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F-theory over a Fano threefold built from A_4-roots

Herbert Clemens and Stuart Raby

In a previous paper, the authors showed the advantages of building a $\mathbb{Z}_2$-action into an F-theory model W_4/B_3, namely the action of complex conjugation on the complex algebraic group with compact real form E_8. The goal of this paper is to construct the Fano threefold B_3 directly from the roots of $SU(5)$ in such a way that the action of complex conjugation is exactly the desired $\mathbb{Z}_2$-action and the quotient of this action on W_4/B_3 and its Heterotic dual have the phenomenologically correct invariants.

1. Introduction

A particular challenge in Heterotic F-theory duality arises when one wishes to transfer a $\mathbb{Z}_2$-action

$$V_3^\vee/B_2^\vee = \frac{V_3/B_2}{\mathbb{Z}_2} \tag{1.1}$$

on a elliptically fibered Calabi-Yau Heterotic threefold V_3/B_2 to a $\mathbb{Z}_2$-action

$$W_4^\vee/B_3^\vee = \frac{W_4/B_3}{\mathbb{Z}_2} \tag{1.2}$$

on an elliptically fibered Calabi-Yau fourfold that becomes the F-theory dual. We have proposed a framework for such a duality in [4]. W_4/B_3 with $B_3 = B_2 \times \mathbb{P}_{[u_0,v_0]}$ with del Pezzo B_2 is defined by a *Tate form*

$$wy^2 = x^3 + a_5xyw + a_4zx^2w + a_3z^2yw^2 + a_2z^3xw^2 + a_0z^5w^3 \tag{1.3}$$

with $a_j, z, \frac{y}{x} \in H^0\left(K_{B_3}^{-1}\right)^{[-1]}$ with respect to the $\mathbb{Z}_2$-action. We will require that W_4/B_3 be defined subject to the condition

$$a_5 + a_4 + a_3 + a_2 + a_0 = 0. \tag{1.4}$$

The condition is equivalent to the condition that W_4/B_3 have a second section τ given by

$$\begin{gathered} x = wz^2 \\ y = wz^3. \end{gathered}$$

Incorporating translation by the difference of τ and the standard section ζ given by

$$\begin{gathered} x = 0 \\ w = 0 \end{gathered}$$

into the $\mathbb{Z}_2$-action allows us to eliminate vector-like exotics in a final paper [5] of this series.

Furthermore B_3 is a $\mathbb{P}^1$-fiber bundle over B_2 on which $\mathbb{Z}_2$ must act equivariantly. One desires such a configuration in order to employ the Wilson line mechanism for symmetry-breaking consistently and simultaneously on both the Heterotic model and its F-theory dual.

The $\mathbb{Z}_2$-action on a V_3 must be free. Furthermore on the F-theory side the $\mathbb{Z}_2$-action must restrict to a free $\mathbb{Z}_2$-action on a distinguished smooth anti-canonical divisor $S_{\mathrm{GUT}} \subseteq B_3$. Therefore it must act skew-symmetrically on the anti-canonical section z defining S_{GUT}. In [4] we showed that, while x and w are symmetric with respect to the $\mathbb{Z}_2$-action on (1.3), y, z, and all the a_j have to be skew-symmetric.

The Heterotic $\mathbb{Z}_2$-action must preserve the initial E_8-symmetry and so the F-theory $\mathbb{Z}_2$-action must preserve initial E_8-symmetry as well. In short, the challenge is to begin with E_8-symmetry on both the Heterotic and F-theory sides and, for successive subgroups $G_{\mathbb{R}} \leq E_8$, to match breaking to $G_{\mathbb{R}}$-symmetry on the Heterotic side with simultaneous breaking to $G_{\mathbb{R}}$-symmetry on the F-theory side throughout, ending with symmetry-breaking to $G_{\mathbb{R}} = SU(3) \times SU(2) \times U(1)$, the so-called Minimal Supersymmetric Standard Model [MSSM].

As we showed in [4] , the necessity that $\mathbb{Z}_2$ must act as

$$\frac{dx}{y} \mapsto -\frac{dx}{y}$$

on the relative one-form of the elliptic fibration W_4/B_3 implies that it must incorporate the central involution

$$-I_8 : \mathfrak{h}_{E_8^{\mathbb{C}}} \to \mathfrak{h}_{E_8^{\mathbb{C}}}$$

on the Cartan subalgebra of the complex algebraic group $E_8^{\mathbb{C}}$ at the outset *without* breaking initial E_8-symmetry on the quotients (1.1) and (1.2).

To achieve this, in [4] we proposed the method of replacing all roots ρ with $-\rho$ via the operation of complex conjugation on the complex algebraic group $E_8^{\mathbb{C}}$ and all relevant subgroups $G_{\mathbb{C}}$, an operation that restricts to the identity on all compact real forms $G_{\mathbb{R}}$. Since all the compact real forms have faithful real matrix representations, this complex conjugation operator will not affect $G_{\mathbb{R}}$-symmetry and will commute with the various symmetry-breaking steps.

The purpose of this paper is to build the appropriate base space B_3 of the elliptically fibered F-theory model W_4/B_3 in such a way that the $\mathbb{Z}_2$-action on B_3 is exactly that induced by the complex conjugation operator on the complex algebraic group $SL(5;\mathbb{C})$. Therefore it will fix the compact real form $SU(5)$ so that the requisite Wilson line can be wrapped simultaneously on the $\mathbb{Z}_2$-actions on the Heterotic and F-theory sides.

1.1. Building B_3 from the action of the Weyl group of $SU(5)$ on its complexified Cartan subalgebra

In fact we will build B_3 as a quotient of the resolution of the projectivization of the graph of
$$(\exp(2\pi i \cdot \alpha) \mapsto \exp(2\pi i \cdot (-\alpha)))$$
on the maximal torus
$$\exp\left(\mathfrak{h}_{SL(5;\mathbb{C})}\right)$$
by an action of the longest element of the Weyl group $W(SL(5;\mathbb{C}))$. This will allow the $\mathbb{Z}_2$-action on B_3 to automatically commute with the action of complex conjugation.

In later sections we will show that B_3 as constructed will have the correct numerical characteristics so that the F-theory model (1.2) will have the desired properties (3-generation, correct chiral invariants, no vector-like exotics, etc.). The transfer of information between the F-theory and its Heterotic dual is the subject of a companion paper [4]. The application to the production of the final phenomenologically consistent F-theory/Heterotic duality is the subject of the final paper [5] in this sequence.

Remark 1. Throughout this paper, we will let
$$\mathbb{P}^{d-1}_{[i_1,\ldots,,i_d]}$$
denote the weighted complex projective $(d-1)$-space with weights $[i_1,\ldots,,i_d]$ and will let
$$\mathbb{P}_{[u_1,\ldots,u_d]}$$
denote the (unweighted) complex projective space with homogeneous coordinates $[u_1,\ldots,,u_d]$.

2. The spectral divisor

The role of the Tate form (1.3) is to break E_8-symmetry to that of the first summand of its maximal sub-group
$$\frac{SU(5)_{gauge} \times SU(5)_{Higgs}}{\mathbb{Z}_5}.$$

The crepant resolution $\tilde{W}_4/B_3$ of W_4/B_3 will have I_5-type fibers over generic points of
$$S_{\mathrm{GUT}} := \{z = 0\} \subseteq B_3.$$

The I_5-fibration over S_{GUT} carries the $SU(5)_{gauge}$-symmetry. $SU(5)_{Higgs}$-symmetry is broken on a five-sheeted branched covering of B_3 given by the lift of

$$\mathcal{C}_{Higgs} := W_4 \cdot \left(\left\{wy^2 = x^3\right\} - \{w = 0\}\right) \tag{2.1}$$

to a divisor $\tilde{\mathcal{C}}_{Higgs} \subseteq \tilde{W}_4$. Its symmetry is broken by assigning non-trivial eigenvalues to the fundamental representation $SU(5)_{Higgs}$ using the spectral construction with respect to the push-forward to B_3 of a line bundle $\mathcal{L}_{Higgs}$ on $\tilde{\mathcal{C}}_{Higgs}$. We see this as follows.

In parallel to the construction for $SU(5)_{gauge}$ in [4], we imbed

$$\begin{array}{ccc} & & \mathfrak{h}^{\mathbb{C}}_{SU(5)_{Higgs}} \\ & & \downarrow^{(c_2,c_3,c_4,c_5)} \\ (S_{\mathrm{GUT}} - \{a_0 = 0\}) & \rightarrow & \frac{\mathfrak{h}^{\mathbb{C}}_{SU(5)_{Higgs}}}{W(SU(5))} \end{array}$$

in such a way that the image of $W_4 - \{a_0 = 0,\, w = 1\}$ in $\mathbb{C}^3 \times \frac{\mathfrak{h}^{\mathbb{C}}_{SU(5)_{Higgs}}}{W(SU(5))}$ is a family of rational double-point surface singularities. The above diagram allows the Casimir operators c_j to operate on the fundamental representation of $SU(5)_{Higgs}$ with eigenvalues that are tracked via a spectral construction [8, 9].

The Tate form (1.3) then records the above geometrically in W_4/B_3 by considering it as a hypersurface in

$$P := \mathbb{P}\left(\mathcal{O}_{B_3} \oplus \mathcal{O}_{B_3}(2N) \oplus \mathcal{O}_{B_3}(3N)\right)$$

with fiber coordinate $[w, x, y]$.

2.0.1. The spectral divisor. We define the map

$$\begin{gathered} \mathbb{P}\left(\mathcal{O}_{B_3} \oplus \mathcal{O}_{B_3}(N)\right) \rightarrow \mathbb{P}\left(\mathcal{O}_{B_3} \oplus \mathcal{O}_{B_3}(2N) \oplus \mathcal{O}_{B_3}(3N)\right) = P \\ [w, t] \mapsto \left[w, x = t^2 w, y = t^3 w\right], \end{gathered}$$

Dividing by w^3 the inverse image of W_4 in $\mathbb{P}\left(\mathcal{O}_{B_3} \oplus \mathcal{O}_{B_3}(N)\right)$ has equation

$$0 = a_5 t^5 + a_4 z t^4 + a_3 z^2 t^3 + a_2 z^3 t^2 + a_0 z^5 \tag{2.2}$$

We next blow up up the locus $\{t = z = 0\}$ in $\mathbb{P}\left(\mathcal{O}_{B_3} \oplus \mathcal{O}_{B_3}(N)\right)$ via

$$\left\{\left|\begin{array}{cc} t & z \\ T & Z \end{array}\right| = 0\right\} \subseteq \mathbb{P}\left(\mathcal{O}_{B_3} \oplus \mathcal{O}_{B_3}(N)\right) \times \mathbb{P}_{[T,Z]} \tag{2.3}$$

in which the proper transform of (2.2) becomes

$$0=a_5T^5+a_4ZT^4+a_3Z^2T^3+a_2Z^3T^2+a_0Z^5 \tag{2.4}$$

defining the *spectral divisor* that we denote as $\mathcal{D}$. In particular, the spectral divisor expands the singular locus $\{x=y=z=0\}$ of W_4. The condition (1.4) implies that homogeneous form in (2.4) is divisible by $Z-T$, that is, the spectral divisor admits a $(4+1)$ factorization.

$$\mathcal{D}=\mathcal{D}^{(4)}+\mathcal{D}^{(1)}\subseteq B_3\times\mathbb{P}_{[T,Z]} \tag{2.5}$$

given by the equation

$$\begin{gathered}0=a_5T^5+a_4ZT^4+a_3Z^2T^3+a_2Z^3T^2+a_0Z^5=\\ \left(a_5T^4+a_{54}T^3Z-a_{20}T^2Z^2-a_0TZ^3-a_0Z^4\right)(T-Z)\end{gathered} \tag{2.6}$$

where $a_{jk}:=a_j+a_k$. The involution $\tilde{\beta}_4/\beta_3$ of W_4/B_3 leaves (2.6) and each of its two factors invariant.[1]

3. The role of the Cartan sub-algebra $\mathfrak{h}_{SL(5;\mathbb{C})_{gauge}}$

3.1. Tracking roots

Again referring to [4] we track roots during symmetry-breaking from E_8 to the maximal subgroup

$$\frac{SU(5)_{gauge}\times SU(5)_{Higgs}}{\mathbb{Z}_5}$$

on the F-theory side by returning to the Tate form

$$wy^2=x^3+a_5xyw+a_4zx^2w+a_3z^2yw^2+a_2z^3xw^2+a_0z^5w^3$$

where we divide both sides by a_0^6, rescale by the rule

$$\begin{aligned}\tfrac{x}{a_0^2}&\to x\\ \tfrac{y}{a_0^3}&\to y\\ \tfrac{z}{a_0}&\to z\end{aligned}$$

[1] This $4+1$ split is often written in terms of the variable $s=Z/T$, e.g. the factor $(Z-T)$ becomes the factor $(s-1)$ used to remove $\mathbf{10}_{\{-4\}}$ states as in formula (70) in [2]. In our case, the $4+1$ split is global.

and define the functions

$$c_j = \frac{a_j}{a_0}$$

on $B_3' := B_3 - \{a_0 = 0\}$. By this rescaling we obtain the equation

$$y^2 = x^3 + c_5 xy + c_4 z x^2 + c_3 z^2 y + c_2 z^3 x + z^5 \tag{3.1}$$

in the variables (x, y, z) parametrized by the 'free' variables (c_2, c_3, c_4, c_5) that we interpret as a family $\mathcal{V}_{gauge}$ of rational double-point surface singularities. (See §4.1 of [4].)

Next by interpreting the c_j as the $SU(5)$ Casimir generators, we pull the family $\mathcal{V}_{gauge}$ back to $\mathfrak{h}_{SL(5;\mathbb{C})_{gauge}}$ by the map

$$(c_2, c_3, c_4, c_5) : \mathfrak{h}_{SL(5;\mathbb{C})_{gauge}} \to \frac{\mathfrak{h}_{SL(5;\mathbb{C})_{gauge}}}{W\left(SL\left(5;\mathbb{C}\right)\right)}$$

where we were able to interpret it as a family of weighted homogeneous polynomials of weight 30 that is therefore also obtained via pull-back from a map to the semi-universal deformation of

$$y^2 = x^3 + z^5. \tag{3.2}$$

Next defining

$$z := \sum_{j=2}^{5} \kappa_j \alpha_j \tag{3.3}$$

for general complex constants κ_j as in §4.1 of [4], we obtained morphisms

$$B_3' \to \frac{\mathfrak{h}_{SL(5;\mathbb{C})}}{W\left(SL\left(5;\mathbb{C}\right)\right)}$$

and

$$W_4' := W_4 \times_{B_3} B_3' \to \mathcal{V}_{gauge}$$

for $B_3' = B_3 - \{a_0 = 0\}$. Further we showed that the complex conjugation operator ι induces equivariant involutions

$$\left(\left(a_0, a_2, a_3, a_4, a_5\right), x, y, z\right) \mapsto \left(\left(-a_0, -a_2, -a_3, -a_4, -a_5\right), x, -y, -z\right) \tag{3.4}$$

on W_4/B_3 and

$$\left(\left(c_2, c_3, c_4, c_5\right), x, y, z\right) \mapsto \left(\left(c_2, -c_3, c_4, -c_5\right), x, -y, z\right)$$

on (3.1). This allowed us in [4] to interpret the equivariant crepant resolution of (3.1) over $\mathfrak{h}_{SL(5;\mathbb{C})}$ as induced by the Brieskorn-Grothendieck equivariant crepant resolution [3, 12] of the semi-universal deformation of (3.2) and thereby track roots and the action

$$\begin{array}{ccc} i\cdot\mathfrak{h}_{SU(5)_{gauge}}\times i\cdot\mathfrak{h}_{SU(5)_{Higgs}} & \xrightarrow{-I_4\times -I_4} & i\cdot\mathfrak{h}_{SU(5)_{gauge}}\times i\cdot\mathfrak{h}_{SU(5)_{Higgs}} \\ \downarrow & & \downarrow \\ i\cdot\mathfrak{h}_{E_8} & \xrightarrow{-I_8} & i\cdot\mathfrak{h}_{E_8} \end{array}$$

of complex conjugation.

3.2. Notation distinguishing Weyl chambers

We will have a single Tate form defining our F-theory model W_4 but initially we will have two desingularizations that we will denote as $\dot{W}_4/\dot{B}_3$ or the 'blue' desingularization and as $\ddot{W}_4/\ddot{B}_3$ or the 'red' desingularization, depending on whether we consider a given Weyl chamber of $SU(5)$ or its negative as the 'positive' Weyl chamber. $\dot{B}_3$ will be related to $\ddot{B}_3$ by a Cremona transformation representing the passage of each root to its negative. Indeed it is the resolution of the graph of that transformation that will determine our ultimate B_3 and the quotient under its induced involution that will be our ultimate $B_3^{\vee}$.

4. $S_4 \subseteq W(SU(5))$

Our strategy is now to identify a group $G \leq W(SU(5))$ such that

$$\frac{\mathfrak{h}^{\mathbb{C}}_{SU(5)}}{G} \supseteq B_3' \subseteq B_3$$

gives rise to a phenomenologically correct F-theory model.

4.1. Building a toric $B_3^{\wedge}$ from $SU(5)$ roots

A set of simple positive roots ordered by the A_4-Dynkin diagram is given by

$$\{\alpha_i = e_i - e_{i-1},\}_{i=1,\dots,4}\,. \tag{4.1}$$

One immediately checks that the permutation of the axes e_j given by the product of transpositions $(e_0 e_4)(e_1 e_3)$ acts as

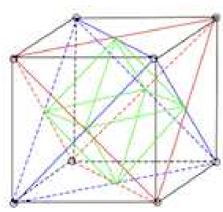
$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} -\alpha_4 & -\alpha_3 & -\alpha_2 & -\alpha_1 \end{pmatrix}$$

and so is the composition of $-I_4$ with the unique symmetry of the A_4-Dynkin diagram. Therefore it is the unique longest element of $W(SU(5))$. This symmetry fixes exactly one of the five axes, namely the axis e_2, and therefore lies the permutation subgroup

$$S_4 = Perm\{e_0, e_1, e_3, e_4\} \hookrightarrow S_5 = Perm\{e_0, e_1, e_2, e_3, e_4\} = W(SU(5))$$

via the identification of the axis e_j of the fundamental representation of $SU(5)$ with the root $e_j - e_2$. We use this fact to construct a root basis for $\mathfrak{h}^{\mathbb{C}}_{SU(5)}$ that is convenient for a toric construction of our 'new' B_3.

We next project the root space along the e_2-axis to obtain the vertices of a 3-dimensional cube. Thus we place e_2 at $(0,0,0)$, the center of the cube below (that we will denote as CUBE) whose eight vertices are $(\pm 1, \pm 1, \pm 1)$. The elements $\{e_0, e_1, e_3, e_4\}$ can be identified with the vertices of the blue tetrahedron inscribed in CUBE



as follows:

$$\begin{array}{rc} e_0: & (1,-1,1) \\ e_{1:} & (1,1,-1) \\ e_3: & (-1,-1,-1) \\ e_4: & (-1,1,1) \end{array}$$

That is, $\{e_0, e_4\}$ are the two 'top' blue vertices and $\{e_1, e_3\}$ are the two 'bottom' blue vertices and their negatives are the four vertices of the red tetrahedron. Circled vertices are those of the polyhedral fan.

4.2. The Weyl group of $SU(5)$ and the Cremona involution as symmetries of the cube

The group of orientation-preserving symmetries of CUBE (or equivalently the orientation-preserving symmetries of the inscribed green octahedron) maps isomorphically to the permutation group of axes $\pm e_j$, that is,

$$S_4 = Perm\left\{\left\{\pm e_0\right\}, \left\{\pm e_1\right\}, \left\{\pm e_3\right\}, \left\{\pm e_4\right\}\right\} \subseteq S_5 = W\left(SU\left(5\right)\right).$$

For example, $\left(\left\{\pm e_0\right\}\left\{\pm e_4\right\}\right)\left(\left\{\pm e_1\right\}\left\{\pm e_3\right\}\right)$ is the rotation of CUBE around the vertical axis through an angle of π. It gives the above longest element of $W\left(SU\left(5\right)\right)$. Rotation of the cube around the diagonal axis through $(1,1,1)$ through an angle of $2\pi/3$ is the cyclic permutation $\left(\left\{\pm e_0\right\}\left\{\pm e_1\right\}\left\{\pm e_4\right\}\right)$. A rotation with axis spanned by the midpoints of a pair of opposite edges only flips the pair of axes given by the endpoints of the edges. We will be especially interested in the commutator subgroup of the involution $\left(\left\{\pm e_0\right\}\left\{\pm e_4\right\}\right)\left(\left\{\pm e_1\right\}\left\{\pm e_3\right\}\right)$ in S_4.

Finally $A_4 \subseteq S_4$ is the subgroup of orientation-preserving symmetries of the cube that preserve the blue tetrahedron and therefore also preserve the red tetrahedron, so that the quotient S_4/A_4 interchanges the two. The full symmetry group of CUBE is then generated by adjoining the central, orientation-reversing element given by reflection through the origin that we denote as C. It is the involution induced by C given by $\left(e_i - e_2 \leftrightarrow e_2 - e_i\right)_{i=0,1,3,4}$ on W_4/B_3 that will the yield the quotient F-theory model $W_4^\vee/B_3^\vee$ with a $\mathbb{Z}_2$-action. It is the $\mathbb{Z}_4$-group generated by the cyclic permutation $\left(\left\{\pm e_0\right\}\left\{\pm e_1\right\}\left\{\pm e_4\right\}\left\{\pm e_3\right\}\right)$ that will lead to an asymptotic $\mathbb{Z}_4$ $\mathbf{R}$ symmetry on the semi-stable degeneration $W_{4,0}/B_{3,0}$ of W_4/B_3. Notice that the Cremona involution $C \notin W\left(SU\left(5\right)\right)$, however C commutes with all the elements of $S_4 \leq W\left(SU\left(5\right)\right)$.

5. Toric geometry of B_3

5.1. Coordinatizing roots

We choose $U(4) \subseteq SU(5)$ via the inclusion

$$\begin{gathered} U(4) \hookrightarrow SU(5) \\ A \mapsto \hat{A} := \begin{pmatrix} \overline{\det A} & 0 \\ 0 & A \end{pmatrix} \end{gathered} \tag{5.1}$$

with maximal torus of $SU(5)$ identified with diagonal unitary matrices A. Logarithms of eigenvalue functions for the restriction of the adjoint representation

$$Ad_{SU(5)} : SU(5) \rightarrow GL(\mathfrak{su}(5) \otimes \mathbb{C})$$

to the maximal torus give the (root) linear operators on the complexified Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}_{SU(5)} = \mathfrak{h}_{SL(5;\mathbb{C})}$. We choose set of roots $\{e_j - e_2\}_{j=0,1,3,4}$ as basis for $\left(\mathfrak{h}^{\mathbb{C}}_{SU(5)}\right)^*$ as in the previous section. Exponentiating we let

$$x, y, z, w$$

denote the corresponding eigenvalue functions.

Remark 2. As is standard in the literature, we have used the letter z to denote the form whose vanishing defines the anti-canonical divisor $S_{\mathrm{GUT}} \subseteq B_3$. In several places below, we will abuse notation by also using each of the letters x, y, z, w to denote one of the homogeneous coordinates $[x, y, z, w]$ of the $\mathbb{P}^3 = \mathbb{P}\left(\mathfrak{h}^{\mathbb{C}}_{A_4}\right)$ where $\mathfrak{h}_{A_4}$ denotes the Cartan subalgebra of $SU(5)$. We trust that the intended meaning of x, y, z, w in each instance will be clear from the context.

Then we can make the identification

$$\begin{aligned} \log x &= e_0 - e_2 \\ \log y &= e_1 - e_2 \\ \log w &= e_3 - e_2 \\ \log z &= e_4 - e_2. \end{aligned} \tag{5.2}$$

giving a basis for the A_4 root lattice. The distinguished Weyl chamber (4.1) is given by the system of positive simple roots

$$\begin{aligned} \alpha_1 &= e_1 - e_0 = \log(y/x) \\ \alpha_2 &= e_2 - e_1 = \log(1/y) \\ \alpha_3 &= e_3 - e_2 = \log w \\ \alpha_4 &= e_4 - e_3 = \log(z/w). \end{aligned} \tag{5.3}$$

(Notice that the set (5.2) of roots is not a set of simple roots for a single Weyl chamber, however it does span the root lattice.) We obtain 24 of the 120 Weyl chambers by the 24 permutations of $\{x, y, z, w\}$ in (5.3). The longest element of the Weyl group is then given by

$$\begin{gathered} y/x \leftrightarrow w/z \\ 1/y \leftrightarrow 1/w \\ w \leftrightarrow y \\ z/w \leftrightarrow x/y. \end{gathered}$$

Passing from roots to their negatives corresponds to the Cremona transformation

$$\begin{gathered} x \leftrightarrow \tfrac{1}{x} \\ y \leftrightarrow \tfrac{1}{y} \\ z \leftrightarrow \tfrac{1}{z} \\ w \leftrightarrow \tfrac{1}{w} \end{gathered} \tag{5.4}$$

that in turn corresponds to the orientation-reversing symmetry of CUBE given by reflection through the origin.

5.2. Tracking symmetry-breaking within the Cartan subalgebra of $E_8^{\mathbb{C}}$

As mentioned above, the Tate form tracks symmetry breaking to

$$\frac{SU(5)_{gauge} \times SU(5)_{Higgs}}{\mathbb{Z}_5} \hookrightarrow E_8.$$

As we have shown in [4] symmetry-breaking must be compatible with the three-dimensional commutative diagram

$$\begin{array}{ccccc} SL(5;\mathbb{C}) & \overset{\dot{\kappa}}{\hookrightarrow} & \dot{E}_8^{\mathbb{C}} & & \\ \uparrow & & \uparrow & \nwarrow & \\ SU(5) & \to & E_8 & & \iota \\ \downarrow & & \downarrow & \swarrow & \\ SL(5;\mathbb{C}) & \overset{\ddot{\kappa}}{\hookrightarrow} & \ddot{E}_8^{\mathbb{C}} & & \end{array} \tag{5.5}$$

where the top row of (5.5) is mapped to the bottom row by the isomorphism ι given by complex conjugation where ι induces the involutions

$$\begin{gathered} -I_4 : \mathfrak{h}_{SL(5;\mathbb{C})} \to \mathfrak{h}_{SL(5;\mathbb{C})} \\ -I_8 : \mathfrak{h}_{E_8^{\mathbb{C}}} \to \mathfrak{h}_{E_8^{\mathbb{C}}} \end{gathered} \tag{5.6}$$

on complex Cartan subalgebras. Identifying maximal tori also identifies

$$\mathfrak{h}_{SL(5;\mathbb{C})_{gauge}} \oplus \mathfrak{h}_{SL(5;\mathbb{C})_{Higgs}} = \mathfrak{h}_{E_8^{\mathbb{C}}} \tag{5.7}$$

as well as inducing the inclusion

$$W\left(SU\left(5\right)_{gauge}\right) \times W\left(SU\left(5\right)_{Higgs}\right) \hookrightarrow W\left(E_8\right).$$

While $-I_8$ is also the longest element the Weyl group $W\left(E_8\right)$, it does not restrict to an element of the Weyl group $W\left(SU\left(5\right)_{gauge}\right)$ or an element of the Weyl group $W\left(SU\left(5\right)_{Higgs}\right)$. However $-I_8$ and the pair of longest elements in $W\left(SU\left(5\right)_{gauge}\right) \times W\left(SU\left(5\right)_{Higgs}\right)$ differ by the involutive symmetries of the two A_4-Dynkin diagrams.

Our strategy will be to build B_3 and its symmetries from the group $W\left(SU\left(5\right)_{Higgs}\right)$ acting on on the right-hand sum in (5.7). Throughout we maintain the relationship with $W\left(E_8\right)$ as per (5.6) and (5.7) so that the $\mathbb{Z}_2$-action on B_3, that we have denoted as β_3 in [4] and [5] and interchangeably as $C_{u,v}$ below, acts as the composition of the complex conjugation $-I_8$ on 5.7 and the longest element of $W\left(SU\left(5\right)_{Higgs}\right)$ on B_3.

5.3. CUBE as a toric polyhedral fan

The standard toric presentation of $\mathbb{P}^3$ is given by a real vector space $N_{\mathbb{R}} = N_{\mathbb{Z}} \otimes \mathbb{R} = \mathbb{R}^3$ with fan equal to the union of four strongly convex rational polyhedral cones $\sigma_{xyz}, \sigma_{xyw}, \sigma_{xzw}, \sigma_{yzw} \subseteq N_{\mathbb{R}}$ such that for the duals

$$S\left(\sigma\right) = \left\{\mathbf{m} \in \mathrm{Hom}\left(N_{\mathbb{Z}}, \mathbb{Z}\right) : \mathbf{m} \cdot \sigma \geq 0\right\}$$

the associated group algebras $\mathbb{C}\left[S\left(\sigma\right)\right]$ are identified with the respective affine rings $\mathbb{C}\left[x/w, y/w, z/w\right]$, $\mathbb{C}\left[x/w, y/z, w/z\right]$, $\mathbb{C}\left[x/y, z/y, w/y\right]$, and $\mathbb{C}\left[y/x, z/x, w/x\right]$. The edges e_x, e_y, e_z, e_w of the fan are identified with the four divisors on $\mathbb{P}^3$ given by the vanishing of the respective variables.

We have two such toric presentations of $\mathbb{P}^3$ in CUBE, one given by the blue tetrahedron that we will denote as

$$\dot{\mathbb{P}} := \mathbb{P}_{[\dot{x},\dot{y},\dot{z},\dot{w}]}$$

and the other given by the red tetrahedron that we will denote as

$$\ddot{\mathbb{P}} = \mathbb{P}_{[\ddot{x},\ddot{y},\ddot{z},\ddot{w}]}.$$

Both toric representations are given with respect to the same toric lattice $N^{\wedge}$, the one generated by either the red four or the blue four vertices of CUBE. $\mathbb{P}_{[\dot{x},\dot{y},\dot{z},\dot{w}]}$ has toric fan given by the vertices of the blue tetrahedron and $\mathbb{P}_{[\ddot{x},\ddot{y},\ddot{z},\ddot{w}]}$ has toric fan given by the vertices of the red tetrahedron.

This allows us to use CUBE to torically represent the resolution of the graph of the Cremona transformation (5.4). Namely the graph of the Cremona transformation is given by the relations

$$\dot{x}\ddot{x} = \dot{y}\ddot{y} = \dot{z}\ddot{z} = \dot{w}\ddot{w} = 1 \tag{5.8}$$

on the Zariski-open subset

$$\{\dot{x}\cdot\dot{y}\cdot\dot{z}\cdot\dot{w}\neq 0\} \times \{\ddot{x}\cdot\ddot{y}\cdot\ddot{z}\cdot\ddot{w}\neq 0\} \subseteq \mathbb{P}_{[\dot{x},\dot{y},\dot{z},\dot{w}]} \times \mathbb{P}_{[\ddot{x},\ddot{y},\ddot{z},\ddot{w}]}.$$

We define

$$\begin{aligned} B_3^{\wedge} := \big\{ \left([\dot{x},\dot{y},\dot{z},\dot{w}]\,,[\ddot{x},\ddot{y},\ddot{z},\ddot{w}]\right) \in \mathbb{P}_{[\dot{x},\dot{y},\dot{z},\dot{w}]} \times \mathbb{P}_{[\ddot{x},\ddot{y},\ddot{z},\ddot{w}]} :\\ \dot{x}\ddot{x} = \dot{y}\ddot{y} = \dot{z}\ddot{z} = \dot{w}\ddot{w} = 1\big\} \end{aligned}$$

as simply the closure of the graph of the Cremona transformation.

$B_3^{\wedge}$ is a toric manifold with respect to the same toric lattice $N^{\wedge}$. The polyhedral fan has vertices at the eight vertices of CUBE together with the six additional points $(\pm 2,0,0)$, $(0,\pm 2,0)$, and $(0,0,\pm 2)$. These fourteen vertices correspond to the fourteen toroidal divisors whose sum is the anticanonical divisor of $B_3^{\wedge}$. The inclusion of cones generate two birational morphisms

$$\begin{aligned} \dot{\pi} : B_3^{\wedge} \to \dot{\mathbb{P}}\\ \ddot{\pi} : B_3^{\wedge} \to \ddot{\mathbb{P}}. \end{aligned}$$

The Cremona involution is given toroidally by the reflection C of CUBE through the origin.

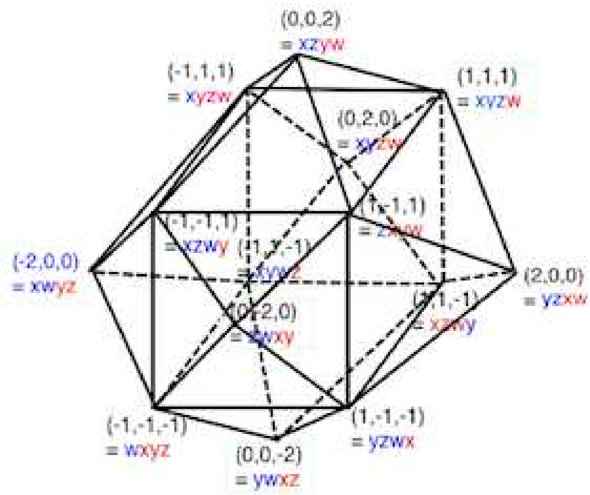
To further describe the toroidal divisors, we denote the red vertices of the fan as $e_{\dot{x}\ddot{y}\ddot{z}\ddot{w}}, e_{\dot{y}\ddot{x}\ddot{z}\ddot{w}}, e_{\dot{z}\ddot{x}\ddot{y}\ddot{w}}, e_{\dot{w}\ddot{x}\ddot{y}\ddot{z}}$ and the blue vertices as $e_{\dot{x}\dot{y}\dot{z}\ddot{w}}, e_{\dot{x}\dot{y}\dot{w}\ddot{z}}, e_{\dot{x}\dot{z}\dot{w}\ddot{y}}$, $e_{\dot{y}\dot{z}\dot{w}\ddot{x}}$. Set

$$e_{\dot{x}\dot{z}\ddot{y}\ddot{w}}$$

the vertex above CUBE that lies on the ray through the origin that bisects the segment joining $e_{\dot{x}\ddot{y}\ddot{z}\ddot{w}}$ and $e_{\dot{z}\ddot{x}\ddot{y}\ddot{w}}$. This same ray bisects the segment joining $e_{\dot{x}\dot{y}\dot{z}\ddot{w}}$ and $e_{\dot{x}\dot{z}\dot{w}\ddot{y}}$. This choice will force the top vertices of the cube to be

$$e_{\dot{x}\ddot{y}\ddot{z}\ddot{w}}, e_{\ddot{x}\dot{y}\ddot{z}\ddot{w}}, e_{\dot{z}\ddot{x}\ddot{y}\ddot{w}}, e_{\ddot{x}\ddot{z}\dot{w}\ddot{y}}.$$

We use the analogous notation for the other five possible $(2,2)$-partitions. We obtain a toroidal fan:



The toroidal divisors are given by the vertices of the polyhedral fan pictured above where the blue-red colorations of the variables in the monomial

$$xyzw$$

correspond to the decorations $\{\dot{}, \ddot{}\}$ in $\mathbb{P}_{[\dot{x},\dot{y},\dot{z},\dot{w}]} \times \mathbb{P}_{[\ddot{x},\ddot{y},\ddot{z},\ddot{w}]}$.

Passing from roots to their negatives corresponds to the orientation-reversing symmetry of the above cube given by reflection through the origin. The reflection also interchanges the blue tetrahedron with the red one. In fact, the full subgroup $S_4 \subseteq W(SU(5))$ of symmetries of CUBE acts the set of decorated monomials, dividing them into sets with an even number of blue variables and sets with an odd number of blue variables. The toroidal divisors are just the restriction to $B_3^{\wedge}$ of the divisors

$$(5.9) \qquad \begin{aligned} E_{\dot{x}\ddot{y}\ddot{z}\ddot{w}} &:= \{\dot{x}=0\} \times \{\ddot{y}=\ddot{z}=\ddot{w}=0\} \subseteq \dot{\mathbb{P}} \times \ddot{\mathbb{P}} \\ E_{\dot{x}\dot{z}\ddot{y}\ddot{w}} &:= \{\dot{x}=\dot{z}=0\} \times \{\ddot{y}=\ddot{w}=0\} \subseteq \dot{\mathbb{P}} \times \ddot{\mathbb{P}} \\ E_{\dot{y}\dot{z}\dot{w}\ddot{x}} &:= \{\dot{y}=\dot{z}=\dot{w}=0\} \times \{\ddot{x}=0\} \subseteq \dot{\mathbb{P}} \times \ddot{\mathbb{P}} \end{aligned}$$

etc., where of course $E.$ is the divisor given in toric notation by $e.$. For example, the toric dictionary gives

$$E_{\dot{x}\ddot{y}\ddot{z}\ddot{w}} \leftrightarrow e_{\dot{x}\ddot{y}\ddot{z}\ddot{w}}$$

and the normal bundle to $E_{\dot{x}\ddot{y}\ddot{z}\ddot{w}}$ in $\tilde{\mathbb{P}}^3$ is

$$\mathcal{O}_{E_{\dot{x}\ddot{y}\ddot{z}\ddot{w}}}(-1,-1)\,.$$

So by adjunction

$$K_{\tilde{\mathbb{P}}^3} \cdot E_{\dot{x}\ddot{y}\ddot{z}\ddot{w}} = \mathcal{O}_{E_{\dot{x}\ddot{y}\ddot{z}\ddot{w}}}(-1,-1)\,. \tag{5.10}$$

If we classify the above components (5.9) to be of type $(1,3)$, $(2,2)$, and $(3,1)$ respectively, there are four divisors of type $(3,1)$, four divisors of type $(1,3)$ and six divisors of type $(2,2)$. All non-empty intersections occur as intersections of a component of type $(2,2)$ with a component of type $(1,3)$ or $(3,1)$ obtained by changing the decoration on exactly one of its four variables, for example

$$E_{\dot{x}\dot{y}\dot{z}\ddot{w}} \cap E_{\dot{x}\dot{y}\ddot{z}\ddot{w}} = \{\dot{x} = \dot{y} = \dot{z} = \ddot{z} = 0\}\,.$$

There are exactly $24 = \frac{2 \cdot 4!}{2}$ such intersections, 12 projecting to a vertex in $\dot{\mathbb{P}}$ and an edge in $\ddot{\mathbb{P}}$ and 12 projecting to a vertex $\ddot{\mathbb{P}}$ and an edge in $\dot{\mathbb{P}}$. The divisors of type $(3,1)$ are the four vertex rays in $N^{\wedge}_{\mathbb{R}}$ in the original toric description of $\mathbb{P}^3$, those of type $(1,3)$ are the rays through the barycenters of the cones and those of type $(2,2)$ are the rays through the barycenters of the faces.

The anti-canonical bundle of $B_3^{\wedge}$ is represented by 14-hedron given by the (reduced) support of the total transform of the tetrahedron in $\dot{\mathbb{P}}$ or in $\ddot{\mathbb{P}}$, that is

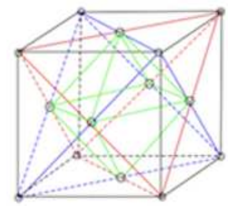
$$\begin{aligned} K^{-1}_{B_3^{\wedge}} := {} & E_{\dot{x}\ddot{y}\ddot{z}\ddot{w}} + E_{\dot{y}\ddot{x}\ddot{z}\ddot{w}} + E_{\dot{z}\ddot{x}\ddot{y}\ddot{w}} + E_{\dot{w}\ddot{x}\ddot{y}\ddot{z}} \\ & + E_{\dot{x}\dot{z}\ddot{y}\ddot{w}} + E_{\dot{x}\dot{y}\ddot{z}\ddot{w}} + E_{\dot{x}\dot{w}\ddot{y}\ddot{z}} + E_{\dot{y}\dot{z}\ddot{x}\ddot{w}} + E_{\dot{y}\dot{w}\ddot{x}\ddot{z}} + E_{\dot{z}\dot{w}\ddot{x}\ddot{y}} \\ & + E_{\dot{y}\dot{z}\dot{w}\ddot{x}} + E_{\dot{x}\dot{z}\dot{w}\ddot{y}} + E_{\dot{x}\dot{y}\dot{w}\ddot{z}} + E_{\dot{x}\dot{y}\dot{z}\ddot{w}}. \end{aligned} \tag{5.11}$$

5.4. Toric quotients of $B_3^{\wedge}$

Next define the 'over-lattice'

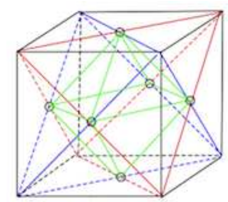
$$N_{\mathbb{Z}} = \left\{ \left(\frac{a}{2} + \frac{b}{2} + \frac{c}{2} \right) : (a,b,c) \in N^{\wedge},\ a+b+c \equiv_2 0 \right\} \supseteq N^{\wedge}$$

inducing a toric quotient of $B_3^{\wedge}$. With respect to the lattice $N_{\mathbb{Z}}$, the polyhedral fan generated by the polyhedral fan for $B_3^{\wedge}$ becomes



where the vertices of the fan are circled.

Now the six red-blue-green crossing points generate $N_{\mathbb{Z}}$. The green octahedron with vertices at the six red-blue-green crossing points



is the toric representation of

$$\mathbb{P}_{u,v} := \mathbb{P}_{[u_0,v_0]} \times \mathbb{P}_{[u_1,v_1]} \times \mathbb{P}_{[u_2,v_2]}$$

where

$$\begin{aligned} [u_0, v_0] &= \left[\frac{xz}{yw}, \frac{yw}{xz}\right] \\ [u_1, v_1] &= \left[\frac{xy}{zw}, \frac{zw}{xy}\right] \\ [u_2, v_2] &= \left[\frac{xw}{yz}, \frac{yz}{xw}\right] \end{aligned}$$

and again circled vertices are those of the polyhedral fan. The toric $\mathbb{P}_{u,v}$ is invariant under the the action of the longest element $((e_0e_4)(e_1e_3))$ of $W(SU(5))$, namely the toric involution given by

$$\begin{array}{ccc} \mathbb{P}_{[u_1,v_1]} \times \mathbb{P}_{[u_2,v_2]} & \overset{((e_0e_4)(e_1e_3))}{\longrightarrow} & \mathbb{P}_{[u_1,v_1]} \times \mathbb{P}_{[u_2,v_2]} \\ ([u_1,v_1],[u_2,v_2]) & \mapsto & ([v_1,u_1],[v_2,u_2]) . \end{array}$$

We will distinguish the 'vertical' $\mathbb{P}^1 = \mathbb{P}_{[u_0,v_0]}$. Thus $\{u_0 = 0\}$ will correspond to the 'top' of the cube and $\{v_0 = 0\}$ will correspond to the 'bottom.'

Our distinguished 'vertical' $\mathbb{P}_{[u_0,v_0]}$ in CUBE is the one spanned by

$$f_{\dot{x}\dot{z}\ddot{y}\ddot{w}} := \frac{1}{2} \cdot e_{\dot{x}\dot{z}\ddot{y}\ddot{w}} \, and \, f_{\dot{y}\dot{w}\ddot{x}\ddot{z}} = -f_{\dot{x}\dot{z}\ddot{y}\ddot{w}}$$

that is, this $\mathbb{P}^1$ is the one corresponding to the partition $(\{x,z\},\{y,w\})$ of $\{x,y,z,w\}$. Analogously we have $\mathbb{P}_{[u_1,v_1]}$ corresponding to the fan vertices

$$f_{\dot{x}\dot{y}\ddot{z}\ddot{w}} \, and \, f_{\dot{z}\dot{w}\ddot{x}\ddot{y}} = -f_{\dot{x}\dot{y}\ddot{z}\ddot{w}},$$

and finally $P_{[u_2,v_2]}$ corresponding to the fan

$$f_{\dot{x}\dot{w}\ddot{y}\ddot{z}} \, and \, f_{\dot{y}\dot{z}\ddot{x}\ddot{w}} = -f_{\dot{x}\dot{w}\ddot{y}\ddot{z}}.$$

The following Lemma then follows immediately from the toroidal picture together with the fact that the Cremona involution on $B_3^{\wedge}$ reverses the decorations on the $xyzw$-monomials, whereas the action of S_4 permutes the variables x, y, z, w.

Lemma 3. *i) The involution*

$$C_{u,v} : \mathbb{P}_{u,v} \to \mathbb{P}_{u,v}$$

induced by the Cremona involution C on $B_3^{\wedge}$ is given by

$$[u_j, v_j] \mapsto [v_j, u_j]$$

for $j = 0, 1, 2$.

ii) The $\mathbb{Z}_4$-action by the cyclic permutation

$$T_0 := (\{\pm e_0\}\{\pm e_1\}\{\pm e_4\}\{\pm e_3\}) \tag{5.12}$$

is a rotation of 90° of the green octahedron shown above in this Subsection around its vertical axis.

iii) The cyclic permutation $(\{\pm e_0\}\{\pm e_1\}\{\pm e_4\}\{\pm e_3\})$ is the $\mathbb{Z}_4$-action on $\mathbb{P}_{[u_1,v_1]} \times \mathbb{P}_{[u_2,v_2]}$ generated by

$$[u_1, v_1], [u_2, v_2] \xrightarrow{T_0} [u_2, v_2], [v_1, u_1]$$

that we also denote as T_0.

iv) Replacing the homogeneous coordinates $[u_0, v_0]$ *for* $\mathbb{P}_{[u_0,v_0]}$ *with the single affine coordinate* $\frac{u_0-v_0}{u_0+v_0}$*, there exists a* $\mathbb{Z}_4$*-action that we denote as* $T_{u,v}$ *on* $B_3 = \mathbb{P}_{[u_0,v_0]} \times B_2$ *defined by*

$$\begin{aligned}(5.13)\qquad &\left(\frac{u_0 - v_0}{u_0 + v_0}, [u_1, v_1], [u_2, v_2]\right)\\ &\xrightarrow{T_{u,v}} \left(i \cdot \left(\frac{u_0 - v_0}{u_0 + v_0}\right), T_0\left([u_1, v_1], [u_2, v_2]\right)\right).\end{aligned}$$

iii) Furthermore

$$\begin{aligned}(5.14)\qquad T_{u,v}^2\left(\tfrac{u_0-v_0}{u_0+v_0}, [u_1, v_1], [u_2, v_2]\right) &= \left(\tfrac{v_0-u_0}{u_0+v_0}, [v_1, u_1], [v_2, u_2]\right).\\ &= C_{u,v}\left(\tfrac{u_0-v_0}{u_0+v_0}, [u_1, v_1], [u_2, v_2]\right).\end{aligned}$$

6. B_2, B_3 and their symmetries

The toroidal blow-up of $\mathbb{P}_{u,v}$ at the eight points of

$$\{u_0v_0 = u_1v_1 = u_2v_2 = 0\}$$

as shown in the previous Section cannot be chosen for B_3 since its anti-canonical line bundle is far from ample. In fact it is given by sections of $\mathcal{O}_{\mathbb{P}_{u,v}}(2,2,2)$ that vanish to second order at the eight vertices $\{u_0v_0 = u_1v_1 = u_2v_2 = 0\}$ and therefore also vanish to first order along all the edges of CUBE. So its anti-canonical linear system will not be basepoint-free which will be necessary for our geometric model. Furthermore it has

$$\left(K^{-1}\right)^3 = -16.$$

On the other hand $K_{\mathbb{P}_{u,v}}^{-1}$ is ample with

$$\left(K_{\mathbb{P}_{u,v}}^{-1}\right)^3 = 48.$$

In our application to W_4/B_3 we will need the linear system $\left|K_{B_3}^{-1}\right|$ basepoint-free and, in particular, for three-generation we will need

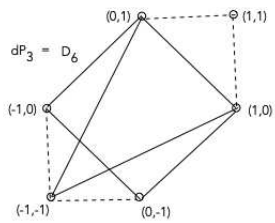
$$(6.1)\qquad \left(K_{B_3}^{-1}\right)^3 = 12.$$

To achieve (6.1), we will modify

$$\mathbb{P}_{B_2} := \mathbb{P}_{[u_1,v_1]} \times \mathbb{P}_{[u_2,v_2]}.$$

Since each blown up point on $\mathbb{P}_{B_2}$ reduces $\left(K_{B_3}^{-1}\right)^3$ by six, we will need to blow up $\mathbb{P}_{B_2}$ at six points. Furthermore, in order that the F-theory model incorporate an eventual $\mathbb{Z}_4$-action that induces asymptotically a $\mathbb{Z}_4$ $\mathbf{R}$-symmetry, the six points will have to comprise the union of two orbits of the $\mathbb{Z}_4$-action.

The fan of the toric representation of $\mathbb{P}_{B_2}$ is the horizontal square of the green octohedron in the above figures also shown as the tilted square in the diagram below. We blow up $\mathbb{P}_{B_2}$ torically by adjoining the vertices $(1,1)$ and $(-1,-1)$ to its fan to obtain the toric fan of D_6. But this polyhedral fan can also be viewed as the toric fan $\mathbb{P}^2 = \mathbb{P}_{[a,b,c]}$ blown up at the three points with only one non-zero coordinate. It is obtained by adjoining the three vertices $(1,1)$,$(-1,0)$ and $(0,-1)$ to the isosceles triangle fan of $\mathbb{P}_{[a,b,c]}$.



In fact this isomorphism is given explicitly by the correspondence

$\mathbb{P}_{[a,b,c]}$	$\mathbb{P}_{B_2}$	Fan vertex
blow up $\{a = b = 0\}$	blow up $\{u_1 = u_2 = 0\}$	$(1,1)$
proper transform $\{a = 0\}$	proper transform $\{u_2 = 0\}$	$(1,0)$
blow up $\{a = c = 0\}$	proper transform $\{v_1 = 0\}$	$(0,-1)$
proper transform $\{c = 0\}$	blow up $\{v_1, v_2 = 0\}$	$(-1,-1)$
blow up $\{b = c = 0\}$	proper transform $\{v_2 = 0\}$	$(-1,0)$
proper transform $\{b = 0\}$	proper transform $\{u_1 = 0\}$	$(0,1)$

We next analyze in detail the construction of the del Pezzo surface B_2 for $B_3 = \mathbb{P}_{[u_0,v_0]} \times B_2$.

6.1. The del Pezzo B_2 and its symmetries

To achieve a Fano $B_3 = B_2 \times \mathbb{P}_{[u_0,v_0]}$ necessary for the F-theory dual, B_2 must be a del Pezzo surface. By Castelnuovo's Rationality Theorem, the $\mathbb{Z}_2$-action on B_2, that we call β_2, cannot be free. On the other hand, β_2 must have at most finite fixpoint set since otherwise the $\mathbb{Z}_2$-action β_3 on B_3 cannot act freely on the ample anti-canonical section $S_{\mathrm{GUT}} \subseteq B_3$, a necessary condition for an F-theory model with Wilson-line symmetry breaking. Now by Table 6 and Figure 10 in [1], there is one and only one sequence of del Pezzo surfaces with involution having finite fixpoint set. These are the four entries in Table 6 that have no entry in either the Σ column nor in the $\mathbf{R}$ column. The sequence is represented in Figure 10 by the left vertical column that begins with $\mathbb{P}_{[u_1,v_1]} \times \mathbb{P}_{[u_2,v_2]}$ and proceeds by blowing up three additional pairs of points to obtain the phenomenologically desirable $dP_7 = D_2$, the standard mathematical notation for the family of del Pezzo surfaces whose anti-canonical divisor has self-intersection 2.[2]

We will also need the $\mathbb{Z}_4$-symmetry inherited from a square root of the longest element of the Weyl group of $SU(5)$ as in Lemma 3 T_0 acts as

$$u_1 \mapsto u_2 \mapsto v_1 \mapsto v_2 \mapsto u_1 \tag{6.2}$$

on the blown up $\mathbb{P}_{[u_1,v_1]} \times \mathbb{P}_{[u_2,v_2]}$. Whether or not B_2 is del Pezzo will, as we see next, depends on on the choice of the orbit of the $\mathbb{Z}_4$-symmetry on $\mathbb{P}_{[u_1,v_1]} \times \mathbb{P}_{[u_2,v_2]}$ generated by T_0, a cyclic element of the Weyl group of $SU(5)$ whose square is its longest element.

Theorem 4. *For generic choice of the orbit of the action* (6.2)*, the resulting B_2 is a del Pezzo surface.*

Proof. The proof will first transfer the $\mathbb{Z}_4$-action from $\mathbb{P}_{[u_1,v_1]} \times \mathbb{P}_{[u_2,v_2]}$ to $\mathbb{P}_{[a,b,c]}$ and then check general position for points of an orbit, that is, no three points of the seven blown up point lie on a line and no six points lie on a conic. The birational passage from $\mathbb{P}_{[u_1,v_1]} \times \mathbb{P}_{[u_2,v_2]}$ to $\mathbb{P}_{[a,b,c]}$ is given as follows. An orbit of the $\mathbb{Z}_4$-action (6.2) on $\mathbb{P}_{[u_1,v_1]} \times \mathbb{P}_{[u_2,v_2]}$ can be written as

$$\begin{aligned}
([u_1,v_1],[u_2,v_2]) &= \left(\left[\tfrac{u_1}{v_1},1\right],\left[\tfrac{u_2}{v_2},1\right]\right)\\
([u_2,v_2],[v_1,u_1]) &= \left(\left[\tfrac{u_2}{v_2},1\right],\left[\tfrac{v_1}{u_1},1\right]\right)\\
([v_1,u_1],[v_2,u_2]) &= \left(\left[\tfrac{v_1}{u_1},1\right],\left[\tfrac{v_2}{u_2},1\right]\right)\\
([v_2,u_2],[u_1,v_1]) &= \left(\left[\tfrac{v_2}{u_2},1\right],\left[\tfrac{u_1}{v_1},1\right]\right)
\end{aligned}$$

that translates to

$$\begin{gathered}
(a,b)\\
\left(b,a^{-1}\right)\\
\left(a^{-1},b^{-1}\right)\\
\left(b^{-1},a\right)
\end{gathered} \tag{6.3}$$

[2]More precisely, 'phenomenologically desirable' equates to '3-generation, one Higgs doublet, and no vector-like exotics.'

that together with $(0,0)$, $(\infty,0)$ and $(0,\infty)$ become the seven blown-up points in $\mathbb{P}_{[a,b,c]}$ by setting $c=1$. One immediately checks that for general choice of a and b the slopes of the four points in (6.3) are distinct and non-zero. So no line containing two of the points that are blown up contains any of the 5 other points that are blown up. Also conics through 6 of the 7 points fall into one of the two cases:

1) Parabolas with horizontal or vertical major axis containing ((0.0)) and so of the form

$$y = cx\,(x-2d) \tag{6.4}$$

containing all four points of (6.3). Therefore

$$\begin{array}{c}(a,b): \ b = ca\,(a-2d) \ \ c = \frac{b}{a(a-2d)}\\ \left(b,a^{-1}\right): \ a^{-1} = cb\,(b-2d) \ \ c = \frac{a^{-1}}{b(b-2d)}\\ \left(a^{-1},b^{-1}\right): \ b^{-1} = ca^{-1}\left(a^{-1}-2d\right) \ \ c = \frac{b^{-1}}{a^{-1}(a^{-1}-2d)}\\ \left(b^{-1},a\right): \ a = cb^{-1}\left(b^{-1}-2d\right) \ \ c = \frac{a}{b^{-1}(b^{-1}-2d)}\end{array}$$

so by high school algebra we obtain a polynomial relationship between a and b

$$\begin{array}{c}\frac{b}{a(a-2d)} = \frac{a^{-1}}{b(b-2d)} : \ \frac{b^3-a}{b^2-1} = 2d\\ \frac{b}{a(a-2d)} = \frac{b^{-1}}{a^{-1}(a^{-1}-2d)} : \ \frac{ba^{-2}-b^{-1}a^2}{ba^{-1}-b^{-1}a} = 2d\\ a\left(b^3-a\right)\left(b^2-a^2\right) = \left(b^2-1\right)\left(b^2-a^4\right).\end{array}$$

2) Hyperbolae with the two coordinate axes as asymptotes and so of the form

$$(x-c)\,(y-d) = e \tag{6.5}$$

containing all four points of (6.3) or

$$xy - (cy+dx) = 0 \tag{6.6}$$

containing three four points of (6.3). In the case of (6.5)

$$\begin{array}{c}(a,b): \ (a-c)\,(b-d) = e\\ \left(b,a^{-1}\right): \ (b-c)\,(1-ad) = ae\\ \left(a^{-1},b^{-1}\right): \ (1-ac)\,(1-bd) = abe\\ \left(b^{-1},a\right): \ (1-bc)\,(a-d) = be.\end{array}$$

So

$$\begin{aligned} a\,(a-c)\,(b-d) &= (b-c)\,(1-ad) \\ ab\,(a-c)\,(b-d) &= (1-ac)\,(1-bd) \\ b\,(a-c)\,(b-d) &= (1-bc)\,(a-d) \end{aligned}$$

and so

$$\begin{aligned} d &= \tfrac{(b-c)-a(a-c)b}{(b-c)a-a(a-c)} \\ d &= \tfrac{1-ac-ab(a-c)b}{(1-ac)b-ab(a-c)} \\ d &= \tfrac{a(1-bc)-b^2(a-c)}{(1-bc)-b(a-c)}. \end{aligned}$$

As above this gives us two linear equations in c with coefficients that are polynomials in (a,b). Again as above solving each for $c$and setting the two expressions for c equal to each other we get a non-trivial polynomial equation in (a,b) that must be satisfied so that this conic passes through the six given points. The case (6.6) is similar. We have three linear equations in (c,d). Use pairs of these to eliminate d and then set the two expression for c equal to each other to get a non-trivial polynomial relation in (a,b) that must be satisfied.

We have therefore shown that if (a,b) does not satisfy any of the above finite number of polynomial equations, then D_6 blown up at the orbit of (a,b) under the $\mathbb{Z}_4$-action is a del Pezzo surface. □

Thus for generic choice of (a,b) the resulting dP_7 is in fact a del Pezzo surface D_2. We define

$$B_2 = D_2$$

with

$$B_3 = \mathbb{P}_{[u_0,v_0]} \times B_2.$$

Then we will consider the specialization of (6.2) under the specialization

$$\left(B_3 = \mathbb{P}_{[u_0,v_0]} \times B_2\right) \Rightarrow \left(B_{3,0} = \left(\mathbb{P}_{[1,a]} \cup \mathbb{P}_{[1,b]}\right) \times B_2\right)$$

to the semi-stable degeneration such that the limit that encodes the structure of the Heterotic model. $T_{u,v}$ will act equivariantly on our degenerating family of Calabi-Yau fourfolds that we will denote $W_{4,\delta}/B_{3,\delta}$ with δ denoting the parameter for the degenerating family of fourfolds. Actually for all $\delta \neq 0$

$$B_3 = B_2 \times \mathbb{P}_{[u_0,v_0]} \cong B_{3,\delta}$$

but at $\delta = 0$, while B_2 remains stationary, we will have the $\mathbb{P}^1$-splitting

$$\mathbb{P}_{[u_0,v_0]} \Rightarrow \mathbb{P}_{[1,a]} \cup \mathbb{P}_{[1,b]}$$

that will force $W_{4,0}$ to split into two components $dP_a/\left(B_2 \times \mathbb{P}_{[1,a]}\right)$ and $dP_b/\left(B_2 \times \mathbb{P}_{[1,b]}\right)$. We will thereby equip the semi-stable limit

$$\begin{aligned} W_{4,0}/B_2 &= \left(dP_a \cup dP_b\right)/B_2 \\ V_3/B_2 &= \left(dP_a \cap dP_b\right)/B_2 \end{aligned} \tag{6.7}$$

with an 'asymptotic $\mathbb{Z}_4$ $\mathbf{R}$-symmetry.' The action of the element of order 2 in $\mathbb{Z}_4$ is simply the $\mathbb{Z}_2$-action β_3 employed in (1.2) and papers [4, 5]. This passage to the semistable limit will be defined and described in detail in Section 9 of this paper.

6.2. Eigenvectors and eigenvalues for the $\mathbb{Z}_4$ symmetry on B_3

By Theorem 4, blowing up D_6 at the four additional points (6.3) of a generic orbit of T_0 yields a del Pezzo D_2 if the four points are in general position. Since D_r for $r < 6$ is no longer toric, we have to leave the family of toric varieties in order to achieve a smooth threefold with the correct numerical invariants. Now any four points on $\mathbb{P}_{[a,b,c]}$ in general position are the base locus of a pencil of conics on which T_0 should act. This action has two fixpoints $q_1(a,b,c)$ and $q_2(a,b,c)$ and the blow-up of D_6 at the four points given us by the smooth divisor

$$\left\{\begin{vmatrix} q_1 & q_2 \\ k & l \end{vmatrix} = 0\right\} \subseteq D_6 \times \mathbb{P}_{[k,l]}. \tag{6.8}$$

T_0 also acts equivariantly on the mapping

$$B_2 \to \mathbb{P}\left(H^0\left(K_{B_2}^{-1}\right)\right) \cong \mathbb{P}^2 \tag{6.9}$$

that turns out to be a double cover with branch locus a smooth quartic plane curve $R \subseteq \mathbb{P}\left(H^0\left(K_{B_2}^{-1}\right)\right)$. We define $q_{[1,0,0]}, q_{[0,1,0]}, q_{[0,0,1]}$ as the conic in the pencil (6.8) containing the point indicated by its respective subscript.

Then

$$\begin{gathered} a\cdot q_{[1,0,0]}\\ b\cdot q_{[0,1,0]}\\ c\cdot q_{[0,0,1]}\end{gathered}$$

are a basis for the anti-canonical linear system of B_2. It turns out that

$$\begin{aligned} t_1&=\log a\\ t_2&=\log b\\ t_3&=\log c\end{aligned}$$

are natural coordinates for $H^0\left(K_{B_2}^{-1}\right)$ so that the final choice of B_2 should be such that, referring to (6.3),

$$B_2=\left\{t_0^2=f_4\left(t_1,t_2,t_3\right)\right\}$$

such that the generator T_0 of the $\mathbb{Z}_4$-action

$$\begin{gathered}(\log a,\log b)\\ (\log b,-\log a)\\ (-\log a,-\log b)\\ (-\log b,\log a)\,,\end{gathered}$$

that is, T_0 should be an automorphism of B_2 the form

$$(t_0;[t_1,t_2,t_3])\mapsto$$
$$\left(\det\begin{pmatrix}0&-1&0\\1&0&0\\0&0&-1\end{pmatrix}\cdot t_0;\left[\begin{pmatrix}t_1&t_2&t_3\end{pmatrix}\begin{pmatrix}0&-1&0\\1&0&0\\0&0&-1\end{pmatrix}\right]\right)$$

This automorphism of B_2 has characteristic polynomial

$$\frac{\lambda^4-1}{\lambda-1}=\lambda^3+\lambda^2+\lambda+1.$$

In fact we can demand that $\mathrm{Aut}B_2$ is the entire symmetry group of CUBE, namely

$$S_4\times\mathbb{Z}_2=signed\,S_3,$$

where

$$B_2=\left\{t_0^2=t_1^4+t_2^4+t_3^4+\alpha\left(t_1^2t_2^2+t_1^2t_3^2+t_2^2t_3^2\right)\right\}.$$

By elementary algebra there are then bitangent lines to R that we will denote as $\{m_1:=c_\alpha t_1+t_2=0\}$ and $\{m_2:=t_1-c_\alpha t_2=0\}$ respectively. Then

$$\begin{aligned} m_1 \circ T_0 &= m_2 \\ m_2 \circ T_0 &= -m_1 \end{aligned}$$

with eigenvectors

$$\begin{aligned} m_{+i} &:= m_1 - i \cdot m_2 \\ m_{-i} &:= m_1 + i \cdot m_2 \end{aligned}$$

indexed by eigenvalues for the action of T_0 on $H^0\left(K_{B_2}^{-1}\right)$. The third eigenvalue must be -1. Let n_{-1} denote the associated eigenvector. In what follows and associated paper, it will be convenient to appeal to two sets of coordinates for $\mathbb{P}\left(H^0\left(K_{B_2}^{-1}\right)\right)$, one being the coordinates $\mathbb{P}_{[n_{-1},m_{+i},m_{-i}]}$ given by eigenvectors and the other being $\mathbb{P}_{[n_0,m_1,m_2]}$ incorporating the defining forms for the bitangent lines defined above. To avoid notational confusion

$$n_0 = n_{-1}$$

defines the same line in $\mathbb{P}\left(H^0\left(K_{B_2}^{-1}\right)\right)$ but the different index indicates which coordinate system we are referring to. Then

$$\begin{aligned} T_0^*\left(n_0\right) &= n_0 \circ T_0 = -n_0 \\ T_0^*\left(m_1\right) &= m_1 \circ T_0 = m_2 \\ T_0^*\left(m_2\right) &= m_2 \circ T_0 = -m_1 \end{aligned} \tag{6.10}$$

and

$$\begin{aligned} T_0^*\left(n_{-1}\right) &= n_{-1} \circ T_0 = -n_0 \\ T_0^*\left(m_{+i}\right) &= m_{+i} \circ T_0 = i \cdot m_{+i} \\ T_0^*\left(m_{-i}\right) &= m_{-i} \circ T_0 = -i \cdot m_1. \end{aligned} \tag{6.11}$$

With respect to the direct-sum decomposition

$$H^0\left(K_{B_2}^{-1}\right) = H^0\left(K_{B_2}^{-1}\right)^{[+1]} \oplus H^0\left(K_{B_2}^{-1}\right)^{[-1]} \tag{6.12}$$

induced by the involution $T_{B_2}^2$ the first summand is generated by a single section n_0 while the second summand is the two-dimensional T_0^2-eigenspace with eigenvalue -1. The second summand is spanned by any two of the four vectors

$$m_1, m_2, m_{+i}, m_{-i}.$$

For $B_3 = B_2 \times \mathbb{P}_{[u_0,v_0]}$, the restriction map

$$H^0\left(\mathcal{O}_{\mathbb{P}_{[n_0,m_1,m_2]}}(1) \boxtimes \mathcal{O}_{\mathbb{P}_{[u_0,v_0]}}(2)\right) \to H^0\left(K_{B_3}^{-1}\right) \tag{6.13}$$

is an isomorphism. Therefore we can write

$$T_{u,v}\left(\frac{u_0 - v_0}{u_0 + v_0}, ([n_{-1}, m_{+i}, m_{-i}])\right) = \left(i \cdot \left(\frac{u_0 - v_0}{u_0 + v_0}\right), T_0\left([n_{-1}, m_{+i}, m_{-i}]\right)\right)$$

on $H^0\left(\mathcal{O}_{\mathbb{P}_{[n_{-1},m_{+i},m_{-i}]}}(1) \boxtimes \mathcal{O}_{\mathbb{P}_{[u_0,v_0]}}(2)\right)$ or equivalently

$$T_{u,v}\left(\frac{u_0 - v_0}{u_0 + v_0}, ([n_0, m_1, m_2])\right) = \left(i \cdot \left(\frac{u_0 - v_0}{u_0 + v_0}\right), T_0\left([n_0, m_1, m_2]\right)\right)$$

on $H^0\left(\mathcal{O}_{\mathbb{P}_{[n_0,m_1,m_2]}}(1) \boxtimes \mathcal{O}_{\mathbb{P}_{[u_0,v_0]}}(2)\right)$. We sum up as follows.

Lemma 5. *i)* $\left|K^{-1}_{\mathbb{P}_{[u_0,v_0]} \times B_2}\right|$ *is basepoint-free. Also*

$$h^0\left(K^{-1}_{\mathbb{P}_{[u_0,v_0]} \times B_2}\right) = 9$$

and

$$h^k\left(K^{-1}_{\mathbb{P}_{[u_0,v_0]} \times B_2}\right) = 0$$

for $k > 0$.

ii) The quotient

$$\left(\mathbb{P}_{[u_0,v_0]} \times B_2\right)^{\vee} = \frac{\mathbb{P}_{[u_0,v_0]} \times B_2}{\{C_{u,v}\}}$$

carries a faithful $\mathbb{Z}_2$*-action* $T_{u,v}$.

iii) Under the branched double cover

$$B_2 \to \mathbb{P}_{[n_0,m_1,m_2]}$$

and the six exceptional curves of D_6 *listed in the Table at the beginning of this Section specialize to* $S_3 \leq \mathrm{Aut}(B_2)$ *and to bitangents of the branch curve* R*. In particular,* $\{m_1 = 0\}$ *and* $\{m_2 = 0\}$ *are bitangents to the branch locus* R *and satisfy*

$$C^*_{u,v}(m_1) = -m_1$$
$$C^*_{u,v}(m_2) = -m_2.$$

Proof. i)

$$K_{B_3}^{-1} = K_{B_2}^{-1} \boxtimes K_{\mathbb{P}_{[u_0,v_0]}}^{-1}.$$

Now use the Künneth formula and the Kodaira Vanishing Theorem.

ii) The $\mathbb{Z}_4$-action on B_3 is the action $T_{u,v}$ as constructed above. Since $C_{u,v}$ and T_{B_2} commute and $T_{u,v}^2 = C_{u,v}$, the action descends to a faithful $\mathbb{Z}_2$-action on the $C_{u,v}$-quotient $B_3^\vee$.

iii) Classical fact deriving from the fact that B_2 is the projection of a cubic surface in $\mathbb{P}^3$ from one of its points. Under this projection plane sections spanned by one of the 27 lines and the center of projection map to bitangent lines to $R \subseteq \mathbb{P}_{[n_0,m_1,m_2]}$ as does the plane section tangent to the cubic at the center of projection. So the claim follows from the corresponding assertion for a cubic surface with S_3 symmetry. □

We have the direct-sum decomposition

$$\left(\pi_{B_3^\vee}\right)_* K_{B_2 \times \mathbb{P}_{[u_0,v_0]}}^{-1} = K_{B_3^\vee}^{-1} \oplus \left(K_{B_3^\vee}^{-1} \otimes \mathcal{O}_{B_3^\vee}\left(\varepsilon_{e,v}\right)\right)$$

and, referring to (5.12) and (5.13), we have the following tables of eigenvectors and values for actions on anti-canonical forms on $B_{3,\delta}$:

Table 1:	$T_{u,v}$	$C_{u,v}$
$h^0\left(K_{B_{3,0}^\vee}^{-1}\right) = 4$		$C_{u,v}(w) = w$ $C_{u,v}(x) = x$ $C_{u,v}(y) = -y$ $C_{u,v}(z) = -z$
$(u_0+v_0)^2 \cdot n_{-1}$	-1	$+1$
$(u_0-v_0)^2 \cdot n_{-1}$	$+1$	$+1$
$\left(u_0^2 - v_0^2\right) \cdot m_{-i}$	-1	$+1$
$\left(u_0^2 - v_0^2\right) \cdot m_{+i}$	$+1$	$+1$

Table 2:	$T_{u,v}$	$C_{u,v}$
$h^0\left(K^{-1}_{B^\vee_{3,0}} \otimes \mathcal{O}_{B^\vee_3}(\varepsilon_{u,v})\right) = 5$		$C_{u,v}(w) = w$ $C_{u,v}(x) = x$ $C_{u,v}(y) = -y$ $C_{u,v}(z) = -z$
$(u_0+v_0)^2 \cdot m_{-i}$	$-i$	-1
$(u_0+v_0)^2 \cdot m_{+i}$	$+i$	-1
$(u_0-v_0)^2 \cdot m_{-i}$	$+i$	-1
$(u_0-v_0)^2 \cdot m_{+i}$	$-i$	-1
$\left(u_0^2-v_0^2\right) \cdot n_{-1} =: z_0$	$-i$	-1

Lemma 6. *The anti-canonical linear system* $\left|H^0\left(K^{-1}_{B^\vee_3}\right)\right|$ *is numerically effective (nef) and big . It has two basepoints*

$$(\{[1,\pm1]\} \times \{1,0,0\}) \in \mathbb{P}_{[(u_0^2+v_0^2),u_0v_0]} \times \mathbb{P}_{[n_0,m_1,m_2]}.$$

Therefore $S^\vee_{\mathrm{GUT}}$ *is a 'hyperelliptic' Enriques surface. The rational mapping*

$$\varphi : S^\vee_{\mathrm{GUT}} \dashrightarrow \mathbb{P}^3$$

induced by the linear system is resolved be a simple blow-up of the two basepoints iyielding a $2-1$ *morphism*

$$\tilde{\varphi} : \widetilde{S^\vee_{\mathrm{GUT}}} \to \mathbb{P}^3$$

onto a smooth quadric surface $\mathbb{P}_{[(u_0^2+v_0^2),u_0v_0]} \times \mathbb{P}_{[m_1,m_2]}$ *where the image the two exceptional curves is* $\{[1,\pm1]\} \times \mathbb{P}_{[m_1,m_2]}$.

Proof. By direct computation with Table 2 just above, $S^\vee_{\mathrm{GUT}}$ is identified as being in case 2c for $n=3$ in Theorem 3.2.2 of [13]. Key to this classification is the fact that in Table 2 all five sections vanish at $([\pm1,1],[1,0,0])$. $\square$

7. Requirements for the Tate form

For the purposes of obtaining phenomenologically consistent numerical data,[3] we utilize the $B^\wedge_3$-divisors $E.$ in (5.9) and their corresponding rays $e.$ in the

[3] "...phenomenologically consistent numerical data" refers to the necessity of having three $\mathbf{10}$ and no $\bar{\mathbf{10}}$-representations over the matter curve $\Sigma^{(4)}_{\mathbf{10}}$, three $\bar{\mathbf{5}}$ and no $\mathbf{5}$-representations over the matter curve $\Sigma^{(41)}_{\bar{\mathbf{5}}}$, as well as having exactly one Higgs doublet over the Higgs curve $\Sigma^{(44)}_{\bar{\mathbf{5}}}$.

lattice $N^{\wedge}$. We let the ray $f.$ denote the same ray in the lattice N and denote the corresponding divisor as $F.$. Recalling that the fan used in the construction of B_3 contains the ray generated by $f_{\dot{x}\dot{z}\dot{y}\ddot{w}}$ as well as its negative $-f_{\dot{x}\dot{z}\dot{y}\ddot{w}}$ corresponding to the divisors $\{u_0 = 0\}$ and $\{v_0 = 0\}$ respectively.

The Tate form of our eventual F-theory model W_4/B_3 is written as

$$y^2w = x^3 + a_5xyw + a_4zx^2w + a_3z^2yw^2 + a_2z^3xw^2 + a_0z^5w^3.$$

The spectral divisor is given by the equation

$$0 = a_5t^5 + a_4t^4z + a_3t^3z^2 + a_2t^2z^3 + a_0z^5$$

where $t = \frac{y}{x}$. By Table 2 above, $\dim H^0\left(K_{B_3}^{-1}\right) = 5$ so the spaces

$$\mathbb{C} \cdot u_0v_0m_1 + \mathbb{C} \cdot \left(v_0^2 + u_0^2\right)m_2 + \mathbb{C} \cdot z$$

and

$$\mathbb{C}{\cdot}a_4 + \mathbb{C}{\cdot}a_3 + \mathbb{C}{\cdot}a_2$$

have a one-dimensional intersection generated by

$$\begin{aligned}\lambda_1 \cdot u_0v_0m_1 + \lambda_2 \cdot \left(v_0^2 + u_0^2\right)m_2 + z \\ = \kappa_4{\cdot}a_4 + \kappa_3{\cdot}a_3 + \kappa_2{\cdot}a_2.\end{aligned}$$

Lemma 7. *In order to have the relation*

$$z = \kappa_5{\cdot}a_5 + \kappa_4{\cdot}a_4 + \kappa_3{\cdot}a_3 + \kappa_2{\cdot}a_2$$

required in §4 of [4], define

$$a_5 = -\left(\lambda_1 \cdot u_0v_0m_1 + \lambda_2 \cdot \left(v_0^2 + u_0^2\right)m_2\right) \tag{7.1}$$

where $\{m_j = 0\}_{j=1,2}$ define a distinguished two-orbit of the action of T_0 on the 28 bitangents to the branch locus of $B_2/\mathbb{P}_{[n_0,m_1,m_2]}$.[4] *This pair of $C_{u,v}$-*

[4]This assumption is critical so that our model satisfies the condition of three-generation

invariant bitangents $\{m_j = 0\}_{j=1,2}$ *then lifts into the image of*

$$F_{\dot{x}\dot{z}\ddot{y}\ddot{w}} = \left\{ \frac{e_{\dot{x}\dot{z}\ddot{y}\ddot{w}}}{2} = 0 \right\} \Rightarrow \{u_0 = 0\}$$

and into

$$F_{\dot{y}\dot{w}\ddot{x}\ddot{z}} = \left\{ \frac{e_{\dot{y}\dot{w}\ddot{x}\ddot{z}}}{2} = 0 \right\} \Rightarrow \{v_0 = 0\} .$$

Also

$$\left(v_0^2 + u_0^2\right) m_j,\ u_0 v_0 m_j \in H^0 \left(K_{B_3}^{-1}\right)^{[-1]} .$$

Therefore, for all allowable choices of $z \subseteq H^0 \left(K_{B_3}^{-1}\right)^{[-1]}$ *there are rational curves*

$$\begin{aligned} F_+ &\subseteq \{z = 0\} \cap F_{\dot{x}\dot{z}\ddot{y}\ddot{w}} \\ F_- &\subseteq \{z = 0\} \cap F_{\dot{y}\dot{w}\ddot{x}\ddot{z}} \end{aligned} \tag{7.2}$$

lying in S_{GUT} *with intersection matrix*

$$\begin{array}{ccc} \cdot & F_+ & F_- \\ F_+ & -2 & 0 \\ F_- & 0 & -2. \end{array}$$

$C_{u,v}$ *interchanges* F_+ *with* F_-. *We choose the remaining* a_j *and* z *and* $t = \frac{y}{x}$ *in the space generated by the forms in Table 2.*

Additionally we choose $t \in H \left(K_{B_3}^{-1}\right)^{[-1]}$ *so that it vanishes on* $\{z = u_0 v_0 = m_2 = 0\}$. *This choice guarantees that the surfaces*

$$\left(S_{\mathrm{GUT}} \cap \left(\{u_0 v_0 = m_2 = 0\}\right)\right) \times \mathbb{P}_{[t,z]} \tag{7.3}$$

lie in the spectral divisor 2.5), in fact in the component $\mathcal{D}^{(4)}$ *of*

$$\mathcal{D} := \left\{ a_5 t^5 + a_4 t^4 z + a_3 t^3 z^2 + a_2 t^2 z^3 + a_0 z^5 = 0 \right\} \subseteq \mathbb{P}_{[t,z]} \times B_3. \tag{7.4}$$

Proof. We need only prove the last statement. We have chosen

$$t = l\left(n_0, m_1, m_2\right) \cdot u_0 v_0 + q\left(u_0, v_0\right) \cdot m_2 \tag{7.5}$$

namely as a generic section containing $\{u_0 v_0 = m_2 = 0\} \subseteq \mathbb{P}_{[u_0,v_0]} \times \mathbb{P}_{[n_0,m_1,m_2]}$ and

$$z = \lambda_1 m_1 \cdot u_0 v_0 + \lambda_2 \cdot \left(v_0^2 + u_0^2\right) m_2 + \sum_{j=2}^{4} \kappa_j \cdot a_j$$

in (7.1). Since $\{u_0v_0 = m_2 = 0\} \subseteq \{z = 0\}$, $\sum_{j=2}^{4} \kappa_j \cdot a_j$ must also vanish there. That is, we can write

$$z = l'(n_0, m_1, m_2) \cdot u_0v_0 \cdot +q'(u_0, v_0) \cdot m_2. \tag{7.6}$$

Since all sections of $H^0(\mathcal{O}_{B_3}(N))$ are pull-backs of sections of

$$H^0\left(\mathcal{O}_{\mathbb{P}_{[u_0,v_0]}\times\mathbb{P}_{[n_0,m_1,m_2]}}(N)\right)$$

under the branched double cover

$$B_3 = \mathbb{P}_{[u_0,v_0]} \times B_2 \to \mathbb{P}_{[u_0,v_0]} \times \mathbb{P}_{[n_0,m_1,m_2]},$$

we first consider (7.4) as an equation over $\mathbb{P}_{[u_0,v_0]} \times \mathbb{P}_{[n_0,m_1,m_2]}$. We consider the blow-up of (7.4) in

$$\mathbb{P}_{[u_0,v_0]} \times \mathbb{P}_{[n_0,m_1,m_2]} \times \mathbb{P}_{[T,Z]}$$

defined by

$$\begin{aligned} t &= T \cdot \xi \\ z &= Z \cdot \xi \end{aligned}$$

so that from (7.4)

$$\left| \begin{array}{cc} a_4t^2z^3 + a_2t^2z^2 + a_0z^5 & 1 \\ -\left(a_5t^5 + a_3t^3z^2\right) & 1 \end{array} \right| = \xi^5 \cdot \left| \begin{array}{cc} a_4T^2Z^3 + a_2T^2Z^2 + a_0Z^5 & 1 \\ -\left(a_5T^5 + a_3T^3Z^2\right) & 1 \end{array} \right|.$$

By (7.5) and (7.6)

$$\left| \begin{array}{cc} q & -l \\ u_0v_0 & m_2 \end{array} \right| = \left| \begin{array}{cc} q' & -l' \\ u_0v_0 & m_2 \end{array} \right| = 0$$

and the fact that the matrix

$$\begin{pmatrix} q & -l \\ q' & -l' \end{pmatrix}$$

is everywhere of rank at least one, the proper transform

$$\begin{aligned} \mathcal{D} := \left\{ \left| \begin{array}{cc} a_4T^2Z^3 + a_2T^2Z^2 + a_0Z^5 & 1 \\ -\left(a_5T^5 + a_3T^3Z^2\right) & 1 \end{array} \right| = 0 \right\} \\ \subseteq \mathbb{P}_{[u_0,v_0]} \times \mathbb{P}_{[n_0,m_1,m_2]} \times \mathbb{P}_{[T,Z]}. \end{aligned} \tag{7.7}$$

contains the surface

$$\{\xi = 0\} \cap \mathcal{D} = \left(\{u_0 v_0 = m_2 = 0\} \times \mathbb{P}_{[T,Z]}\right) \\ \subseteq \left(\mathbb{P}_{[u_0,v_0]} \times \mathbb{P}_{[n_0,m_1,m_2]}\right) \times \mathbb{P}_{[T,Z]}$$

projecting to

$$\{u_0 v_0 = m_2 = 0\} \subseteq \{z = 0\} \subseteq \left(\mathbb{P}_{[u_0,v_0]} \times \mathbb{P}_{[n_0,m_1,m_2]}\right).$$

With respect to the branched double cover

$$\mathbb{P}_{[u_0,v_0]} \times B_2 \to \left(\mathbb{P}_{[u_0,v_0]} \times \mathbb{P}_{[n_0,m_1,m_2]}\right)$$

we have a cartesian diagram

$$\begin{array}{ccc} \mathcal{D} & \to & B_3 \times \mathbb{P}_{[T,Z]} \\ \downarrow & & \downarrow \\ \mathcal{D}' & \to & \left(\mathbb{P}_{[u_0,v_0]} \times \mathbb{P}_{[n_0,m_1,m_2]}\right) \times \mathbb{P}_{[T,Z]}. \end{array}$$

The pull-back of the the exceptional set

$$\{\xi = 0\} \cap \mathcal{D}' \subseteq \left(\mathbb{P}_{[u_0,v_0]} \times \mathbb{P}_{[n_0,m_1,m_2]}\right) \times \mathbb{P}_{[T,Z]}$$

to $B_3 \times \mathbb{P}_{[T,Z]}$ is the reducible surface

$$\left(F_\pm \cup F_\pm^{opp}\right) \times \mathbb{P}_{[T,Z]} \subseteq B_3 \times \mathbb{P}_{[T,Z]} \tag{7.8}$$

where $F_\pm \cup F_\pm^{opp}$ denote the components of $\{m_2 = 0\}$ in $\{u_0 = 0\} \times B_2$ and $\{v_0 = 0\} \times B_2$ respectively. So the spectral divisor

$$\mathcal{D} := \left\{a_5 T^5 + a_4 T^4 Z + a_3 T^3 Z^2 + a_2 T^2 Z^3 + a_0 Z^5 = 0\right\} \\ \subseteq B_3 \times \mathbb{P}_{[T,Z]} \tag{7.9}$$

contains the four-component divisor

$$\left(F_+ \times \mathbb{P}_{[T,Z]}\right) + \left(F_- \times \mathbb{P}_{[T,Z]}\right) + \left(F_+^{opp} \times \mathbb{P}_{[T,Z]}\right) + \left(F_-^{opp} \times \mathbb{P}_{[T,Z]}\right)$$

entirely supported in $S_{\mathrm{GUT}} \times \mathbb{P}_{[T,Z]}$. Therefore the divisor

$$\left(F_+ \times \mathbb{P}_{[T,Z]}\right) - \left(F_- \times \mathbb{P}_{[T,Z]}\right) \tag{7.10}$$

on $\mathcal{D}$ pushes forward to the trivial divisor on B_3 but restricts to a non-trivial divisor on the threefold $S_{\mathrm{GUT}} \times \mathbb{P}_{[T,Z]}$. Furthermore the difference

$(F_+ - F_-)$ defines a non-trivial line bundle on S_{GUT} that has degree zero with respect to the polarization N. Additionally $\beta_3 = C_{u,s}$ exchanges F_+ with F_-. These numerics will allow us to define a Higgs line bundle with the correct numerical invariants. □

To prevent the existence of vector-like exotics in our F-theory model, we have required

$$a_5 + a_4 + a_3 + a_2 + a_0 = 0 \tag{7.11}$$

so that the section of P/B_3 given by

$$\begin{aligned} x &= z^2 w \\ y &= z^3 w \end{aligned} \tag{7.12}$$

lies in W_4. Furthermore this additional assumption forces the spectral divisor $\mathcal{D}$ given by (7.4) to become reducible, with one component given by $t = z$, and the other component of degree 4. More precisely

$$\begin{gathered} a_5 t^5 + a_4 t^4 z + a_3 t^3 z^2 + a_2 t^2 z^3 + a_0 z^5 = \\ (t - z)\left(a_5 t^4 + a_{54} t^3 z - a_{20} t^2 z^2 - a_0 z^3 t - a_0 z^4\right) \end{gathered}$$

where $a_{54} = a_5 + a_4$ and $a_{20} = a_2 + a_0$, etc.

Referring to (3.4) and [5], the Higgs curve $\Sigma_{\bar{\mathbf{5}}}^{(44)}$ is derived from the surface in $\mathcal{D}^{(4)}$ defined by common solutions to the $C_{u,v}$-equivariant system of equations

$$\begin{gathered} a_5 t^4 - a_{20} t^2 z^2 - a_0 z^4 = 0 \\ a_{54} t^2 - a_0 z^2 = 0. \end{gathered} \tag{7.13}$$

It doubly covers the surface in B_3 defined by the resultant equation that, using $a_{54320} = 0$, reduces to

$$\begin{vmatrix} a_4 & -a_5 \\ a_3 + a_0 & a_3 \end{vmatrix} = 0 \tag{7.14}$$

with branch locus defined by the restriction of the divisor class N.

The matter curve $\Sigma_{\bar{\mathbf{5}}}^{(41)}$ is given by the common solutions to

$$a_{420} = z = 0.$$

The other matter curve $\Sigma_{\mathbf{10}}^{(4)}$ is given by

$$a_5 = z = 0.$$

7.1. Numerology of divisors on S_{GUT}

Furthermore (6.1) implies that the genus of $Z := N \cdot S_{\mathrm{GUT}}$ is 7 and

$$\{u_0 = 0\} \cdot N^2 = \{v_0 = 0\} \cdot N^2 = 2 \tag{7.15}$$

where, as above, we denote $K_{B_3}^{-1} = \mathcal{O}_{B_3}(N)$.

Proposition 8. *The line bundle $\mathcal{O}_{B_3}(N)$ is ample and the line bundle $\mathcal{O}_{B_3}(2N)$ is very ample.*

Proof. The line bundle $\mathcal{O}_{B_3}(N)$ is the pull-back of the very ample line bundle $\mathcal{O}_{\mathbb{P}_{[u_0,v_0]}}(2) \boxtimes \mathcal{O}_{\mathbb{P}_{[n_0,m_1,m_2]}}(1)$ on $\mathbb{P}_{[u_0,v_0]} \times \mathbb{P}_{[n_0,m_1,m_2]}$. By Tables 1&2 above, the pull-back map

$$H^0\left(\mathcal{O}_{\mathbb{P}_{[u_0,v_0]}\times\mathbb{P}_{[n_0,m_1,m_2]}}(N)\right) \to H^0\left(\mathcal{O}_{B_3}(N)\right)$$

is an isomorphism. However the injective pull-back map

$$H^0\left(\mathcal{O}_{\mathbb{P}_{[u_0,v_0]}\times\mathbb{P}_{[n_0,m_1,m_2]}}(2N)\right) \to H^0\left(\mathcal{O}_{B_3}(2N)\right)$$

has a one-dimensional cokernel generated by the quartic ramification locus of the branched double cover $B_2/\mathbb{P}_{[n_0,m_1,m_2]}$ and therefore contains sections that separate the two sheets. $\square$

To understand the use of these surfaces, one must consider the image $\mathcal{O}_{B_3}(2N)$ is very ample.

$$\mathcal{C}_{Higgs} = \mathcal{C}_{Higgs}^{(4)} \cup \tilde{\tau}(B_3) \subseteq \tilde{W}_4$$

of the spectral divisor $\mathcal{D}$ in the canonical crepant resolution $\tilde{W}_4$ constructed above. $S_{\mathrm{GUT}} \subseteq B_3$ has canonical lifting

$$\tilde{S}_{\mathrm{GUT}} \subseteq \mathcal{C}_{Higgs} \subseteq \tilde{W}_4$$

by means of which the Higgs line bundle is pushed forward to a line bundle on the divisor $\mathcal{C}_{Higgs}$.

The push-forward of multiples of this line bundle on $S_{\mathrm{GUT}} \subseteq B_3$ will be denoted as

$$\mathcal{O}_{S_{\mathrm{GUT}}}\left(m\left(F_{+}-F_{-}\right)\right).$$

The push-forward to $S_{\mathrm{GUT}} \subseteq B_3$ of the restriction of the Higgs line bundle $\mathcal{L}_{Higgs}$ to $\tilde{S}_{\mathrm{GUT}}$ will then become

$$\mathcal{O}_{S_{\mathrm{GUT}}}\left(N+m\left(F_{+}-F_{-}\right)\right)$$

for appropriate m where N is the anti-canonical divisor of B_3 (again restricted to S_{GUT}). This twisting of the restriction of $K_{B_3}^{-1}$ by this degree-zero line bundle is exactly the modification that removes first cohomology of $\mathcal{L}_{Higgs}$ on the matter curves and, by a classical theorem in the theory of Prym varieties, drops the number of Higgs doublets to one.

Also in our eventual F-theory model, $S_{\mathrm{GUT}} = \{z=0\}$ will be linearly equivalent to N, matter curves will have divisor class

$$Z := S_{\mathrm{GUT}} \cdot N$$

and the Higgs curve Z_2 on S_{GUT} will have class $2Z$.

Since B_2 is a del Pezzo surface, its anti-canonical bundle is ample as is the anti-canonical bundle

$$K_{B_2}^{-1} \boxtimes K_{[u_0,v_0]}^{-1}$$

allowing us to apply the Kodaira Vanishing Theorem repeated in what follows. The cohomology sequence for the short exact sequence

$$0 \rightarrow \mathcal{O}_{B_3} \rightarrow \mathcal{O}_{B_3}(N) \rightarrow \mathcal{O}_{S_{\mathrm{GUT}}}(Z) \rightarrow 0 \tag{7.16}$$

shows that the forms in the left-hand column of Tables 1 and 2 above together form a basis for $H^0\left(\mathcal{O}_{B_3}(N)\right)$, that

$$\frac{H^0\left(\mathcal{O}_{B_3}(N)\right)}{\mathbb{C}\cdot z} \cong H^0\left(\mathcal{O}_{S_{\mathrm{GUT}}}(Z)\right),$$

that $h^1\left(\mathcal{O}_{B_3}(N)\right) = h^1\left(\mathcal{O}_{S_{\mathrm{GUT}}}(Z)\right)$, and that

$$h^2\left(\mathcal{O}_{B_3}(N)\right) = h^2\left(\mathcal{O}_{S_{\mathrm{GUT}}}(Z)\right) = 0.$$

The cohomology sequence for the short exact sequence

$$0 \rightarrow \mathcal{O}_{S_{\mathrm{GUT}}}(-Z) \rightarrow \mathcal{O}_{S_{\mathrm{GUT}}} \rightarrow \mathcal{O}_Z \rightarrow 0$$

shows that $h^1(\mathcal{O}_{S_{\mathrm{GUT}}}(-Z)) = h^1(\mathcal{O}_{S_{\mathrm{GUT}}}(Z)) = 0$ so that $h^1(\mathcal{O}_{B_3}(N)) = 0$ as well.

For any smooth curve Z_n linearly equivalent to nZ, the cohomology sequence associated to the short exact sequence

$$0 \to \mathcal{O}_{S_{\mathrm{GUT}}}(-(n-1)Z) \to \mathcal{O}_{S_{\mathrm{GUT}}}(Z) \to \mathcal{O}_{Z_n}(Z) \to 0$$

and Kodaira vanishing and Serre duality show that

$$H^0(\mathcal{O}_{Z_n}(Z)) \cong H^0(\mathcal{O}_{S_{\mathrm{GUT}}}(Z)) \cong H^2(\mathcal{O}_{S_{\mathrm{GUT}}}(-Z))^* \tag{7.17}$$

and

$$H^1(\mathcal{O}_{Z_n}(Z)) \cong H^2(\mathcal{O}_{S_{\mathrm{GUT}}}(-(n-1)Z)) \cong H^0(\mathcal{O}_{S_{\mathrm{GUT}}}((n-1)Z))^*$$

and that all these groups have rank seven for $n=1$ and eight for $n=2$.

Finally, for $F_{\cdot}$ equal to F_+ or F_-, consider

$$0 \to \mathcal{O}_{B_3}(N - F_{\cdot}) \to \mathcal{O}_{B_3}(N) \to \mathcal{O}_{F_{\cdot}}(S_{\mathrm{GUT}} \cap F_{\cdot}) \to 0$$

$$0 \to \mathcal{O}_{S_{\mathrm{GUT}}}(Z - (F_{\cdot} \cap S_{\mathrm{GUT}})) \to \mathcal{O}_{S_{\mathrm{GUT}}}(Z) \to \mathcal{O}_{F_{\cdot} \cap S_{\mathrm{GUT}}}(Z) \to 0$$

and

$$0 \to \mathcal{O}_{S_{\mathrm{GUT}}}(Z) \to \mathcal{O}_{S_{\mathrm{GUT}}}(Z + (F_{\cdot} \cap S_{\mathrm{GUT}})) \to \mathcal{N}_{(F_{\cdot} \cap S_{\mathrm{GUT}})|S_{\mathrm{GUT}}}(Z \cap F_{\cdot}) \to 0$$

where $\mathcal{N}_{(F_{\cdot} \cap S_{\mathrm{GUT}})|S_{\mathrm{GUT}}}$ denotes the normal bundle of the rational curve $F_{\cdot} \cap S_{\mathrm{GUT}}$ in S_{GUT} so that

$$\mathcal{N}_{(F_{\cdot} \cap S_{\mathrm{GUT}})|S_{\mathrm{GUT}}} \cong \mathcal{O}_{(F_{\cdot} \cap S_{\mathrm{GUT}})}(-2).$$

8. $\mathbb{Z}_2$-quotients $B_2^{\vee}$ and $B_3^{\vee}$ and their invariants

Now the involution $C_{u,v}$ acts freely on S_{GUT} and as

$$\mathcal{O}_{S_{\mathrm{GUT}}}(N + m(F_+ - F_-)) \mapsto \mathcal{O}_{S_{\mathrm{GUT}}}(N + m(F_- - F_+))$$

Returning to

$$\pi_{S_{\mathrm{GUT}}^{\vee}} : S_{\mathrm{GUT}} \to S_{\mathrm{GUT}}^{\vee} := \frac{S_{\mathrm{GUT}}}{\{C_{u,v}\}}$$

and letting $\mathcal{O}_{S_{\mathrm{GUT}}^{\vee}}(\varepsilon_{u,v})$ denote the non-trivial flat (orbifold) line bundle quotient of the trivial bundle induced by the fixpoint-free action of the involution $C_{u,v}$, we have the splitting

$$\left(\pi_{S_{\mathrm{GUT}}^{\vee}}\right)_{*}\left(\mathcal{O}_{S_{\mathrm{GUT}}}(N)\right)=\mathcal{O}_{S_{\mathrm{GUT}}^{\vee}}(N)^{[+1]}\oplus\mathcal{O}_{S_{\mathrm{GUT}}^{\vee}}(N)^{[-1]}$$

where

$$\mathcal{O}_{S_{\mathrm{GUT}}^{\vee}}(N)^{[-1]}=\mathcal{O}_{S_{\mathrm{GUT}}^{\vee}}(N)^{[+1]}\otimes\mathcal{O}_{S_{\mathrm{GUT}}^{\vee}}(\varepsilon_{u,v}).$$

8.1. Cohomology of the Higgs bundle the Higgs curve

We are now ready for the computation of the cohomology

$$H^{i}\left(\Sigma_{\bar{\mathbf{5}}}^{(44)};\mathcal{L}_{\bar{\mathbf{5}}}^{(44)}\right)=H^{i}\left(\check{\Sigma}_{\bar{\mathbf{5}}}^{(44)};\mathcal{L}_{Higgs}^{\vee,[+1]}\right)\oplus H^{i}\left(\check{\Sigma}_{\bar{\mathbf{5}}}^{(44)};\mathcal{L}_{Higgs}^{\vee,[-1]}\right)$$

of the Higgs line bundle $\mathcal{L}_{S_{\mathrm{GUT}}}=\mathcal{O}_{S_{\mathrm{GUT}}}\left(N+m\left(F_{+}-F_{-}^{opp}\right)\right)$ restricted to the Higgs curve $\Sigma_{\bar{\mathbf{5}}}^{(44)}$ as defined in (7.14). We have seen in (7.17) above that on S_{GUT} that the restriction map

$$H^{0}\left(\mathcal{O}_{S_{\mathrm{GUT}}}(N)\right)\rightarrow H^{0}\left(\mathcal{O}_{\Sigma_{\bar{\mathbf{5}}}^{(44)}}(N)\right)$$

is an isomorphism. This fact has the important corollary that, by Tables 1 & 2 above, every section of $H^{0}\left(\mathcal{O}_{S_{\mathrm{GUT}}}(N)\right)$ is the pullback of a section of $H^{0}\left(\mathcal{O}_{\mathbb{P}_{[u_0,v_0]}\times\mathbb{P}_{[n_0,m_1,m_2]}}(N)\right)$, i.e. for the reduced image $T_{\mathrm{GUT}}\subseteq\mathbb{P}_{[u_0,v_0]}\times\mathbb{P}_{[n_0,m_1,m_2]}$ of $S_{\mathrm{GUT}}\subseteq B_3$

$$H^{0}\left(\mathcal{O}_{S_{\mathrm{GUT}}}(N)\right)=\rho^{*}H^{0}\left(\mathcal{O}_{T_{\mathrm{GUT}}}(N)\right).$$

Therefore for $m>0$ we have the commutative diagram

$$\begin{array}{ccc}
H^{0}\left(\mathcal{O}_{T_{\mathrm{GUT}}}(N-m\cdot\{m_2=0\})\right) & \hookrightarrow & H^{0}\left(\mathcal{O}_{T_{\mathrm{GUT}}}(N)\right)\\
\updownarrow= & & \updownarrow=\\
H^{0}\left(\mathcal{O}_{S_{\mathrm{GUT}}}\left(N-m\cdot\left(F_{\pm}+F_{\pm}^{opp}\right)\right)\right) & \hookrightarrow & H^{0}\left(\mathcal{O}_{S_{\mathrm{GUT}}}(N)\right)\\
\updownarrow= & & \updownarrow=\\
H^{0}\left(\mathcal{O}_{S_{\mathrm{GUT}}}(N-m\cdot F_{\pm})\right) & \hookrightarrow & H^{0}\left(\mathcal{O}_{S_{\mathrm{GUT}}}(N)\right)\\
\updownarrow= & & \updownarrow=\\
H^{0}\left(\mathcal{O}_{\Sigma_{\bar{\mathbf{5}}}^{(44)}}\left(\Sigma_{\bar{\mathbf{5}}}^{(44)}\cdot(N-m\cdot F_{\pm})\right)\right) & \hookrightarrow & H^{0}\left(\mathcal{O}_{\Sigma_{\bar{\mathbf{5}}}^{(44)}}\left(\Sigma_{\bar{\mathbf{5}}}^{(44)}\cdot N\right)\right)
\end{array}$$

where all the vertical maps are isomorphisms. Since $\mathcal{O}_{\Sigma_{\bar{\mathbf{5}}}^{(44)}}\left(\Sigma_{\bar{\mathbf{5}}}^{(44)} \cdot N\right)$ is a theta characteristic and

$$C_{u,v}^{*}\left(\mathcal{O}_{\Sigma_{\bar{\mathbf{5}}}^{(44)}}\left(\Sigma_{\bar{\mathbf{5}}}^{(44)} \cdot (m \cdot (F_{+} - F_{-}))\right)\right) = \mathcal{O}_{\Sigma_{\bar{\mathbf{5}}}^{(44)}}\left(\Sigma_{\bar{\mathbf{5}}}^{(44)} \cdot (m \cdot (F_{-} - F_{+}))\right)$$

we are exactly in the situation of Step II of Lemma 1 in [11]. When $m = 2$ the spaces on the left in the inclusion diagram are zero since no non-zero section of $H^{0}\left(\mathcal{O}_{T_{\mathrm{GUT}}}(N)\right)$ vanishes to second-order on $\{m_2 = 0\}$. Therefore the eight points of $2\left(\Sigma_{\bar{\mathbf{5}}}^{(44)} \cdot \left(F_{-} + F_{-}^{opp}\right)\right)$ impose the independent conditions required in Step II for the proof of the Lemma. We start from $m = 1$ where

$$\begin{gathered} H^{0}\left(\mathcal{O}_{\Sigma_{\bar{\mathbf{5}}}^{(44)}}\left(\Sigma_{\bar{\mathbf{5}}}^{(44)} \cdot (N + (F_{+} - F_{-}))\right)\right) = \\ H^{0}\left(\mathcal{O}_{\Sigma_{\bar{\mathbf{5}}}^{(44)}}\left(\Sigma_{\bar{\mathbf{5}}}^{(44)} \cdot (N - F_{-})\right)\right) = \\ H^{0}\left(\mathcal{O}_{S_{\mathrm{GUT}}}(N - F_{-})\right) = H^{0}\left(\mathcal{O}_{\mathbb{P}_{[u_0, v_0]}}(2)\right) \cdot m_2. \end{gathered}$$

Since $F_{-} \cdot \Sigma_{\bar{\mathbf{5}}}^{(44)} = 2$ the two points impose independent conditions on $H^{0}\left(\mathcal{O}_{S_{\mathrm{GUT}}}(N - F_{-})\right)$ so we conclude by Step II of Lemma 1 in [11] that

$$\begin{aligned} 1 &= h^{0}\left(\mathcal{O}_{\Sigma_{\bar{\mathbf{5}}}^{(44)}}\left(\Sigma_{\bar{\mathbf{5}}}^{(44)}(N - 2F_{-})\right)\right) \\ &= h^{0}\left(\mathcal{O}_{\Sigma_{\bar{\mathbf{5}}}^{(44)}}\left(\Sigma_{\bar{\mathbf{5}}}^{(44)} \cdot (N + 2(F_{+} - F_{-}))\right)\right) \end{aligned} \tag{8.1}$$

confirmed by the fact that only one section of $H^{0}\left(\mathcal{O}_{S_{\mathrm{GUT}}}(N - F_{-})\right)$, namely $v_0^2 \cdot m_2$ vanishes additionally to order 2 at the points of $\left(\Sigma_{\bar{\mathbf{5}}}^{(44)} \cap F_{-}\right)$ and that the single generator of $H^{0}\left(\mathcal{O}_{\Sigma_{\bar{\mathbf{5}}}^{(44)}}\left(\Sigma_{\bar{\mathbf{5}}}^{(44)} \cdot (N + 2(F_{+} - F_{-}))\right)\right)$ is given by

$$\frac{u_0^2}{v_0^2} m_2 \in H^{0}\left(\Sigma_{\bar{\mathbf{5}}}^{(44)}; \mathcal{L}_{\bar{\mathbf{5}}}^{(44)}\right) \tag{8.2}$$

and the single generator of $H^{0}\left(\mathcal{O}_{\Sigma_{\bar{\mathbf{5}}}^{(44)}}\left(\Sigma_{\bar{\mathbf{5}}}^{(44)} \cdot (N + 2(F_{-} - F_{+}))\right)\right)$ is given by

$$\frac{v_0^2}{u_0^2} m_2 \in H^{0}\left(\Sigma_{\bar{\mathbf{5}}}^{(44)}; C_{u,v}^{*}\left(\mathcal{L}_{\bar{\mathbf{5}}}^{(44)}\right)\right)$$

To compute the symmetric/anti-symmetric decomposition of this generator on $\check{\Sigma}_{\bar{\mathbf{5}}}^{(44)}$ in

$$\mathcal{L}_{Higgs}^{\vee,[+1]} \oplus C_{u,v}^{*}\left(\mathcal{L}_{Higgs}^{\vee,[+1]}\right) = \mathcal{L}_{Higgs}^{\vee,[+1]} \oplus \mathcal{L}_{Higgs}^{\vee,[-1]}$$

we consider local sections $(\gamma_1 \oplus \delta_1)$ and $(\gamma_2 \oplus \delta_2)$ of

$$\mathcal{L}_{Higgs} \oplus C^*_{u,v}\left(\mathcal{L}_{Higgs}\right)$$

at points p_1 and $p_2 = C_{u,v}(p_1)$ and write the expression

$$\left(\gamma_1 - C^*_{u,v}(\delta_2), \delta_1 - C^*_{u,v}(\gamma_2)\right).$$

If this expression is $(0,0)$ we have a symmetric section, that is, the pull-back of a local section of $\mathcal{L}^{\vee,[+1]}_{Higgs}$ at $p \in \check{\Sigma}^{(44)}_{\overline{\mathbf{5}}}$ with inverse image $\{p_1, p_2\}$. On the other hand, if

$$\left(\gamma_1 + C^*_{u,v}(\delta_2), \delta_1 + C^*_{u,v}(\gamma_2)\right) = (0,0)$$

the section is antisymmetric.

In our case

$$\begin{array}{ll} \gamma_1 = v_0^2 m_2 & \gamma_2 = u_0^2 m_2 \\ \delta_1 = u_0^2 m_2 & \delta_2 = v_0^2 m_2. \end{array}$$

Since $C^*_{u,v}(m_2) = -m_2$ the symmetric summand

$$\begin{aligned} \gamma_1 + C^*_{u,v}(\delta_2) &= v_0^2 m_2 + C^*_{u,v}\left(v_0^2 m_2\right) \\ &= v_0^2 m_2 + \left(-v_0^2 m_2\right) = 0 \end{aligned}$$

is zero and the unique non-zero section

$$\begin{aligned} \gamma_1 - C^*_{u,v}(\delta_2) &= v_0^2 m_2 - C^*_{u,v}\left(v_0^2 m_2\right) \\ &= v_0^2 m_2 - \left(-v_0^2 m_2\right) = 2 v_0^2 m_2, \end{aligned}$$

the image of $2 m_2 v_0^2 \in \rho^* H^0\left(\mathcal{O}_{T_{\text{GUT}}}(N)\right)$, generates the anti-symmetric summand.

Lemma 9. *We select $m = 2$ for our final refinement of the definition of the Higgs line bundle*

$$\mathcal{L}_{Higgs} = \mathcal{O}_{\mathcal{D}_5}\left(N + 2\left(F_+ - F_-\right)\right)$$

on the spectral variety $\mathcal{D}_5 = \mathcal{D}_4 \cup \mathcal{D}_1$. .There is a unique non-zero section (8.2) of the Higgs line bundle on the Higgs curve $\Sigma^{(44)}_{\overline{\mathbf{5}}}$. Its image under the direct sum decomposition

$$H^i\left(\Sigma^{(44)}_{\overline{\mathbf{5}}}; \mathcal{L}^{(44)}_{\overline{\mathbf{5}}}\right) = H^i\left(\check{\Sigma}^{(44)}_{\overline{\mathbf{5}}}; \mathcal{L}^{\vee,[+1]}_{Higgs}\right) \oplus H^i\left(\check{\Sigma}^{(44)}_{\overline{\mathbf{5}}}; \mathcal{L}^{\vee,[-1]}_{Higgs}\right) \tag{8.3}$$

lies in the anti-symmeric summand.

9. Asymptotic $\mathbb{Z}_4$ R-symmetry

9.1. Asymptotic Tate form

We next examine an 'asymptotic' $\mathbb{Z}_4$ R-symmetry for these constructions. For this we will have to lift the $\mathbb{Z}_4$-action on B_3 described in Tables 1&2 above by lifting it to a $\mathbb{Z}_4$-action on the semi-stable degeneration $W_{4,0}$ introduced in (6.7). The lifting is described by

Table 1^{asymp}:	$T_{u,v}$	$C_{u,v}$
$h^0\left(K_{B_{3,0}^{\vee}}^{-1}\right)=4$	$T_{u,v}(w)=w$ $T_{u,v}(x)=-x$ $T_{u,v}(y)=iy$	$C_{u,v}(w)=w$ $C_{u,v}(x)=x$ $C_{u,v}(y)=-y$ $C_{u,v}(z)=-z$
$(u_0+v_0)^2\cdot n_{-1}$	-1	$+1$
$(u_0-v_0)^2\cdot n_{-1}$	$+1$	$+1$
$\left(u_0^2-v_0^2\right)\cdot m_{-i}$	-1	$+1$
$\left(u_0^2-v_0^2\right)\cdot m_{+i}$	$+1$	$+1$

Table 2^{asymp}:	$T_{u,v}$	$C_{u,v}$
$h^0\left(K_{B_{3,0}^{\vee}}^{-1}\otimes\mathcal{O}_{B_{3,0}^{\vee}}(\varepsilon_{u,v})\right)=5$	$T_{u,v}(w)=w$ $T_{u,v}(x)=-x$ $T_{u,v}(y)=iy$	$C_{u,v}(w)=w$ $C_{u,v}(x)=x$ $C_{u,v}(y)=-y$ $C_{u,v}(z)=-z$
$(u_0+v_0)^2\cdot m_{-i}$	$-i$	-1
$(u_0+v_0)^2\cdot m_{+i}$	$+i$	-1
$(u_0-v_0)^2\cdot m_{-i}$	$+i$	-1
$(u_0-v_0)^2\cdot m_{+i}$	$-i$	-1
$\left(u_0^2-v_0^2\right)\cdot n_{-1}=:z_0$	$-i$	-1

In the F-theory models we are proposing, the semi-stable degeneration

$$W_{4,\delta}\Rightarrow W_{4,0}=dP_a\cup dP_b$$

to the union of two bundles of del Pezzo surfaces over B_2 we must introduce deformations

$$\begin{aligned}a_{j,\delta}&=\delta a_j+(1-\delta)\,a_{j,0}\\ z_\delta&=\delta z+(1-\delta)\,z_0\\ t_\delta&=\delta t+(1-\delta)\,t_0\end{aligned}$$

of the Tate form (1.3) such that $a_{j,0}, z_0, t_0$ all are constrained to lie in the three-dimensional $(-i)$-eigenspace for $T_{u,v}$ spanned by the $(-i)$-eigenvectors for $T_{u,v}$ given in Table 2^{asymp} and so that the action of $T_{u,v}$ on $[w, x, y]$ is also as indicated in Table 2^{asymp}. These conditions will insure the existence of an asymptotic $\mathbb{Z}_4$ **R**-symmetry as we discuss next.

First of all

$$[u_0, v_0] \mapsto [u_0, v_0] \begin{bmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{bmatrix} \tag{9.1}$$

that is

$$T_{u,v}^* \left(\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \right) = \begin{bmatrix} u_0 \circ T_{u,v} \\ v_0 \circ T_{u,v} \end{bmatrix} = \begin{bmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{bmatrix} \cdot \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$$

so that $T_{u,v}^* (u_0 \cdot v_0) = \frac{1}{2} \left(u_0^2 + v_0^2 \right)$ and $\left(T_{u,v}^* \right) \left(\frac{u_0^2+v_0^2}{2} \right) = u_0 v_0$. Next from (6.10) $T_{u,v}^* (m_1) = m_2$ and $\left(T_{u,v}^* \right) (m_2) = -m_1$.

We will require that

$$z_0 = \left(u_0^2 - v_0^2 \right) \cdot n_{-1}$$

so that

$$\begin{gathered} T_{u,v}^* (z_0) = -\left(\left(\tfrac{1+i}{2} u_0 + \tfrac{1-i}{2} v_0 \right)^2 - \left(\tfrac{1-i}{2} u_0 + \tfrac{1+i}{2} v_0 \right)^2 \right) \cdot n_{-1} \\ -\left(\tfrac{i}{2} \left(u_0^2 - v_0^2 \right) - \tfrac{i}{2} \left(-\left(u_0^2 - v_0^2 \right) \right) \right) \cdot n_{-1} = \\ -i \cdot z_0 \end{gathered}$$

As well we require that

$$a_{5,0} = u_0 v_0 \cdot m_1 + i \cdot \left(\frac{u_0^2 + v_0^2}{2} \right) m_2$$

so that

$$\begin{gathered} \left(T_{u,v}^* \right) (a_{5,0}) \\ = \left(\tfrac{u_0^2+v_0^2}{2} \right) m_2 - i \cdot u_0 v_0 \cdot m_1 \\ = -i \cdot \left(u_0 v_0 \cdot m_1 + i \cdot \left(\tfrac{u_0^2+v_0^2}{2} \right) m_2 \right) \\ = -i \cdot a_{5.0}. \end{gathered}$$

Then, referring to (7.1)

$$\begin{gathered} a_{5,\delta} \in \mathbb{C} \cdot \left(u_0 v_0 \cdot m_1 + i \cdot \left(\frac{u_0 + v_0}{2} \right) m_2 \right) \\ + \delta \cdot \left(\mathbb{C} \left(v_0^2 + u_0^2 \right) + \mathbb{C} u_0 v_0 \right) m_2 + \mathbb{C} \cdot z_\delta \end{gathered}$$

and the necessary twisting divisor $F_+ - F_-$ lie in S_{GUT} for all δ.

To achieve a $\mathbb{Z}_4$ **R**-symmetry in the limiting $W_{4,0}$ we first note that in the restriction Tate form to the Heterotic model, the sections a_j, z, $\frac{y}{x} \in H^0\left(K_{B_3}^{-1}\right)$ specialize by definition to sections

$$a_{j,0},\ z_0,\ \frac{y_0}{x_0} \in H^0\left(K_{B_2}^{-1}\right) = \mathbb{C}\cdot m_{+i} + \mathbb{C}\cdot m_{-i} + \mathbb{C}\cdot n_{-1}$$

whose variation in the fiber variable $a = \frac{u_0 - v_0}{u_0 + v_0}$ of B_3/B_2 is absorbed in the dP_9 fiber while $[n_{-1}, m_{+i}, m_{-i}]$ in (6.11) are the coordinates of the base of the Heterotic model $\tilde{V}_3/B_2$. Secondly the action of $T_{u,v}$ on $H^0\left(\mathcal{O}_{\mathbb{P}_{[u_0,v_0]}}(2)\right)$, that is, on the fibers of $B_{3,0}/B_2$ has eigenvalues $+i$, $+1$, and -1. Then , so

$$\begin{aligned} q_{+1}(u_0, v_0) &= u_0 v_0 + \tfrac{1}{2}\left(u_0^2 + v_0^2\right) \\ q_{-1}(u_0, v_0) &= u_0 v_0 - \tfrac{1}{2}\left(u_0^2 + v_0^2\right) \\ q_{+i}(u_0, v_0) &= u_0^2 - v_0^2. \end{aligned}$$

Then the $-i$-eigenspace in Table 2 is spanned by the three vectors

$$q_{+i}\cdot n_{-1},\ q_{+1}\cdot m_{-i},\ q_{-1}\cdot m_{+i}.$$

$\left(T_{u,v}^*\right)^2 (u_0 \cdot v_0) = C_{u,v}^* (u_0 \cdot v_0) = v_0 \cdot u_0$ as required for compatibility with the definition of the Higgs bundle for all δ.

Proceeding in this way we can arrange so that the action of $T_{u,v}$ on the Heterotic model is given by the action of

$$\begin{aligned} a_{j,0} \circ T_{u,v} &= (-i)\cdot a_{j,0}\ \ j = 2,3,4,5 \\ a_{0,0} &= -\left(a_{2,0} + a_{3,0} + a_{4,0} + a_{5,0}\right). \\ z_0 &= \kappa_{2,0}\cdot a_{2,0} + \kappa_{3,0}\cdot a_{3,0} + \kappa_{4,0}\cdot a_{4,0} + \kappa_{5,0}\cdot a_{5,0}. \end{aligned}$$

Notice again that these definitions imply that $T_{u,v}$ acts on the Heterotic model as $T_{u,v}$ on V_3/B_2 and as (9.1) on the dP_9-bundles.

Lemma 10. *Under the above assumptions, the specialization of the Tate form and $B_{2,\delta}$ (1.3) to $\delta = 0$ is taken to minus itself under the action of $T_{u,v}$. We will therefore say that our F-theory model satisfies an asymptotic $\mathbb{Z}_4$ **R**-symmetry.*

Proof. One checks directly using (1.3) and the eigenvalues in the $T_{u,v}$-column in the above Table 2^{asymp} that the Tate form is taken to minus itself. $\square$

Therefore $T_{u,v}$ takes the holomorphic four-form on the semi-stable limit $W_{4,0}$ to minus itself [7]. As a consequence $T_{u,v}$ acts trivially on the global section of $K_{S_{\mathrm{GUT}}}$. Furthermore $T_{u,v}$ commutes with $C_{u,v}$ since the action of $T_{u,v}^2$ coincides with the action of $C_{u,v}$. Thus $T_{u,v}$ will be our candidate for the asymptotic $\mathbb{Z}_4$ **R**-symmetry on the quotient Calabi-Yau fourfold

$$W_{4,0}^{\vee}/B_{3,0}^{\vee} := \frac{W_{4,0}/B_{3,0}}{\{C_{u,v}\}}.$$

9.2. Charges for $\mathbb{Z}_4$-R symmetry [10]

The generator $T_{u,v}$ of our asymptotic $\mathbb{Z}_4$ **R**-symmetry on $W_{4,0}$ and $W_{4,0}^{\vee}$ has the defining equations of $\Sigma_{\mathbf{10}}^{(4)}$ and $\Sigma_{\bar{\mathbf{5}}}^{(41)}$ as $-i$-eigenvectors and the defining equation of $\Sigma_{\bar{\mathbf{5}}}^{(44)}$ as (-1)-eigenvector. Therefore, since z_0 has eigenvalue $(-i)$, we apply these values in [5] to obtain the following table:

TABLE 3: $T_{u,v}$	Equation	$T_{u,v}$-charge	states
matter fields on $\frac{\Sigma_{\mathbf{10}}^{(4)}}{\{C_{u,v}\}}$	$a_5 = z = 0$	-1	$H^0\left(\frac{\Sigma_{\mathbf{10}}^{(4)}}{\{C_{u,v}\}}; \mathcal{L}_{Higgs}^{\vee,[\pm 1]}\right)$
matter fields on $\frac{\Sigma_{\bar{\mathbf{5}}}^{(41)}}{\{C_{u,v}\}}$	$a_{420} = z = 0$	-1	$H^0\left(\frac{\Sigma_{\bar{\mathbf{5}}}^{(41)}}{\{C_{u,v}\}}; \mathcal{L}_{Higgs}^{\vee,[\pm 1]}\right)$
Higgs fields on $\frac{\Sigma_{\bar{\mathbf{5}}}^{(44)}}{\{C_{u,v}\}}$	$\begin{vmatrix} a_4 & -a_5 \\ a_3 + a_0 & a_3 \end{vmatrix} = z = 0$	$+i$	$H^0\left(\frac{\Sigma_{\bar{\mathbf{5}}}^{(44)}}{\{C_{u,v}\}}; \mathcal{L}_{Higgs}^{\vee,[-1]}\right)$ $H^1\left(\frac{\Sigma_{\bar{\mathbf{5}}}^{(44)}}{\{C_{u,v}\}}; \mathcal{L}_{Higgs}^{\vee,[-1]}\right)$
bulk matter on $\frac{S_{\mathrm{GUT}}}{\{C_{u,v}\}}$	$z = 0$	$-i$	$H^2\left(K_{\frac{S_{\mathrm{GUT}}}{\{C_{u,v}\}}}\right)$

10. Conclusion

Rather than inventing a base-space B_3 and fine-tuning it to yield the right invariants for a phenomenologically consistent F-theory, we have adopted the philosophy that the representation theory required by the physics will dictate the base space for the Tate form and ensuing F-theory. Perhaps surprisingly, the representation theory contains almost completely within itself one and only one phenomenologically consistent F-theory. The detailed presentation of that model, based on the construction and analysis of the B_3 presented in this paper, is the subject of the companion paper [5].

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Department of Mathematics, The Ohio State University
Columbus, OH 43210-1174, USA
E-mail address: `clemens.43@osu.edu`
E-mail address: `raby.1@osu.edu`