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Transforming Stäckel Hamiltonians of Benenti type to polynomial form

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In this paper we discuss two canonical transformations that turn Stäckel separable Hamiltonians of Benenti type into polynomial form: transformation to Viète coordinates and transformation to Newton coordinates. Transformation to Newton coordinates has been applied to these systems only very recently and in this paper we present a new proof that this transformation indeed leads to polynomial form of Stäckel Hamiltonians of Benenti type. Moreover we present all geometric ingredients of these Hamiltonians in both Viète and Newton coordinates.

1. Introduction

The aim of this paper is to investigate two canonical transformations of the phase space to coordinates in which the so called Stäckel separable systems of Benenti type attain a polynomial form, as well as to present all geometric objects, related with such systems (the pseudo-Riemannian metric tensor and its Killing tensors as well as the conformal Killing tensor, present in the Hamiltonians of the system) in these new coordinates.

Stäckel systems constitute an important family of quadratic in momenta Hamiltonian systems that are separable, in the sense of Hamilton-Jacobi theory, in orthogonal coordinates. These systems were introduced by Paul Stäckel in [10], where he presented the conditions for separability of Hamilton-Jacobi equation of a natural Hamiltonian system (that is a system of the form $H = K + V$ where K is a quadratic in momenta form and V is a potential defined on the underlying configurational space of the system) in orthogonal coordinates, see for example [12] for a comprehensive review of this subject. Stäckel systems can most conveniently be obtained from the separation relations [11] that are linear in the Hamiltonians H_i and quadratic in momenta μ_i. Further specifications of ingredients in these separation relations lead to so called Benenti systems (see the next section for all the necessary definitions and details).

The obtained Stäckel (or Benenti) Hamiltonians H_j, as well as their geometric components, are usually given by complicated rational functions, if written in the canonical coordinates in which they were originally created through separation relations. In literature, two maps turning Benenti systems into polynomial form are known: the map to the so called Viète coordinates [2] and the map to the so called Newton coordinates [7], the second map discovered only recently.

In [7] the authors set and solved the problem of constructing an integrable polynomial hierarchy of Hamiltonian dynamical systems on C^{2N} using symmetric powers of plane algebraic curves of the form

$$\sum_{j=1}^{n} \lambda^{n-j} H_j = F(\lambda, \mu) \tag{1}$$

where F is thus an arbitrary polynomial in λ and μ. More specifically, the authors construct a canonical map that transforms these hierarchies of Hamiltonian dynamical systems to a polynomial form. In this paper we present our own proof of the polynomiality of these systems. We do it in the special, but important, case of systems - called Benenti systems - depending quadratically on the momenta variables (so that $F = \frac{1}{2} f(\lambda)\mu^2 - \varphi(\lambda)$), using geometric methods and the direct map between the Viète and Newton coordinates. We also present the explicit form of all the geometric structures that are present in the Benenti Hamiltonians in Newton coordinates. These results are new.

2. Stäckel systems

Consider a $2n$-dimensional manifold $\mathcal{M}$ equipped with a Poisson bracket π. Suppose also that $(\lambda, \mu) = (\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n)$ are global Darboux coordinates on $\mathcal{M}$ (i.e, $\{\lambda_i, \lambda_j\} = \{\mu_i, \mu_j\} = 0$ for all $i, j = 1, \ldots, n$ while $\{\lambda_i, \mu_j\} = \delta_{ij}$). A set of algebraic equations of the form

$$\varphi_i(\lambda_i, \mu_i, H_1, \ldots, H_n) = 0, \quad i = 1, \ldots, n \tag{2}$$

is called separation relations if it is globally solvable (except possibly for a union of lower dimensional submanifolds) with respect to the parameters H_i.

Among all possible separations relations (2), a natural subclass consists of the separation relations that are linear in the Hamiltonians H_k:

$$\sum_{k=1}^{n} S_{ik}(\lambda_i,\mu_i)H_k=\psi_i(\lambda_i,\mu_i),\quad i=1,\dots,n. \tag{3}$$

Here S_{ik} and ψ_i are arbitrary smooth functions of two arguments (λ_i,μ_i). The relations (3) are called the *generalized Stäckel separation relations* and the related dynamical systems, obtained by solving (3) with respect to H_k, are called the *generalized Stäckel systems.* The matrix $S=[S_{ik}(\lambda_i,\mu_i)]$ is called a *generalized Stäckel matrix.* Although the restriction to separation relations linear in H_k seems to be very strong, it appears that an overwhelming majority of all separable systems considered in the literature falls into various subclasses of this class. The most important class of systems in (3) is the class of *classical Stäckel systems*, that is systems with the matrix S being a Stäckel matrix (so that $S_{ik}=S_{ik}(\lambda_i)$) and with ψ_i being *quadratic* in momenta μ:

$$S_{ik}(\lambda_i,\mu_i)=S_{ik}(\lambda_i),\quad \psi_i(\lambda_i,\mu_i)=\frac{1}{2}f_i(\lambda_i)\mu_i^2-\varphi_i(\lambda_i),$$

so that the separation relations (3) attain the form

$$\varphi_i(\lambda_i)+\sum_{k=1}^{n} S_{ik}(\lambda_i)H_k=\tfrac{1}{2}f_i(\lambda_i)\mu_i^2,\quad i=1,\dots,n. \tag{4}$$

The relations (4) are called *Stäckel separation relations.* A particular Stäckel system is thus defined by a choice of the Stäckel matrix $S_{ik}(\lambda_i)$ and by a choice of $2n$ functions f_i and φ_i. Solving the relations (4) with respect to H_k we obtain n quadratic in momenta functions (Hamiltonians) on $\mathcal{M}$

$$H_r=\frac{1}{2}\mu^T A_r\mu+V_r(\lambda),\quad r=1,\dots,n, \tag{5}$$

where A_r are $n\times n$ matrices given by

$$A_r=\operatorname{diag}\left(f_1(\lambda_1)\left(S^{-1}\right)_{r1},\dots,f_n(\lambda_n)\left(S^{-1}\right)_{rn}\right),\quad r=1,\dots,n.$$

As the Hamiltonians (5) are defined through separation relations, they are in involution with respect to the canonical Poisson bracket on $\mathcal{M}$.

There is a natural geometric interpretation of Stäckel systems given by (5). If we factorize A_r as $A_r = K_r G$, where

$$G = A_1 = \operatorname{diag}\left(f_1(\lambda_1)\left(S^{-1}\right)_{11}, \dots, f_n(\lambda_n)\left(S^{-1}\right)_{1n}\right)$$

and

$$K_r = \operatorname{diag}\left(\frac{\left(S^{-1}\right)_{r1}}{\left(S^{-1}\right)_{11}}, \dots, \frac{\left(S^{-1}\right)_{rn}}{\left(S^{-1}\right)_{1n}}\right), \qquad r = 1, \dots, n$$

(so that $K_1 = I$) then we can interpret the matrix G as a contravariant form of a metric tensor on a manifold $\mathcal{Q}$ such that $\mathcal{M} = T^*\mathcal{Q}$ is the cotangent bundle to $\mathcal{Q}$. The corresponding covariant metric tensor will be denoted by g so that $gG = I$. It can be shown that the matrices K_r are then $(1,1)$-Killing tensors of the metric G. For a fixed Stäckel matrix S we have thus the whole family of metrics G parametrized by n arbitrary functions f_i of one variable λ_i. The tensors K_r are then Killing tensors for any metric from this family. Thus, the Stäckel Hamiltonians H_r in (5) are geodesic Hamiltonians of a Liouville integrable system in the Riemannian space $(\mathcal{M}, g)$. Further, due to the linearity of the separation relations (4), the functions $V_r(\lambda)$ on $\mathcal{Q}$ are defined by the following separation relations

$$\sum_{k=1}^{n} S_{ik}(\lambda_i) V_k = -\varphi_i(\lambda_i), \qquad i = 1, \dots, n,$$

and are called in literature separable potentials on $\mathcal{Q}$.

3. Stäckel systems of Benenti type

From now on we restrict ourselves to the case the Stäckel matrix S in (4) is of the very particular form $S_{ij} = \lambda_i^{n-j}$ or explicitly:

$$S = \begin{pmatrix} \lambda_1^{n-1} & \lambda_1^{n-2} & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_n^{n-1} & \lambda_n^{n-2} & \dots & 1 \end{pmatrix} \tag{6}$$

thus being a Vandermonde matrix. The corresponding Stäckel systems are thus defined by separation relations of the form

$$\varphi_i(\lambda) + \sum_{j=1}^{n} \lambda_i^{n-j} H_j = \frac{1}{2} f_i(\lambda_i)\mu_i^2 \quad i = 1, \dots, n, \tag{7}$$

and are called in literature *Benenti systems*. Benenti systems have been studied much in literature recently, see for example [1, 3] and references therein.

The inverse of S as given by (6) is given by the following lemma.

Lemma 1. *If S is the $n \times n$ Vandermonde matrix given by $S_{ij} = \lambda_i^{n-j}$ then*

$$\left[S^{-1}\right]_{ij} = -\frac{1}{\Delta_j}\frac{\partial \rho_i}{\partial \lambda_j},$$

where

$$\rho_i = (-1)^i \sigma_i(\lambda), \quad \Delta_j = \prod_{k \neq j} (\lambda_j - \lambda_k)$$

and where $\sigma_r(\lambda)$ are elementary symmetric polynomials.

By definition

$$\sigma_i(\lambda) = \sum_{1 \leq j_1 < \ldots < j_i \leq n} \lambda_{j_1} \ldots \lambda_{j_i}, \quad i = 1, \ldots, n,$$

so that

$$\sigma_0 = 1, \ \ \sigma_1 = \sum_{i=1}^{n} \lambda_i, \ \ \sigma_2 = \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j, \ \ \ldots, \ \ \sigma_n = \prod_{i=1}^{n} \lambda_i.$$

Lemma 1 can be proved by a direct calculation. By this lemma, solving (7) with respect to H_r yields n functions (Hamiltonians) H_r on $\mathcal{M}$

$$\begin{aligned} H_r &= -\frac{1}{2}\sum_{i=1}^{n} \frac{\partial \rho_r}{\partial \lambda_i}\frac{f_i(\lambda_i)\mu_i^2}{\Delta_i} + V_r(\lambda) \\ &\equiv \frac{1}{2}\mu^T K_r G \mu + V_r(\lambda), \qquad r = 1, \ldots, n \end{aligned} \tag{8}$$

called *Benenti Hamiltonians*. Thus, for Benenti Hamiltonians the metric tensor G is given by

$$G = \operatorname{diag}\left(\frac{f_1(\lambda_1)}{\Delta_1}, \ldots, \frac{f_n(\lambda_n)}{\Delta_n}\right)$$

while the Killing tensors K_r are given by

$$K_r = -\operatorname{diag}\left(\frac{\partial \rho_r}{\partial \lambda_1}, \ldots, \frac{\partial \rho_r}{\partial \lambda_n}\right) \quad r = 1, \ldots, n. \tag{9}$$

From now and in what follows, we further assume that all f_i are equal, and likewise all φ_i:

$$f_i := f, \quad \varphi_i := \varphi \tag{10}$$

so that all the Hamiltonians (8) are generated by the single *separation curve*:

$$\varphi(\lambda) + \sum_{j=1}^{n} \lambda^{n-j} H_j = \frac{1}{2} f(\lambda) \mu^2 \tag{11}$$

and are given explicitly by:

$$\begin{aligned} H_r &= -\frac{1}{2} \sum_{i=1}^{n} \frac{\partial \rho_r}{\partial \lambda_i} \frac{f(\lambda_i) \mu_i^2}{\Delta_i} + V_r(\lambda) \\ &\equiv \frac{1}{2} \mu^T K_r G \mu + V_r(\lambda), \qquad r = 1, \ldots, n \end{aligned} \tag{12}$$

and thus the metric tensor G is now given by

$$G = \operatorname{diag}\left(\frac{f(\lambda_1)}{\Delta_1}, \ldots, \frac{f(\lambda_n)}{\Delta_n} \right).$$

Of particular interest is the case $f(\lambda_i) = \lambda_i^m$ with $m \in \mathbb{Z}$. In such a case the metric tensor G will be denoted by G_m:

$$G_m = \operatorname{diag}\left(\frac{\lambda_1^m}{\Delta_1}, \ldots, \frac{\lambda_n^m}{\Delta_n} \right), \; m \in \mathbb{Z}.$$

Of course, if f is a Laurent polynomial

$$f(\lambda) = \sum_{\alpha \in A} a_\alpha \lambda_i^\alpha, \tag{13}$$

where $A \subset \mathbb{Z}$ is a finite set, then

$$G = \sum_{\alpha \in A} a_\alpha G_\alpha. \tag{14}$$

It can be shown that the metric G_m is flat for $m \in \{0, \ldots, n\}$ and of constant curvature for $m = n + 1$ (the same is true for f being a polynomial in λ of order m). Moreover

$$G_m = L^m G_0, \;\; G_0 = \operatorname{diag}\left(\frac{1}{\Delta_1}, \ldots, \frac{1}{\Delta_n}\right), \tag{15}$$

where

$$L = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$$

is a $(1,1)$-tensor called *special conformal Killing tensor* [8]. It can be shown [5] that all K_r can be calculated from the formula

$$K_1 = I, \quad K_r = \sum_{k=0}^{r-1} \rho_k L^{r-1-k}, \quad r = 2, \ldots, n. \tag{16}$$

In order to illustrate the form of separable potentials $V_r(\lambda)$ in the Benenti case, we further assume that φ is a Laurent sum of the form

$$\varphi(\lambda) = \sum_{\alpha \in A} c_\alpha \lambda_i^\alpha, \tag{17}$$

where $A \subset \mathbb{Z}$ is a finite set and c_α are some real constants. The Benenti separation relations (7) become

$$\sum_{\alpha \in A} c_\alpha \lambda_i^\alpha + \sum_{j=1}^{n} \lambda_i^{n-j} H_j = \frac{1}{2} f(\lambda_i) \mu_i^2, \quad i = 1, \ldots, n \tag{18}$$

and due to their linearity we have

$$V_r = \sum_{\alpha \in A} c_\alpha V_r^{(\alpha)},$$

where $V_r^{(\alpha)}$ are so called *basic separable potentials*. By linearity of (18), the potentials $V_r^{(\alpha)}$ satisfy the relations

$$\lambda_i^\alpha + \sum_{r=1}^{n} V_r^{(\alpha)} \lambda_i^{n-r} = 0, \quad i = 1, \ldots, n$$

and, again by Lemma 1, they are given by

$$V_r^{(\alpha)} = \sum_{i=1}^{n} \frac{\partial \rho_r}{\partial \lambda_i} \frac{\lambda_i^\alpha}{\Delta_i}, \quad r = 1, \ldots, n.$$

The basic separable potentials $V_r^{(\alpha)}$ can be explicitly constructed by the following formula [5]:

$$V^{(\alpha)} = R^{\alpha} V^{(0)}, \qquad V^{(\alpha)} = (V_1^{(\alpha)}, \ldots, V_n^{(\alpha)})^T, \tag{19}$$

where

$$R = \begin{pmatrix} -\rho_1 & 1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ \vdots & 0 & 0 & 1 \\ -\rho_n & 0 & 0 & 0 \end{pmatrix} \tag{20}$$

and $V^{(0)} = (0, \ldots, 0, -1)^T$. The first n basic potentials are trivial

$$V_k^{(\alpha)} = -\delta_{k,n-\alpha}, \qquad \alpha = 0, \ldots, n-1.$$

The first nontrivial positive potential is

$$V^{(n)} = (\rho_1, \ldots, \rho_n)^T$$

and higher potentials are more complicated polynomials in q_i. The first negative potential is

$$V^{(-1)} = \left(\frac{1}{\rho_n}, \ldots, \frac{\rho_{n-1}}{\rho_n} \right)^T$$

and the higher negative potentials are more complicated rational functions of all ρ_i. Note also that the recursion formulas (19)-(20) are not tensorial; they look the same in any coordinate system.

4. Polynomial form of Benenti systems

As we saw in the previous section, even the relatively simple Benenti Hamiltonians are complicated rational functions when expressed in the separation variables (λ, μ). In this section we demonstrate two canonical maps that under certain conditions transform Benenti Hamiltonians (12) to a polynomial form.

4.1. Benenti systems in Viète coordinates

Suppose that we change the position coordinates on the base manifold $\mathcal{Q}$ through the map

$$q_i = \rho_i(\lambda) \quad i = 1, \ldots, n, \tag{21}$$

where, as in Lemma 1, $\rho_i(\lambda) = (-1)^i \sigma_i(\lambda)$. This map induces the map (point transformation) on $T^*\mathcal{Q}$:

$$p = \left(J_V^{-1}\right)^T \mu, \tag{22}$$

where J_V is the Jacobian of the map (21):

$$(J_V)_{ij} = \frac{\partial \rho_i}{\partial \lambda_j}. \tag{23}$$

Let us find an explicit form of (22). To do this we need the following lemma.

Lemma 2. *Denote by k_i the i-th column of an $n \times n$ nondegenerate matrix A:*

$$A = (k_1|k_2|\ldots|k_n)$$

and by r_j the j-th row of its inverse

$$A^{-1} = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}.$$

Then, if $\alpha_i \in \mathbb{R}$ for $i = 1, \ldots, n$

$$(\alpha_1 k_1|\alpha_2 k_2|\ldots|\alpha_n k_n)^{-1} = \begin{pmatrix} r_1/\alpha_1 \\ r_2/\alpha_2 \\ \vdots \\ r_n/\alpha_n \end{pmatrix}.$$

This elementary lemma follows from the fact that $r_i k_j = \delta_{ij}$. An analogous lemma is of course true if we consider rows of A instead of its columns.

Combining lemmas 1 and 2 we obtain that

$$\left(J_V^{-1}\right)_{ij} = -\frac{\lambda_i^{n-j}}{\Delta_i} \tag{24}$$

and thus the map (22) can be written as

$$p_i = \ -\sum_{k=1}^{n} \frac{\lambda_k^{n-i}}{\Delta_k}\mu_k, \quad i = 1, \ldots, n. \tag{25}$$

The coordinates (q, p) defined by (21) and (25) are called *Viète coordinates*. To summarize, the map $(\lambda, \mu) \to (q, p)$ from separation coordinates to Viète coordinates is given by

$$q_i = \rho_i(\lambda), \quad p_i = -\sum_{k=1}^{n} \frac{\lambda_k^{n-i}}{\Delta_k}\mu_k, \quad i = 1, \ldots, n. \tag{26}$$

Being a point transformation, the map (26) is a canonical map which means that Viète coordinates are Darboux (canonical) coordinates as well:

$$\{q_i, q_j\} = \{p_i, p_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}.$$

Let us now investigate the structure of Benenti Hamiltonians (12) in Viète coordinates (q, p). The Hamiltonians (12) are of course written in tensor form so that in Viète coordinates

$$H_r(q, p) = \frac{1}{2}p^T K_r(q)G(q)p + V_r(q), \qquad r = 1, \ldots, n \tag{27}$$

where, by transformation laws for tensors,

$$K_r(q) = J_V K_r \left(J_V\right)^{-1}, \ \ G(q) = J_V G \left(J_V\right)^T. \tag{28}$$

The first formula in (28) yields, after some calculation

$$(K_r(q))^i_j = \begin{cases} q_{i-j+r-1}, & i \leq j \text{ and } r \leq j \\ \\ -q_{i-j+r-1}, & i > j \text{ and } r > j \\ \\ 0 & \text{otherwise} \end{cases}. \tag{29}$$

Here and throughout the whole section we use the convention that $q_0 = 1$ and $q_k = 0$ for $k < 0$ and for $k > n$. Thus, all the $K_r(q)$ are linear in q-variables.

Further, for the monomial case $f(\lambda_i) = \lambda_i^m$ with $m \in \{0, \ldots, n+1\}$ we can obtain from the second formula in (28) that

(30)

$$
\begin{aligned}
G_m^{ij}(q) &= \begin{cases} q_{i+j+m-n-1}, & i,j = 1,\ldots,n-m \\ -q_{i+j+m-n-1}, & i,j = n-m+1,\ldots,n \qquad m = 0,\ldots,n \\ 0 \quad \text{otherwise} & \end{cases} \\
G_m^{ij}(q) &= q_i q_j - q_{i+j}, \qquad i,j = 1,\ldots,n, \qquad m = n+1.
\end{aligned}
$$

The formulas (29) and (30) can alternatively be obtained with the help of the special conformal Killing tensor L by using the formulas (16) and (15), respectively, and the fact that the tensor L can be easily calculated in Viète coordinates through tensor transformation law $L(q) = J_V L\,(J_V)^{-1}$. We obtain

$$L_j^i(q) = -\delta_j^1 q_i + \delta_j^{i+1}$$

that is

$$
(31) \qquad L(q) = \begin{pmatrix} -q_1 & 1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ \vdots & 0 & 0 & 1 \\ -q_n & 0 & 0 & 0 \end{pmatrix}.
$$

Note therefore that L happens to have the same form in q-coordinates as the recursion matrix (20). This seems to be a pure coincidence without any deeper meaning; we stress again that R in (20) is not a tensor. In any case, due to the fact that all the entries in L are linear in q_i we see that all the entries in G_m are linear in q_i for $m = 0, \ldots, n+1$, quadratic in q_i for $m = n+1$ and higher order polynomials for higher m. Moreover, by (29), all entries in $K_r(q)$ are linear in q_i. Using all these facts and the formula (14) we obtain the following important corollary:

Corollary 3. *If f is a polynomial in (13), then the geodesic parts of Benenti Hamiltonians (27) have a polynomial form. Moreover, if the right hand side of (17) is a polynomial, then by the recursive relations (19)-(20) also the potentials V_r in the Benenti Hamiltonians (12) are in this case polynomials in q_i. Thus, in such a case, the whole Hamiltonians $H_r(q,p)$ (and not just their geodesic parts) are polynomials.*

Example 4. Consider the case $n = 2$, $f(\lambda) = 1$ (i.e. a purely monomial situation with $m = 0$ in (30), so that $G = G_0$) and $\varphi(\lambda) = \lambda^3$. Then the separation curve (18) becomes

$$\lambda^3 + \lambda H_1 + H_2 = \frac{1}{2}\mu^2$$

and yields the Hamiltonians H_i in the explicit form

$$\begin{aligned} H_1 &= \frac{1}{2(\lambda_1 - \lambda_2)}(\mu_1^2 - \mu_2^2) - (\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2) \\ H_2 &= \frac{1}{2(\lambda_1 - \lambda_2)}(\lambda_1\mu_2^2 - \lambda_2\mu_1^2) + \lambda_1\lambda_2(\lambda_1 + \lambda_2) \end{aligned}$$

so both Hamiltonians are rational functions of separation coordinates (λ, μ). The above Hamiltonians have exactly the form (12) with the metric

$$G = G_0 = \operatorname{diag}\left(\frac{1}{\Delta_1}, \frac{1}{\Delta_2}\right),$$

and with the Killing tensors (9) given explicitly by:

$$K_1 = I, \;\; K_2 = -\operatorname{diag}(\lambda_2, \lambda_1).$$

The map (26) to Viète coordinates has the explicit form:

$$\begin{aligned} &q_1 = -(\lambda_1 + \lambda_2), \quad q_2 = \lambda_1\lambda_2, \\ &p_1 = \frac{1}{\lambda_2 - \lambda_1}(\lambda_1\mu_1 - \lambda_2\mu_2), \qquad p_2 = \frac{1}{\lambda_2 - \lambda_1}(\mu_1 - \mu_2). \end{aligned}$$

An elementary calculations shows that H_i in these variables attain the form

$$\begin{aligned} H_1(q,p) &= \frac{1}{2}q_1p_2^2 + p_1p_2 - q_1^2 + q_2 \\ H_2(q,p) &= \frac{1}{2}p_1^2 + q_1p_1p_2 + \frac{1}{2}q_1^2p_2^2 - \frac{1}{2}p_2^2q_2 - q_1q_2 \end{aligned}$$

which is in agreement with (30) and (29). Explicitly:

$$G_0(q) = \begin{pmatrix} 0 & 1 \\ 1 & q_1 \end{pmatrix}, \; K_1(q) = I, \;\; K_2(q) = \begin{pmatrix} 0 & 1 \\ -q_2 & q_1 \end{pmatrix}.$$

Thus, the Hamiltonians H_r become polynomial in Viète coordinates (q, p).

Example 5. Consider the case $n = 3$, $f(\lambda) = \lambda$ (so that $m = 1$ in (30) and thus $G = G_1$) and $\varphi(\lambda) = \lambda^5$. Then the separation curve (18) becomes

$$\lambda^5 + \lambda^2 H_1 + \lambda H_2 + H_3 = \frac{1}{2}\lambda\mu^2.$$

Solving the corresponding separation coordinates yields the Benenti Hamiltonians (12) with the metric $G_1 = LG_0$ with

$$L = \operatorname{diag}\left(\lambda_1, \lambda_2, \lambda_3\right)$$

so that

$$G_1 = LG_0 = \operatorname{diag}\left(\frac{\lambda_1}{\Delta_1}, \frac{\lambda_2}{\Delta_2}, \frac{\lambda_3}{\Delta_3}\right)$$

and with the Killing tensors (9) given explicitly by

$$\begin{aligned} &K_1 = I, \quad K_2 = \operatorname{diag}\left(\lambda_2 + \lambda_3, \lambda_1 + \lambda_3, \lambda_1 + \lambda_2\right), \\ &K_3 = -\operatorname{diag}\left(\lambda_2\lambda_3, \lambda_1\lambda_3, \lambda_1\lambda_2\right), \end{aligned}$$

while the potentials $V_r = V_r^{(5)}$ have the form

$$\begin{aligned} V_1^{(5)} &= \lambda_1^3 + \lambda_2^3 + \lambda_3^3 + \lambda_1^2\lambda_2 + \lambda_1^2\lambda_3 + \lambda_1\lambda_2^2 + \lambda_1\lambda_3^2 + \lambda_2^2\lambda_3 + \lambda_2\lambda_3^2 + \lambda_1\lambda_2\lambda_3, \\ V_2^{(5)} &= \lambda_1^3\lambda_2 + \lambda_1^3\lambda_3 + \lambda_1^2\lambda_2^2 + 2\lambda_1^2\lambda_2\lambda_3 + \lambda_1^2\lambda_3^2 + \lambda_1\lambda_2^3 + 2\lambda_1\lambda_2^2\lambda_3 \\ &\quad + 2\lambda_1\lambda_2\lambda_3^2 + \lambda_1\lambda_3^3 + \lambda_2^3\lambda_3 + \lambda_2^2\lambda_3^2 + \lambda_2\lambda_3^3, \\ V_3^{(5)} &= \lambda_1\lambda_2\lambda_3\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3\right). \end{aligned}$$

The map (26) to Viète coordinates has now the form

$$q_1 = -(\lambda_1 + \lambda_2 + \lambda_3), \quad q_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3, \quad q_3 = -\lambda_1\lambda_2\lambda_3$$

and

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \left(J_V^{-1}\right)^T \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}$$

with J_V and J_V^{-1} given by (4.1) and (24) respectively. Explicitly

$$J_V = \begin{pmatrix} -1 & -1 & -1 \\ \lambda_2 + \lambda_3 & \lambda_1 + \lambda_3 & \lambda_1 + \lambda_2 \\ -\lambda_2\lambda_3 & -\lambda_1\lambda_3 & -\lambda_1\lambda_2 \end{pmatrix}$$

and

$$J_V^{-1} = -\begin{pmatrix} \frac{\lambda_1^2}{\Delta_1} & \frac{\lambda_1}{\Delta_1} & \frac{1}{\Delta_1} \\ \frac{\lambda_2^2}{\Delta_2} & \frac{\lambda_2}{\Delta_2} & \frac{1}{\Delta_2} \\ \frac{\lambda_2^2}{\Delta_3} & \frac{\lambda_2}{\Delta_3} & \frac{1}{\Delta_3} \end{pmatrix}.$$

An elementary calculation shows that H_i in these variables attain the form

$$\begin{aligned} H_1(q,p) &= \frac{1}{2}q_1p_2^2 + p_1p_2 - \frac{1}{2}q_3p_3^2 + q_1^3 - 2q_1q_2 + q_3, \\ H_2(q,p) &= \frac{1}{2}p_1^2 - \frac{1}{2}q_2p_2^2 - \frac{1}{2}q_1q_3p_3^2 + q_1p_1p_2 - q_3p_2p_3 + q_1^2q_2 - q_1q_3 - q_2^2, \\ H_3(q,p) &= -\frac{1}{2}q_3p_2^2 - \frac{1}{2}q_2q_3p_3^2 - q_3p_1p_3 - q_1q_3p_2p_3 + q_1^2q_3 - q_2q_3, \end{aligned}$$

which is in agreement with (30) and (29). Explicitly:

$$\begin{aligned} G_0(q) &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & q_1 \\ 1 & q_1 & q_2 \end{pmatrix}, \, K_1(q) = I, \, K_2(q) = \begin{pmatrix} 0 & 1 & 0 \\ -q_2 & q_1 & 1 \\ -q_3 & 0 & q_1 \end{pmatrix}, \\ K_3(q) &= \begin{pmatrix} 0 & 0 & 1 \\ -q_3 & 0 & q_1 \\ 0 & -q_3 & q_2 \end{pmatrix}, \end{aligned}$$

while the tensor L attains the form as in (31):

$$L(q) = \begin{pmatrix} -q_1 & 1 & 0 \\ -q_2 & 0 & 1 \\ -q_3 & 0 & 0 \end{pmatrix}.$$

Note again that the Hamiltonians H_r become polynomial in Viète coordinates (q,p).

4.2. Benenti systems in Newton coordinates

The second method of turning Benenti Hamiltonian systems (12) into a polynomial form is by using Newton coordinates. A general method, involving Newton coordinates, suitable for arbitrary algebraic separation curve of the form (1), has been discovered by V. M. Buchstaber and A. V. Mikhailov [7] only quite recently. In this section we present our own proof the polynomiality result for the important case of Benenti Hamiltonians (12), i.e. Hamiltonians generated from the Benenti separation curve (11), using a geometric approach. We also investigate in detail the structure of Benenti Hamiltonians (12) in Newton coordinates.

Consider the following map (consisting of a sequence of Newton polynomials) on the base manifold $\mathcal{Q}$:

$$Q_i = \frac{1}{i}\sum_{s=1}^{n} \lambda_s^i. \tag{32}$$

This map induces the map on $T^*\mathcal{Q}$:

$$P = \left(J_N^{-1}\right)^T \mu, \tag{33}$$

where $P = (P_1, \ldots, P_n)^T$ and J_N is the Jacobian of the map (32),

$$(J_N)_{ij} = \frac{\partial Q_i}{\partial \lambda_j} = \lambda_j^{i-1}.$$

Thus, $J_N = V^T$, where V is the Vandermonde matrix, but different from S:

$$V = \begin{pmatrix} 1 & \lambda_1 & \ldots & \lambda_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & & \lambda_n^{n-1} \end{pmatrix}. \tag{34}$$

This also means that (33) leads to $P = V^{-1}\mu$.

Lemma 6. *In the above notation*

$$\left(V^{-1}\right)_{ij} = -\frac{1}{\Delta_j}\frac{\partial \rho_{n-i+1}}{\partial \lambda_j}.$$

The reader should compare this lemma with Lemma 1. Thus, the map (32) induces the following map on $T^*\mathcal{Q}$

$$Q_i = \frac{1}{i}\sum_{s=1}^{n} \lambda_s^i, \;\; P_i = -\sum_{j=1}^{n} \frac{1}{\Delta_j}\frac{\partial \rho_{n-i+1}}{\partial \lambda_j}\mu_j, \;\; i = 1, \ldots, n \tag{35}$$

and we call the coordinates (Q, P) *Newton coordinates* on $\mathcal{M}$. The reader should compare this map with the map (26). Again, since the map $(\lambda, \mu) \to (Q, P)$ is a point transformation map on $T^*\mathcal{Q}$, the Newton coordinates

(Q, P) are Darboux (canonical) coordinates, that is

$$\{Q_i, Q_j\} = \{P_i, P_j\} = 0, \quad \{Q_i, P_j\} = \delta_{ij}.$$

Let us now investigate the structure of Benenti Hamiltonians (12) in (Q, P)-coordinates. The Hamiltonians (12) are written in tensor form and thus

$$H_r(Q, P) = \frac{1}{2} P^T K_r(Q) G(Q) P + V_r(Q), \qquad r = 1, \ldots, n. \tag{36}$$

In the monomial case, i.e., when $f(\lambda) = \lambda^m$ we have

$$H_r(Q, P) = \frac{1}{2} P^T K_r(Q) L^m(Q) G_0(Q) P + V_r(Q), \quad r = 1, \ldots, n. \tag{37}$$

Let us now investigate the structure of (36) and in particular (37), in Newton coordinates. Due to tensor transformation laws, $L(Q)$, $K_r(Q)$ and $G(Q)$ are given by

$$L(Q) = J_N L \left(J_N\right)^{-1}, \quad K_r(Q) = J_N K_r \left(J_N\right)^{-1} \tag{38}$$

and by

$$G(Q) = J_N G \left(J_N\right)^T. \tag{39}$$

In order to express explicitly the right hand sides of (38) and (39) we need to invert the map $\lambda \to Q$ given by (32), which is in general not algebraically invertible. Let us thus consider the map $q \to Q$ between the Viète coordinates (26) and the Newton coordinates. In the recent paper [4] it is proven that this map is given by

$$Q_r = -\frac{1}{r} \sum_{k=1}^{r} V_k^{(n+r-k)}(q), \quad r = 1, \ldots, n, \tag{40}$$

where $V_k^{(\alpha)}(q)$ are the basic separable potentials as given by (19)-(20). Below we present a theorem in which we extend the understanding of the above formula.

Theorem 7. *The map $q \to Q$ as given by (40) has the following form:*

$$Q_r(q) = -q_r + \tau_r^{(r-1)}(q_1, \ldots, q_{r-1}), \quad r = 1, \ldots, n, \tag{41}$$

where $\tau_r^{(\alpha)}$ denotes a polynomial of order α and where $\tau_1^{(0)} = 0$. The map $q \to Q$ is algebraically invertible, with the inverse map of the form

$$q_r(Q) = -Q_r + \eta_r^{(r-1)}(Q_1, \ldots, Q_{r-1}), \quad r = 1, \ldots, n, \tag{42}$$

where $\eta_r^{(\alpha)}$ denotes a polynomial of order α with $\eta_1^{(0)} = 0$. Moreover, neither $\tau_r^{(\alpha)}$ nor $\eta_r^{(\alpha)}$ depend on n.

One proves this theorem by direct calculations, using the properties of basic separable potentials $V_k^{(\alpha)}$. This theorem can also be deduced from the well-known Newton's identities (Girard-Newton formulae) between elementary symmetric polynomials $\sigma_r(\lambda)$ and the r-th power sums $\sum_{s=1}^{n} \lambda_s^r$ (see for example [9], Exercise 8, Ch.1., Sec.2, on p. 28). This theorem means that both the map $q \to Q$ and its inverse $Q \to q$ are polynomial maps and moreover that the transformation between the first n variables, i.e. between $q_1, \ldots, q_n$ and $Q_1, \ldots, Q_n$, does not change after increasing n to $n+1$. Expicitly, the first few expressions in both maps are

$$\begin{aligned}
Q_1 &= -q_1, \\
Q_2 &= -q_2 + \frac{1}{2} q_1^2, \\
Q_3 &= -q_3 - \frac{1}{3} q_1^3 + q_2 q_1, \\
Q_4 &= -q_4 + \frac{1}{4} q_1^4 - q_1^2 q_2 + q_3 q_1 + \frac{1}{2} q_2^2 \\
&\ \vdots
\end{aligned}$$

for the map $Q \to q$ and

$$\begin{aligned}
q_1 &= -Q_1, \\
q_2 &= -Q_2 + \frac{1}{2}Q_1^2, \\
q_3 &= -Q_3 - \frac{1}{6}Q_1^3 + Q_2Q_1, \\
q_4 &= -Q_4 + \frac{1}{24}Q_1^4 - \frac{1}{2}Q_1^2Q_2 + Q_3Q_1 + \frac{1}{2}Q_2^2 \\
&\vdots
\end{aligned}$$

for the inverse map $q \to Q$. These maps are simply Newton's identities but written in the variables q_r and Q_r rather than, as it is usually done, in the variables σ_r and $\sum_{s=1}^{n} \lambda_s^r$. It is now possible to calculate the tensor L in the Newton coordinates Q. After some calculations we obtain:

(43)
$$L(Q) = J_N L \left(J_N\right)^{-1} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ -q_n(Q) & -q_{n-1}(Q) & -q_{n-2}(Q) & \dots & -q_1(Q) \end{pmatrix},$$

or, equivalently

$$L(Q)_j^i = -q_{n-j+1}(Q)\delta_n^i + \delta_{j-1}^i, \quad i,j = 1,\dots,n,$$

where the functions $q_i(Q)$ are given by (42). Thus, the entries of $L(Q)$ are polynomials, and the same is of course true for any positive power $L^m(Q)$ of $L(Q)$.

Let us now calculate the Killing tensors K_r in Newton coordinates Q. We will do it by transforming $K_r(q)$, as given by (29), to Q variables, by the formula $K_r(Q) = J_{VN}K_r(q)\left(J_{VN}\right)^{-1}$, where J_{VN} is the Jacobian transformation from Viète coordinates to Newton coordinates. First we find that

$$(J_{VN})_{i,j} = \sum_{s=0}^{n} q_s (J_{VN})_{i-s,j} - q_1 q_{i-s} + q_i, \quad i,j = 1,\dots,n,$$

with $(J_{VN})_{1,j} = -1$, $(J_{VN})_{2,j} = q_1$, $(J_{VN})_{3,j} = -q_1^2 + q_2$ for any fixed j. This also yields that

$$(J_{VN})_{i,j}^{-1} = -q_{i-j}, \quad i,j = 1,\dots,n.$$

Note that this last result also means that the map (41) can now be extended to the whole manifold $\mathcal{M} = T^*\mathcal{Q}$ by completing it with the map between the canonical momenta:

$$P_i = \left[(J_{VN})^{-1} \right]^T_{ij} p_j = -\sum_{j=1}^{n} q_{j-i} p_j, \quad i = 1, ..., n. \tag{44}$$

After some calculations we obtain that

$$(K_r(Q))^i_j = \begin{cases} q_{i-j+r-1}(Q), & i-j \leq 0 \text{ and } r \leq n-i+1 \\ -q_{i-j+r-1}(Q), & i-j > 0 \text{ and } r > n-i+1 \\ 0 \quad \text{otherwise} \end{cases}, \tag{45}$$

cf. (29). Thus, since all $q_i(Q)$ by (42) are polynomials then all the entries of $K_r(Q)$ are polynomials in Q_i as well. Finally, let us consider $G_0(Q)$, i.e. the metric G_0 in Newton coordinates, by transforming $G_0(q)$, as given by (30), into Q variables, by the transformation formula $G_0(Q) = J_{VN} G_0(q) \left(J_{VN}\right)^T$.

Lemma 8. *The metric G_0 in Newton coordinates (32) attains the form of a lower-triangular Hankel matrix given by the recursive formulas*

$$G_0(Q)_{i,j} = \begin{cases} -\sum\limits_{s=1}^{n} q_s(Q) \left(G_0\right)_{i-s,j} + q_1(Q) q_{i-1}(Q) - q_i(Q), & i \geq j \\ 0, & i < j \end{cases} \tag{46}$$
$$\text{for } i, j = 3, \ldots, n,$$

with $G_0(Q)_{1,j} = 1, G_0(Q)_{2,j} = -q_1$, and $G_0(Q)_{3,j} = q_1^2 - q_2$ for arbitrary fixed j.

As a consequence, the metric $G_m(Q)$ also attains the form of a lower-triangular Hankel matrix. This can be verified using induction with respect to m in

$$G_m(Q)_{i,j} = L(Q)^i_j G_{m-1}(Q)_{i,j}.$$

Taking into account the formulas (43), (45) and Lemma 8 we obtain a corollary that is an analogue of Corollary 3 in case of Newton coordinates.

Corollary 9. *If f in (13) is a polynomial, then the geodesic parts of Benenti Hamiltonians $H_r(Q, P)$ in (36) have in Newton coordinates (35) a* *polynomial form. Moreover, if the right hand side of (17) is a pure polynomial, then the potentials $V_r(Q)$ in the Benenti Hamiltonians (36) are in this case also polynomials. Thus, in such a case, all the Hamiltonians $H_r(Q,P)$ (and not just their geodesic parts) are polynomials.*

Let us now present some examples.

Example 10. We proceed in the same setting as in Example 4, i.e. we consider the case $n=2$, $f(\lambda)=1$ (so that $m=0$) and $\varphi(\lambda)=\lambda^3$, but in Newton coordinates. The map (41)-(44) reads now

$$\begin{gathered} Q_1=-q_1, \quad Q_2=\frac{1}{2}q_1^2-q_2, \\ P_1=-p_1-q_1p_2, \quad P_2=-p_2 \end{gathered}$$

and it transforms the Hamiltonians from Example 26 to the form

$$\begin{aligned} H_1(Q,P) &= \frac{1}{2}P_2^2Q_1+P_1P_2-Q_2-\frac{1}{2}Q_1^2, \\ H_2(Q,P) &= -\frac{1}{4}P_2^2Q_1^2+\frac{1}{2}P_2^2Q_2+\frac{1}{2}P_1^2+\frac{1}{2}Q_1^3-Q_1Q_2, \end{aligned}$$

which is in agreement with (46) and (45). Explicitly:

$$G_0(Q)=\begin{pmatrix} 0 & 1 \\ 1 & Q_1 \end{pmatrix}, \; K_1(Q)=I, \; K_2(Q)=\begin{pmatrix} -Q_1 & 1 \\ Q_2-\frac{1}{2}Q_1^2 & 0 \end{pmatrix}.$$

Moreover, L becomes

$$L(Q)=\begin{pmatrix} 0 & 1 \\ Q_2-\frac{1}{2}Q_1^2 & Q_1 \end{pmatrix}.$$

Example 11. We now consider Example 5 in Newton coordinates i.e. the case $n=3$, $m=1$ and $\varphi(\lambda)=\lambda^5$. As $n=3$ the map (41) is now

$$Q_1=-q_1, \quad Q_2=\frac{1}{2}q_1^2-q_2, \quad Q_3=-\frac{1}{3}q_1^3+q_1q_2-q_3, \tag{47}$$

and its inverse (42) is

$$q_1=-Q_1, \quad q_2=\frac{1}{2}Q_1^2-Q_2, \quad q_3=-\frac{1}{6}Q_1^3+Q_1Q_2-Q_3.$$

The map (44) between momenta is

$$
\begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} = \begin{pmatrix} -1 & -q_1 & -q_2 \\ 0 & -1 & -q_1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}, \tag{48}
$$

with the inverse

$$
\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} -1 & -Q_1 & -\frac{1}{2}Q_1^2 - Q_2 \\ 0 & -1 & -Q_1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}.
$$

The map (47)-(48) transforms the Hamiltonians $H_r(q,p)$ in Example 5 to the form

$$
\begin{aligned}
H_1(Q,P) &= \frac{1}{2}P^T \begin{pmatrix} 0 & 1 & Q_1 \\ 1 & Q_1 & \frac{1}{2}Q_1^2 + Q_2 \\ Q_1 & \frac{1}{2}Q_1^2 + Q_2 & Q_3 + Q_1Q_2 + \frac{1}{6}Q_1^3 \end{pmatrix} P + V_1^{(5)}(Q) \\
H_2(Q,P) &= \frac{1}{2}P^T \begin{pmatrix} 1 & 0 & Q_2 - \frac{1}{2}Q_1^2 \\ 0 & Q_2 - \frac{1}{2}Q_1^2 & Q_3 - \frac{1}{3}Q_1^3 \\ Q_2 - \frac{1}{2}Q_1^2 & Q_3 - \frac{1}{2}Q_1^3 & -\frac{1}{12}Q_1^4 - Q_1^2Q_2 + Q_3Q_1 + Q_2^2 \end{pmatrix} P \\
&\quad + V_2^{(5)}(Q) \\
H_3(Q,P) &= \frac{1}{2}P^T \begin{pmatrix} 0 & 0 & \frac{1}{6}Q_1^3 - Q_2Q_1 + Q_3 \\ 0 & \frac{1}{6}Q_1^3 - Q_2Q_1 + Q_3 & \frac{1}{6}Q_1^4 - Q_2Q_1^2 + Q_3Q_1 \\ \frac{1}{6}Q_1^3 - Q_2Q_1 + Q_3 & \frac{1}{6}Q_1^4 - Q_2Q_1^2 + Q_3Q_1 & \begin{matrix} \frac{1}{12}Q_1^5 - \frac{1}{3}Q_1^3Q_2 + \frac{1}{2}Q_3Q_1^2 \\ -Q_1Q_2^2 + Q_3Q_2 \end{matrix} \end{pmatrix} P \\
&\quad + V_3^{(5)}(Q),
\end{aligned}
$$

where

$$
\begin{aligned}
V_1^{(5)}(Q) &= -\frac{1}{6}Q_1^3 - Q_1Q_2 - Q_3, \\
V_2^{(5)}(Q) &= Q_1^2Q_2 - Q_1Q_3 - Q_2^2 + \frac{1}{12}Q_1^4, \\
V_3^{(5)}(Q) &= -\frac{1}{12}Q_1^5 + \frac{1}{3}Q_1^3Q - \frac{1}{2}Q_1^2Q_3 + Q_1Q_2^2 - Q_2Q_3,
\end{aligned}
$$

which is again in agreement with (46) and (45). Explicitly:

$$
G_0(Q) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & Q_1 \\ 1 & Q_1 & \frac{1}{2}Q_1^2 + Q_2 \end{pmatrix}, \quad K_1(Q) = I,
$$

$$
K_2(Q) = \begin{pmatrix} -Q_1 & 1 & 0 \\ 0 & -Q_1 & 1 \\ \frac{1}{6}Q_1^3 - Q_1Q_2 + Q_3 & Q_2 - \frac{1}{2}Q_1^2 & 0 \end{pmatrix},
$$

$$
K_3(Q) = \begin{pmatrix} \frac{1}{2}Q_1^2 - Q_2 & -Q_1 & 1 \\ \frac{1}{6}Q_1^3 - Q_1Q_2 + Q_3 & 0 & 0 \\ 0 & \frac{1}{6}Q_1^3 - Q_1Q_2 + Q_3 & 0 \end{pmatrix},
$$

and

$$
L(Q) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{6}Q_1^3 - Q_1Q_2 + Q_3 & Q_2 - \frac{1}{2}Q_1^2 & Q_1 \end{pmatrix}.
$$

5. Conclusions

In this paper we have considered Benenti Hamiltonian systems generated by a single separation curve (11), i.e. given by the separation relations (7) together with the assumption (10). In the case of these systems we have presented a new geometric version of Buchstaber and Mikhailov results [7]: we have shown the polynomiality of these Hamiltonian systems in Newton coordinates and also presented explicit form of all the geometric objects, associated with Benenti Hamiltonians, in these coordinates.This has been done by constructing and analysing the map between the Viète and Newton coordinates.

A natural questions that arises is whether it is possible to extend our construction to the case that the Hamiltonians H_i are not generated by a single separation curve but by the more general separation relations (7) without the assumption (10), i.e. with different f_i and φ_i, and perhaps for even more general separation relations. This will be a subject of another research paper.

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