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On the complex affine structures of SYZ fibration of del Pezzo surfaces

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Given any smooth cubic curve $E \subseteq \mathbb{P}^2$, we show that the complex affine structure of the special Lagrangian fibration of $\mathbb{P}^2 \setminus E$ constructed by Collins–Jacob–Lin [12] coincides with the affine structure used in Carl–Pomperla–Siebert [15] for constructing mirror. Moreover, we use the Floer-theoretical gluing method to construct a mirror using immersed Lagrangians, which is shown to agree with the mirror constructed by Carl–Pomperla–Siebert.

1. Introduction

Mirror symmetry is a duality between the symplectic geometry of a Calabi–Yau manifold X and the complex geometry of its mirror $\check{X}$. With the help of mirror symmetry, one can achieve a lot of enumerative invariants of Calabi–Yau manifolds, which are a priori hard to compute.

To construct the mirror for a Calabi–Yau manifold, Strominger–Yau–Zaslow proposed the following conjectures [39]: First of all, a Calabi–Yau manifold X near the large complex structure limit admits a special Lagrangian fibration. This is one of the very few geometric descriptions of Calabi–Yau manifolds. Second, the mirror $\check{X}$ of X can be constructed as the dual torus fibration of X. Third, the Ricci-flat metric on X is closed to the semi-flat metric, with corrections coming from the holomorphic discs with boundaries on special Lagrangian torus fibres.

For a long time, Strominger–Yau–Zaslow conjecture serves a guiding principle for mirror symmetry. Many of its implications are proved as the building blocks for understanding mirror symmetry. For instance, it provides a geometric way of realizing the homological mirror functor [34]. However, there is very few progress on the original conjecture itself. Only very few examples of special Lagrangian fibrations are known due to technical difficulties of knowing explicit form of Ricci-flat metric. From the conjecture, one need to know the Ricci-flat metric for the existence of special Lagrangian fibration. While the explicit form of the Ricci-flat metric would involve the correction from the holomorphic discs. To retrieve such information, one need to know the boundary conditions, which are provided by the special Lagrangian torus fibres. Thus, the special Lagrangian fibration, the Ricci-flat metric and the correction from holomorphic discs form an iron triangle and firmly linked to each other. Actually, all the examples in the literature are either with respect to the flat metric or the hyperKähler rotation of the holomorphic Lagrangian fibrations. Furthermore, one usually can only track the hyperKähler manifold via Torelli type theorem after hyperKähler rotation rather than writing down the explicit equation.

To get around the analytic difficulties, Kontsevich–Soibelman [29], Gross–Siebert [25] developed the algebraic alternative to construct the mirror families using rigid analytic spaces. One takes the dual intersection complex B of the maximal degenerate Calabi–Yau varieties, there is a natural integral affine structures with singularities on B. By studying the scattering diagrams on B, one can reconstruct the Calabi–Yau family near the large complex structure limit. It is a folklore theorem that the affine manifold B is the base for the Strominger–Yau–Zaslow conjecture, while the support of the scattering diagrams are the projection of the holomorphic discs with boundaries on special Lagrangian torus fibres. There are many success of understanding mirror symmetry via this algebraic approach.

On the other hand, one can use Lagrangian Floer theory to construct mirrors and prove homological mirror symmetry. Fukaya [18] has proposed family Floer homology which was further developed by Tu [40] and Abouzaid [2, 3]. The family Floer mirror is constructed as the set of Maurer–Cartan elements for the A_∞ structures of the Lagrangian torus fibres quotient by certain equivalences. As Lagrangian torus fibres bound Maslov index zero holomorphic discs, the Maurer–Cartan elements will jump and induces nontrivial gluing of charts. It is expected that such jumps behave the same way as the cluster transformations associate to the ones in the scattering diagram.

A symplectic realization of the SYZ mirror construction was first illustrated in some inspiring examples by Auroux [1]. Using symplectic geometry, the SYZ mirror construction was realized for toric Calabi–Yau manifolds [14] by Chan, Leung and the first named author. They have interesting mirror maps and Gromov-Witten theory. The mirror construction for blowing-up of toric hypersurfaces was realized by Abouzaid-Auroux-Katzarkov [4]. Fukaya–Oh–Ohta–Ono [20–22] developed the Floer-theoretical construction in great detail for compact toric manifolds, which generalize and strengthen the result of Cho–Oh [11] for toric Fano manifolds.

In all these cases, the mirrors constructed in symplectic geometry coincide with the ones produced from Gross–Siebert program. The holomorphic discs can be written down explicitly and no scattering of Maslov index zero discs occur.

Singular SYZ fibers are the sources of Maslov index zero holomorphic discs and quantum corrections. In [7, 8], Cho, Hong and the first named author found a way to construct a localized mirror of a Lagrangian immersion by solving the Maurer–Cartan equation for the formal deformations coming from immersed sectors. Moreover, gluing between the local mirror charts based on Fukaya isomorphisms was developed in [9]. Applying to singular fibers, it gives a canonical (partial) compactification of the SYZ mirror by gluing the local mirror charts of singular fibers with those of regular tori [27].

In general, it is difficult to explicitly compute the Floer theoretical mirror. Maslov index zero discs can glue to new families of Maslov index zero discs, which is analogue of scattering or wall-crossing in Gross–Siebert program. It is in general complicated to control the scattering of Maslov index zero discs.

With the assumption that the Lagrangian fibration is special, one can have extra control of the locus of torus fibres bounding holomorphic discs. They form affine lines with respect to the complex affine structure. In particular, this allows us to study a version of open Gromov–Witten invariants defined by the third author and identified them with the tropical disc counting [31–33].

It is reasonable to expect that the Gross–Siebert mirror and the Floer-theoretical mirror are equivalent. The first step toward such statement is to identify the affine manifolds with singularities of the SYZ fibration and the one used in the Gross–Siebert program.

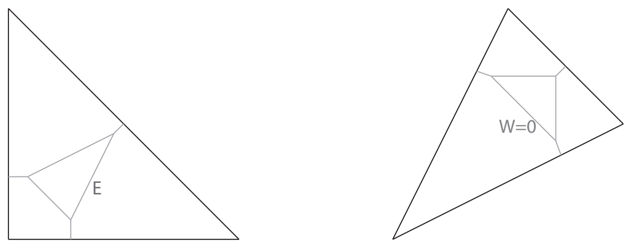

Figure 1: The moment-map polytope of $\mathbb{P}^2$ and its dual.

Conjecture 1.1. *Let X_t be a family of Calabi–Yau toric degeneration X_0 and X_t admits a special Lagrangian fibration. Then the limit of the complex affine structures of the special Lagrangian fibration coincides with the affine structures on the dual intersection complex of X_0.*

In this paper, we will establish first such a statement for the case of $\mathbb{P}^2$.

Theorem 1.2 (=Theorem 3.11). *Conjecture 1.1 holds for the SYZ fibration of $X = \mathbb{P}^2 \setminus E$, where E is a smooth cubic curve.*

The Gross-Siebert type mirror construction of $\mathbb{P}^2 \setminus E$ is done by Carl–Pomperla–Siebert [15] and has the following description. First, take the toric variety $\mathbb{P}^2/\mathbb{Z}_3$, whose moment-map polytope is dual to that of $\mathbb{P}^2$, see Figure 1. We have the meromorphic function $W = z + w + 1/zw$ on $\mathbb{P}^2/\mathbb{Z}_3$. The pole divisor of W is the sum of the three toric divisors. The zero divisor of W intersects with the pole divisor at three points. We blow up $\mathbb{P}^2/\mathbb{Z}_3$ at these three points, so that W induces an elliptic fibration. (We can further blow up the three orbifold points of $\mathbb{P}^2/\mathbb{Z}_3$ to make the total space smooth.) Finally we delete the strict transform of the three toric divisors (which is the fiber at ∞) and this defines the mirror space. The Landau–Ginzburg superpotential is the elliptic fibration map induced by W. It is also worth noticing that the theorem is also achieved by Pierrick Beausseau with a different approach [36]. We refer the readers for the inspiring heuristic discussion there about such an expectation from a different point of view.

For the family Floer mirror, it is glued from torus charts, which are the deformation spaces of Lagrangian torus fibers. Due to scattering of Maslov zero holomorphic discs, there are infinitely many walls and chambers in this case, and each chamber corresponds to a torus chart.

On the other hand, in the Fano situation of this paper, we can use the method in [9, 27] to construct a $\mathbb{C}$-valued mirror. The special Lagrangian fibration on $\mathbb{P}^2 \setminus E$ [12] has three singular fibers which are nodal tori. Instead of the (infinitely many) torus fibers, we take the monotone moment-map torus together with three monotone Lagrangian immersions (in place of the singular SYZ fibers), and glue their deformation spaces together to construct the mirror.

Theorem 1.3. *For $\mathbb{P}^2 - E$, the Floer-theoretical mirror glued from the deformation spaces of the monotone moment-map torus and the three monotone Lagrangian immersions coincides with the Carl–Pomperla–Siebert mirror described above.*

More precisely, the gluing construction has to be carried out over the Novikov field

$$\Lambda := \left\{ \sum_{i=0}^{\infty} a_i \mathbf{T}^{A_i} \mid a_i \in \mathbb{C}, A_i \in \mathbb{R} \text{ and increases to } +\infty \right\}$$

so that the Lagrangian deformation spaces have the correct topology and dimension. See Remark 4.10. After we glue up a space over Λ using Lagrangian Floer theory, we restrict to $\mathbb{C}$ to get a $\mathbb{C}$-valued mirror.

Outline of the paper

In Section 2, we review the geometry of the special Lagrangian fibration on $\mathbb{P}^2 \setminus E$ and the complex affine structure induced from the special Lagrangian fibration in Section . We also describe the affine manifold which is used to construct for mirror in [15]. In Section 3, we first explain how to use hyperKähler rotation to reduce the problem to relative periods of an extremal rational elliptic surface, where the geometry can be very explicit. Then we verified various properties of the relative periods for the proof of the main theorem. In Section 4, we carry out the Floer theoretical construction and show that it agrees with Carl–Pomperla–Siebert mirror.

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2. SYZ fibration on del Pezzo surfaces

We will first review the results in [12]: Let Y be a del Pezzo surface or a rational elliptic surface, $D \in |-K_Y|$ be a smooth anti-canonical divisor and $X = Y \backslash D$. There exists a meromorphic volume form Ω on with simple pole along D which is unique up to a $\mathbb{C}^*$-scaling. Therefore, one can view X as a log Calabi–Yau surface. Moreover, Tian–Yau proved the following theorem:

Theorem 2.1 ([41]). *There exists an exact complete Ricci-flat metric ω_{TY} on X.*

We will assume that $2\omega_{TY}^2 = \Omega \wedge \bar{\Omega}$ after a suitable scaling of Ω.

Definition 2.2. *Let X be a complex manifold with a holomorphic volume form Ω and a Ricci-flat metric ω. A half dimensional submanifold L is a special Lagrangian with respect to (ω, Ω) if $\omega|_L = 0$ and $\mathrm{Im}\Omega|_L = 0$.*

It is conjectured by Yau and also Auroux [2] that there exists a special Lagrangian fibration on X. The conjecture is proved by Colllins–Jacob–Lin earlier.

Theorem 2.3 ([12]). *The log Calabi–Yau surface X admits a special Lagrangian fibration $\pi : X \to B_{SYZ}$ with respect to ω_{TY}.*

Although the proof of the existence of special Lagrangian fibration in [12] still largely use the hyperKähler structure, an important difference from the earlier examples is that one knows which complex structure can support the special Lagrangian fibration. Moreover, one can use algebraic geometry to understand the complex structure after the hyperKähler rotation.

Theorem 2.4 ([12]). *With the above notation and $d = (-K_Y)^2$. Let $\check{X}$ denotes the underlying topological space of X with Kähler form and holomorphic* *volume form*

$$\begin{aligned} \check{\omega} &= \operatorname{Re}\Omega \\ \check{\Omega} &= \omega - \sqrt{-1}\cdot \operatorname{Im}\Omega. \end{aligned} \tag{2.1}$$

Then $\check{X}$ admits an elliptic fibration and compactifiation to a rational elliptic surface $\check{Y}$ by adding an I_d singular fibre over ∞.

$$\begin{tikzcd} \check{X} \arrow[r, hook] \arrow[d] & (\check{Y}, I_d) \arrow[d] \\ \mathbb{C} \arrow[r, hook] & (\mathbb{P}^1, \infty) \end{tikzcd}$$

From the asymptotic behavior of $\check{\Omega}$, one has

Proposition 2.5. [13] *The holomorphic* 2*-form $\check{\Omega}$ on $\check{X}$ coincide with the meromorphic* 2*-form on $\check{Y}$ with simply pole along the fibre over ∞.*

In particular, the rational elliptic surface $\check{Y}$ has singular configuration $I_9I_1^3$ for the case $Y = \mathbb{P}^2$ [12]. The extremal rational elliptic surfaces have no deformation and thus can be identified by explicit equation. In the case of $Y = \mathbb{P}^2$, $\check{X}$ can actually be realized as the fibrewise compactification of the Landau–Ginzburg mirror

$$\begin{aligned} W\colon (\mathbb{C}^*)^2 &\longrightarrow \mathbb{C} \\ (t_1, t_2) &\mapsto t_1 + t_2 + \frac{1}{t_1 t_2}. \end{aligned} \tag{2.2}$$

It is straight-forward to check that W has three critical values $\lambda_0, \lambda_1, \lambda_2$ and the cross-ratio with ∞ is fixed. Thus, we may assume that $\lambda_i = 3\zeta^i$, where $\zeta = \exp(2\pi i/3)$. The fibres of W are three-punctured elliptic curves. By computing the global monodromy which is conjugating to $\begin{pmatrix} 1 & 9 \\ 0 & 1 \end{pmatrix}$, the Lefschetz fibration $W : (\mathbb{C}^*)^2 \to \mathbb{C}$ can be compactified to such an extremal rational elliptic surface by adding three sections and an I_9 singular fibre at infinity.

There is a $\mathbb{Z}_3$-action $(x, y) \mapsto (\zeta x, \zeta y)$ on $(\mathbb{C}^*)^2$ which induces a $\mathbb{Z}_3$-action on the base $\mathbb{P}^1$ permuting the three critical values. Let E_0 be the fibre over $0 \in \mathbb{P}^1$ which is fixed by the $\mathbb{Z}_3$-action.

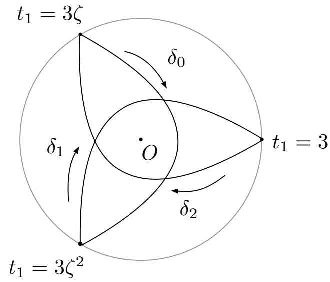

Figure 2: The vanishing cycles in E_0.

Lemma 2.6 (cf. [5, Lemma 3.1]**).** *We can choose a basis for* $\mathrm{H}_1(E_0, \mathbb{Z}) \simeq \mathbb{Z}^2$ *and orientations for the vanishing cycles* $[V_0]$, $[V_1]$, $[V_2]$ *of* λ_0, λ_1, λ_2 *such that* $[V_0]$, $[V_1]$ *and* $[V_2]$ *are represented by* $(-2, -1)$, $(1, -1)$ *and* $(1, 2)$ *respectively and the vanishing cycle from* ∞ *along the curve* $\overline{\infty O}$ *in* FIGURE 4 *is represented by* $(0, 1)$. *In particular, we have* $[V_0] + [V_1] + [V_2] = 0$.

We will describe the orientation explicitly in §3.3 **(C)**.

Given a special Lagrangian fibration $X \to B_{SYZ}$ with respect to (ω, Ω), we will denote L_q for the fibre over $q \in B_{SYZ}$. Let B_0 be the complement of discriminant locus, then there exists an integral affine structure on B_0 [26] which we will now explain below: Choose a reference fibre L_{q_0} and basis $e_i \in \mathrm{H}_1(L_{q_0})$. For a nearby torus fibre L_q and a path ϕ connecting q and q_0, let C_i be the union of the parallel transport of e_i along ϕ. Then the complex affine coordinate $f_i(q)$ of q is defined to be

$$f_i(q) := \int_{C_i} \operatorname{Im} \Omega, \tag{2.3}$$

which is well-defined since L_q, L_{q_0} are special Lagrangians. It is straightforward to check that for a different choice of the basis and paths, the transition function falls in $\mathrm{GL}(n, \mathbb{Z}) \rtimes \mathbb{R}^n$, where $n = \dim_{\mathbb{R}} L_q$. Thus, B_0 is an integral affine manifold and we say B_{SYZ} is an integral affine manifold with singularities $\Delta = B_{SYZ} \backslash B_0$. The above integral affine structure is usually known as the complex affine structure of the special Lagrangian fibration in the context of mirror symmetry.

3. Equivalence of the two affine structures

From now on, we concentrate on the case $Y = \mathbb{P}^2$ with the Landau–Ginzburg potential function (2.2). Recall that $\mathbb{P}^2$ is defined by the polytope $\Delta = \mathrm{Conv}\{(-1,-1),(2,-1),(-1,2)\}$. Let $\nabla = \Delta^\vee$ be the dual polytope. We denote by $\mathbf{P}_\nabla$ the toric variety defined by ∇ and by $\widetilde{\mathbf{P}}_\nabla \to \mathbf{P}_\nabla$ the maximal projective crepant partial resolution of $\mathbf{P}_\nabla$, which is a resolution in the present case.

We denote by q the coordinate of the target space of the potential function W in (2.2) and regard $(W - q\cdot 1)$ as a *holomorphic section* of the anti-canonical bundle over $\widetilde{\mathbf{P}}_\nabla$. Precisely, the monomials t_1, t_2, $t_1^{-1}t_2^{-1}$ correspond to the integral points $(1,0),(0,1),(-1,-1)$ in ∇ and the monomial $t_1^0t_2^0 = 1$ corresponds to the integral point $(0,0)$. The subvariety $\{W - q\cdot 1 = 0\} \subset \widetilde{\mathbf{P}}_\nabla$ gives the desired compactification of our fiber $W^{-1}(q)$. The family $\{W - q\cdot 1 = 0\}$ is a pencil spanned by the section 1 and $t_1 + t_2 + t_1^{-1}t_2^{-1}$, and can be extended to a family over $\mathbb{P}^1$. It is straightforward to check that the sections 1 and $t_1 + t_2 + t_1^{-1}t_2^{-1}$ intersect at three points. Blowing-up the base locus gives a morphism $\check{Y} \to \mathbb{P}^1$. The fiber at $\infty \in \mathbb{P}^1$ is a union of proper transforms of toric divisors in $\widetilde{\mathbf{P}}_\nabla$, which is a I_9 fiber. For simplicity, the proper transform of the I_9 fiber in $\check{Y}$ is also denoted by I_9.

Let $\check{X} := \check{Y} \setminus I_9$. First of all, it is clear that

$$\mathrm{H}_4(\check{Y},\mathbb{Z}) \cong \mathbb{Z},\ \mathrm{H}_2(\check{Y},\mathbb{Z}) \cong \mathbb{Z}^{10},\ \text{and } \mathrm{H}_0(\check{Y},\mathbb{Z}) \cong \mathbb{Z}.$$

Secondly, from the Poincaré duality for orientable manifolds, we have

$$\mathrm{H}_k(\check{X},\mathbb{Z}) \cong \mathrm{H}_{\mathrm{c}}^{4-k}(\check{X},\mathbb{Z}) \cong \mathrm{H}^k(\check{X},\mathbb{Z}),\ \forall\, k.$$

Finally, let U be the preimage of a small neighborhood around $\infty \in \mathbb{P}^1$ under $\check{Y} \to \mathbb{P}^1$. I_9 is a retract of U. Utilizing the Mayer–Vietoris resolution for simple normal crossing varieties, one can easily derive

$$\mathrm{H}^2(I_9,\mathbb{Z}) \cong \mathbb{Z}^9,\ \mathrm{H}^1(I_9,\mathbb{Z}) \cong \mathbb{Z},\ \text{and } \mathrm{H}^0(I_9,\mathbb{Z}) \cong \mathbb{Z}.$$

Consider the Mayer–Vietoris sequence assicoated to the pair $(U, \check{X})$, we can show that

$$\mathrm{H}^2(\check{X},\mathbb{C}) \cong \mathrm{H}_2(\check{X},\mathbb{C}) \cong \mathbb{C}^2$$

We put $E_q = \{W - q\cdot 1 = 0\} \subset \check{Y}$. It follows that $\mathrm{H}_2(\check{X},\mathbb{Z})$ is generated by the class of $S^1 \times S^1 \subset (\mathbb{C}^*)^2$ and the class of E_q.

From the construction, the standard toric form

$$\frac{\mathrm{d}t_1}{t_1} \wedge \frac{\mathrm{d}t_2}{t_2}$$

on $(\mathbb{C}^*)^2$ extends to a *meromorphic* form on $\widetilde{\mathbf{P}}_\nabla$ with simple poles along the union of toric divisors. Via the pullback further to $\check{Y}$, we obtain a meromorphic 2-form which has simple poles along I_9.

In what follows, we set

$$\check{\Omega} = \sqrt{-1} \cdot \frac{\mathrm{d}t_1}{t_1} \wedge \frac{\mathrm{d}t_2}{t_2}.$$

The meromorphic top form $\check{\Omega}$ has the property that

$$\operatorname{Re} \check{\Omega}\big|_{\mathrm{H}_2(\check{X},\mathbb{Z})} \equiv 0.$$

We can represent $\check{\Omega}$ in a different way, which turns out to be useful in the sequel. It follows from (2.2) that

$$\begin{aligned} \mathrm{d}q &= \mathrm{d}t_1 + \mathrm{d}t_2 - \frac{t_1 \mathrm{d}t_2 + t_2 \mathrm{d}t_1}{(t_1 t_2)^2} \\ &= \left(1 - \frac{1}{t_1^2 t_2}\right) \mathrm{d}t_1 + \left(1 - \frac{1}{t_1 t_2^2}\right) \mathrm{d}t_2. \end{aligned}$$

A direct calculation gives

$$\mathrm{d}q \wedge \mathrm{d}t_1 = \frac{1 - t_1 t_2^2}{t_2} \frac{\check{\Omega}}{\sqrt{-1}}$$

and therefore

$$\check{\Omega} = -\sqrt{-1} \cdot \frac{t_2}{1 - t_1 t_2^2} \mathrm{d}q \wedge \mathrm{d}t_1 \tag{3.1}$$

provided that $1 - t_1 t_2^2 \neq 0$.

3.1. The affine structure of Carl–Pumperla–Siebert

The construction of the mirror of a del Pezzo surface relative to a smooth anti-canonical divisor is studied in [15]. We will describe the affine manifold with singularities used in [15] below: The underlying space is $\mathbb{R}^2$ topologically.

There are three singularities with local monodromy conjugate to $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ locating at $A' = (0, -\frac{1}{2})$, $B' = (\frac{1}{2}, \frac{1}{2})$ and $C' = (-\frac{1}{2}, 0)$. To cooperate with the standard affine structure of $\mathbb{R}^2$ for computation convenience, they introduce cuts and the affine transformation as follows: let

$$\begin{aligned}
l_1^+ &= \left\{\left(\frac{1}{2}, y\right) \,\middle|\, y \geq \frac{1}{2}\right\}\\
l_1^- &= \left\{\left(x, \frac{1}{2}\right) \,\middle|\, x \geq \frac{1}{2}\right\}\\
l_2^+ &= \left\{\left(x, -\frac{1}{2}\right) \,\middle|\, x \geq 0\right\}\\
l_2^- &= \left\{\left(-t, -\frac{1}{2} - t\right) \,\middle|\, t \geq 0\right\}\\
l_3^+ &= \left\{\left(-\frac{1}{2} - t, -t\right) \,\middle|\, t \geq 0\right\}\\
l_3^- &= \left\{\left(-\frac{1}{2}, y\right) \,\middle|\, y \geq 0\right\}
\end{aligned}$$

Disgard the sector bounded by l_i^+, l_i^-, then glue the cuts by the affine transformations $\begin{pmatrix}2 & 1\\ -1 & 0\end{pmatrix}$, $\begin{pmatrix}-1 & 4\\ -1 & 3\end{pmatrix}$, $\begin{pmatrix}-1 & 1\\ -4 & 3\end{pmatrix}$ respectively and one reaches the affine manifold in [15]. See FIGURE 3 above.

3.2. A hyperKähler rotation trick

To compute the complex affine coordinates, generally one needs to compute the relative periods for the imaginary part of the holomorphic volume form on X. Technically, it is not computable generally due to the fact that the special Lagrangian fibration is never explicit. The advantage of the work of Collins–Jacob–Lin is that one knows both the explicit equation of X and $\check{X}$. From (2.3) and Theorem 2.4, one can compute the complex affine coordinates via the geometry on $\check{X}$ with a particular phase for $\check{\Omega}$. From Mayer–Vietoris sequence, one has $\mathrm{H}_2(\check{X}, \mathbb{Z}) \cong \mathrm{H}_2(X, \mathbb{Z}) \cong \mathbb{Z}^2$. Since ω_{TY} is exact, we have $\omega_{TY}|_{\mathrm{H}_2(X,\mathbb{Z})} \equiv 0$. Because of the existence of the compact special Lagrangian tori, we have $\Omega|_{\mathrm{H}_2(X,\mathbb{Z})} \neq 0$. Therefore, the phase of $\check{\Omega}$ need to be chosen such that

$$\mathrm{Re}\, \check{\Omega}\big|_{\mathrm{H}_2(\check{X},\mathbb{Z})} \equiv 0$$

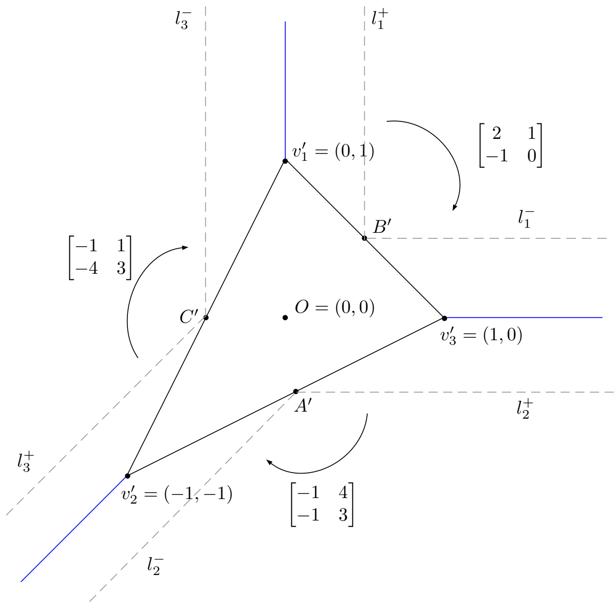

Figure 3: The affine plane given in [15].

from (2.1). To sum up, we have the following lemma:

Lemma 3.1. *We resume the notation introduced in the last paragraph in §2. The complex affine structure of the special Lagrangian fibration in Theorem 2.3 for $\mathbb{P}^2$ can be computed via*

$$f_i(q) := \int_{C_i} \operatorname{Im} \check{\Omega}$$

on the extremal rational elliptic surface $\check{Y}$ with singular configuration $I_9 I_1^3$. Here $\check{\Omega}$ is the meromorphic volume form on $\check{Y}$ with simple pole along the I_9 fibre and $\operatorname{Re}\check{\Omega}|_{\mathrm{H}_2(\check{X},\mathbb{Z})} \equiv 0$.

3.3. Relative periods on $\check{Y}$

We analyze the relative periods on $\check{Y}$ in more detail in this subsection. The fiber over q in $\check{Y} \to \mathbb{P}^1$ is denoted by E_q.

(A) Vanishing cycles. Note that E_3, $E_{3\zeta}$ and $E_{3\zeta^2}$ are all the singular fibers in $\check{X} \to \mathbb{C}$ and each of which is of type I_1. According to [5], after a parallel transport to E_0, the vanishing cycle for $E_{3\zeta^j}$ can be represented by a cycle V_j in E_0 such that the image of V_j under the projection $(t_1, t_2) \mapsto t_1$ is given by the arcs δ_j drawn in FIGURE 2.

Note that the cycle class $[V_j] \in \mathrm{H}_1(E_0, \mathbb{Z})$ is only defined up to sign at this moment because we have not fixed the orientation yet. However, once the orientation of V_0 is determined, the $\mathbb{Z}/3\mathbb{Z}$-action will uniquely determine the orientations for V_1 and V_2.

Remark 3.2. *Suppose the orientation of V_j is given with respect to the $\mathbb{Z}/3\mathbb{Z}$-action. We can accordingly choose an integral basis $\{c, d\} \subset \mathrm{H}_1(E_0, \mathbb{Z})$ such that the vanishing cycles $[V_0]$, $[V_1]$ and $[V_2]$ are represented by $-2c - d$, $c - d$ and $c + 2d$, respectively. Also note that the presentations are chosen with respect to the $\mathbb{Z}/3\mathbb{Z}$-action on the q-plane $\mathbb{C}_q$.*

(B) Lefschetz thimbles. For each j, we define a simply connected domain

$$W_j := \mathbb{C} \setminus \cup_{k \neq j} \{q : q = r\zeta^k \text{ with } r \geq 3\}. \tag{3.2}$$

For $q \in W_j$, let γ be a smooth curve joint q and $3\zeta^j$ contained in W_j. Let $V_j \subset E_0$ be the representatives described in **(A)**. Then the *Lefschetz thimble* of V_j along γ, which is denoted by $\Gamma_j^\gamma(q)$, is the union of the parallel transport of V_j along the cycle γ. Precisely,

$$\Gamma_j^\gamma(q) := \cup_{q' \in \gamma} V_j^{(q')}, \tag{3.3}$$

where $V_j^{(q')}$ is the parallel transport of V_j along any curve in W_j connecting 0 and q and then from q to q' along γ.

We shall mention that different representatives $\tilde{V}_j$ give different Lefschetz thimbles $\tilde{\Gamma}_j^\gamma(q)$ and also that the Lefschetz thimble *does* depend on the choice of the curve connecting 0 and q. One proves that in any case their difference is a *coboundary*. Consequently, by Stokes' theorem, the integral

$$\int_{\Gamma_j^\gamma(q)} \check{\Omega} \tag{3.4}$$

is independent of the choice of the representatives and the curve connecting 0 and q. However, it is defined only up to a sign at this moment because of the orientation.

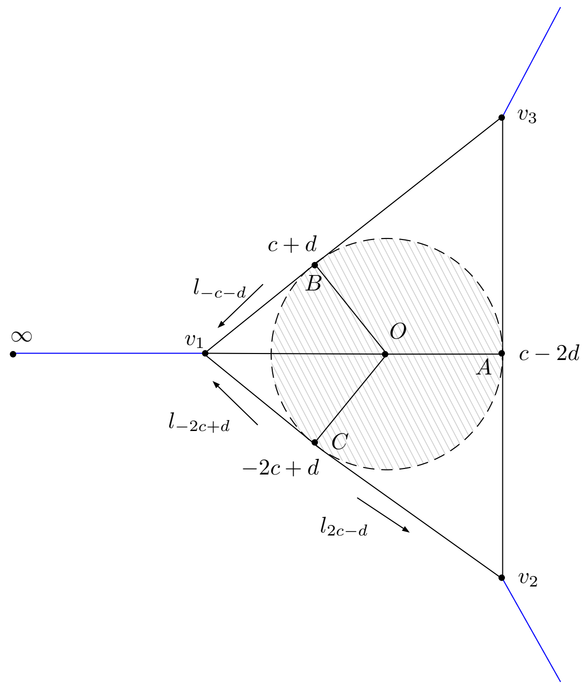

Figure 4: The orientation of the cycles. The oriented line segments $\vec{\gamma}_0$, $\vec{\gamma}_1$, and $\vec{\gamma}_2$ defined in (3.6) are the oriented line segments $\overrightarrow{AO}$, $\overrightarrow{BO}$ and $\overrightarrow{CO}$. For other notation, see the paragraph **(D)**.

(C) Orientations. Recall that δ_j is an oriented arc with orientation drawn in Figure 2. From (2.2), we can solve

$$t_2^{\pm} = \left((q-t_1) \pm \sqrt{(q-t_1)^2 - 4/t_1}\right)/2. \tag{3.5}$$

The orientation of V_0 is chosen in the following manner. First we note that V_0 is set-theoretically equal to the union of the graph of the holomorphic functions t_2^+ and t_2^- along the arc δ_0 defined in (3.5). For the graph of t_2^+, we take the induced orientation from δ_0. For the graph of t_2^-, we shall take the induced orientation from $-\delta_0$. This pins down an orientation of V_j.

For each j, let

$$\vec{\gamma}_j := \left\{q \colon q = r\zeta^j,\ 0 \le r \le 3\right\} \tag{3.6}$$

be an oriented (from $3\zeta^j$ towards 0) line segment (consult Figure 4).

We denote by $\vec{\Gamma}^{\gamma_j}_j(0)$ the Lefschetz thimble $\Gamma^{\gamma_j}_j(0)$ with the induced orientation from the S^1-bundle structure. Precisely, when restricting on $\{q\colon q = r\zeta^j,\ \ 0 \leq r < 3\}$, $\Gamma^{\gamma_j}_j(0)$ becomes an S^1-bundle whose fibers are equipped with an orientation coming from V_j. Therefore, it has an induced orientation which can be extended to the whole $\Gamma^{\gamma_j}_j(0)$.

Remark 3.3. *It follows from the construction that the orientations for* $\vec{\Gamma}^{\gamma_j}_j(0)$ *are compatible with the* $\mathbb{Z}/3\mathbb{Z}$*-action.*

Proposition 3.4. *We have, for all* j*,*

$$\int_{\vec{\Gamma}^{\gamma_j}_j(0)} \operatorname{Im} \check{\Omega} \in \mathbb{R}_+. \tag{3.7}$$

Corollary 3.5. *We have*

$$\lim_{q \to -\infty} \operatorname{Im} \int_{\vec{\Gamma}^{\gamma_0}_0(q)} \check{\Omega} = \infty. \tag{3.8}$$

Proof. We defer the proofs of Proposition 3.4 and Corollary 3.5 in Appendix A. □

Once we pin down the orientation of V_0, the orientations of V_j are also uniquely determined. We then pick an integral basis $\{c, d\} \subset \mathrm{H}_1(E_0, \mathbb{Z})$ such that the vanishing cycles $[V_0]$, $[V_1]$, and $[V_2]$ are represented by $-2c - d$, $c - d$, and $c + 2d$, respectively.

(D) Affine structures. We describe the affine structure on $\mathbb{C}$ using the elliptic fibration $\check{X} \to \mathbb{C}$. Consider the set

$$\Xi_1 := \left\{ q \in \mathbb{C} : \operatorname{Im} \int_{\Gamma^{\gamma}_1(q)} \check{\Omega} = 0,\ \gamma : \text{curve contained in } \mathbb{C} \setminus \{A, C\} \text{ joining } B \text{ and } q \right\}. \tag{3.9}$$

The condition that the imaginary part is equal to zero is independent of the choice of the curve γ since the monodromy matrices are real. The set Ξ_1 is well-defined. Notice that the set $\Xi_1 \setminus \{B\}$ has two connected components.

Working on the simply connected domain $V = W_1 \cap W_2 \cap W_3$, we can define the locus

$$l_{-c+d} := \left\{ \int_{\Gamma_1^\gamma(q)} \check{\Omega} \in \mathbb{R}_+ \text{ for any parameterized curve} \right.$$
$$\left. \gamma \subset V \text{ joining } B \text{ and } q \right\}.$$

Similarly we can define another curve $l_{c+2d} \subset \Xi_2$ by requiring that

$$l_{c+2d} := \left\{ \int_{\Gamma_2^\gamma(q)} \check{\Omega} \in \mathbb{R}_- \text{ for any parameterized curve} \right.$$
$$\left. \gamma \subset V \text{ joining } C \text{ and } q \right\}.$$

Remark 3.6. *The above definition explains the notation in* FIGURE 4*. However, we have not verified the validity of the intersection point v_1 in* FIGURE 4*. We will prove that the curves $\overline{O\infty}$, l_{-c+d} and l_{c+2d} intersect at one point, where $\overline{O\infty}$ denotes the negative real axis.*

We resume the notation given in this subsection and in FIGURE 4 and FIGURE 2 without recalling it. To describe the affine structure a little bit more, we need to study the integration over the Lefschetz thimbles.

Definition 3.7. *Let $q \in \overline{O\infty}$. We denote by $\vec{\sigma}_0(q)$ the oriented curve $\overline{OA} \cup \overline{Oq}$ from A toward q, $\vec{\sigma}_1(q)$ the oriented curve $\overline{OB} \cup \overline{Oq}$ from A toward q and by $\vec{\sigma}_2(q)$ the oriented curve $\overline{OC} \cup \overline{Oq}$ from A toward q (cf.* FIGURE 5*).*

For simplicity, we will write $\Gamma_j(q) := \Gamma_j^{\vec{\sigma}_j(q)}(q)$ (see (3.3)*), i.e., $\Gamma_0(q)$ is the Lefschetz thimble of the cycle V_0 along $\vec{\sigma}_0(q)$, $\Gamma_1(q)$ is the Lefschetz thimble of the cycle V_1 along $\vec{\sigma}_1(q)$ and $\Gamma_2(q)$ is the Lefschetz thimble of the cycle V_2 along $\vec{\sigma}_2(q)$.*

Lemma 3.8. *The map $\phi_0(q) = \bar{q}$ fixing O and A but exchanging B and C is an automorphism of the affine structure.*

Proof. The affine lines are mapped to affine lines via ϕ_0. □

In particular, the lemma implies that the fixed locus of ϕ_0, an arc from A to infinity passing through O and another arc from A to infinity without passing through O, are affine lines.

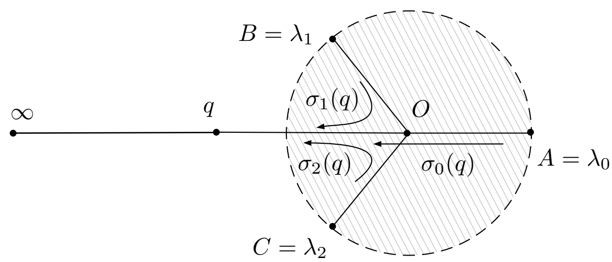

Figure 5: The curves $\sigma_j(q)$.

Corollary 3.9. *The induced map* $(\phi_0)_*\colon \mathrm{H}_1(E_0, \mathbb{Z}) \to \mathrm{H}_1(E_0, \mathbb{Z})$ *under the basis* $\{c, d\}$ *is given by*

$$\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Proof. Note that ϕ_0 is the complex conjugation. We have $(\phi_0)_*(-2c - d) = 2c + d$ and $(\phi_0)_*(c - d) = -c - 2d$ by our choice of orientations, which yields the corollary. $\square$

Lemma 3.10. *We have*

$$\int_{\Gamma_1(q)-\Gamma_2(q)} \check{\Omega} \in \mathbb{R}, \ \forall\ q \in \overline{O\infty}.$$

Proof. The lemma is proved by using the $\mathbb{Z}/2\mathbb{Z}$-symmetry ϕ_0. Since $\Gamma_1(q) - \Gamma_2(q)$ is invariant under ϕ_0, we have

$$\int_{\Gamma_1(q)-\Gamma_2(q)} \check{\Omega} = \int_{(\phi_0^{-1})_*(\Gamma_1(q)-\Gamma_2(q))} \phi_0^*\check{\Omega} = \int_{\Gamma_1(q)-\Gamma_2(q)} \overline{\check{\Omega}}$$

and the conclusion holds. $\square$

We conclude this paragraph by proving the validity of the existence of the triple intersection point v_1 in FIGURE 4. Let us write

$$\begin{aligned} F_1(q) :&= \int_{\Gamma_1(q)} \operatorname{Im}\check{\Omega} \\ &= \int_{\Gamma_1(q)-\Gamma_1(0)} \operatorname{Im}\check{\Omega} + \int_{\Gamma_1(0)} \operatorname{Im}\check{\Omega} \\ &= \int_{\Gamma_1(q)-\Gamma_1(0)} \operatorname{Im}\check{\Omega} + \int_{\Gamma_0(0)} \operatorname{Im}\check{\Omega}. \end{aligned} \tag{3.10}$$

The last equality in (3.10) holds by the $\mathbb{Z}/3\mathbb{Z}$-symmetry on the q-plane.

For $q \leq 0$,

$$\Gamma_1(q) - \Gamma_1(0) = \cup_{q' \in \overline{qO}} V_1^{(q')}. \tag{3.11}$$

Consequently, utilizing Lemma 3.10 and the relation $2V_1 = -V_0 - (V_2 - V_1)$, we get

$$\begin{aligned} \int_{\Gamma_1(q)-\Gamma_1(0)} \operatorname{Im} \check{\Omega} &= \int_{\cup_{q' \in \overline{qO}} V_1^{(q')}} \operatorname{Im} \check{\Omega} \\ &= -\frac{1}{2} \int_{\Gamma_0(q)-\Gamma_0(0)} \operatorname{Im} \check{\Omega} - \frac{1}{2} \int_{\Gamma_2(q)-\Gamma_1(q)} \operatorname{Im} \check{\Omega} \\ &= -\frac{1}{2} \int_{\Gamma_0(q)-\Gamma_0(0)} \operatorname{Im} \check{\Omega}, \end{aligned} \tag{3.12}$$

where the last equality comes from (3.10). Then (3.10) is transformed into

$$F_1(q) = -\frac{1}{2} \int_{\Gamma_0(q)} \operatorname{Im} \check{\Omega} + \frac{3}{2} \int_{\Gamma_0(0)} \operatorname{Im} \check{\Omega}, \text{ for } q \leq 0.$$

From Corollary 3.5, we see that $F_1(q) \to -\infty$ when $q \to -\infty$. Together with $F_1(0) > 0$, there exists some $v_1 \in \overline{O\infty}$ such that

$$F_1(v_1) = 0.$$

Then v_1 is the triple intersection point we are looking for.

3.4. Proof of the Main Theorem

We will prove the main theorem

Theorem 3.11. *There exists an affine isomorphism between B_{SYZ} and B_{CPS}.*

Recall that the base B_{SYZ} of the special Lagrangian fibration for $\mathbb{P}^2 \backslash E$ can be topologically identified with $\mathbb{C}$. There are with three singularities of affine structures.

Lemma 3.12. *Assume that the vanishing cycle has class (p,q), then the monodromy across the branch cut is* $\begin{pmatrix} 1-pq & p^2 \\ -q^2 & 1+pq \end{pmatrix}$.

Proof. From Picard–Lefschetz formula, the monodromy is conjugate to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and the vanishing cycle is invariant under the monodromy. $\square$

To write down the affine structure, one needs to introduce one branch cut from each of the three singularities to infinity. From Lemma 2.6 and Lemma 3.12, the affine transformations along the cuts are $\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}$, $\begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix}$ in counter-clockwise order. To compare to B_{CPS}, we have further requirements on the branch cuts.

Lemma 3.13. *There exists an affine ray emanating from each of the three singularities such that its tangent is in the monodromy invariant direction at infinity.*

Proof. We will explain the cut emanating from A and the other two are similar. From Lemma 3.8 and Corollary 3.9, the set $\{3 < q < \infty\}$ is an affine line defined by $d = (V_2 - V_1)/3$, the cycle invariant under ϕ_0, which is the vanishing cycle at the infinity by Lemma 2.6. $\square$

Proof. (of the main theorem) To match the affine structure with B_{CPS}, we will take the branch cuts to be the affine rays in Lemma 3.13. Recall that the orientations of vanishing cycles V_0, V_1, V_2 are chosen as in Lemma 2.6 so that they respect the $\mathbb{Z}_3$-symmetry on the q-plane. $\square$

Recall we have the identification $\mathrm{H}_2(X, E_0) \cong \mathbb{Z}^2$ from Lemma 2.6. We identify $A, B, C \in B_{SYZ}$ with $A', B', C' \in B_{CPS}$ and $v_1, v_2, v_3 \in B$ with v_1', v_2', v_3', there is an induced affine isomorphism from the affine triangle $v_1v_2v_3$ in B_{SYZ} to $v_1'v_2'v_3'$ in B_{CPS}. Since the affine transformation acrossing the cut in B_{SYZ} and B_{CPS} are the same from Lemma 3.12, the affine isomorphisms glue to an affine isomorphism $B_{SYZ} \cong B_{CPS}$.

4. Floer-theoretical gluing construction of mirror geometry

In the previous section, we have well understood the affine structure associated to the special Lagrangian fibration on $\mathbb{P}^2 - E$, where E is a smooth elliptic curve. In this section, we construct the Floer theoretical mirror of $\mathbb{P}^2$ relative to E, which is a direct application of the gluing method developed in [9, 27].

The strategy is the following. The special Lagrangian fibration has exactly three singular fibers. Each of these is a nodal torus pinched at one point. However, these singular fibers are located in different energy levels, in the sense that the pseudo-isomorphisms between their formal deformations involve Novikov parameter. The resulting mirror would be defined over Λ.

To simplify the situation, we take the following Lagrangians instead of the special Lagrangian fibers. We take a monotone moment-map fiber of $\mathbb{P}^2$, and use symplectic reduction by $\mathbb{S}^1$ to construct three monotone immersed Lagrangians, which play the role of the above three singular fibers. We consider the weakly unobstructed deformation spaces of these Lagrangians, and glue them together via quasi-isomorphisms in the Fukaya category.

Using these monotone Lagrangians, the gluing relations will be defined over $\mathbb{C}$, and hence we can reduce to a $\mathbb{C}$-valued mirror. Moreover, the construction of [9] produces a mirror functor from the Fukaya category to the mirror matrix factorization category $\mathrm{Fuk}(\mathbb{P}^2) \to \mathrm{MF}(\check{X}_{\mathbb{C}}, \tilde{W})$, which induces a derived equivalence [10].

4.1. The Lagrangian objects

Let $\mathbf{L}_0$ be the monotone moment-map torus fiber of $\mathbb{P}^2$ equipped with the toric Kähler form, whose fan is generated by e_1, e_2 and $e_3 = -e_1 - e_2$, where $\{e_1, e_2\} \subset \mathfrak{t} \cong \pi_1(\mathbf{L}_0)$ is the standard basis. Consider flat connections on $\mathbf{L}_0$, whose holonomies along the loops $e_1, e_2 \in \pi_1(\mathbf{L}_0)$ are given by z_1, z_2 respectively. Let $z_3 = 1/z_1 z_2$ which is the holonomy along e_3. Denote these flat connections by $\nabla^{(z_1,z_2)}$.

The flat connections over a Lagrangian are taken over Λ_0, with holonomies $z_i \in \Lambda_0^\times$, where

$$\Lambda_0 := \left\{ \sum_{i=0}^{\infty} a_i \mathbf{T}^{A_i} \mid a_i \in \mathbb{C}, A_i \geq 0 \text{ and increases to } +\infty \right\}$$

is the Novikov ring, and

$$\Lambda_0^\times := \left\{ \sum_{i=0}^{\infty} a_i \mathbf{T}^{A_i} \in \Lambda_0 \mid A_0 = 0 \text{ and } a_0 \neq 0 \right\}$$

is the group of invertible elements. This ensures the Floer theory for the Lagrangian decorated by a flat connection is convergent over Λ.

Following [1], we can 'push in' one of the corners of the moment map polytope. Namely, let $\mathbb{C}^2_{(i)}$ be the standard coordinate charts and $X^{(i)}, Y^{(i)}$ the corresponding inhomogeneous coordinates for $i = 0, 1, 2$. Denote the

T^2-moment map by

$$\mu : \mathbb{P}^2 \to \mathbf{t}^* \text{ with } \mu^{-1}(\{0\}) = \mathbf{L}_0.$$

Here the toric Kähler form is taken such that the moment map image is the triangle with vertices $(-1,-1), (-1,2), (2,-1)$ (in the basis $\{e_1^\vee, e_2^\vee\} \subset \mathbf{t}^*$).

Consider the $\mathbb{S}^1$-action in each direction $e_{i+2} - e_{i+1}$ (where the subscript is mod 3). The corresponding moment map is $(\mu, e_{i+2} - e_{i+1})$. Moreover, the function $X^{(i)} \cdot Y^{(i)}$ is invariant under this $\mathbb{S}^1$-action and gives a complex coordinate $\zeta = \zeta^{(i)}$ on the reduced space $\mathbb{C}^2_{(i)} /\!\!/ \mathbb{S}^1_{e_{i+2}-e_{i+1}}$. Using this symplectic reduction, one obtains the following Lagrangian torus fibration.

Proposition 4.1 ([23, 24]). *For any $c \in \mathbb{C}$, $((\mu, e_{i+2} - e_{i+1}), |X^{(i)} \cdot Y^{(i)} - c|)$ defines a Lagrangian fibration on $\mathbb{C}^2_{(i)} = \mathbb{P}^2 - D_i^T$ where D_i^T is the toric divisor corresponding to e_i.*

When $c = 0$, this is just isomorphic to the Lagrangian fibration given by the moment map.

We shall take the following Lagrangian objects. In the reduced space $\mathbb{C}^2_{(i)} /\!\!/ \mathbb{S}^1_{e_{i+2}-e_{i+1}}$, $\mathbf{L}_0$ is given by a circle of radius $r > 0$ centered at $\zeta = 0$. Moreover $(\mu, e_{i+2} - e_{i+1}) = 0$ for $\mathbf{L}_0$. For each $i = 1, 2, 3$, we take

$$\mathbf{L}_i := \{|X^{(i)} \cdot Y^{(i)} - r| = r, (\mu, e_{i+2} - e_{i+1}) = 0\} \subset \mathbb{P}^2$$

which is the singular fiber of the above Lagrangian fibration (for $c = r > 0$). $\mathbf{L}_i$ is an immersed two-sphere with a single nodal point. We denote the immersion by $\iota_i : \mathbb{S}^2 \to \mathbb{P}^2$ whose image is $\mathbf{L}_i$.

For each $i = 1, 2, 3$, we also have the Chekanov torus

$$\mathbf{L}'_i := \{|X^{(i)} \cdot Y^{(i)} - (3r/2)| = r, (\mu, e_{i+2} - e_{i+1}) = 0\} \subset \mathbb{P}^2.$$

See Figure 6.

Proposition 4.2. *For t sufficiently small, the Lagrangians $\mathbf{L}_i$ and $\mathbf{L}'_i$ lie in $\mathbb{P}^2 - E_t$ where $E_t = \{xyz + t(x^3 + y^3 + z^3) = 0\} \subset \mathbb{P}^2$.*

Proof. E_t lies in a neighborhood of the union of toric divisors $xyz = 0$. After intersecting with the moment-map level set $\{(\mu, e_{i+2} - e_{i+1}) = 0\}$, it is a compact set whose image in $\mathbb{C} \cup \{\infty\} \cong \mathbb{P}^1$ under $X^{(i)} \cdot Y^{(i)}$ consists of two connected components, one is a compact simply connected region near 0 (but does not contain 0), and one is a compact neighborhood of ∞. For t small, these two regions are disjoint from the base circles of $\mathbf{L}_i$ and $\mathbf{L}'_i$. $\square$

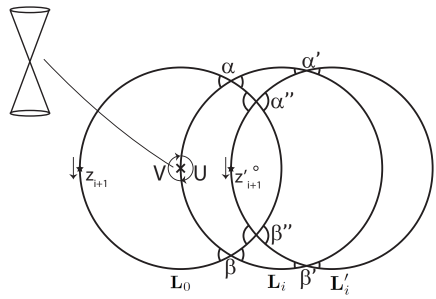

Figure 6: The images of the Lagrangians in the reduced space.

As explained above, we have the flat connections $\nabla^{(z_1,z_2)}$ on $\mathbf{L}_0$. Now we parametrize the flat connections on the Chekanov tori $\mathbf{L}_i'$ by fixing the following trivialization of the conic fibrations.

The conic fibration of $X^{(i)} \cdot Y^{(i)}$ restricted to $\mathbb{C}^2_{(i)} - \{Y^{(i)} = 0\}$ is trivial, and $Y^{(i)} \in \mathbb{C}^\times$ serves as the fiber coordinate. The map

$$((X^{(i)}Y^{(i)} - 3r/4)/|X^{(i)}Y^{(i)} - 3r/4|, Y^{(i)}/|Y^{(i)}|)$$

gives an identification of $\mathbf{L}_0$ and $\mathbf{L}_i'$ with T^2. Thus $e_1, e_2, e_3 \in \pi_1(\mathbf{L}_0)$ can be identified as elements in $\pi_1(\mathbf{L}_i')$.

Let's denote the holonomy of a flat connection over $\mathbf{L}_i'$ along e_{i+1} by z_{i+1}', and that along the monodromy invariant direction $(e_{i+2} - e_{i+1})$ by w_{i+1}'. We shall consider the objects $\left(\mathbf{L}_i', \nabla^{(z_{i+1}', w_{i+1}')}\right)$. For $\mathbf{L}_0$, the holonomy of a flat connection along $(e_{i+2} - e_{i+1})$ is denoted by w_{i+1}, which equals to $z_{i+2}z_{i+1}^{-1}$.

In conclusion, we shall consider the objects $(\mathbf{L}_0, \nabla^{(z_1,z_2)})$, $\left(\mathbf{L}_i', \nabla^{(z_{i+1}', w_{i+1}')}\right)$, and the Lagrangian immersions $\mathbf{L}_i$ for $i = 1, 2, 3$.

4.2. The Floer theoretical mirror

We construct a mirror out of the objects $\mathbf{L}_0$ and $\mathbf{L}_i$. This gives a nice application of the gluing method in [7, 27].

We take a Morse model for the Lagrangian Floer theory. Pearl trajectories, which are formed by holomorphic discs components together with gradient flow lines of a fixed Morse function, were developed in [6, 35] for the deformation theory of monotone Lagrangians. In [19], the Morse model was developed to general situations using a homotopy between the Morse complex and the singular chain complex. There is also a slightly different formulation in [16]. Such a Morse model was further developed to apply to a G-equivariant setting in [28,30]. Fixing the choice of a Morse function f on a Lagrangian L and perturbation datum for the pearl trajectories, an A_∞ structure $\{m_k : k \in \mathbb{Z}_{\geq 0}\}$ is constructed on the space of chains $\mathcal{F}(L)$ generated by critical points of f. Moreover, given a degree-one chain $b \in \mathcal{F}^1(L)$, one has the deformed A_∞ structure $\{m_k^b : k \in \mathbb{Z}_{\geq 0}\}$ [20]. L can also be decorated by flat connections ∇, which produce $\{m_k^{(L,\nabla)} : k \in \mathbb{Z}_{\geq 0}\}$.

The holomorphic discs bounded by the torus $\mathbf{L}_0 \subset \mathbb{P}^2$ were known due to the classification by [11]. Moreover, $(\mathbf{L}_0, \nabla^{(z_1,z_2)})$ are weakly unobstructed [20], namely,

$$m_0^{(\mathbf{L}_0,\nabla^{(z_1,z_2)})} = W \cdot \mathbf{1}_{\mathbf{L}_0}$$

where $\mathbf{1}_{\mathbf{L}_0}$ is the unit. The disc potential is given by

$$W = T^{A/3}\left(z_1 + z_2 + \frac{1}{z_1 z_2}\right) = T^{A/3}\left(z_1 + z_2 + z_3\right)$$

where A is the area of the line class in $\mathbb{P}^2$.

For the grading of the Lagrangians, for each $i = 1, 2, 3$, we consider the anti-canonical divisor

$$D_i := \{X^{(i)} Y^{(i)} = 3r/4\} \cup \{z^{(i)} = 0\}$$

(where $z^{(i)}$ is the homogeneous coordinate that defines the toric divisor $D_i^T = \{z^{(i)} = 0\}$).

Lemma 4.3. *$\mathbf{L}_0$, $\mathbf{L}_i$ and $\mathbf{L}_i'$ are graded Lagrangians in the complement $\mathbb{P}^2 - D_i$.*

Proof. $\mathbf{L}_0$, $\mathbf{L}_i$ and $\mathbf{L}_i'$ are isotopic to special Lagrangian fibers with respect to the holomorphic volume form $dX^{(i)} \wedge dY^{(i)}/(X^{(i)}Y^{(i)} - 3r/4)$ defined on $\mathbb{P}^2 - D_i$, and hence they are graded. □

Then the Maslov index formula of [1, 11] can be applied and one has the following.

Proposition 4.4 ([1, 11])**.** *The Maslov index of a disc β bounded by $\mathbf{L}_0, \mathbf{L}_i, \mathbf{L}_i'$ equals to $\mu(\beta) = 2\,\beta \cdot D_i$.*

Now we fix a choice of Morse functions on the Lagrangians. In above we have fixed an identification of $\mathbf{L}_0$ and $\mathbf{L}_i'$ with the standard T^2. Let's take a perfect Morse function on T^2 such that the unstable circles of the two degree-one critical points are dual to the $\mathbb{S}^1$-orbits in the directions of e_1 and e_2 respectively. By abuse of notation, we also denote these two degree-one critical points by e_1, e_2. The maximum and minimum points are denoted by $\mathbf{1}$ and e_{12} respectively.

For the immersed Lagrangians $\mathbf{L}_i$, the choice of Morse functions is more subtle and we proceed as follows. First, consider the immersed generators for the Floer theory. The domain of the immersion is $\mathbb{S}^2$. The inverse image of the transverse self-nodal point consists of two points $q_1, q_2 \in \mathbb{S}^2$. The branch jumps $q_1 \to q_2$ and $q_2 \to q_1$ are denoted by U_i and V_i respectively. See FIGURE 6. By using the grading in Lemma 4.3, it is easy to see the following.

Lemma 4.5. *Both* $U_i, V_i \in \mathrm{CF}(\mathbf{L}_i, \mathbf{L}_i)$ *have* $\deg = 1$.

We use U_i, V_i for the Maurer-Cartan deformations of $\mathbf{L}_i$. By using a $\mathbb{Z}_2$-symmetry, they can be shown to be unobstructed:

Lemma 4.6 ([27, Lemma 3.3])**.** $u_iU_i + v_iV_i \in \mathrm{CF}(\mathbf{L}_i, \mathbf{L}_i)$ *are bounding cochains for* $\mathbf{L}_i \subset \mathbb{P}^2 - D_i$*, namely,* $m_0^{u_iU_i+v_iV_i} = 0$*, where*

$$(u_i, v_i) \in \Lambda_0^2 - \{val(u_iv_i) = 0\}.$$

It is important to take $\mathrm{val}(u_iv_i) > 0$, since there are constant polygons at the nodal point (whose number of U_i corners must equal to the number of V_i corners to go back to the same branch) contributing to the Floer theory of $\mathbf{L}_i$. This ensures Novikov convergence of $m_0^{u_iU_i+v_iV_i}$.

We construct isomorphisms between $(\mathbf{L}_0, \nabla^{(z_1,z_2)})$ and $(\mathbf{L}_i, u_iU_i + v_iV_i)$ under suitable gluing relations between (z_1, z_2) and (u_i, v_i). Observe that $\mathbf{L}_i$ intersects cleanly with $\mathbf{L}_0$ (or $\mathbf{L}_i'$) at two circle fibers (α, β) (or (α', β')) over the two intersection points of the base loci $|\zeta| = r$ and $|\zeta - r| = r$ (or $|\zeta - 3r/2| = r$) in the ζ-plane. Similarly, $\mathbf{L}_0$ intersects with $\mathbf{L}_i'$ at two circles (α'', β''). We fix a perfect Morse function on each of these circles. The maximum and minimum points are denoted by $\alpha \otimes \mathbf{1}, \alpha \otimes \mathbf{m}$ respectively (and similar for $\beta, \alpha', \beta', \alpha'', \beta''$, where $\mathbf{m}$ stands for 'minimum').

$$\mathrm{CF}(\mathbf{L}_i, \mathbf{L}_0) = \mathrm{Span}_\Lambda\{\alpha \otimes \mathbf{1}, \alpha \otimes \mathbf{m}, \beta \otimes \mathbf{1}, \beta \otimes \mathbf{m}\}$$

which have degrees $0, 1, 1, 2$ respectively. We can also regard them as generators of $\mathrm{CF}(\mathbf{L}_0, \mathbf{L}_i)$, and they have degrees $1, 2, 0, 1$ respectively.

By the projection to the complex ζ-plane, one can deduce the following (see [27, Section 3.3]), which is important for computing $m_1^{\mathbf{L}_i,\mathbf{L}_0}(\alpha \otimes \mathbf{1})$ and $m_1^{\mathbf{L}'_i,\mathbf{L}_i}(\alpha' \otimes \mathbf{1})$.

Lemma 4.7. *In $\mathbb{P}^2 - D_i$, $\mathbf{L}_i$ and $\mathbf{L}_0$ (or similarly $\mathbf{L}_i$ and $\mathbf{L}'_i$) bound exactly two non-constant Maslov-two holomorphic polygons that have output to $\beta \otimes \mathbf{1}$ (or $\beta' \otimes \mathbf{1}$). One of them has corners at α, β (or α', β'). The other has corners at α, β, V (or α', β', U).*

The Morse function on $\mathbf{L}_i$ that we choose is the following. The boundaries of the above two holomorphic polygons in $\mathbf{L}_i$ give two curved segments. We take a perfect Morse function on the domain $\mathbb{S}^2$ of $\mathbf{L}_i$ such that the two critical points lie in $\mathbb{S}^2 - \{q_1, q_2\}$, and the two flow lines connecting q_1, q_2 to the minimum are distinct and do not intersect with any of these curve segments.

Then we have the following isomorphisms between the Lagrangian branes.

Theorem 4.8. *$\alpha \otimes \mathbf{1} \in \mathrm{CF}((\mathbf{L}_0, \nabla^{(z_1,z_2)}), (\mathbf{L}_i, u_i U_i + v_i V_i))$ is an isomorphism if and only if $v_i = z_{i+1}^{-1}$ and $u_i v_i = 1 + z_i^{-1} z_{i+1}^{-2}$ where the subscripts are mod 3.*

Proof. Fix $i = 1, 2, 3$. First we consider

$$\alpha'' \otimes \mathbf{1} \in \mathrm{CF}((\mathbf{L}_0, \nabla^{(z_{i+1},w_{i+1})}), (\mathbf{L}'_i, \nabla^{(z'_{i+1},w'_{i+1})}))$$

between the tori. $m_{1,u}(\alpha'' \otimes \mathbf{1})$ has degree $\deg(\alpha'' \otimes \mathbf{1}) + 1 - \mu(u) \geq 0$ where $\mu(u)$ is the Chern-Weil Maslov index of the strip class u. Since $\alpha'' \otimes \mathbf{1}$ has degree zero and the minimal Maslov index for $\mathbf{L}_0$ and $\mathbf{L}_i$ is zero, $m_1(\alpha'' \otimes \mathbf{1})$ is merely contributed by strips with Chern-Weil Maslov index zero. We have $\mu(u) = 2u \cdot D_i$. Thus any u which contributes to $m_1(\alpha'' \otimes \mathbf{1})$ does not intersect with D_i. We have

$$m_1(\alpha'' \otimes \mathbf{1}) = (1 - w'_{i+1} w_{i+1}^{-1}) \alpha'' \otimes \mathbf{m} + (1 + w_{i+1} - z'_{i+1} z_{i+1}^{-1} s) \beta'' \otimes \mathbf{1}$$

where the first term is contributed by the two flow lines from $\alpha'' \otimes \mathbf{1}$ to $\alpha'' \otimes \mathbf{m}$, and the second term is contributed from the holomorphic strips from $\alpha'' \otimes \mathbf{1}$ to $\beta'' \otimes \mathbf{1}$ [27, 37, 38]. Hence the cocycle condition $m_1(\alpha'' \otimes \mathbf{1}) = 0$ implies $w'_{i+1} = w_{i+1} = z_{i+1}^{-1} z_{i+2} = z_i^{-1} z_{i+1}^{-2}$ and $z'_{i+1} = z_{i+1}(1 + w_{i+1})$. Moreover, the strips also give $m_2(\beta'' \times \mathbf{1}, \alpha'' \times \mathbf{1}) = \mathbf{1}_{\mathbf{L}_0}$ and $m_2(\alpha'' \times \mathbf{1}, \beta'' \times \mathbf{1}) = \mathbf{1}_{\mathbf{L}'_i}$. Thus $\alpha'' \times \mathbf{1}$ is an isomorphism if and only if the above relations hold.

Now we consider $m_1(\alpha \otimes \mathbf{1})$ and $m_1(\alpha' \otimes \mathbf{1})$. We have

$$m_1(\alpha \otimes \mathbf{1}) = (u_i - z'_{i+1})\beta \otimes \mathbf{1} + (w_{i+1} - f(u_i v_i))\alpha \otimes \mathbf{m}$$
$$m_1(\alpha' \otimes \mathbf{1}) = (v_i - z_{i+1}^{-1})\beta' \otimes \mathbf{1} + (w'_{i+1} - g(u_i v_i))\alpha \otimes \mathbf{m}$$

for some series f and g. Requiring them to be zero implies $u_i = z'_{i+1}$, $v_i = z_{i+1}^{-1}$, $w_{i+1} = f(u_i v_i), w'_{i+1} = g(u_i v_i)$. It easily follows that α and α' are isomorphisms under the above relations.

Since $m_2(\alpha, \alpha') = \alpha''$, $\alpha'' \otimes \mathbf{1}$ is also an isomorphism under the above relations. Thus $w'_{i+1} = w_{i+1} = z_i^{-1} z_{i+1}^{-2}$ and $z'_{i+1} = z_{i+1}(1 + w_{i+1})$, implying $f(u_i v_i) = g(u_i v_i)$ and $u_i v_i = 1 + w_{i+1}$. Result follows. □

According to the above theorem, the formal deformation spaces of $\mathbf{L}_0$ and $\mathbf{L}_i$ for $i = 1, 2, 3$ are glued by the transitions $v_i = z_{i+1}^{-1}$ and $u_i v_i = 1 + z_i z_{i+1}^2$. We denote the resulting space by $\check{X}$. It consists of the chart $(\Lambda_0^\times)^2$ coming from the torus $\mathbf{L}_0$, and the charts $(\Lambda_0^2)_{(i)} - \{\mathrm{val}\,(u_i v_i) = 0\}$ coming from the immersed sphere $\mathbf{L}_i$ for $i = 1, 2, 3$.

$\check{X}$ is defined over Λ. On the other hand, note that the transition functions do not involve the Novikov parameter $\mathbf{T}$. This is because the base circles of $\mathbf{L}_0$, $\mathbf{L}_i$ and $\mathbf{L}'_i$ in the reduced space are taken to be the same size, so that the symplectic areas of strips are the same. The $\mathbb{C}$-valued part of $\check{X}$ is denoted by $\check{X}_{\mathbb{C}}$, which is the union of the $\mathbb{C}$-valued parts of the charts of $\check{X}$.

Remark 4.9. *The $\mathbb{C}$-valued part of the chart $(\Lambda_0^2)_{(i)} - \{val(u_i v_i) = 0\}$ of the immersed Lagrangian $\mathbf{L}_i$ is the singular conic*

$$\{(u_i, v_i) \in \mathbb{C}^2 : u_i v_i = 0\} = \{(u_i, 0) : u_i \in \mathbb{C}^\times\} \cup \{(0, v_i) : v_i \in \mathbb{C}^\times\} \cup \{(0, 0)\}$$

whose valuation is $\{(0, +\infty)\} \cup \{(+\infty, 0)\} \cup \{(+\infty, +\infty)\}$. Note that this subset is disconnected under the non-Archimedian topology. Moreover, the $\mathbb{C}$-valued part of the gluing region with the torus chart $(\mathbb{C}^\times)^2 \subset (\Lambda_0^\times)^2$ is $\{(0, v_i) : v_i \in \mathbb{C}^\times\}$. This is not of the correct complex dimension. Thus we first work over Λ to construct the mirror, and then we can restrict to $\mathbb{C}$ to get the $\mathbb{C}$-valued mirror.

Remark 4.10. *In the above Floer theoretical construction, the mirror is simply glued from one torus chart $(\mathbb{C}^\times)^2$ and three charts coming from immersed spheres. On the other hand, the corresponding cluster variety consists of infinitely many torus charts.*

4.3. Identification with the Carl–Pomperla–Siebert mirror

Now we show that the resulting geometry from the above construction agrees with the Carl–Pomperla–Siebert mirror. This gives Theorem 1.3.

Proposition 4.11. *$\check{X}_{\mathbb{C}}$ is the blowing up at three points in the three toric divisors of the toric variety whose fan has the rays generated by $(2,-1),(-1,2)$ and $(-1,-1)$, with the strict transform of the toric divisors removed. $W_{\mathbb{C}} = z_1 + z_2 + \frac{1}{z_1 z_2}$ on $(\mathbb{C}^\times)^2$ extends to be a proper elliptic fibration on $\check{X}_{\mathbb{C}}$ with three I_1 singular fibers.*

Proof. The blowing up of the toric chart $\mathbb{C}_{(V)} \times \mathbb{C}^\times_{(Z)}$ at $(V,Z) = (0,-1)$ has local charts $\mathbb{C}^2_{(U,V)} - \{UV = 1\}$ and $\mathbb{C}^2_{(\tilde{V},\tilde{Z})} - \{\tilde{Z} = 1\}$ with the change of coordinates $V = \tilde{V}\tilde{Z}$ and $U = \tilde{V}^{-1}$ (where $\tilde{Z} = Z + 1$). The strict transform of the toric divisor $\{V = 0\} \subset \mathbb{C}_{(V)} \times \mathbb{C}^\times_{(Z)}$ is given by $\tilde{V} = 0$, and its complement in the blowing-up is identified with the chart $u_i v_i = 1 + z_i^{-1} z_{i+1}^{-2}$ of $\check{X}_{\mathbb{C}}$ via $\tilde{Z} = 1 + z_i^{-1} z_{i+1}^{-2}$, $U = u_i$, $V = v_i$. The open torus orbit $\mathbb{C}^\times_{(V)} \times \mathbb{C}^\times_{(Z)}$ is identified with the torus chart of $\mathbf{L}_0$ by $V = z_{i+1}^{-1}$ and $Z = z_i^{-1} z_{i+1}^{-2}$. This gives the identification between $\check{X}_{\mathbb{C}}$ and the blowing-up.

We already know that W on $(\mathbb{C}^\times)^2_{(z_1,z_2)}$ gives a fibration whose generic fibers are three-punctured elliptic curves. W has three critical values, whose fibers are 3-punctured A_1 singular fibers. Below, we see that the partial compactification by the immersed charts $\mathbb{C}^2_{(i)} - \{u_i v_i = 1\}$ exactly fill in the punctures in all elliptic fibers.

Consider a fiber $W = c$ for $c \in \mathbb{C}$. For the chart $\mathbb{C}^2_{(i)} - \{u_i v_i = 1\}$, $u_i v_i = 1 + z_{i+1}^{-1} z_{i+2}$. Thus $z_{i+2} = (u_i v_i - 1) z_{i+1} = (u_i v_i - 1) v_i^{-1} = u_i - v_i^{-1}$. Then

$$W = z_{i+1} + z_{i+2} + \frac{1}{z_{i+1} z_{i+2}} = u_i + v_i^2 (u_i v_i - 1)^{-1}$$

in the chart. The fiber is given by

$$u_i(u_i v_i - 1) + v_i^2 = c(u_i v_i - 1).$$

The partial compactification coming from this chart is $v_i = 0$. Thus it adds the point $(u_i, v_i) = (c, 0)$ to the fiber. In other words, the coordinate axes $v_i = 0$ are sections of the fibration of W. The partial compactification adds in these three sections which are exactly the union of three punctures of the elliptic fibers. $\square$

We note that the meromorphic functions u_i for $i = 1, 2, 3$ satisfy the following explicit equation.

Proposition 4.12.

$$(u_1^3 + u_2^3 + u_3^3) + 2u_1u_2u_3 - \sum_{i=1}^{3}(u_1^2u_2 + u_1u_2^2) = 0.$$

Proof. We have

$$z_0z_1z_2 = 1. \tag{4.1}$$

Moreover,

$$u_i = z_{i+1}(1 + z_{i+1}^{-1}z_{i+2}) = z_{i+1} + z_{i+2}. \tag{4.2}$$

We compute u_i^3, $u_i^2u_{i+1}, u_iu_{i+1}^2$ and $u_1u_2u_3$ using (4.2). It turns out the variables z_0, z_1, z_2 can all be eliminated and we obtain the resulting equation. □

Appendix A. The proof of Proposition 3.4

We resume the notation introduced in §3. Abusing the notation, for $q \in \{r\zeta^j : r \leq 3\}$, let γ_j be the line segment connecting q and $3\zeta^j$ and $\Gamma_j^{\gamma_j}(q)$ to denote the set-theoretic union

$$\bigcup_{q' \in \gamma_j} V_j^{(q')}. \tag{A.1}$$

We also denote by $\vec{\gamma}_j$ the line segment γ_j equipped with an orientation from $3\zeta^j$ towards to q and by $\vec{\Gamma}_j^{\gamma_j}(q)$ the set $\Gamma_j^{\gamma_j}(q)$ with the induced orientation as in . The integral

$$\int_{\vec{\Gamma}_j^{\gamma_j}(q)} \check{\Omega} \tag{A.2}$$

becomes a function in $q \in W_j$. For simplicity, we put

$$G(q) := \int_{\vec{\Gamma}_0^{\gamma_0}(q)} \check{\Omega}, \text{ where } q \in (-\infty, 3] \subset \mathbb{C}. \tag{A.3}$$

Proposition 3.4 is an immediate consequence of the following lemma.

Lemma A.1.
$$\int_{\vec{\Gamma}_0^{\gamma_0}(q)} \check{\Omega} \in \sqrt{-1}\mathbb{R}_+ \ \textit{for } q \in (-\infty, 3].$$

Proof. Using $\check{\Omega}$ is d-closed and independent of q, we compute

$$\frac{\mathrm{d}G(q)}{\mathrm{d}q} = \int_{\partial\vec{\Gamma}_0^{\gamma_0}(q)} \iota_{\frac{\partial}{\partial q}} \check{\Omega}.$$

From the construction, $\partial\vec{\Gamma}_0^{\gamma_0}(q)$ is equal to $V_0^{(q)}$ as *oriented cycles*.

Recall that $E_q \cap (\mathbb{C}^*)^2 := \{(t_1, t_2) \in (\mathbb{C}^*)^2 \colon t_1 + t_2 + (t_1 t_2)^{-1} = q\}$. Let $\delta_0(q)$ be the image of $V_0^{(q)}$ under the projection

$$(\mathbb{C}^*)^2 \to \mathbb{C}^*,\ (t_1, t_2) \mapsto t_1. \tag{A.4}$$

For $q \in (-\infty, 3]$, $E_q \cap (\mathbb{C}^*)^2 \to \mathbb{C}^*$ admits three ramifications: only one of them lies on the real axis and the other two are symmetric with respect to the real axis, denoted by x and $\bar{x}$. Here we assume that $\mathrm{Im}(x) > 0$. x and $\bar{x}$ are connected through $\delta_0(q)$. We equip $\delta_0(q)$ with an orientation going from x to $\bar{x}$. Note that $\delta_0(0) \equiv \delta_0$ as oriented cycles.

We can write $\partial\vec{\Gamma}_0^{\gamma_0}(q) = V_0^{(q)} = \partial\Gamma^+(q) \cup \partial\Gamma^-(q)$, union of the graph of t_2^+ and the graph of t_2^- along $\delta_0(q)$ as in the paragraph **(C)**. Then

$$\begin{aligned} \frac{\mathrm{d}G(q)}{\mathrm{d}q} &= \int_{\partial\vec{\Gamma}_0^{\gamma_0}(q)} \iota_{\frac{\partial}{\partial q}} \check{\Omega} \\ &= \int_{\partial\Gamma^+(q)} \iota_{\frac{\partial}{\partial q}} \check{\Omega} + \int_{\partial\Gamma^-(q)} \iota_{\frac{\partial}{\partial q}} \check{\Omega} \\ &= \int_{\delta_0(q)} \frac{\sqrt{-1}}{\sqrt{(q-t_1)^2 - 4/t_1}} \frac{\mathrm{d}t_1}{t_1} + \int_{-\delta_0(q)} \frac{-\sqrt{-1}}{\sqrt{(q-t_1)^2 - 4/t_1}} \frac{\mathrm{d}t_1}{t_1}. \end{aligned} \tag{A.5}$$

We explain the third equality above. Restricting on $\partial\Gamma^+(q)$ or $\partial\Gamma^-(q)$ and making use of the equation $t_1^2 t_2 + t_1 t_2^2 + 1 = q t_1 t_2$, we obtain

$$\begin{aligned} \iota_{\frac{\partial}{\partial q}} \check{\Omega}\Big|_{\partial\Gamma^\pm(q)} &= -\sqrt{-1} \cdot \frac{t_2^\pm}{q \cdot t_1 t_2^\pm - t_1^2 t_2^\pm - 2t_1 (t_2^\pm)^2} \mathrm{d}t_1 \\ &= -\sqrt{-1} \cdot \frac{1}{q - t_1 - 2t_2^\pm} \frac{\mathrm{d}t_1}{t_1}. \end{aligned} \tag{A.6}$$

Also from (3.5), we see that

$$2t_2^\pm + t_1 - q = \mp\sqrt{(q-t_1)^2 - 4/t_1}. \tag{A.7}$$

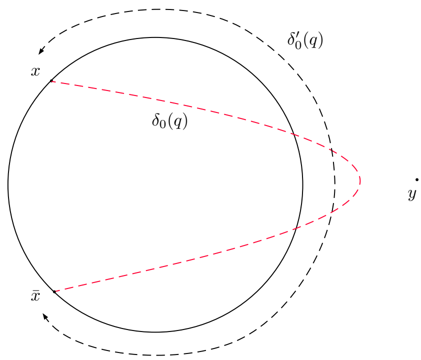

Figure A1: The deformed contour $\delta_0'(q)$.

Together with the induced orientation on $\partial\Gamma^{\pm}(q)$, we arrive at the desired equality. Note that the branched cut of $\sqrt{z}$ in (A.7) is chosen such that

$$\sqrt{z} = \exp\left(\frac{1}{2}\log z\right), \quad \log z = r + \sqrt{-1}\theta \text{ with } \theta \in [0, 2\pi).$$

Since both of the integrands in (A.5) are holomorphic, we can deform the cycle $\delta_0(q)$ a little bit. We have the following two cases: **(a)** $0 < q < 3$ and **(b)** $q < 0$.

For the case **(a)**, we can deform $\delta_0(q)$ into a *circular* arc $\delta_0'(q)$ joining the end points x and $\bar{x}$ without touching the third ramification point y, where the integrands have a pole (cf. FIGURE A1).

Moreover, on the circular arc, we have

$$\operatorname{Im} \sqrt{(q - t_1)^2 - 4/t_1} \geq 0. \tag{A.8}$$

Therefore,

$$\int_{\delta_0'(q)} \frac{\sqrt{-1}}{\sqrt{(q - t_1)^2 - 4/t_1}} \frac{\mathrm{d}t_1}{t_1}$$

has negative imaginary part and so does (A.5). This implies that the imaginary part of $G(q)$ decreases.

For the case **(b)**, we can deform $\delta_0(q)$ into the contour $\delta_0''(q)$ (cf. FIGURE A2). By symmetry, it suffices to compute the integral over (I) $\sim$ (III). The equation (A.8) still holds for (I) and (III). On the contour (II), with the

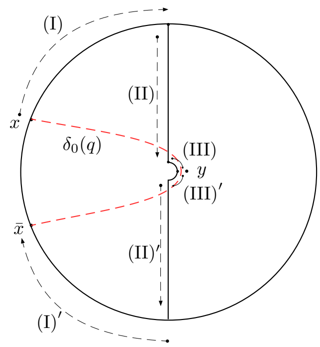

Figure A2: The deformed contour $\delta_0''(q)$, which is the union of (I) $\sim$ (III) and (I)$'$ $\sim$ (III)$'$.

parameterization $t_1 = \sqrt{-1} \cdot r$,

$$\begin{aligned}(q - \sqrt{-1}r)^2 + 4\frac{\sqrt{-1}}{r} &= q^2 - 2qr\sqrt{-1} - r^2 + \frac{4\sqrt{-1}}{r} \\ &= (q^2 - r^2) + \sqrt{-1}\left(\frac{4}{r} - 2qr\right)\end{aligned}$$

has positive imaginary part if $r > 0$, which guarantees that $\sqrt{(q - t_1)^2 - 4/t_1}$ has *positive real part* if $r > 0$. Also we have $\mathrm{d}t_1/t_1 = \mathrm{d}r/r$. These implies again that

$$\int_{(\mathrm{II})} \frac{\sqrt{-1}}{\sqrt{(q-t_1)^2 - 4/t_1}} \frac{\mathrm{d}t_1}{t_1}$$

has negative imaginary part.

We deduce from above that in both cases, (A.5) has negative imaginary parts. Together with the fact $G(3) = 0$, it follows that $G(q) \in \sqrt{-1} \cdot \mathbb{R}_+$ for $q < 3$. □

Corollary A.2. *We have* $G(0) \in \sqrt{-1} \cdot \mathbb{R}_+$.

Proof. This immediately follows from Lemma A.1. □

Corollary A.3. $\lim_{q \to -\infty} G(q) = \sqrt{-1} \cdot \infty$.

Proof. Assume $q < 0$. We adapt the notation in Figure A2. To compute the integral (A.5), as in the proof of Lemma A.1, we can deform the path $\delta_0(q)$ to $\delta_0''(q)$. We put $r := |x|$.

Note that

$$\frac{-\sqrt{-1}}{\sqrt{(q-t_1)^2-4/t_1}}\frac{\mathrm{d}t_1}{t_1}$$

has negative imaginary part on the whole (II). In particular, we have

$$\mathrm{Im}\int_{\delta_0''(q)}\frac{\sqrt{-1}}{\sqrt{(q-t_1)^2-4/t_1}}\frac{\mathrm{d}t_1}{t_1}\le \mathrm{Im}\int_{\mathcal{C}}\frac{\sqrt{-1}}{\sqrt{(q-t_1)^2-4/t_1}}\frac{\mathrm{d}t_1}{t_1}, \tag{A.9}$$

where $\mathcal{C}$ is the (clockwise oriented) contour

$$re^{i\theta} \text{ with } \theta\in\left[\frac{3\pi}{4},\frac{\pi}{2}\right].$$

It suffices to estimate the right hand side of (A.9). On $\mathcal{C}$, we have

$$(q-t_1)^2-4/t_1\sim(q-t_1)^2=q^2\left(1-\frac{r}{q}e^{i\theta}\right)^2 \tag{A.10}$$

provided $|q|$ is large enough. In the meanwhile, $r/|q|\sim 1$. It is not hard to see that

$$\mathrm{Im}\left(\frac{1}{\sqrt{(q-t_1)^2-4/t_1}}\right)\ge\frac{\kappa}{|q|},$$

for some positive constant κ. Since $\mathcal{C}$ is clockwise oriented, we have

$$\mathrm{Im}\int_{\delta_0''(q)}\frac{\sqrt{-1}}{\sqrt{(q-t_1)^2-4/t_1}}\frac{\mathrm{d}t_1}{t_1}\le-\frac{\kappa\cdot\mathrm{length}(\mathcal{C})}{|q|}=:-\frac{\kappa'}{|q|}. \tag{A.11}$$

This shows that

$$\frac{\mathrm{d}(\mathrm{Im}\,G(q))}{\mathrm{d}q}\le\frac{\kappa'}{q}\text{ for all } q\ll 0 \tag{A.12}$$

and therefore $\lim_{q\to-\infty}\mathrm{Im}\,G(q)=\infty$. □

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