_3,y_4)\,dy_1dy_2dy_3dy_4\\
&-\ldots
\end{aligned}$$

We then have the following.

[31]Notice this is almost the same pairing as $\langle f,g\rangle_{2\varkappa}$ on $H_Y^{(1)} = C^\infty(Y)[[\hbar^{1/2}]]$ but without the prefactor $\frac{1}{(\det 2\varkappa)^{\frac{1}{2}}}$.

[32]Here superscript S_n denotes functions invariant under the natural action of S_n on $C_n^\circ(Y)$, i.e. symmetric functions.

Proposition 7.4. *Denote the pairing on the Fock space by $\langle \cdot, \cdot \rangle_F$. Then, there is a canonical isomorphism*

$$(H_Y, \langle \cdot, \cdot \rangle_{2\varkappa}) \cong \left(F_+(Y), \frac{1}{\det(2\varkappa)^{\frac{1}{2}}} \langle \cdot, \cdot \rangle_F \right).$$

Proof. First, we observe that, if we ignore the determinant factor in the pairing in Definition 7.1, then there is an obvious isomorphism from H_Y onto $L^2(D'(S^1), \mu_{2\varkappa})[[\hbar^{1/2}]]$ that respects orthogonal decomposition, where the orthogonal decomposition of H_Y comes from the normal ordering as discussed above and the one on $L^2(D'(S^1), \mu_{2\varkappa})[[\hbar^{1/2}]]$ comes from Wiener chaos decomposition. Now, the proposition follows from Remark 7.3. □

For us it will be convenient to use the following normalization of Gaussian measures.

Definition 7.5. *We define the "unnormalized" Gaussian measure corresponding to a symmetric positive operator A to be*

$$\mu'_A = \frac{\mu_A}{\det(A)^{1/2}}.$$

Here we assume $\det A$ *exists in the zeta-regularized sense.*

Remark 7.6. *Given a cobordism $(\Sigma, \partial_L \Sigma, \partial_R \Sigma)$ let $Y = \partial\Sigma = \partial_L \Sigma \sqcup \partial_R \Sigma$. Then the associated space of boundary states H_Y satisfies*

$$H_Y \cong \mathrm{HS}(H_{\partial_L \Sigma}, H_{\partial_R \Sigma}),$$

where HS *denotes Hilbert-Schmidt operators (this is a standard property of the tensor product of Hilbert spaces). Given the three compact Riemannian 1-manifolds Y_L, Y, Y_R, the pairing $H_Y \otimes H_Y \to \mathbb{R}[[\hbar^{1/2}]]$ extends to the composition map $H_{Y_L \sqcup Y} \otimes H_{Y \sqcup Y_R} \to H_{Y_L \sqcup Y_R}$.*

We now define the target category as follows.

Definition 7.7. *The category* $\mathbf{Hilb}^{\mathrm{form}}$ *is the category where*

- *objects are real Hilbert spaces tensored with $\mathbb{R}[[\hbar^{1/2}]]$,*
- *morphisms are $\mathbb{R}[[\hbar^{1/2}]]$-linear Hilbert-Schmidt operators.*

7.3. Proof of functoriality

As was explained in the discussion above, we need to slightly adjust the partition function to account for the pairing on the space of boundary states:

Definition 7.8. *Let $\Sigma \equiv (\Sigma, \partial_L\Sigma, \partial_R\Sigma)$ be a cobordism and τ be a tadpole function on Σ. For a 1-dimensional manifold Y, denote $\varkappa := \sqrt{\Delta_Y + m^2}$. The* functorial partition function *of Σ is*

$$\overline{Z^\tau_\Sigma}[\tilde{\eta}] = e^{\frac{1}{2}\int_{\partial\Sigma} \tilde{\eta}\varkappa\tilde{\eta}\,\mathrm{dVol}_{\partial\Sigma}} Z^\tau_\Sigma[\tilde{\eta}]. \tag{7.4}$$

Proposition 7.9. *We have $\overline{Z^\tau_\Sigma} \in \mathbf{Hilb}^{\mathrm{form}}(H_{\partial_L\Sigma}, H_{\partial_R\Sigma})$.*

Proof. By Remark 7.6, it is enough to show that $\overline{Z^\tau_\Sigma} \in H_{\partial\Sigma}$. Notice that

$$\overline{Z^\tau_\Sigma}[\tilde{\eta}] = \frac{1}{\det(\Delta_\Sigma + m^2)^{\frac{1}{2}}} e^{-\frac{1}{2}\int_{\partial\Sigma} \tilde{\eta}S\tilde{\eta}\,\mathrm{dVol}_{\partial\Sigma}} Z^{\mathrm{pert},\tau}_\Sigma[\tilde{\eta}]$$

where $S = D_\Sigma - \varkappa$ and $Z^{\mathrm{pert},\tau}_\Sigma$ is given by summing over all diagrams with no boundary-boundary edges. Let Γ be a Feynman diagram with n boundary vertices and no boundary-boundary edges. By Proposition 3.17, we have $F(\Gamma) \in H^{(n)}_{\partial\Sigma}$. Denote $(H_{\partial\Sigma})_{k/2}$ the order $k/2$ part in $\hbar$ of $H_{\partial\Sigma}$. Then we have that $e^{-\frac{1}{2}\int_{\partial\Sigma} \tilde{\eta}S\tilde{\eta}\,\mathrm{dVol}_{\partial\Sigma}} F(\Gamma) \in (H_{\partial\Sigma})_{\ell(\Gamma)}$, because the operator $\delta = \varkappa^{-1}S$ is trace-class (see Proposition A.3). Since at any order in $\hbar$ there are only finitely many diagrams with no boundary-boundary edges, we conclude that $\overline{Z^\tau_\Sigma} \in H_{\partial\Sigma}$. $\square$

The gluing formula can then be re-interpreted as the fact that, if τ is a local assignment of tadpole functions, then partition functions $\overline{Z^\tau}$ assemble into a functor.

Theorem 7.10. *Let τ_Σ be a local assignment of tadpole functions. The assignment*

$$\overline{Z^\tau}\colon \mathbf{Riem}^2 \to \mathbf{Hilb}^{\mathrm{form}}$$

given on objects by

$$\overline{Z^\tau}(Y \times (-\epsilon, \epsilon)) = H_Y$$

and on morphisms by

$$\overline{Z^\tau}(\Sigma) = \overline{Z^\tau_\Sigma}$$

is a functor.

Proof. Let $\Sigma = \Sigma_L \cup_Y \Sigma_R$ be a Riemannian cobordism. Since $\mathbf{Riem}^2$ is a semicategory, we actually only have to check the composition rule

$$\overline{Z^\tau}(\Sigma_L \cup_Y \Sigma_R) = \overline{Z^\tau}(\Sigma_L) \circ \overline{Z^\tau}(\Sigma_R).$$

We denote $Y_L = \partial_L \Sigma = \partial_L \Sigma_L, Y = \partial_R \Sigma_L = \partial_L \Sigma_R, Y_R = \partial_R \Sigma = \partial_R \Sigma_R$. To see this, recall that (this is Remark 7.6) composition of morphisms $F_1 \colon H_1 \to H_2$, $F_2 \colon H_2 \to H_3$ in the category $\mathbf{Hilb}^{\mathrm{form}}$ is given by extension of the pairing on H_2. On the other hand, this pairing is given by integrating against the Gaussian measure $\mu'_{2\varkappa}$ in the L^2 space corresponding to H_2:

$$\begin{aligned}
\overline{Z^\tau}(\Sigma_L) \circ \overline{Z^\tau}(\Sigma_R) &= \int_{D'(Y)} \overline{Z^\tau}(\Sigma_L)\overline{Z^\tau}(\Sigma_R) d\mu'_{2\varkappa} \\
&= e^{\frac{1}{2}\int_{Y_L} \tilde{\eta}\varkappa\tilde{\eta}\,\mathrm{dVol}_{Y_L}} e^{\frac{1}{2}\int_{Y_R} \tilde{\eta}\varkappa\tilde{\eta}\,\mathrm{dVol}_{Y_R}} \int_{D'(Y)} e^{\int_Y \tilde{\eta}\varkappa\tilde{\eta}\,\mathrm{dVol}_Y} Z^\tau_{\Sigma_L} Z^\tau_{\Sigma_R} d\mu'_{2\varkappa} \\
&= e^{\frac{1}{2}\int_{Y_L} \tilde{\eta}\varkappa\tilde{\eta}\,\mathrm{dVol}_{Y_L}} e^{\frac{1}{2}\int_{Y_R} \tilde{\eta}\varkappa\tilde{\eta}\,\mathrm{dVol}_{Y_R}} e^{-S_0(\phi^{\Sigma_L}_{\tilde{\eta}_L})} e^{-S_0(\phi^{\Sigma_R}_{\tilde{\eta}_R})} \\
&\int_{D'(Y)} \widehat{Z^\tau_{\Sigma_L}}\,\widehat{Z^\tau_{\Sigma_R}} e^{-\frac{1}{2}\int_Y \tilde{\eta}(S_L+S_R)\tilde{\eta}\,\mathrm{dVol}_Y} d\mu'_{2\varkappa}
\end{aligned} \tag{7.5}$$

Here $S_X = D_{\Sigma_X} - \varkappa$ for $X \in \{L, R\}$. Since $\varkappa^{-1}S_X$ is a trace-class operator by Proposition A.3, by known results on Gaussian measures (see e.g. [25, Chapter 7] or [4]),we have $e^{-\frac{1}{2}\int_Y \tilde{\eta}(S_L+S_R)\tilde{\eta}\,\mathrm{dVol}_Y} \mu'_{2\varkappa} = \mu'_{D_{\Sigma_L,\Sigma_R}}$ as measures on $D'(S^1)$.[33] However, integration against the latter Gaussian measure is just the pairing defined in Definition 3.8. So, expression (7.5) can be rewritten as

$$e^{\frac{1}{2}\int_{\partial\Sigma} \tilde{\eta}\varkappa\tilde{\eta}} e^{-S_0(\phi^{\Sigma_L}_{\tilde{\eta}_L})} e^{-S_0(\phi^{\Sigma_R}_{\tilde{\eta}_R})} \langle \widehat{Z^\tau_{\Sigma_L}}, \widehat{Z^\tau_{\Sigma_R}} \rangle_{\Sigma_L, Y, \Sigma_R}.$$

Hence the gluing formula 6.1 implies the composition law for $\overline{Z^\tau}$. $\square$

Remark 7.11. *The Hilbert-Schmidt norm of a partition function of a Riemannian cobordism Σ admits the following interpretation: its square is the partition function of the closed "doubled surface" $\widetilde{\Sigma} = \Sigma \cup_{\partial\Sigma} \overline{\Sigma}$ (assuming*

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[33] With our convention for the Gaussian measure, this can be proven similarly to Theorem 7.3 in [25] by showing the Fourier transforms of the two measures are equal.

the glued metric on $\widetilde{\Sigma}$ is smooth):

$$||\overline{Z^\tau}(\Sigma)||^2_{HS} = Z^{\widetilde{\tau}}(\widetilde{\Sigma})$$

Here we endow $\widetilde{\Sigma}$ with the tadpole function $\widetilde{\tau} = \tau * \tau$ – the gluing of τ (some a priory fixed tadpole function) on Σ and its reflection on the second copy, $\overline{\Sigma}$.

7.4. Another proof of functoriality

For the reader's convenience, here we give another proof of Theorem 7.10, under an extra assumption on admissible cobordisms. It is a direct proof using perturbation theory and not relying on infinite-dimensional measure theory.

Assumption 7.12. *For each boundary component Y of the cobordism Σ, the operator $\delta = \varkappa^{-1} D_\Sigma - 1$ on Y has the operator norm*

$$||\delta|| < 1 \tag{7.6}$$

(i.e. eigenvalues of δ are in the interval $(-1, 1)$).

Remark 7.13. *(a) Cobordisms satisfying Assumption 7.12 form a* subcategory *of* $\mathbf{Riem}^2$.[34] *We denote this subcategory* $\mathbf{Riem}^2_{||\delta||<1}$.

(b) Assumption 7.12 is not vacuous. E.g., a cylinder of height H, cf. (A.19), satisfies it iff $H > \frac{c}{m}$ where $c = \operatorname{arccoth} 2 \approx 0.5493$. Thus, short cylinders *fail the assumption.*
Another example: a spherical sector satisfies the assumption if the cone angle satisfies $\phi < c'$, with $c' \approx 0.9023\pi$. For $c' < \phi < \pi$, the assumption fails for mR in a certain interval (R is the sphere radius).

(c) As implied by (a) and (b), if we attach to the boundaries of any surface Σ sufficiently long cylinders, the resulting surface will satisfy Assumption 7.12.

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[34]The reason is that if $\widetilde{\Sigma} = \Sigma \cup_{Y'} \Sigma'$ and Y is a boundary component of Σ disjoint from Y', then we have $D_\Sigma > D_{\widetilde{\Sigma}} > 0$ (we mean the YY block of both Dirichlet-to-Neumann operators; an inequality of operators $A > B$ means that $A - B$ is a positive operator). This inequality follows from the gluing formula for Green's functions (Proposition 4.2) upon taking the second normal derivative. Therefore, $\delta_\Sigma > \delta_{\widetilde{\Sigma}} > -1$. Thus, if $||\delta_\Sigma|| < 1$, then also $||\delta_{\widetilde{\Sigma}}|| < 1$.

Let $\Sigma = \Sigma_L \cup_Y \Sigma_R$ be a Riemannian cobordism cut into two by Y. Let $S_i = D_{\Sigma_i} - \varkappa$, with $i \in \{L, R\}$. We assume that Σ_L, Σ_R satisfy Assumption 7.12.

Lemma 7.14. *(i) For $\Psi \in H_Y$ any state on Y, one has the following comparison of expectation values (7.2) and (3.12):*

$$\left\langle \Psi e^{-\frac{1}{2}\int_Y \widetilde{\eta}(S_L+S_R)\widetilde{\eta}\,\mathrm{dVol}_Y} \right\rangle_{2\varkappa} = \left\langle \Psi \right\rangle_{\Sigma_L,Y,\Sigma_R} \tag{7.7}$$

(ii) For $\Psi_1, \Psi_2 \in H_Y$ any pair of states on Y, one has the following comparison of pairings (7.1) and (3.14):

$$\langle \Psi_1 e^{-\frac{1}{2}\int_Y \widetilde{\eta} S_L(\widetilde{\eta})\,\mathrm{dVol}_Y}, \Psi_2 e^{-\frac{1}{2}\int_Y \widetilde{\eta} S_R(\widetilde{\eta})\,\mathrm{dVol}_Y} \rangle_{2\varkappa} = \left\langle \Psi_1, \Psi_2 \right\rangle_{\Sigma_L,Y,\Sigma_R} \tag{7.8}$$

Proof. For (i), assume that Ψ is given by a wave function $\psi(y_1, \ldots, y_n)$. The l.h.s. of (7.7) evaluates to

$$\begin{aligned} &\frac{1}{\det(2\varkappa)^{1/2}} \sum_{\mathfrak{m}\in\mathfrak{M}_n} \int_{C_n^\circ(Y)} dy_1 \cdots dy_n \, \psi(y_1, \ldots, y_n) \cdot \\ &\quad \cdot \prod_{\{i,j\}\in\mathfrak{m}} \Big(\sum_{k=0}^{\infty} \big(-(2\varkappa)^{-1}(S_L+S_R)\big)^k (2\varkappa)^{-1} \Big)(y_i, y_j) \\ &\quad \cdot \exp\Big(\sum_{p=1}^{\infty} \frac{1}{2p} \operatorname{tr}\big(-(2\varkappa)^{-1}(S_L+S_R)\big)^p \Big) \end{aligned}$$

The sum over k – the "dressed boundary propagator" – evaluates to $(2\varkappa + S_L + S_R)^{-1} = K$, the Green's function of D_{Σ_L,Σ_R}. The sum over p evaluates to

$$\begin{aligned} -\frac{1}{2}\operatorname{tr}\log(1 + (2\varkappa)^{-1}(S_L+S_R)) &= -\frac{1}{2}\log\det(1 + (2\varkappa)^{-1}(S_L+S_R)) \\ &= -\frac{1}{2}\log\frac{\det(D_{\Sigma_L,\Sigma_R})}{\det(2\varkappa)} \end{aligned}$$

Here $\det(1 + \cdots)$ is understood as a Fredholm determinant. Therefore, the l.h.s. of (7.7) coincides with the r.h.s.

Here the convergence of the sum over k and the sum over p, relies on two "smallness" properties of the operator $\delta^{\mathrm{tot}} = (2\varkappa)^{-1}(S_L + S_R) = \frac{1}{2}(\delta_L + \delta_R)$:

- the trace-class property $\operatorname{tr}\delta^{\text{tot}} < \infty$ (we have it by Proposition A.3) which is needed for the individual terms in the sum over p to be well-defined and
- the property $||\delta^{\text{tot}}|| < 1$ (implied by Assumption 7.12 for Σ_L, Σ_R) needed for the convergence of the sums over k and p.

Part (ii) follows trivially from (i) by setting $\Psi = \Psi_1 \odot \Psi_2$. $\square$

Proof of Theorem 7.10 under Assumption 7.12 . Functoriality of $\overline{Z}$ restricted to $\mathbf{Riem}^2_{||\delta||<1}$ follows from the gluing formula have already proven (Theorem 6.1) and from Lemma 7.14. Indeed, for any Σ we have

$$\overline{Z^{\tau}_{\Sigma}} = e^{-\frac{1}{2}\int_{\partial\Sigma} \widetilde{\eta} S(\widetilde{\eta})\, \mathrm{dVol}_{\partial\Sigma}} \widehat{Z^{\tau}_{\Sigma}}$$

Therefore,

$$\begin{aligned}
\left\langle \overline{Z}_{\Sigma_L}, \overline{Z}_{\Sigma_R} \right\rangle_{2\varkappa} &= e^{-\frac{1}{2}\int_{Y_L} \widetilde{\eta}_L S_L(\widetilde{\eta}_L)\, \mathrm{dVol}_{Y_L}} e^{-\frac{1}{2}\int_{Y_R} \widetilde{\eta}_R S_R(\widetilde{\eta}_R)\, \mathrm{dVol}_{Y_R}} \cdot \\
&\quad \cdot \left\langle e^{-\frac{1}{2}\int_Y \widetilde{\eta} S_L(\widetilde{\eta})\, \mathrm{dVol}_Y} \widehat{Z}_{\Sigma_L} \,,\, e^{-\frac{1}{2}\int_Y \widetilde{\eta} S_R(\widetilde{\eta})\, \mathrm{dVol}_Y} \widehat{Z}_{\Sigma_R} \right\rangle_{2\varkappa} \\
&\underset{\text{Lemma 7.14}}{=} e^{-\frac{1}{2}\int_{Y_L} \widetilde{\eta}_L S_L(\widetilde{\eta}_L)\, \mathrm{dVol}_{Y_L}} e^{-\frac{1}{2}\int_{Y_R} \widetilde{\eta}_R S_R(\widetilde{\eta}_R)\, \mathrm{dVol}_{Y_R}} \left\langle \widehat{Z}_{\Sigma_L} \,,\, \widehat{Z}_{\Sigma_R} \right\rangle_{\Sigma_L, Y, \Sigma_R} \\
&\underset{\text{Theorem 6.1}}{=} \overline{Z}_{\Sigma}
\end{aligned}$$

Here we are suppressing the tadpoles in the notations. Thus, $\overline{Z}$ satisfies the gluing formula with respect to $\langle \cdot, \cdot \rangle_{2\varkappa}$, which proves functoriality. $\square$

Remark 7.15. *The fact that in the perturbative approach we needed an additional assumption (7.6) on cobordisms while this assumption was not needed in the measure-theoretic approach of Section 7.3 can be modeled on on the following toy example. Consider a 1-dimensional Gaussian integral*

$$\begin{aligned}
\int_{-\infty}^{\infty} dx\, e^{-(1+C)x^2/2} = \int dx\, e^{-x^2/2} e^{-Cx^2/2} &= \int dx\, e^{-x^2/2} \sum_{k=0}^{\infty} \frac{(-C)^k}{2^k k!} x^{2k} \\
\text{“}=\text{”} &\sum_{k=0}^{\infty} \frac{(-C)^k}{2^k k!} \int dx\, e^{-x^2/2} x^{2k} \\
&\underset{\text{Wick's lemma}}{=} \sqrt{2\pi} \sum_{k=0}^{\infty} \frac{(-C)^k}{2^k k!} (2k-1)!! \\
&= \sqrt{2\pi}\left(1 - \frac{1}{2}C + \frac{1}{2!}\frac{1}{2}\frac{3}{2}C^2 - \cdots\right) \\
&= \sqrt{2\pi}(1+C)^{-\frac{1}{2}}
\end{aligned}$$

Here the l.h.s. defined measure-theoretically makes sense for any $C > -1$ whereas the sum after " $=$ " is only convergent for $-1 < C < 1$, i.e. an additional restriction on C arises. The point here is that in the equality " $=$ " we are interchanging an integral and a sum which is only valid under this additional restriction on C.

Remark 7.16. *One can remove the restrictive Assumption 7.12 in the perturbative proof of functoriality by considering the following "mixed" picture for the pairing $\langle\cdot,\cdot\rangle_{2\varkappa}$. One can*[35] *split functions on Y into the span of eigenfunctions of δ with eigenvalues ≥ 1 and the span of eigenfunctions with eigenvalues in the interval $(-1,1)$:*

$$C^\infty(Y) = [C^\infty(Y)]_{||\delta||\geq 1} \oplus [C^\infty(Y)]_{||\delta||<1}$$

Here the first term on the right is a finite-dimensional *vector space. Then, one can define the pairing $\langle\cdot,\cdot\rangle_{2\varkappa}$ as a combination of a finite-dimensional measure-theoretic Gaussian integral over $[C^\infty(Y)]_{||\delta||\geq 1}$ and a perturbatively defined, via Wick contractions, Gaussian integral over $[C^\infty(Y)]_{||\delta||<1}$ where one does not have a convergence problem.*

8. Discussion and outlook

In this paper we have defined the perturbative partition function of two-dimensional scalar field theory as a formal power series, and shown that it satisfies an Atiyah-Segal type gluing relation. In particular, this shows that the perturbatively defined path integral in our model satisfies a crucial property expected from the path integral – a Fubini-type theorem.

To obtain this result we used gluing formulae for the zeta-regularized determinants and the Green's function of the Helmholtz operator, together with some combinatorics of Feynman diagrams. Naturally, one is led to the expectation that similar techniques will allow to prove gluing formulae for other theories.

As explained above, the gluing pairing can be thought of as a mathematical definition of a functional integral over boundary fields. One can think of this functional integral as an expectation value with respect to a non-local boundary theory. A similar perspective was advocated in [13],[14].

Similar results for first-order gauge theories in BV formalism have been obtained by Cattaneo, Reshetikhin and the second author in [9]. We choose a slightly different way to define the partition function on a manifold with

[35] This strategy is inspired by the proof of Theorem 7.3 in [25].

boundary.[36] However, we expect the two approaches to be ultimately equivalent. We plan to explore this relation in the future.

We have also proven that the perturbative quantization in our model gives rise to a functor from the category of Riemannian 2-cobordisms to the category of Hilbert spaces and Hilbert-Schmidt operators.

The cutting-gluing formula for partition functions underlying the functoriality result relies on the careful treatment of tadpole diagrams and their interaction with locality.

The following questions naturally arise from our treatment of scalar theory.

I. Compare with the treatment in the first order formalism. In particular: is the non-local "gluing theory" on the boundary the effective theory for some local gluing theory that arises in the first order formalism?

II. The adjusted partition function $\overline{Z}$ entering in the functorial formulation modifies the standard partition function Z (corresponding to quantization with Dirichlet polarization on the boundary) by a factor $e^{\frac{1}{2\hbar}\int_{\partial\Sigma} \mathrm{dVol}_{\partial\Sigma}\eta\varkappa(\eta)}$. It begs an interpretation in terms of a new "Helmholtz" polarization imposed on the boundary, where $\partial_n\phi - \varkappa(\phi)$ is fixed on $\partial\Sigma$. This new polarization can be seen as a complex polarization on the boundary phase space, whereas Dirichlet condition gives a real polarization; the two are connected by a Segal-Bargmann transform which can be represented by a partition function of a short cylinder with Dirichlet polarization on one side and Helmholtz condition on the other side.[37]

III. It would be very interesting to extend our treatment of 2-dimensional scalar theory to allow cutting and gluing with corners. Correspondingly, we expect the functorial picture to generalize to a fully extended FQFT out of an appropriate Riemannian cobordism 2-category. In topological case, this formalism is known from Baez-Dolan-Lurie [3],[40]. A related question is enrichment of the theory by defects supported on strata.

IV. A big open problem is the compatibility of renormalization with locality in more general setting and in higher-dimensional theories. In

[36] We use the (unique) harmonic extension of the boundary field, while [9] uses a discontinuous extension of the boundary fields, i.e. it drops to zero immediately outside the boundary.

[37] In the context of Chern-Simons theory, (generalized) Segal-Bargmann transform via attaching a cylinder with appropriate boundary polarizations is studied in the paper in preparation [10].

particular, it would be natural to try to extend our treatment of scalar theory to higher dimension (but restricting the potential p to be renormalizable, e.g. $p(\phi) = \phi^4$ in dimension ≤ 4 or $p(\phi) = \phi^3$ in dimension ≤ 6) and studying renormalization and RG flow there.

V. This paper and [28],[9] suggest that there is certain algebraic structure on Feynman graphs which is responsible for Atiyah-Segal type gluing formulae for perurbative partition functions. One can consider graph-valued partition function (in the spirit of LMO invariant or Kontsevich-Kuperberg-Thurston-Lescop construction) and one expects this version of partition function to be an idempotent (or, dually, a group-like element) w.r.t. the gluing operation on graphs.[38]

Appendix A. Examples

In this section we provide some explicit examples of determinants, tadpole functions, and gluing formulae. Even though the main focus of this paper is two-dimensional scalar field theory, we consider also 1-dimensional examples, where answers are simpler and more explicit. All the constructions in this paper are valid, with minor adjustments, also for 1-dimensional scalar field theory.

A.1. One-dimensional examples

A.1.1. Interval. Denote $A^{DD}_{m,l} = (-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + m^2)$ with Dirichlet boundary conditions. The Green's function of the operator $A^{DD}_{m,l}$ can be explicitly computed and yields

$$G(x,y) = \frac{1}{m} \frac{\sinh mx \sinh m(l-y)\theta(y-x) + \sinh m(l-x) \sinh my\theta(x-y)}{\sinh ml},$$

where θ is the Heaviside function, which leads to the tadpole function[39]

$$\tau(x) = G(x,x) = \frac{\sinh mx \sinh m(l-x)}{m \sinh ml}.$$

[38] A similar problem is currently being investigated in [33].

[39] Here $\theta(0) = 1/2$. Notice that the Green's function is actually continuous across the diagonal.

The zeta-regularized determinant of A can be computed[40] as

$$\det A^{DD}_{m,l} = \frac{2\sinh ml}{m}.$$

Notice that in the limit as $m \to 0$ we obtain $2l$ - for Dirichlet boundary conditions, the operator $A^{DD}_{0,L}$ has no kernel and we obtain its nonzero determinant. In particular, we see that the tadpole is consistent (in the weak sense, Definition 5.5) with zeta-regularization: We have

$$\frac{\mathrm{d}}{\mathrm{d}m^2}\log\det(A^{DD}_{m,l}) = \frac{1}{4m}\left(\frac{l\cosh ml}{\sinh ml} - \frac{1}{m}\right) = \int_0^l \tau(x)dx.$$

Now consider the gluing of the two intervals $I_{l_1} = [0, 1_1]$ and $I_{l_2} = [l_1, l_1 + l_2]$ over the the point $Y = \{l\}$. Then the Dirichlet-to-Neumann operator along Y is given by

$$D_{I_{l_1},Y,I_{l_2}} : \eta \mapsto m(\coth ml_1 + \coth ml_2)\eta = m\frac{\sinh m(l_1+l_2)}{\sinh ml_1 \sinh ml_2}\eta.$$

Then we compute

$$\begin{aligned}
&\det(A^{DD}_{ml_1})\det(A^{DD}_{ml_2})\det\frac{1}{2}D_{I_{l_1},Y,I_{l_2}}\\
&= \left(\frac{2}{m}\sinh ml_1\right)\left(\frac{2}{m}\sinh ml_2\right)\frac{1}{2}\left(m\coth ml_1 + \coth ml_2\right)\\
&= \frac{2}{m}\sinh m(l_1+l_2) = \det A^{DD}_{m,l_1+l_2}.
\end{aligned}$$

The factor $\frac{1}{2}$ which appears here - in contrast to the gluing formula (4.3) - arises because we are gluing 1-dimensional determinants. It is a correctional factor in the gluing formula for determinants that is present in *odd* dimensions,[41] see [38]. Finally, let us consider the gluing of the tadpole function. We will check that for $x < l_1$, $\tau_{l_1+l_2}(x) = \tau_{l_1}(x) * \tau_{l_2}(x)$ (the other cases are

[40] See e.g. [48, Example 3, p.220] for a derivation.

[41] This factor hints at the fact that in odd dimensions one should normalize the "measure" of the path integral accordingly. The correct normalization of the path integral on spacetimes with boundaries should be such that it is compatible with gluing. A similar discussion for factors in gluing of partition functions in abelian BF theory is given in [9].

similar). Indeed, the left hand side is

$$\tau_{l_1+l_2}(x) = \frac{\sinh mx \sinh m(l_1+l_2-x)}{m \sinh m(l_1+l_2)}$$

while the right hand side is

$$\begin{aligned}
&\tau_{l_1}(x) + \left(\frac{\mathrm{d}}{\mathrm{d}y}\bigg|_{y=l_1} G(x,y) \right)^2 D^{-1}_{I_{l_1},Y,I_{l_2}} \\
&= \frac{\sinh mx \sinh m(l_1-x)}{m \sinh ml_1} + \left(\frac{\sinh mx}{\sinh ml_1} \right)^2 \frac{\sinh ml_1 \sinh ml_2}{m \sinh m(l_1+l_2)} \\
&= \left(\frac{\sinh mx}{m} \right) \frac{\sinh m(l_1-x) \sinh m(l_1+l_2) + \sinh mx \sinh ml_2}{\sinh ml_1 \sinh m(l_1+l_2)} \\
&= \tau_{l_1+l_2}(x).
\end{aligned}$$

A.1.2. Circle. Consider a circle of length l, and let $A_{m,l} = -\mathrm{d}^2/\mathrm{d}x^2 + m^2$. The spectrum of this operator is $\lambda_k = (2\pi k/l)^2 + m^2, k \in \mathbb{Z}$. Then one can compute the determinant as

$$\det A_{m,l} = 4 \sinh^2 \frac{ml}{2}. \tag{A.1}$$

For $x, y \in \mathbb{R}$ denote $d(x,y) = l \left\{ \frac{x-y}{l} \right\}$, then the Green's function can be expressed as

$$G(x,y) = \frac{1}{2m} \frac{\cosh m(d(x,y) - l/2)}{\sinh \frac{ml}{2}}$$

and the tadpole function is

$$\tau(x) = G(x,x) = \frac{1}{2m} \coth \frac{ml}{2}.$$

Again, one immediately verifies that the tadpole function is (weakly) consistent with zeta-regularization, namely

$$\frac{1}{2m}\frac{\mathrm{d}}{\mathrm{d}m} \log\det A_{m,l} = \frac{2}{2m}\frac{\mathrm{d}}{\mathrm{d}m} \log \sinh \frac{ml}{2} = \frac{l}{2m} \coth \frac{ml}{2} = \int_{S^1} \tau(x)\, dx$$

Next, we want to glue a circle out of two arcs I_1, I_2 of length l_1, l_2 along the interface $Y = \{p, q\}$ (see Figure A1). The corresponding Dirichlet-to-

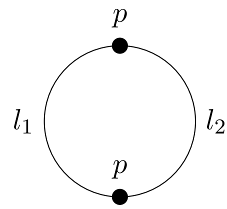

Figure A1. Gluing a circle from two intervals of length l_1, l_2.

Neumann operator is the sum of the two operators:

$$D_N(\eta_p, \eta_q) = m \begin{pmatrix} \coth ml_1 + \coth ml_2 & -\left(\frac{1}{\sinh ml_1} + \frac{1}{\sinh ml_2}\right) \\ -\left(\frac{1}{\sinh ml_1} + \frac{1}{\sinh ml_2}\right) & \coth ml_1 + \coth ml_2 \end{pmatrix} \begin{pmatrix} \eta_p \\ \eta_q \end{pmatrix}$$

and straightforward computation shows that its determinant is

$$\det D_N = \frac{4m^2}{\sinh ml_1 \sinh ml_2} \sinh^2 \frac{m(l_1 + l_2)}{2}$$

Therefore, we obtain that the product of determinants is

$$\begin{aligned} &\det A^{DD}_{m,l_1} \det A^{DD}_{m,l_1} \det \frac{1}{2} D_N \\ &= \frac{2\sinh ml_1}{m} \frac{2\sinh ml_2}{m} \frac{m^2}{\sinh ml_1 \sinh ml_2} \sinh^2 \frac{m(l_1+l_2)}{2} \\ &= 4\sinh^2 \frac{m(l_1+l_2)}{2} = \det A_{m,l_1+l_2} \end{aligned}$$

where again, the factor $\frac{1}{2}$ in the Dirichlet-to-Neumann operator turns out to be correct[42] according to the gluing formula [38].

Let us also check the gluing of tadpoles. Again, we will check the case where $x \in I_1$. Then, the gluing formula for the tadpole reads

$$\tau_{I_1} * \tau_{I_2}(x) = \tau_{I_1}(x) + \left(-\frac{\mathrm{d}}{\mathrm{d}\nu} G_{I_1}(x,p) \quad \frac{\mathrm{d}}{\mathrm{d}\nu} G_{I_1}(x,q)\right) D_N^{-1} \begin{pmatrix} -\frac{\mathrm{d}}{\mathrm{d}\nu} G_{I_1}(x,p) \\ \frac{\mathrm{d}}{\mathrm{d}\nu} G_{I_1}(x,q) \end{pmatrix}$$

[42] The general formula says that the prefactor is $e^{\log 2(\zeta_{\Delta_Y}(0) + \dim \ker \Delta_Y)}$, where Δ_Y is the Laplacian on the interface. If $\dim Y = 0$ we have $\Delta_Y = 0$, and hence $\zeta_{\Delta_Y} = 0$ and $\dim \ker \Delta_Y = \dim C^\infty(Y) = |Y|$. In this case $|Y| = 2$ - hence the prefactor 4.

where matrix multiplication replaces integration. The inverse of the Dirichlet-to-Neumann operator can be explicitly computed and yields

$$D_N^{-1} = \begin{pmatrix} \frac{1}{2}\coth m(l_1+l_2)/2 & \frac{\sinh ml_1+\sinh ml_2}{4\sinh^2 m(l_1+l_2)/2} \\ \frac{\sinh ml_1+\sinh ml_2}{4\sinh^2 m(l_1+l_2)/2} & \frac{1}{2}\coth m(l_1+l_2)/2 \end{pmatrix}$$

A straightforward computation then shows

$$\begin{aligned} \tau_{I_1} * \tau_{I_2}(x) &= \frac{\sinh mx \sinh m(l_1-x)}{m\sinh ml_1} \\ &\quad + \frac{1}{4\sinh ml_1}\Big(2\cosh m(l_1-2x) \\ &\qquad + (\cosh ml_1 - \cosh ml_2)\sinh^{-2}\frac{m(l_1+l_2)}{2}\Big) \\ &= \frac{1}{2m}\coth\frac{m(l_1+l_2)}{2} = \tau(x) \end{aligned}$$

A.2. Two-dimensional examples

Now let us turn to two-dimensional examples. The main tool that we will use is the heat kernel of the Laplacian $K_\Delta(t,x,y)$ and the heat kernel for the corresponding Helmholtz operator, $K_{\Delta+m^2} = e^{-m^2t}K_\Delta(t,x,y)$. We recall some formulae for heat kernels of standard metrics. On the real line, the heat kernel is

$$K_\Delta^{\mathbb{R}}(t,x,y) = \frac{1}{\sqrt{4\pi t}}\, e^{\frac{-(x-y)^2}{4t}}.$$

From this, one can infer the heat kernel on the circle of length L through periodic summation:

$$K_\Delta^{S^1}(t,x,y) = \frac{1}{\sqrt{4\pi t}}\sum_{k=-\infty}^{\infty} e^{\frac{-(x-y-kL)^2}{4t}} \tag{A.2}$$

and the heat kernel on an interval of length L with Dirichlet boundary conditions (through image charges):

$$K_\Delta^{DD}(t,x,y) = \frac{1}{\sqrt{4\pi t}}\sum_{k=-\infty}^{\infty} e^{\frac{-(x-y-2L)^2}{4t}} - e^{\frac{-(x+y-2kL)^2}{4t}} \tag{A.3}$$

In addition, we recall the fact that the heat kernel of the Laplacian of a product metric is the product of the heat kernels of the Laplacians associated to the two metrics.

A.2.1. Torus. First, we consider a torus $T \equiv T_{L_1,L_2}$ of circumferences L_1 and L_2. Then the heat kernel is given by

$$K_\Delta^T(t,(x,x'),(y,y')) = \frac{1}{4\pi t} \sum_{k,l=-\infty}^{\infty} e^{\frac{-(x-x'-kL_1)^2}{4t}} e^{\frac{-(y-y'-lL_2)^2}{4t}}$$

Its restriction to the diagonal reads

$$\theta_\Delta^T(t,(x,y)) \equiv \theta_\Delta^T(t) = \frac{1}{4\pi t} \sum_{k,l=-\infty}^{\infty} e^{\frac{-(kL_1)^2}{4t}} e^{\frac{-(lL_2)^2}{4t}}$$

which is conveniently expressed in terms of the Jacobi theta function

$$\vartheta(z,\tau) = \sum_{k=-\infty}^{\infty} \exp(\pi i k^2 \tau + 2\pi i k z) \tag{A.4}$$

as

$$\theta_\Delta^T(t,(x,y)) \equiv \theta_\Delta^T(t) = \frac{1}{4\pi t} \vartheta\left(0, \frac{iL_1^2}{4\pi t}\right) \vartheta\left(0, \frac{iL_2^2}{4\pi t}\right).$$

The heat kernel of $A = \Delta + m^2$ is then given by

$$\theta_A^T(t) = \frac{e^{-m^2 t}}{4\pi t} \vartheta\left(0, \frac{iL_1^2}{4\pi t}\right) \vartheta\left(0, \frac{iL_2^2}{4\pi t}\right).$$

Extracting the divergence at $t = 0$, we write

$$\theta_A^T(t) = \frac{e^{-m^2 t}}{4\pi t} + e^{-m^2 t} h(t)$$

where $h(t) = \frac{1}{4\pi t}\left(\vartheta\left(0, \frac{iL_1^2}{4\pi t}\right)\vartheta\left(0, \frac{iL_2^2}{4\pi t}\right) - 1\right)$ falls off like $e^{-C/t}$ as $t \to 0$. The local zeta function of A is the Mellin transform of this object:

$$\zeta_A^{loc}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{e^{-m^2 t}}{4\pi t}(1 + h(t))dt$$

For $\operatorname{Re} s > 1$, the first term is given by

$$\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{e^{-m^2 t}}{4\pi t} = \frac{\Gamma(s-1)}{4\pi\Gamma(s) m^{2(s-1)}} = \frac{1}{4\pi(s-1)m^{2(s-1)}}.$$

The second integral can be explicitly given in terms ofa modified Bessel function of the second kind $K_s(x)$, thus the analytically continued zeta function is

$$\zeta_A(s)^{loc} = \frac{1}{4\pi(s-1)m^{2(s-1)}} + \frac{m^{1-s}}{2\pi\Gamma(s)} \sum_{k,l\neq 0} b_{k,l}^{s-1} K_{1-s}(2mb_{k,l})$$

where $b_{k,l} = \sqrt{k^2L_1^2 + l^2L_2^2}$. The first term has a pole at $s=1$, while the second term is an entire function of s. The tadpole function is the finite part of $\zeta_A(s)^{loc}$ at $s=1$:

$$\tau^{reg} = -\frac{\log m^2}{4\pi} + \frac{1}{2\pi}\sum_{k,l\neq 0} K_0(2mb_{k,l})$$

and the logarithm of the zeta-regularized determinant is

$$\log\det A = \zeta_A'(0) = \frac{L_1L_2}{4\pi} m^2(\log m^2 - 1) + \frac{mL_1L_2}{2\pi}\sum_{k,l\neq 0} \frac{K_1(2mb_{k,l})}{b_{k,l}}.$$

One immediately verifies $\frac{\mathrm{d}}{\mathrm{d}m^2}\log\det A = -\int_T \tau^{reg}\,\mathrm{dVol}_T$ from the relation $\frac{\mathrm{d}}{\mathrm{d}x} x^s K_s(x) = -x^s K_{s-1}(x)$.

A.2.2. Cylinder. Let us consider a cylinder C of circumference L and height H. Then, the heat kernel is given by

$$K^C_\Delta(t,(x,y),(x',y')) = K^{S^1}_\Delta(t,x,x')K^{DD}_\Delta(t,y,y').$$

The restriction to the diagonal is given by, restricting to the diagonal in (A.2),(A.3),

$$\theta^C_\Delta(t,(x,y)) = \frac{1}{4\pi t}\sum_{k=-\infty}^{\infty} e^{-\frac{(2kL)^2}{4t}} \sum_{l=-\infty}^{\infty} e^{\frac{-(lH)^2}{4t}} - e^{\frac{-(2y-2lH)^2}{4t}} \tag{A.5}$$

Using the Jacobi theta function (A.4) we can express (A.5) as

$$\theta^C_\Delta(t,(x,y)) = \frac{1}{4\pi t}\vartheta\left(0,\frac{iL^2}{4\pi t}\right)\left(\vartheta\left(0,\frac{iH^2}{\pi t}\right) - e^{\frac{-y^2}{t}}\vartheta\left(\frac{yH}{\pi i t},\frac{iH^2}{\pi t}\right)\right) \tag{A.6}$$

Recall that the theta function has the modular transform $\vartheta(z/\tau,-1/\tau) = \alpha\vartheta(z,\tau)$, where $\alpha = (-i\tau)^{\frac{1}{2}}\exp(\pi i z^2/\tau)$. Setting $\tau = \frac{4\pi i t}{H^2}, z = \frac{y}{H}$, we can rewrite (A.6) as

$$\theta^C_\Delta(t,(x,y)) = \frac{1}{2LH}\vartheta\left(0,\frac{4\pi it}{L^2}\right)\left(\vartheta\left(0,\frac{\pi it}{H^2}\right) - \vartheta\left(\frac{y}{H},\frac{\pi it}{H^2}\right)\right) \tag{A.7}$$

Integrating over $(x,y) \in C$ we obtain

$$\Theta^C_\Delta(t) := \int_C *\theta^C_\Delta = \frac{1}{2}\vartheta\left(0,\frac{4\pi it}{L^2}\right)\left(\vartheta\left(0,\frac{\pi it}{H^2}\right) - 1\right)$$

which we rewrite using the modular transform as
(A.8)

$$\Theta^C_\Delta(t) = \frac{LH}{4\pi t}\vartheta\left(0,\frac{iL^2}{4\pi t}\right)\left(\vartheta\left(0,\frac{iH^2}{\pi t}\right) - \frac{\sqrt{\pi t}}{H}\right) = \frac{LH}{4\pi t} - \frac{L}{4\sqrt{\pi t}} + h(t)$$

where

$$h(t) = \frac{LH}{4\pi t}\left(\vartheta\left(0,\frac{iL^2}{4\pi t}\right)\vartheta\left(0,\frac{iH^2}{\pi t}\right) - 1\right) - \frac{L}{4\sqrt{\pi t}}\left(\vartheta\left(0,\frac{iL^2}{4\pi t}\right) - 1\right)$$

satisfies $h(t) \simeq e^{-C/t}/t$ as $t \to 0$. Now consider the operator $\mathrm{A} := \Delta_C + m^2$, it heat kernel is given by $\theta^C_{\mathrm{A}} = e^{-m^2 t}\theta_\Delta$. The local zeta function of A is its Mellin transform,

$$\zeta_{\mathrm{A}}(s,(x,y)) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-m^2 t}\theta_\Delta dt.$$

To investigate its behavior at $s = 0, 1$, we define $g(t) = e^{-m^2 t}/(4\pi t)$ and then write

$$\zeta(s,(x,y)) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{e^{-m^2 t}}{4\pi t}dt + \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-m^2 t}\left(\theta_\Delta - \frac{1}{4\pi t}\right)dt.$$

The second integral converges absolutely at $s = 1$. The first integral can be explicitly computed (for $\operatorname{Re} s > 1$) and yields

$$\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{e^{-m^2 t}}{4\pi t}dt = \frac{\Gamma(s-1)}{4\pi\Gamma(s)m^{2(s-1)}} = \frac{1}{(s-1)}\cdot\frac{1}{4\pi m^{2(s-1)}}$$

The zeta-regularized tadpole function is given by

$$\begin{aligned}\tau^{reg}_{\mathrm{A}}(x,y) &= \lim_{s\to 1}\frac{\mathrm{d}}{\mathrm{d}s}(s-1)\zeta_{\mathrm{A}}(s,(x,y)) \\ &= -\frac{\log m^2}{4\pi} + \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-m^2 t}\left(\theta^C_\Delta - \frac{1}{4\pi t}\right)dt\end{aligned} \tag{A.9}$$

which θ^C_Δ given by (A.6). The zeta function of A is

$$\zeta_{\mathrm{A}}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-m^2 t} \Theta_\Delta dt$$

and using the decompositon (A.8) we write

$$\begin{aligned}\zeta_{\mathrm{A}}(s) = &\frac{LH}{(s-1)} \cdot \frac{1}{4\pi m^{2(s-1)}} - \frac{L}{4\sqrt{\pi}} \cdot \frac{\Gamma(s-1/2)}{\Gamma(s)\, m^{2s-1}} \\ &+ \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-m^2 t} h(t) dt\end{aligned}$$

Here, the last integral converges absolutely for any $s \in \mathbb{C}$. The logarithm of the zeta-regularized determinant is given by

$$\log\det \mathrm{A} = \zeta'_{\mathrm{A}}(0) = \frac{LH}{4\pi} m^2 (\log m^2 - 1) + \frac{Lm}{2} + \int_0^\infty t^{-1} e^{-m^2 t} h(t) dt$$

Gluing two cylinders into a cylinder. Next, we will investigate gluing of two cylinders into a longer cylinder. For this, consider the Dirichlet-to-Neumann operator D_H, say, on the lower end of a cylinder of circumference L and height H, with Helmholtz operator A_H (Circumference L and mass m are fixed). It is easiest to determine the Dirichlet-to-Neumann operator by its action on the basis of $C^\infty(S^1)$ given by $\eta_n(x) = \exp(2\pi i n x/L)$. The unique function $\phi_{\eta_n}(x, y)$ satisfying $A_H \phi_{\eta_n} = 0$, $\phi_{\eta_n}(x, 0) = \eta_n(x)$ and $\phi_{\eta_n}(x, H) = 0$ is

$$\phi_{\eta_n}(x, y) = \eta_n(x) \frac{\sinh(H-y)\omega_n}{\sinh H\omega_n},$$

where $\omega_n = \sqrt{m^2 + (2\pi n/L)^2}$. Taking the y derivative at $y = 0$ we find

$$D_H(\varphi_{\eta_n}) = \varphi_{\eta_n} \omega_n \coth H\omega_n.$$

When gluing two cylinders, the relevant Dirichlet-to-Neumann operator is $D_{H_1} + D_{H_2}$, which has eigenvalues $\lambda_n = \omega_n(\coth H_1\omega_n + \coth H_2\omega_n)$. To compute its zeta-regularized determinant, recall that $\det_\zeta(K_1 K_2) = \det_\zeta K_1 \det(K_2)$ if K_2 is the identity plus a trace class operator (and hence has a well-defined Fredholm determinant). In this example, we let $K_2 \colon \eta_n \mapsto \frac{1}{2}(\coth H_1\omega_n + \coth H_2\omega_n)\eta_n$, which is identity plus trace class and $K_1 \colon \eta_n \mapsto 2\omega_n \eta_n$. The zeta determinant of K_1 is the square root of the zeta-determinant of the Helmholtz operator[43] on S^1 given in (A.1): $\det K_1 = 2\sinh mL/2$.

[43]Taking squares commutes with zeta-regularized products. On the other hand, multiplying all terms by a constant a multiplies the zeta-regularized product $a^{\zeta(0)}$,

Thus,

$$\det(D_{H_1} + D_{H_2}) = 2\sinh\frac{mL}{2}\prod_{n\in\mathbb{Z}}\frac{\coth H_1\omega_n + \coth H_2\omega_n}{2}.$$

The gluing formula for zeta-regularized determinants thus implies the interesting identity

$$\begin{aligned}\log\det A_{H_1+H_2} &- \log\det A_{H_1} - \log\det A_{H_2}\\ &= -\frac{Lm}{2} + \int_0^\infty t^{-1}e^{-m^2t}(h_{H_1+H_2}(t) - h_{H_1}(t) - h_{H_2}(t))dt\\ &= \log\det(D_{H_1} + D_{H_2})\\ &= \log\left(2\sinh\frac{mL}{2}\right) + \sum_{n\in\mathbb{Z}}\log\left(\frac{\coth H_1\omega_n + \coth H_2\omega_n}{2}\right)\end{aligned}$$

and one can check numerically that his formula holds.
We can extend this numerical check to tadpoles. The value of the glued tadpole $\tau_{H_1} * \tau_{H_2}$ at some point $(x,y)\in C_{H_1}$ is

$$\begin{aligned}\tau_{H_1}(x,y) + \int_{S^1\times S^1} &\partial_\nu G((x,y),(x',0))(D_{H_1}+D_{H_2})^{-1}\\ &\cdot (x',x'')\partial_\nu G((x,y),(x'',0))dx'dx''.\end{aligned}$$

Here the tadpole function on the cylinder is given by (A.9); $\partial_\nu G$ is the normal derivative of the Green's function in the second argument. To compute the second term, we expand the normal derivative in Fourier modes

$$\partial_\nu G((x,y),(x',0)) = \int_0^\infty dt\frac{e^{-m^2t}}{\sqrt{4\pi t^3}L}\left(\sum_{n=-\infty}^{\infty} e^{-\frac{4t\pi^2n^2}{L^2}}\varphi_n(x')\right)g(t,y),$$

where

$$g(t,y) = e^{-\frac{y^2}{4t}}\left(y\,\vartheta\left(\frac{yH}{\pi it},\frac{iH^2}{\pi t}\right) - \frac{2H_1}{2\pi i}\vartheta'\left(\frac{yH}{\pi it},\frac{iH^2}{\pi t}\right)\right).$$

The inverse of the Dirichlet-to-Neumann operator is given by

$$\varphi_n(x)\mapsto(\omega_n(\coth(H_1\omega_n) + \coth(H_2\omega_n)))^{-1}\varphi_n(x)$$

---

where ζ is the zeta function of the sequence. One can check that in this case $\zeta(0)=0$, using e.g. the results of [48].

so that we obtain

$$\tau_{H_1} * \tau_{H_2} = \tau_{H_1} + \int_{[0,\infty]^2} dtdu \frac{e^{-m^2(t+u)}}{\sqrt{4\pi u^3}\sqrt{4\pi t^3}L^2} \cdot \sum_{n=-\infty}^{\infty} \frac{e^{-\frac{4(t+u)\pi^2 n^2}{L^2}}}{\omega_n(\coth(H_1\omega_n) + \coth(H_2\omega_n))} g(t,y)g(u,y).$$

Again one can check numerically that this equals $\tau_{H_1+H_2}(y)$.

A.2.3. Sphere. Consider a sphere of radius R.

Green's function. The Green's function for the Helmholtz operator on a sphere is:

$$G_{S^2}(x,y) = \frac{1}{4\cos\pi\left(\frac{1}{4} - (mR)^2\right)^{\frac{1}{2}}} \cdot {}_2F_1\Big(\alpha_1, \alpha_2; 1; \cos^2\frac{d(x,y)}{2R}\Big) \tag{A.10}$$

with $\alpha_{1,2}$ the roots of the quadratic equation $\alpha^2 - \alpha + (mR)^2 = 0$ and $d(x,y)$ is the geodesic distance in the sphere metric; ${}_2F_1$ is the hypergeometric function.

Note that the Green's function on a hemisphere can be obtained from (A.10) by the image charge method:

$$G_{H^+}(x,y) = G_{S^2}(x,y) - G_{S^2}(x,\hat{y})$$

where $\hat{y}$ is the reflection of y through the equatorial plane.

Remark A.1. *The asymptotics $m \to 0$ of the Green's function (A.10) reads*

$$G_{S^2}(x,y) \underset{m\to 0}{\sim} \frac{1}{\mathrm{Area}(S^2)\cdot m^2} \underbrace{-\frac{1}{2\pi}\left(\log\sin\frac{d(x,y)}{2R} + \frac{1}{2}\right)}_{G^{\mathrm{fin}}} + \mathrm{O}(m^2R^2)$$

The finite part G^{fin} here is the propagator of the massless scalar theory on S^2. It satisfies

$$\Delta G^{\mathrm{fin}} = \delta(x,y) - c \tag{A.11}$$

with the constant $c = \frac{1}{\mathrm{Area}(S^2)}$.[44]

[44] The constant shift by $-c$ is related to the zero-mode of Δ. More precisely, the integral operator defined by G^{fin} inverts Δ only on the orthogonal complement of constant functions and vanishes on constant functions.

In fact, as one can show from examining the $t \to \infty$ asymptotics of the heat kernel, this behavior is universal: for any surface Σ, the asymptotics $m \to 0$ of the Green's function is $G(x,y) \underset{m\to 0}{\sim} \frac{1}{\mathrm{Area}(\Sigma)\cdot m^2} + G^{\mathrm{fin}}(x,y) + \mathrm{O}(m^2)$ with $G^{\mathrm{fin}}(x,y)$ some function satisfying the (A.11) with $c = \frac{1}{\mathrm{Area}(\Sigma)}$.

Tadpole. Next, note that the zeta-regularized tadpole on the sphere can be computed from $y \to x$ asymptotics of the Green's function (A.10) (by first calculating the point-splitting tadpole and then using Corollary 5.21) and yields

$$\tau^{\mathrm{reg}} = \frac{\log R^2 - \psi(\alpha_1) - \psi(\alpha_2)}{4\pi} \tag{A.12}$$

where $\psi(z) = \frac{d}{dz}\log\Gamma(z)$ is the digamma function.

Determinant. Helmholtz operator on the sphere of radius R has eigenvalues $\lambda_l = \frac{l(l+1)}{R^2} + m^2$ with multiplicities $2l+1$, for $l = 0, 1, 2, \ldots$. The corresponding zeta-regularized determinant is:

$$\begin{aligned} \det(\Delta + m^2) &= \Big(\prod_{l\geq 0} \Big(\frac{l(l+1)}{R^2} + m^2\Big)^{2l+1} \Big)_{\mathrm{reg}} \\ &= R^{-2\zeta_{\bar{A}}(0)} \Big(\prod_{l\geq 0} (l(l+1) + m^2R^2)^{2l+1} \Big)_{\mathrm{reg}} \\ &= R^{-2(\frac{1}{3} - m^2R^2)} \mathbb{F}(m^2R^2) \end{aligned} \tag{A.13}$$

Here we denoted $\bar{A} = R^2(\Delta + m^2)$ the rescaled Helmholtz operator; we denoted its regularized determinant by $\mathbb{F}(m^2R^2)$. We are using the property of zeta-regularized determinants $\det(c\bar{A}) = c^{\zeta_{\bar{A}}(0)}\det(\bar{A})$. The value $\zeta_{\bar{A}}(0) = \frac{1}{3} - m^2R^2$ is calculated straightforwardly.[45]

In fact, we can determine the function $\mathbb{F}(z)$ in (A.13) explicitly from knowing the zeta-regularized tadpole on the sphere (A.12), by integrating the tadpole against mass (cf. Corollary 5.16). Indeed, setting $R = 1$, we

[45]Indeed, for $\mathrm{Re}(s) > 1$ one has $\zeta_{\bar{A}}(s) = \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1} e^{-m^2R^2t} f(t)$, where $f(t) = \sum_{l=0}^\infty (2l+1)e^{-l(l+1)t}$. Using Euler-Maclaurin formula, one obtains the asymptotics $f(t) \sim \frac{1}{t} + \frac{1}{3} + \mathrm{O}(t)$ at $t \to 0$. Thus, the analytic continuation of $\zeta_{\bar{A}}(s)$ to $s = 0$ is: $\zeta_{\bar{A}}(0) = \lim_{s\to 0} \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1} e^{-m^2R^2t}(f(t) - \frac{1}{t} - \frac{1}{3}) + \frac{(m^2R^2)^{1-s}}{s-1} + \frac{1}{3}(m^2R^2)^{-s} = \frac{1}{3} - m^2R^2$.

obtain

$$\begin{aligned} \log \mathbb{F}(z) &= \int^z dm^2\, 4\pi\, \tau(m^2)\big|_{R=1} \\ &= C + \log z - \int_0^z dm^2 \left(\psi(\alpha_1) + \psi(\alpha_2) + \frac{1}{m^2}\right) \end{aligned} \tag{A.14}$$

where $\alpha_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - m^2}$ are the roots of the equation $\alpha^2 - \alpha + m^2 = 0$. The constant in (A.14) is

$$C = \log \det{}' \Delta\big|_{R=1} = \frac{1}{2} - 4\zeta'(-1)$$

– the logarithm of the determinant of the Laplacian on the unit sphere (with zero-mode excluded), for which we quote the result from [34].

Function (A.14) has the following asymptotics at $z \to 0$ and at $z \to \infty$:

$$\log \mathbb{F}(z) \underset{z\to 0}{\sim} \log z + C + \mathrm{O}(z), \quad \log \mathbb{F}(z) \underset{z\to\infty}{\sim} -z\log z + z + \frac{1}{3}\log z + o(1)$$

The integral (A.14) can be evaluated in terms of Barnes G-function (a.k.a. "double Gamma function"), yielding

$$\begin{aligned} \log \mathbb{F}(z) = C - 2z &- \log\left(\frac{1}{\pi}\cos\pi\left(\frac{1}{4} - z\right)^{\frac{1}{2}}\right) \\ &+ 2\log\left(G(\alpha_1)G(\alpha_2)\right)\Big|_{\alpha_{1,2} = \frac{1}{2} \pm \left(\frac{1}{4} - z\right)^{\frac{1}{2}}} \end{aligned}$$

Putting together the result for the determinant, we have:

Lemma A.2. *The zeta-regularized determinant of the Helmholtz operator* $\Delta + m^2$ *on the sphere of radius* R *is:*

$$\begin{aligned} \det(\Delta + m^2) = {} & e^{\frac{1}{2} - 4\zeta'(-1)} R^{-2\left(\frac{1}{3} - m^2R^2\right)} \cdot \\ & \cdot \frac{\pi e^{-2m^2R^2}}{\cos\pi\left(\frac{1}{4} - m^2R^2\right)^{\frac{1}{2}}}\; G(\alpha_1)^2 G(\alpha_2)^2\Big|_{\alpha_{1,2} = \frac{1}{2} \pm \left(\frac{1}{4} - m^2R^2\right)^{\frac{1}{2}}} \end{aligned}$$

In particular, in the limit $m \to 0$, the determinant behaves as

$$\det(\Delta + m^2) \underset{m\to 0}{\sim} e^C R^{-2\cdot\frac{1}{3}} \cdot m^2R^2$$

Therefore, at zero mass, the determinant with excluded zero-mode on a sphere of radius R is

$$\det{}'\Delta = e^C R^{-2\cdot\frac{1}{3}+2} = e^C R^{\frac{4}{3}}$$

A.3. Dirichlet-to-Neumann operators: explicit examples

A.3.1. Example: disk. For Σ a disk of radius R with flat metric, the Dirichlet-to-Neumann operator acts diagonally in the basis $\phi_n(\theta) = e^{in\theta}$ in the space of L^2 functions on the boundary circle (with θ the polar angle):

$$D_\Sigma:\ \phi_n(\theta) \mapsto \lambda_n \cdot \phi_n(\theta)$$

with $n \in \mathbb{Z}$ and with eigenvalues

$$\lambda_n = m\frac{I_n'(mR)}{I_n(mR)} = \frac{m}{2}\,\frac{I_{n+1}(mR)+I_{n-1}(mR)}{I_n(mR)} \tag{A.15}$$

where I_n is the modified Bessel's function. This follows from the fact that the general solution of Helmholtz equation on the disk can be written, via separation of variables in polar coordinates, as $\phi(\theta, r) = \sum_{n=-\infty}^{\infty} c_n \phi_n(\theta) I_n(mr)$ with c_n constant coefficients.

A.3.2. Example: hemisphere. Consider Σ a hemisphere of radius R with standard metric. The Dirichlet-to-Neumann operator again acts diagonally in the basis of functions $\phi_n(\theta) = e^{in\theta}$, with θ the polar angle parameterizing the equator (the boundary of the hemishpere):

$$D_\Sigma:\ \phi_n(\theta) \mapsto \lambda_n \cdot \phi_n(\theta)$$

with $n \in \mathbb{Z}$ and eigenvalues

$$\lambda_n = \frac{2}{R}\,\frac{\Gamma\left(\frac{n+1+\alpha_1}{2}\right)\Gamma\left(\frac{n+1+\alpha_2}{2}\right)}{\Gamma\left(\frac{n+\alpha_1}{2}\right)\Gamma\left(\frac{n+\alpha_2}{2}\right)} \tag{A.16}$$

with $\alpha_{1,2}$ the two roots of the quadratic equation $\alpha^2 - \alpha + (mR)^2 = 0$. One proves this similarly to (A.15) – from the separation of variables for the Helmholtz equation in spherical coordinates.

We remark that the zeta-regularized determinant of D_Σ for a hemisphere can calculated explicitly, yielding

$$\det{}_{\mathrm{reg}} D_\Sigma = \underbrace{\det{}_{\mathrm{reg}}(\varkappa)}_{2\sinh(\pi m R)} \cdot \prod_{n=-\infty}^{\infty} \frac{\lambda_n}{\omega_n} = 2\cos\pi\left(\frac{1}{4}-(mR)^2\right)^{1/2} \tag{A.17}$$

where $\omega_n = (\frac{n^2}{R^2}+m^2)^{1/2}$ are the eigenvalues of the operator $\varkappa = (\Delta+m^2)^{1/2}\big|_{\partial\Sigma}$. Here to compute the second factor in the middle expression (the Fredholm determinant of $\varkappa^{-1}D_\Sigma$), the crucial observation is that (A.16) can be written in the form $\lambda_n = \frac{2}{R}\frac{f_{n+1}}{f_n}$, which allows one to compute the finite product $\prod_{n=-N}^N \frac{\lambda_n}{\lambda_n^\varkappa} = 2^{2N+1}\frac{f_{N+1}}{f_{-N}}\prod_{n=-N}^N((n+imR)(n-imR))^{-1/2}$ – this is a certain combination of Gamma functions, and the limit $N\to\infty$ can be evaluated straightforwardly.

By the BFK gluing formula for determinants, the expression (A.17) appears as a ratio of the determinant of the Helmholtz operator on a sphere S^2 and the product of determinants of the Helmholtz operators on the upper and lower hemispheres H^+, H^-:

$$\begin{aligned}
\frac{\det_{\mathrm{reg}}(\Delta+m^2)_{S^2}}{\det_{\mathrm{reg}}(\Delta+m^2)_{H^+}\cdot\det_{\mathrm{reg}}(\Delta+m^2)_{H^-}} &= \frac{\left(\prod_{l\geq 0}\left(\frac{l(l+1)}{R^2}+m^2\right)^{2l+1}\right)_{\mathrm{reg}}}{\left(\left(\prod_{l\geq 0}\left(\frac{l(l+1)}{R^2}+m^2\right)^{l}\right)_{\mathrm{reg}}\right)^2} \\
&= \left(\prod_{l\geq 0}\left(\frac{l(l+1)}{R^2}+m^2\right)\right)_{\mathrm{reg}} \\
&\underset{\mathrm{BFK}}{=} \det{}_{\mathrm{reg}}(D_{H^+}+D_{H^-}) = 2\cos\pi\left(\frac{1}{4}-(mR)^2\right)^{1/2}
\end{aligned}$$

A.3.3. How far are the Dirichlet-to-Neumann operators from the square root of Helmholtz operator on the boundary? The operator $\varkappa = (\Delta+m^2)^{1/2}$ on a circle is diagonalized the basis $\phi_n(\theta)$ with eigenvalues

$$\omega_n = \left(\frac{n^2}{R^2}+m^2\right)^{1/2} \tag{A.18}$$

Using the results (A.15, A.16) and the case of the cylinder of height H considered in Section A.2.2, we have the following $n\to\infty$ asymptotics for the ratio of the n-th eigenvalue of the Dirichlet-to-Neumann operator on a disk/hemisphere/cylinder[46] to ω_n:

$$\frac{\lambda_n^{\text{disk}}}{\omega_n} \underset{n\to\infty}{\sim} 1 - \frac{(mR)^2}{2n^3} + \mathrm{O}(n^{-4}),$$
$$\frac{\lambda_n^{\text{hemisphere}}}{\omega_n} \underset{n\to\infty}{\sim} 1 - \frac{(mR)^2}{4n^4} + \mathrm{O}(n^{-5}), \tag{A.19}$$
$$\frac{\lambda_n^{\text{cylinder}}}{\omega_n} = \coth H\omega_n \underset{n\to\infty}{\sim} 1 + \mathrm{O}(n^{-\infty})$$

Thus, $\varkappa^{-1} D_\Sigma = 1 + \delta$ is the identity plus a pseudodifferential operator δ of negative order N, where:

surface	disk	hemisphere	cylinder
N	-3	-4	$-\infty$

The result for the hemisphere can be further generalized to a result for the spherical sector with cone angle ϕ. In this case, we have[47]

$$\frac{\lambda_n^{\text{spherical sector}}}{\omega_n} \underset{n\to\infty}{\sim} 1 - \frac{(mR)^2 \cos\phi\, \sin^2\phi}{2n^3} + \frac{(mR)^2(1+3\cos 2\phi)\sin^2\phi}{8n^4} + \mathrm{O}(n^{-5}) \tag{A.20}$$

where $\omega_n = \left(\frac{n^2}{R^2 \sin^2\phi} + m^2\right)^{1/2}$ – the eigenvalues of $\varkappa$ on the boundary circle. Thus:

(a) In the case $\phi \neq \frac{\pi}{2}$, δ has order -3, while at $\phi = \frac{\pi}{2}$ (the case of a hemisphere) the coefficient of n^{-3} in (A.20) vanishes and δ degenerates to order -4.

(b) If Σ_1, Σ_2 are two complementary spherical sectors (which glue into a full sphere), (A.20) implies that in the sum $\delta_1 + \delta_2$, the order -3 term cancels out, leaving an order -4 pseudodifferential operator.

For a general surface, we have the following result:

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[46] In the case of a cylinder, we mean the block of Dirichlet-to-Neumann operator connecting one of the boundary circles to itself.

[47] Explicitly, the eigenvalues of the Dirichlet-to-Neumann operator are given by $R^{-1}\frac{d}{d\phi}\log P^n_{-\alpha}(\cos\phi)$ with α either root of $\alpha^2 - \alpha + (mR)^2 = 0$ and with $P^\mu_\nu(z)$ the Legendre function. The relevant asymptotics of the Legendre function was obtained in [44].

Proposition A.3. *Let Σ be a surface with smooth boundary, then $S = D_\Sigma - \varkappa$ is a pseudodifferential operator of order at most -2. In particular, $\delta = \varkappa^{-1} S = \varkappa^{-1} D_\Sigma - 1$ is a pseudodifferential operator of order at most -3, and a fortiori a trace-class operator.*[48]

Proof. We will adapt the proof of [56, Proposition C.1], where for $m = 0$, it is shown that $D_\Sigma - \sqrt{\Delta_{\partial\Sigma}} = B$, where B is an order at most 0 pseudodifferential operator with principal symbol

$$\sigma(B_0)(x, \xi) = \frac{1}{2}\left(\operatorname{Tr} A_{\partial\Sigma} - \frac{\langle A^*_{\partial\Sigma}\xi, \xi\rangle}{\langle \xi, \xi\rangle}\right).$$

Here $A_{\partial\Sigma}$ is the Weingarten map. Adapting slightly the proof in loc. cit. we can see this is true also in the massive case. Since $T^*_x\partial\Sigma$ is one dimensional, $A_{\partial\Sigma}$ is just multiplication by a real number and the expression on the right hand side vanishes trivially. This shows that S is an operator of order at most -1. In fact, the proof of [56, Proposition C.1] can be used to analyze the full symbol $p_{D_\Sigma}(x, \xi)$ of D_Σ (see also [36, Proposition 1.1]). In particular, it can be shown that

$$p_{D_\Sigma}(x, \xi) = |\xi| + m^2/2|\xi| + \mathrm{O}(|\xi|^{-2}).$$

Also, the full symbol $p_\varkappa(x, \xi)$ of $\varkappa$ has similar behavior:

$$p_\varkappa(x, \xi) = |\xi| + m^2/2|\xi| + \mathrm{O}(|\xi|^{-2}).$$

This shows that $D_\Sigma - \varkappa$ is an operator of order at most -2 which completes the proof. □

Also, for any Σ with a product metric near the boundary, δ is a smoothing operator (i.e. one has $N = -\infty$), see [38].

Explicit examples above, plus the expectation that the singular part of the integral kernel of δ must be universally expressed in terms of local metric characteristics (curvature of Σ and extrinsic curvature of the boundary at the point), suggest the following.

Conjecture A.4.

(i) Let Σ be a smooth surface with smooth boundary, endowed with a Reimannian metric. If the boundary of Σ is totally geodesic, then the

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[48] Recall that a pseudodifferential operator on a circle of order $N < -1$ is automatically trace-class.

operator $\delta = \varkappa^{-1} D_\Sigma - 1$ *is a PDO (pseudodifferential operator) of order* -4.

(ii) *If a surface* Σ *is cut by a* 1*-manifold* Y *into two surfaces* Σ_L, Σ_R*, then* $\frac{1}{2}(\delta_L + \delta_R) = \frac{1}{2\varkappa_Y} D_{\Sigma_L,\Sigma_R} - 1$ *is a PDO of order* -4*. Here we are not assuming that* Y *is geodesic.*

Remark A.5. *For comparison, we comment on the behavior of the operator* $S = D_\Sigma - \varkappa$ *in the case* $m = 0$ *(on two-dimensional surfaces).*[49]

(a) *The operator* D_Σ *is conformally invariant in the massless case, and likewise for the operators* $\varkappa$ *and* S*. More explicitly, let* $f : \Sigma \to \Sigma'$ *be a conformal diffeomorphism,* $f^* g_{\Sigma'} = \Omega \cdot g_\Sigma$*, with* g_Σ*,* $g_{\Sigma'}$ *the two metrics and* Ω *the Weyl factor. Then, for* Y *a boundary component of* Σ*, we have*

$$D_\Sigma = (\Omega|_Y)^{\frac{1}{2}} (f|_Y)^* D_{\Sigma'}, \qquad \varkappa_Y = (\Omega|_Y)^{\frac{1}{2}} (f|_Y)^* \varkappa_{Y'},$$
$$S_\Sigma = (\Omega|_Y)^{\frac{1}{2}} (f|_Y)^* S_{\Sigma'}$$

(b) S *is a smoothing operator for any* Σ.[50]

(c) *For* Σ *homeomorphic to a disk (with any metric),* $S = 0$.[51]

Appendix B. Trace of stress-energy tensor and tadpole. Trace anomaly

In this appendix we start by giving an interpretation of the tadpole in terms of the stress-energy tensor T in free massive scalar theory – as the vacuum expectation value of the trace $\mathrm{tr}\, T$. We then compare the trace of quantum stress-energy tensor, defined in terms of the variation of the partition function w.r.t. a Weyl transformation of the metric, with the expectation value of the trace of the classical stress-energy tensor (the variation of the classical action w.r.t. a Weyl transformation) and calculate the difference between the two (the "trace anomaly"), in free and then in interacting theory.

[49] In the massless case, we prefer to talk about S rather than $\delta = \varkappa^{-1} D_\Sigma - 1 = \varkappa^{-1} S$, since the latter is not everywhere defined, due to $\varkappa$ being invertible only on the orthogonal complement of constants on a given boundary component.

[50] By (a), one can straighten the metric near the boundary into a cylindrical one by a conformal mapping; and for a cylindrical metric near the boundary, S is smoothing.

[51] Using (a), by Riemann mapping theorem, this case reduces to the case of the standard unit disk where the explicit computation is straightforward, e.g. as $m \to 0$ limit of (A.15), see also [56].

In the massive free scalar theory, the *classical* stress-energy tensor, defined as the variational derivative of the action w.r.t. the metric, is given by

$$T_{\mathrm{cl}}(x) = \frac{2}{\sqrt{\det g}} \frac{\delta S_\Sigma}{\delta g^{-1}(x)} = d\phi \otimes d\phi - g\left(\frac{1}{2}\langle d\phi, d\phi\rangle + \frac{m^2}{2}\phi^2\right)$$

where g is the metric and g^{-1} its inverse; $\langle , \rangle$ is the Hodge inner product. In particular, its trace is $\mathrm{tr}\, T_{\mathrm{cl}} = -m^2\phi^2$. Thus, the normalized expectation value is:

$$\big\langle \mathrm{tr}\, T_{\mathrm{cl}}(x)\big\rangle = -\hbar m^2 \tau(x) \tag{B.1}$$

So, in a free massive theory, the tadpole[52] is proportional to the expectation value of the trace of the stress-energy tensor.

Consider the $m \to 0$ asymptotics of (B.1) averaged over the surface:

$$\int_\Sigma d^2x \,\big\langle \mathrm{tr}\, T_{\mathrm{cl}}(x)\big\rangle = -\hbar m^2 \int_\Sigma d^2x\, \tau(x) \underset{m\to 0}{\sim} -\hbar + \mathrm{O}(m^2) \tag{B.2}$$

The reason for that is that the averaged tadpole behaves as

$$\begin{aligned} \int_\Sigma d^2\tau(x) &= \frac{d}{dm^2} \log \underbrace{\det(\Delta + m^2)}_{m^2 \widetilde{\det}(\Delta+m^2)} = \frac{1}{m^2} + \log \widetilde{\det}(\Delta + m^2) \\ &\underset{m\to 0}{\sim} \frac{1}{m^2} + \mathrm{const} + \mathrm{O}(m^2) \end{aligned} \tag{B.3}$$

Here we denoted $\widetilde{\det}(\Delta + m^2)$ the zeta-regularized determinant of $\Delta + m^2$ with the lowest eigenvalue $\lambda = m^2$ excluded from the regularized product. Thus, the asymptotics $1/m^2$ of the averaged tadpole – and hence the constant asymptotics in (B.2) – are due to the lowest eigenvalue of $\Delta + m^2$.

One also has a local version of the result (B.3):

$$\tau(x) \underset{m\to 0}{\sim} \frac{1}{\mathrm{Area}(\Sigma)\cdot m^2} + \mathrm{O}(1).$$

It follows from (5.7) and the large-time asymptotics of the heat kernel $\theta_\Delta(x,t) \underset{t\to\infty}{\sim} \frac{1}{\mathrm{Area}(\Sigma)}$ (cf. Remark A.1).

[52]Throughout this section, a “tadpole” means a “zeta-regularized tadpole.”

Next, recall that the *quantum* stress-energy tensor is defined as a variational derivative of the partition function of the theory w.r.t. the metric (rather than the expectation value of the classical stress-energy tensor):

$$\langle T_q(x) \rangle = -\hbar \frac{2}{\sqrt{\det g}} \frac{\delta \log Z_\Sigma}{\delta g^{-1}(x)}$$

Correspondingly, its trace measures the reaction of the partition function to an infinitesimal Weyl rescaling of metric $g \to e^\sigma \cdot g$ (with $\sigma \in C^\infty(\Sigma)$ a Weyl scaling factor):

$$\langle \operatorname{tr} T_q(x) \rangle = \hbar \frac{2}{\sqrt{\det g}} \frac{\delta}{\delta \sigma}\Big|_{\sigma=0} \log Z_\Sigma^{g \to e^\sigma \cdot g} \tag{B.4}$$

Averaging over the surface, we get

$$\int_\Sigma d^2x \, \langle \operatorname{tr} T_q(x) \rangle = 2\hbar \, \frac{d}{d\sigma}\Big|_{\sigma=0} \log Z_\Sigma^{g \to e^\sigma \cdot g} \tag{B.5}$$

with σ a constant (point-independent) Weyl scaling factor.

Remark B.1. *Recall (see e.g. [18, Section 5.4.2]) that in conformal field theory the classical stress-energy tensor is traceless,* $\operatorname{tr} T_{\mathrm{cl}} = 0$*, whereas on the quantum level one has the "trace anomaly" (or "Weyl anomaly"): the trace of the stress-energy tensor has an expectation value proportional to the central charge c and the scalar curvature K:*[53]

$$\langle \operatorname{tr} T_q \rangle^{\mathrm{CFT}} = \frac{c\hbar}{24\pi} K \tag{B.6}$$

This corresponds to the following behavior of the partition function of a CFT under finite Weyl transformations of metric (see e.g. [53], [19]):

$$Z^{\mathrm{CFT}}_{\Sigma, e^\sigma \cdot g} = e^{\frac{c}{48\pi} \int_\Sigma \frac{1}{2} d\sigma \wedge * d\sigma + K\sigma \mathrm{dVol}_\Sigma} \; Z^{\mathrm{CFT}}_{\Sigma, g}$$

In particular, free massless scalar field is a conformal theory with central charge $c = 1$ *and, according to (B.6), should satisfy*

$$\langle \operatorname{tr} T_q \rangle^{\mathrm{CFT}}_{m=0} = \frac{\hbar}{24\pi} K \tag{B.7}$$

[53] In this section we use the notation K rather than R for the scalar curvature, to avoid the mix up with the radius of the sphere.

B.1. Trace of quantum vs. classical stress-energy tensor on a sphere

Consider the example of Σ a sphere of radius R. The partition function of the free massive scalar theory is $Z = \det(\Delta + m^2)^{-\frac{1}{2}}$ with the determinant given by (A.13).

Note that (A.13) implies that

$$\left(R^2 \frac{\partial}{\partial R^2} - m^2 \frac{\partial}{\partial m^2}\right) \log\det(\Delta + m^2) = -\frac{1}{3} + m^2 R^2$$

Comparing with (B.5) (where a global Weyl rescaling is tantamount to changing the radius of the sphere), with (B.1) and with Corollary 5.16, we obtain

$$\int_{S^2} d^2x \left\langle \operatorname{tr} T_q \right\rangle = \int_{S^2} d^2x \left\langle \operatorname{tr} T_{\mathrm{cl}} \right\rangle + \hbar \left(\frac{1}{3} - m^2 R^2\right)$$

Or in the non-averaged form (here we can use that $\left\langle \operatorname{tr} T_q \right\rangle$, $\left\langle \operatorname{tr} T_{\mathrm{cl}} \right\rangle$ must be constant functions in the case of a sphere, due to isometries acting transitively):

$$\left\langle \operatorname{tr} T_q \right\rangle = \left\langle \operatorname{tr} T_{\mathrm{cl}} \right\rangle + \hbar \left(\frac{1}{12\pi R^2} - \frac{m^2}{4\pi}\right) \tag{B.8}$$

We interpret the second term in the r.h.s. as the trace anomaly in the free massive scalar theory.

B.2. Trace anomaly for a general surface (and in the interacting theory)

The following result is a generalization of (B.8) to an arbitrary closed surface.

Lemma B.2. *For the free massive scalar theory on a closed surface Σ, the trace of the quantum stress-energy tensor at a point x is*

$$\left\langle \operatorname{tr} T_q(x) \right\rangle = \left\langle \operatorname{tr} T_{\mathrm{cl}}(x) \right\rangle + \hbar \frac{1}{4\pi} \left(\frac{K(x)}{6} - m^2\right) \tag{B.9}$$

with $K(x)$ the scalar curvature. The first term on the r.h.s. is $\left\langle \operatorname{tr} T_{\mathrm{cl}}(x) \right\rangle = -\hbar m^2 \tau(x)$ with $\tau(x)$ the zeta-regularized tadpole (5.7).

We interpret the second term on the r.h.s. of (B.9) as the trace anomaly.

Proof. For $\sigma \in C^\infty(\Sigma)$, denote $A_\sigma = e^{-\sigma}\Delta + m^2$ the Helmholtz operator for the Weyl-rescaled metric $e^\sigma g$. First note that $\mathrm{tr}\, e^{-tA_{\epsilon\sigma}} = \mathrm{tr}\, e^{-tA} + \epsilon t\, \mathrm{tr}\, \sigma \Delta e^{-tA} + \mathrm{O}(\epsilon^2)$ (cf. the proof of Lemma 5.17). Using this, we have

$$\begin{aligned}
\frac{d}{d\epsilon}\Big|_{\epsilon=0} \log\det A_{\epsilon\sigma} &= -\frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathrm{tr}\, e^{-tA_{\epsilon\sigma}} \\
&= -\frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \cdot t\ \mathrm{tr}\, \sigma \left(-\frac{\partial}{\partial t} - m^2\right) e^{-tA} \\
&= -\frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty dt\ \left(s\, t^{s-1} - m^2 t^s\right) \mathrm{tr}\, \sigma e^{-tA} \\
&= -\frac{d}{ds}\Big|_{s=0} \int_\Sigma d^2x\, \sigma(x)(s\zeta_A(s,x) - m^2 s\zeta_A(s+1,x)) \\
&= \int_\Sigma d^2x\, \sigma(x)(-\zeta_A(0,x) + m^2\tau(x))
\end{aligned}$$

Comparing with (B.4) and using the definition of the partition function in free theory $Z = \det^{-\frac{1}{2}} A$, we have

$$\langle T_q(x)\rangle = -\hbar m^2 \tau(x) + \hbar\zeta_A(0,x) = \langle T_{\mathrm{cl}}(x)\rangle + \hbar\zeta_A(0,x) \tag{B.10}$$

We find the value $\zeta_A(0,x)$ from the small-t expansion of the heat kernel (5.5):

$$\begin{aligned}
\zeta(0,x) = \lim_{s\to 0}\Bigg(&\frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1}\, e^{-m^2 t} \left(\theta_\Delta(t,x) - \frac{1}{4\pi t} - \frac{1}{4\pi}\frac{K(x)}{6}\right) \\
&+ \frac{1}{4\pi}\frac{m^{-2(s-1)}}{s-1} + \frac{1}{4\pi}\frac{K(x)}{6} m^{-2s}\Bigg) \\
= \ &\frac{1}{4\pi}\left(\frac{K(x)}{6} - m^2\right)
\end{aligned} \tag{B.11}$$

Putting (B.10) and (B.11) together, we obtain the statement. □

Remark B.3 (On the massless limit and comparison to CFT). *In the limit $m \to 0$, (B.9) becomes*

$$\left\langle \mathrm{tr}\, T_q(x)\right\rangle = -\frac{\hbar}{\mathrm{Area}(\Sigma)} + \frac{\hbar}{4\pi}\frac{K(x)}{6}$$

which seems to contradict the known result (B.7) from CFT. The explanation is that the trace of the quantum stress-energy tensor in the $m \to 0$

limit of the massive theory and in the massless theory differ by a shift due to different normalizations of partition functions. A reasonable normalization/regularization of the partition function of the massless free scalar field CFT is:

$$Z^{\mathrm{CFT}}_{m=0} = \lim_{m\to 0}\left[\frac{\det(\Delta+m^2)}{m^2\mathrm{Area}(\Sigma)}\right]^{-1/2} = \mathrm{Area}(\Sigma)^{\frac{1}{2}}\lim_{m\to 0}\widetilde{\det}(\Delta+m^2)^{-\frac{1}{2}} = \mathrm{Area}(\Sigma)^{\frac{1}{2}}\left(\det{}'\Delta\right)^{-\frac{1}{2}} \tag{B.12}$$

where $\widetilde{\det}$ *is the determinant with the lowest eigenvalue* $\lambda = m^2$ *excluded.*[54] *The extra factor* $\mathrm{Area}(\Sigma)^{\frac{1}{2}}$ *here accounts, upon taking the variation w.r.t. Weyl transformations, for the difference*

$$\left\langle \operatorname{tr} T_q\right\rangle^{\mathrm{CFT}} = \lim_{m\to 0}\left\langle \operatorname{tr} T_q\right\rangle + \frac{\hbar}{\mathrm{Area}(\Sigma)}$$

Therefore, (B.9) and (B.7) are in agreement and not in contradiction.

Lemma B.2 admits the following generalization to the non-free case.

Proposition B.4. *Consider the massive scalar theory with interaction potential* $p(\phi) = \sum_{n\geq 0}\frac{p_n}{n!}\phi^n$ *on a closed surface* Σ*. The trace of the quantum*

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[54]For example, for $\Sigma = T_\tau = \mathbb{C}/(\mathbb{Z}\oplus\tau\mathbb{Z})$ a torus with modular parameter τ, the extra factor $\mathrm{Area}^{\frac{1}{2}}$ in (B.12) restores the invariance of the partition function under the modular transformation $\tau\to -\tau^{-1}$. Indeed, by Kronecker's limit formula (see e.g. [20], [48]), $\det{}'\Delta = (\mathrm{Im}\tau)^2|\eta(\tau)|^4$, with $\eta(\tau)$ the Dedekind's eta function. This determinant is not invariant under $\tau\to-\tau^{-1}$. However, $\frac{1}{\mathrm{Area}(T_\tau)}\det{}'\Delta = (\mathrm{Im}\tau)^{-1}\det{}'\Delta$ is invariant, and thus the partition function with normalization (B.12) is also modular invariant.

As a side note, in the operator formalism, the partition function for a torus $Z^{\mathrm{CFT}}_{\mathrm{op}} = \operatorname{tr}_{H_{S^1}} q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}$, with $q = e^{2\pi i\tau}$, is in fact ill-defined for the massless scalar CFT, due to a continuum of primary fields/states (vertex operators). However, it can be regularized by allowing the scalar field to take values in a target circle of radius r (see e.g. [18]). For a large target radius, this regularized partition function behaves as $Z^{\mathrm{CFT}}_{\mathrm{op}}(r) \underset{r\to\infty}{\sim} r\cdot Z^{\mathrm{CFT}}$ where the coefficient on the right is the partition function (B.12).

stress energy, defined via (B.4), satisfies

$$\left\langle \operatorname{tr} T_q(x) \right\rangle = \left\langle \operatorname{tr} T_{\mathrm{cl}}(x) + \hbar \frac{1}{4\pi} \left(\frac{K(x)}{6} - m^2 - \frac{\partial^2}{\partial \phi^2} p(\phi) \right) \right\rangle \tag{B.13}$$

at any point $x \in \Sigma$. Here $\operatorname{tr} T_{\mathrm{cl}} = -m^2\phi^2 - 2\,p(\phi)$ is the trace of the classical stress-energy tensor. On the r.h.s., $\langle \cdots \rangle$ means the normalized expectation value (one-point correlation function) in the interacting theory.

Proof. Consider the expression

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \log Z_\Sigma^{g \to e^{\epsilon\sigma} g} \tag{B.14}$$

On one hand, by definition (B.4), it is equal to

$$\frac{1}{2\hbar} \int_\Sigma d^2x\, \sigma(x) \left\langle \operatorname{tr} T_q(x) \right\rangle \tag{B.15}$$

On the other hand,

$$\log Z_\Sigma = -\frac{1}{2} \log\det(\Delta + m^2) + \sum_{\Gamma\,\text{connected}} \frac{\hbar^{-\chi(\Gamma)}}{|\operatorname{Aut}(\Gamma)|} F_\Gamma \tag{B.16}$$

where the sum on the right is over connected Feynman graphs. Taking the derivative of this expression w.r.t. a Weyl transform, we get the following contributions to (B.14):

(i) Derivative hits $-\frac{1}{2}\log\det(\Delta + m^2)$. This gives the contribution

$$\int_\Sigma d^2x\, \sigma(x) \Big(\underbrace{-\frac{m^2}{2}\tau(x)}_{(\mathrm{i})'} + \underbrace{\frac{1}{2}\frac{1}{4\pi}\left(\frac{K(x)}{6} - m^2\right)}_{(\mathrm{i})''} \Big)$$

as we obtained in Lemma B.2.

(ii) Derivative hits dVol_Σ in F_Γ corresponding to one of the vertices of Γ (recall that the Riemannian volume form contains $\sqrt{\det g}$ which scales with Weyl transformations). This gives the contribution[55]

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathrm{dVol}_x^{g \to e^{\epsilon\sigma} g} = \sigma(x)\mathrm{dVol}_x$$

[55] Here and below when we say "contribution $\cdots$" while discussing the element Ξ of Feynman diagrams, we mean "the contribution to (B.14) where one instance of Ξ is replaced by $\frac{d}{d\epsilon}\big|_{\epsilon=0} \Xi^{g \to e^{\epsilon\sigma} g} = \cdots$."

(iii) Derivative hits one of the edges (Green's functions) in F_Γ – but not one of the short loops. This gives the contribution[56]

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} G^{g\to e^{\epsilon\sigma}g}(x_1,x_2) = \int_\Sigma d^2x\, G(x_1,x)(-m^2\sigma(x))G(x,x_2) \tag{B.17}$$

(iv) Derivative hits a short loop, giving the contribution[57]

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \tau^{g\to e^{\epsilon\sigma}g}(x) = \underbrace{\int_\Sigma d^2y\, G(x,y)(-m^2\sigma(y))G(y,x)}_{(\mathrm{iv})'} + \underbrace{\frac{1}{4\pi}\sigma(x)}_{(\mathrm{iv})''} \tag{B.18}$$

Next, we note that

- Contributions of type (ii) above sum up to the expectation value

$$\hbar^{-1}\int_\Sigma d^2x\,\sigma(x)\,\big\langle -p(\phi(x))\big\rangle \tag{B.19}$$

- Contributions of types (iii), (iv)' and (i)' sum up to

$$\hbar^{-1}\int_\Sigma d^2x\,\sigma(x)\big\langle -\frac{m^2}{2}\phi^2(x)\big\rangle \tag{B.20}$$

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[56]Indeed, using the formula for the derivative of the inverse of an operator M_ϵ in a parameter, $\frac{d}{d\epsilon}M_\epsilon^{-1} = -M_\epsilon^{-1}(\frac{d}{d\epsilon}M_\epsilon)M_\epsilon^{-1}$, we have:

$$\begin{aligned}
&\frac{d}{d\epsilon}\Big|_{\epsilon=0}\int \mathrm{dVol}_{x_2} G(x_1,x_2)f(x_2)\Big|_{g\to e^{\epsilon\sigma}g}\\
&\quad = -\int d^2x\int d^2x_2 G(x_1,x)(-\sigma(x)\Delta_x)G(x,x_2)f(x_2)\\
&\quad = -m^2\int d^2x\int d^2x_2 G(x_1,x)\sigma(x)G(x,x_2)f(x_2) + \int d^2x_2 G(x_1,x_2)\sigma(x_2)f(x_2)
\end{aligned}$$

where the last term is due to the variation of dVol_{x_2} while the first is due to the variation of the Green's function itself. Here $f(x_2)$ is an arbitrary test function and we used that $\Delta_x G(x,x_2) = -m^2G(x,x_2) + \delta(x,x_2)$.

[57]This is proven easiest by analyzing the point-split tadpole (5.19) where the first term $G(x,x')$ reacts to the Weyl transform according to (B.17) and the variation of the singular subtraction $\frac{1}{2\pi}\log d(x,x')$ yields the second term in (B.18). Then we recall that τ and τ^{split} differ by a universal constant, see Corollary 5.21; thus, their variational derivatives coincide.

- Contributions (iv)" sum up to

$$\int_\Sigma d^2x\,\sigma(x)\Big\langle -\frac{1}{4\pi}\underbrace{\sum_n \frac{p_n}{n!}\binom{n}{2}\phi^{n-2}(x)}_{\frac{1}{2}\frac{\partial^2}{\partial\phi^2}p(\phi)}\Big\rangle \tag{B.21}$$

Finally, we note that (B.19) and (B.20) together yield $\frac{1}{2\hbar}\int d^2x\,\sigma(x)\,\big\langle T_{\mathrm{cl}}(x)\big\rangle$, while (B.21) and (i)" above together yield

$$\frac{1}{2}\int d^2x\,\sigma(x)\,\frac{1}{4\pi}\Big\langle\frac{K(x)}{6} - m^2 - \frac{\partial^2}{\partial\phi^2}p(\phi)\Big\rangle$$

Thus, the sum of these two expressions is (B.15), which finishes the proof. □

In Lemma B.2 and in Proposition B.4, one can allow Σ to be a surface with boundary and x an *interior* point. Both results continue to hold in this case, where now l.h.s. and r.h.s. are understood as functions of the boundary field η (the proof is adapted straightforwardly).

Note that the second derivative of the interaction $\frac{\partial^2}{\partial\phi^2}p(\phi)$ that we see in the trace anomaly in (B.13), we have encountered before in the context of RG flow, see (5.14), (5.18).

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