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Asymptotics of the Banana Feynman amplitudes at the large complex structure limit

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Recently Bönisch-Fischbach-Klemm-Nega-Safari [3] discovered, via numerical computation, that the leading asymptotics of the l-loop Banana Feynman amplitude at the large complex structure limit can be described by the Gamma class of a degree $(1,\dots,1)$ Fano hypersurface F in $(\mathbb{P}^1)^{l+1}$. We confirm this observation by using a Gamma-conjecture type result [10] for F.

1. Introduction

The *l-loop Banana Feynman amplitude* (see [16, (8.1)-(8.2)], [3, (2.1)]) is the integral

$$\mathcal{F}(q,t) = \int_{(\mathbb{R}_{>0})^l} \frac{1}{t-\phi_q(y)} \frac{dy_1\cdots dy_l}{y_1\cdots y_l}$$

where ϕ_q is the Laurent polynomial

$$\phi_q(y) = (q_1 y_1 + \cdots + q_l y_l + q_{l+1})(y_1^{-1} + \cdots + y_l^{-1} + 1).$$

The parameters q_i, denoted by ξ_i^2 in [3], are the squares of the internal masses and t is the square of the external momentum. When t is a large positive number, the integrand has a pole along the hypersurface $(\phi_q(y) = t)$ and the integral diverges. We regularize the integral by means of analytic continuation: we know that the integral converges for $t < 0$, and it can then be analytically continued to the complex plane (of t) minus the branch cut $[T,\infty)$, where $T := (\sum_{i=1}^{l+1} \sqrt{q_i})^2 = \min\{\phi_q(y) : y \in (\mathbb{R}_{>0})^l\}$.

The Feynman amplitude can be regarded as a relative period of the mixed Hodge structure of the pair $(\mathbb{P}_\Delta \setminus M_{q,t}, \partial\mathbb{P}_\Delta \setminus M_{q,t})$, where $\mathbb{P}_\Delta$ is an l-dimensional toric variety such that $\phi_q^{-1}(t) \subset (\mathbb{C}^\times)^l$ is compactified to an anticanonical hypersurface $M_{q,t} \subset \mathbb{P}_\Delta$ (intersecting every toric stratum properly) and $\partial\mathbb{P}_\Delta = \mathbb{P}_\Delta \setminus (\mathbb{C}^\times)^l$ is the toric boundary. As such, it satisfies *inhomogeneous* Picard-Fuchs differential equations with respect to the parameters q and t, which extend the Picard-Fuchs equations for $M_{q,t} = \overline{\phi_q^{-1}(t)}$. We refer the reader to [1–3, 13, 16] and references therein for differential equations, Hodge-theoretic and arithmetic aspects of the Feynman amplitudes.

In the present notes, we study the asymptotics of the Banana Feynman amplitude $\mathcal{F}(q,t)$ near the large complex structure limit $t = \infty$ (or equivalently $q_1 = \cdots = q_{l+1} = 0$) of $M_{q,t}$.

Theorem 1. *Let F be a degree $(1,\dots,1)$ Fano hypersurface in $(\mathbb{P}^1)^{l+1}$ and let $p_1,\dots,p_{l+1} \in H^2(F)$ denote the hyperplane classes pulled back from $(\mathbb{P}^1)^{l+1}$. For $q_1,\dots,q_{l+1},t > 0$, we have*

$$\mathcal{F}(q, t \mp \mathtt{i}0) \sim \frac{1}{t}\int_F e^{-p\log(q/t)} \cup \widehat{\Gamma}_F \cdot e^{\pm\pi\mathtt{i}c_1(F)}\Gamma(1-c_1(F)) \quad \text{as } t \to \infty$$

where $p\log(q/t) = \sum_{i=1}^{l+1} p_i \log(q_i/t)$ and the sign depends on whether we perform the analytic continuation anti-clockwise or clockwise from a negative real t.

This follows from the power series expansion of $\mathcal{F}(q,t)$ we give in Theorem 8 below. The class $\widehat{\Gamma}_F \in H^*(F)$ in the theorem is the *Gamma class* [9, 11, 14] of the tangent bundle TF; it is explicitly given as

$$\widehat{\Gamma}_F = \frac{\Gamma(1+p_1)^2\cdots\Gamma(1+p_{l+1})^2}{\Gamma(1+p_1+\cdots+p_{l+1})} = \frac{e^{-2\gamma c_1(F)}}{\Gamma(1+c_1(F))}$$

where $\Gamma(1+z) = \int_0^\infty e^{-t}t^z dt$ is the Euler Γ-function (when evaluating it at a cohomology class, we take its Taylor expansion) and $\gamma = 0.577\cdots$ is the Euler constant. We also note that $c_1(F) = p_1 + \cdots + p_{l+1}$.

Remark 2. Kerr [12, Example 9.10] outlined another way to evaluate the asymptotics of the l-loop Banana Feynman integral.

Remark 3. Theorem 1 confirms the numerical computation by Bönisch-Fischbach-Klemm-Nega-Safari [3, §3-4]. By taking the imaginary and the real parts of Theorem 1, we get the following asymptotics as $t \to \infty$:

$$\Im\mathcal{F}(q, t - \mathtt{i}0) \sim \frac{1}{t}\int_F e^{-p\log(q/t)} \cup \frac{\prod_{j=1}^{l+1}\Gamma(1+p_j)^2}{\Gamma(1+c_1(F))^2}\pi c_1(F)$$

$$= \frac{\pi}{t}\int_W e^{-p\log(q/t)} \cup \widehat{\Gamma}_W \tag{4}$$

$$\Re\mathcal{F}(q, t - \mathtt{i}0) \sim \frac{1}{t}\int_F e^{-p\log(q/t)} \cup \cos(\pi c_1(F))\frac{\Gamma(1-c_1(F))}{\Gamma(1+c_1(F))}e^{-2\gamma c_1(F)} \tag{5}$$

where $W \subset F$ is an anticanonical hypersurface, i.e. the intersection of two degree $(1, \dots, 1)$ hypersurfaces in $(\mathbb{P}^1)^{l+1}$; this is a mirror of $M_{q,t}$. These asymptotics coincide[1] with [3, (3.18); (4.19), (4.20)].

Remark 6. By the reflection principle, the imaginary part of $\mathcal{F}(q, t - \mathtt{i}0)$ with $t > 0$ can be understood as the difference $\frac{1}{2\mathtt{i}}(\mathcal{F}(q, t - \mathtt{i}0) - \mathcal{F}(q, t + \mathtt{i}0))$ of two analytic continuations. This can then be equated with the residue integral

$$\pi\int_{\phi_q^{-1}(t)\cap(\mathbb{R}_{>0})^l}\operatorname{Res}\left(\frac{1}{t-\phi_q(y)}\frac{dy_1\cdots dy_l}{y_1\cdots y_l}\right)$$

over the vanishing cycle $\phi_q^{-1}(t) \cap (\mathbb{R}_{>0})^l \subset M_{q,t}$. The Calabi-Yau Gamma conjecture [8, 10] predicts that the asymptotics of such vanishing periods should be given by the Gamma class of the mirror partner W of $M_{q,t}$, as in (4); in the case at hand this has been proved in [10, Theorem 5.7]. On the other hand, the asymptotics (5) of the real part of $\mathcal{F}(q, t)$ discovered in [3] is slightly beyond the scope of the Calabi-Yau Gamma conjecture; it is related to (a degeneration of) the mixed Hodge structure (see also the recent work [7]). In this paper, we will derive this from the *Fano* Gamma conjecture [5, 6, 9–11].

Remark 7. We can interpret $\widehat{\Gamma}_F \cdot \Gamma(1 - c_1(F))$ as the Gamma class of the total space K_F of the canonical bundle of F. See [2] for the relation to local mirror symmetry.

2. Proof of the asymptotics

Theorem 1 follows immediately from the following result (compare [10, Proposition 5.1]).

[1] The factor $e^{-2\gamma c_1(F)}$ is missing in the second expression of [3, (4.20)].

Theorem 8. *Let $q_1, \dots, q_{l+1}$ be positive real numbers. For $t \ll 0$, we have*

$$\mathcal{F}(q,t) = \frac{1}{t} \int_F I_W(q/(-t), -1) \cup \widehat{\Gamma}_F \cdot \Gamma(1 - c_1(F))$$

where $I_W(q,z)$ is the cohomology-valued hypergeometric series

$$I_W(q,z) = e^{p \log q / z} \sum_{d=(d_1,\dots,d_{l+1}) \in \mathbb{N}^{l+1}} \frac{\prod_{k=1}^{d_1+\cdots+d_{l+1}} (p_1 + \cdots + p_{l+1} + kz)^2}{\prod_{i=1}^{l+1} \prod_{k=1}^{d_i} (p_i + kz)^2} q^d$$

with $p \log q = \sum_{i=1}^{l+1} p_i \log q_i$ and $q^d = q_1^{d_1} \cdots q_{l+1}^{d_{l+1}}$.

Remark 9. The hypergeometric series $I_W(q,z)$ is the Givental I-function [4] for the anticanonical hypersurface $W \subset F$, which is mirror to $M_{q,t}$. Here we regard it as a function taking values in $H^*(F)$, rather than in $H^*(W)$.

A crucial observation [17, p.41] is the fact that the Laurent polynomial $\phi_q(y)$ is a mirror of the Fano manifold F. The Givental mirror [4, p.150, equation $(**)$] of the $(1,\dots,1)$-hypersurface $F \subset (\mathbb{P}^1)^{l+1}$ is given by the oscillatory integral

$$\int_{C \subset \{u_1 + \cdots + u_{l+1} = 1\}} e^{-(\frac{q_1}{u_1} + \cdots + \frac{q_{l+1}}{u_{l+1}})} \frac{d \log u_1 \cdots d \log u_{l+1}}{d(u_1 + \cdots + u_{l+1})}.$$

By the Przyjalkowsky change of variables [15]

$$u_1 = \frac{y_1}{1 + y_1 + \cdots + y_l}, \quad \cdots \quad u_l = \frac{y_l}{1 + y_1 + \cdots + y_l},$$
$$u_{l+1} = \frac{1}{1 + y_1 + \cdots + y_l},$$

the above oscillatory integral can be rewritten as

$$\int_{C'} e^{-(1+y_1+\cdots+y_l)(\frac{q_1}{y_1} + \cdots + \frac{q_l}{y_l} + q_{l+1})} \frac{dy_1 \cdots dy_l}{y_1 \cdots y_l}.$$

The phase function equals the Laurent polynomial $-\phi_q(y)$ after the change of variables $y_i \to y_i^{-1}$. Therefore, the Gamma-conjecture type result [10, Theorem 5.7] implies that we have

$$\int_{(\mathbb{R}_{>0})^l} e^{-\phi_q(y)} \frac{dy_1 \cdots dy_l}{y_1 \cdots y_l} = \int_F I_F(q, -1) \cup \widehat{\Gamma}_F \tag{10}$$

for $q_1, \ldots, q_{l+1} > 0$, where I_F is the Givental I-function [4] for F

$$I_F(q,z) = e^{p \log q/z} \sum_{d=(d_1,\ldots,d_{l+1}) \in \mathbb{N}^{l+1}} \frac{\prod_{k=1}^{d_1+\cdots+d_{l+1}} (p_1 + \cdots + p_{l+1} + kz)}{\prod_{i=1}^{l+1} \prod_{k=1}^{d_i} (p_i + kz)^2} q^d.$$

We substitute rq_i for q_i in the equation (10) and perform the Laplace transformation with respect to r. We find

$$\text{(11)} \quad \int_0^\infty e^{rt} dr \int_{(\mathbb{R}_{>0})^l} e^{-\phi_{rq}(y)} \frac{dy_1 \cdots dy_l}{y_1 \cdots y_l} = \int_0^\infty e^{rt} dr \int_F I_F(rq, -1) \cup \widehat{\Gamma}_F$$

for $t < 0$. A similar computation appeared in [10, Section 5.1]. Using $\phi_{rq}(y) = r\phi_q(y)$ and performing the integration in r first[2], the left-hand side yields the Feynman amplitude

$$\begin{aligned} &\int_{(\mathbb{R}_{>0})^l} \left(\int_0^\infty e^{(t-\phi_q(y))r} dr \right) \frac{dy_1 \cdots dy_l}{y_1 \cdots y_l} \\ &\qquad = - \int_{(\mathbb{R}_{>0})^l} \frac{1}{t - \phi_q(y)} \frac{dy_1 \cdots dy_l}{y_1 \cdots y_l} = -\mathcal{F}(q,t). \end{aligned}$$

The right-hand side can be computed termwise, using

$$\int_0^\infty e^{rt} \prod_{i=1}^{l+1} (rq_i)^{d_i - p_i} dr = \Gamma(1 + \textstyle\sum_{i=1}^{l+1}(d_i - p_i)) \dfrac{q^{d-p}}{(-t)^{1+\sum_{i=1}^{l+1}(d_i - p_i)}}.$$

Note that the coefficient of q^d in the series $I_F(q,-1)$ has the norm bounded by $C^{1+|d|}/|d|!$ for some $C > 1$, where $|d| = d_1 + \cdots + d_{l+1}$. From this it follows that, for a sufficiently negative $t \ll 0$, we can interchange the sum over d and the integral and that the right-hand side of (11) converges; in particular the left-hand side also does. This proves Theorem 8.

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[2]This is legitimate, because the integrand is a positive continuous function.

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