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Spectrally determined singularities in a potential with an inverse square initial term

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We study the inverse spectral problem for Bessel type operators with potential $v(x)$: $H_\kappa = -\partial_x^2 + \frac{k}{x^2} + v(x)$. The potential is assumed smooth in $(0, R)$ and with an asymptotic expansion in powers and logarithms as $x \to 0^+, v(x) = O(x^\alpha), \alpha > -2$. Specifically we show that the coefficients of the asymptotic expansion of the potential are spectrally determined. This is achieved by computing the expansion of the trace of the resolvent of this operator which is spectrally determined and elaborating the relation of the expansion of the resolvent with that of the potential, through the singular asymptotics lemma.

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1. Introduction

Let $R > 0, \alpha > -2$ and v be a real valued function, smooth in $C_0^\infty((0, R))$ that has an asymptotic expansion together with all its derivatives as $x \to 0^+$

of the form

$$v(x) \sim x^{\alpha} \sum_{n=0}^{\infty} v_n x^{k_n}$$

where $\{k_n\}$ is an increasing sequence of positive real numbers. This function appears in the singular Sturm-Liouville operator $H = -\partial_x^2 + \frac{\kappa}{x^2} + v(x)$ on $(0, R)$ with Dirichlet boundary conditions at $0, R$. In this paper we examine whether the spectrum of this Sturm-Liouville operator uniquely determines the sequence of asymptotic coefficients $\{v_n\}$ of the potential function.

We prove that this is indeed the case provided that $v_0 \neq 0$; this is achieved by the explicit construction of the spectral invariants that in turn determine the asymptotic coefficients of the potential function v from the spectrum. The function

$$f_H(\eta) = \eta^2 Tr(1 + \eta^2 H)^{-1}$$

is defined for small $\eta > 0$ and is of course spectrally determined. The $\eta \to 0$ asymptotic expansion of f will provide the desired spectral invariants: the singularity of the Sturm -Liouville operator H manifests itself in the appearance of coefficients in the $\eta \to 0^+$ asymptotic expansion of $f_H(\eta)$ that depend polynomially on the coefficients $\{v_n\}$ of v. Precisely we have that this asymptotic expansion is of the form

$$(1) \qquad f_H(\eta) \sim \sum_{n=0}^{\infty} A_n \eta^{2n+1} + \eta^4 \Big(\sum_{n=-1}^{\infty} B_n \eta^{2n+1} \log \eta + \sum_{n=0}^{\infty} C_n \eta^{l_n} \Big)$$

where l_n is real but not an odd integer. Evidently it contains apart from pure odd powers also new terms, odd powers with logarithms as well as pure powers other than the odd ones. The sequences of asymptotic coefficients B_n, C_n of the asymptotic expansion of f are polynomials in the asymptotic coefficients of the potential $\{v_n\}$; it is exactly this fact that allows us to determine recursively these $\{v_n\}$ from the "heat invariants" that are equivalent to the $\eta \to 0^+$ asymptotic coefficients of $f_H(\eta)$. The exact formulation of this tehorem is the following:

Theorem. *Let χ be a smooth cutoff function, $\chi \equiv 1$ in a neighborhood of the origin, then the function $f_{\chi,H}(\eta)$ has an asymptotic expansion of the form*

$$f_{\chi,H}(\eta) \sim \sum_{k=0}^{\infty} A_{2k+1} \eta^{2k+1} + \sum_{n=0}^{\infty} B_n \eta^{z_n} + \sum_{k=0}^{\infty} C_{2k} \eta^{2k} + \sum_{k=0}^{\infty} D_{2k+1} \eta^{2k+1} \log \eta$$

where $z_n \in \mathbf{R} \setminus \mathbf{Z}, B_k, C_k, D_k$ *are homogeneous polynomials in the asympotic coefficients* $\{v_n\}$ *of the potential of degree* z *if we define* $deg(v_n) = k_n + 2$.

Inverse spectral results of the form we described above appeared first in [4],[5],[6], [10],[9]. Actually in [9] certain coefficients are calculated in the course of calculations of operator determinats and in the rest the generic case is studied exhaustively. The general case requires the treatment of certain exceptional cases. This treatment is provided here and leads to the full result.

The conical singularities have been treated extensively in the literature for diverse purposes giving rise to diverse results and calculations. We refer to the first papers that dealt with asymptotics of either the heat [12] or the wave kernel [13] on spaces with such singularities.

Possible applications of these results are indicated already in [6]. We recall these briefly here. Potentials of this type arise in the wave equation for a vibrating rod of variable cross section, when the cross-sectional area of the rod vanishes quartically (as a function of the distance from the end of the rod) at one point. In the same spirit we could determine the profile of a surface of revolution with a conical singularity on the axis, asymptotically to all orders, from the spectrum of Laplacian restricted to functions with polar symmetry.

We indicate here briefly two applications in physics. Furthermore the present study exhausts the determination of a confining potential which is hydrogen atom-like at short distance [15], from the complete set of bound state energies. This is indicated in [4] but only for the case of the $s-$wave. Hence the present study fills in the higher angular momemtum instances for potentials of asymptotic expansion of the form given above. We will present here briefly the construction in [4]. Confinement is expressed by Dirichlet boundary conditions on the surface of a sphere. The Hamiltonian with a radial potential is

$$H = -\Delta_3 + v(|x|)$$

where $v \in \Gamma^{\infty}((0,R))$ with $v(x) = O(x^{\alpha}), \alpha > -2$ as $x \to 0$. The boundary condition on the domain of H is $\psi(R) = 0$. The $l-$th spherical harmonic Hamiltonian is unitarily equivalent to

$$H_0 = -\partial_r^2 + \frac{l(l+1)}{r^2} + v(r)$$

which is an an unbounded operator on $L^2((0,R); dr)$. The boundary conditions inherited from the 3-dimensional problem are $f(0) = f(R) = 0$. A physically meaningful question is whether the potential v could be determined from the mass spectrum of the bound states associated to a given spherical harmonic because both the mass (the energy) and the angular momentum (the order of the spherical harmonic) are measurable in the laboratory.

The study we perform answers this affirmatively when the potential is a real analytic function on $(0, R)$ and that near 0 is written as a convergent series of the following form

$$v(x) = x^{-2} \sum_{k,j} v_{k,j} x^{\alpha_k} \log^j x,$$

for $\{\alpha_k\}$ an increasing sequence of positive real numbers. If the function v is not of this form then we can only obtain the full asymptotic expansion of the potential at small distances. Moreover we can deduce several corollaries of the preceding theorem that are interesting in physics. For instance availing ourselves of the results of [8] on irregular singularities, we state the following:

Corollary. *Let $H = -\partial_x^2 + v(x) + P(x)$ be an oprator on $\mathbf{R}_+$ with Dirichlet boundary conditions. Let v be a real valued analytic function, rapidly decreasing at ∞, while near 0 it is given by a convergent series of the form*

$$v(x) = x^{-2} \sum_{k,j} v_{k,j} x^{\alpha_k} \log^j x$$

for $\{\alpha_k\}$ an increasing sequance of positive real numbers. The function P is given and is of the form

$$P(x) = \sum_{n=0}^{N} a_k x^{\beta_n}$$

for $0 \leq \beta_0 < \cdots < \beta_N$ and $a_N > 0$. Then the spectrum of H determines v.

The preceding result incorporates for instance the case of quantum particle systems trapped in magnetic fields. An immediate application of the preceding corollary refers to the 2-dimensional magnetic trap [1], due to a vanishing magnetic field at ∞. Precisely, let $0 < \epsilon < 1$ and (r, θ) be the coordinates in $\mathbf{R}^2$ and consider the quanutm motion in a magnetic field of the type $B = (2 - \epsilon) r^{-\epsilon}$. The Hamilotnian for a particle subject to such a magnetic field as well as to an unknown radial potential $v(r)$ has the form

$$H = -\partial_r^2 + \frac{1}{r^2}\partial_\theta^2 + r^{2(1-\epsilon)} + 2ir^{-\epsilon}\partial_\theta + v(r)$$

and restricting to angular momentum m we obtain an 1-dimensional operator of the above type provided that $v(r)$ is of the required form. We notice also the example from [17] of a quantum particle moving in a magnetic field with fixed angular momentum in the direction of the field. Similar conclusions could be obtained in that case as well. The results in [2] allow us to incorporate the case of a pair of opposite charges with fixed relative angular momentum moving in a homogeneous magnetic field and interacting through an unknown potential that we wish to determine spectrally when the latter satisfies the preceding assumptions.

Actually we insert a cutoff function $\chi, supp\chi \subset [0, R)$ in order to deal the operator in the half line. The $\eta \to 0$ asymptotic expansion of

$$f_{\tilde{H}}(\eta) = \eta^2 tr(1 + \eta^2 \tilde{H})^{-1}$$

- $\tilde{H}$ is the operator in $(0, \infty)$- coincides $mod(\eta^\infty)$ terms with that of f_H.

The paper is organised as follows: we start with the neccessary facts concerning the operator domain, the operator estimates and we construct the resolvent of the unperturbed operator. Next we give the existence and the precise form of the asymptotic expansion for the operator function we introduced above. We conclude with the calculation of the asymptotic coefficients that provide the recursive relations that determine the potential asymptotic coefficients. The solution of these require certain elementary estimates that are included.

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2. Operator domain and operator estimates

2.1. Function spaces for singular heat expansions

We will consider operators of the form $H = H_\kappa + v$, for $H_\kappa = -\partial_x^2 + \frac{\kappa}{x^2}$ and v is a real valued potential function belongs to the space $\Gamma^\infty(0, R)$ of functions that have an asymptotic expansion as $x \to 0^+$ together with their derivatives. This space is described as follows that are encountered in the resolvent expansions that we are going to deal with.

The space $\Gamma^\infty(\mathbf{R}_+)$ consists of the $C^\infty(\mathbf{R}_+)-$ functions that have asymptotic expansions as $x \to 0^+$, together with all their derivatives, in complex powers of x and integral powers of $\log x$. Precisely, $f \in \Gamma^\infty(\mathbf{R}_+)$ if $f \in C^\infty(\mathbf{R}_+)$ and there exists a $S : \mathbf{C} \to \mathbf{Z}_+$ such that $\sum_{\Re z < a} S(z) < \infty$ for each $a \in \mathbf{R}$, and

$$\partial_x^m f(x) = \sum_{\Re z \leq a} \sum_{\mathbf{Z}_+ \ni j < S(z)} f_{z,j} \partial_x^m (x^z \log^j x) + O(x^{a+\delta-m})$$

for some $\delta = \delta_a > 0$, some $f_{z,j} \in \mathbf{C}$ and all $a \in \mathbf{R}$. S is called the asymptotic character of the expansion. Define the following differential operators on $C^\infty(0,\infty)$, for given $S : \mathbf{C} \to \mathbf{Z}_+$ as described above

$$P_z[S] = \prod_{\Re z' \leq \Re z} (x\partial_x - z')^{S(z')},$$

$$\overline{P_z}[S] = P_z[S](x\partial_x - z)^{S(z')} = \prod_{\Re z' \leq \Re z} (x\partial_x - z')^{S(z')}$$

The following proposition is proved in [7]; though it is elementary it provides criterion that shades light on these spaces

Proposition 1. *$f \in \Gamma^\infty(0,\infty)$ iff $f \in C^\infty(0,\infty)$ and there is an asymptotic character such that $P_z[S]f(x) = O(x^{z-\varepsilon})$ for all $z \in \mathbf{C}$ and $\varepsilon > 0$.*

The space $\Gamma^\infty((\mathbf{R}_+)^2)$ consists of all the functions $f \in C^\infty((\mathbf{R})^2)$ for which there exist S_1, S_2 such that

$$(x\partial_x)^{s_1}(y\partial_y)^{s_2}\overline{P^x_{z_1}}[S_1]\overline{P^y_{z_2}}[S_2]f(x,y) = O(x^{\Re z_1+\delta_1}y^{\Re z_2+\delta_2})$$

for all $z_1, z_2 \in \mathbf{C}, s_1, s_2 \in \mathbf{Z}_+, (x,y)$ in any compact neighborhood of $\partial(\mathbf{R}^2_+)$ and some δ_j depending on z_1, z_2.
Evidently (S_1, S_2) is analogously the asymptotic character of $f \in \Gamma^\infty(\mathbf{R}^2_+)$. If $f \in \Gamma^\infty(\mathbf{R}_+)$, we let $D_{k,j}f(0)$ denote the coefficient of $x^k \log^j x$ in the expansion of f as $x \to 0$.

Now let $\alpha > -2, \{k_n\}$ be an increasing sequence of positive real numbers then we have that $S(z) = 1$ for $z = k_n + \alpha$ and $S(z) = 0$ otherwise. equivalently, the potential function has the following asymptotic expansion as $x \to 0$

$$v(x) \sim x^\alpha \sum_{n=0}^{\infty} v_n x^{k_n}$$

Actually for $0 < x < \epsilon$ and $\alpha > -2$ the obvious estimate $x^\alpha \leq \epsilon x^{-2} + B_\epsilon$ implies for $\phi \in C_0^\infty(\mathbf{R}_+)$ and $c < 1$ that

$$\|V\phi\|_{L^2} \leq c\|H_\kappa\phi\|_{L^2} + b\|\phi\|_{L^2}$$

which in view of the Kato-Rellich theorem reduces the domain questions of H to those for H_κ.
Through the Hardy inequality we conclude that if $\kappa \geq -\frac{1}{4}$ then the operator H_κ is positive and hence possesses at least one self adjoint extension, the *Friedrichs' extension*. Recall that if $\kappa \geq \frac{3}{4}$, then Weyl's criterion implies that the operator H_κ is essentially self-adjoint and hence the Friedrichs' is the only extension. The domain of self adjointness consists of the functions

$$\begin{aligned}\mathcal{D}(H) &= \{\phi \in L^2(\overline{\mathbf{R}_+}), \|\partial_x^2\phi\|_{L^2} < \infty, \|x^{-2}\phi\|_{L^2} < \infty\} \\ &\subset L^2(\overline{\mathbf{R}_+}; \frac{dx}{x^4}) \cap H^2(\overline{\mathbf{R}_+}).\end{aligned}$$

The Sobolev embedding theorem implies that these are L^2-functions with absolutely continuous first derivative. Additionally, since $\phi \in \mathcal{D}(H)$ then $\|\frac{\phi}{x^2}\| < \infty$ which in turn expresses the Dirichlet boundary conditions, $\phi(0) = \lim_{x\to 0}\phi(x) = 0$.

2.2. The resolvent

Let $H = -\partial_x^2 + v$ be a Shrödinger operator, with real valued potential function. Clearly, it is formally symmetric. If ϕ, ψ are the unique elements of $kerH$ that are integrable at $0, \infty$ respectively then the resolvent of H is given by

$$R_\lambda(x,y) = \Theta(x-y)\frac{\psi(x,\lambda)\phi(y,\lambda)}{W_x(\phi(\lambda),\psi(\lambda))} + \Theta(y-x)\frac{\phi(x,\lambda)\psi(y,\lambda)}{W_x(\phi(\lambda),\psi(\lambda))}$$

where W is the Wronskian of the solutions of $(H-\lambda)\phi = 0$. In the case of the operator $H_{\kappa,\alpha} = -\partial_x^2 + \frac{\kappa}{x^2} + \frac{\alpha}{x}$ the solutions are expressed through the confluent hypergeometric functions [19], [16]:

$$\phi(x,\lambda) = M_{\mu,\nu}(2x\sqrt{-\lambda}), \qquad \psi(x,\lambda) = W_{\mu,\nu}(2x\sqrt{-\lambda})$$

where the indices are $\mu = \frac{\alpha}{\sqrt{-\lambda}}$ and $\nu = \sqrt{\kappa + \frac{1}{4}}$. Their Wronskian [1] is

$$W = \frac{2\sqrt{-\lambda}\,\Gamma(2\nu+2)}{\Gamma(\nu-\mu+1)}.$$

[1]Since our study is effected in the sector of $|\Im\lambda| \leq \frac{1}{\epsilon}(\Re\lambda + \epsilon)$ then we assume that $\nu - \gamma > -\frac{1}{2}$

The confluent hypergeometric functions coincide for $\alpha = 0$ with the Bessel functions:

$$M_{0,\nu}(x\sqrt{-\lambda}) = 2^{2\nu}\Gamma(\nu+1)\sqrt{x}I_\nu(\frac{x}{2}\sqrt{-\lambda}),$$
$$W_{0,\nu}(x\sqrt{-\lambda}) = \sqrt{\frac{x}{\pi}}K_\nu(\frac{x}{2}\sqrt{-\lambda}),$$

the resolvent of $H_{\kappa,\alpha}$ could be obtained from that of H_κ by Neumann series. The latter will be used in the operator estimates that follow. Therefore we form the Whittaker Green's function:

$$R_\lambda(x,y) = \Theta(x-y)\frac{\Gamma(\nu-\mu+\frac{1}{2})\Gamma(\nu-\mu+1)M_{\mu,\nu}(2y\sqrt{-\lambda})W_{-\mu,\nu}(2x\sqrt{-\lambda})}{2\sqrt{-\lambda}\Gamma(2\nu+2)\Gamma(\nu+\mu+\frac{1}{2})}$$
$$+\Theta(y-x)\frac{\Gamma(\nu-\mu+\frac{1}{2})\Gamma(\nu-\mu+1)M_{\mu,\nu}(2y\sqrt{-\lambda})W_{-\mu,\nu}(2x\sqrt{-\lambda})}{2\sqrt{-\lambda}\Gamma(2\nu+2)\Gamma(\nu+\mu+\frac{1}{2})}$$

2.3. Operator estimates

We denote by $R^0_\lambda = (\lambda - H_\kappa)^{-1}$ the resolvent of H_κ. The positivity of H_κ implies that

1. The operator norm is

$$\|R^0_\lambda\|_{L^2} = O(|\lambda|^{-1})$$

for $|\lambda| \to \infty$ uniformly in the cone $|\Im\lambda| \geq \frac{1}{\epsilon}(\Re\lambda + \epsilon)$:

Since $R^0_\lambda L^2(\overline{\mathbf{R}_+}) \subset \mathcal{D}(H_\kappa)$ we choose then $\phi \in L^2(\overline{\mathbf{R}_+})$: $R^0_\lambda\phi = \psi$. It follows that for $k \leq -\epsilon$:

$$\|(\lambda - H_\kappa)\psi\|^2_{L^2} = |\lambda - k|^2\|\psi\|^2_{L^2} + \|(k-H_\kappa)\psi\|^2_{L^2} + 2(\Re(k-\lambda)(\psi, H_\kappa\psi))$$
$$\geq |(\lambda - k)|^2\|\psi\|^2_{L^2}$$

and the estimate follows.

The resolvent R^0_λ is represented by the kernel for $\sigma = \frac{1}{\sqrt{-\lambda}}$:

$$R_\sigma(x,y) = (xy)^{1/2}[\Theta(x-y)K_\nu(\frac{x}{\sigma})I_\nu(\frac{y}{\sigma}) + \Theta(y-x)I_\nu(\frac{x}{\sigma})K_\nu(\frac{y}{\sigma})].$$

We have the following classical estimates for the classical trace norms $||A||_k := tr(|A|^k)^{1/k}$:

2. Let $\phi \in C_0^\infty(\mathbf{R})$ then $||R_\lambda^0\phi||_1 = O(|\lambda|^{-1/2})$ for λ in the cone $|\Im\lambda| \geq \frac{1}{\epsilon}(\Re\lambda + \epsilon)$.

Proof: For $\lambda \leq -\epsilon$ and ν in the above cone it holds that

$$||R_\nu^0\phi||_1 \leq ||(1 + (\lambda - \nu)R_\lambda^0)||_{L^2}^{-1}||R_\lambda^0\phi||_1$$

as well as that for $\sigma^2 = -\frac{1}{\lambda})$:

$$||R_\lambda^0\phi||_1 = \int_0^\infty dx x\phi(x)(K_\nu I_\nu)(\frac{x}{\sigma}).$$

Hence we have the required estimate for

$$\int_0^\infty dx x\phi(x)(K_\nu I_\nu)(\frac{x}{\sigma}) \leq \sigma||x\phi||_{L^2}||K_\nu I_\nu||_{L^2}.$$

3. Let $\alpha \in \overline{\mathbf{R}_-}, \alpha + d \leq 2$ then

$$\|x^{-\alpha}\partial_x^d R_\lambda^0\|_{L^2} = O(|\lambda|^{\frac{\alpha+d}{2}-1})$$

Proof. This follows for $d = 0$ from the inequality given in the beginning while for $d = 2$ it results from the fact that $-\partial_x^2 \leq H_{\kappa+\frac{1}{4}}$ and therefore

$$\|\partial_x^2\phi\| \leq \|H_{\kappa+\frac{1}{4}}R_\lambda^0\phi\|^2 \leq (\|\phi\|_{L^2}^2 + |\lambda|^2\|R_\lambda^0\phi\|_{L^2}^2).$$

Furthermore for $d = 1$ we have for $\psi = R_\lambda^0\phi$ that

$$\begin{aligned}(x^{-\alpha}\partial_x\psi, x^{-\alpha}\partial_x\psi) &= (x^{-2\alpha}\psi, -\partial_x^2\psi) + 2\alpha(\partial_x\psi, x^{2(\alpha-\frac{1}{2}}\psi) \\ &\leq 2\delta\|x^{2\alpha}\psi\|_{L^2}^2 + 2\delta^{-1}\|\partial_x^2\psi\|_{L^2}^2 \\ &\quad + 2\epsilon\|x^{-(2\alpha+1)}\psi\|_{L^2}^2 + 2\epsilon^{-1}\|\partial_x\psi\|_{L^2}^2\end{aligned}$$

the latter choosing $\delta = |\lambda|^{\alpha-\frac{1}{2}}$ and $\epsilon = |\lambda|^{\frac{\alpha}{2}}$ gives the required estimate.

2.4. The Neumann series

Let $\alpha > -2, k_0 = 0$ and $\{k_n\}_{n=1}^\infty$ an increasing sequence of positive real numbers. The potential v has the asymptotic expansion at the origin of the form:

$$v(x) \sim x^\alpha \sum_{n=0}^\infty v_n x^{k_n}.$$

In the sequel we will distinguish two cases:

- **Case I.** Let $\alpha = -1$ and $v_0 \neq 0$.
  In this case the distributional trace of the resolvent $R_\lambda(H)$, $f_H(\frac{1}{\sqrt{-\lambda}}) := Tr(R_\lambda(H))$ is computed using Neumann series around $H_{\kappa,\alpha} = -\partial_x^2 + \frac{\kappa}{x^2} + \frac{\alpha}{x}$:

$$f_H(\frac{1}{\sqrt{-\lambda}}) = \sum_{j=0}^{N} Tr(R_\lambda(H_{\kappa,\alpha})(v - \frac{\alpha}{x}))^j R_\lambda(H_{\kappa,\alpha})$$
$$+ Tr(R_\lambda(H_{\kappa,\alpha})(v - \frac{\alpha}{x}))^{N+1} R_\lambda(H).$$

- **Case II.** Let $\alpha > -1$ and $v_0 \neq 0$.
  The Neumann series is based on the resolvent of the usual Bessel operator H_κ. The case of $-2 < \alpha < -1$ is treated within the frame of case B.

The Neumann series around the Bessel operator, due to its behaviour under scaling, is used in order to obtain the general form of the asymptotic expansion.

3. The asymptotic expansion

In this paragraph we 'll establish the existence and the precise form of the asymptotic expansion of the distributional trace of the resolvent $R_\lambda(H)$ of the operator $H = -\partial_x^2 + \frac{\kappa}{x^2} + v(x)$ by means of a Neumann series based on the resolvent $R_\lambda(H_\kappa$ of $H_\kappa = -\partial_x^2 + \frac{\kappa}{x^2}$, for a cut off function $\chi \in C_0^\infty(\mathbf{R}_+)$, $\chi \equiv 1$ in a neighbourhood of zero :

$$f_{\chi,H}(\frac{1}{\sqrt{-\lambda}}) := Tr(\chi R_\lambda(H)) = \sum_{j=1}^{N} Tr(\chi(R_\lambda(H_\kappa)v)^j R_\lambda(H_\kappa)) + O(|\lambda|^{\frac{N(\alpha-2)}{2}})$$

Denote by $I_j(\frac{1}{\sqrt{-\lambda}}) = Tr(\chi(R_\lambda(H_\kappa)v)^j R_\lambda(H_\kappa))$. Actually we have the following

Theorem. *Let χ be a smooth cutoff function, $\chi \equiv 1$ in a neighborhood of the origin, then the function $f_{\chi,H}(\eta)$ has an asymptotic expansion of the form*

$$f_{\chi,H}(\eta) \sim \sum_{k=0}^{\infty} A_{2k+1}\eta^{2k+1} + \sum_{n=0}^{\infty} B_n \eta^{z_n} + \sum_{k=0}^{\infty} C_{2k}\eta^{2k} + \sum_{k=0}^{\infty} D_{2k+1}\eta^{2k+1}\log\eta$$

where $z_n \in \mathbf{R} \setminus \mathbf{Z}, B_k, C_k, D_k$ *are homogeneous polynomials in the asympotic coefficients* $\{v_n\}$ *of the potential of degree* z *if we define* $deg(v_n) = k_n + 2$.

The proof will be achieved in two steps: first we prove the existence by appealing to the singular asymptotics lemma that we recall below and in the second step we use a scaling argument in conjuction with the asymptotics of the resolvent of the Bessel operator $H_\kappa = -\partial_x^2 + \frac{\kappa}{x^2}$ which as it is well known it is represented by the kernel :

$$R_\sigma(x,y) = (xy)^{\frac{1}{2}}[\Theta(x-y)K_\nu(\frac{x}{\sigma})I_\nu(\frac{y}{\sigma}) + \Theta(y-x)I_\nu(\frac{x}{\sigma})K_\nu(\frac{y}{\sigma})].$$

Step 1. The trace

$$I_j = tr(\chi R_\lambda(H_\kappa)(vR_\lambda(H_\kappa))^{j-1})$$

is the $j+1$-tuple integral

$$\int_0^\infty dx \int_0^\infty dx_1 \cdots \int_0^\infty dx_j \chi(x) R_\lambda(x,x_1)\dots R_\lambda(x_j,x)$$

which is the sum of integrals, each one for an ordering of the the variables $x, x_1, \dots, x_j$ hence for $x \geq x_1 \geq \cdots \geq x_j$:

$$\int_0^\infty dx \int_0^\infty dx_1 \cdots \int_0^{x_{j-1}} dx_j x^2 \chi(x) x_1^2 v(x_1) \dots x_j^2 v(x_j)(K_\nu(\frac{x}{\sigma})I_\nu(\frac{x_j}{\sigma}))^2 \times \prod_{i=1}^{j-1} K_\nu(\frac{x_i}{\sigma})I_\nu(\frac{x_i}{\sigma})$$

and we perform the coordinate change (blow up)

$$x_1 = xt_1, \dots, x_j = x_{j-1}t_{j-1}$$

and

$$x_1 = x\theta_1, \dots, x_j = x\theta_{j-1}, \theta_k = t_1 \dots t_k$$

that leads to a sum of integrals of the form:

$$\int_0^\infty dx x^{2j} \chi(x) \int_0^1 \frac{d\theta_1}{\theta_1} \cdots \int_0^1 \frac{d\theta_j}{\theta_j} \prod_{k=1}^{j} v(x\theta_j)\theta_j^2 K_\nu^2(\frac{x}{\sigma}) I_\nu^2(\frac{x}{\sigma}\theta_j) \prod_{k=1}^{j-1} (K_\nu I_\nu)(\frac{x}{\sigma}\theta_k)$$

which is finally of the form suggested from the SAL

$$\mathcal{I}_j(\sigma) = \int_0^\infty \frac{dx}{x} \chi(x) x^3 \cdot F(\xi, x)$$

where

$$F(\xi, x) := \int_0^1 \frac{d\theta_1}{\theta_1} \cdots \int_0^1 \frac{d\theta_j}{\theta_j} (\mathcal{B}_\nu(\xi, \theta_j) \mathcal{R}_{j\nu}(\xi; \theta_1, \cdots, \theta_{j-1}) v_j(x; \theta_1, \cdots, \theta_j).$$

if we set

$$\mathcal{B}_\nu(\xi, \theta_j) := (K_\nu(\frac{1}{\xi}) I_\nu(\frac{\theta_j}{\xi}))^2, \qquad \mathcal{R}^j_\nu(\xi, \theta_1, \cdots, \theta_{j-1}) := \prod_{i=1}^{j-1} (K_\nu I_\nu)(\frac{\theta_i}{\xi})$$

and

$$v_j(x; \theta_1, \cdots, \theta_j) := \prod_{i=1}^{j} (x\theta_i)^2 v(x\theta_i) = O(x^{j(\alpha+2)})$$

Now we see that $v_j \in \Gamma^\infty((\mathbf{R}_+))$ with asymptotic character $\underbrace{S + \cdots + S}_{j-times}$ since $v \in \Gamma^\infty(\mathbf{R}_+)$ has character S, $v(x) = O(x^\alpha), \alpha > -2$. Furthemore the character of $(\mathcal{Q}_\nu \cdot \mathcal{P}^j_\nu)(\xi, \theta_1, \cdots, \theta_j)$ is $S(k) = 1$ for $k \in j + 1 + 2\mathbf{Z}_+$. The estimates for the Bessel functions as $\xi \to 0$:

$$K_\nu(\frac{1}{\xi}) I_\nu(\frac{\theta}{\xi}) = O(\xi), \qquad K_\nu(\frac{\theta}{\xi}) I_\nu(\frac{\theta}{\xi}) = O(\xi).$$

From these it follows that $F(\xi, x) \in \Gamma^\infty((\mathbf{R}_+)^2)$ and also that

$$|\overline{P}^x_z F(\xi, x)| \leq (x\xi)^{\Re z + \delta} h_z(\xi)$$

where we have that

$$\overline{P}^x_z = \prod_{Rez' \leq Rez} (x\partial_x - z')^{S(z')}, \qquad \int_1^\infty h_z \frac{dz}{z} < \infty.$$

Then we conclude that

$$\mathcal{I}_j(\sigma) \sim \sigma^3 \sum_n^\infty B^{(j)}_n \sigma^{z_n} + \sigma^{j+1} (\sum_{k=0}^\infty A^{(j)}_k \sigma^{2k} + \sum_{k=0}^\infty C^{(j)}_k \sigma^{2k+1} + \sum_{k=0}^\infty D^{(j)}_k \sigma^{2k} (\log \sigma))$$

where the $z_n \notin j+1+2\mathbf{Z}_+$ and the $B_z^{(j)}, C_k^{(j)}, D_k^j$ are polynomials in the coefficients of the potential of the potential of degree $k+j$ if we define $degv_z = z+2$. These suffice for the B's in the expansion.

Step 2. We have to establish the nature of the non classical terms logarithmic terms. This will be achieved by induction on the ordered couples (j,k) where j is the order of the term in the Neumann series and k is order in the asymptotic expansion of the potential. For $k=\alpha$ the assertion follows from above: everything depends on v_0; indeed $j=0$ follows from [5]. Assume that $k>\alpha$. Then consider

$$I_j = Tr(\chi(R_\lambda(H_\kappa)v)^j R_\lambda(H_\kappa)$$

Notice that up to $O(\lambda^{-\infty})$ we could embody the cutoff function in the potentials V. Accordingly let $N \in \mathbf{Z}, k_N \geq -2$ and split the potential as $v = v_N + R_N$, where $v_N(x) \sim x^\alpha \sum_{n=0}^N v_n x^{k_n}$. The following commutator identities for $k_r \geq \frac{k_n}{2}-1, r<n$ and $\epsilon = k_n - k_r > 0$ give that

$$x^{k_n}R_\lambda(H_\kappa) = x^{k_r}R_\lambda(H_\kappa)x^\epsilon + 2\epsilon x^{k_r}R_\lambda(H_\kappa)(x^{\epsilon-2}(x\partial_x + \frac{\epsilon-1}{2\epsilon})R_\lambda(H_\kappa),$$
$$x\partial_x R_\lambda(H_\kappa) = R_\lambda(H_\kappa)x\partial_x - 2R_\lambda(H_\kappa)^2 H_\kappa$$

and in turn allow us to vary the number of the $R_\lambda(H_\kappa)$ factors. This in view of the fact that the asymptotic expansion of $I_\nu K_\nu(x)$ as $x\to\infty$ contains only odd powers, in view of $\lambda\partial_\lambda R_\lambda(H_0) = -2\sigma\partial_\sigma R_\sigma$, leading to the polynomial dependence of the coefficients of the logarithms and the even powers:

$$I_j(\eta) \sim \sum_{k=0}^\infty A_{2k+1}\eta^{-2k-1} + \sum_{n=0}^\infty B_{z_n}\eta^{-z_n} + \sum_{k=0}^\infty C_{2k}\eta^{-2k} + \sum_{k=0}^\infty D_k\eta^{-k-4}\log\eta$$

where $z_n \in \mathbf{R}\setminus\mathbf{Z}, z>-2$ and the B_z, C_k, D_k are homogeneous polynomials in the asympotic coefficients $\{v_n\}_{n=0}^\infty$ of v of degree z if we define $deg(v_z) = z+2$. Then summing up we obtain an expansion of the same form and we appeal to the results in par. 5 of [4] in order to keep the odd terms in the log's.

In conclusion we supply the asymptotics of

$$G_t(\xi) := K_\nu(\frac{1}{\xi})I_\nu(\frac{t}{\xi})$$

as $\xi \to 0$ uniformly in $0 \leq t \leq 1$. For this we calculate the Mellin transform using the integral representation:

$$G_t(\xi) = K_\nu(\frac{1}{\xi}) I_\nu(\frac{t}{\xi}) = \int_0^\infty \frac{x^2}{x^2 + \xi^{-2}} J_\nu(tx) J_\nu(x) \frac{dx}{x}.$$

The Hankel transforms [20] allows us to obtain further:

$$G_t(\xi) \frac{(2t)^\nu}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})} \int_0^\infty \frac{x^{\nu+2}}{x^2 + \xi^{-2}} \int_0^1 \frac{J_\nu(x\sqrt{1 + t^2 - 2ut})}{(1 + t^2 - 2ut)^{\frac{\nu}{2}}} du \frac{dx}{x}.$$

From this we get that the Mellin transform is

$$\widehat{G}_t(s) = -\frac{2^{-s+2\nu-2}\Gamma(\nu - \frac{s}{2})\Gamma(-\frac{s}{2})}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})}(1 + t)^{s+\nu} \times \int_0^1 \eta^{\nu-\frac{1}{2}}(1 - \eta)^{\nu-\frac{1}{2}}(1 - \frac{2t}{(1 + t)^2}\eta)^{s-\nu} d\eta.$$

A Taylor expansion then leads to the series

$$\widehat{G}_t(s) = -(1 + t)^{s-2\nu} \frac{2^{-s+2\nu-2}\Gamma(\frac{s}{2} + \frac{1}{2})\Gamma(\nu - \frac{s}{2})\Gamma(-\frac{s}{2})}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})} \times \sum_{k=0}^\infty \frac{(-1)^k I_k(t)}{k!(\Gamma(\frac{s}{2} + \frac{1}{2} - k)} \frac{(2t)^k}{(1 + t)^{2k}}$$

where

$$I_k(t) = \int_0^1 \eta^{\nu-\frac{1}{2}+k} (\frac{1 - \eta}{1 - \frac{2t}{(1+t)^2}\eta})^{\nu-\frac{1}{2}} d\eta.$$

This integral satisfies that $I_k = O(k^{-\frac{1}{2}-\delta})$ as $k \to \infty$, uniformly in t, where $\delta = \frac{1}{2}$ for $1 > \epsilon > 0, \nu \geq \frac{1}{2}$ and $\delta = \nu$ for $\nu < \frac{1}{2}$. Stirling' s formula then suggests that the summand behaves as $O(k^{-\frac{s}{2}-1-\delta})$ and hence it is uniformly convergent for $0 < t < 1$. Finally we conclude that the asymptotic expansion of $g_t(\xi)$ as $\xi \to 0$ contains only odd powers of ξ. [2]

4. The recursions for the coefficients of the expansion

We will consider the two cases indicated in section 2.4 as follows:

[2] In order to extend meromorphically it out of the strip $-1 + \nu < \Re s < \nu$ we differentiate with respect to $\zeta = \frac{2t}{(1+t)^2}$. The poles as $t \to 1$ remain located there.

4.1. Case I

The first order term in the Neumann series is reduced $mod(\lambda^{-\infty})$, to

$$I_1(\frac{1}{\sqrt{-\lambda}}) = tr(R_\lambda(H_{\kappa,v_0})(v - \frac{v_0}{x})R_\lambda(H_{\kappa,v_0})\chi)$$
$$= -\partial_\lambda tr(R_\lambda(H_{\kappa,v_0})(v - \frac{v_0}{x})\chi)$$

Setting $\sigma = \frac{1}{\sqrt{-\lambda}}$ we arrive through the formula

$$I_1(\sigma) = \frac{\sigma^2}{2}\sigma\partial_\sigma \tilde{I}_1(\sigma)$$

at the integral

$$\tilde{I}_1(\sigma) = \sigma \cdot \int_0^\infty \mathcal{G}_{\mu,\nu}(\frac{2}{y})V(x)\frac{dx}{x} \tag{2}$$

where

$$V(x) = x(v(x) - \frac{v_0}{x}), \qquad \mathcal{G}_{\mu,\nu}(\frac{2}{y}) = \frac{\Gamma(\nu - \mu + \frac{1}{2})}{2\Gamma(2\nu+1)}M_{\mu,\nu}W_{\mu,\nu}(\frac{2}{y}),$$

for $M_{\mu,\nu}, W_{\mu,\nu}$ being the Whittaker functions $y = \frac{\sigma}{x}, \mu = \frac{v_0 x y}{2}$. This Green's function could be simplified in the form that we 'll use in the sequel

$$\mathcal{G}_{\mu,\nu}(y) := \frac{\tilde{\mathcal{G}}_{\mu,\nu}(y)}{\alpha_\nu(\mu)}$$

where

$$\tilde{\mathcal{G}}_{\mu,\nu}(y) = y^{2\nu+1}\tilde{W}_{\mu,\nu}(y)\tilde{M}_{\mu,\nu}(y)$$

for the integrals

$$\tilde{M}_{\mu,\nu}(y) = \int_{-1}^1 e^{-y(1-\theta_1)}(1-\theta_1^2)^{\nu-\frac{1}{2}}(\frac{1-\theta_1}{1+\theta_1})^\mu d\theta_1,$$

$$\tilde{W}_{\mu,\nu}(y) = \int_0^\infty e^{-2y\theta_2}(\theta_2(1+\theta_2)^{\nu-\frac{1}{2}}(\frac{1+\theta_2}{\theta_2})^{-\mu}d\theta_2,$$

$$\alpha_\nu(\mu) = \Gamma(\nu - \mu + \frac{1}{2})\Gamma(\nu + \mu + \frac{1}{2}).$$

The study of the $\sigma \to 0$ asymptotics of the above intgeral will provide the inverse spectral result. Indeed the Singular Asymptotics lemma provides the asymptotic coefficients of the integral $\sigma \to 0$. For brevity we introduce the notation

$$\mathcal{C}_{j,\nu}(y) = \partial_\mu^j \mathcal{G}_{\mu,\nu}(\frac{2}{y})|_{\mu=0}, \qquad c_{j,\nu}(y) = \partial_\mu^j \tilde{\mathcal{G}}_{\mu,\nu}(\frac{2}{y})|_{\mu=0}.$$

Then we distinguish between the following cases:

A. $2\nu \in \mathbf{Z}_+, \nu = n + \frac{1}{2}$. For $l = 0, \dots$ we appeal to the logarithmic terms

$$\begin{aligned} D_{2l+1,1}\tilde{I}_1(0) &= -\frac{1}{(2l)!} D^y_{2l,0} \partial_x^{2l}|_{x=0}[\mathcal{G}_{\mu,\nu}(\frac{2}{y}) \cdot V(x)] \\ &= -\sum_{j=0}^{2l} (\frac{v_0}{2})^j [(D^y_{2l,0}(y^j \mathcal{C}_{j,\mu}(y))] v_{2l-j-1} \end{aligned}$$

and also to the even powers that are given

$$D_{2l,0}\tilde{I}_1(0) = -\sum_{j=0}^{2l} (\frac{v_0^j}{2^j j!}) [u^y_{2l,0}(y^j \mathcal{C}_{j,\mu}(y))] v_{2l-j-1}.$$

Actually for the inverse spectral result we'd like the explicit forms:

$$\begin{aligned} D_{2l+3,1}\tilde{I}_1(0) &= B^0_{2l+2}(\nu) v_{2l+2} - (l+1) B^1_{2l+2}(\nu) v_0 v_{2l+1} \\ &\quad - (l+1)(l+\frac{1}{2}) B^2_{2l+2}(\nu) v_0^2 v_{2l} + P_{2l+2}(v) \end{aligned}$$

and also that

$$\begin{aligned} D_{2n+2j+2,0}\tilde{I}_1(0) &= C^0_{2j+2n+1}(\nu) v_{2n+2j+1} \\ &\quad + (j+n+\frac{1}{2}) C^1_{2j+2n+1}(\nu) v_0 v_{2n+2j} + Q_{2n+2j+1}(v) \end{aligned}$$

where we have denoted the coefficients

$$B^j_k(n) := D^y_{k,0}(y^j \mathcal{C}_{j,\nu}) = D_{k-j,0}\mathcal{C}_{j,\nu}, \qquad C^j_k(\nu) := u^y_{k,0}(y^j \mathcal{C}_{j,\nu}) = u_{k-j,0}(\mathcal{C}_{j,\nu})$$

and the P, Q are the polynomials suggested precedingly by the general form of the asymptotic expansion. Notice that if $l = 0, \dots, n$ then we 'll see in the sequel that:

$$B^0_{2l}(\nu) \neq 0, \qquad C^0_{2l+1}(\nu) \neq 0$$

therefore coefficients $v_k, k = 1, \dots, 2n+1$ are determined immediately. For the remaining we appeal to the pair of equations; hence we have to establish, provided that $v_0 \neq 0$, that the determinant:

$$\Delta_k(\nu) = B^1_{2n+2k+2}(\nu)C^1_{2n+2k+1}(\nu) - B^2_{2n+2k+2}(\nu)C^0_{2n+2k+1}(\nu)$$

does not vanish.

B. $2\nu \in \mathbf{R} \setminus \mathbf{Z}_+$. The inverse spectral result requires again the logarithmic odd powers as above, which are provided from the first order terms

$$D_{2l+3,1}\tilde{I}_1(0) = D^{(0)}_{2l+3}(\nu)v_{2l+1} + P_{2l+2}(V)$$

as well as the the pure powers $\alpha \in \mathbf{R}, \alpha > -2, \alpha \neq 0, 2, \ldots$:

$$D_{\alpha,0}\tilde{I}_1(0) = u^y_{\alpha,0}[(\bar{P}_\alpha \mathcal{G}_{\mu,\nu}(\frac{2}{y}) \cdot V]|_{x=0}.$$

In this formula $\bar{P}_\alpha = \prod_{z\leq\alpha}(x^z x\partial_x x^{-z})$ and hence we obtain the set of equations

$$D_{\alpha+2,0}\tilde{I}_1(0) = C^0_{\alpha+2}(\nu)v_{\alpha+1} + C^1_{\alpha+2}(\nu)v_0 v_{\alpha+1} + C^2_{\alpha+2}(\nu)v_0^2 v_\alpha + Q_{\alpha+2}(v).$$

Notice that if $\alpha \neq 2\nu + 2j + 1$ then $C^0_{\alpha+2} \neq 0$ and hence v_α is determined by the first terms for $\alpha \neq 2\nu + k$. These in particular contain the coefficients for $\alpha = 2\nu + 2j + 1, j = 0, \ldots$.

$$\begin{aligned} D_{2\nu+2l+4,0}\tilde{I}_1(0) &= C^1_{2\nu+2l+3}(\nu)v_0 v_{2\nu+2l+2} + C^2_{2\nu+2l+3}(\nu)v_0^2 v_{2\nu+2l+1} \\ &\quad + Q_{2\nu+2l+3}(v), \\ D_{2\nu+2l+3,0}\tilde{I}_1(0) &= C^0_{2\nu+2l+2}(\nu)v_{2\nu+2l+2} + C^1_{2\nu+2l+2}(\nu)v_0 v_{2\nu+2l+1} \\ &\quad + Q_{2\nu+2l+2}(v) \end{aligned}$$

where Q are the same polynomials that appear also above. Again we have to stablish that determinant:

$$\Delta_k(\nu) = C^1_{2\nu+2k+2}(\nu)C^1_{2\nu+2k+1}(\nu) - C^2_{2\nu+2k+2}(\nu)C^0_{2\nu+2k+1}(\nu)$$

of the coefficients of the preceding system of equations is nonvanishing.

The Mellin Transforms of $C_{j,\nu}$. Finally all these asymptotic coefficients coincide with the coefficients of the Laurent expansion of the Mellin transform

$$\hat{c}_{j,\nu}(s)=\int_0^\infty x^s c_{j,\nu}(x)\frac{dx}{x}$$

as it is explained in the appendix on the Singular Asymptotics Lemma. In the sequel we' ll employ the identification provided by the identities:

$$(3)\qquad D_{k,0}(c_{j,\nu})(0)=Res_{s=-k}\hat{c}_{j,\nu}\qquad u_{k,0}(c_{j,\nu})=\hat{c}_{j,\nu}(-k)$$

The function $c_{j,\nu}(y)$ is written under the following change of variables

$$\theta_1=2\overline{\theta}_1-1,\qquad \theta_2=\frac{(1-\overline{\theta_1})\overline{\theta_2}}{1-\overline{\theta}_2}$$

in the form:

$$c_{j,\nu}(y)=\int_0^1\int_0^1\frac{d\theta_1 d\theta_2}{1-\theta_2}(\theta_1\theta_2)^{\nu-\frac{1}{2}}(1-\theta_1\theta_2)^{\nu-\frac{1}{2}}(\frac{1-\theta_1}{1-\theta_2})^{2\nu-1}$$
$$\times\log^j(\frac{1-\theta_1\theta_2}{\theta_1\theta_2})\exp(-\frac{2}{y}\frac{1-\theta_1}{1-\theta_2}).$$

Further we perform the change of variables

$$(0,1)\times(0,1)\ni(\theta_1,\theta_2)\mapsto(\xi,\eta)\in(0,1)\times(0,\infty),$$

$$\xi=\theta_1\theta_2,\qquad \eta=\frac{1-\theta_1}{1-\theta_2},$$

$$\frac{d\theta_1 d\theta_2}{1-\theta_2}=\frac{d\xi d\eta}{2r(\xi,\eta)(1+\eta)}$$

where $r(\xi,\eta)=(1-\frac{4\xi\eta}{(1+\eta)^2})^{\frac{1}{2}}$. In these coordinates the preceding integral is written as:

$$c_{j,\nu}(y)=\int_0^\infty\frac{\eta^{2\nu+1}e^{-\frac{2\eta}{y}}d\eta}{1+\eta}\int_0^1\frac{d\xi}{r(\xi,\eta)}[\xi(1-\xi)]^{\nu-\frac{1}{2}}\log^j(\frac{\xi}{1-\xi}).$$

In the sequel we 'll derive its asymptotics as $y\to 0$. To that end, we perform the change of variable $\tilde{\xi}=\frac{\xi}{1-\xi}$ to obtain finally by abusing notation for the function $r(\xi,\eta)$ that

$$c_{j,\nu}(y) = \int_0^\infty \frac{e^{-\frac{2\eta}{y}}\eta^{2\nu+1}d\eta}{1+\eta}\int_0^\infty \frac{\xi^{\nu-\frac{1}{2}}d\xi}{r(\xi,\eta)}\frac{\log^j \xi}{(1+\xi)^{2\nu+1}}.$$

By a Taylor expansion of the function $r(\xi,\eta)$ we obtain the Mellin transform of the function $c_{j,\nu}(s)$:

$$\widehat{c}_{j,\nu}(s) = (-1)^j 2^{s-2\nu-1}\Gamma(-s+2\nu+1)\sum_{l=0}^{\infty} d_l^j(\nu)\Gamma(s+n+1)\Gamma(n-s)$$

where

$$d_l^0(\nu) = \frac{(2l+1)B(l+\nu+\frac{1}{2},\nu+\frac{1}{2})}{(\Gamma(l+1))^2}$$
$$d_l^1(\nu) = d_l^0(\nu)[\psi(l+\nu+\frac{1}{2})-\psi(\nu+\frac{1}{2})],$$
$$d_l^2(\nu) = d_l^0(\nu)[(\psi(l+\nu+\frac{1}{2})-\psi(\nu+\frac{1}{2}))^2-\psi'(l+\nu+\frac{1}{2})-\psi'(\nu+\frac{1}{2})]$$

These combined with Stirilng's formula suggests that the series converges absolutely for $0<\Re s<1$ and represents a meromorphic function with poles at the integral points. Using the Laurent expansion of the Gamma function we obtain that for $l<k$

$$\begin{aligned} 2^{s-1}\Gamma(-s+2\nu+1)\Gamma(s+l+1)\Gamma(l-s) = \\ \frac{(-1)^{k-l}\Gamma(l+k+1)\Gamma(k+2\nu+1)}{2^{k-1}\Gamma(k-l)} \\ \times [\frac{1}{s+k} - (\psi(l+k+1)+\psi(k-l+1) \\ -\log 2+\psi(2\nu+k+1)+O(s+k)] \end{aligned}$$

and for $l\geq k$

$$\begin{aligned} &2^{s-1}\Gamma(-s+2\nu+1)\Gamma(s+l+1)\Gamma(l-s) \\ &\quad = 2^{-(k+1)}\Gamma(k+2\nu+1)\Gamma(l-k+1)\Gamma(l+k+1)+O(s+k). \end{aligned}$$

Furthermore let $N = [2\nu]$ then the Taylor expansions at $-(2\nu + k), k \in \mathbf{Z}_+$ give that for $l \leq N + k$:

$$\begin{aligned} &2^{s-1}\Gamma(-s + 2\nu + 1)\Gamma(s + l + 1) \cdot \Gamma(l - s) \\ &\quad = \frac{\Gamma(k + 4\nu + 1)\Gamma(2\nu + k + l + 1)}{\sin 2\nu\pi\Gamma(2\nu + k - l)} + O(s + 2\nu + k) \end{aligned}$$

and for $l \leq N + k$ we have

$$\begin{aligned} &2^{s-1}\Gamma(-s + 2\nu + 1)\Gamma(s + l + 1) \cdot \Gamma(l - s) \\ &\quad = 2^{-(2\nu+k+1)}\Gamma(k + 4\nu + 1)\Gamma(l + 1 - 2\nu + k)\Gamma(2\nu + k + l) \\ &\qquad + O(s + 2\nu + k) \end{aligned}$$

The identities. At this point notice that since

$$\begin{aligned} \tilde{M}_{0,\nu}(x) &= 2^{\nu}\Gamma(\nu + \frac{1}{2})\sqrt{\pi}x^{-\nu}e^{-x}I_{\nu}(x), \\ \tilde{W}_{0,\nu}(x) &= \frac{2^{-\nu}\Gamma(\nu + \frac{1}{2})}{\sqrt{\pi}}x^{-\nu}e^{x}K_{\nu}(x) \end{aligned}$$

then

$$\mathcal{C}_{0,\nu}(y) = yI_{\nu}(y)K_{\nu}(y)$$

and this suggests further that

$$\widehat{\mathcal{C}}_{0,\nu}(s) = \frac{\Gamma(\frac{1}{2} - \frac{s}{2} + \nu)\Gamma(\frac{1}{2} - \frac{s}{2})\Gamma(\frac{s}{2})}{4\sqrt{\pi}\Gamma(\frac{1}{2} + \nu + \frac{s}{2})}.$$

The general form of the asymptotic expansion suggests the absence of logarithms in the odd powers that in turn implies the identities $B^{j}_{2m+1-j}(\nu) = 0$. In parallel the exact form of $\widehat{\mathcal{C}}_{0,\nu}$ suggests that for $\nu = n + \frac{1}{2}$ then $B^{0}_{k}(n + \frac{1}{2}) = 0$ for $k = -2(n + l)$ whereas $C^{0}_{2\nu+2k+3}(\nu) = 0$. We 'll employ later these identities but prior to that we study the resulting coefficients.

The asymptotic coefficients in the recurrence relation. We treat the preceding cases separately.

A. $\nu = n + \frac{1}{2}$**.** The required coefficients are given by

$$B^{j}_{k+j+1}(\nu) = \frac{(-1)^{k}\Gamma(2n + k + j + 3)}{2^{2n+k+j+1}}[\beta^{j}_{k+j,0}(\nu) - \beta^{j}_{k+j,1}(\nu)]$$

where for $j = 1, 2$

$$\beta^1_{2k+1,0}(\nu) = \sum_{l=0}^{k} d^j_{2l}(\nu)\frac{\Gamma(2k+2+2l)}{\Gamma(2k+2-2l)}$$

$$\beta^1_{2k+1,1}(\nu) = \sum_{l=0}^{k} d^j_{2l+1}(\nu)\frac{\Gamma(2k+2l+3)}{\Gamma(2k+1-2l)}$$

$$\beta^2_{2k+2,0}(\nu) = \sum_{l=0}^{k+1} d^j_{2l}(\nu)\frac{\Gamma(2k+3+2l)}{\Gamma(2k+3-2l)}$$

$$\beta^2_{2k+2,1}(\nu) = \sum_{l=0}^{k+1} d^j_{2l+1}(\nu)\frac{\Gamma(2k+2l+4)}{\Gamma(2k+3-2l)}$$

as well as that for δ_{ij} the usual Kronecker symbol:

$$\begin{aligned} C^j_{k+j+1}(\nu) = -\frac{(-1)^k\Gamma(k+2n+j+2)}{2^{k+j+2n+3}}&[\gamma^j_{k+j+1,0}(\nu) - \gamma^j_{k+j+1,1}(\nu) \\ &+ (1-\delta_{j0})[\gamma^0_{k+j+1,0}(\nu) - \gamma^j_{k+j+1,1}(\nu)] \\ &+ (\psi(k+j+2n+3) - \log 2) + r_{k+j+1}(\nu)] \end{aligned}$$

where now for $j = 0, 1$

$$\gamma^0_{2k+2,0}(\nu) = \sum_{l=0}^{k+1} d^j_{2l}(\nu)\frac{\Gamma(2k+2l+4)}{\Gamma(2k+3-2l)}\big(\psi(2l+2k+3) + \psi(2k-2l+3)\big)$$

$$\gamma^0_{2k+2,1}(\nu) = \sum_{l=0}^{k+1} d^j_{2l+1}(\nu)\frac{\Gamma(2k+2l+4)}{\Gamma(2k+3-2l)}\big(\psi(2l+2k+4) + \psi(2k+3-2l)\big)$$

$$\gamma^1_{2k+1,0}(\nu) = \sum_{l=0}^{k} d^j_{2l}(\nu)\frac{\Gamma(2k+2l+3)}{\Gamma(2k+2-2l)}\big(\psi(2l+2k+2) + \psi(2k-2l+2)\big)$$

$$\gamma^0_{2k+1,1}(\nu) = \sum_{l=0}^{k} d^j_{2l+1}(\nu)\frac{\Gamma(2k+2l+3)}{\Gamma(2k+2-2l)}\big(\psi(2l+2k+3) + \psi(2k+2-2l)\big)$$

$$r_m(\nu) = \sum_{l=0}^{\infty} d^j_{l+m}(\nu)\Gamma(l+1)\Gamma(l+2m+1)$$

We will use the above in order to get $B^1_{2n+2k+2}(\nu)$, $B^2_{2n+2k+2}(\nu)$, $C^0_{2n+2k+1}(\nu)$, $C^1_{2n+2k+1}(\nu)$.

II. $2\nu \notin \mathbf{Z}_+$**.** Let $N = [2\nu]$ then we get that:

$$C^j_{2\nu+k+j}(\nu) = \frac{\Gamma(k+4\nu+j+1)}{2^{4\nu+k+j+1}} \times [\frac{\pi}{\sin 2\nu\pi}(-1)^k\big(\gamma^j_{N+k+j,0}(\nu) - \gamma^j_{N+k+j,1}(\nu)\big) + \rho_{N+k+j+1}(\nu)]$$

where

$$\gamma^j_{N+k+j,0}(\nu) = \sum_{l=0}^{[\frac{N+k+j}{2}]} d^j_{2l}(\nu)\frac{\Gamma(2\nu+k+2l+j+2)}{\Gamma(2\nu+k+j-2l)}$$

$$\gamma^j_{N+k+j,1}(\nu) = \sum_{l=0}^{[\frac{N+k+j}{2}]} d^j_{2l+1}(\nu)\frac{\Gamma(2\nu+k+2l+j+3)}{\Gamma(2\nu+k+j-2l-1)}$$

$$\rho_{N+k+j+1}(\nu) = \sum_{l=0}^{\infty} d^j_{l+N+k+j+2}(\nu)\Gamma(l+N-2\nu+1) \times \Gamma(l+2\nu+N+2k+2j+1)$$

The preceding formulae will provide $C^0_{2\nu+2k+2}(\nu)$, $C^1_{2\nu+2k+1}(\nu)$, $C^1_{2n+2k+2}(\nu)$, $C^2_{2n+2k+2}(\nu)$.

Certain elementary estimates. An application of the steepest descent provides the following elementary estimate for the Γ-function, for $\delta \in (0, \frac{1}{2})$, $x \geq 1$:

$$c_\delta(x+1)^x e^{-(1-\frac{4\delta^2}{2-\delta^2})} \leq \Gamma(x) \leq C_\delta(x+1)^x e^{-(1-\delta^2)x}$$

while the Euler-MacLaurin formula gives for the ψ, ψ'-functions and $a > 0, k \in \mathbf{Z}_+$:

$$\pi_{21}(a) < \psi(k+a+1) - \log(k+1) + \frac{1}{6(k+a)} < \pi_{11}(a),$$

$$\pi_{12}(a) < \psi'(k+a+1) - \frac{k-1}{(a+1)(k+a)} + \frac{1}{3(k+a)^2} < \pi_{22}(a)$$

as well as that

$$\pi_{11}(a) = \psi(a) + \frac{1}{a} + \frac{4a+3}{6(a+1)^2}, \pi_{21}(a) = \psi(a) + \frac{1}{a} + \frac{2a+3}{6(a+1)^2},$$

$$\pi_{12}(a) = \psi'(a) - \frac{1}{a^2} + \frac{3a+5}{6(a+1)^3} + \frac{a+6}{8(a+1)},$$

$$\pi_{22}(a) = \psi'(a) - \frac{1}{a^2} + \frac{3a+5}{6(a+1)^3} - \frac{a+6}{8(a+1)},$$

Actually we have the elementary inequality for $t > 0$,

$$min(a,b) \leq \frac{e^{at}-1}{e^{bt}-1} \leq max(a,b)$$

which is used after the

$$n^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^s e^{-nt} \frac{dt}{t}$$

These imply that for $\beta_\delta < 1$:

$$\frac{\kappa_1(\delta,\nu)\beta_\delta^l}{(l+1)^{2l+\nu+\frac{1}{2}}} \leq d_l^0(\nu) \leq \frac{\kappa_2(\delta,\nu)}{\beta_\delta^l (l+1)^{2l+\nu+\frac{1}{2}}}.$$

that allow us to obtain that for $\zeta_\delta(m) = 2m + \delta^2 + 3$, $\eta_1(\delta) = \frac{2\delta^2-15\delta-8}{2-\delta^2}$, $\eta_2(\delta) = \frac{2\delta^2-7\delta+14}{2-\delta^2}$ the estimates

$$\kappa_1(\delta) < \beta_{m,l}^0(\nu) \leq \frac{\kappa_2(\delta,\nu)e^{m\zeta_\delta(m)}}{m^{2\nu}\zeta_\delta(m)^{\nu+\frac{1}{2}}}(1 + O(\zeta^{\nu+\frac{1}{2}} m^{2\nu} e^{-\zeta_\delta(m)m})),$$

$$\kappa_1(\nu) < \gamma_{m,l}^0(\nu) \leq \frac{\kappa_2(\delta,\nu)e^{m\zeta_\delta(m)}}{m^{2\nu}\zeta_\delta(m)^{\nu+\frac{1}{2}}}(1 + O(\zeta^{\nu+\frac{1}{2}} m^{2\nu} e^{-\zeta_\delta(m)m}))$$

$$\frac{\kappa_2(\delta,\nu)e^{(\log 4 - 2 - \eta_2(\delta))m}}{(1+m)^{\nu+\frac{5}{2}}} < r_m(\nu), \rho_m(\nu) < \frac{\kappa_1(\delta,\nu)e^{(2+\log 4-\eta_1(\delta))m}}{(1+m)^{\nu+\frac{5}{2}}}$$

At this point we remark that the preceding identities are expressed in this notation in the form

$$\beta_{2k+1,0}^0(n+\frac{1}{2}) = \beta_{2k+1,1}^0(n+\frac{1}{2}),$$

$$\frac{\pi}{\sin 2\nu\pi}(-1)^k\big(\gamma_{N+k+j,0}^0(\nu) - \gamma_{N+k+j,1}^0(\nu)\big) + \rho_{N+k+j+1}(\nu) = 0$$

The determinants of the recurrence relation are given respectively by

$$\Delta_k(\nu) = [B_{2n+2k+2}^1(\nu)C_{2n+2k+1}^1(\nu) - B_{2n+2k+2}^2(\nu)C_{2n+2k+1}^0(\nu)](n+k+\frac{1}{2})v_0^2,$$

$$\Delta_k(\nu) = [C_{2\nu+2k+2}^1(\nu)C_{2\nu+2k+1}^1(\nu) - C_{2\nu+2k+2}^2(\nu)C_{2\nu+2k+1}^0(\nu)]v_0^2.$$

The identities in conjuction with the elementary inequality

$$\varepsilon(1 - x^{\varepsilon}) \leq \log x \leq \varepsilon(x^{\varepsilon} - 1)$$

allows to exlcude the annihilation of the determinants.

4.2. Case II. The coefficients of the expansion

In the preceding section we assumed that $v_0 \neq 0$ when the 'initial power' $\alpha = -1$. However we mentioned that when the initial power $\alpha > -1$ then we should use the Neumann series around the Bessel operator H_κ. The preceding paragraph suggests that certain terms are excluded from the first order perturbational terms and hence signify that the potential is decomposed as:

$$v(x) = x^{2\nu} v_0 + x^{2\nu} u(x) + \tilde{v}$$

where $u \in C_0^\infty(\overline{\mathbf{R}_+})$ and as $x \to 0$, for $u_{2j} = v_{2\nu+2j}$ then $u(x) \sim \sum_{j=1}^\infty u_{2j} x^{2j}$. Expanding the trace we concentrate on the contribution of the first three terms, abbreviating $R_\lambda \equiv R_\lambda(H_\kappa)$ to obtain $mod(\lambda^{-\infty})$:

$$\begin{aligned} Tr(R_\lambda \cdot v R_\lambda \cdot v) = {} & 2u_0 Tr(R_\lambda x^{2\nu} R_\lambda \cdot \tilde{v}) + 2u_0 Tr(R_\lambda x^{2\nu} R_\lambda x^{2\nu} \cdot u) \\ & + u_0^2 Tr(R_\lambda x^{2\nu} R_\lambda \cdot x^{2\nu}) + 2Tr(R_\lambda \cdot \tilde{v} R_\lambda \cdot x^{2\nu} u) \\ & + Tr(R_\lambda \cdot \tilde{v} R_\lambda \cdot \tilde{v}) + Tr(R_\lambda \cdot x^{2\nu} u R_\lambda \cdot x^{2\nu} u). \end{aligned}$$

However, there we should assume that $u_0 = v_{2\nu} \neq 0$; otherwise we have through the commutator identity

$$x^{2j} R_\lambda = R_\lambda x^{2j} + 2j R_\lambda D_{2j-3} x^{2(j-1)} R_\lambda$$

which by iteration allows us to exhaust all the $x-$ powers at the cost of R_λ-powers. At this point the scaling argument restricts the contribution to a given order only to that from the first term. Therefore, the traces are essentially comprised in the general form

$$I(\sigma) := Tr(R_\sigma \cdot x^{2\nu} \cdot R_\sigma \cdot \mathcal{V})$$

where $\mathcal{V}$ stands for $x^{2\nu}u, \tilde{v}$. The identity

$$\partial_x x^{\nu+1} I_{\nu+1}(x) = x^{\nu+1} I_\nu(x) x$$

suggests further that

$$\partial_x x^{2\nu+2}(I^2_{\nu+1}(x) + I^2_\nu(x)) = 2(2\nu+1)x^{2\nu+1}I^2_\nu(x)$$

and an integration of the latter reduce the study to that of integrals of the form for $\xi = \frac{\sigma}{x}$:

$$I(\sigma) = \frac{\sigma^2}{2}\int_0^\infty \frac{dx}{x}x^{2\nu+2}\mathcal{V}(x)\mathcal{F}(\xi)$$

where

$$\mathcal{F}(\xi) = \xi^2\frac{1}{2\nu+1}(\xi^{-2}B_0(\frac{1}{\xi}) + B_1(\frac{1}{\xi}))$$

having set that

$$B_j(\xi) := (\xi^j K_\nu I_{\nu+j})(\xi)\cdot(\xi^j K_\nu I_{\nu+j})(\xi),$$

The Singular asymptotics lemma suggests that if $2\nu+\alpha \notin \mathbf{Z}$ $v_\alpha \neq 0$ is a nonvanishing asymptotic coefficient of the "potential " $\mathcal{V}$ then

$$D_{2\nu+\alpha+2,0}I(0) = C_{2\nu+\alpha+2}(\nu)u_0v_\alpha + P_{2\nu+\alpha+2}(v)$$

while if $2\nu+\alpha \in \mathbf{Z}$ then

$$D_{2\nu+\alpha+2,1}I(0) = B_{2\nu+\alpha+2}(\nu)u_0v_\alpha + Q_{2\nu+\alpha+2}(v).$$

For these we need the Mellin tranfrom of $\mathcal{F}(\xi)$ which are calculated in the next paragraph since

$$C_{2\nu+\alpha+2}(\nu) = u_{2\nu+\alpha+2,0}(\mathcal{F}) = \widehat{\mathcal{F}}(-2\nu-\alpha-2),$$

$$B_{2\nu+\alpha+2}(\nu) = D_{2\nu+\alpha+2,0}\mathcal{F} = Res_{s=-(2\nu+\alpha+2)}(\widehat{\mathcal{F}}).$$

4.3. Bessel function formulae

We give in some detail the calculation of the Mellin transform of the functions: $\mathcal{B}_0, \mathcal{B}_1$. The calculations of the Mellin transforms are executed using the integral representation for $j=0,1$

$$B_j(\xi)=\xi^j K_\nu I_{\nu+j}(\xi)=\int_0^\infty \frac{x^{j+2}}{x^2+\xi^2}J_{\nu+j}(x)J_\nu(x)\frac{dx}{x}$$

and the classical Weber-Schaftheitlin integral[3]. The convolution formula allows thus to obtain for:

$$\widehat{\mathcal{B}_j}(s)=\frac{(\Gamma(j+\nu-\frac{s}{2}))^2}{8\sqrt{\pi}\Gamma(\frac{1}{2}+\nu)^2(2\pi i)}\cdot\int_{\lambda-i\infty}^{\lambda+i\infty}F_1(w)F_2(w)F_3(w)F_4(w)dw$$

setting

$$F_1(w):=\frac{\Gamma(\frac{w-s}{2})\Gamma(-\frac{w}{2}+\nu+j)}{\Gamma(-\frac{s}{2}+\nu+j)},\qquad F_2(w):=\frac{\Gamma(-\frac{w}{2})\Gamma(\frac{w-s}{2}+\nu+j)}{\Gamma(-\frac{s}{2}+\nu+j)},$$

$$F_3(w):=\frac{\Gamma(\frac{s-w+1}{2})\Gamma(\frac{1}{2}+\nu)}{\Gamma(\frac{s-w}{2}+1+\nu)},\qquad F_4(w):=\frac{\Gamma(\frac{w+1}{2})\Gamma(\frac{1}{2}+\nu)}{\Gamma(\frac{w}{2}+1+\nu)}.$$

Next we study the meromorphic properties of the function $\widehat{B_j}$: the resultant formula implies in the domain $D_3=\{(u,v,w)\in(\overline{\mathbf{R}_+})^3/w\le 1, uv\ge w\}$ that:

$$\widehat{\mathcal{B}_j}(s)=\frac{(\Gamma(j+\nu-\frac{1}{2}s))^2}{\sqrt{\pi}(\Gamma(\frac{1}{2}+\nu))^2}\cdot I_j(s)$$

where

$$I_j(s):=\int_{D_3}\frac{dudvdw}{uvw}[u^{-s}((1+u^2)^{-\nu+\frac{s}{2}})][v^{-s+2\nu+j}(1+v^2)^{-\nu-j+\frac{s}{2}}]$$
$$\cdot\, w^{s+1}(1-w^2)^{\nu-1/2}(\frac{w}{uv})\cdot(1-\frac{w^2}{u^2v^2})^{\nu-1/2}$$

This is written further through a Taylor development as

$$I_j(s)=\sum_{l=0}^{\infty}\frac{\Gamma(\frac{s}{2}+\frac{3}{2}+l)}{l!\Gamma(\nu+\frac{1}{2}-l)\Gamma(\frac{s}{2}+\nu+2+l)}I_{jl}(s)$$

---

[3]These are condensed in one formula:

$$\int_0^\infty x^{j-s}J_{\nu+j}(x)J_\nu(x)\frac{dx}{x}=\frac{\Gamma(\frac{s}{2}+\frac{1}{2})\Gamma(-\frac{s}{2}+\nu+j)}{\Gamma(\frac{1}{2}s+1)\Gamma(\frac{s}{2}+1+\nu)}$$

if we introduce the integral

$$I_{jl}(s) := \int\int_{uv\geq 1} \frac{dudv}{uv}[u^{-s-2l-1}(1+u^2)^{-\nu+\frac{s}{2}})] \cdot [v^{-s+2\nu+j-2l}(1+v^2)^{-\nu-j+\frac{s}{2}}][u^{2\nu}v^{-(2\nu+1)}+1].$$

This represents a holomorphic function in the left half plane $\Re s < 0$ positive on the negative real axis. Furthermore the substitution $u^2 = \frac{1}{1-\xi} - 1, v^2 = \frac{1}{\eta\xi} - 1$ leads to the familiar type of integral [4]

$$I_{jl}(s) = \frac{1}{4}\int_0^1\int_0^1 \frac{d\xi d\eta}{\xi\eta}\xi^{-\frac{s}{2}-\frac{1}{2}+\frac{j}{2}}\eta^{l+\frac{j}{2}}(1-\xi)^{l+\nu-\frac{1}{2}}(1-\eta\xi)^{-\frac{s}{2}+\nu-l+\frac{j}{2}-1} \cdot [1+\frac{\eta^{\nu+\frac{1}{2}}\xi^{2\nu+\frac{1}{2}}}{(1-\xi)^{\nu}(1-\eta\xi)^{\nu+\frac{1}{2}}}]$$

The following facts allow us to conclude that there exists $\alpha > -2$ such that one of the coefficients $B_{2\nu+\alpha+2}(\nu)$ or $C_{2\nu+\alpha+2}(\nu)$ is non zero. These are indeed checked easily:

1) *Notice that* $|I_1(s)| \leq \frac{1}{2}|I_0(s)|$.
2) *For* $s \in \mathbf{C}, \Re s < 0, l \in \mathbf{N}$ *we observe that* $|I_{jl}(s)| \leq I_{j0}(\Re s)$*; precisely*

$$\frac{1}{(1+M)^{2l+2\nu+1}} \leq \frac{I_{jl}(s)}{I_{j0}(s)} \leq 1 + \frac{1}{(1+M)^{2l+1+s}}$$

3) *Actually as* $l \to \infty$ *then*

$$\frac{\Gamma(\frac{s}{2}+\frac{3}{2}+l)}{l!\Gamma(\nu+\frac{1}{2}-l)\Gamma(\nu+2+\frac{s}{2}+l)} = O(l^{-1-\nu}).$$

The series converges absolutely for all $s \in \mathbf{C}$.

4) *The function* $I(s)$ *has poles at the negative of an odd integer.*

In the terminology introduced already we have the following lemma that has appeared in various forms in the literature, [4], [6],[7],[12],[3] as as well as Melrose refers to it as the push forward lemma.

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[4]If $s = 4\nu + j$ or $s = 6\nu + j - 1$ this integral is computed explicitly. We omit the result since this case falls out of our goals.

Theorem. (The singular asymptotics lemma) Let $f(y,x) \in \Gamma^\infty(\mathbf{R}^2_+)$. Let (S,S') be the asymptotic character of f. Suppose that f has compact $x-$support and

$$|\overline{P^x_z}[S']f(y,x)| \le (xy)^{\Re z+\delta} h_z(y)$$

for some $\delta = \delta_z > 0$ and h_z satisfying $\int_0^1 h_z(\frac{1}{t})\frac{dt}{t} < \infty$. Let

$$F(s) = \int_0^\infty f(\frac{s}{x}, x)\frac{dx}{x}.$$

Then $F \in \Gamma^\infty(\mathbf{R}_+)$ and $S_1 + S_2$ is an asymptotic character of F. The asymptotic coefficients of F are given by

$$D_{k,j}F(0) = \sum_{r=j}^{S(k)-1} \frac{r!}{j!} u^x_{k,r-j} D^y_{k,r} f(y,x)|_{y=0} + \sum_{r=j}^{S'(k)-1} \frac{r!}{j!} u^y_{k,r-j} D^x_{k,r} f(y,x)|_{x=0}$$
$$- \sum_{r=0}^{j-1} \frac{(j-r-1)!r!}{j!} D^x_{k,r} D^y_{k,j-1-r} f(y,x)|_{x=y=0}$$

where $u_{k,j}$ is a linear functional on $\Gamma^\infty(\mathbf{R}_+)$ defined as follows. Let S be an asymptotic character of f. Let $l \ge S(k)$ and $D_z = x\partial_x - z$. Let

$$r_k(f)(x) = f(x) - \sum_{\Re z \le \Re k, z \ne k} \sum_{j \in \mathbf{Z}_+} D_{zj} f(0) x^z \log^j x.$$

We define

$$u_{k,j}(f) = \frac{1}{(j+l)!} \int_0^\infty x^{-k}(-\log^{j+l} x) r_k(D^l_k f)(x)\frac{dx}{x}.$$

It can be readily checked that $u_{kj}(f)$ is independent of l and therefore of S, for S large enough.

The computations are in fact facilitated through Mellin transforms because of the formula, [7]:

$$u_{k,j}(f) = \hat{f}_j(-k), \qquad D_{k,j} = \hat{f}_{-j-1}(-k)$$

where $f_l(z_0)$ is the coefficient of the $(z-z_0)^j$ in the Laurent expansion of the meromorphic extension of the Mellin transform:

$$\hat{f}(z) = \int_0^\infty f(x) x^z \frac{dx}{x}$$

(which is defined for $Rez \gg 0$ if f is of bounded support) around $z = z_0$.

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