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On gerbe duality and relative Gromov-Witten theory

Xiang Tang and Hsian-Hua Tseng

We formulate and study an extension of gerbe duality to relative Gromov-Witten theory.

1. Introduction

In this short note, we propose a conjecture about relative Gromov-Witten theory on a general noneffective Deligne-Mumford stack.

Throughout this note, we work over $\mathbb{C}$. Let G be a finite group, $\mathcal{B}$ a smooth (proper) Deligne-Mumford stack, and

$$\pi : \mathcal{Y} \to \mathcal{B}$$

a G-gerbe. In [10], the *dual* of $\pi : \mathcal{Y} \to \mathcal{B}$ is a pair $(\widehat{\mathcal{Y}}, c)$ where $\widehat{\mathcal{Y}}$ is a disconnected stack with an étale map

$$\widehat{\pi} : \widehat{\mathcal{Y}} = \coprod_i \widehat{\mathcal{Y}}_i \to \mathcal{B}$$

and c is a $\mathbb{C}^*$-valued 2-cocycle on $\widehat{\mathcal{Y}}$.

We briefly recall the construction of $\widehat{\mathcal{Y}}$ and c, and refer the readers to [14] for the detail. We focus on describing $\mathcal{Y}$ locally. A chart of $\mathcal{B}$ looks like a quotient of $\mathbb{C}^n$ by a finite group Q acting linearly on $\mathbb{C}^n$. And a G-gerbe over $[\mathbb{C}^n/Q]$ can be described as follows. There is a group extension

$$1 \longrightarrow G \longrightarrow H \longrightarrow Q \longrightarrow 1.$$

The group H acts on $\mathbb{C}^n$ via its homomorphism to Q. Locally a chart of $\mathcal{Y}$ over $\mathcal{B}$ looks like

$$[\mathbb{C}^n/H] \longrightarrow [\mathbb{C}^n/Q].$$

To construct the dual $\widehat{\mathcal{Y}}$, we consider the space $\widehat{G}$, the (finite) set of isomorphism classes of irreducible G-representations. As G is a normal subgroup of H, H acts on G by conjugation, which naturally gives an H action on $\widehat{G}$. Furthermore, as G acts by conjugation, the G action on $\widehat{G}$ is trivial. Therefore, the quotient group Q acts on $\widehat{G}$. The dual space $\widehat{\mathcal{Y}}$ locally looks like $[(\widehat{G}\times\mathbb{C}^n)/Q]$. We notice that the dual $\widehat{\mathcal{Y}}$ has a canonical map to the quotient $\widehat{G}/Q$, and therefore is a disjoint union of stacks over $\widehat{G}/Q$.

The construction of c is from the Clifford theory of induced representations [7]. Given an irreducible G-representation ρ on V_ρ, we want to introduce an H representation. Let $[\rho]$ be the corresponding point in $\widehat{G}$. Recall that the finite group Q acts on $\widehat{G}$. We denote $Q_{[\rho]}$ to be the stabilizer group of Q action on $[\widehat{G}]$ at the point $[\rho]$. For any $q\in Q_{[\rho]}$, define $q(\rho)$, a representation of G on V_ρ, by

$$q(\rho)(g)=\rho\big(q^{-1}(g)\big).$$

As q fixes $[\rho]$ in $\widehat{G}$, $q(\rho)$, as a G-representation, is equivalent to ρ. Therefore, there is an intertwining operator T^ρ_q on V_ρ such that

$$T^\rho_q\rho=q(\rho)T^\rho_q.$$

In general, the operators $\{T^\rho_q\}_{q\in Q_{[\rho]}}$ fail to satisfy

$$T^\rho_q\circ T^\rho_{\rho'}=T^\rho_{qq'}.$$

By Schur's lemma, we can check that there is a number $c^{[\rho]}(q,q')\in\mathbb{C}^*$, such that

$$T^\rho_q\circ T^\rho_{\rho'}=c^{[\rho]}(q,q')T^\rho_{qq'}.$$

In [14], we explained that these functions $c^{[\rho]}(q,q')$ glue to a globally defined $\mathbb{C}^*$-gerbe over the dual $\widehat{\mathcal{Y}}$.

The authors of [10] propose a gerbe duality principle which suggests that there is an equivalence between the geometry of $\mathcal{Y}$ and that of the pair $(\widehat{\mathcal{Y}},c)$. Several aspects of such an equivalence have been proven in [14]. In [14, Conjecture 1.8], the following conjecture is also explicitly formulated:

Conjecture 1.1. As generating functions, the genus g Gromov-Witten theory of $\mathcal{Y}$ is equal to the genus g Gromov-Witten theory[1] of $(\widehat{\mathcal{Y}},c)$,

$$GW_g(\mathcal{Y})=GW_g(\widehat{\mathcal{Y}},c).$$

Conjecture 1.1 has been proven in increasing generalities, see [3], [4], [5], and [15]. In particular, we have obtained the following theorem.

[1]It is also called c-twisted Gromov-Witten theory of $\widehat{\mathcal{Y}}$.

Theorem 1.2. *([15, Theorem 1.1]) When $\mathcal{Y}$ is a banded G-gerbe over $\mathcal{B}$, Conjecture 1.1 holds true.*

A toric Deligne-Mumford stack $\mathcal{Y}$ is a banded gerbe over an effective DM stack $\mathcal{B}$, see e.g. [9]. As a corollary to Theorem 1.2, we can compute the Gromov-Witten theory of $\mathcal{Y}$ in term of the (twisted) Gromov-Witten theory of the dual toric DM stack $\widehat{\mathcal{Y}}$.

From the perspective of Gromov-Witten theory, Gromov-Witten theory *relative* to a divisor is important. Therefore it is natural to consider an extension of Conjecture 1.1 to the relative setting. Let

$$D \subset \mathcal{B}$$

be a smooth (irreducible) divisor. The inverse images

$$\mathcal{D} := \pi^{-1}(D) \subset \mathcal{Y}, \quad \widehat{\mathcal{D}} := \widehat{\pi}^{-1}(D) \subset \widehat{\mathcal{Y}}$$

are smooth divisors. The c-twisted Gromov-Witten theory of the relative pair $(\widehat{\mathcal{Y}}, \widehat{\mathcal{D}})$ can be defined using the construction of [2], [11], [12], and [13]. A natural extension of Conjecture 1.1 is the following

Conjecture 1.3. As generating functions, the genus g Gromov-Witten theory of $(\mathcal{Y}, \mathcal{D})$ is equal to the genus g Gromov-Witten theory of $((\widehat{\mathcal{Y}}, \widehat{\mathcal{D}}), c)$. Symbolically,

$$GW_g(\mathcal{Y}, \mathcal{D}) = GW_g((\widehat{\mathcal{Y}}, \widehat{\mathcal{D}}), c).$$

The purpose of this note is to present some evidence to Conjecture 1.3.

2. Evidence to Conjecture 1.3

By taking the divisor D to be empty, we see that Conjecture 1.3 implies Conjecture 1.1. Next, we explain how to derive Conjecture 1.3 from the full strength of Conjecture 1.1.

Let $r \geq 1$. We consider the stack of r-th roots of $\mathcal{Y}$ along $\mathcal{D}$, denoted by

$$\mathcal{Y}_{\mathcal{D},r}.$$

Consider also the stack of r-th roots of $\mathcal{B}$ along D, denoted by

$$\mathcal{B}_{D,r}.$$

Let $\rho : \mathcal{B}_{D,r} \to B$ be the natural map. Consider the Cartesian diagram:

$$\begin{array}{ccc} \rho^*\mathcal{Y} & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \pi \\ \mathcal{B}_{D,r} & \xrightarrow{\rho} & B. \end{array}$$

The pull-back $\rho^*\mathcal{Y} \to \mathcal{B}_{D,r}$ is a G-gerbe. By functoriality property of root constructions, there is a natural map

$$\mathcal{Y}_{\mathcal{D},r} \to \rho^*\mathcal{Y},$$

which is an isomorphism. Therefore we have

$$GW_g(\mathcal{Y}_{D,r}) = GW_g(\rho^*\mathcal{Y}). \tag{2.1}$$

Since $\rho^*\mathcal{Y} \to \mathcal{B}_{D,r}$ is a G-gerbe, applying Conjecture 1.1, we have

$$GW_g(\rho^*\mathcal{Y}) = GW_g(\widehat{\rho^*\mathcal{Y}}, c'), \tag{2.2}$$

where $(\widehat{\rho^*\mathcal{Y}}, c')$ is the dual pair of the gerbe $\rho^*\mathcal{Y}$. By construction of the dual pair, we have

$$\widehat{\rho^*\mathcal{Y}} = \rho^*\widehat{\mathcal{Y}}$$

and $c' = \rho^*c$. Here $\rho^*\widehat{\mathcal{Y}}$ fits in the Cartesian diagram

$$\begin{array}{ccc} \rho^*\widehat{\mathcal{Y}} & \longrightarrow & \widehat{\mathcal{Y}} \\ \downarrow & & \downarrow \widehat{\pi} \\ \mathcal{B}_{D,r} & \xrightarrow{\rho} & B. \end{array}$$

By functoriality property of root constructions, we have $\rho^*\widehat{\mathcal{Y}} \simeq \widehat{\mathcal{Y}}_{\widehat{\mathcal{D}},r}$. Therefore

$$GW_g(\widehat{\rho^*\mathcal{Y}}, c') = GW_g(\widehat{\mathcal{Y}}_{\widehat{\mathcal{D}},r}, \rho^*c). \tag{2.3}$$

Combining (2.1)–(2.3), we have

$$GW_g(\mathcal{Y}_{D,r}) = GW_g(\widehat{\mathcal{Y}}_{\widehat{\mathcal{D}},r}, \rho^*c). \tag{2.4}$$

The left hand side of (2.4), $GW_g(\mathcal{Y}_{D,r})$, is a polynomial in r for r large. Furthermore, taking the r^0-coefficient, we obtain the relative Gromov-Witten invariants:

$$\mathrm{Coeff}_{r^0} GW_g(\mathcal{Y}_{D,r}) = GW_g(\mathcal{Y}, \mathcal{D}). \tag{2.5}$$

This follows from the arguments of [16], suitably extended to the setting of Deligne-Mumford stacks, see [17], [6].

The right hand side of (2.4), $GW_g(\widehat{\mathcal{Y}}_{\widehat{\mathcal{D}},r}, \rho^* c)$, is also a polynomial in r for r large. Taking the r^0-coefficient, we obtain the relative Gromov-Witten invariants:

$$\mathrm{Coeff}_{r^0} GW_g(\widehat{\mathcal{Y}}_{\widehat{\mathcal{D}},r}, \rho^* c) = GW_g((\widehat{\mathcal{Y}}, \widehat{\mathcal{D}}), c). \tag{2.6}$$

This again follows from a suitable extension of [16] as in [17], [6]. Note that the $\rho^* c$-twist plays no role in applying the arguments of [16], as they are essentially done at the level of virtual cycles (see [8]), while $\rho^* c$-twist takes place at insertions.

Combining (2.5) and (2.6), we arrive at Conjecture 1.3.

Remark 2.1. In genus 0, the argument about polynomiality in r can be replaced by (a suitable extension of) the arguments of [1].

The following result follows directly from Theorem 1.2 and the above discussion.

Theorem 2.1. *When $\mathcal{Y}$ is a banded G-gerbe over $\mathcal{B}$, as generating functions, the genus g Gromov-Witten theory of $(\mathcal{Y}, \mathcal{D})$ is equal to the genus g Gromov-Witten theory of $((\widehat{\mathcal{Y}}, \widehat{\mathcal{D}}), c)$. Symbolically,*

$$GW_g(\mathcal{Y}, \mathcal{D}) = GW_g((\widehat{\mathcal{Y}}, \widehat{\mathcal{D}}), c).$$

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Department of Mathematics, Washington University
St. Louis, MO 63130, USA
E-mail address: xtang@math.wustl.edu

Department of Mathematics, Ohio State University
Columbus, OH 43210, USA
E-mail address: hhtseng@math.ohio-state.edu