

Gauge fixing and regularity of axially symmetric and axistationary second order perturbations around spherical backgrounds

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Perturbation theory in geometric theories of gravitation is a gauge theory of symmetric tensors defined on a Lorentzian manifold (the background spacetime). The gauge freedom makes uniqueness problems in perturbation theory particularly hard as one needs to understand in depth the process of gauge fixing before attempting any uniqueness proof. This is the first paper of a series of two aimed at deriving an existence and uniqueness result for rigidly rotating stars to second order in perturbation theory in General Relativity. A necessary step is to show the existence of a suitable choice of gauge and to understand the differentiability and regularity properties of the resulting gauge tensors in some “canonical form”, particularly at the centre of the star. With a wider range of applications in mind, in this paper we analyse the gauge fixing and regularity problem in a more general setting. In particular we tackle the problem of the Hodge-type decomposition into scalar, vector and tensor components on spheres of symmetric and axially symmetric tensors with finite differentiability down to the origin, exploiting a strategy in which the loss of differentiability is as low as possible. Our primary interest, and main result, is to show that stationary and axially symmetric second order perturbations around static and spherically symmetric background configurations can indeed be rendered in the usual “canonical form” used in the literature while losing *only* one degree of differentiability and keeping all relevant quantities bounded near the origin.

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1. Introduction

Perturbation theory in metric theories of gravity is one of the fundamental tools to tackle many realistic problems in relativistic astrophysics, ranging from slowly rotating stars to the emission of gravitational waves from binary systems in certain limits. Perturbation theory is, in essence, a theory of symmetric tensors defined on a Lorentzian manifold (the background spacetime). From a structural point of view its main particularity is that the theory is not only covariant, but also gauge invariant. If the perturbation theory is developed to order k , the number of symmetric tensors K_N is also k and the gauge freedom involves k vector fields (see [5] for the explicit gauge transformation law at every level k).

The gauge freedom is at the same time a feature and a nuisance of the theory. Among its positive consequences, the gauge freedom can often be exploited to simplify the problem under consideration (in much the same way as in electromagnetism). On the other hand, the gauge freedom is always there, so any solution of a problem immediately gives rise to the whole class of gauge related solutions that are, a priori, equally valid. An immediate consequence is that uniqueness problems in perturbation theory become much harder, since one needs to understand in depth the process of gauge fixing before attempting any uniqueness proof.

We encountered this difficulty in full when we started the project of proving a rigorous existence and uniqueness result for slowly rotating stars to second order in perturbation theory. As already mentioned, one of the necessary steps was to show the existence of a suitable choice of gauge and to understand the differentiability and regularity properties of the resulting gauge fixed tensors, particularly at the centre of the star. Despite the vast literature available on choice of gauges, specifically in our setup of perturbations around spherically symmetric backgrounds, this problem had never been addressed before rigorously. It turned out that the problem is considerable harder than one could have expected a priori. In this paper we report on our results on this subject. Although our primary motivation for this work still lies on the existence and uniqueness problem for slowly rotating stars, the existence of gauges that we analyse in this paper have a much wider range of applicability and are interesting on their own, independently of the original application we have in mind. This, combined with the length and level of complication we have encountered, justifies presenting the results in a separate paper.

Our specific interest is to understand the problem of gauge fixing and the properties of the resulting “canonical form” for stationary and axially symmetric perturbations around static and spherically symmetric background configurations with a regular centre. We restrict the perturbations to the so-called orthogonally transitive case and we go to second order in perturbation theory. It is important to emphasize that, for the sake of generality and particularly to apply the results in our subsequent work on slowly rotating stars, we need to work with finite differentiability, and in fact we want to keep our differentiability requirements as low as possible. This requirement is one of the main sources of complication in our arguments.

We tackle the problem in two separate steps, each of which requires fewer assumptions on the background. The first step is concerned with orthogonally transitive stationary and axially symmetric perturbations. Here the background need not admit any extra symmetry, i.e. the results apply for general backgrounds admitting a stationary and axially symmetric orthogonally transitive action. This step is not particularly complicated and we deal with it in Section 4. We first analyse the case of general backgrounds admitting an orthogonally transitive Abelian group action of any dimension and perturbations up to second order that “inherit” these background symmetries (the precise definition of this notion is given in Definition 2.1). However, it is only in the case of orthogonally transitive stationary and axially symmetric perturbations that we can ensure the gauge transformation keeps the differentiability of the perturbation tensors *also* on the axis. The

main result for this step is Proposition 4.5 where *block*-canonical forms for the first and second order perturbations tensors are given.

The second step is considerable harder. Here the background is assumed to be spherically symmetric (but not necessarily static). Perturbations around spherically symmetric backgrounds have been studied extensively in the literature and in many different areas (see e.g. [3, 4, 9, 12, 17] and references therein). A very common choice of gauge is to assume that the angular-angular part of the perturbation tensor is proportional to the standard metric on the sphere. This is for instance one of the defining properties of the Regge-Wheeler (RW) gauge [18], but it is shared by many other gauge fixing procedures. By far the argument most widely used to justify that such a choice of gauge is possible is to decompose the perturbation tensor into scalar, vector and tensor spherical harmonics. Then, for each mode it is easy to construct a gauge vector that transforms the perturbation tensor into the desired form. Despite its simplicity, this argument falls short to provide an existence proof of the gauge vector because that would require showing that the collection of gauge vectors at each mode corresponds to the mode decomposition of a gauge vector. In other words, one must show that the mode series converges. This is not a simple problem.

The second approach is based on using the Hodge-type scalar-vector-tensor (SVT) decomposition of symmetric tensors on the sphere [24]. The approach of replacing the spherical harmonic mode decomposition by SVT decompositions to study perturbations around spherical background has been used in the literature (see e.g. [8, 10] and [14], where the full set of perturbations are expressed in terms of functions on the sphere). Its use to show existence of a suitable gauge vector in four spacetime dimensions and for first order perturbation tensors can be summarized as follows. For spacetime dimension two, the SVT decomposition is applied on two-dimensional spheres and it is a well-known fact that in the two-dimensional sphere the only traceless and transverse symmetric tensor is the zero tensor. Thus, the SVT decomposition takes a simple form that involves only a scalar and a vector field (the explicit form appears in (5.28) below). This, combined with the gauge transformation law, makes it immediate to show that a gauge vector exists such that the angular-angular part of the first order perturbation tensor can be made proportional to the standard metric on the sphere. This approach however does not cover all our needs either, even at the first order level. The main difficulty lies at the centre, i.e. at the point(s) where the spheres defined as the surfaces of transitivity of the spherical action on the spacetime degenerate to points. The SVT decomposition is well-understood on each sphere, but we need to deal with a two-parameter family of spheres

that *degenerate to a point*. This prevents us from using directly the standard results on SVT decomposition to show existence of the appropriate gauge vector. It should be emphasized, however, that the SVT decomposition is a very important guiding principle for our approach to the problem.

Another important source of complication is our need to use finite differentiability. As we shall see along the text, it is a fact that rendering the perturbation tensor into a canonical form typically lowers the differentiability. Given that we want to keep the differentiability requirements as low as we can, it becomes necessary to find a good strategy where the loss of differentiability is as low as possible. This prevents us even from adapting directly the standard methods on Hodge decomposition on the sphere in the domain away from the origin. Indeed, these methods (see e.g. [11]) work by obtaining second order elliptic equations for each one of the components arising in the Hodge decomposition. To derive these equations, one needs to take two derivatives of the original tensor. These derivatives are then regained by standard elliptic regularity (working e.g. in Hölder spaces). However, when dealing with a two parameter family of problems as in our case, the loss of two derivatives in the coordinates that label the spheres cannot be regained by elliptic regularity.

The method we follow (see Theorem 5.4) consists in writing directly a system of coupled first order PDE on each sphere. While we are not aware of any general theorem that gives existence, we can exploit the axial symmetry of the perturbations to transform the system of PDE into a decoupled system of ODE. This strategy allows us to achieve a loss of only one derivative (away from the origin). Although we have no proof that this loss is optimal, we do have strong indication that it cannot be improved in general. The method that we follow introduces an important complication at the axis of symmetry where the ODEs become singular. In fact, most of the technical work in this paper is devoted to understanding the existence and regularity at the axis of the solutions of these ODE as well as to understand the regularity with respect to transversal directions away from the two-sphere and, very particularly, the behaviour of the solutions near the origin. We devote Appendix B to study all these issues.

It turns out that the behaviour near the origin is complicated. Our main result in this respect is that all the relevant quantities stay bounded near the origin. However, we do not show that the perturbation tensor is even continuous at the origin (let alone differentiable). Again we have no proof that our result is optimal, but we strongly suspect that it is not possible to write the perturbation tensor in canonical form and not to lose a great deal of regularity at the origin.

This is in fact one of the points we want to stress in this paper. In the physics literature it is a rather common belief that, as long as the number of restrictions matches the number of free functions in the gauge transformation, the process of restricting the gauge and writing the resulting tensor in some predetermined form comes at essentially no cost. The analysis in this paper shows very clearly that these issues are very delicate and that exploiting the gauge in order to transform the perturbation tensors into some useful form may easily spoil some other desired properties (such as continuity or differentiability at certain places). Only by knowing precisely how much deterioration is generated, can one decide whether using the canonical form is convenient (or even possible) for the specific problem under consideration.

1.1. Main result

The main result, Theorem 6.3, can be stated roughly as follows. Consider a static and spherically symmetric spacetime (M, g) with g of class C^{n+1} with $n \geq 2$, with timelike integrable Killing vector field (KVF) ξ , and single out a generator of an axial symmetry η , so that

$$g = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + \mathcal{R}^2(r) (d\theta^2 + \sin^2 \theta d\phi^2), \quad \xi = \partial_t, \quad \eta = \partial_\phi.$$

Now, given any stationary (ξ) and axially (η) symmetric (and orthogonally transitive) C^{n+1} perturbation to second order around (M, g) there exists a gauge transformation that yield first and second order perturbation tensors K_1^Ψ and K_2^Ψ that are C^{n-1} and C^{n-2} outside the origin, respectively, and can be written as

$$\begin{aligned} K_1^\Psi &= -4e^{\nu(r)} h^{(1)}(r, \theta) dt^2 - 2\omega^{(1)}(r, \theta) \mathcal{R}^2(r) \sin^2 \theta dt d\phi + 4e^{\lambda(r)} m^{(1)}(r, \theta) dr^2 \\ &\quad + 4k^{(1)}(r, \theta) \mathcal{R}^2(r) (d\theta^2 + \sin^2 \theta d\phi^2) + 4e^{\lambda(r)} \partial_\theta f^{(1)}(r, \theta) \mathcal{R}(r) dr d\theta, \\ K_2^\Psi &= \left(-4e^{\nu(r)} h^{(2)}(r, \theta) + 2\omega^{(1)^2}(r, \theta) \mathcal{R}^2(r) \sin^2 \theta \right) dt^2 \\ &\quad - 2\omega^{(2)}(r, \theta) \mathcal{R}^2(r) \sin^2 \theta dt d\phi + 4e^{\lambda(r)} m^{(2)}(r, \theta) dr^2 \\ &\quad + 4k^{(2)}(r, \theta) \mathcal{R}^2(r) (d\theta^2 + \sin^2 \theta d\phi^2) + 4e^{\lambda(r)} \partial_\theta f^{(2)}(r, \theta) \mathcal{R}(r) dr d\theta \end{aligned}$$

outside the axis of symmetry. Moreover, the result provides full control of the differentiability and boundedness properties of the functions involved. Let us stress again the fact that this result does not ensure the continuity of either tensor K_1^Ψ or K_2^Ψ at the origin. We can prove, however, that the gauge vectors extend continuously to zero at the origin.

The gauge freedom involved in the above forms is found and discussed in Section 6.3.

1.2. Plan of the paper

The structure of the paper is as follows. Section 2 is devoted to produce the necessary definitions on perturbations that inherit some of the symmetries present in the background. It serves us also to fix the differentiability of the perturbation scheme and the perturbation tensors. In Section 3 we analyse the structure that symmetric 2-covariant tensors invariant under the axial symmetry must have in a convenient (partly Cartesian) class of charts in the presence of the axis. The results are presented in Lemma 3.6, which generalises the well known results on the form of the metric in axially symmetric spaces, see e.g. [6]. In parallel, Section 4 deals with orthogonally transitive perturbation schemes, that is, perturbations that inherit the two-dimensional group of isometries acting orthogonally transitively admitted by the background (but arbitrary otherwise), to second order. The result for stationary and axisymmetric orthogonally transitive spacetimes is given in Proposition 4.5.

Next we retake the results from Section 3 and particularise to axial perturbations around spherically symmetric backgrounds. In particular, we prove in Theorem 5.4 the existence of the decomposition on the sphere of symmetric axially symmetric tensors (of finite differentiability), down to the behaviour of the decomposition at the origin. That result is then (partially) used to prove Proposition 5.6, which states the existence of a gauge vector that renders the first order perturbation tensor in some convenient form, while keeping control of the differentiability properties and behaviour at the origin of the relevant quantities. The analogous, but much more involved result, for second order is presented in Proposition 5.7.

We finally combine in Section 6 all those results to build the proof of the main results of this paper, in the form of Proposition 6.2 leading to Theorem 6.3.

Let us stress that our work here is purely geometric, we do not make use of any field equations. For the same reason, we do not make any consideration as to the physical meaning of the perturbation.

We have tried to write down this work as self-contained as possible, leaving the more technical work for the Appendices. The control of the differentiability (specially on the axis) and boundedness near the origin of the relevant components of the perturbation tensors requires several results on radially symmetric functions which we state and prove in Appendix A.

Although these results should be essentially known, they are not easily found in the literature in the form we need. Finally, the building block in showing existence of gauges is Lemma 5.3. Establishing this result requires some rather long technical work which is left to Appendix B.

1.3. Notation

Given the various setups considered in this work, we have been compelled to introduce a substantial amount of notation. We will fix most of the notation along the way, fundamentally at the start of the sections in which the relevant frameworks are introduced, in particular in Section 5 and Appendix B. Nevertheless, we fix here some basic notation that will be used from the start.

A C^{n+1} spacetime (M, g) is a k -dimensional ($k \geq 2$) orientable C^{n+2} manifold M endowed with a time-oriented Lorentzian metric g of class C^{n+1} . We assume $n \geq 2$ unless otherwise stated. Scalar products of two vector fields X, Y with the metric g will be denoted both by $g(X, Y)$ and $\langle X, Y \rangle$. We say that a geometric object is “smooth” when it has maximum differentiability allowed by the background. A function f defined on an open dense subset \mathcal{U}' of some neighbourhood $\mathcal{U} \in M$ is said to be $C^m(\mathcal{U})$ if it can be extended to all \mathcal{U} with this property.

We will use the usual square bracket notation $[m]$ for the integer part of $m \in \mathbb{R}$. We also use Landau’s big- O and little- o notation with its standard meaning.

2. Definition of perturbation scheme and symmetry preserving perturbations

The construction of a spacetime perturbation relies on a one-parameter family of C^{n+1} ($n \geq 2$) spacetimes $(M_\varepsilon, \hat{g}_\varepsilon)$, where ε takes values in an open interval $I_0 \subset \mathbb{R}$ containing zero, from where we single out the background $(M, g) := (M_0, \hat{g}_0)$, diffeomorphically identified through a gauge ψ_ε so that

$$(2.1) \quad \psi_\varepsilon : M \rightarrow M_\varepsilon,$$

and ψ_0 is the identity. The diffeomorphisms ψ_ε are assumed to be C^{n+2} for each ε . This allows us to define a family of metrics g_ε of class C^{n+1} on M related to \hat{g}_ε by $g_\varepsilon := \psi_\varepsilon^*(\hat{g}_\varepsilon)$. We further assume that this family of metrics is at least C^2 in ε (to guarantee we can go to second order) and that ε -derivatives do not affect the differentiability class. Define also the tensor

$g_\varepsilon^{\sharp\mu\nu}$ by $g_\varepsilon^\sharp := d\psi_\varepsilon^{-1}(\hat{g}_\varepsilon^\sharp)$ where $\hat{g}_\varepsilon^\sharp$ is the contravariant metric associated to \hat{g}_ε . Note that g_ε^\sharp is the contravariant metric associated to g_ε .

The first and second order perturbation tensors K_1 and K_2 on (M, g) are obtained from g_ε as follows

$$(2.2) \quad K_1 := \left. \frac{dg_\varepsilon}{d\varepsilon} \right|_{\varepsilon=0}, \quad K_2 := \left. \frac{d^2g_\varepsilon}{d\varepsilon^2} \right|_{\varepsilon=0},$$

while the derivatives of the contravariant metrics are ¹

$$(2.3) \quad \left. \frac{dg_\varepsilon^{\sharp\alpha\beta}}{d\varepsilon} \right|_{\varepsilon=0} = -K_1^{\alpha\beta}, \quad \left. \frac{d^2g_\varepsilon^{\sharp\alpha\beta}}{d\varepsilon^2} \right|_{\varepsilon=0} = -K_2^{\alpha\beta} + 2K_1^{\alpha\mu}K_{1\mu}{}^\beta.$$

The identification by ψ_ε is highly non-unique and its freedom can be realized by taking into consideration an ε -dependent diffeomorphism $\Omega_\varepsilon : M \rightarrow M$ in M before applying ψ_ε . The new identification is $\psi_\varepsilon^g := \psi_\varepsilon \circ \Omega_\varepsilon$ and introduces a new family of tensors $g_\varepsilon^g = \psi_\varepsilon^{g*}(\hat{g}_\varepsilon) = \Omega_\varepsilon^*(g_\varepsilon)$ on M with corresponding first and second order perturbation tensors K_1^g and K_2^g . We again assume that Ω_ε are C^{m+2} diffeomorphisms with C^2 dependence in ε and that ε -derivatives do not change the differentiability class. In terms of the first and second order (spacetime) *gauge vectors* V_1 and V_2 , defined as follows

$$(2.4) \quad V_1 := \left. \frac{\partial\Omega_\varepsilon}{\partial\varepsilon} \right|_{\varepsilon=0}, \quad V_2 := \left. \frac{\partial V_\varepsilon}{\partial\varepsilon} \right|_{\varepsilon=0}, \quad V_\varepsilon := \left. \frac{\partial(\Omega_{\varepsilon+h} \circ \Omega_\varepsilon^{-1})}{\partial h} \right|_{h=0},$$

the relation between K_1^g, K_2^g and K_1, K_2 is given by [5, 13]

$$(2.5) \quad K_1^g = K_1 + \mathcal{L}_{V_1}g,$$

$$(2.6) \quad K_2^g = K_2 + \mathcal{L}_{V_2}g + 2\mathcal{L}_{V_1}K_1^g - \mathcal{L}_{V_1}\mathcal{L}_{V_1}g.$$

Since the background manifold is C^{n+2} and the metric is C^{n+1} , the natural differentiability class preserved by these gauge transformations is as follows. At first order the gauge vector V_1 is C^{n+1} and the perturbation tensor K_1 is C^n . At second order V_2 is C^n and K_2 is C^{n-1} . We will therefore incorporate this assumption into our definitions:

¹All Greek indices in this paper are raised and lowered with the background metric g and its inverse.

A *perturbation* is, by definition, the family $(M_\varepsilon, \hat{g}_\varepsilon)$ with all the possible identifications $\{\psi_\varepsilon\}$ related to each other by a gauge transformation. The intrinsic gauge freedom of a perturbation is a major source of complication, as a specific problem may have very different forms in different gauges. A selection of a class of gauges leads to what we call a *perturbation scheme*, namely a triple $(M_\varepsilon, \hat{g}_\varepsilon, \{\psi_\varepsilon\})$, where $\{\psi_\varepsilon\}$ denotes the selected class of gauges. A *perturbation scheme* will be said of class C^{n+1} when the family \hat{g}_ε is C^{n+1} , the perturbation tensors K_1, K_2 are, respectively, C^n and C^{n-1} and the gauge vectors V_1, V_2 are, respectively, C^{n+1} and C^n .

In many specific problems one often needs to preserve some of the symmetries of the background along the perturbation. This leads to the notion of “inheritance of symmetries” which we define next.

Definition 2.1. Let $(M_\varepsilon, \hat{g}_\varepsilon, \{\psi_\varepsilon\})$ be a perturbation scheme whose background spacetime (M, g) admits a Killing vector field ξ . The perturbation scheme is said to **inherit the (local) symmetry generated by ξ** whenever for all $\varepsilon \in I_0$ and all $\psi_\varepsilon \in \{\psi_\varepsilon\}$, the vector field $\hat{\xi}_\varepsilon := d\psi_\varepsilon(\xi)$ is a Killing vector of $(M_\varepsilon, \hat{g}_\varepsilon)$.

Remark 2.2. Of course, this definition just recovers the usual idea that the perturbation admits a symmetry when so does the family g_ε , since

$$\mathcal{L}_\xi g_\varepsilon = \psi_\varepsilon^*(\mathcal{L}_{\hat{\xi}_\varepsilon} \hat{g}_\varepsilon) = 0.$$

Note that the notion of “inheriting a (local) isometry” depends not only on the perturbation itself, but also on the perturbation scheme. Indeed, given a perturbation scheme $(M_\varepsilon, \hat{g}_\varepsilon, \{\psi_\varepsilon\})$ one may construct other perturbation schemes belonging to the same perturbation where the symmetry is not inherited. This is because a fixed vector field ξ which is Killing for all g_ε will not, in general, be a Killing of $\Omega_\varepsilon^*(g_\varepsilon)$. Thus, demanding the existence of a perturbation scheme where a symmetry is inherited is useful both to restrict geometrically the family of perturbations and *also* to restrict the class of allowed gauges.

Remark 2.3. If a perturbation scheme inherits a collection of (local) isometries of the background that form a subalgebra \mathbb{A}_0 , then each $(M_\varepsilon, \hat{g}_\varepsilon)$ admits the same algebra, because push-forwards preserve commutation relations [20]. For the same reason, \mathbb{A}_0 must leave invariant the family of tensors g_ε on M .

Besides inheriting (local) isometries, in many cases some other geometrical aspects concerning the orbit of the (local) group are also required to be preserved. In the following two sections we consider axially symmetric perturbations and “orthogonal transitive” perturbations independently. Our final aim will be the construction of a convenient perturbation scheme for stationary and axially symmetric perturbations that inherit also the geometric property of being “orthogonal transitive.”

3. Axially symmetric perturbations

In this section we recall the concept of axial symmetry and introduce the definition of axially symmetric perturbations as a particular instance of the general notion of “symmetry inheritance”. Our main result of the section is Lemma 3.6 where we explore the consequences of axial symmetry on the structure of symmetric two-covariant tensors. This result will play a relevant role in subsequent sections, where the background is taken to be spherically symmetric.

The definition of axial symmetry is standard (see e.g. [2, 7, 16]):

Definition 3.1. A spacetime (M, g) is axially symmetric whenever there is an effective realization of the one-dimensional torus T into M that is an isometry and such that the set of fixed points is non-empty.

We denote by η the Killing vector field defined by this realization assuming that the torus T has been parametrized with the standard 2π -periodicity angle. First consequences from the definition are that the set of fixed points, where $\eta = 0$ and which we call the axis \mathcal{A} , is a codimension two Lorentzian and time-oriented surface [7, 16]. Furthermore, \mathcal{A} is autoparallel and for any point $p \in \mathcal{A}$ there is a neighbourhood of p such that $\eta^2 := \langle \eta, \eta \rangle$ is non-negative and zero only at points on the axis (a priori this property may fail sufficiently far away from the axis). Moreover

$$(3.1) \quad \lim_{\eta^2 \rightarrow 0} \frac{\langle \nabla \eta^2, \nabla \eta^2 \rangle}{4\eta^2} = 1.$$

This is the so-called *regular axis property*, from where the usual *elementary flatness* around the axis can be inferred.

We can particularize Definition 2.1 to the case of axial symmetry and introduce the notion of axially symmetric perturbation scheme. To be explicit:

Definition 3.2. A perturbation scheme $(M_\varepsilon, \hat{g}_\varepsilon, \{\psi_\varepsilon\})$ of an axially symmetric background spacetime (M, g) with axial vector η is an **axially symmetric perturbation scheme** if it inherits the axial symmetry in the sense of Definition 2.1.

In this setup $\hat{\eta}_\varepsilon = d\psi_\varepsilon(\eta)$ is a Killing vector for each $(M_\varepsilon, \hat{g}_\varepsilon, \{\psi_\varepsilon\})$, with axis $\hat{\mathcal{A}}_\varepsilon$ defined by the points at which $\hat{\eta}_\varepsilon = 0$, i.e. $\hat{\mathcal{A}}_\varepsilon = \{p_\varepsilon \in M_\varepsilon; \hat{\eta}_\varepsilon|_{p_\varepsilon} = 0\}$. By the invertibility of $\hat{\eta}_\varepsilon = d\psi_\varepsilon(\eta)$ the points p_ε that satisfy $\hat{\eta}_\varepsilon|_{p_\varepsilon} = 0$ are those $p_\varepsilon = \psi_\varepsilon(p)$ such that $\eta|_p = 0$, and therefore ψ_ε simply maps the axis at the background \mathcal{A} to their corresponding $\hat{\mathcal{A}}_\varepsilon$.

Since the Killing equations $\mathcal{L}_\eta g_\varepsilon = 0$ hold for all ε , we necessarily have, up to second order,

$$(3.2) \quad \mathcal{L}_\eta K_1 = 0, \quad \mathcal{L}_\eta K_2 = 0.$$

The regular axis property (3.1) holds at each ε -component, that is, the function

$$\hat{\Lambda}_\varepsilon := \frac{\hat{g}_\varepsilon^{\alpha\beta} \partial_\alpha(\hat{\eta}_\varepsilon^2) \partial_\beta(\hat{\eta}_\varepsilon^2)}{4\hat{\eta}_\varepsilon^2},$$

where $\hat{\eta}_\varepsilon^2 := \hat{g}_\varepsilon(\hat{\eta}_\varepsilon, \hat{\eta}_\varepsilon)$, must be 1 at $\hat{\mathcal{A}}_\varepsilon$. Therefore, by construction, the pull-back $\Lambda_\varepsilon := \psi_\varepsilon^*(\hat{\Lambda}_\varepsilon) = g_\varepsilon^{\alpha\beta} \partial_\alpha(\eta_\varepsilon^2) \partial_\beta(\eta_\varepsilon^2) / 4\eta_\varepsilon^2$, where $\eta_\varepsilon^2 := \psi_\varepsilon^*(\hat{\eta}_\varepsilon^2) = g_\varepsilon(\eta, \eta)$, must attain 1 at \mathcal{A} , i.e.

$$\lim_{\eta^2 \rightarrow 0} \Lambda_\varepsilon = 1.$$

Since $\lim_{\eta^2 \rightarrow 0} \Lambda_0 = 1$ (regular background configuration), the *regular axis property* on the perturbation scheme translates, to second order, onto the fact that

$$\Lambda^{(1)} := \left. \frac{d\Lambda_\varepsilon}{d\varepsilon} \right|_{\varepsilon=0}, \quad \Lambda^{(2)} := \left. \frac{d^2\Lambda_\varepsilon}{d\varepsilon^2} \right|_{\varepsilon=0},$$

satisfy

$$(3.3) \quad \lim_{\eta \rightarrow 0} \Lambda^{(1)} = 0, \quad \lim_{\eta \rightarrow 0} \Lambda^{(2)} = 0.$$

Since

$$\left. \frac{d\eta_\varepsilon^2}{d\varepsilon} \right|_{\varepsilon=0} = K_1(\eta, \eta), \quad \left. \frac{d^2\eta_\varepsilon^2}{d\varepsilon^2} \right|_{\varepsilon=0} = K_2(\eta, \eta),$$

and recalling (2.3) a straightforward calculation shows

$$(3.4) \quad \Lambda^{(1)} = \frac{1}{4\eta^2} \left(-K_1(\nabla\eta^2, \nabla\eta^2) + 2\langle \nabla(K_1(\eta, \eta)), \nabla\eta^2 \rangle - \frac{1}{\eta^2} K_1(\eta, \eta) \langle \nabla\eta^2, \nabla\eta^2 \rangle \right)$$

and

$$(3.5) \quad \Lambda^{(2)} = \frac{1}{4\eta^2} \left\{ -K_2(\nabla\eta^2, \nabla\eta^2) + 2\langle \nabla(K_2(\eta, \eta)), \nabla\eta^2 \rangle - \frac{1}{\eta^2} K_2(\eta, \eta) \langle \nabla\eta^2, \nabla\eta^2 \rangle + 2(K_1 \cdot K_1)(\nabla\eta^2, \nabla\eta^2) - 4K_1(\nabla(K_1(\eta, \eta)), \nabla\eta^2) + 2\langle \nabla(K_1(\eta, \eta)), \nabla(K_1(\eta, \eta)) \rangle \right\} - \frac{2}{\eta^2} \Lambda^{(1)} K_1(\eta, \eta).$$

Summarizing we have shown the following result.

Lemma 3.3. *Consider an axially symmetric perturbation scheme $(M_\varepsilon, \hat{g}_\varepsilon, \{\psi_\varepsilon\})$ for an axially symmetric background spacetime (M, g) with axial Killing vector η . Denote $\eta^2 := \langle \eta, \eta \rangle$. Then, the first and second order perturbation tensors K_1 and K_2 satisfy (3.2), the axis of symmetry of the perturbation coincides with the background axis of symmetry \mathcal{A} and the quantities $\Lambda^{(1)}$ and $\Lambda^{(2)}$, given by (3.4) and (3.5) respectively, vanish there.*

Since in this paper we will be concerned with perturbations up to second order it makes sense to relax the definition of axially symmetric perturbation and impose conditions only up to this order. This leads to the following definition.

Definition 3.4. A perturbation scheme $(M_\varepsilon, \hat{g}_\varepsilon, \{\psi_\varepsilon\})$ of an axially symmetric background spacetime (M, g) is a **second order axially symmetric perturbation** if it satisfies the outcome of Lemma 3.3.

The defining property of an axially symmetric perturbation scheme is that the axial Killing vector of the background is mapped to an axial Killing vector of $(M_\varepsilon, \hat{g}_\varepsilon)$. Except in very special circumstances the spacetime $(M_\varepsilon, \hat{g}_\varepsilon)$, $\varepsilon \neq 0$ will admit only one axial Killing vector $\hat{\eta}_\varepsilon$, so $d\psi_\varepsilon(\eta)$ is forced to be this unique axial Killing field. In exceptional circumstances,

where additional axial symmetries are present, the axially symmetric perturbation scheme becomes automatically larger. This additional freedom, however, is of a trivial nature and can be removed by any a priori identification of an axial Killing at every $\varepsilon \neq 0$. In the next lemma we identify restrictions on the perturbation vectors that arise from either the uniqueness of the axial symmetry of $(M_\varepsilon, \hat{g}_\varepsilon)$, $\varepsilon \neq 0$ or, in the exceptional cases, of having identified one axial Killing at each ε . This result will be useful in [15] where uniqueness issues are discussed.

Lemma 3.5. *In the setup of Lemma 3.3, if the class $\{\psi_\varepsilon\}$ is such that the axial Killing vector $\hat{\eta}_\varepsilon = d\psi_\varepsilon(\eta)$ of $(M_\varepsilon, \hat{g}_\varepsilon)$ is independent of $\psi_\varepsilon \in \{\psi_\varepsilon\}$, then the gauge vectors V_1 and V_2 defined by a change of gauge within the class satisfy*

$$[V_1, \eta] = 0, \quad [V_2, \eta] = 0.$$

Proof. Let ψ_ε and ψ_ε^g belong to the class $\{\psi_\varepsilon\}$. Recalling the definition $\Omega_\varepsilon = \psi_\varepsilon^{-1} \circ \psi_\varepsilon^g$, the assumption of the lemma implies (in fact, is equivalent to) both $\boldsymbol{\eta} = \Omega_\varepsilon^{-1*}(\boldsymbol{\eta})$ and $\boldsymbol{\eta} = \Omega_\varepsilon^*(\boldsymbol{\eta})$, for all $\varepsilon \in I_0$. From the expression of V_ε in (2.4) and the definition of Lie derivative we get $\mathcal{L}_{V_\varepsilon}\boldsymbol{\eta} = 0$. Evaluating at $\varepsilon = 0$ yields $\mathcal{L}_{V_{\varepsilon=0}}\boldsymbol{\eta} = \mathcal{L}_{V_1}\boldsymbol{\eta} = 0$. Taking the derivative at $\varepsilon = 0$ and using (see e.g. the proof of Lemma 1 in [13])

$$\frac{d}{d\varepsilon}\mathcal{L}_{V_\varepsilon}\boldsymbol{\eta} = \mathcal{L}_{\frac{dV_\varepsilon}{d\varepsilon}}\boldsymbol{\eta},$$

the definition of V_2 in (2.4) gives $\mathcal{L}_{V_2}\boldsymbol{\eta} = 0$. □

In order to explore the consequences of Definition 3.4 we need to study the restrictions imposed by equation (3.2). The following lemma applies to arbitrary symmetric tensors invariant under an axial Killing and may have independent interest. For related results in the particular case when K is the spacetime (background) metric see [6].

Lemma 3.6. *Let (M, g) be a k -dimensional ($k \geq 2$) axially symmetric spacetime with axial Killing η and axis \mathcal{A} . Let K be a symmetric 2-covariant tensor satisfying $\mathcal{L}_\eta K = 0$ and C^m ($m \geq 1$) on a neighbourhood $\mathcal{U}_\mathcal{A} \subset M$ of a portion of \mathcal{A} . By restricting $\mathcal{U}_\mathcal{A}$ if necessary we take $\mathcal{U}_\mathcal{A}$ invariant under η and admitting global coordinates $\{x, y, w^u\}$, where $u = 3, \dots, k$ if $k \geq 3$ or $w^u = \emptyset$ otherwise, such that $\eta = x\partial_y - y\partial_x$. Let $U \subset \mathbb{R}^k$ be the set where $\{x, y, w^u\}$ take values. Then, using the notation $\|x\| := \sqrt{x^2 + y^2}$ and*

$K_{xx} := K(\partial_x, \partial_x)$, etc, it follows that, on $U \setminus \{\|x\| = 0\}$:

$$(3.6) \quad 2K_{xx} = \mathfrak{A}_1(\|x\|, w^u) + \mathfrak{A}_2(\|x\|, w^u) \frac{x^2 - y^2}{\|x\|^2} - \mathfrak{A}_3(\|x\|, w^u) \frac{2xy}{\|x\|^2},$$

$$(3.7) \quad 2K_{yy} = \mathfrak{A}_1(\|x\|, w^u) - \mathfrak{A}_2(\|x\|, w^u) \frac{x^2 - y^2}{\|x\|^2} + \mathfrak{A}_3(\|x\|, w^u) \frac{2xy}{\|x\|^2},$$

$$(3.8) \quad 2K_{xy} = \mathfrak{A}_2(\|x\|, w^u) \frac{2xy}{\|x\|^2} + \mathfrak{A}_3(\|x\|, w^u) \frac{x^2 - y^2}{\|x\|^2},$$

$$(3.9) \quad K_{xu} = \frac{1}{\|x\|^2} (\mathfrak{A}_{\rho u}(\|x\|, w^u)x + \mathfrak{A}_{\eta u}(\|x\|, w^u)y),$$

$$(3.10) \quad K_{yu} = \frac{1}{\|x\|^2} (-\mathfrak{A}_{\eta u}(\|x\|, w^u)x + \mathfrak{A}_{\rho u}(\|x\|, w^u)y),$$

$$(3.11) \quad K_{uv} = \mathfrak{A}_{uv}(\|x\|^2, w^u),$$

where, for each $\mathfrak{B} \in \{1, 2, 3, \rho u, \eta u, uv\}$, the functions $\mathfrak{A}_{\mathfrak{B}}(\rho, w^u)$ are defined on the domain $U_\eta := \{\rho \in \mathbb{R}_{\geq 0}, w^u \in \mathbb{R}^{k-2}(\rho, 0, w^u) \in U\} \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^{k-2}$. Moreover,

$$(3.12) \quad \mathfrak{A}_{\mathfrak{B}}(\rho, w^u) = \mathcal{P}_{\mathfrak{B}}(\rho^2, w^u) + \Phi_{\mathfrak{B}}^{(m)}(\rho, w^u),$$

where each $\mathcal{P}_{\mathfrak{B}}(\rho^2, w^u)$ is a polynomial of degree $[\frac{m}{2}]$ in ρ^2 , $\Phi_{\mathfrak{B}}^{(m)}$ is $o(\rho^m)$ and $\mathcal{P}_{\mathfrak{B}}, \Phi_{\mathfrak{B}}^{(m)}$ are C^m on U_η . Furthermore $\mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_{\rho u}, \mathcal{P}_{\eta u}$ vanish at $\rho = 0$ (for all w^u).

Remark 3.7. Several expressions above look undetermined at $\|x\| = 0$. The factors $(x^2 - y^2)/\|x\|^2$ and $xy/\|x\|^2$ are bounded but have no limit as $\|x\| \rightarrow 0$. However, this is only apparent because $\mathfrak{A}_2, \mathfrak{A}_3, \mathfrak{A}_{\rho u}, \mathfrak{A}_{\eta u}$ vanish as $\|x\| \rightarrow 0$. Therefore, the expressions are valid in the whole U .

Proof. The equation $\mathcal{L}_\eta K = 0$ takes the following explicit form in components

$$(3.13) \quad \eta(K_{xx}) = -2K_{xy}, \quad \eta(K_{yy}) = 2K_{xy}, \quad \eta(K_{xy}) = K_{xx} - K_{yy},$$

$$(3.14) \quad \eta(K_{xu}) + K_{yu} = 0, \quad \eta(K_{yu}) - K_{xu} = 0,$$

$$(3.15) \quad \eta(K_{uv}) = 0.$$

K being at least C^1 and η vanishing on the axis these equations readily imply that $K_{xx} - K_{yy}, K_{xy}, K_{xu}, K_{yu}$ all vanish at $\|x\| = 0$. Moreover, the

following consequences are easily obtained

$$\begin{aligned} \eta(K_{xx} + K_{yy}) &= 0, & \eta(xK_{xu} + yK_{yu}) &= 0, \\ \eta(yK_{xu} - xK_{yu}) &= 0, & \eta(K_{uv}) &= 0, \end{aligned}$$

so $K_{xx} + K_{yy}, xK_{xu} + yK_{yu}, yK_{xu} - xK_{yu}$ and K_{uv} are radially symmetric in the variables $\{x, y\}$ and C^m in U . By Lemma A.1 in Appendix A, there exist $\mathfrak{A}_1(\rho, w^u), \mathfrak{A}_{\rho u}(\rho, w^u), \mathfrak{A}_{\eta u}(\rho, w^u)$ and $\mathfrak{A}_{uv}(\rho, w^u)$ defined on U_η and satisfying (3.12) such that

$$\begin{aligned} K_{xx} + K_{yy} &= \mathfrak{A}_1(\|x\|, w^u), & K_{uv} &= \mathfrak{A}_{uv}(\|x\|, w^u), \\ yK_{xu} - xK_{yu} &= \mathfrak{A}_{\eta u}(\|x\|, w^u), & xK_{xu} + yK_{yu} &= \mathfrak{A}_{\rho u}(\|x\|, w^u). \end{aligned}$$

Now, $\mathfrak{A}_{\rho u}(\rho, w^u)$ and $\mathfrak{A}_{\eta u}(\rho, w^u)$ vanish at $\rho = 0$ (for all w^u), so the corresponding polynomials $\mathcal{P}_{\rho u}, \mathcal{P}_{\eta u}$ have the same property. From the definitions it is clear that (3.9), (3.10) and (3.11) hold. Next, define

$$\begin{aligned} A_2 &:= \frac{x^2 - y^2}{\|x\|^2}(K_{xx} - K_{yy}) + \frac{2xy}{\|x\|^2}2K_{xy}, \\ A_3 &:= -\frac{2xy}{\|x\|^2}(K_{xx} - K_{yy}) + \frac{x^2 - y^2}{\|x\|^2}2K_{xy} \end{aligned}$$

in $U \setminus \{\|x\| = 0\}$. The vanishing of $K_{xx} - K_{yy}$ and K_{xy} at the axis shows that these functions extend continuously to zero at $\|x\| = 0$. Moreover, since

$$\eta\left(\frac{x^2 - y^2}{\|x\|^2}\right) = -2\frac{2xy}{\|x\|^2}, \quad \eta\left(\frac{2xy}{\|x\|^2}\right) = 2\frac{x^2 - y^2}{\|x\|^2},$$

we have $\eta(A_2) = 0$ and $\eta(A_3) = 0$, and therefore A_2, A_3 are radially symmetric in $\{x, y\}$. Define the corresponding traces $\mathfrak{A}_2, \mathfrak{A}_3 : U_\eta \rightarrow \mathbb{R}$ by $A_2 = \mathfrak{A}_2(\|x\|, w^u)$ and $A_3 = \mathfrak{A}_3(\|x\|, w^u)$. Obviously these functions satisfy $\mathfrak{A}_2(0, w^u) = \mathfrak{A}_3(0, w^u) = 0$. Directly from the definitions $f := K_{xx} - K_{yy}$ takes the form $f = \mathfrak{A}_2(\|x\|, w^u)\frac{x^2 - y^2}{\|x\|^2} - \mathfrak{A}_3(\|x\|, w^u)\frac{2xy}{\|x\|^2}$ in $U \setminus \{\|x\| = 0\}$. So far we thus have (3.6) and (3.7), and the first equation in (3.13) leads to (3.8). It only remains to show that \mathfrak{A}_2 and \mathfrak{A}_3 admit the decomposition (3.12). Define the functions

$$\begin{aligned} s_2(x, w^u) &:= f(x, y = 0, w^u) = \mathfrak{A}_2(\text{abs}(x), w^u), \\ s_3(x, w^u) &:= -\frac{1}{\sqrt{2}}f(x, y = x, w^u) = \mathfrak{A}_3(\text{abs}(x), w^u). \end{aligned}$$

First of all, the fact that f is C^m (in U) implies $s_2(x, w^u)$ and $s_3(x, w^u)$ are C^m functions of their arguments, particularly in the domain $x \geq 0$. Setting $\rho = \text{abs}(x)$ we have that $\mathfrak{A}_2(\rho, w^u) = s_2(\rho, w^u)$ and $\mathfrak{A}_3(\rho, w^u) = s_3(\rho, w^u)$, and by Lemma A.1 particularised for $p = 1$, $\mathfrak{A}_2, \mathfrak{A}_3$ are C^m functions in U_η with the structure given in (3.12). \square

4. “Orthogonally transitive” perturbations

Recall that an Abelian G_2 (local) group of isometries on a spacetime (M, g) is **orthogonally transitive** if, except possibly on a subset with empty interior, the orbits of the local isometry are two-dimensional, non-null and their orthogonal spaces form an integrable distribution (equivalently, almost everywhere in M there exists a foliation by immersed non-null surfaces orthogonal to the orbits everywhere). By definition we say that this property is inherited by a perturbation scheme if the background admits an Abelian orthogonally transitive G_2 local action which is inherited on each $(M_\varepsilon, \hat{g}_\varepsilon)$ and the corresponding (local) orbits are orthogonally transitive.

4.1. Orthogonally transitive actions

The results of this subsection are essentially known. We include them for completeness since they will be needed later.

Consider a C^{n+1} ($n \geq 0$) pseudo-Riemannian manifold (M, g) of dimension k and arbitrary signature. Let $\{\xi_i\}$ $i, j, l, \dots = 1, \dots, s < k$ be a collection of (linearly independent) Killing vector fields that form an algebra, i.e. such that

$$[\xi_i, \xi_j] = C_{ij}^l \xi_l.$$

For all $p \in M$ define $\Pi|_p = \text{span}(\xi_1|_p, \dots, \xi_s|_p)$. This is a vector subspace of $T_p M$ of dimension at most s . Define $M' := \{p \in M; \dim(\Pi|_p) = s\}$. Linear independence of the Killing vectors implies M' is dense in M . We make the following two assumptions:

- (i) For all $p \in M'$, the metric g restricted to $\Pi|_p$ is non-degenerate.
- (ii) For all $p \in M'$, the g -orthogonal complement $\Pi^\perp|_p$ defines an integrable distribution of M' .

Note that by (i) we have $T_pM = \Pi|_p \oplus \Pi^\perp|_p$ for all $p \in M'$. Since the distribution $\{\Pi^\perp|_p\}$ is defined as the set of vectors in the kernel of all² $\xi_i := g(\xi_i, \cdot)$, the Fröbenius theorem states that (ii) can be written equivalently in the following two ways

$$\begin{aligned} \text{(ii')} \quad & d\xi_{i_1} \wedge \xi_{i_2} \wedge \cdots \wedge \xi_{i_{s+1}} = 0 \quad \forall i_1, \dots, i_{s+1} = 1, \dots, s, \\ \text{(ii'')} \quad & d\xi_i = \sum_{j=1}^s \xi_j \wedge \Sigma_{ij} \end{aligned}$$

where Σ_{ij} are smooth one-forms on M' .

We note the following facts. If V, W are vector fields everywhere orthogonal to $\{\xi_i\}$ (or, in other words, taking values in $\Pi^\perp|_p$, for all $p \in M$) then $[\xi_i, V]$ and $[V, W]$ have the same property. The proof is by direct computation (α, β, \dots are indices of M and ∇ is the Levi-Civita derivative of g):

$$\begin{aligned} \xi_{i\alpha}[\xi_j, V]^\alpha &= \xi_{i\alpha} \left(\xi_j^\beta \nabla_\beta V^\alpha - V^\beta \nabla_\beta \xi_j^\alpha \right) = -V_\alpha \left(\xi_j^\beta \nabla_\beta \xi_i^\alpha - \xi_i^\beta \nabla_\beta \xi_j^\alpha \right) \\ &= -V_\alpha [\xi_j, \xi_i]^\alpha = 0 \end{aligned}$$

and

$$\begin{aligned} \xi_{i\alpha}[V, W]^\alpha &= \xi_{i\alpha} \left(V^\beta \nabla_\beta W^\alpha - W^\beta \nabla_\beta V^\alpha \right) = V^\beta W^\alpha \left(-\nabla_\beta \xi_{i\alpha} + \nabla_\alpha \xi_{i\beta} \right) \\ &= d\xi_i(W, V) = \left(\sum_{j=1}^s \xi_j \wedge \Sigma_{ij} \right)(W, V) = 0. \end{aligned}$$

Define $\gamma_{ij} := g(\xi_i, \xi_j)$. The next lemma introduces a set of closed one-forms that will then allow us to introduce suitable local coordinates.

Lemma 4.1. *In the setup above, assume further that the Lie algebra $\{\xi_i\}$ is Abelian. Then there exists smooth closed one-forms ζ^i on M' such that*

$$(4.1) \quad \xi_i = \gamma_{ij} \zeta^j.$$

Proof. By condition (i), γ_{ij} is invertible on M' . Let γ^{ij} be the inverse (i.e. such that $\gamma^{ik} \gamma_{kj} = \delta_j^i$) and define $\zeta^i := \gamma^{ij} \xi_j$. These are C^{n+1} one-forms on M' obviously satisfying (4.1). It remains to show that $d\zeta^i = 0$. Observe first

²We use boldface to denote the metrically related one-form associated to a vector field.

that $\zeta^i(\xi_j) = \delta_j^i$. We use the definition of exterior differential

$$d\zeta^i(X, Y) = X(\zeta^i(Y)) - Y(\zeta^i(X)) - \zeta^i([X, Y])$$

where X, Y are arbitrary vector fields in M' . To prove $d\zeta^i = 0$ it suffices to show that the right-hand side of this expression vanishes in the following three cases, (a) $X = \xi_i, Y = \xi_j$, (b) $X = \xi_i, Y = W$ and (c) $X = V, Y = W$, where V, W are everywhere orthogonal to $\{\xi_j\}$. Now

$$\begin{aligned} (a) \quad & d\zeta^i(\xi_j, \xi_l) = \xi_j(\delta_l^i) - \xi_l(\delta_j^i) - \zeta^i([\xi_j, \xi_l]) = 0 \\ (b) \quad & d\zeta^i(\xi_j, V) = -V(\delta_j^i) - \zeta^i([\xi_j, V]) = 0 \\ (c) \quad & d\zeta^i(V, W) = -\zeta^i([V, W]) = 0, \end{aligned}$$

where in (a) we used the assumption that algebra is Abelian and in (b), (c) the fact that $[\xi_j, V]$ and $[V, W]$, along with V, W are all orthogonal to ξ_j . □

Corollary 4.2. *If M' is simply connected, then there exist smooth functions z^i on M' such that*

$$(4.2) \quad \xi_i(z^j) = \delta_i^j, \quad \xi_i = \gamma_{ij} dz^j.$$

Proof. By simply connectedness, closed is equivalent to exact. Define z^j by $\zeta^j = dz^j$ and (4.2) are immediate. □

It is also a well-known fact that under assumptions (i) and (ii) there exist local coordinates near any point in M' in which the metric separates in blocks. The argument is standard, but we include it for completeness. Fix $p \in M'$ and let Σ_p be the integrable manifold of the distribution Π^\perp containing p . Select any local coordinate system $\{x^a\}$ on Σ_p near p and extend the functions x^a as constants along the integral manifolds of the distribution Π . By construction $\xi^i(x^a) = 0$. It is immediate that $\{x^a, z^i\}$ defines a local coordinate system in a neighbourhood of p . By the first of (4.2) we have $\xi^i = \partial_{z^i}$, so the metric in these coordinates only depends on $\{x^a\}$. By the second in (4.2) the cross components g_{ia} of the metric vanish and $g_{ij} = \gamma_{ij}$. The block diagonal structure follows

$$g = \gamma_{ij}(x^c) dx^i dx^j + h_{ab}(x^c) dx^a dx^b.$$

4.2. “Orthogonally transitive” perturbation scheme

Let $(M_\varepsilon, \hat{g}_\varepsilon, \{\psi_\varepsilon\})$ be a perturbation scheme that inherits an orthogonally transitive Abelian G_s (local) group of isometries generated by $\text{span}\{\xi_i\}$, $i, j = 1, \dots, s$, ($1 \leq s < \dim(M)$) on (M, g) . The next proposition shows that the first and second order metric perturbation tensors can be taken as block diagonal in M' . Later we show that in the stationary and axisymmetric case the transformation does not lower the differentiability of the perturbation tensors in the whole M , that is, including the axis.

Proposition 4.3. *Let $(M_\varepsilon, \hat{g}_\varepsilon, \{\psi_\varepsilon\})$ be a perturbation scheme of class C^{n+1} ($n \geq 2$) that inherits an Abelian G_s , $1 \leq s < \dim(M)$ (local) isometry group that acts orthogonally transitively. Let M' be the open and dense subset where the orbits of the Abelian (local) isometry group are non-null and have dimension s and assume that M' is simply connected. Then, for each $g_\varepsilon = \psi_\varepsilon(\hat{g}_\varepsilon)$, $\psi_\varepsilon \in \{\psi_\varepsilon\}$ there exist first and second order gauge vectors V_1 and V_2 such that the corresponding transformed K_1^g and K_2^g are of class $C^n(M')$ and $C^{n-1}(M')$ respectively and block diagonal, i.e. take form*

$$\begin{aligned} K_1 &= K_{1ij}(x^c)dx^i dx^j + K_{1ab}(x^c)dx^a dx^b, \\ K_2 &= K_{2ij}(x^c)dx^i dx^j + K_{2ab}(x^c)dx^a dx^b \end{aligned}$$

in any local coordinate system $\{x^\alpha\} = \{x^i, x^a\}$ in M' adapted to ξ_i and to the orthogonal transitivity of the background, i.e. where

$$(4.3) \quad g = \gamma_{ij}(x^c)dx^i dx^j + h_{ab}(x^c)dx^a dx^b, \quad \xi_i = \partial_{x^i}.$$

Proof. By Lemma 4.1 and Corollary 4.2, there exists smooth scalar functions $z_\varepsilon^i : M' \rightarrow \mathbb{R}$ satisfying $\xi_i(z_\varepsilon^j) = \delta_i^j$ such that the one-form fields $\xi_i^\varepsilon := g_\varepsilon(\xi_i, \cdot)$ take the form

$$\xi_i^\varepsilon = \gamma_{ij}^\varepsilon dz_\varepsilon^j, \quad \gamma_{ij}^\varepsilon := g_\varepsilon(\xi_i, \xi_j).$$

Let $\{x^\alpha\}$ be any local coordinate where the background metric takes the form (4.3). The condition $\xi_i(z_\varepsilon^j) = \delta_i^j$ implies the existence of scalar functions $u_\varepsilon^j(x^a)$ such that $z_\varepsilon^j = x^j + u_\varepsilon^j(x^a)$. The form of ξ_i^ε forces that the metric g_ε in these coordinates takes the form

$$g_\varepsilon = \gamma_{ij}^\varepsilon(dx^i + \partial_a u_\varepsilon^i dx^a)(dx^j + \partial_b u_\varepsilon^j dx^b) + h_{ab}^\varepsilon dx^a dx^b.$$

Since $\xi_i = \partial_{x^i}$ are Killing vectors of this metric, $\gamma_{ij}^\varepsilon(x^a)$ and $h_{ab}^\varepsilon(x^c)$. Moreover, the equality $g = g_{\varepsilon=0}$ implies

$$\gamma_{ij}^\varepsilon|_{\varepsilon=0} = \gamma_{ij}, \quad h_{ij}^\varepsilon|_{\varepsilon=0} = h_{ij}, \quad u_\varepsilon^i|_{\varepsilon=0} = 0.$$

Computing the first and second order perturbation tensors, one finds after a simple calculation

$$\begin{aligned} K_1 &= K_{1ij}dx^i dx^j + 2\gamma_{ij}\partial_a U^j dx^i dx^a + K_{1ab}dx^a dx^b, \\ K_2 &= K_{2ij}dx^i dx^j + 4K_{1ij}\partial_a U^j dx^i dx^a + 2\gamma_{ij}\partial_a U^i \partial_b U^j dx^a dx^b \\ &\quad + 2\gamma_{ij}\partial_a W^i dx^i dx^a + K_{2ab}dx^a dx^b, \end{aligned}$$

where

$$\begin{aligned} K_{1ij} &:= \left. \frac{d\gamma_{ij}^\varepsilon}{d\varepsilon} \right|_{\varepsilon=0}, & K_{1ab} &:= \left. \frac{dh_{ab}^\varepsilon}{d\varepsilon} \right|_{\varepsilon=0}, & U^i &:= \left. \frac{du_\varepsilon^i}{d\varepsilon} \right|_{\varepsilon=0}, \\ K_{2ij} &:= \left. \frac{d^2\gamma_{ij}^\varepsilon}{d\varepsilon^2} \right|_{\varepsilon=0}, & K_{2ab} &:= \left. \frac{d^2h_{ab}^\varepsilon}{d\varepsilon^2} \right|_{\varepsilon=0}, & W^i &:= \left. \frac{d^2u_\varepsilon^i}{d\varepsilon^2} \right|_{\varepsilon=0}. \end{aligned}$$

We consider the gauge vectors $V_1 = -U^i(x^a)\partial_i$ and $V_2 = -W^i(x^a)\partial_i$. It is immediate that

$$\mathcal{L}_{V_1}g = -2\gamma_{ij}\partial_a U^j dx^i dx^a$$

and hence $K_1^g = K_1 + \mathcal{L}_{V_1}g = K_{1ij}dx^i dx^j + K_{1ab}dx^a dx^b$ is block diagonal, as claimed. For the second order perturbation we use the form (2.6). A simple calculation gives

$$\mathcal{L}_{V_1}K_1^g = -2K_{1ij}^\varepsilon\partial_a U^j dx^i dx^a, \quad \mathcal{L}_{V_1}\mathcal{L}_{V_1}g = 2\gamma_{ij}\partial_a U^i \partial_b U^j dx^a dx^b$$

and we conclude that $K_2 = K_{2ij}dx^i dx^j + K_{2ab}dx^a dx^b$, i.e. block diagonal. We have performed the computation to change the gauge in local coordinates, but it is clear that both V_1 and V_2 are globally defined on M' and smooth because they are intrinsically defined by

$$(4.4) \quad \begin{aligned} V_1 &= - \left. \frac{dz_\varepsilon^i}{d\varepsilon} \right|_{\varepsilon=0} \xi_i = - \left. \frac{du_\varepsilon^i}{d\varepsilon} \right|_{\varepsilon=0} \xi_i, \\ V_2 &= - \left. \frac{d^2z_\varepsilon^i}{d\varepsilon^2} \right|_{\varepsilon=0} \xi_i = - \left. \frac{d^2u_\varepsilon^i}{d\varepsilon^2} \right|_{\varepsilon=0} \xi_i. \end{aligned}$$

□

Remark 4.4. This proposition includes as a particular case the situation when the perturbation scheme inherits a static (local) isometry, i.e. an orthogonally transitive one-parameter (local) isometry group with timelike orbits.

At the level of generality of Proposition 4.3 we can only establish the differentiability of V_1 and V_2 on M' , i.e. away from points where the distribution $\{\Pi_p\}$ degenerates. The reason lies in the non-invertibility of γ_{ij} at the complement of M' . Extending the differentiability of the gauge vectors to the whole of M is a delicate issue which, however, must be addressed in our problem. Indeed, an important step of our argument will be applying Lemma 3.6 on perturbation tensors written in block diagonal form, for which it is crucial that these tensors are differentiable everywhere, including the axis. To accomplish this we will exploit the fact that the metrics \hat{g}_ε are stationary and axisymmetric. Using the results in Lemma 3.6 we will be able to analyse the behaviour of the one-forms ζ_ε^i defined in Lemma 4.1 near the axis. This will be sufficient, via integration of $dz_\varepsilon^i = \zeta_\varepsilon^i$, to show via (4.4) that the gauge vectors V_1 and V_2 are differentiable everywhere.

We need several well-known facts (see e.g. [21]) about stationary and axisymmetric group actions. Recall that this is a spacetime isometry generated by an axial Killing vector η , see Section 3, assumed to be spacelike everywhere outside the axis, and a timelike Killing vector ξ . The group is necessarily Abelian and the points where the group orbits are two-dimensional and non-null (i.e. M') are determined by $\det \gamma = \langle \xi, \xi \rangle \langle \eta, \eta \rangle - \langle \xi, \eta \rangle^2 < 0$. Given that $\eta^2 = \langle \eta, \eta \rangle$ is non-negative and zero only at points of the axis \mathcal{A} , $\det \gamma = 0$ also defines \mathcal{A} . The construction of a stationary and axially symmetric perturbation scheme implies all the above for each ε , in particular for $\det \gamma^\varepsilon$. Lemma 3.3 ensures that the axis of the perturbation, i.e. the axis at each ε , coincides with \mathcal{A} , and therefore we have that $\det \gamma^\varepsilon$ vanishes on \mathcal{A} and only there.

Proposition 4.5. *Let $(M_\varepsilon, \hat{g}_\varepsilon, \{\psi_\varepsilon\})$ be a perturbation scheme of class C^{n+1} ($n \geq 2$) that inherits a stationary and axisymmetric isometry group that acts orthogonally transitively. Assume that $M' = M \setminus \mathcal{A}$ is simply connected. Then, for each $g_\varepsilon = \psi_\varepsilon(\hat{g}_\varepsilon)$, $\psi_\varepsilon \in \{\psi_\varepsilon\}$ there exist first and second order gauge vectors V_1 and V_2 such that the corresponding transformed K_1^g and K_2^g are of class $C^n(M)$ and $C^{n-1}(M)$ respectively and block diagonal, i.e. take form*

$$\begin{aligned}
 K_1 &= K_{1ij}(x^c)dx^i dx^j + K_{1ab}(x^c)dx^a dx^b, \\
 K_2 &= K_{2ij}(x^c)dx^i dx^j + K_{2ab}(x^c)dx^a dx^b
 \end{aligned}$$

in any local coordinate system $\{x^\alpha\} = \{x^i, x^a\}$ in M' adapted to ξ_i and to the orthogonal transitivity of the background, i.e. where

$$(4.5) \quad g = \gamma_{ij}(x^c)dx^i dx^j + h_{ab}(x^c)dx^a dx^b, \quad \xi_i = \partial_{x^i}.$$

Proof. We are in the setting of Proposition 4.3 with $s = 2$, $\xi_1 = \xi$ and $\xi_2 = \eta$. We define ξ_i^ε as in this proposition and $\zeta_\varepsilon^i := \gamma^{\varepsilon ij}\xi_j^\varepsilon$ on $M \setminus \mathcal{A}$ (cf. Lemma 4.1). The core of the proof is understanding in detail the behaviour of the closed one-forms ζ_ε^i as we approach \mathcal{A} , for which we will use the results in Lemma 3.6. Introducing coordinates $\{x, y, z, t\}$ in a neighbourhood $\mathcal{U}_\mathcal{A} \subset M$ of the axis such that $\eta = x\partial_y - y\partial_x$ and $\xi = \partial_t$, we may apply Lemma 3.6 with $K = g_\varepsilon$, $m = n + 1$ and $w^u = \{z, t\}$ ($u = 3, 4$). As in this lemma, we call $U \subset \mathbb{R}^4$ the set where $\{x, y, z, t\}$ take values.

We shall use expressions (3.6)-(3.11) with $K \rightarrow g_\varepsilon$ and $\mathfrak{U} \rightarrow \mathfrak{U}^\varepsilon$. Since g_ε is stationary, $\mathfrak{U}_{\mathfrak{B}}^\varepsilon$ do not depend on t , i.e. they are functions $\mathfrak{U}_{\mathfrak{B}}^\varepsilon(\|x\|, z)$, where $\|x\|^2 := x^2 + y^2$. Their traces satisfy (3.12) and are defined on the domain $D_\eta := \{\rho \in \mathbb{R}_{\geq 0}, z \in \mathbb{R}; (\rho, 0, z, t) \in U\} \subset \mathbb{R}_{\geq 0} \times \mathbb{R}$. Lemma A.1 in Appendix A ensures that $\mathfrak{U}_{\mathfrak{B}}^\varepsilon(\|x\|, z)$ are $C^{n+1}(\mathcal{U}_\mathcal{A})$. From on on, we write $\mathfrak{U}_{\mathfrak{B}}^\varepsilon$ when we refer to $\mathfrak{U}_{\mathfrak{B}}^\varepsilon(\|x\|, z)$, i.e. as functions on $\mathcal{U}_\mathcal{A}$.

The analysis of ζ_ε^i relies on the following:

Claim: For $\mathfrak{C} \in \{2, 3, \rho t, \rho z, \eta t, \eta z\}$, the functions $\tilde{\mathfrak{U}}_\mathfrak{C}^\varepsilon(\rho, z) := \frac{1}{\rho}\mathfrak{U}_\mathfrak{C}^\varepsilon(\rho, z)$ and $\frac{1}{\rho}\tilde{\mathfrak{U}}_\mathfrak{C}^\varepsilon\tilde{\mathfrak{U}}_{\mathfrak{C}'}$ admit the expansion (we drop a label ε in the right-hand side for simplicity)

$$(4.6) \quad \tilde{\mathfrak{U}}_\mathfrak{C}^\varepsilon(\rho, z) = \rho \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \rho^{2k} P_{\mathfrak{C}k}(z) + \tilde{\Phi}_\mathfrak{C}^{(n)}(\rho, z),$$

$$(4.7) \quad \frac{1}{\rho}\tilde{\mathfrak{U}}_\mathfrak{C}^\varepsilon\tilde{\mathfrak{U}}_{\mathfrak{C}'}$$

where $P_{\mathfrak{C}k}(z)$, $P_{\mathfrak{C}'k}(z)$ are C^{n+1} functions of z and $\tilde{\Phi}_\mathfrak{C}^{(n)}(\rho, z)$, $\tilde{\Phi}_{\mathfrak{C}'}^{(n)}(\rho, z)$ are C^m and $o(\rho^n)$ with respect to ρ and C^{m+1} with respect to z .

Proof of the claim: Lemma 3.6 with $m = n + 1$ establishes that $\mathfrak{U}_\mathfrak{C}^\varepsilon(\rho, z)$ vanishes at the axis and admits an expansion

$$\mathfrak{U}_\mathfrak{C}^\varepsilon(\rho, z) = \rho^2 \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \rho^{2k} P_{\mathfrak{C}k}(z) + \Phi_\mathfrak{C}^{(n+1)}(\rho, z),$$

where $P_{\mathfrak{C}^k}(z)$ are C^{n+1} functions of z and $\Phi_{\mathfrak{C}}^{(n+1)}$ are $C^{n+1}(D_\eta)$ and $o(\rho^{n+1})$. Setting $\tilde{\Phi}_{\mathfrak{C}}^{(n)}(\rho, z) := \frac{1}{\rho}\Phi_{\mathfrak{C}}^{(n+1)}(\rho, z)$, the expansion (4.6) follows. The function $\tilde{\Phi}_{\mathfrak{C}}^{(n)}(\rho, z)$ is $o(\rho^n)$ and C^n with respect to ρ by item (iii) of Lemma B.7 applied to the one-dimensional case. Moreover, this function is C^{n+1} in z , since, being $o(\rho^n)$, all partial derivatives with respect to z extend to $\rho = 0$, where they vanish. For the product function, it is immediate from the definition that

$$\begin{aligned} \frac{1}{\rho}\tilde{\mathfrak{A}}_{\mathfrak{C}}^\varepsilon\tilde{\mathfrak{A}}_{\mathfrak{C}'}^\varepsilon(\rho, z) &= \rho \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \rho^{2k} P_{\mathfrak{C}^k}(z) + \rho \sum_{k=\lfloor \frac{n+1}{2} \rfloor}^{2\lfloor \frac{n-1}{2} \rfloor} \rho^{2k} P_{\mathfrak{C}^k}(z) \\ &\quad + \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \rho^{2k} \left(P_{\mathfrak{C}^k}(z)\tilde{\Phi}_{\mathfrak{C}'}^{(n)} + P_{\mathfrak{C}'^k}(z)\tilde{\Phi}_{\mathfrak{C}}^{(n)} \right) \\ &\quad + \frac{1}{\rho}\tilde{\Phi}_{\mathfrak{C}}^{(n)}(\rho, z)\tilde{\Phi}_{\mathfrak{C}'}^{(n)}(\rho, z). \end{aligned}$$

The last terms define $\tilde{\Phi}_{\mathfrak{C}\mathfrak{C}'}^{(n)}(\rho, z)$ and the expansion (4.7) follows. The property that $\tilde{\Phi}_{\mathfrak{C}\mathfrak{C}'}^{(n)}(\rho, z)$ is C^{n+1} in z is clear, as it holds for each term. Concerning the property of being C^n in ρ and $o(\rho^n)$, this is immediate for the first two terms. For the last term, it follows from its product structure, as shown in Corollary B.4 of Appendix B.

Combining this claim with Lemma A.1, we also conclude that the functions $\tilde{\mathfrak{A}}_{\mathfrak{C}}^\varepsilon\tilde{\mathfrak{A}}_{\mathfrak{C}'}^\varepsilon(\|x\|, z)$ are $C^n(\mathcal{U}_A)$. We are now ready to compute ζ_ε^i .

Recall that $\gamma_{ij}^\varepsilon = g_\varepsilon(\xi_i, \xi_j)$. Using the explicit expressions for g_ε given in Lemma 3.6, a straightforward calculation gives

$$\det \gamma^\varepsilon = \frac{1}{2}\|x\|^2 \left((\mathfrak{A}_1^\varepsilon - \mathfrak{A}_2^\varepsilon)\mathfrak{A}_{tt}^\varepsilon - 2\tilde{\mathfrak{A}}_{\eta t}^\varepsilon\tilde{\mathfrak{A}}_{\eta t}^\varepsilon \right) := \|x\|^2 D_{\gamma^\varepsilon}.$$

From the above, D_{γ^ε} is $C^n(\mathcal{U}_A)$ and its restriction to any value of $\|x\|$ is C^{n+1} in z . Moreover, the value of D_{γ^ε} on the axis is $D_{\gamma^\varepsilon}|_{\|x\|=0}(z) = \mathfrak{A}_1^\varepsilon(0, z)\mathfrak{A}_{tt}^\varepsilon(0, z)$. This is not zero because $\mathfrak{A}_{tt}^\varepsilon = g_\varepsilon(\xi, \xi) < 0$, c.f. (3.11), while (3.6) restricted to $x = y = 0$ provides $2g_\varepsilon(\partial_x, \partial_x)|_{\|x\|=0} = \mathfrak{A}_1(0, z)$, so that $\mathfrak{A}_1(0, z) > 0$. Since $\det \gamma^\varepsilon$ vanishes only at the axis, it follows that D_{γ^ε} is nowhere zero on \mathcal{U}_A . Therefore the inverse $D_{\gamma^\varepsilon}^{-1} := 1/D_{\gamma^\varepsilon}$ is also $C^n(\mathcal{U}_A)$, and by Lemma A.1, the corresponding trace function $D_{\gamma^\varepsilon}^{-1}(\rho, z)$ must have

the form

$$(4.8) \quad D_{\gamma^\varepsilon}^{-1}(\rho, z) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \rho^{2k} P_{\gamma^\varepsilon k}(z) + \tilde{\Phi}_{\gamma^\varepsilon}^{(n)}(\rho, z),$$

where $P_{\gamma^\varepsilon 0}(z)$ is nowhere zero, $P_{\gamma^\varepsilon k}(z)$ are C^{n+1} in z , and $\tilde{\Phi}_{\gamma^\varepsilon}^{(n)}(\rho, z)$ are C^{n+1} in z , C^n in ρ and $o(\rho^n)$. From the definition of ζ_ε^i (see Lemma 4.1), a direct computation gives

$$\begin{aligned} \zeta_\varepsilon^1 &= dt + \frac{1}{\|x\|} \mathfrak{Z}_{\varepsilon\rho}^1(xdx + ydy) + \mathfrak{Z}_{\varepsilon z}^1 dz, \\ \zeta_\varepsilon^2 &= d \arctan \frac{y}{x} + \frac{1}{\|x\|} \mathfrak{Z}_{\varepsilon\rho}^2(xdx + ydy) + \mathfrak{Z}_{\varepsilon z}^2 dz, \end{aligned}$$

where

$$\begin{aligned} \mathfrak{Z}_{\varepsilon\rho}^1(\|x\|, z) &:= \frac{1}{2D_{\gamma^\varepsilon}} \left((\mathfrak{U}_1^\varepsilon - \mathfrak{U}_2^\varepsilon) \tilde{\mathfrak{U}}_{\rho t}^\varepsilon + \mathfrak{U}_3^\varepsilon \tilde{\mathfrak{U}}_{\eta t}^\varepsilon \right), \\ \mathfrak{Z}_{\varepsilon z}^1(\|x\|, z) &:= \frac{1}{2D_{\gamma^\varepsilon}} \left((\mathfrak{U}_1^\varepsilon - \mathfrak{U}_2^\varepsilon) \mathfrak{U}_{tz}^\varepsilon - 2\tilde{\mathfrak{U}}_{\eta t}^\varepsilon \tilde{\mathfrak{U}}_{\eta z}^\varepsilon \right), \\ \mathfrak{Z}_{\varepsilon\rho}^2(\|x\|, z) &:= \frac{1}{2D_{\gamma^\varepsilon}} \left(\tilde{\mathfrak{U}}_3^\varepsilon \mathfrak{U}_{tt}^\varepsilon + 2\frac{1}{\|x\|} \tilde{\mathfrak{U}}_{\eta t}^\varepsilon \tilde{\mathfrak{U}}_{\rho t}^\varepsilon \right), \\ \mathfrak{Z}_{\varepsilon z}^2(\|x\|, z) &:= \frac{1}{2D_{\gamma^\varepsilon}} \frac{1}{\|x\|} \left(\mathfrak{U}_{tz}^\varepsilon \tilde{\mathfrak{U}}_{\eta t}^\varepsilon - \mathfrak{U}_{tt}^\varepsilon \tilde{\mathfrak{U}}_{\eta z}^\varepsilon \right). \end{aligned}$$

We can now integrate $dz_\varepsilon^i = \zeta_\varepsilon^i$. Using $\|x\|d\|x\| = xdx + ydy$, the functions z_ε^i take the form (the integrability conditions are ensured by Corollary 4.2)

$$\begin{aligned} z_\varepsilon^1 &= t + \int_0^{\|x\|} \mathfrak{Z}_{\varepsilon\rho}^1(s, z) ds + \int \mathfrak{Z}_{\varepsilon z}^1(0, z) dz =: x^1 + u_\varepsilon^1, \\ z_\varepsilon^2 &= \arctan \frac{y}{x} + \int_0^{\|x\|} \mathfrak{Z}_{\varepsilon\rho}^2(s, z) ds + \int \mathfrak{Z}_{\varepsilon z}^2(0, z) dz =: x^2 + u_\varepsilon^2, \end{aligned}$$

where we have set $x^1 = t$ and $x^2 = \arctan(y/x) = \varphi$. By (4.4), the gauge vectors are constructed from u_ε^i , so we only need to care about the integral terms and show that they define $C^{n+1}(\mathcal{U}_A)$ functions.

From Lemma 3.6 and the claim above it follows that $\mathfrak{Z}_{\varepsilon z}^i(0, z)$ are C^{n+1} functions of z , so the two integrals over z are C^{n+2} functions of z . Concerning

the ρ -integrals, (3.12), (4.8) and the claim imply that

$$\mathfrak{Z}_{\varepsilon\rho}^i(\rho, z) = \rho \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \rho^{2k} P_{\varepsilon k}^i(z) + \tilde{\Phi}_{\varepsilon}^{i(n)}(\rho, z),$$

where $P_{\varepsilon k}^i(z)$ and $\tilde{\Phi}_{\varepsilon}^{i(n)}$ are C^{n+1} with respect to z and $\tilde{\Phi}_{\varepsilon}^{i(n)}$ is C^n in ρ and $o(\rho^n)$. As a result,

$$Z_{\varepsilon}^i(\rho, z) := \int_0^{\rho} \mathfrak{Z}_{\varepsilon\rho}^i(s, z) ds = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} P_{\varepsilon k}^i(z) \frac{1}{2k+1} \rho^{2(k+1)} + \int_0^{\rho} \tilde{\Phi}_{\varepsilon}^{i(n)}(s, z) ds$$

is C^{n+1} in the domain D_{η} . The first term is a polynomial in ρ^2 , while the last integral is $o(\rho^{n+1})$ (e.g. by the mean value theorem). Lemma A.1 ensures that $Z_{\varepsilon}^i(\|x\|, z)$ is $C^{n+1}(\mathcal{U}_A)$. Applying the definition (4.4) we conclude that the gauge vectors V_1 and V_2 are $C^{n+1}(\mathcal{U}_A)$. □

Given a pertubation scheme that inherits a stationary and axisymmetric orthogonally transitive group action, we shall always consider the related perturbation scheme whose existence is proved in this proposition. The corresponding metric perturbation tensors will be denoted K_1 and K_2 (without the g superindex).

Remark 4.6. When the codimension of the Abelian (local) group action is two, the metric $h^{\varepsilon}(x^c)_{\text{ab}} dx^a dx^b$ is two-dimensional. Given any point $p \in M'$ and a sufficiently small neighbourhood U_p of p there exist coordinates $\{x^i, x^a\}$ adapted to the Killings where the background metric is not only block diagonal, but also with h_{ab} diagonal. By restricting U_p if necessary, it is also true that there exists a change of coordinates $\{x^i, x_{\varepsilon}^a(x^b)\}$ where $h_{\text{ab}}^{\varepsilon}$ is also diagonal for all ε . This map can be considered as a local diffeomorphism and hence as a local gauge transformation. In the transformed gauge one has, by construction, that $K_{1\text{ab}}^g$ and $K_{2\text{ab}}^g$ are both diagonal. It should be emphasized however, that in general this local gauge transformation does not exist globally, so it is unjustified to assume that $K_{1\text{ab}}$ and $K_{2\text{ab}}$ are diagonal in some atlas. Generally speaking, this limitation is present even in the most favourable situation when a global coordinate system $\{x^{\alpha}\}$ adapted to the Killings exists on M' in which the background metric is both block diagonal and with diagonal h_{ab} .

5. Axially symmetric perturbations on spherically symmetric backgrounds

From now on we restrict ourselves to the case in which the spacetime dimension is four and the background spacetime is spherically symmetric. Before introducing some convenient definitions and notation on spherically symmetric spacetimes we start by fixing some notation on the unit sphere $(\mathbb{S}^2, g_{\mathbb{S}^2})$.

5.1. Spherically symmetric backgrounds

Let Y^a ($a = 1, 2, 3$) be the spherical harmonics with $\ell = 1$ on the sphere. More specifically, Y^a is defined as the restriction of the Cartesian coordinates of \mathbb{R}^3 to the unit sphere, i.e.

$$Y^1 = \sin \theta \cos \phi, \quad Y^2 = \sin \theta \sin \phi, \quad Y^3 = \cos \theta.$$

We will refer to $\{x^A\} = \{\theta, \phi\}$ as the standard angular spherical coordinates, in which the spherical unit metric reads $g_{\mathbb{S}^2} = d\theta^2 + \sin^2 \theta d\phi^2$. Let us denote by \bar{d} the exterior differential on \mathbb{S}^2 , and by \bar{D}_A the covariant derivative on $(\mathbb{S}^2, g_{\mathbb{S}^2})$. The spherical harmonics Y^a satisfy $\bar{D}_A \bar{D}_B Y^a = -Y^a g_{\mathbb{S}^2 AB}$ and the six dimensional algebra of conformal Killing vectors on \mathbb{S}^2 is spanned by $\{\bar{D}_A Y^a (= \bar{d}Y^a)\}$ (proper conformal Killings) and $\{\epsilon_{AB} \bar{D}^B Y^a (= -\star_{\mathbb{S}^2} \bar{d}Y^a)\}$ (Killing vectors) where ϵ_{AB} is the volume form of $(\mathbb{S}^2, g_{\mathbb{S}^2})$. In standard spherical coordinates we choose the orientation so that the Hodge dual $\star_{\mathbb{S}^2}$ acts as $\star_{\mathbb{S}^2} d\phi = -1/\sin \theta d\theta$ and $\star_{\mathbb{S}^2} d\theta = \sin \theta d\phi$.

Definition 5.1. A (four-dimensional) spacetime (M, g) is **spherically symmetric** if it admits an $SO(3)$ group of isometries acting transitively on spacelike surfaces (which may degenerate to points). Denote by $\mathcal{C} \subset M$ the set of fixed points of the group action (which may be empty).

As it is known [22], every connected component of \mathcal{C} is the image of a timelike geodesic which is a closed set in M . The set $M \setminus \mathcal{C}$, the principal part of M , is dense in M . Standard results [19] show that the surfaces of transitivity of the group (orbits) $SO(3)$ generating the spherical symmetry in $(M \setminus \mathcal{C}, g)$ are spheres $\mathbb{S}^2(\subset \mathbb{R}^3) \rightarrow S_r \subset M$, admit a family of orthogonal (and thus timelike) surfaces S_\perp (the integrable distribution of assumption (ii) in subsection 4.1), and $M \setminus \mathcal{C}$ is diffeomorphic to a warped product $S_\perp \times_{\mathcal{R}_\perp} \mathbb{S}^2$. Therefore, on $M \setminus \mathcal{C}$ there exist coordinates $\{x^I, x^A\}$ for which

g takes the form

$$(5.1) \quad g = g_{\perp}(x^I)_{JK} dx^J dx^K + \mathcal{R}_{\perp}^2 g_{\mathbb{S}^2}{}_{AB} dx^A dx^B,$$

where g_{\perp} is Lorentzian and \mathcal{R}_{\perp} is a positive function $\mathcal{R}_{\perp} : S_{\perp} \rightarrow \mathbb{R}$. The function \mathcal{R}_{\perp} is the restriction to S_{\perp} of a function $\mathcal{R} : M \setminus \mathcal{C} \rightarrow \mathbb{R}$ invariant under the $SO(3)$ group and which extends to \mathcal{C} as $\mathcal{R}(\mathcal{C}) = 0$. The function \mathcal{R} measures the area $4\pi\mathcal{R}^2$ of the orbits of the $SO(3)$ group, which are thus spheres $(S_r, \mathcal{R}^2 g_{\mathbb{S}^2})$ endowed with the metric $\mathcal{R}^2 g_{\mathbb{S}^2}$.

Any vector field V on $M \setminus \mathcal{C}$ can be decomposed as $V = V_{\perp} + V_{\parallel}$, where $V_{\parallel} = V^A \partial_A$ is tangent to S_r and V_{\perp} is tangent to its g -orthogonal complement, S_{\perp} . For tangent vectors $X = X_{\parallel}$ we will use $\overline{\mathbf{X}} := g_{\mathbb{S}^2}(X_{\parallel}, \cdot)$ to distinguish $\overline{\mathbf{X}} = \overline{X}_A dx^A$ from $\mathbf{X} = X_A dx^A = g(X, \cdot)$, so that $\mathbf{X} = \mathcal{R}^2 \overline{\mathbf{X}}$.

For later use, it is convenient to express the Lie derivative of any 2-covariant symmetric tensor T along any vector of the form $s_{\parallel} = s^A(x^B) \partial_A$. In coordinates $\{x^I, x^A\}$ this is

$$(5.2) \quad \begin{aligned} \mathcal{L}_{s_{\parallel}} T = & s^A \partial_A T_{IJ} dx^I dx^J + 2(s^B \partial_B T_{IA} + T_{IB} \partial_A s^B) dx^I dx^A \\ & + (\mathcal{L}_{s_{\parallel}} \overset{\circ}{T})_{AB} dx^A dx^B, \end{aligned}$$

where we denote by $\overset{\circ}{T}$ the “full tangent part” to S_r of T , i.e. $\overset{\circ}{T} = T_{AB} dx^A dx^B$. The application of this expression to $T = g$, and taking into account that $\dot{g} = \mathcal{R}^2 g_{\mathbb{S}^2}$, yields, for any pair of vectors X, Y , the following equality on $M \setminus \mathcal{C}$

$$(5.3) \quad \mathcal{L}_{s_{\parallel}} g(X, Y) = \mathcal{R}^2 \mathcal{L}_{s_{\parallel}} g_{\mathbb{S}^2}(X_{\parallel}, Y_{\parallel}).$$

Let us now single out one axial Killing vector η of the $so(3)$ algebra, which after a convenient rotation of the spherical coordinates can be set to be $\eta = \partial_{\phi}$. We denote the corresponding axis by \mathcal{A} . Observe that $\mathcal{C} \subset \mathcal{A}$. Using the above, the axial Killing vector η , since $\eta = \eta_{\parallel}$, defines an axial Killing vector on the unit sphere, $\eta = \eta^A \partial_A$, and we take the labels a on Y^a so that the rotation generated by η has axis along x^3 . By doing that we are saying that the maps $\mathbb{S}^2(\subset \mathbb{R}^3) \rightarrow S_r \subset M$ are such that $x^3 = \pm 1$ are mapped onto $\mathcal{A} \cap S_r$. We thus have, by construction, $\overline{\eta}_A := g_{\mathbb{S}^2}{}_{AB} \eta^B = \epsilon_{AB} \overline{D}^B Y^3$, this is $\overline{\eta} = -\star_{\mathbb{S}^2} \overline{d}Y^3$. We denote by ι_{\parallel} the conformal Killing vector on the sphere given by

$$\overline{\iota} := \star_{\mathbb{S}^2} \overline{\eta} = \overline{d}Y^3 = dY^3,$$

where in the last equality we have used the fact that for any function $f(x^A)$ we have $\overline{d}f = df$.

We finally define the vector ι on (M, g) tangent to each sphere and satisfying $\iota = \iota_{\parallel}$. Clearly $\langle \iota, \iota \rangle = \langle \eta, \eta \rangle$ and $\langle \iota, \eta \rangle = 0$ by construction. The vector ι is thus defined on $M \setminus \mathcal{C}$ and has the same differentiability as η there. However, since $\eta = 0$ on \mathcal{A} (including $\mathcal{C} \subset \mathcal{A}$), ι extends continuously to \mathcal{C} by setting $\iota|_{\mathcal{C}} = 0$. In spherical coordinates $x^A = \{\theta, \phi\}$ we just have $\eta = \partial_{\phi}$ and $\iota = -\sin \theta \partial_{\theta}$. Notice also that $\iota = \mathcal{R}^2 dY^3$.

Consider a spherically symmetric C^{m+2} spacetime (M, g) with C^{m+1} metric ($n \geq 2$). Using the construction above we take a spherically symmetric neighbourhood $\mathcal{U} \subset M$ that may contain a connected component \mathcal{C}_0 of \mathcal{C} . We make the following assumption on the existence of Cartesian coordinates with suitable differentiability.

Assumption S₁: We assume the existence of a C^{m+2} coordinate chart that maps \mathcal{U} onto $U = U^3 \times I$ where I is an open interval and $U^3 \subset \mathbb{R}^3$ is a radially symmetric domain. Moreover, $\mathcal{C}_0 \cap \mathcal{U}$ (if non empty) is mapped to³ $\{0_3\} \times I \subset U$ and using Cartesian coordinates $x^i = \{x, y, z\}$ for \mathbb{R}^3 and $\{t\}$ along I , the metric g takes the form

$$(5.4) \quad g = -e^{\nu} dt^2 + 2\mu dt(x_i dx^i) + v(x_i dx^i)^2 + \chi \delta_{ij} dx^i dx^j,$$

where ν, μ, v and χ are C^{n+1} functions of $\{x, y, z, t\}$ and radially symmetric in $\{x, y, z\}$.

Without loss of generality, we demand that the corresponding spherical coordinates $\{r, \theta, \phi\}$ defined by

$$(5.5) \quad \begin{aligned} x &= rY^1 = r \sin \theta \cos \phi, \\ y &= rY^2 = r \sin \theta \sin \phi, \\ z &= rY^3 = r \cos \theta \end{aligned}$$

are such that $\{\theta, \phi\}$ correspond to the above $\mathbb{S}^2 \rightarrow S_r$ construction.

As usual we identify geometric objects on \mathcal{U} with their representation in this chart. We introduce $|x| := \sqrt{x^2 + y^2 + z^2}$ (which corresponds to the spherical coordinate r above) and observe that $\mathcal{C}_0 \cap \mathcal{U}$ corresponds to the set of points with vanishing $|x|$. A consequence of assumption S₁ is that χ does not vanish on \mathcal{U} (as g would degenerate where $\chi = 0$), and the function \mathcal{R} takes the form $\mathcal{R}^2 = \chi |x|^2$. Therefore, for any function $f : \mathcal{U} \rightarrow \mathbb{R}$, $\mathcal{R}^2 f \in O(|x|^l)$ is equivalent to $\chi f \in O(|x|^{l-2})$, and hence to $f \in O(|x|^{l-2})$. Analogously, if $\mathcal{R}^2 f \in o(|x|^l)$ then $f \in o(|x|^{l-2})$.

³We denote the origin of \mathbb{R}^p by 0_p , see Appendix B.

In this chart the axial Killing vector η reads $\eta = x\partial_y - y\partial_x$ and its square norm $\eta^2 := \langle \eta, \eta \rangle$ is $\eta^2 = (\mathcal{R}^2/|x|^2)(x^2 + y^2) = \chi\|x\|^2$. The (piece of) axis of symmetry $\mathcal{A} \cap \mathcal{U}$ is thus located at $\mathcal{A} \cap \mathcal{U} = \{p \in \mathcal{U}; x(p) = y(p) = 0\}$.

In the following (see also Appendix B) we extend the little-o notation in terms of a limit on the axis \mathcal{A} : For any positive function g defined on $\mathcal{U} \setminus \mathcal{A}$ we set (observe we are using boldface in \mathfrak{o})

$$f \in \mathfrak{o}(g) \iff \lim_{\|x\| \rightarrow 0} fg^{-1} = 0.$$

Note that for any function $f \in \mathfrak{o}(\|x\|^l)$ we have that $f/(\sqrt{\eta^2})^l$ vanishes on all $a \in \mathcal{A}$.

We define $\hat{n} = \frac{1}{|x|}x^i\partial_i$ as the radial vector (normal to each S_r) outside the origin, which in spherical coordinates reads $\hat{n} = \partial_r$. It is convenient to introduce the following smooth vector field defined on \mathcal{U} :

$$(5.6) \quad \hat{\rho} := x\partial_x + y\partial_y$$

and simply use the shorthand $\hat{z} := \partial_z$. In terms of these objects, the following expressions hold on $\mathcal{U} \setminus \mathcal{C}_0$

$$(5.7) \quad \iota = -\frac{1}{|x|}(z\hat{\rho} - \|x\|^2\hat{z}),$$

$$(5.8) \quad \hat{n} = \frac{1}{|x|}(\hat{\rho} + z\hat{z}).$$

In addition to being C^{n+1} on $U \setminus \mathcal{C}_0$, it is clear that \hat{n} is bounded near the origin and ι extends continuously to \mathcal{C}_0 as $\iota|_{\mathcal{C}_0} = 0$. An immediate consequence of (5.7)-(5.8) is

$$(5.9) \quad |x|\hat{\rho} = \|x\|^2\hat{n} - z\iota \quad \text{on } \mathcal{U} \setminus \mathcal{C}_0.$$

When convenient we will use $\{w^u\}$ to refer to $\{z, t\}$.

5.2. Axially symmetric perturbations on spherically symmetric backgrounds

In this subsection we prepare the stage that will allow us to prove Propositions 5.6 and 5.7 below. The aim of the propositions is to look (at first and second order respectively) for a change of gauge that takes the perturbation tensors K_1 and K_2 , in the form given in Proposition 4.5, to a form in which

their angular part is proportional to the metric on the sphere, while keeping the block structure. This is part of the very well known Regge-Wheeler gauge. As we shall see, the gauge vectors V_1 and V_2 that take us from K_1 and K_2 to the gauged K_1^g and K_2^g with the desired properties satisfy suitable differential equations with inhomogeneous terms. Since one of our aims is to understand whether or not the regularity properties (and differentiability) of K_1 and K_2 are kept in K_1^g and K_2^g , it becomes necessary to study the properties of those inhomogeneous terms, as well as the effect they have on the regularity of the gauge vectors via the differential equations that are satisfied.

We start by establishing several facts of the original perturbation tensors that will play a role in determining the properties of the inhomogeneous terms. These first results that follow are stated for general symmetric tensors invariant under the axial symmetry, which will be denoted by K and taken to be of class C^m . Later on, these results will be applied to K_1 and K_2 , of class C^n and C^{n-1} respectively, by letting m take the values n and $n - 1$ correspondingly.

Lemma 5.2. *Let K be a symmetric 2-covariant C^m tensor in \mathcal{U} for $m \geq 1$ satisfying $\mathcal{L}_\eta K = 0$. Then,*

- 1) *the functions $K(\hat{n}, \iota)$ and $K(\hat{n}, \hat{n})$ are $C^m(\mathcal{U} \setminus \mathcal{C}_0)$ and radially symmetric in $\{x, y\}$. Moreover, $K(\hat{n}, \iota) \in \mathfrak{o}(\|x\|)$ and $K(\hat{n}, \hat{n})$ is bounded near \mathcal{C}_0 .*
- 2) *the functions $K(\eta, \eta)/\|x\|^2$ and $K_u^{(m)} := \frac{|x|^2}{\|x\|^2} K(\iota, \iota)$ are $C^m(\mathcal{U})$ and radially symmetric in $\{x, y\}$. Moreover, $K_u^{(m)} \in O(\|x\|^2)$.*
- 3) *the function $K(\eta, \partial_u)$, as well as q_-, q_+, q_\times, q defined by*

$$(5.10) \quad q_- := \frac{\mathcal{R}^2}{2\eta^2} \{K(\iota, \iota) - K(\eta, \eta)\}, \quad q_+ := \frac{\mathcal{R}^2}{2\eta^2} \{K(\iota, \iota) + K(\eta, \eta)\},$$

$$(5.11) \quad q_\times := |x| \frac{\mathcal{R}^2}{\eta^2} K(\iota, \eta), \quad q := |x|^2 K(\hat{n}, \iota)$$

are all $C^m(\mathcal{U})$, radially symmetric in $\{x, y\}$ and have the following structure

$$(5.12) \quad K(\eta, \partial_u) = \|x\|^2 \sum_{k=0}^{[m/2]-1} \|x\|^{2k} P_k^u(w^u) + \Phi^{u(m)},$$

$$(5.13) \quad q_- = \sum_{k=1}^{[m/2]+1} \|x\|^{2k} P_{0k}^- + z\Phi_{10}^{-(m)} + \|x\|^2 \Phi_{01}^{-(m)},$$

$$(5.14) \quad q_+ = z^2 P_{20}^+ + \sum_{k=1}^{[m/2]+1} \|x\|^{2k} P_{0k}^+ + z \Phi_{10}^{+(m)} + \|x\|^2 \Phi_{01}^{+(m)},$$

$$(5.15) \quad q_\times = z^2 \|x\|^2 P_{21}^\times + \sum_{k=2}^{[m/2]+1} \|x\|^{2k} P_{0k}^\times + z^2 \Phi_{20}^{\times(m)} + \|x\|^2 \Phi_{01}^{\times(m)},$$

$$(5.16) \quad q = z \|x\|^2 P_{11}^* + \sum_{k=2}^{[m/2]+1} \|x\|^{2k} P_{0k}^* + z^2 \Phi_{20}^{*(m)} + \|x\|^2 \Phi_{01}^{*(m)},$$

where all P functions are C^m functions of $\{z, t\}$ and $\Phi^{(m)} \in C^m(U)$ are radially symmetric in $\{x, y\}$ and $\mathbf{o}(\|x\|^m)$.

In particular, q and q_- are $\mathbf{o}(\|x\|)$, $q \in O(|x|^3)$ and $q_- \in O(|x|^2)$. Moreover, if $m \geq 2$ we also have $\hat{n}(q_-) \in \mathbf{o}(\|x\|)$.

Proof. The neighbourhood \mathcal{U} (and its corresponding U) and K fit the assumptions of Lemma 3.6. From the definitions in (5.6) it is straightforward to obtain (with obvious notation when the subindices u, v refer to $w^u = \{z, t\}$, and \mathfrak{A}_1 with no arguments stands for $\mathfrak{A}_1(\|x\|, w^u)$ on \mathcal{U} , etc)

$$(5.17) \quad 2K(\eta, \eta) = \|x\|^2(\mathfrak{A}_1 - \mathfrak{A}_2), \quad 2K(\hat{\rho}, \hat{\rho}) = \|x\|^2(\mathfrak{A}_1 + \mathfrak{A}_2),$$

$$K(\hat{\rho}, \hat{z}) = \mathfrak{A}_{\rho z},$$

$$(5.18) \quad K(\hat{z}, \hat{z}) = K_{zz} = \mathfrak{A}_{zz}, \quad K(\eta, \partial_u) = -\mathfrak{A}_{\eta u}$$

on U . Combining this with (5.7)-(5.8) lead to

$$(5.19) \quad K(\hat{n}, \iota) = -\frac{z\|x\|}{|x|^2} \frac{1}{2} \|x\|(\mathfrak{A}_1 + \mathfrak{A}_2 - 2\mathfrak{A}_{zz}) + \frac{1}{|x|^2} (\|x\|^2 - z^2) \mathfrak{A}_{\rho z},$$

$$(5.20) \quad K(\hat{n}, \hat{n}) = \frac{\|x\|^2}{|x|^2} \frac{\mathfrak{A}_1 + \mathfrak{A}_2}{2} + 2 \frac{z}{|x|} \frac{\mathfrak{A}_{\rho z}}{|x|} + \frac{z^2}{|x|^2} \mathfrak{A}_{zz},$$

$$(5.21) \quad K_u^{(m)} := \frac{|x|^2}{\|x\|^2} K(\iota, \iota) = \frac{1}{2} z^2 (\mathfrak{A}_1 + \mathfrak{A}_2) - 2z \mathfrak{A}_{\rho z} + \|x\|^2 \mathfrak{A}_{zz}.$$

By Lemma 3.6 the functions $\mathfrak{A}_{\mathfrak{B}}(\rho, w^u)$, $\mathfrak{B} \in \{1, 2, 3, \rho u, \eta u, uv\}$ admit an expansion

$$(5.22) \quad \mathfrak{A}_{\mathfrak{B}}(\rho, z, t) = \sum_{k=0}^{[m/2]} \rho^{2k} P_{\mathfrak{B}k}(z, t) + \Phi_{\mathfrak{B}}^{(m)}(\rho, z, t),$$

where $P_{\mathfrak{B}k}(z, t)$ are C^m functions of z, t and $\Phi_{\mathfrak{B}}^{(m)}(\rho, z, t)$ are $o(\rho^m)$ and C^m functions in the domain $U_\eta := \{\rho \in \mathbb{R}_{\geq 0}, (z, t) \in \mathbb{R}^2; (\rho, 0, z, t) \in U\} \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^2$. Moreover $P_{20}, P_{30}, P_{\rho u 0}, P_{\eta u 0}$ are identically zero because the functions $\mathfrak{A}_2, \mathfrak{A}_3, \mathfrak{A}_{\rho u}, \mathfrak{A}_{\eta u}$ vanish at $\rho = 0$. By Lemma A.1 in Appendix A, all the functions $\mathfrak{A}_{\mathfrak{B}}(\|x\|, z, t)$ are $C^m(\mathcal{U})$ (and are obviously radially symmetric in $\{x, y\}$).

Point 1 of the lemma follows from (5.19) and (5.20). The right-hand sides are clearly $C^m(\mathcal{U} \setminus \mathcal{C}_0)$. Since $z\|x\|/|x|^2$ vanishes on $\|x\| = 0$, the first term in (5.19) is $\mathfrak{o}(\|x\|)$. Boundedness of $\|x\|^2/|x|^2$ and $z^2/|x|^2$ plus the fact that $\mathfrak{A}_{\rho z} \in \mathfrak{o}(\|x\|)$ (because $P_{\rho z 0} = 0$) implies that the second term is also $\mathfrak{o}(\|x\|)$. Therefore $K(\hat{n}, \iota) \in \mathfrak{o}(\|x\|)$. Concerning $K(\hat{n}, \hat{n})$, the only term that needs inspection is the second one, since the rest are clearly bounded. Taking into account that, by Lemma B.5 in Appendix B, $\mathfrak{A}_{\rho z} \in \mathfrak{o}(\|x\|)$ implies $\mathfrak{A}_{\rho z} \in \mathfrak{o}(|x|)$, we conclude that $K(\hat{n}, \hat{n})$ is bounded near \mathcal{C}_0 .

Point 2 is immediate from the first in (5.17) and (5.21), after using again that $\mathfrak{A}_{\rho z} \in \mathfrak{o}(|x|)$.

Concerning point 3, we compute q_-, q_+, q_\times, q . Directly from the definitions (5.10)-(5.11) one finds, using that $\mathcal{R}^2/\eta^2 = |x|^2/\|x\|^2$,

$$(5.23) \quad 2q_- = z^2\mathfrak{A}_2 - 2z\mathfrak{A}_{\rho z} + \|x\|^2\mathfrak{A}_{zz} - \frac{1}{2}\|x\|^2(\mathfrak{A}_1 - \mathfrak{A}_2),$$

$$(5.24) \quad 2q_+ = z^2\mathfrak{A}_1 - 2z\mathfrak{A}_{\rho z} + \|x\|^2\mathfrak{A}_{zz} + \frac{1}{2}\|x\|^2(\mathfrak{A}_1 - \mathfrak{A}_2),$$

$$(5.25) \quad q_\times = -\frac{1}{2}z(\|x\|^2 + z^2)\mathfrak{A}_3 - (\|x\|^2 + z^2)\mathfrak{A}_{\eta z},$$

$$(5.26) \quad q = -\frac{1}{2}z\|x\|^2(\mathfrak{A}_1 + \mathfrak{A}_2 - 2\mathfrak{A}_{zz}) + (\|x\|^2 - z^2)\mathfrak{A}_{\rho z}.$$

Inserting (5.22) in these expressions as well as in the expression for $K(\eta, \partial_u)$ given in (5.18) yields (5.12)-(5.16) after a simple rearranging of terms. The right-hand sides of (5.12)-(5.16) satisfy the requirements of Lemma A.1 so $K(\eta, \partial_u), q_-, q_+, q_\times, q$ are $C^m(\mathcal{U})$ as claimed.

For the remaining properties, the fact that $q, q_- \in \mathfrak{o}(\|x\|), q \in O(|x|^3)$ and $q_- \in O(|x|^2)$ are immediate from (5.16) and (5.13). It remains to show that $\hat{n}(q_-) \in \mathfrak{o}(\|x\|)$ when $m \geq 2$. Observe this is a stronger result than the one provided by Lemma B.7 for a general C^1 and $\mathfrak{o}(\|x\|)$ function. The reason behind its validity is the special structure of q_- , (5.13), which we rewrite here as

$$q_- = \sum_{k=1}^{[m/2]+1} \|x\|^{2k} P_{0k}^- + \Phi_{q_-}^{(m)},$$

where $\Phi_{q^-}^{(m)}$ is $C^m(U)$, $\mathbf{o}(\|x\|^m)$ and radially symmetric in $\{x, y\}$. Applying \hat{n} yields

$$\hat{n}(q_-) = \sum_{k=1}^{\lfloor m/2 \rfloor + 1} \left(2k \frac{\|x\|}{|x|} \|x\|^{2k-1} P_{0k}^- + \|x\|^{2k} \frac{z}{|x|} \partial_z P_{0k}^- \right) + \hat{n}(\Phi_{q^-}^{(m)}).$$

Boundedness of $\frac{\|x\|}{|x|}$, $\frac{z}{|x|}$, P_{0k}^- and $\partial_z P_{0k}^-$ imply that the terms in the summation are $\mathbf{o}(\|x\|)$. Lemma B.7, since $m \geq 2$ by assumption, ensures $\hat{n}(\Phi_{q^-}^{(m)}) \in \mathbf{o}(\|x\|)$. The result follows. \square

Our next lemma is concerned with the solutions of a class of differential equations that will arise in the process of changing the gauge and on the differentiability and regularity properties of the corresponding gauge vector. Most of the technical work is developed in Appendix B, to which we refer.

Lemma 5.3. *Consider the vector field in $\mathcal{U} \setminus \mathcal{C}_0$*

$$S_{(a)} = \frac{1}{\mathcal{R}^{2a}} \gamma^\iota$$

for $a \in \mathbb{R}$, where $\gamma : \mathcal{U} \rightarrow \mathbb{R}$ satisfies the equation

$$(5.27) \quad \iota(\gamma) = \sum_{(k,l) \in \mathcal{V}} \|x\|^{2k} z^l P_{lk}(z, t) + \sum_{(k',l') \in \mathcal{V}'} \|x\|^{2k'} z^{l'} \Phi_{l'k'}^{(m)}(x) =: \mathcal{Q}$$

on \mathcal{U} . Here, $\mathcal{V}, \mathcal{V}'$ are finite subsets of $\mathbb{N} \times \mathbb{N}$ satisfying, respectively, $\mathcal{V} \subset \{k \geq 1\} \times \{l \geq 0\}$ and $\mathcal{V}' \subset \{2k' + l' \geq 1\}$, P_{lk} are C^m in their arguments and $\Phi_{l'k'}^{(m)}$ are $C^m(U)$ and $\mathbf{o}(\|x\|^m)$ for $m \geq 1$. Then there exists an axially symmetric solution $\gamma \in C^m(\mathcal{U} \setminus \mathcal{C}_0) \cap C^0(\mathcal{U})$, and consequently the vector $S_{(a)}$ is $C^m(\mathcal{U} \setminus \mathcal{C}_0)$.

Moreover, if $b := \min_{\mathcal{V}} \{2k + l\} \leq \min_{\mathcal{V}'} \{2k' + l'\} + m =: c + m$ then $\gamma \in O(|x|^b)$ and $\hat{n}(\gamma) \in O(|x|^{b-1})$, whereas if $b > c + m$ then $\gamma \in o(|x|^{c+m})$ and $\hat{n}(\gamma) \in o(|x|^{c+m-1})$. On top of that, if $c + m \geq b \geq 2a$ or $b > c + m \geq 2a - 1$ then $S_{(a)}$ can be extended continuously to \mathcal{C}_0 by setting $S_{(a)}|_{\mathcal{C}_0} = 0$.

Proof. As shown in Corollary B.10 in Appendix B there exists an axially symmetric solution γ of (5.27) which is $C^m(\mathcal{U} \setminus \mathcal{C}_0)$ and extends continuously to \mathcal{C}_0 , where it vanishes. By the same corollary, if $b \leq c + m$ then $\gamma \in O(|x|^b)$ and $\hat{n}(\gamma) \in O(|x|^{b-1})$, and if $b > c + m$ then $\gamma \in o(|x|^{c+m})$ and $\hat{n}(\gamma) \in o(|x|^{c+m-1})$. Clearly $S_{(a)} = \gamma / \mathcal{R}^{2a} \iota$ for any a is a C^m vector field

in $\mathcal{U} \setminus \mathcal{C}_0$. To obtain its behaviour at \mathcal{C}_0 we simply analyse the application of $S_{(a)}$ to the Cartesian coordinate functions using (5.7) and (5.6), which provide

$$\begin{aligned} S_{(a)}(x) &= -\chi^{-a} \frac{1}{|x|^{1+2a}} \gamma x z, \\ S_{(a)}(y) &= -\chi^{-a} \frac{1}{|x|^{1+2a}} \gamma y z, \\ S_{(a)}(z) &= \chi^{-a} \frac{1}{|x|^{1+2a}} \gamma \|x\|^2 \end{aligned}$$

after using $\mathcal{R}^2 = \chi |x|^2$. Recall that $\chi \in C^{n+1}(U)$ and does not vanish anywhere in U . If $\gamma \in O(|x|^b)$ (case $c + m \geq b$) the three components are $O(|x|^{b+1-2a})$ and hence their limits at \mathcal{C}_0 vanish under the hypothesis $b \geq 2a$. If $\gamma \in o(|x|^{c+m})$ (case $b > c + m$) the three components are $o(|x|^{c+m+1-2a})$, which thus vanish as $|x| \rightarrow 0$ if $c + m \geq 2a - 1$. In both cases we conclude that $S_{(a)}$ can be extended continuously to \mathcal{C}_0 as $S_{(a)}|_{\mathcal{C}_0} = 0$. \square

5.2.1. Decomposition on spheres of symmetric axially symmetric tensors. Given the above results we are ready to prove an intermediate but important result that is the core of the existence of the gauge we look for at first order. We present it as an independent result on the decomposition on spherically symmetric spaces of symmetric axially symmetric tensors into scalar, vector and tensor components. The importance of this result lies on the fact that it determines not only the existence (known) but also the differentiability of the decomposition and the behaviour of such decomposition around the origin. We use the notation and definitions from subsection 5.1 regarding spherically symmetric spaces.

Theorem 5.4. *Let $m \geq 1$ and (M, g) be a spherically symmetric C^{m+2} background with a C^{m+1} metric satisfying assumption S_1 . Let K be a symmetric 2-covariant C^m tensor on $\mathcal{U} \subset M$ satisfying $\mathcal{L}_\eta K = 0$. There exists a vector V tangent to the spheres S_r which is C^m on $\mathcal{U} \setminus \mathcal{C}_0$ and extends continuously to \mathcal{C}_0 , where it vanishes, and a function $k \in C^m(\mathcal{U} \setminus \mathcal{C}_0) \cap C^0(\mathcal{U})$ and $O(|x|^2)$ such that $\overset{\circ}{K}$, namely the tangent-tangent part to the spheres of K , decomposes as*

$$(5.28) \quad \overset{\circ}{K}_{AB} = \overline{D}_A V_B + \overline{D}_B V_A + k g_{S^2 AB}.$$

The vector V is given explicitly by $V = \alpha \iota + \beta \eta$ with $C^m(\mathcal{U} \setminus \mathcal{C}_0)$ functions α and β , both bounded near \mathcal{C}_0 .

Remark 5.5. The proof of the theorem also shows that the tensor

$$N := K - \mathcal{L}_{\alpha\iota}g$$

satisfies $N(\iota, \iota) = N(\eta, \eta)$ and $N(\iota, \eta) = K(\iota, \eta)$.

Proof. We start by showing that (5.28) is equivalent to

$$(5.29) \quad H := K - \mathcal{L}_Vg$$

satisfying

$$(5.30) \quad H(\iota, \iota) = H(\eta, \eta), \quad H(\iota, \eta) = 0.$$

Indeed, the contraction of (5.29) with any pair of vectors $X = X_{\parallel}$, $Y = Y_{\parallel}$, i.e. tangent to the spheres,

$$H(X, Y) = K(X, Y) - \mathcal{L}_Vg(X, Y),$$

is equivalent, term by term and on each sphere, to

$$\mathring{H}(X_{\parallel}, Y_{\parallel}) = \mathring{K}(X_{\parallel}, Y_{\parallel}) - \mathcal{R}^2 \mathcal{L}_{V_{\parallel}}g_{\mathbb{S}^2}(X_{\parallel}, Y_{\parallel})$$

after using (5.3) and the fact that $V = V_{\parallel}$. In index notation, this is in turn equivalent to

$$\begin{aligned} \mathring{H}_{AB} &= \mathring{K}_{AB} - \mathcal{R}^2 \mathcal{L}_{V_{\parallel}}g_{\mathbb{S}^2}{}_{AB} \\ &= \mathring{K}_{AB} - \mathcal{R}^2 (\overline{D}_A \overline{V}_B + \overline{D}_B \overline{V}_A) \\ &= \mathring{K}_{AB} - \overline{D}_A V_B - \overline{D}_B V_A, \end{aligned}$$

and the equivalence between (5.28) and (5.30) follows.

Consider $V = \alpha\iota + \beta\eta$. We start with the following identity for any 2-covariant tensor T

$$(5.31) \quad \begin{aligned} \mathcal{L}_V T &= \alpha \mathcal{L}_{\iota} T + \beta \mathcal{L}_{\eta} T + d\alpha \otimes T(\iota, \cdot) + T(\cdot, \iota) \otimes d\alpha \\ &\quad + d\beta \otimes T(\eta, \cdot) + T(\cdot, \eta) \otimes d\beta, \end{aligned}$$

which applied to g renders

$$(5.32) \quad \mathcal{L}_V g = \alpha \mathcal{L}_{\iota} g + d\alpha \otimes \iota + \iota \otimes d\alpha + d\beta \otimes \eta + \eta \otimes d\beta$$

after using η is Killing. Therefore H takes the form

$$H = K - \alpha \mathcal{L}_\iota g - d\alpha \otimes \iota - \iota \otimes d\alpha - d\beta \otimes \eta - \eta \otimes d\beta.$$

We use now that ι is the conformal Killing vector on the sphere with $\iota = \mathcal{R}^2 dY^3$, so that using (5.3),

$$(5.33) \quad \begin{aligned} \mathcal{L}_\iota g(X, Y) &= \mathcal{R}^2 \mathcal{L}_{\iota_\parallel} g_{\mathbb{S}^2}(X_\parallel, Y_\parallel) \\ &= -2\mathcal{R}^2 Y^3 g_{\mathbb{S}^2}(X_\parallel, Y_\parallel) = -2Y^3 \langle X_\parallel, Y_\parallel \rangle \end{aligned}$$

(recall $\mathring{g} = \mathcal{R}^2 g_{\mathbb{S}^2}$ and $g = g_\perp + \mathring{g}$), and therefore

$$(5.34) \quad \begin{aligned} H(X, Y) &= K(X, Y) + 2Y^3 \alpha \langle X_\parallel, Y_\parallel \rangle - X(\alpha) \iota(Y_\parallel) \\ &\quad - \iota(X_\parallel) Y(\alpha) - X(\beta) \eta(Y_\parallel) - \eta(X_\parallel) Y(\beta). \end{aligned}$$

Hence, in particular,

$$(5.35) \quad H(\eta, \eta) = K(\eta, \eta) + 2Y^3 \alpha \langle \eta, \eta \rangle - 2\eta(\beta) \eta(\eta),$$

$$(5.36) \quad H(\iota, \iota) = K(\iota, \iota) + 2Y^3 \alpha \langle \iota, \iota \rangle - 2\iota(\alpha) \iota(\iota),$$

$$(5.37) \quad H(\iota, \eta) = K(\iota, \eta) - \bar{\iota}(\iota) \eta(\alpha) - \iota(\beta) \bar{\eta}(\eta).$$

The equations in (5.30) are therefore equivalent to

$$(5.38) \quad 0 = K(\eta, \eta) - K(\iota, \iota) + 2(\iota(\alpha) - \eta(\beta)) \eta^2,$$

$$(5.39) \quad 0 = K(\iota, \eta) - (\eta(\alpha) + \iota(\beta)) \eta^2,$$

respectively, after using $\iota(\iota) = \eta(\eta) = \langle \eta, \eta \rangle =: \eta^2$. These equations clearly imply $\eta(\iota(\alpha)) + \eta(\eta(\beta)) = 0$ and $\eta(\iota(\beta)) + \eta(\eta(\alpha)) = 0$. Since $[\eta, \iota] = 0$ it suffices to consider α and β such that $\eta(\alpha) = 0$ and $\eta(\beta) = 0$, and the equations become

$$(5.40) \quad 0 = K(\eta, \eta) - K(\iota, \iota) + 2\iota(\alpha) \eta^2,$$

$$(5.41) \quad 0 = K(\iota, \eta) - \iota(\beta) \eta^2.$$

We thus have two separate ODEs, one for α and one for β . We deal first with equation (5.40), which can be cast as

$$(5.42) \quad \iota(\mathcal{R}^2 \alpha) = q_-$$

with q_- given by (5.10). Lemma 5.2 ensures that q_- satisfies the requirements of the right-hand side \mathcal{Q} of equation (5.27) of Lemma 5.3 with

$b = \min_{\mathcal{V}}\{2k + l\} = 2$ and $c = \min\{2k' + l'\} = 1$. Note that Lemma 5.2 also ensures $q_- \in \mathbf{o}(\|x\|)$ and $O(|x|^{\frac{\mathcal{V}}{2}})$. By setting $\gamma = \mathcal{R}^2\alpha$ and $\mathcal{Q} = q_-$, Lemma 5.3 thus establishes there exists an axially symmetric solution $\mathcal{R}^2\alpha \in C^m(\mathcal{U} \setminus \mathcal{C}_0) \cap C^0(\mathcal{U})$ of (5.42), and consequently the vector $V_l := \alpha_l = \mathcal{R}^{-2}\gamma_l$ is $C^m(\mathcal{U} \setminus \mathcal{C}_0)$. Moreover, since we have $b(=2) \leq c + m$ given that $m \geq 1$, $\mathcal{R}^2\alpha$ is also $O(|x|^2)$, and since $b \geq 2a$ for $a = 1$, the vector $V_l = \mathcal{R}^{-2}\gamma_l$ extends continuously to \mathcal{C}_0 , where it vanishes. Note that $\mathcal{R}^2\alpha \in O(|x|^2)$ implies $\alpha \in C^m(\mathcal{U} \setminus \mathcal{C}_0)$ is bounded near \mathcal{C}_0 , as claimed. The use of (5.37) with $\beta = 0$ (and $\eta(\alpha) = 0$) together with the above proves, in particular, Remark 5.5.

We next analyse in an analogous manner equation (5.41), which we write as

$$(5.43) \quad \iota(|x| \mathcal{R}^2\beta) = q_{\times}$$

with q_{\times} given by (5.11). Lemma 5.2 ensures now that q_{\times} satisfies the requirements of \mathcal{Q} in Lemma 5.3 with $b = 4$ and $c = 2$. By setting $\gamma = |x| \mathcal{R}^2\beta$ and $\mathcal{Q} = q_{\times}$ Lemma 5.3 thence ensures that $|x| \mathcal{R}^2\beta \in C^m(\mathcal{U} \setminus \mathcal{C}_0) \cap C^0(\mathcal{U})$ and $O(|x|^4)$ if $m \geq 2$ (because then $b \leq c + m$) and $o(|x|^3)$ if $m = 1$ (because $b > c + m$). This implies, in any case, that $\beta \in C^m(\mathcal{U} \setminus \mathcal{C}_0)$ can be continuously extended to \mathcal{C}_0 , where it vanishes. As a result, the vector $V_{\eta} := \beta_{\eta}$ is $C^m(\mathcal{U} \setminus \mathcal{C}_0)$ and can be continuously extended to \mathcal{C}_0 as zero.

Clearly, the vector $V = V_l + V_{\eta}$ satisfies both conditions in (5.30) and thence the outcome of the theorem.

Finally, a straightforward calculation shows, since $\eta(\alpha) = 0$, that $\overline{D}^A V_A = \overline{D}^A V_{lA} = \mathcal{R}^2(\iota(\alpha) - 2\alpha Y^3)$, from where it is direct to arrive at

$$(5.44) \quad k = \frac{1}{2} \text{tr}_{\mathbb{S}^2} \overset{\circ}{K} - \overline{D}^A V_{lA} = \frac{\mathcal{R}^2}{\eta^2} K(\eta, \eta) + 2\mathcal{R}^2\alpha Y^3$$

after using (5.40). Since $\mathcal{R}^2/\eta^2 = |x|^2/\|x\|^2$, we can use point 2 of Lemma 5.2 to conclude that $\frac{\mathcal{R}^2}{\eta^2} K(\eta, \eta)$ is $C^m(\mathcal{U})$ and $O(|x|^2)$, while we have from the above that $\mathcal{R}^2\alpha \in C^m(\mathcal{U} \setminus \mathcal{C}_0) \cap C^0(\mathcal{U})$ and $O(|x|^2)$. Therefore $k \in C^m(\mathcal{U} \setminus \mathcal{C}_0) \cap C^0(\mathcal{U})$ and $O(|x|^2)$. □

5.2.2. Choice of gauge at first order. We are ready to show that given any axially symmetric perturbation there exists gauge vectors that render the full angular part of the perturbations in some convenient manner. At this point we could use the previous theorem in full in order to achieve a perturbation tensor that is proportional to the unit sphere metric (at first order). However, for our purposes we will only need the partial result given

by its Remark 5.5, leaving aside the $\{\theta, \phi\}$ crossed term, which makes things simpler. That is because later we will focus on perturbations that inherit an orthogonally transitive two-dimensional group of isometries, in which case the crossed term vanishes from the beginning, and thus it suffices to take $\beta = 0$.

Proposition 5.6. *Let $n \geq 1$ and (M, g) be a spherically symmetric C^{n+2} background with a C^{n+1} metric satisfying assumption S_1 . Let K_1 be a $C^n(\mathcal{U})$ first order perturbation tensor satisfying $\mathcal{L}_\eta K_1 = 0$. Then there exists a $C^n(\mathcal{U} \setminus \mathcal{C}_0)$ first order gauge vector $V_1 = -\tilde{\alpha}\iota$, that extends continuously to zero on \mathcal{C}_0 , such that the corresponding gauge transformed tensor K_1^g (which is automatically $C^{n-1}(\mathcal{U} \setminus \mathcal{C}_0)$) satisfies*

$$(5.45) \quad K_1^g(\eta, \eta) = K_1^g(\iota, \iota),$$

where

$$(5.46) \quad K_1^g(\eta, \eta) = K_1(\eta, \eta) + 2 \frac{z}{|x|} \tilde{\alpha} \eta^2.$$

In addition

$$(5.47) \quad K_1^g(\iota, \eta) = K_1(\iota, \eta),$$

$$(5.48) \quad K_1^g(\iota, \nu_1) = K_1(\iota, \nu_1) - \nu_1(\tilde{\alpha})\eta^2,$$

$$(5.49) \quad K_1^g(\eta, \nu_1) = K_1(\eta, \nu_1),$$

$$(5.50) \quad K_1^g(\nu_1, \nu_2) = K_1(\nu_1, \nu_2),$$

for any vectors ν_1 and ν_2 orthogonal to the spheres S_r . The function $\tilde{\alpha}$ is axially symmetric, of class $C^n(\mathcal{U} \setminus \mathcal{C}_0)$ and bounded near \mathcal{C}_0 , and $|x| \hat{n}(\tilde{\alpha}) \in C^{n-1}(\mathcal{U} \setminus \mathcal{C}_0)$ is also bounded near \mathcal{C}_0 .

Moreover, the function $K_1^g(\eta, \eta)/\eta^2$ is $C^n(\mathcal{U} \setminus \mathcal{C}_0)$, axially symmetric and bounded near \mathcal{C}_0 , and the function

$$q_1^g := |x|^2 K_1^g(\iota, \hat{n}) \quad \text{outside } \mathcal{C}_0, \quad q_1^g(\mathcal{C}_0) = 0$$

is $C^{n-1}(\mathcal{U} \setminus \mathcal{C}_0) \cap C^0(\mathcal{U})$, $\mathbf{o}(\|x\|)$ and takes the form

$$(5.51) \quad q_1^g = z\|x\|^2 P_{11}^* + \sum_{k=2}^{[n/2]+1} \|x\|^{2k} P_{0k}^* + \|x\|^2 \Phi_{01}^{*(n)} + z^2 \Phi_{20}^{*(n)} + \|x\|^2 \Gamma_1^{(n-1)}$$

where the P^* functions are C^n functions of $\{z, t\}$, $\Phi^{*(n)} \in C^n(\mathcal{U})$ are radially symmetric in $\{x, y\}$ and $\mathbf{o}(\|x\|^n)$ and $\Gamma_1^{(n-1)} \in C^{n-1}(\mathcal{U} \setminus \mathcal{C}_0) \cap C^0(\mathcal{U})$ is radially symmetric in $\{x, y\}$ and $O(|x|)$.

Proof. Let $\tilde{\alpha}$ be the function α of Theorem 5.4 applied to $K = K_1$ and $n = m$, and define the vector $V_1 = -\tilde{\alpha}v$. From that theorem we know that $\tilde{\alpha}$ is $C^n(\mathcal{U} \setminus \mathcal{C}_0)$ and bounded near the origin. By Remark 5.5 we have that $K_1^g = K_1 + \mathcal{L}_{V_1}g$ satisfies (5.45) and (5.47).

For later use we note that (5.34) with $H = K_1^g$, $K = K_1$, $\alpha = \tilde{\alpha}$ and $\beta = 0$ yields, for any any pair of vectors X, Y ,

$$(5.52) \quad K_1^g(X, Y) = K_1(X, Y) + 2Y^3\tilde{\alpha}\langle X_{\parallel}, Y_{\parallel} \rangle - X(\tilde{\alpha})\iota(Y_{\parallel}) - \iota(X_{\parallel})Y(\tilde{\alpha}).$$

The equation satisfied by $\tilde{\alpha}$ is (from (5.42))

$$(5.53) \quad \iota(\mathcal{R}^2\tilde{\alpha}) = q_{1-}, \quad q_{1-} := \frac{\mathcal{R}^2}{2\eta^2} \{K_1(\iota, \iota) - K_1(\eta, \eta)\}.$$

$$(5.54) \quad \iff \quad 0 = K_1(\eta, \eta) - K_1(\iota, \iota) + 2\iota(\tilde{\alpha})\eta^2.$$

As discussed in the proof of Theorem 5.4 (for $K = K_1$, $m = n$ and $q_- = q_{1-}$), (5.53) satisfies the requirements of Lemma 5.3 with $\gamma = \mathcal{R}^2\tilde{\alpha}$, $\mathcal{Q} = q_{1-}$ and $m = n \geq 1$, with $b = 2$ and $c = 1$.

Equations (5.46) and (5.48)-(5.50) follow immediately from (5.52) after using $\eta(\tilde{\alpha}) = 0$, $\iota(\iota) = \langle \eta, \eta \rangle =: \eta^2$. The claim that $K_1^g(\eta, \eta)/\eta^2$ is $C^n(\mathcal{U} \setminus \mathcal{C}_0)$ and bounded near the centre follows from equation (5.46), since $\tilde{\alpha}$ (by Theorem 5.4) and $K_1(\eta, \eta)/\eta^2$ (by point 2 in Lemma 5.2 applied to $K = K_1$ and $m = n$) are both $C^n(\mathcal{U} \setminus \mathcal{C}_0)$ and bounded near the centre.

Concerning the properties of q_1^g , we use (5.48) with $\nu_1 = \hat{n}$, to get (recall that $\eta^2 = \chi\|x\|^2$)

$$(5.55) \quad q_1^g := |x|^2 K_1^g(\iota, \hat{n}) = |x|^2 K_1(\iota, \hat{n}) + \|x\|^2 \Gamma_1^{(n-1)},$$

$$\Gamma_1^{(n-1)} := -\chi |x|^2 \hat{n}(\tilde{\alpha}).$$

By point 3 in Lemma 5.2 (with $K = K_1$, $m = n$), the first term extends to a $C^n(\mathcal{U})$ function admitting an expression of the form (5.16). For the second term, we compute (using $\mathcal{R}^2 = \chi|x|^2$),

$$(5.56) \quad |x| \hat{n}(\tilde{\alpha}) = |x| \hat{n}(\gamma/\mathcal{R}^2) = \frac{1}{\chi} \left(\frac{\hat{n}(\gamma)}{|x|} - \frac{\gamma}{|x|^2} \left(|x| \hat{n}(\chi) \frac{1}{\chi} + 2 \right) \right).$$

Since $\gamma \in O(|x|^2)$ and $\hat{n}(\gamma) \in O(|x|^1)$ by virtue of Lemma 5.3 (since $c + m \geq b = 2$, see above), and the term $|x|\hat{n}(\chi)/\chi = \frac{1}{\chi}x^i\partial_i\chi$ is $C^n(U)$ (because $\chi \in C^{n+1}(U)$ and does not vanish on \mathcal{U}), we conclude that $|x|\hat{n}(\tilde{\alpha})$ is bounded near \mathcal{C}_0 , as claimed. This implies in particular that $\Gamma_1^{(n-1)}$ is $O(|x|)$. Given that $\tilde{\alpha} \in C^n(\mathcal{U} \setminus \mathcal{C}_0)$ it follows immediately that $\Gamma_1^{(n-1)}$ (extended at the centre with the value zero) is $C^{n-1}(\mathcal{U} \setminus \mathcal{C}_0) \cap C^0(\mathcal{U})$ and $O(|x|)$. We conclude that the function q_1^g in (5.55) is $C^{n-1}(\mathcal{U} \setminus \mathcal{C}_0) \cap C^0(\mathcal{U})$, and takes the form (5.51), from where it is direct to check that, since $n \geq 1$, q_1^g is also $\mathbf{o}(\|x\|)$. \square

5.2.3. Choice of gauge at second order. We now perform an analogous, but more involved, procedure for the second order perturbation.

Proposition 5.7. *Assume the setting of Proposition 5.6 and restrict $n \geq 2$. Let K_2 be a $C^{n-1}(\mathcal{U})$ second order perturbation tensor satisfying $\mathcal{L}_\eta K_2 = 0$. Then, there exists a $C^{n-1}(\mathcal{U} \setminus \mathcal{C}_0)$ second order gauge vector $V_2 = -\tilde{v}\iota$, that extends continuously to zero on \mathcal{C}_0 , such that the corresponding gauge transformed tensor K_2^g (which is immediately $C^{n-2}(\mathcal{U} \setminus \mathcal{C}_0)$) satisfies*

$$(5.57) \quad K_2^g(\eta, \eta) = K_2^g(\iota, \iota).$$

In addition,

$$(5.58) \quad K_2^g(\iota, \eta) = K_2(\iota, \eta) - 2\iota(\tilde{\alpha}K_1(\iota, \eta)),$$

$$(5.59) \quad K_2^g(\iota, \nu_1) = K_2(\iota, \nu_1) - \nu_1(\tilde{v})\eta^2 - 2\iota(\tilde{\alpha}K_1(\iota, \nu_1)) + \tilde{\alpha}\eta^2\nu_1(\tilde{\alpha}) \\ - 2\nu_1(\tilde{\alpha})K_1(\iota, \iota) + \tilde{\alpha}(K_1^g + K_1)(\iota, [\iota, \nu_1]) \\ + 3\iota(\tilde{\alpha})\nu_1(\tilde{\alpha})\eta^2 - 4\tilde{\alpha}\frac{\tilde{v}}{|x|}\eta^2\nu_1(\tilde{\alpha}),$$

$$(5.60) \quad K_2^g(\eta, \nu_1) = K_2(\eta, \nu_1) - 2\tilde{\alpha}\iota(K_1(\eta, \nu_1)) - 2\nu_1(\tilde{\alpha})K_1(\eta, \iota) \\ + \tilde{\alpha}(K_1^g + K_1)(\eta, [\iota, \nu_1]),$$

$$(5.61) \quad K_2^g(\nu_1, \nu_2) = K_2(\nu_1, \nu_2) - 2\tilde{\alpha}\iota(K_1(\nu_1, \nu_2)) + 2\nu_1(\tilde{\alpha})\nu_2(\tilde{\alpha})\eta^2 \\ + \tilde{\alpha}(K_1^g + K_1)([\iota, \nu_1], \nu_2) + \tilde{\alpha}(K_1^g + K_1)(\nu_1, [\iota, \nu_2]) \\ - 2\nu_1(\tilde{\alpha})K_1(\iota, \nu_2) - 2\nu_2(\tilde{\alpha})K_1(\iota, \nu_1),$$

for any vectors ν_1 and ν_2 orthogonal to the spheres S_r . The function \tilde{v} is axially symmetric, $C^{n-1}(\mathcal{U} \setminus \mathcal{C}_0)$ and bounded near \mathcal{C}_0 .

Moreover, the function $K_2^g(\eta, \eta)/\eta^2$ is $C^{n-1}(\mathcal{U} \setminus \mathcal{C}_0)$ and bounded near \mathcal{C}_0 , and the function

$$q_2^g := |x|^2 K_2^g(\iota, \hat{n}) \quad \text{outside } \mathcal{C}_0, \quad q_2^g(\mathcal{C}_0) = 0$$

is $C^{n-2}(\mathcal{U} \setminus \mathcal{C}_0) \cap C^0(\mathcal{U})$ and $\mathfrak{o}(\|x\|)$ and takes the form

$$(5.62) \quad q_2^g = z\|x\|^2 P_{11}^{**} + \sum_{k=2}^{[(n-1)/2]+1} \|x\|^{2k} P_{0k}^{**} + \|x\|^2 \Phi_{01}^{**(n-1)} + z^2 \Phi_{20}^{**(n-1)} + \|x\|^2 \Gamma_2^{(n-2)} - \iota(2\tilde{\alpha}q_1)$$

where $q_1 := |x|^2 K_1(\iota, \hat{n})$, all the P^{**} functions are C^{n-1} functions of $\{z, t\}$, $\Phi^{**(n-1)} \in C^{n-1}(\mathcal{U})$ are radially symmetric in $\{x, y\}$ and $\mathfrak{o}(\|x\|^{n-1})$ and $\Gamma_2^{(n-2)} \in C^{n-2}(\mathcal{U} \setminus \mathcal{C}_0) \cap C^0(\mathcal{U})$ is radially symmetric in $\{x, y\}$ and $O(|x|)$. Moreover, the function $\iota(\tilde{\alpha}q_1)$ is $C^{n-1}(\mathcal{U} \setminus \mathcal{C}_0)$, $\mathfrak{o}(\|x\|)$ (in particular, it admits a continuous extension to \mathcal{C}_0 with value zero).

Proof. By Proposition 5.6 there exists a first order gauge vector $V_1 = -\tilde{\alpha}\iota$, where $\tilde{\alpha} \in C^n(\mathcal{U} \setminus \mathcal{C}_0)$ satisfies (5.54), so that (5.45) holds. The second order gauge transformation (2.6) can be rewritten (upon combining with (2.5)) as

$$(5.63) \quad K_2^g = K_2 + \mathcal{L}_{V_2}g + \mathcal{L}_{V_1}M, \quad M := K_1^g + K_1.$$

Consider the second order gauge vector $V_2 = -\tilde{\nu}\iota$. Immediate consequences of (5.31) (with $\beta = 0$) and (5.33) are

$$\begin{aligned} \mathcal{L}_{V_2}g(X, Y) &= 2Y^3\tilde{\nu}\langle X_{\parallel}, Y_{\parallel} \rangle - X(\tilde{\nu})\iota(Y_{\parallel}) - Y(\tilde{\nu})\iota(X_{\parallel}), \\ \mathcal{L}_{V_1}M(X, Y) &= -\tilde{\alpha}\iota(M(X, Y)) + \tilde{\alpha}M([\iota, X], Y) + \tilde{\alpha}M(X, [\iota, Y]) \\ &\quad - X(\tilde{\alpha})M(\iota, Y) - M(X, \iota)Y(\tilde{\alpha}), \end{aligned}$$

where to get the second we also used the Leibniz rule for the Lie derivative. Thus, (5.63) is

$$(5.64) \quad K_2^g(X, Y) = K_2(X, Y) + 2Y^3\tilde{\nu}\langle X_{\parallel}, Y_{\parallel} \rangle - X(\tilde{\nu})\iota(Y_{\parallel}) - Y(\tilde{\nu})\iota(X_{\parallel}) - \tilde{\alpha}\iota(M(X, Y)) + \tilde{\alpha}M([\iota, X], Y) + \tilde{\alpha}M(X, [\iota, Y]) - X(\tilde{\alpha})M(\iota, Y) - M(X, \iota)Y(\tilde{\alpha}).$$

The explicit form of M follows from Proposition 5.6, which gives

$$(5.65) \quad M(\eta, \eta) = 2K_1(\eta, \eta) + 2Y^3\tilde{\alpha}\eta^2,$$

$$(5.66) \quad M(\iota, \iota) = 2K_1(\iota, \iota) + 2Y^3\tilde{\alpha}\eta^2 - 2\eta^2\iota(\tilde{\alpha}),$$

$$(5.67) \quad M(\iota, \eta) = 2K_1(\iota, \eta), \quad M(\iota, \nu_1) = 2K_1(\iota, \nu_1) - \nu_1(\tilde{\alpha})\eta^2,$$

$$(5.68) \quad M(\eta, \nu_1) = 2K_1(\eta, \nu_1), \quad M(\nu_1, \nu_2) = 2K_1(\nu_1, \nu_2),$$

where, to get the second, we have used (5.54). Inserting these into (5.64) and using $[\iota, \eta] = 0$, $\eta(\tilde{\alpha}) = 0$, a straightforward computation yields (5.58)-(5.61), as well as

$$(5.69) \quad K_2^g(\iota, \iota) = K_2(\iota, \iota) + 2Y^3\tilde{\nu}\eta^2 - 2\eta^2\iota(\tilde{\nu}) - 2\tilde{\alpha}\iota(K_1(\iota, \iota)) \\ + \tilde{\alpha}\iota(-2Y^3\tilde{\alpha}\eta^2 + 2\eta^2\iota(\tilde{\alpha})) - 4\iota(\tilde{\alpha})K_1(\iota, \iota) \\ - 4Y^3\tilde{\alpha}\iota(\tilde{\alpha})\eta^2 + 4\eta^2\iota(\tilde{\alpha})^2,$$

$$(5.70) \quad K_2^g(\eta, \eta) = K_2(\eta, \eta) + 2Y^3\tilde{\nu}\eta^2 - 2\tilde{\alpha}\iota(K_1(\eta, \eta)) - 2\tilde{\alpha}\iota(\tilde{\alpha}Y^3\eta^2).$$

We may now find the explicit form of equation (5.57). Subtracting (5.69) and (5.70) one finds, after trivial rearrangements, that (5.57) becomes

$$(5.71) \quad 0 = K_2(\eta, \eta) - K_2(\iota, \iota) - 2\tilde{\alpha}\iota(K_1(\eta, \eta) - K_1(\iota, \iota)) \\ + 2\eta^2\iota(\tilde{\nu}) - \tilde{\alpha}\iota(2\eta^2\iota(\tilde{\alpha})) + 4\iota(\tilde{\alpha})K_1(\iota, \iota) \\ + 4Y^3\tilde{\alpha}\iota(\tilde{\alpha})\eta^2 - 4\eta^2\iota(\tilde{\alpha})^2.$$

In order to analyse this equation, it turns out to be convenient to introduce an the auxiliary $\tilde{\zeta} := \mathcal{R}^2\tilde{\nu} + \tilde{\alpha}q_{1-}$, where q_{1-} was introduced in (5.53) and it is $C^n(\mathcal{U})$ and $O(|x|^2)$ by Lemma 5.2 applied to $K = K_1$, $m = n$, $q_- = q_{1-}$. Using the first equation in (5.53), a straightforward computation shows that (5.71) can be rewritten as

$$(5.72) \quad \iota(\tilde{\zeta}) = q_{2-} + \frac{1}{\mathcal{R}^2}q_{1-}(q_{1-} - 2q_{1+}),$$

where

$$q_{1+} := \frac{\mathcal{R}^2}{2\eta^2} \{K_1(\iota, \iota) + K_1(\eta, \eta)\}, \quad q_{2-} := \frac{\mathcal{R}^2}{2\eta^2} \{K_2(\iota, \iota) - K_2(\eta, \eta)\}.$$

The right-hand side of (5.72) is invariant under η , so it suffices to look for $\tilde{\zeta}$ satisfying $\eta(\tilde{\zeta}) = 0$. It is convenient to decompose $\tilde{\zeta} = \tilde{\zeta}_{(0)} + \tilde{\zeta}_{(1)}$ and split (5.72) into the two equations

$$(5.73) \quad \iota(\tilde{\zeta}_{(0)}) = q_{2-},$$

$$(5.74) \quad \iota(\mathcal{R}^2\tilde{\zeta}_{(1)}) = q_{1-}(q_{1-} - 2q_{1+}).$$

The splitting is such that the right-hand side is $C^{n-1}(\mathcal{U})$ in the first equation and $C^n(\mathcal{U})$ in the second. Concerning the first equation Lemma 5.2, applied to $K = K_2$ and $m = n - 1 \geq 1$, tells us that q_{2-} has the form (5.13) and thus satisfies the requirements of Lemma 5.3 with $b = \min_{\vee} \{2k + l\} = 2$ and

$c = \min_{\mathcal{V}}\{2k' + l'\} = 1$, and thence $b \leq c + m$. Applying the lemma, there exists a solution $\tilde{\zeta}_{(0)}$ satisfying

$$(5.75) \quad \tilde{\zeta}_{(0)} \in C^{m-1}(\mathcal{U} \setminus \mathcal{C}_0) \cap C^0(\mathcal{U}), \quad \tilde{\zeta}_{(0)} \in O(|x|^2), \quad \hat{n}(\tilde{\zeta}_{(0)}) \in O(|x|).$$

Regarding equation (5.74) we first obtain the structure of its right-hand side. From Lemma 5.2 applied to $K = K_1$ (specifically from expressions (5.13) and (5.14) for q_{1-} and q_{1+}) it follows

$$\begin{aligned} q_{1-}(q_{1-} - 2q_{1+}) &= z^2 \sum_{k=1}^{[n/2]+1} \|x\|^{2k} \check{P}_{2k} + \sum_{k=2}^{2[n/2]+2} \|x\|^{2k} \check{P}_{0k} \\ &\quad + \|x\|^4 \check{\Phi}_{02}^{(n)} + z^2 \check{\Phi}_{20}^{(n)} + z\|x\|^2 \check{\Phi}_{11}^{(n)}, \end{aligned}$$

where the functions \check{P} are C^m in $\{z, t\}$ and $\check{\Phi}^{(n)} \in C^m(\mathcal{U})$ are radially symmetric in $\{x, y\}$ and $\mathbf{o}(\|x\|^n)$. Thus, the right-hand side of (5.74) satisfies the hypotheses of Lemma 5.3 with $m = n \geq 2$, $b = \min\{2k + l\} = 4$ and $c = \min_{\mathcal{V}}\{2k' + l'\} = 2$, so that $b \leq c + m$. By this lemma, there exists a solution $\mathcal{R}^2 \tilde{\zeta}_{(1)}$ of (5.74) satisfying

$$\begin{aligned} \mathcal{R}^2 \tilde{\zeta}_{(1)} &\in C^n(\mathcal{U} \setminus \mathcal{C}_0) \cap C^0(\mathcal{U}), \quad \mathcal{R}^2 \tilde{\zeta}_{(1)} \in O(|x|^4), \quad \hat{n}(\mathcal{R}^2 \tilde{\zeta}_{(1)}) \in O(|x|^3) \\ \implies \tilde{\zeta}_{(1)} &\in C^n(\mathcal{U} \setminus \mathcal{C}_0) \cap C^0(\mathcal{U}), \quad \tilde{\zeta}_{(1)} \in O(|x|^2), \quad \hat{n}(\tilde{\zeta}_{(1)}) \in O(|x|). \end{aligned}$$

Combining this and (5.75), there is a solution $\tilde{\zeta}$ of (5.72) satisfying

$$(5.76) \quad \tilde{\zeta} \in C^{n-1}(\mathcal{U} \setminus \mathcal{C}_0) \cap C^0(\mathcal{U}), \quad \tilde{\zeta} \in O(|x|^2), \quad \hat{n}(\tilde{\zeta}) \in O(|x|).$$

This, together with the fact that $\tilde{\alpha} \in C^n(\mathcal{U} \setminus \mathcal{C}_0)$ is bounded (by Proposition 5.6), and q_{1-} is $C^n(\mathcal{U})$ and $O(|x|^2)$, imply that

$$(5.77) \quad \tilde{v} = \mathcal{R}^{-2}(\tilde{\zeta} - \tilde{\alpha}q_{1-}) \quad \text{is} \quad C^{m-1}(\mathcal{U} \setminus \mathcal{C}_0) \quad \text{and bounded near} \quad \mathcal{C}_0$$

(note that \tilde{v} is not necessarily defined at the origin). Since ι extends continuously to the centre as the zero vector, boundedness of \tilde{v} implies that $V_2 = -\tilde{v}\iota$ extends continuously to the centre with the value zero.

We deal now with the properties of $K_2^g(\eta, \eta)/\eta^2$. First, we rewrite equation (5.70) as

$$(5.78) \quad \frac{1}{\|x\|^2} K_2^g(\eta, \eta) = \frac{1}{\|x\|^2} K_2(\eta, \eta) + 2 \frac{z}{|x|} \chi \tilde{v} - 2\tilde{\alpha} \frac{1}{\|x\|^2} \iota(K_1(\eta, \eta)) - 2 \frac{1}{\|x\|^2} \frac{1}{|x|} \iota(\tilde{\alpha} z \eta^2).$$

We want to check that all terms in the right-hand side are $C^{m-1}(\mathcal{U} \setminus \mathcal{C}_0)$ and bounded near \mathcal{C}_0 . For the first term this is immediate from item 2 of Lemma 5.2 applied to K_2 and $m = n - 1 \geq 1$. For the second, it follows from (5.77) and the properties of χ . Concerning the third term, point 2 in Lemma 5.2 implies that $K_1(\eta, \eta)$ admits a decomposition of the form

$$2K_1(\eta, \eta) = \|x\|^2 (P_{10}(z, t) + \Phi_\eta),$$

with $P_{10}(x, t)$ is a C^n function of its variables and Φ_η is $C^n(\mathcal{U})$ and $\mathbf{o}(\|x\|)$. Computing the derivative we get

$$2\iota(K_1(\eta, \eta)) = \|x\|^2 \left\{ -2 \frac{z}{|x|} (P_{10} + \Phi_\eta) + \frac{\|x\|^2}{|x|} \partial_z P_{10} + \iota(\Phi_\eta) \right\}$$

after taking into account that $\iota(\|x\|) = -\frac{z}{|x|} \|x\|$, $\iota(z) = \|x\|^2 / |x|$ and $\iota(t) = 0$. By Lemma B.8 the function Φ_η satisfies $\iota(\Phi_\eta) \in \mathbf{o}(\|x\|)$. It is clear that all terms in brackets are $C^{n-1}(\mathcal{U} \setminus \mathcal{C}_0)$ and bounded near \mathcal{C}_0 , so the same holds for $\|x\|^{-2} \iota(K_1(\eta, \eta))$ and we conclude that the third term in (5.78) is $C^{n-1}(\mathcal{U} \setminus \mathcal{C}_0)$ and bounded. Finally, the last term reads

$$\frac{1}{\|x\|^2 |x|} \iota(\tilde{\alpha} z \eta^2) = \frac{q_{1-}}{|x|^2} \frac{z}{|x|} + \tilde{\alpha} \frac{\eta^2}{|x|^2} - 2\chi \tilde{\alpha} \frac{z^2}{|x|^2},$$

after using (5.53). All three terms are $C^n(\mathcal{U} \setminus \mathcal{C}_0)$, and since $q_{1-} \in O(|x|^2)$ the first term is, like the rest, bounded near \mathcal{C}_0 . Summarizing, (5.78) implies that $K_2^g(\eta, \eta)/\|x\|^2$ is $C^{n-1}(\mathcal{U} \setminus \mathcal{C}_0)$ and bounded near the origin. The same holds for $K_2^g(\eta, \eta)/\eta^2$ given the properties of χ .

We consider now $|x|^2 K_2^g(\hat{n}, \iota)$, for which we need to analyse (5.59) for $\nu_1 = \hat{n}$. Using $[\iota, \hat{n}] = 0$ we obtain, after simple rearranging,

$$(5.79) \quad |x|^2 K_2^g(\iota, \hat{n}) = |x|^2 K_2(\iota, \hat{n}) + \|x\|^2 (\Gamma_A + \Gamma_B) - 2\iota \left(\tilde{\alpha} |x|^2 K_1(\iota, \hat{n}) \right),$$

where we have defined

$$(5.80) \quad \Gamma_A := \chi |x| \left\{ -|x| \hat{n}(\tilde{v}) + |x| \tilde{\alpha} \iota(\hat{n}(\tilde{\alpha})) + |x| \hat{n}(\tilde{\alpha})(3\iota(\tilde{\alpha}) - 4\tilde{\alpha}Y^3) \right\},$$

$$(5.81) \quad \Gamma_B := -2 \frac{|x|^2}{\|x\|^2} \hat{n}(\tilde{\alpha}) K_1(\iota, \iota).$$

The first term in (5.79) is a $C^{n-1}(\mathcal{U})$ function by virtue of Lemma 5.2 with $K = K_2$, $m = n - 1$. In fact, it corresponds to the function q in that lemma, so it admits an expansion of the form

$$(5.82) \quad |x|^2 K_2(\iota, \hat{n}) = z \|x\|^2 P_{11}^{**} + \sum_{k=2}^{[(n-1)/2]+1} \|x\|^{2k} P_{0k}^{**} + \|x\|^2 \Phi_{01}^{**(n-1)} + z^2 \Phi_{20}^{**(n-1)}$$

with all properties stated in the Proposition. For the second term we use an analogous procedure as in Proposition 5.6. We may use (5.56) replacing $\tilde{\alpha}$ by \tilde{v} and using the corresponding $\gamma = \mathcal{R}^2 \tilde{v} = \tilde{\zeta} - \tilde{\alpha} q_{1-}$, so that

$$(5.83) \quad \begin{aligned} |x| \hat{n}(\tilde{v}) &= \frac{1}{\chi} \left(\frac{\hat{n}(\tilde{\zeta} - \tilde{\alpha} q_{1-})}{|x|} - \frac{\tilde{\zeta} - \tilde{\alpha} q_{1-}}{|x|^2} \left(|x| \hat{n}(\chi) \frac{1}{\chi} + 2 \right) \right) \\ &= \frac{1}{\chi} \left(\frac{\hat{n}(\tilde{\zeta})}{|x|} - |x| \hat{n}(\tilde{\alpha}) \frac{q_{1-}}{|x|^2} - \frac{\tilde{\alpha} \hat{n}(q_{1-})}{|x|} - \frac{\tilde{\zeta} - \tilde{\alpha} q_{1-}}{|x|^2} \left(|x| \hat{n}(\chi) \frac{1}{\chi} + 2 \right) \right). \end{aligned}$$

On the other hand, we compute

$$(5.84) \quad \begin{aligned} |x| \iota(\hat{n}(\tilde{\alpha})) &= |x| \hat{n}(\iota(\tilde{\alpha})) = |x| \hat{n} \left(\frac{q_{1-}}{\mathcal{R}^2} \right) \\ &= \frac{1}{\chi} \left(\frac{\hat{n}(q_{1-})}{|x|} - \frac{q_{1-}}{|x|^2} \left(|x| \hat{n}(\chi) \frac{1}{\chi} + 2 \right) \right). \end{aligned}$$

All terms in (5.83) and (5.84) are bounded near the origin as a consequence of (5.76), together with the facts (we also use that $\mathbf{o}(\|x\|) \implies o(|x|)$, see Lemma B.5)

$$\begin{aligned} q_{1-} &\in O(|x|^2), & \hat{n}(q_{1-}) &\in \mathbf{o}(\|x\|) \\ & & & \text{(Lemma 5.2 for } K = K_1, m = n \geq 2, q_- = q_{1-}), \\ \tilde{\alpha} \quad \text{and} \quad |x| \hat{n}(\tilde{\alpha}) &\text{ bounded} & & \text{(Proposition 5.6).} \end{aligned}$$

Boundedness of the last term in brackets in (5.80) is immediate. The property that all terms in Γ_A are $C^{n-2}(\mathcal{U} \setminus \mathcal{C}_0)$ is obvious. Thus, we conclude

that

$$\Gamma_A \in C^{n-2}(\mathcal{U} \setminus \mathcal{C}_0), \quad \Gamma_A \in O(|x|)$$

so that, in particular this function extends continuously to the centre with the value zero. Concerning Γ_B , point 2 of Lemma 5.2 applied to $K = K_1$ and $m = n \geq 2$, together with the above properties of $|x| \hat{n}(\tilde{\alpha})$, ensure Γ_B is $C^{n-1}(\mathcal{U} \setminus \mathcal{C}_0)$ and $O(|x|)$. The function $\Gamma_2^{(n-2)}$ in the statement of the Proposition is simply $\Gamma_2^{(n-2)} := \Gamma_A + \Gamma_B$ and obviously also satisfies $\Gamma_2^{(n-2)} \in C^{n-2}(\mathcal{U} \setminus \mathcal{C}_0) \cap C^0(\mathcal{U})$ and $O(|x|)$.

It only remains to show that the last term in (5.79), which is clearly $C^{n-1}(\mathcal{U} \setminus \mathcal{C}_0)$, can also be extended continuously to \mathcal{C}_0 . Recall first that the function $q_1 := |x|^2 K_1(\iota, \hat{n})$ is (by Lemma 5.2 applied to $K = K_1$, $m = n$) $C^n(\mathcal{U})$ and $\mathfrak{o}(\|x\|)$. Computing the derivative, and using $\iota(\tilde{\alpha}) = q_{1-}/\mathcal{R}^2$, one finds

$$(5.85) \quad \iota \left(\tilde{\alpha} |x|^2 K_1(\iota, \hat{n}) \right) = \chi^{-1} \frac{q_{1-}}{|x|^2} q_1 + \tilde{\alpha} \iota(q_1),$$

Since $q_{1-} \in O(|x|^2)$ the first term extends continuously to \mathcal{C}_0 where it vanishes. For the second, we apply Lemma B.8 to q_1 for $l = 1$ (recall that $n \geq 2$) to conclude that $\iota(q_1) \in \mathfrak{o}(\|x\|)$. Since $\tilde{\alpha}$ is bounded near the origin, it follows that $\tilde{\alpha} \iota(q_1)$ is $\mathfrak{o}(\|x\|)$ and hence extends continuously to \mathcal{C}_0 with the value zero. Note that the form (5.62), given that $n \geq 2$, implies that q_2^g is $\mathfrak{o}(\|x\|)$. This completes the proof. \square

6. General stationary and axisymmetric perturbation scheme on spherically symmetric backgrounds

In this section we combine the results in Section 4.1 for orthogonally transitive actions and those in Section 5 involving spherically symmetric backgrounds to construct a stationary and axisymmetric perturbation scheme on spherically symmetric backgrounds. We first show the existence of gauge vectors that render the first and second order perturbation tensors in the standard forms assumed in the literature. Subsection 6.2 is devoted to discussing uniqueness properties of these gauge vectors and the last subsection to studying the gauge freedom left in those forms.

While in the previous section the spherical background was arbitrary, here we restrict ourselves to the static case, since this is what we shall need in [15]. We start by making explicit the definition of static and spherically

symmetric spacetime that we use and then impose the global assumptions on the background that will be needed.

Definition 6.1. A (four-dimensional) spacetime (M, g) is static and spherically symmetric if it admits an $SO(3)$ group of isometries acting transitively on spacelike surfaces (which may degenerate to points), and a Killing vector ξ which is timelike everywhere, commutes with the generators of $SO(3)$ and is orthogonal to the $SO(3)$ orbits.

It is easy to check that such ξ is necessarily hypersurface orthogonal, which justifies the name “static” in the definition.

Our global assumption is the following.

Assumption H₁: M is diffeomorphic to $U^3 \times I$ where $I \subset \mathbb{R}$ is an open interval and U^3 is a radially symmetric domain of \mathbb{R}^3 , which may or may not contain the origin, with the orbits of the Killing ξ along the I factor and $SO(3)$ acting in the standard way on U^3 . In addition, in the cartesian coordinates $\{x, y, z, t\}$ of $U^3 \times I$, the metric g takes the form

$$g = -e^\nu dt^2 + v(x_i dx^i)^2 + \chi \delta_{ij} dx^i dx^j$$

with ν, v, χ are C^{n+1} functions of the coordinates x, y, z and radially symmetric.

Note that assumption H₁ implies assumption S₁, so all the results in the previous section hold. Observe also that the set of fixed points of the $SO(3)$ action (the centre of symmetry) is either empty or $\mathcal{C}_0 := \{0_3\} \times I$.

The Lorentzian signature of g implies that both χ and $\chi + v|x|^2$ are positive everywhere, so we may define a C^{n+1} function on M by $e^\lambda = \chi + v|x|^2$. We also introduce the non-negative function $\mathcal{R} \in C^{n+1}(M \setminus \mathcal{C}_0)$ defined by $\mathcal{R}^2 = \chi|x|^2$. It is clear that this function can be extended continuously to \mathcal{C}_0 , where it vanishes.

From $\{x, y, z, t\}$ we may define standard spherical coordinates $\{r, \theta, \phi\}$, see (5.5), so that $r = |x|$. The set $\{r, \theta, \phi, t\}$ is a coordinate system in $M \setminus \mathcal{A}$ with r taking values in (a, b) with $0 \leq a < b \leq +\infty$. Note that U^3 is a ball if and only if $\mathcal{C}_0 \neq \emptyset$, and if and only if $a = 0$.

The functions $\nu, \lambda, \mathcal{R}$ are radially symmetric so, when expressed in the spherical coordinates depend only on r . We write $\nu(r), \lambda(r), \mathcal{R}(r)$ (i.e. making explicit the argument r) when we refer to this representation of the

functions. We finally note that the metric g on $M \setminus \mathcal{A}$ in spherical coordinates takes the form

$$(6.1) \quad g = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + \mathcal{R}^2(r) (d\theta^2 + \sin^2 \theta d\phi^2), \quad \xi = \partial_t.$$

Consider the smooth vector fields $\{\partial_x, \partial_y, \partial_z, \partial_t\}$ and the vector field $\hat{n} = \frac{1}{|x|}(x\partial_x + y\partial_y + z\partial_z) = \partial_r$, which is smooth outside \mathcal{C}_0 . The functions ν, λ satisfy

$$e^\nu = g(\partial_t, \partial_t) \quad \text{in } M, \quad e^\lambda = g(\hat{n}, \hat{n}) \quad \text{in } M \setminus \mathcal{C}_0,$$

and in fact can also be defined by these expressions in the respective domains.

Any of the Killing vectors $\eta_a, a = 1, 2, 3$, of the $so(3)$ algebra together with the static Killing ξ generate an Abelian G_2 group of isometries which acts orthogonally transitively on timelike surfaces outside the axis of rotation \mathcal{A} . Following Section 5 we choose $\eta = \partial_\phi$, without loss of generality. Observe that, under our assumptions, $M \setminus \mathcal{A}$ is a simply connected manifold.

Consider a perturbation scheme $(M_\varepsilon, \hat{g}_\varepsilon, \{\psi_\varepsilon\})$ of class C^{n+1} around this background that inherits the orthogonally transitive stationary and axisymmetric action generated by $\{\xi, \eta\}$. By Proposition 4.5 there is a choice of gauge that preserves the differentiability (i.e. C^n for K_1 and C^{n-1} for K_2) such that

$$(6.2) \quad K_1 = K_{1ij}(x^c) dx^i dx^j + K_{1a6}(x^c) dx^a dx^b,$$

$$(6.3) \quad K_2 = K_{2ij}(x^c) dx^i dx^j + K_{2a6}(x^c) dx^a dx^b,$$

in the coordinates $\{t, r, \theta, \phi\}$ where the metric is given by (6.1) and $\xi = \partial_t, \eta = \partial_\phi$. The metric has the form (4.5) with $\{x^i\} = \{t, \phi\}, \{x^a\} = \{r, \theta\}$. Observe that the coordinates used in Section 5.1 correspond now to $\{x^I\} = \{t, r\}$ and $\{x^A\} = \{\theta, \phi\}$. We define for convenience the unit vector $n := -e^{-\lambda/2} \hat{n}$ outside the origin, where it is of class C^{n+1} by construction. In spherical coordinates we have $n = -e^{-\lambda/2} \partial_r$.

From now on we let $(M_\varepsilon, \hat{g}_\varepsilon, \{\psi_\varepsilon\})$ to denote the **maximal** perturbation scheme (of the given perturbation) where this holds. Our aim is to show that the perturbation tensors K_1 and K_2 , assumed to be C^n and C^{n-1} tensors respectively, can be rendered in the forms found in the literature, at the cost of (i) losing their differentiability by one outside the origin (we refer to the discussion of this point in the Introduction) but *keeping a crucial property of boundedness at the origin*, and (ii) restricting the gauge freedom, of course.

Proposition 6.2. *Let (M, g) be a static and spherically symmetric background, with g of class C^{n+1} , with $n \geq 2$, satisfying assumption H_1 . Let K_1 and K_2 be first and second order perturbation tensors of class C^n and C^{n+1} respectively, satisfying (6.2) and (6.3). Then, there exists gauge vectors V_1 and V_2 , that extend continuously to zero at \mathcal{C}_0 , such that the gauge transformed tensors K_1^Ψ and K_2^Ψ are respectively C^{n-1} and C^{n-2} outside the origin and satisfy*

$$(6.4) \quad K_1^\Psi = K_1^\Psi{}_{ij}(x^\epsilon) dx^i dx^j + K_1^\Psi{}_{ab}(x^\epsilon) dx^a dx^b,$$

$$(6.5) \quad K_2^\Psi = K_2^\Psi{}_{ij}(x^\epsilon) dx^i dx^j + K_2^\Psi{}_{ab}(x^\epsilon) dx^a dx^b,$$

on $M \setminus \mathcal{A}$ together with

$$(6.6) \quad K_1^\Psi{}_{AB} dx^A dx^B = 4k^{(1)} \mathcal{R}^2 g_{\mathbb{S}^2}, \quad K_2^\Psi{}_{AB} dx^A dx^B = 4k^{(2)} \mathcal{R}^2 g_{\mathbb{S}^2},$$

where the functions $k^{(1)}$ and $k^{(2)}$, invariant under η and ξ , are defined $M \setminus \mathcal{C}_0$ by

$$(6.7) \quad 4k^{(1)} = \frac{1}{\eta^2} K_1^\Psi(\eta, \eta), \quad 4k^{(2)} = \frac{1}{\eta^2} K_2^\Psi(\eta, \eta)$$

and satisfy $k^{(1)} \in C^n(M \setminus \mathcal{C}_0)$ and $k^{(2)} \in C^{n-1}(M \setminus \mathcal{C}_0)$, and are bounded near \mathcal{C}_0 . Moreover,

- 1) the 1-form $K_1^\Psi(\xi, \cdot)$ equals $K_1(\xi, \cdot)$ on $M \setminus \mathcal{C}_0$ and thus extends to a 1-form of class $C^n(M)$.
- 2) $K_1^\Psi(n, n)$ is $C^n(M \setminus \mathcal{C}_0)$ and bounded near \mathcal{C}_0 .
- 3) the 1-form $K_2^\Psi(\xi, \cdot)$ is of class $C^{n-1}(M \setminus \mathcal{C}_0)$ and bounded near \mathcal{C}_0 . It is given by

$$(6.8) \quad K_2^\Psi(\xi, \cdot) = K_2(\xi, \cdot) - 2\tilde{\alpha} \mathcal{L}_\iota K_1(\xi, \cdot).$$

In particular

$$(6.9) \quad \begin{aligned} K_2^\Psi(\xi, \eta) &= K_2(\xi, \eta) - 2\tilde{\alpha} \iota(K_1(\xi, \eta)) \\ &= K_2(\xi, \eta) - 2\tilde{\alpha} \left\{ \|x\|^2 \mathcal{P}_n + \iota(\Phi^{(n)}) \right\}, \end{aligned}$$

where $\mathcal{P}_n \in C^{n-1}(M \setminus \mathcal{C}_0)$ and bounded near \mathcal{C}_0 and $\Phi^{(n)}$ is $C^n(M)$ and $\Phi^{(n)} \in \mathfrak{o}(\|x\|^n)$.

- 4) $K_2^\Psi(n, n)$ is $C^{n-1}(M \setminus \mathcal{C}_0)$ and bounded near \mathcal{C}_0 .

5) *the functions*

$$(6.10) \quad q_1^\Psi = |x|^2 K_1^\Psi(\iota, \hat{n}) \quad \text{outside } \mathcal{C}_0, \quad q_1^\Psi = 0 \text{ on } \mathcal{C}_0$$

$$(6.11) \quad q_2^\Psi = |x|^2 K_2^\Psi(\iota, \hat{n}) \quad \text{outside } \mathcal{C}_0, \quad q_2^\Psi = 0 \text{ on } \mathcal{C}_0$$

are $C^{n-1}(M \setminus \mathcal{C}_0)$ and $C^{n-2}(M \setminus \mathcal{C}_0)$ respectively, and both $C^0(M)$ and $\mathfrak{o}(\|x\|)$, and take the forms given by (5.51) and (5.62) respectively.

Proof. The proof is based on Propositions 5.6 and 5.7. The vectors ξ, η, ι are smooth on $M \setminus \mathcal{C}_0$ (the first two actually everywhere). From the coordinate expressions $\xi = \partial_t, \eta = \partial_\phi, \iota = -\sin\theta\partial_\theta$, valid on a dense set of $M \setminus \mathcal{C}_0$, it follows immediately that $\{\xi, \eta, \iota\}$ commute with each other on $M \setminus \mathcal{C}_0$. It is also clear that $\xi(\eta^2) = 0$. Since, by assumption, K_1, K_2 satisfy $\mathcal{L}_\xi K_1 = \mathcal{L}_\xi K_2 = 0$, the Lie derivative along ξ of equation (5.54) yields $\eta^2 \xi(\iota(\tilde{\alpha})) = 0$ and therefore $\xi(\iota(\tilde{\alpha})) = 0$ outside the axis \mathcal{A} . Using $[\xi, \iota] = 0$, this is equivalent to $\iota(\xi(\tilde{\alpha})) = 0$ on $M \setminus \mathcal{A}$. This ensures that the solution $\tilde{\alpha}$ of (5.54) can be constructed such that it satisfies $\xi(\tilde{\alpha}) = 0$ outside the axis. Since $\tilde{\alpha} \in C^n(M \setminus \mathcal{C}_0)$, we actually have $\xi(\tilde{\alpha}) = 0$ everywhere on $M \setminus \mathcal{C}_0$. Similarly, the Lie derivative along ξ of (5.71) shows that \tilde{v} can be constructed so that it satisfies $\xi(\tilde{v}) = 0$ on $M \setminus \mathcal{C}_0$. We assume these choices from now on.

The block diagonal form of (6.2)-(6.3) is equivalent to

$$(6.12) \quad K_1(\iota, \eta) = 0, \quad K_1(\iota, \xi) = 0, \quad K_1(\eta, n) = 0, \quad K_1(n, \xi) = 0,$$

$$(6.13) \quad K_2(\iota, \eta) = 0, \quad K_2(\iota, \xi) = 0, \quad K_2(\eta, n) = 0, \quad K_2(n, \xi) = 0,$$

outside the axis. However, since K_1 and K_2 are $C^n(M)$ and $C^{n-1}(M)$ respectively, (6.12) and (6.13) also hold on $M \setminus \mathcal{C}_0$. Applying the gauge transformation discussed in Propositions 5.6, 5.7 and denoting the corresponding gauge transformed tensors as $K_1^\Psi = K_1^g, K_2^\Psi = K_2^g$, it follows directly from (5.47)-(5.50) (applied to $\nu_1 = \xi, \nu_2 = n$) that

$$(6.14) \quad K_1^\Psi(\iota, \eta) = 0, \quad K_1^\Psi(\iota, \xi) = 0, \quad K_1^\Psi(\eta, n) = 0, \quad K_1^\Psi(n, \xi) = 0.$$

Similarly, (5.58)-(5.61) yield, after using $[\iota, n] = 0$,

$$(6.15) \quad K_2^\Psi(\iota, \eta) = 0, \quad K_2^\Psi(\iota, \xi) = 0, \quad K_2^\Psi(\eta, n) = 0, \quad K_2^\Psi(n, \xi) = 0.$$

This proves in particular that the block diagonal form claimed in (6.4)-(6.5) holds.

Define $k^{(1)}, k^{(2)}$ as in (6.7). By virtue of Propositions 5.6 and 5.7 these functions are bounded near the centre and, respectively, $C^n(M \setminus \mathcal{C}_0)$ and $C^{n-1}(M \setminus \mathcal{C}_0)$. Using spherical coordinates $x^A = \{\theta, \phi\}$ and noting that $\iota = -\sin \theta \partial_\theta = -\frac{\|x\|}{|x|} \partial_\theta$, on $M \setminus \mathcal{A}$ we have

$$\begin{aligned} K_1^\Psi{}_{AB} dx^A dx^B &= K_1^\Psi(\partial_\theta, \partial_\theta) d\theta^2 + K_1^\Psi(\partial_\phi, \partial_\phi) d\phi^2 \\ &= \frac{|x|^2}{\|x\|^2} K_1^\Psi(\iota, \iota) d\theta^2 + K_1^\Psi(\eta, \eta) d\phi^2 \\ &= \frac{|x|^2}{\|x\|^2} K_1^\Psi(\eta, \eta) (d\theta^2 + \sin^2 \theta d\phi^2) = \frac{\mathcal{R}^2}{\eta^2} K_1^\Psi(\eta, \eta) g_{\mathbb{S}^2}, \end{aligned}$$

where the first equality is a consequence of $K_1^\Psi(\iota, \eta) = 0$, and the third follows from (5.45) (and that $|x|^2 / \|x\|^2 = \mathcal{R}^2 / \eta^2$). A similar calculation is valid for K_2^Ψ . This establishes (6.6).

We focus now on $K_1^\Psi(\xi, \cdot)$ and $K_2^\Psi(\xi, \cdot)$. For the first we apply (5.52) to $\xi = \partial_t$ and any Y to get

$$(6.16) \quad K_1^\Psi(\xi, Y) = K_1(\xi, Y) - \xi(\tilde{\alpha})\iota(Y_\parallel) = K_1(\xi, Y),$$

after using $\xi_\parallel = 0$ and $\xi(\tilde{\alpha}) = 0$. Thus, $K_1^\Psi(\xi, \cdot) = K_1(\xi, \cdot)$, and the first claim follows. At second order, (5.64) applied to ξ and ∂_{x^μ} gives, after taking into account $[\xi, \iota] = 0$, $\xi(\tilde{v}) = 0$, $\xi(\tilde{\alpha}) = 0$, $K_1(\iota, \xi) = 0$ and $M(\xi, \cdot) = 2K_1(\xi, \cdot)$ (by (6.16)),

$$(6.17) \quad K_2^\Psi(\xi, \partial_{x^\mu}) = K_2(\xi, \partial_{x^\mu}) - 2\tilde{\alpha}(\mathcal{L}_\iota K_1)(\xi, \partial_{x^\mu}).$$

The first term at the right is clearly $C^{n-1}(M)$. By Proposition 5.6, $\tilde{\alpha}$ is $C^{n-1}(M \setminus \mathcal{C}_0)$ and bounded near \mathcal{C}_0 , and thence also is the second term. In particular we have $K_2^\Psi(\xi, \eta) = K_2(\xi, \eta) - 2\tilde{\alpha}\iota(K_1(\xi, \eta))$. By expression (5.12) in Lemma 5.2 for $K = K_1$, $m = n$ and $u = t$ one has (we drop a superindex t to simplify the notation)

$$K_1(\xi, \eta) = \|x\|^2 \sum_{k=0}^{[n/2]-1} \|x\|^{2k} P_k(t, z) + \Phi^{(n)} := \|x\|^2 \tilde{\mathcal{P}} + \Phi^{(n)}$$

and hence

$$K_2^\Psi(\xi, \eta) = K_2(\xi, \eta) - 2\tilde{\alpha} \left\{ -2Y^3 \|x\|^2 \tilde{\mathcal{P}} + \|x\|^2 \iota(\tilde{\mathcal{P}}) + \iota(\Phi^{(n)}) \right\},$$

where

$$\iota(\tilde{\mathcal{P}}) = \sum_{k=0}^{[n/2]-1} \left\{ -2kY^3 \|x\|^{2k} P \right\} + \frac{\|x\|^2}{|x|} \partial_z \tilde{\mathcal{P}}$$

is clearly $C^{n-1}(M \setminus \mathcal{C}_0)$ and bounded near \mathcal{C}_0 . With the appropriate definition of \mathcal{P}_n , expression (6.9) and all the properties listed in item 3 follow. For $K_1^\Psi(n, n)$ and $K_2^\Psi(n, n)$, equations (5.50) and (5.61) with $\nu_1 = \nu_2 = n$ lead to

$$\begin{aligned} (6.18) \quad K_1^\Psi(n, n) &= K_1(n, n), \\ K_2^\Psi(n, n) &= K_2(n, n) - 2\tilde{\alpha}(\mathcal{L}_\iota K_1)(n, n) \\ &\quad + 2(|x|n(\tilde{\alpha}))^2 \frac{\eta^2}{|x|^2} - 4|x|n(\tilde{\alpha}) \frac{1}{|x|} K_1(\iota, n) \end{aligned}$$

where for the second we used $[n, \iota] = 0$. Lemma 5.2 ensures that $K_1(n, n)$ is $C^n(M \setminus \mathcal{C}_0)$ and bounded near \mathcal{C}_0 , so point 2 holds. Concerning the second order, all terms in (6.18) are clearly $C^{n-1}(M \setminus \mathcal{C}_0)$. Lemma 5.2 applied to K_1 , $\mathcal{L}_\iota K_1$, and K_2 , and recalling $e^{-\lambda}$ is bounded near \mathcal{C}_0 , ensures $K_2(n, n)$, $\mathcal{L}_\iota K_1(n, n)$ are bounded near \mathcal{C}_0 , and that $K_1(\iota, n)$ is $\mathfrak{o}(\|x\|)$. Therefore $|x|^{-1} K_1(\iota, n)$ is bounded by Lemma B.5. Since $|x|n(\tilde{\alpha})$ (by Proposition 5.6) and $\eta^2/|x|^2$ are also bounded, point 4 is proved.

Finally, the functions $|x|^2 K_1^\Psi(\hat{n}, \iota)$ and $|x|^2 K_2^\Psi(\hat{n}, \iota)$ are $C^{n-1}(M \setminus \mathcal{C}_0)$ and $C^{n-2}(M \setminus \mathcal{C}_0)$ by virtue of Propositions 5.6 and 5.7 respectively. Those propositions also provide the explicit forms of $q_1^\Psi = q_1^g$ and $q_2^\Psi = q_2^g$, and their behaviour near the axis and the origin as indicated. \square

6.1. Canonical form of the perturbations

By (6.4)-(6.5) and (6.6), the perturbation tensors K_1^Ψ , K_2^Ψ are determined by five functions each. Two of them, $k^{(1)}$ and $k^{(2)}$ have already been defined in the Proposition. Before entering into the main result of the paper we introduce the functions that will determine the remaining parts (four components respectively) of K_1^Ψ and K_2^Ψ .

Let us stress first the fact that given any 1-form \mathbf{X} satisfying $\mathbf{X}(\hat{n}) = \mathbf{X}(\iota) = 0$ we have $\mathbf{X}(\hat{\rho}) = 0$ (see (5.9)) and, as a consequence, the set of equations $\{-xF = \mathbf{X}(\partial_y), yF = \mathbf{X}(\partial_x)\}$ is compatible and defines a unique function F outside the axis $x = y = 0$. The 1-forms of the form $\mathbf{X} = K(\xi, \cdot)$ with K any of the tensors K_1 , K_2 , K_1^Ψ , K_2^Ψ satisfy that property. In the following we use $\xi = \partial_t$.

Regarding K_1^Ψ , and recalling point 1 of Proposition 6.2, the expressions

$$(6.19) \quad h^{(1)} := -\frac{1}{4}e^{-\nu}K_1^\Psi(\partial_t, \partial_t),$$

$$(6.20) \quad -x\chi\omega^{(1)} = K_1^\Psi(\partial_t, \partial_y), \quad y\chi\omega^{(1)} = K_1^\Psi(\partial_t, \partial_x),$$

define, respectively, $h^{(1)} \in C^n(M)$ and $\omega^{(1)} : M \setminus \mathcal{A} \rightarrow \mathbb{R}$. Combining (6.20) with $\eta = x\partial_y - y\partial_x$ yields

$$-\eta^2\omega^{(1)} = K_1^\Psi(\partial_t, \eta)$$

so that, in particular, $\omega^{(1)}$ is axially symmetric. Since the right-hand sides of (6.20) are $C^n(M)$ we may apply Lemma A.3 in Appendix A to conclude that ω extends to a $C^{n-1}(M)$ function.

The third function is $m^{(1)} \in C^n(M \setminus \mathcal{C}_0)$ defined by

$$(6.21) \quad \begin{aligned} m^{(1)} &:= \frac{1}{4} \left\{ K_1^\Psi \alpha + e^{-\nu} K_1^\Psi(\partial_t, \partial_t) - 8k^{(1)} \right\} \\ &= \frac{1}{4} \left\{ K_1^\Psi \alpha - 4h^{(1)} - 8k^{(1)} \right\}. \end{aligned}$$

Finally, we define $f^{(1)} : M \setminus \mathcal{C}_0 \rightarrow \mathbb{R}$ as

$$(6.22) \quad f^{(1)} := e^{-\lambda} \frac{1}{2\sqrt{\chi}} \frac{1}{|x|^3} \Upsilon_1 + \beta_1,$$

where β_1 is any radially symmetric $C^{n-1}(M \setminus \mathcal{C}_0)$ function bounded near \mathcal{C}_0 and Υ_1 any axially symmetric solution to the equation

$$(6.23) \quad \iota(\Upsilon_1) = q_1^\Psi$$

satisfying the outcome of Corollary B.11 for $m = n$. We will show below that this Corollary does in fact apply.

As regards to K_2^Ψ we analogously define $\{h^{(2)}, m^{(2)}\}$ on $M \setminus \mathcal{C}_0$ and $\omega^{(2)}$ on $M \setminus \mathcal{A}$ by

$$(6.24) \quad h^{(2)} := -\frac{1}{4}e^{-\nu} \left(K_2^\Psi(\partial_t, \partial_t) - 2\eta^2\omega^{(1)2} \right),$$

$$(6.25) \quad -x\chi\omega^{(2)} = K_2^\Psi(\partial_t, \partial_y), \quad y\chi\omega^{(2)} = K_2^\Psi(\partial_t, \partial_x),$$

$$(6.26) \quad m^{(2)} := \frac{1}{4} \left\{ K_2^\Psi \alpha + e^{-\nu} K_2^\Psi(\partial_t, \partial_t) - 8k^{(2)} \right\},$$

while we define $f^{(2)}$ on $M \setminus \mathcal{C}_0$ as

$$(6.27) \quad f^{(2)} := e^{-\lambda} \frac{1}{2\sqrt{\chi}} \frac{1}{|x|^3} \Upsilon_2 + \beta_2,$$

where β_2 is any radially symmetric $C^{n-2}(M \setminus \mathcal{C}_0)$ function bounded near \mathcal{C}_0 and Υ_2 is any axially symmetric solution to the equation

$$(6.28) \quad \iota(\Upsilon_2) = q_2^\Psi$$

satisfying the outcome of Corollary B.11 with $m = n - 1$. As before, we will show below that this Corollary may be applied. We emphasize that, unlike $h^{(1)}$ and $\omega^{(1)}$, we cannot guarantee that $h^{(2)}$ and $\omega^{(2)}$ can be extended differentiably to M . This is because $K_2^\Psi(\xi, \cdot)$ is not known to be $C^{n-1}(M)$ due to the presence of the last term in (6.8).

For later use we observe that (6.25) implies

$$(6.29) \quad -\eta^2 \omega^{(2)} = K_2^\Psi(\partial_t, \eta),$$

and that the functions $m^{(1)}$ and $m^{(2)}$ satisfy (on $M \setminus \mathcal{C}_0$)

$$(6.30) \quad m^{(1)} = \frac{1}{4} K_1^\Psi(n, n) = \frac{1}{4} e^{-\lambda} K_1^\Psi(\hat{n}, \hat{n}),$$

$$(6.31) \quad m^{(2)} = \frac{1}{4} K_2^\Psi(n, n) = \frac{1}{4} e^{-\lambda} K_2^\Psi(\hat{n}, \hat{n}).$$

Proposition 6.2 already determines the regularity restrictions on the functions $k^{(1)}$ and $k^{(2)}$ implied by the differentiability of K_1 and K_2 . The next theorem, which is the main result in this work, combines Propositions 4.5 and 6.2 in order to, firstly, establish rigorously that first and second order perturbation tensors of finite differentiability and preserving the axial symmetry admit a gauge transformation that puts them in a canonical form and, secondly, to provide detailed information on the differentiability and regularity properties of suitably defined function components.

Theorem 6.3 (Main Theorem). *Let (M, g) be a static and spherically symmetric background satisfying assumption H_1 , with g of class C^{n+1} with $n \geq 2$, given in spherical coordinates by (6.1). Let us be given first and second order perturbation tensors K_1 and K_2 of class $C^n(M)$ and $C^{n-1}(M)$ respectively satisfying (6.2) and (6.3), where $\{x^i\} = \{t, \phi\}$ and $\{x^\alpha\} = \{r, \theta\}$ and $\xi = \partial_t, \eta = \partial_\phi$.*

Then, there exists gauge vectors V_1 and V_2 , that extend continuously to zero at \mathcal{C}_0 , such that the gauge transformed tensors K_1^Ψ and K_2^Ψ are of class $C^{n-1}(M \setminus \mathcal{C}_0)$ and $C^{n-2}(M \setminus \mathcal{C}_0)$ respectively, and define the functions $\{h^{(1)}, m^{(1)}, k^{(1)}, \omega^{(1)}, f^{(1)}, h^{(2)}, m^{(2)}, k^{(2)}, \omega^{(2)}, f^{(2)}\}$ as above that satisfy the following properties:

- (a.1) $h^{(1)}$ is $C^n(M)$,
- (a.2) $\omega^{(1)}$ is $C^{n-1}(M)$,
- (a.3) the vector field $\omega^{(1)}\eta$ is $C^n(M \setminus \mathcal{C}_0)$,
- (a.4) $m^{(1)}$ and $k^{(1)}$ are $C^n(M \setminus \mathcal{C}_0)$ and bounded near \mathcal{C}_0 ,
- (a.5) $f^{(1)}$ is $C^{n-1}(M \setminus \mathcal{C}_0)$, bounded near \mathcal{C}_0 , $C^n(S_r)$ on all spheres S_r , and moreover $\partial_\theta f^{(1)}$ is C^{n-1} outside the axis and extends continuously to $\mathcal{A} \setminus \mathcal{C}_0$, where it vanishes, and $\hat{n}(f^{(1)})$ and $\partial_t f^{(1)}$ are $C^{n-1}(S_r)$ on all spheres S_r ,
- (b.1) $h^{(2)}$ is $C^{n-1}(M \setminus \mathcal{C}_0)$ and bounded near \mathcal{C}_0 ,
- (b.2) $\omega^{(2)}$ is $C^{n-2}(M \setminus \mathcal{C}_0)$ and bounded near \mathcal{C}_0 ,
- (b.3) the vector field $\omega^{(2)}\eta$ is $C^{n-1}(M \setminus \mathcal{C}_0)$,
- (b.4) $m^{(2)}$ and $k^{(2)}$ are $C^{n-1}(M \setminus \mathcal{C}_0)$ and bounded near \mathcal{C}_0 ,
- (b.5) $f^{(2)}$ is $C^{n-2}(M \setminus \mathcal{C}_0)$, bounded near \mathcal{C}_0 , $C^{n-1}(S_r)$ on all spheres S_r and moreover $\partial_\theta f^{(2)}$ is C^{n-2} outside the axis and extends continuously to $\mathcal{A} \setminus \mathcal{C}_0$, where it vanishes, and $\hat{n}(f^{(2)})$ and $\partial_t f^{(2)}$ are $C^{n-2}(S_r)$ on all spheres S_r .

In terms of these functions K_1^Ψ and K_2^Ψ take the following form on $M \setminus \mathcal{A}$

$$(6.32) \quad K_1^\Psi = -4e^{\nu(r)}h^{(1)}(r, \theta)dt^2 - 2\omega^{(1)}(r, \theta)\mathcal{R}^2(r)\sin^2\theta dtd\phi \\ + 4e^{\lambda(r)}m^{(1)}(r, \theta)dr^2 + 4k^{(1)}(r, \theta)\mathcal{R}^2(r)(d\theta^2 + \sin^2\theta d\phi^2) \\ + 4e^{\lambda(r)}\partial_\theta f^{(1)}(r, \theta)\mathcal{R}(r)drd\theta,$$

$$(6.33) \quad K_2^\Psi = \left(-4e^{\nu(r)}h^{(2)}(r, \theta) + 2\omega^{(1)^2}(r, \theta)\mathcal{R}^2(r)\sin^2\theta\right) dt^2 \\ - 2\omega^{(2)}(r, \theta)\mathcal{R}^2(r)\sin^2\theta dtd\phi + 4e^{\lambda(r)}m^{(2)}(r, \theta)dr^2 \\ + 4k^{(2)}(r, \theta)\mathcal{R}^2(r)(d\theta^2 + \sin^2\theta d\phi^2) \\ + 4e^{\lambda(r)}\partial_\theta f^{(2)}(r, \theta)\mathcal{R}(r)drd\theta.$$

Proof. Expressions (6.32) and (6.33) follow directly from Proposition 6.2 and the definitions (6.19)-(6.25) after taking into account that $\|x\|^2 d\phi = |x|^2 \sin^2\theta d\phi = xdy - ydx$ and $\mathcal{R}^2 = \chi|x|^2$ and $\eta^2 = \chi\|x\|^2$.

It remains to analyse the differentiability and boundedness near \mathcal{C}_0 of the various functions. Points (a.1) and (a.2) have already been proved when $h^{(1)}$ and $\omega^{(1)}$ were introduced in (6.19) and (6.20). Point (a.3) follows immediately from

$$\omega\eta = - \left(K_1^\Psi(\partial_t, \cdot) + 4e^\nu h^{(1)} dt \right)$$

since the right-hand side is a $C^n(M)$ 1-form (by point 1 in Proposition 6.2). For point (a.4), the statement on $k^{(1)}$ has been established in Proposition 6.2. By point 2 of that proposition the right-hand side of (6.30), and thus $m^{(1)}$, also is.

Establishing (a.5) needs some additional work. $f^{(1)}$ being defined by (6.22), we directly have from (6.23) that

$$\partial_\theta f^{(1)} = - \frac{1}{\|x\|} e^{-\lambda} \frac{1}{2\sqrt{\chi}} \frac{1}{|x|^2} q_1^\Psi$$

outside the axis. Given that q_1^Ψ is $C^{n-1}(\mathcal{U} \setminus \mathcal{C}_0) \cap C^0(\mathcal{U})$ and $\mathbf{o}(\|x\|)$ (see Proposition 6.2), $\partial_\theta f^{(1)} \in C^{n-1}(\mathcal{U} \setminus \mathcal{A})$ and can be extended continuously to $\mathcal{A} \setminus \mathcal{C}_0$, where it vanishes. Regarding $f^{(1)}$ itself, we must analyse the solutions Υ_1 of equation (6.23). By Proposition 6.2, q_1^Ψ is given by (5.51), so the right-hand side of (6.23) matches the right-hand side of (B.28) with $b = \min\{2k + l\} = 3$, $c = \min_{\nu'}\{2k' + l'\} = 2$, and $m = n \geq 2$, so that

$$3 \leq b \leq c + n,$$

and Corollary B.11 ensures there exists a solution $\gamma_0 = \Upsilon_1$ which is $C^{n-1}(M \setminus \mathcal{C}_0) \cap C^0(M)$, $C^n(S_r)$ on all S_r , and $O(|x|^3)$, and moreover $\hat{n}(\Upsilon_1)$ and $\partial_t \Upsilon_1$ are $C^{n-1}(S_r)$ on all S_r . Since $\chi \in C^{n+1}(M)$ and nowhere zero, it follows that $f^{(1)}$ defined by (6.22) has the properties listed in item (a.5).

Next, we consider the second order quantities. The definitions of $h^{(2)}$ and $\omega^{(2)}$ imply

$$(6.34) \quad K_2^\Psi(\partial_t, \cdot) = (-4e^\nu h^{(2)} + 2\eta^2 \omega^{(1)}) dt - \omega^{(2)} \eta^2 d\phi.$$

Point 3 in Proposition 6.2 ensures that the left-hand side of (6.34) is $C^{n-1}(M \setminus \mathcal{C}_0)$ and bounded near \mathcal{C}_0 . Using that $\omega^{(1)}$ is $C^{n-1}(M)$, (b.1) follows after contraction with ∂_t .

Expressions (6.25) define an axially symmetric function $\omega^{(2)} : M \setminus \mathcal{A} \rightarrow \mathbb{R}$. Since the right-hand sides of (6.25) are $C^{n-1}(M \setminus \mathcal{C}_0)$, Lemma A.3 (with

n replaced by $n - 1$) implies that $\omega^{(2)} \in C^{n-2}(M \setminus \mathcal{C}_0)$. This proves the differentiability claim in (b.2). To show boundedness near the origin we use an auxiliary function \mathcal{W}_0 defined in terms of the original second order perturbation K_2 (before it has been gauge transformed into K_2^Ψ). Specifically, \mathcal{W}_0 is defined by

$$(6.35) \quad -x\chi\mathcal{W}_0 = K_2(\partial_t, \partial_y), \quad y\chi\mathcal{W}_0 = K_2(\partial_t, \partial_x).$$

Since K_2 is of class $C^{n-1}(M)$, it follows from Lemma A.3 that \mathcal{W}_0 is $C^{n-2}(M)$. It is also clear that (6.35) imply

$$(6.36) \quad -\eta^2\mathcal{W}_0 = K_2(\partial_t, \eta).$$

The function $\omega^{(2)}$ can be written in terms of \mathcal{W}_0 . Inserting (6.36) and (6.9) into (6.29) yields

$$-\eta^2\omega^{(2)} = -\eta^2\mathcal{W}_0 - 2\tilde{\alpha} \left\{ \|x\|^2\mathcal{P}_n + \iota(\Phi^{(n)}) \right\}$$

with $\mathcal{P}_n \in C^{n-1}(M \setminus \mathcal{C}_0)$ and bounded near \mathcal{C}_0 , and $\Phi^{(n)} \in C^n(M)$ and $\mathbf{o}(\|x\|^n)$. Therefore, outside the axis we have

$$\omega^{(2)} = \mathcal{W}_0 + 2\tilde{\alpha} \left\{ \frac{\|x\|^2}{\eta^2}\mathcal{P}_n + \frac{1}{\eta^2}\iota(\Phi^{(n)}) \right\}.$$

This proves that $\omega^{(2)}$ is bounded near the origin because $\mathcal{W}_0 \in C^{n-2}(M)$, the functions $\tilde{\alpha}$ and $\|x\|^2\mathcal{P}_n/\eta^2 = \mathcal{P}_n/\chi$ are both bounded near \mathcal{C}_0 and, finally, $\iota(\Phi^{(n)})/\eta^2$ vanishes at $\|x\| = 0$ by virtue of Lemma B.8, which ensures $\iota(\Phi^{(n)}) \in \mathbf{o}(\|x\|^2)$ given that $n \geq 2$. With this we have established point (b.2).

For (b.3) we simply note that (6.34) can be rewritten as

$$\omega^{(2)}\boldsymbol{\eta} = - \left(K_2^\Psi(\partial_t, \cdot) + (4e^\nu h^{(2)} - 2\eta^2\omega^{(1)})dt \right)$$

and we have already seen that the right-hand side is a C^{n-1} one-form outside \mathcal{C}_0 .

As for (b.4), Proposition 6.2 already ensures $k^{(2)}$ is $C^{n-1}(M \setminus \mathcal{C}_0)$ and bounded near \mathcal{C}_0 , and point 4 of the same proposition states that the right-hand side of (6.31), and thus $m^{(2)}$, also is.

We finally focus on $f^{(2)}$, defined by (6.27). Directly from (6.28) we can write, outside the axis,

$$\partial_\theta f^{(2)} = -\frac{1}{\|x\|} e^{-\lambda} \frac{1}{2\sqrt{\chi}} \frac{1}{|x|^2} q_2^\Psi.$$

Given that q_2^Ψ is $C^{n-2}(\mathcal{U} \setminus \mathcal{C}_0) \cap C^0(\mathcal{U})$ and $\mathbf{o}(\|x\|)$, as stated by Proposition 6.2, then $\partial_\theta f^{(2)} \in C^{n-2}(\mathcal{U} \setminus \mathcal{A})$ and can be extended continuously to $\mathcal{A} \setminus \mathcal{C}_0$, where it vanishes. Regarding $f^{(2)}$, analogously as for $f^{(1)}$, we must analyse the solutions Υ_2 of equation (6.28). This time the form of the inhomogeneous term of the equation q_2^Ψ , given by (5.62), renders (6.28) explicitly as

$$\iota(\Upsilon_2 + 2\tilde{\alpha}q_1) = q_2 + \|x\|^2 \Gamma_2^{(n-2)}.$$

Given the form of q_2 (c.f. the right hand side of (5.82)) and the properties of $\Gamma_2^{(n-2)}$ in Proposition 5.7, the right-hand side of the equation matches the right-hand side of (B.28) with $b = \min_{\nu} \{2k + l\} = 3$, $c = \min_{\nu'} \{2k' + l'\} = 2$, and $m = n - 1 \geq 1$, so that

$$3 \leq b \leq c + n - 1$$

and Corollary B.11 shows that there exists a solution $\gamma_0 = \Upsilon_2 + 2\tilde{\alpha}q_1$ which is $C^{n-2}(M \setminus \mathcal{C}_0) \cap C^0(M)$, $C^{n-1}(S_r)$ on all S_r , and $O(|x|^3)$, and moreover $\hat{n}(\Upsilon_2 + 2\tilde{\alpha}q_1)$ and $\partial_t(\Upsilon_2 + 2\tilde{\alpha}q_1)$ are $C^{n-2}(S_r)$ on all S_r . Since $\tilde{\alpha}q_1$ is $C^{n-1}(M \setminus \mathcal{C}_0) \cap C^0(M)$ and $q_1 \in O(|x|^3)$ (the latter because of Lemma 5.2 with $K = K_1$ and $q = q_1$), we obtain $\Upsilon_2 \in C^{n-2}(M \setminus \mathcal{C}_0) \cap C^0(M)$ and $O(|x|^3)$, and $\hat{n}(\Upsilon_2)$ and $\partial_t \Upsilon_2$ are $C^{n-2}(S_r)$ on all S_r . Therefore $f^{(2)}$ satisfies the properties listed in point (b.5). \square

The previous theorem makes the hypotheses that K_1 and K_2 are of the form (6.2) and (6.3). This assumption is well-justified in the present setup because it holds automatically for any perturbation scheme that inherits an orthogonally transitive, stationary and axially symmetric action, as proved in Proposition 4.5. We state the corresponding result as a corollary, where we also add the property that the gauge vectors V_1 and V_2 whose existence has been proved are axially symmetric, tangent to S_r and orthogonal to η , and extend continuously to zero at the centre (see Propositions 5.6 and 5.7).

Corollary 6.4. *Let (M, g) be a static and spherically symmetric background satisfying assumption H_1 , with g of class C^{n+1} , with $n \geq 2$, given in spherical coordinates by (6.1). Let us be given a C^{n+1} maximal perturbation scheme $(M_\varepsilon, \hat{g}_\varepsilon, \{\psi_\varepsilon\})$ inheriting the orthogonally transitive stationary*

and axisymmetric action generated by $\{\xi = \partial_t, \eta = \partial_\phi\}$. Then, the outcome of Theorem 6.3 holds.

Moreover, the gauge vectors V_1 and V_2 whose existence has been granted and that transform any perturbation tensors K_1, K_2 in this perturbation scheme into the form given by Theorem 6.3, commute with η , are tangent to S_r and orthogonal to η , and extend to zero at \mathcal{C}_0 .

This corollary ensures that there is a change of gauge that takes any (orthogonally transitive) stationary and asymmetric perturbation (to second order) of a static and spherically symmetric configuration to the usual forms (6.32)-(6.33), at the expense of losing one derivative and deteriorating the regularity at the origin. In the following we discuss the gauge properties compatible with (6.32)-(6.33).

6.2. On uniqueness of gauges

So far all the results have focused on the existence of the gauge vectors V_1 and V_2 that take from (6.2) and (6.3) to the canonical form of Theorem 6.3. In this subsection we discuss their uniqueness properties for fixed transformed tensors K_1^Ψ and K_2^Ψ . We start with a general result based on the symmetries of the background and of the perturbation tensors.

Lemma 6.5. *Consider a static and spherically symmetric background admitting no further local isometries and denote by ξ its static and η one of its axial Killing vectors. Let K_1 and K_2 be C^2 perturbation tensors invariant under ξ and η on this background. If K'_1 and K'_2 are obtained from K_1 and K_2 by a gauge transformation defined by axially symmetric gauge vectors V_1 and V_2 (i.e. that commute with η), then the most general such gauge vectors are given by*

$$\tilde{V}_1 = V_1 + A_1\xi + B_2\eta, \quad \tilde{V}_2 = V_2 + A_2\xi + B_2\eta, \quad A_1, A_2, B_1, B_2 \in \mathbb{R}.$$

Proof. We start with the first order. From (2.5) and, by assumption,

$$K'_1 = K_1 + \mathcal{L}_{V_1}g, \quad K'_2 = K_2 + \mathcal{L}_{V_2}g.$$

This leads to $\mathcal{L}_{V_1 - \tilde{V}_1}g = 0$ and therefore $\tilde{V}_1 - V_1$ is any Killing ζ_1 of the background. By assumption ζ_1 commutes with η and given that the Killing algebra of the background is $\mathbb{R} \oplus so(3)$ it must be $\zeta_1 = A_1\xi + B_1\eta$ with

$A_1, B_1 \in \mathbb{R}$. At second order we have, from (2.6) using (2.5), and by assumption

$$K'_2 = K_2 + \mathcal{L}_{V_2}g + \mathcal{L}_{V_1}(K_1^g + K_1), \quad K'_2 = K_2 + \mathcal{L}_{\tilde{V}_2}g + \mathcal{L}_{\tilde{V}_1}(K_1^g + K_1).$$

Therefore $0 = \mathcal{L}_{V_2 - \tilde{V}_2}g - \mathcal{L}_{\zeta_1}(K'_1 + K_1) = \mathcal{L}_{V_2 - \tilde{V}_2}g$, where in the second equality we use that K'_1 and K_1 are invariant under ξ and η . The claim follows. □

This result combined with Lemma 3.5 allows us to complement Corollary 6.4 with the following uniqueness result.

Remark 6.6. In the setup of Corollary 6.4, if the background admits no further local isometries and the perturbation scheme $(M_\varepsilon, \hat{g}_\varepsilon, \{\psi_\varepsilon\})$ is restricted so that the inherited axial Killing vector $\hat{\eta}_\varepsilon = d\psi_\varepsilon(\eta)$ is independent of the choice $\psi_\varepsilon \in \{\psi_\varepsilon\}$, then the gauge vectors V_1 and V_2 are both unique up to the addition of a Killing vector of the background that commutes with η . We emphasize that this condition on the perturbation scheme is no restriction at all if $\hat{g}_\varepsilon, \varepsilon \neq 0$, admits only one axial symmetry.

6.3. Gauge freedom

In this subsection we investigate the gauge freedom to second order that respects the *form* of the first and second order perturbation tensors, as given in the main theorem. Our aim is to find the most general gauge transformation respecting this form under the additional condition that the first order perturbation tensor takes a very special simple form (corresponding to a pure rotation). This is interesting for two reasons. Firstly, because in the setup of [15], this form of the first order perturbation tensor will in fact be a consequence of the gravitational field equations for a rotating fluid ball. And secondly because, as we will discuss later in more detail, our result will include as a particular case the most general gauge transformation that respects the structure of the canonical form of a general perturbation tensor *at first order*.

We start with a general lemma concerning gauge vectors and symmetries.

Lemma 6.7. *Let (M, g) be a spacetime with Killing algebra \mathbb{A} . Let K be a symmetric two-covariant tensor invariant under a subalgebra $\mathbb{A}_0 \subset \mathbb{A}$. Then the tensor $K' := K + \mathcal{L}_s g$, with $s \in \mathfrak{X}(M)$ is also invariant under \mathbb{A}_0 if and*

only if

$$(6.37) \quad [\zeta, s] \in \mathbb{A} \quad \forall \zeta \in \mathbb{A}_0.$$

Assume now $\mathbb{A} = \text{span}\{\xi\} \oplus \text{span}\{\eta_a\}$ with $\{\eta_a\}$ a basis of an $so(3)$ Killing algebra and $[\xi, \eta_a] = 0$, $a = 1, 2, 3$. Let $\mathbb{A}_0 = \text{span}\{\xi, \eta\}$ with $\eta := \eta_1$. Then there exists $\widehat{\eta} \in so(3)$ such that $s = \widehat{s} + \widehat{\eta}$ with \widehat{s} satisfying

$$(6.38) \quad [\xi, \widehat{s}] = C\eta + D\xi, \quad C, D \in \mathbb{R},$$

$$(6.39) \quad [\eta, \widehat{s}] = A\eta + B\xi, \quad A, B \in \mathbb{R}.$$

If, in addition, the orbits of η are closed, then $A = B = 0$.

Proof. Taking the Lie derivative of K' along $\zeta \in \mathbb{A}_0$ and imposing invariance

$$0 = \mathcal{L}_\zeta K' = \mathcal{L}_\zeta K + \mathcal{L}_\zeta \mathcal{L}_s g = \mathcal{L}_{[\zeta, s]} g + \mathcal{L}_s \mathcal{L}_\zeta g = \mathcal{L}_{[\zeta, s]} g$$

so $[\zeta, s]$ is a Killing vector field and (6.37) is established. For the second statement recall that the structure constants of $so(3)$ are

$$[\eta_a, \eta_b] = \epsilon_{ab}{}^c \eta_c \quad a, b, c = 1, 2, 3$$

where $\epsilon_{ab}{}^c$ is the antisymmetric symbol. From (6.37) we have $[\eta, s] = A^a \eta_a + B\xi$ and $[\xi, s] = C^a \eta_a + D\xi$, $A^a, B, C^a, D \in \mathbb{R}$. Commuting ξ with $[\eta, s]$ and using the Jacobi identity

$$\begin{aligned} 0 &= [\xi, A^a \eta_a + B\xi] = [\xi, [\eta, s]] \\ &= [[\xi, \eta], s] + [\eta, [\xi, s]] = [\eta, C^a \eta_a + D\xi] = C^a \epsilon_{1a}{}^b \eta_b, \end{aligned}$$

which is equivalent to $C^2 = C^3 = 0$. Define $\widehat{\eta} = \epsilon_{1c}{}^d A^d \eta_c$ and

$$\widehat{s} := s - \widehat{\eta}.$$

Setting $C := C^1$, (6.38) follows. Moreover,

$$\begin{aligned} [\eta, \widehat{s}] &= [\eta, s - \epsilon_{1c}{}^d A^d \eta_c] \\ &= B\xi + A^a \eta_a - \epsilon_{1c}{}^d A^d \epsilon_{1c}{}^a \eta_a = B\xi + A^a \eta_a - (\delta_d^a - \delta_1^a \delta_{d1}) A^d \eta_a \\ &= B\xi + A^a \eta_a - A^a \eta_a + \eta_1 A^1 = B\xi + A^1 \eta_1, \end{aligned}$$

which is (6.39) after setting $A := A^1$.

Assume now that the orbits of η are closed⁴ (we stay away from the axis). Since η , ξ and \widehat{s} form a 3-dimensional algebra \mathbb{A}_3 determined by (6.38)-(6.39) plus an Abelian subalgebra for $\{\eta, \xi\}$, the argument in Section 3 in [2]⁵ shows that \widehat{s} must also commute with the cyclic η . Therefore (6.39) must hold with $A = B = 0$. \square

Remark 6.8. The results (6.37), (6.38) and (6.39) are purely local, so that they apply also to spacetimes admitting no isometry group action (e.g. because only a portions of the spacetimes is considered).

Lemma 6.7 and the last argument of its proof also implies the following Corollary.

Corollary 6.9. *Let (M, g) be a spacetime admitting a two-dimensional Abelian Killing algebra \mathbb{A} . Write $\mathbb{A} = \text{span}\{\xi, \eta\}$ and assume that η has closed orbits. Then $\mathcal{L}_s g$ is invariant under \mathbb{A} if and only if*

$$(6.40) \quad [\eta, s] = 0,$$

$$(6.41) \quad [\xi, s] = C\eta + D\xi, \quad C, D \in \mathbb{R}.$$

This result may have implications for studying gauge invariance properties of perturbations in general stationary and axially symmetric spacetimes.

From now on, however, we restrict ourselves to static and spherically symmetric spacetimes as defined in Definition 6.1 and Assumption H_1 with $n \geq 0$, and assume we have selected a Killing vector η in the $so(3)$ Killing algebra and have chosen spherical coordinates adapted to η , so that, away from the axis of η , the metric takes the form (6.1). In particular, for the rest of the section the functions $\nu, \lambda, \mathcal{R}$ will be functions of only one variable r . For simplicity we relax our convention of writing the argument explicitly, i.e. $\nu(r)$, etc. since no confusion will arise.

Define \widetilde{s} by

$$(6.42) \quad \widehat{s} = Ct\eta + Dt\xi + \widetilde{s},$$

and note that, from (6.38)-(6.39) with $A = B = 0$,

$$(6.43) \quad [\eta, \widetilde{s}] = [\xi, \widetilde{s}] = 0.$$

⁴Note that this condition is automatic when suitable global conditions, such as e.g. assumption H_1 , are imposed.

⁵As noted in the reference, although the text refers to Killing vectors, the fact that the corresponding local group of transformations are isometries is not used in the proof.

The difference tensor $K' - K = \mathcal{L}_s g$ has the form

$$\begin{aligned}
 (6.44) \quad K' - K &= \mathcal{L}_s g = \mathcal{L}_{\bar{s}} g = \mathcal{L}_{Ct\eta + Dt\xi + \bar{s}} g \\
 &= C(dt \otimes \boldsymbol{\eta} + \boldsymbol{\eta} \otimes dt) + D(dt \otimes \boldsymbol{\xi} + \boldsymbol{\xi} \otimes dt) + \mathcal{L}_{\bar{s}} g \\
 &= -2De^\nu dt^2 + 2C\mathcal{R}^2 \sin^2 \theta dt d\phi + \mathcal{L}_{\bar{s}} g.
 \end{aligned}$$

The form of \bar{s} can be restricted further under additional assumptions on the form of $K - K'$, i.e. $\mathcal{L}_s g$.

Proposition 6.10. *Let (M, g) be a static and spherically symmetric space-time satisfying assumption H_1 for $n \geq 0$, with a selection of axial Killing vector field η and s a vector field on M . Assume*

- (i) $\mathcal{L}_s g$ is invariant under $\text{span}\{\xi, \eta\}$.
- (ii) $\mathcal{L}_s g$ satisfies, in the coordinates $\{t, \phi, r, \theta\}$,

$$(\mathcal{L}_s g)_{ia} = 0$$

where $\{x^i\} = \{t, \phi\}$ and $\{x^a\} = \{r, \theta\}$.

(A) Then

$$(6.45) \quad s = Ct\eta + Dt\xi + \bar{s} + \zeta \quad C, D \in \mathbb{R}$$

where $\bar{s} = \bar{s}^a(x^b)\partial_{x^a}$ and ζ is any Killing vector field of g .

(B) If, in addition to (i) and (ii), $\mathcal{L}_s g$ has $x^A = \{\theta, \phi\}$ components of the form

$$(6.46) \quad (\mathcal{L}_s g)_{AB} dx^A dx^B = W(x^a) (d\theta^2 + \sin^2 \theta d\phi^2)$$

for some function $W(x^a)$, then there exists $\mathcal{Y}(r, \theta)$ and $\alpha(r)$ such that

$$(6.47) \quad \bar{s} = 2\mathcal{Y}(r, \theta)\partial_r + 2\alpha(r) \sin \theta \partial_\theta, \quad \text{and}$$

$$\begin{aligned}
 (6.48) \quad \mathcal{L}_s g &= -2e^\nu (\mathcal{Y}\nu_{,r} + D) dt^2 + 2C\mathcal{R}^2 \sin^2 \theta dt d\phi \\
 &\quad + 4e^\lambda \left(\mathcal{Y}_{,r} + \frac{1}{2}\mathcal{Y}\lambda_{,r} \right) dr^2 \\
 &\quad + 4\mathcal{R}^2 \left(\mathcal{Y} \frac{\mathcal{R}_{,r}}{\mathcal{R}} + \alpha(r) \cos \theta \right) (d\theta^2 + \sin^2 \theta d\phi^2) \\
 &\quad + 4\mathcal{R}e^\lambda \left(\frac{\mathcal{Y}_{,\theta}}{\mathcal{R}} + \mathcal{R}e^{-\lambda} \alpha_{,r} \sin \theta \right) dr d\theta.
 \end{aligned}$$

(C) If, in addition to (i) and (ii), $\mathcal{R}_{,r}$ and $\nu_{,r}$ are not zero on dense subsets and $\mathcal{L}_s g$ has only component in $\{t, \phi\}$, i.e. exists $Z(x^a)$ such that

$$\mathcal{L}_s g = 2Z(x^a)\mathcal{R}^2 \sin^2 \theta dt d\phi,$$

then

$$(6.49) \quad s = Ct\eta + \zeta, \quad \text{and} \quad \mathcal{L}_s g = 2C\mathcal{R}^2 \sin^2 \theta dt d\phi, \quad \text{i.e.} \quad Z(x^a) = C.$$

Proof. We have already shown, from Lemma 6.7 and (6.42), that $s = \hat{\eta} + Ct\eta + Dt\xi + \tilde{s}$ for some \tilde{s} satisfying (6.43). This imposes that the components \tilde{s}^α of \tilde{s} only depend on x^b . Moreover, (6.44) implies $\mathcal{L}_{\tilde{s}}g(\partial_{x^i}, \partial_{x^a}) = 0$ due to assumption (ii). Explicitly

$$0 = \tilde{s}^\mu \partial_\mu g_{ia} + g_{ij} \partial_a \tilde{s}^j + g_{ab} \partial_i \tilde{s}^b = g_{ij} \partial_a \tilde{s}^j \quad \implies \quad \partial_a \tilde{s}^j = 0,$$

and therefore \tilde{s}^j are in fact constants. Thus, there exist constants a, b such that $\tilde{s} = a\eta + b\xi + \bar{s}$ with $\bar{s} = \bar{s}^a(x^b)\partial_{x^a}$. Defining $\zeta = \hat{\eta} + a\eta + b\xi$, expression (6.45) follows.

In case (B), i.e. under the assumption (6.46), we need to impose that $(\mathcal{L}_{\bar{s}}g)_{\phi\phi} - \sin^2 \theta (\mathcal{L}_{\bar{s}}g)_{\theta\theta} = 0$. Explicitly

$$0 = \bar{s}^a \partial_a g_{\phi\phi} - \sin^2 \theta \left(\bar{s}^a \partial_a g_{\theta\theta} + 2g_{\theta\theta} \partial_\theta \bar{s}^\theta \right) = 2\mathcal{R}^2 \sin^2 \theta \left(\frac{\cos \theta}{\sin \theta} \bar{s}^\theta - \partial_\theta \bar{s}^\theta \right).$$

Integrating, there exists $\alpha(r)$ such that $\bar{s}^\theta = 2\alpha(r) \sin \theta$. Letting $2\mathcal{Y}(r, \theta) := \bar{s}^r$ we obtain (6.47). Moreover, a direct computation gives

$$\begin{aligned} \mathcal{L}_{\bar{s}}g &= -2\mathcal{Y}e^\nu \nu_{,r} dt^2 + 4e^\lambda \left(\mathcal{Y}_{,r} + \frac{1}{2}\mathcal{Y}\lambda_{,r} \right) dr^2 \\ &\quad + 4\mathcal{R}^2 \left(\mathcal{Y} \frac{\mathcal{R}_{,r}}{\mathcal{R}} + \alpha(r) \cos \theta \right) (d\theta^2 + \sin^2 \theta d\phi^2) \\ &\quad + 4\mathcal{R}e^\lambda \left(\frac{\mathcal{Y}_{,\theta}}{\mathcal{R}} + \mathcal{R}e^{-\lambda} \alpha_{,r}(r) \sin \theta \right) dr d\theta, \end{aligned}$$

which inserted into (6.44) yields (6.48).

Case (C) is obviously a particular case of (B), so (6.47) and (6.48) hold. This already implies that $Z = C$. To show the rest of (6.49) we use the fact that the coefficients in (6.48) all vanish, except the $\{t, \phi\}$. The dr^2

component restricts $\mathcal{Y}(r, \theta)$ to be

$$\mathcal{Y} = e^{-\lambda/2}\Theta(\theta)$$

for some function $\Theta(\theta)$. The $d\theta^2$ component forces, after using that $\mathcal{R}_{,r}$ is non-zero on a dense set, that

$$\alpha(r) = \frac{\alpha_0 \mathcal{R}_{,r} e^{-\lambda/2}}{\mathcal{R}}, \quad \mathcal{Y} = -\alpha_0 e^{-\lambda/2} \cos \theta, \quad \alpha_0 \in \mathbb{R}.$$

The dt^2 then gives, using that $\nu_{,r}$ is non-zero almost everywhere, $\alpha_0 = 0$ and $D = 0$, and (6.49) follows. \square

We have now the ingredients to analyse in detail the complete gauge freedom that respects the form of the perturbation tensors achieved in our main Theorem 6.3 assuming that the first order perturbation tensor has the specific form given by (6.50) below. As already said, this assumption turns out not to be restrictive at all in the setting of [15] where slowly rotating fluid balls with fixed central pressure are considered.

Even more, the next proposition determines the most general gauge transformation that leaves invariant the form of a *general* first order perturbation tensor in canonical form. Indeed, the gauge transformation law of second order perturbation tensors reduces to the first order one when K_1 is identically zero (see (2.5)-(2.6)). Since the case $K_1 \equiv 0$ is covered by the proposition (by simply setting $\omega^{(1)} = 0$) it follows that the most general gauge transformation that leaves invariant the form of a first order perturbation K_1 defined by the right-hand side of (6.51) (with $\omega^{(1)} = 0$) is given by (6.54)-(6.58) with corresponding gauge vector (6.53).

Proposition 6.11 (Gauge freedom). *In the setup of Theorem 6.3 we assume further that $\mathcal{R}_{,r}(r)$ and $\nu_{,r}(r)$ do not vanish identically on open sets and that the first and second order perturbation tensors take the form*

$$(6.50) \quad K_1 = -2\omega(r, \theta)\mathcal{R}^2(r) \sin^2 \theta dt d\phi.$$

$$(6.51) \quad K_2 = \left(-4e^{\nu(r)}h(r, \theta) + 2\omega^2(r, \theta)\mathcal{R}^2(r) \sin^2 \theta \right) dt^2 + 4e^{\lambda(r)}m(r, \theta)dr^2 \\ + 4k(r, \theta)\mathcal{R}^2(r) (d\theta^2 + \sin^2 \theta d\phi^2) + 4e^{\lambda(r)}\partial_\theta f(r, \theta)\mathcal{R}(r)drd\theta \\ - 2\mathcal{W}(r, \theta)\mathcal{R}^2(r) \sin^2 \theta dt d\phi.$$

Then a first order gauge vector V_1 preserves the form of K_1 (i.e. there is ω^g such that $K_1^g := K_1 + \mathcal{L}_{V_1}g$ is given by (6.50) with $\omega \rightarrow \omega^g$) if and only if,

up to the addition of a Killing vector of the background,

$$(6.52) \quad V_1 = Ct\partial_\phi, \quad C \in \mathbb{R}, \quad \text{and then} \quad \omega^g = \omega - C.$$

For V_1 as in (6.52), the second order gauge vector V_2 preserves the form of (6.51) if and only if

$$(6.53) \quad \begin{aligned} V_2 &= At\partial_t + Bt\partial_\phi + 2\mathcal{Y}(r, \theta)\partial_r + 2\alpha(r)\sin\theta\partial_\theta + \zeta, \\ A, B &\in \mathbb{R}, \quad \zeta \text{ Killing vector of } g, \end{aligned}$$

and K_2^g takes the form (6.51) with the coefficients h, m, k, f transformed to

$$(6.54) \quad h^g = h + \frac{1}{2}A + \frac{1}{2}\mathcal{Y}\nu_{,r},$$

$$(6.55) \quad m^g = m + \mathcal{Y}_{,r} + \frac{1}{2}\mathcal{Y}\lambda_{,r},$$

$$(6.56) \quad k^g = k + \mathcal{Y}\frac{\mathcal{R}_{,r}}{\mathcal{R}} + \alpha(r)\cos\theta,$$

$$(6.57) \quad f^g = f + \frac{\mathcal{Y}}{\mathcal{R}} - \mathcal{R}e^{-\lambda}\alpha_{,r}\cos\theta + \beta(r),$$

$$(6.58) \quad \mathcal{W}^g = \mathcal{W} - B,$$

where the arbitrary function $\beta(r)$ arises because K_2^g only involves $\partial_\theta f^g$.

Remark 6.12. This proposition determines the most general gauge transformation that leaves the form of the perturbation tensors (6.50) and (6.51). One could analyse also which restrictions on the gauge vectors are required to ensure that the gauge transformed tensors also satisfy the regularity and boundedness properties of Theorem 6.3. This is however not needed for the applications we have in mind.

Proof. The first order gauge transformation (2.5) imposes $\mathcal{L}_{V_1}g = K_1^g - K_1$, so we may apply part (C) of Proposition 6.10 to $s = V_1$ and (6.52) follows immediately. For the second part, we let $V_1 := Ct\partial_\phi$. The second order gauge transformation (2.6) takes the form

$$K_2^g = K_2 + \mathcal{L}_{V_2}g + \mathcal{L}_{V_1}(2K_1^g - \mathcal{L}_{V_1}g).$$

The tensor in parenthesis is $2K_1^g - \mathcal{L}_{V_1}g = -2\mathcal{R}^2(2\omega - C)\sin^2\theta dt d\phi$ and its Lie derivative along $Ct\partial_\phi$ is immediately computed to be

$$\mathcal{L}_{Ct\partial_\phi}(2K_1^g - \mathcal{L}_{V_1}g) = -2\mathcal{R}^2C(2\omega - C)\sin^2\theta dt^2.$$

Thus the equation that V_2 must satisfy is

$$\begin{aligned}
 (6.59) \quad \mathcal{L}_{V_2}g &= K_2^g - K_2 + 2\mathcal{R}^2C(2\omega - C)\sin^2\theta dt^2 \\
 &= \left(-4e^{\nu(r)}(h^g - h) + 2\mathcal{R}^2\sin^2\theta(\omega^{g^2} - \omega^2 + C(2\omega - C)) \right) dt^2 \\
 &\quad + 4e^{\lambda(r)}(m^g - m)dr^2 + 4\mathcal{R}^2(r)(k^g - k)(d\theta^2 + \sin^2\theta d\phi^2) \\
 &\quad + 4\mathcal{R}(r)e^{\lambda(r)}\partial_\theta(f^g - f)drd\theta - 2\mathcal{R}^2(r)(\mathcal{W}^g - \mathcal{W})\sin^2\theta dtd\phi \\
 &= -4e^{\nu(r)}(h^g - h)dt^2 + 4e^{\lambda(r)}(m^g - m)dr^2 \\
 &\quad + 4\mathcal{R}^2(r)(k^g - k)(d\theta^2 + \sin^2\theta d\phi^2) \\
 &\quad + 4\mathcal{R}(r)e^{\lambda(r)}\partial_\theta(f^g - f)drd\theta - 2\mathcal{R}^2(r)(\mathcal{W}^g - \mathcal{W})\sin^2\theta dtd\phi,
 \end{aligned}$$

where in the third equality we inserted (6.51) and the corresponding expression for K_2^g and the cancellations in the last equality follow from $\omega^g = \omega - C$. Thus, we may apply part (B) of Proposition (6.10) with $s = V_2$. Expression (6.53) follows directly from (6.45) and (6.47), after renaming $C \rightarrow A$ and $D \rightarrow B$ while (6.54)-(6.58) are obtained by comparing (6.48) with (6.59) after the same redefinition for C and D . \square

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Appendix A. Differentiability of radially symmetric functions

In this Appendix we analyse the relationship between the differentiability properties of a radially symmetric function and of its trace (see below). We expect several of these results to be known, but they do not seem to be easily accessible in the literature, at least in the specific form that we need. The starting point is, however, well-known (see Lemma 3.1 in [1]). We include a proof for completeness.

We set $B^p \subset \mathbb{R}^p$ to be the open ball of radius $a_0 > 0$ centered at the origin.

Lemma A.1. *Let $q : B^p \rightarrow \mathbb{R}$ be radially symmetric, i.e. such that there exists $\mathfrak{q} : [0, a_0) \rightarrow \mathbb{R}$ (the trace of q) with $q(x) = \mathfrak{q}(|x|)$. Then q is a $C^n(B^p)$ ($n \geq 0$) function if and only if \mathfrak{q} is $C^n([0, a_0))$ (i.e. up to the inner boundary) and all its odd derivatives up to order n vanish at zero. Equivalently, if and only if*

$$(A.1) \quad \mathfrak{q}(r) = \mathcal{P}_n(r^2) + \Phi^{(n)}(r),$$

where \mathcal{P}_n is a polynomial of degree $\lfloor \frac{n}{2} \rfloor$ and $\Phi^{(n)}$ is $C^n([0, a_0))$ and satisfies $\Phi^{(n)}(r) = o(r^n)$.

Proof. Evaluating on the line $x = (x_1, 0, \dots)$ we have $q(x_1, 0, \dots) = \mathfrak{q}(\text{abs}(x_1))$. The left-hand side is a C^n even function on $(-a_0, a_0)$, so all its odd derivatives (up to order n) vanish at zero. It follows from the equality that \mathfrak{q} is $C^n([0, a_0))$ up to boundary and that all odd derivatives (up to order n) vanish at zero. A Taylor expansion then gives (A.1). The converse follows by a simple computation. \square

Remark A.2. This result can be applied to functions $q(x^i)$ which are C^m in all variables and radially symmetric only in a subset of the coordinates. A similar remark will apply for the remaining results in this Appendix.

Lemma A.1 will be used in several ways. Our first application is the following statement.

Lemma A.3. *Consider the space $\mathbb{R}^2 \times \mathbb{R}^q$ ($q \geq 0$) coordinated by $\{x_1, x_2, w\}$ and let W be an open and connected neighbourhood of the axis $\mathcal{A} := \{x_1 = x_2 = 0\}$, radially symmetric in $\{x_1, x_2\}$. Let $q : W \setminus \mathcal{A} \rightarrow \mathbb{R}$ be radially symmetric in $\{x_1, x_2\}$ and assume that x_1q and x_2q extend to $C^n(W)$ functions, $n \geq 1$. Then q extends to a $C^{n-1}(W)$ function.*

Remark A.4. The result on the differentiability of q is sharp. Consider the function $|x|^\alpha$ with $1 < \alpha < 2$. It is easy to check that $x_1|x|^\alpha$ and $x_2|x|^\alpha$ are $C^2(\mathbb{R}^2)$ while $|x|^\alpha$ (which is $C^1(\mathbb{R}^2)$ in agreement with the lemma) is not $C^2(\mathbb{R}^2)$.

Proof. Set $x = (x_1, x_2)$ and define $\mathfrak{q}(r, w)$ by $q(x, w) = \mathfrak{q}(|x|, w)$. Let $q_1 := x_1q$, $q_2 := x_2q$. By assumption q_1 is $C^n(W)$, in particular when restricted to

the line $\{x_1 \geq 0, x_2 = 0\}$. Moreover,

$$q_1(x_1, x_2 = 0, w) = x_1 \mathfrak{q}(x_1, w).$$

Since the left-hand side is C^n up to and including the boundary $x_1 = 0$ the same holds for $r\mathfrak{q}(r, w)$.

On the other hand, the function $Q := x_1 q_1 + x_2 q_2 = |x|^2 q$ is radially symmetric (in $\{x_1, x_2\}$) and $C^n(W)$. Moreover $Q(x, w) = r^2 \mathfrak{q}(r, w)|_{r=|x|}$. Given that $r^2 \mathfrak{q}(r, w)$ is C^n up to boundary, Lemma A.1 implies that (we also use the fact that, by construction, Q vanishes at the origin)

$$(A.2) \quad r^2 \mathfrak{q}(r, w) = r^2 \mathcal{P}_n(r^2) + \Phi^{(n)}(r, w)$$

where $\mathcal{P}_n(u)$ is identically zero for $n = 1$ and a polynomial in u of degree $\lfloor \frac{n}{2} - 1 \rfloor$ when $n \geq 2$ (with coefficients that are C^n functions of w) and $\Phi^{(n)}(r, w)$ is C^n up to boundary and $o(r^n)$. Now, $\frac{\Phi^{(n)}}{r} = r\mathfrak{q}(r, w) - r\mathcal{P}_n$. Since $r\mathfrak{q}(r, w)$ is C^n it follows that $\frac{\Phi^{(n)}}{r}$ is C^n and $o(r^{n-1})$ and therefore admits a Taylor expansion of the form

$$\frac{\Phi^{(n)}(r, w)}{r} = \alpha(w)r^n + \widehat{\Phi}^{(n)}(r, w),$$

where the remainder $\widehat{\Phi}^{(n)}(r, w)$ is C^n and $o(r^n)$. Inserting this back into (A.2) yields $\mathfrak{q}(r, w) = \mathcal{P}_n(r^2) + \alpha(w)r^{n-1} + \widehat{\Phi}^{(n)}/r$. The last term is C^{n-1} and $o(r^{n-1})$ by item (iii) of Lemma B.7 applied to the one-dimensional case,⁶ and we have found, after renaming $\widehat{\Phi}^{(n-1)}(r, w) := \widehat{\Phi}^{(n)}/r$,

$$\mathfrak{q}(r, w) = \mathcal{P}_n(r^2) + \alpha(w)r^{n-1} + \widehat{\Phi}^{(n-1)}(r, w).$$

If n is odd, we can apply Lemma A.1 to conclude that $q(x, w) = \mathfrak{q}(|x|, w)$ is $C^{n-1}(W)$. If n is even, we still need to show that $\alpha(w) = 0$ before we can apply that lemma. Let k be defined by $n = 2k$ (note that $k \geq 1$). The polynomial $\mathcal{P}_n(r^2)$ plays no role in the argument, so it can be set to zero without

⁶We stress that all the results in Appendix B up to and including Lemma B.7 are independent of Appendix A. Proving the one-dimensional result here to avoid quoting a result from Appendix B would be redundant.

loss of generality (we redefine $q(x) - \mathcal{P}_n(|x|^2)$ as $q(x)$). Let us compute

$$\begin{aligned}
 \text{(A.3)} \quad (\partial_{x_1})^{2k-1}(q_2) &= (\partial_{x_1})^{2k-1}(x_2q) \\
 &= x_2\alpha(w) (\partial_{x_1})^{2k-1}|x|^{2k-1} \\
 &\quad + x_2(\partial_{x_1})^{2k-1}(\widehat{\Phi}^{(2k-1)}(|x|, w)).
 \end{aligned}$$

The term $(\partial_{x_1})^{2k-1}(\widehat{\Phi}^{(2k-1)})$ is convergent as $|x| \rightarrow 0$ (this claim is a particular case of Lemma B.3 in Appendix B). However, the first term does not converge at zero unless $\alpha(w) = 0$. This can be shown explicitly as follows. A simple induction argument based on $\partial_{x_1}|x| = x_1/|x|$ shows that, for all $l \leq 2k - 1, k \geq 1$,

$$(\partial_{x_1})^l |x|^{2k-1} = \sum_{i=0}^l |x|^{2k-1-l-i} x_1^i a_i, \quad a_i \in \mathbb{R}, \quad a_l \neq 0.$$

In particular, $(\partial_{x_1})^{2k-1}|x|^{2k-1}$ evaluated on the path $x(r) = \beta r$ with $\beta = (\beta_1, \dots, \beta_p)$ a constant unit vector yields $\lim_{r \rightarrow 0} (\partial_{x_1})^{2k-1}|x|^{2k-1} = \sum_{i=0}^{2k-1} \beta_1^i a_i$, and the limit depends on the path. Since, by assumption, q_2 is $C^{2k}(W)$ (in particular $C^{2k-1}(W)$ as well) the only possibility that the right hand side in (A.3) has a limit as $|x| \rightarrow 0$ is $\alpha(w) = 0$ and we may apply Lemma A.1 to $q(x, w)$ to conclude the proof. \square

Lemma A.5. *Let $f : B^p \rightarrow \mathbb{R}$ be radially symmetric. Assume that f is $C^n(B^p)$ and $o(|x|^n)$ with $n \geq 0$, then the function $\frac{x_i}{|x|}f(x)$ ($i = 1, \dots, p$) is also $C^n(B^p)$ and $o(|x|^n)$.*

Proof. That $\frac{x_i}{|x|}f$ is $o(|x|^n)$ follows immediately from the boundedness of $x_i/|x|$ near the origin. Let $\mathbf{f} : [0, a_0) \rightarrow \mathbb{R}$ be the trace of f . By construction $\mathbf{f}(r)$ is $C^n([0, a_0))$ and $o(r^n)$.

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_p)$, $\alpha_j \in \mathbb{N}$ and use the natural notion of (partial) order $\beta \leq \alpha$ iff all $\alpha_i \leq \beta_i$. We use the multiindex notation $\partial^\alpha := \frac{\partial^{\alpha_1}}{\partial(x_1)^{\alpha_1}} \cdots \frac{\partial^{\alpha_p}}{\partial(x_p)^{\alpha_p}}$, and $x^\alpha := x_1^{\alpha_1} \cdots x_p^{\alpha_p}$. We write, as usual, $|\alpha| = \sum_{j=1}^p \alpha_j$. Define the i -th 1-addition over α by $\alpha + 1_i := (\alpha_1, \dots, \alpha_i + 1, \dots, \alpha_m)$. A simple induction argument (with induction parameter $|\alpha|$, note that $|\alpha + 1_i| = |\alpha| + 1$ for all i) and the fact that, when acting on any radially

symmetric functions, $\partial_j f = \frac{x^j}{r} \frac{df(r)}{dr} \Big|_{r=|x|}$ implies that, as long as $|\alpha| \leq n$,

$$(A.4) \quad \partial^\alpha \left(\frac{x^i}{|x|} f(x) \right) = \sum_{a=0}^{|\alpha|} \frac{f^{(a)}(|x|)}{|x|^{|\alpha|-a}} \left(\sum_{\beta \leq \alpha + 1_i} b_{a\beta}^\alpha \frac{x^\beta}{|x|^{|\beta|}} \right),$$

where $b_{a\beta}^\alpha$ are constants (which may vanish) and $f^{(a)}$ denotes the a -th derivative of f . The properties of f imply that $f^{(a)}$ is $C^{n-a}([0, a_0])$ and $o(r^{n-a})$. Thus $\frac{f^{(a)}(|x|)}{|x|^{|\alpha|-a}}$ is $o(|x|^{n-|\alpha|})$, hence $o(1)$ since $|\alpha| \leq n$. Moreover $x^\beta/|x|^{|\beta|}$ is bounded in a neighbourhood of the origin. We conclude from (A.4) that $\partial^\alpha \left(\frac{x^i}{|x|} f(x) \right)$ converges to zero at the origin, which proves the lemma. \square

Appendix B. Existence and regularity of a singular differential equation on spheres

The fundamental aim of this Appendix is to establish Lemma B.9 below. In arbitrary dimension $p \geq 1$ let $B^p \subset \mathbb{R}^p$ be an open ball centered at the origin $\{0_p\}$. Consider a natural number (possibly zero) q and let $V^q \subset \mathbb{R}^q$ be an open connected neighbourhood in \mathbb{R}^q . Define also $V := B^p \times V^q \subset \mathbb{R}^{p+q}$, where we make the usual identification of $\mathbb{R}^p \times \mathbb{R}^q$ and \mathbb{R}^{p+q} . Letting $\pi := \mathbb{R}^{p+q} \rightarrow \mathbb{R}^p$ be the projection into the first factor, we set $\|x\| := \|\pi(x)\|_{\mathbb{R}^p}$, where $\|\cdot\|_{\mathbb{R}^p}$ is the standard euclidean norm in \mathbb{R}^p . We introduce the axis $\mathcal{A} := \{0_p\} \times V^q$ and note that, also, $\mathcal{A} = \{x \in V; \|x\| = 0\}$.

The full set of coordinates in V will be denoted by $x_\mu, \mu = 1, \dots, p+q$, while $a = 1, \dots, p$ coordinate the first \mathbb{R}^p factor. We will use a special name u to refer to points in the first factor \mathbb{R}^p and w to refer to points in the second factor \mathbb{R}^q , so that a point in V reads $x = (u, w)$. In particular $(0_p, w)$ corresponds to a point in \mathcal{A} . Multiindices (see proof of Lemma A.5) will include the following special notation to refer to the part related to \mathbb{R}^p : any multiindex $\alpha = (\alpha_1, \dots, \alpha_{p+q})$ will be separated as $\alpha = \alpha_u \oplus \alpha_w$ where $\alpha_u = (\alpha_1, \dots, \alpha_p)$ is a multiindex in \mathbb{R}^p and α_w in \mathbb{R}^q . Observe that $|\alpha| = |\alpha_u| + |\alpha_w|$.

Following the splitting of V we extend the usual little-o notation in terms of a limit on the axis \mathcal{A} : For any positive function g defined on $V \setminus \mathcal{A}$ we set (observe that the notation for \mathbf{o} is in boldface)

$$f \in \mathbf{o}(g) \iff \lim_{\|x\| \rightarrow 0} fg^{-1} = 0.$$

We need a simple result concerning Taylor expansions of C^l functions with $\mathbf{o}(\|x\|^l)$ behaviour. Let us first recall that given a C^l ($l \geq 1$) function $f : V \rightarrow \mathbb{R}$ the expansion in u in the \mathbb{R}^p factor around 0_p provides⁷

$$(B.1) \quad f(x) = \sum_{|\alpha_u| \leq l} \frac{1}{\alpha_u!} \partial^{\alpha_u} f(0_p, w) x^{\alpha_u} + \sum_{|\alpha_u|=l} Q_{\alpha_u}(x) x^{\alpha_u},$$

where the remainder

$$Q_{\alpha_u}(x) = \frac{|\alpha_u|}{\alpha_u!} \int_0^1 (1-t)^{|\alpha_u|-1} \partial^{\alpha_u} f(tu, w) dt - \frac{1}{\alpha_u!} \partial^{\alpha_u} f(0_p, w)$$

is $C^0(V)$ and satisfies $Q_{\alpha_u}(0_p, w) = 0$.

Lemma B.1 (Taylor). *Let $f : V \rightarrow \mathbb{R}$ be C^l and $\mathbf{o}(\|x\|^l)$ with $l \geq 1$. Then $\partial^\alpha f(a) = 0$ for all $a \in \mathcal{A}$ and $|\alpha| \leq l$. Consequently, in particular,*

$$f(x) = \sum_{|\alpha_u|=l} R_{\alpha_u}(x) x^{\alpha_u},$$

where

$$R_{\alpha_u}(x) = \frac{|\alpha_u|}{\alpha_u!} \int_0^1 (1-t)^{|\alpha_u|-1} \partial^{\alpha_u} f(tu, w) dt$$

is $C^0(V)$ and $R_{\alpha_u}(a) = 0$ for all $a \in \mathcal{A}$.

Proof. We start by proving that $\partial^{\alpha_u} f(a) = 0$ for all $a \in \mathcal{A}$ and $|\alpha_u| \leq l$. Assume this conclusion is not true and let $s \leq l$ be the smallest number for which there exists $a_0 \in \mathcal{A}$ and a multiindex α_{u0} with $|\alpha_{u0}| = s$ satisfying $\partial^{\alpha_{u0}} f(a_0) \neq 0$. Define the (not all zero) constants $C_{\alpha_u} := \frac{1}{\alpha_u!} \partial^{\alpha_u} f(a_0)$ for all $|\alpha_u| = s$. Then, writing $a_0 = (0_p, w_0)$, we have from (B.1) that at all points $x = (u, w_0)$

$$\frac{f(x)}{\|x\|^s} = \sum_{|\alpha_u|=s} C_{\alpha_u} \frac{x^{\alpha_u}}{\|x\|^s} + \sum_{s < |\alpha_u| \leq l} \frac{1}{\alpha_u!} \partial^{\alpha_u} f(0_p, w_0) \frac{x^{\alpha_u}}{\|x\|^s} + \sum_{|\alpha_u|=l} Q_{\alpha_u}(x) \frac{x^{\alpha_u}}{\|x\|^s}.$$

The left-hand side tends to zero when $\|x\| \rightarrow 0$ by assumption. The same is clearly true for the second and third terms of the right-hand side. Taking

⁷As usual we set $\alpha_u! = \alpha_1! \cdots \alpha_p!$.

the limit along a path $x = (\beta r, w_0)$, $r \rightarrow 0$ with any $\beta \in \mathbb{R}^p$ of unit norm, it must be that

$$\sum_{|\alpha_u|=s} C_{\alpha_u} \beta^{\alpha_u} = 0, \quad \forall \beta \in \mathbb{R}^p.$$

This polynomial in β can vanish identically only if all C_{α_u} vanish, which is a contradiction. Therefore, we have that $\partial^{\alpha_u} f(0_p, w) = 0$ for $0 \leq |\alpha_u| \leq l$, and for all w . This implies that $\partial^{\alpha_w} \partial^{\alpha_u} f(0_p, w) \equiv \partial^\alpha f(0_p, w) = 0$ for $0 \leq |\alpha| \leq l$ with $\alpha = \alpha_u \oplus \alpha_w$, as claimed. \square

Remark B.2. Any f as in Lemma B.1 is, in particular, C^k and $\mathbf{o}(\|x\|^k)$ for $0 \leq k \leq l$, so the following is also true

$$(B.2) \quad f(x) = \sum_{|\alpha_u|=k} R_{\alpha_u}(x) x^{\alpha_u} \quad \text{for any } 0 \leq k \leq l.$$

Note also that R_{α_u} with $|\alpha_u| \leq l - 1$ can be differentiated. In particular

$$(B.3) \quad \partial_\mu R_{\alpha_u}(x) = \frac{|\alpha_u|}{\alpha_u!} \int_0^1 (1-t)^{|\alpha_u|-1} t^{\epsilon_\mu} \partial_\mu \partial^{\alpha_u} f(tu, w) dt,$$

with $\epsilon_\mu = 1$ if $\mu \leq p$ and 0 otherwise,

so that $\partial_\mu R_{\alpha_u}(a) = 0$ for all $a \in \mathcal{A}$. Observe that since $f \in \mathbf{o}(\|x\|^l)$, (B.2) implies

$$\lim_{\|x\| \rightarrow 0} \sum_{|\alpha_u|=k} \frac{R_{\alpha_u}(x)}{\|x\|^{l-|\alpha_u|}} \frac{x^{\alpha_u}}{\|x\|^{|\alpha_u|}} = 0 \quad \text{for any } 0 \leq k \leq l.$$

The application of the β -path method as in the previous proof implies that for any $\beta \in \mathbb{R}^p$

$$\lim_{\|x\| \rightarrow 0} \sum_{|\alpha_u|=k} \frac{R_{\alpha_u}(x)}{\|x\|^{l-|\alpha_u|}} \beta^{\alpha_u} = 0 \quad \text{for any } 0 \leq k \leq l,$$

and therefore $R_{\alpha_u}(x) \in \mathbf{o}(\|x\|^{l-|\alpha_u|})$. In summary,

$$\partial_\mu R_{\alpha_u}(a) = 0, \quad R_{\alpha_u}(x) \in \mathbf{o}(\|x\|^{l-|\alpha_u|}) \quad \forall \alpha_u \text{ with } |\alpha_u| \leq l - 1.$$

Lemma B.3. Let $f : V \rightarrow \mathbb{R}$ be C^l with $l \geq 1$ and let $\alpha = \alpha_u \oplus \alpha_w$ be any multiindex satisfying $0 \leq |\alpha| \leq l$.

$$\bullet f \in \mathfrak{o}(\|x\|^l) \implies \partial^\alpha f \in \mathfrak{o}(\|x\|^{l-|\alpha_u|}) \quad (\implies \partial^\alpha f \in \mathfrak{o}(\|x\|^{l-|\alpha|}))$$

Proof. The proof is based on the following two facts, which we establish first. The first one is that for any $f^{(n)} \in C^n(V)$ and $\mathfrak{o}(\|x\|^n)$ with $n \geq 1$ it is true that $\partial_\mu f^{(n)} \in \mathfrak{o}(\|x\|^{n-1})$ and the second is that when $\mu > p$ the decay gets improved to $\partial_\mu f^{(n)} \in \mathfrak{o}(\|x\|^n)$. We show the first claim by applying (B.2) in Remark B.2 to $f^{(n)}$ and $k = n - 1$ to write

$$f^{(n)}(x) = \sum_{|\alpha_u|=n-1} R_{\alpha_u}(x)x^{\alpha_u}.$$

Differentiating and dividing by $\|x\|^{n-1}$ yields

$$(B.4) \quad \frac{\partial_\mu f^{(n)}(x)}{\|x\|^{n-1}} = \sum_{|\alpha_u|=n-1} \left(\partial_\mu R_{\alpha_u}(x) \frac{x^{\alpha_u}}{\|x\|^{n-1}} + R_{\alpha_u}(x) \partial_\mu x^{\alpha_u} \frac{1}{\|x\|^{n-1}} \right).$$

We consider now the limit of this equation to any $a \in \mathcal{A}$. The limit of the first term in the right hand side vanishes as a consequence of $\partial_\mu R_{\alpha_u}(a) = 0$ (Remark B.2) given that $\frac{x^{\alpha_u}}{\|x\|^{n-1}}$ are bounded (because $|\alpha_u| = n - 1 \geq 0$). The limit of the second term vanishes because $R_{\alpha_u}(x) \in \mathfrak{o}(\|x\|)$ (again by Remark B.2), and $\partial_\mu x^{\alpha_u} / \|x\|^{n-2}$ are bounded, so that $R_{\alpha_u}(x) \frac{\partial_i x^{\alpha_u}}{\|x\|^{n-2}} \frac{1}{\|x\|} \in \mathfrak{o}(\|x\|^0)$. Hence

$$\lim_{\|x\| \rightarrow 0} \frac{\partial_\mu f^{(n)}}{\|x\|^{n-1}} = 0,$$

i.e. $\partial_\mu f^{(n)} \in \mathfrak{o}(\|x\|^{n-1})$, as claimed. Obviously this function is also $C^{n-1}(V)$, which allows us to repeat the process as long as $n - 1 \geq 1$. We continue with the second claim, i.e. that $\lim_{\|x\| \rightarrow 0} (\partial_\nu f^{(n)}) / \|x\|^n = 0$ for any $\nu > p$. Let such ν be fixed and introduce the notation $h_{(\nu)}$ for $(\dots, 0_{\nu-1}, h, 0_{\nu+1}, \dots)$. The fundamental theorem of calculus gives, at any point $w_0 \in V^q$,

$$(B.5) \quad f^{(n)}(u, w_0 + h_{(\nu)}) = f^{(n)}(u, w_0) + \int_0^h \partial_\nu f^{(n)}(u, w_0 + s_{(\nu)}) ds.$$

If we define

$$\mathcal{F}_\nu(h, u) := \frac{1}{h\|x\|^n} \int_0^h \partial_\nu f^{(n)}(u, w_0 + s_{(\nu)}) ds,$$

equation (B.5) divided by $h\|x\|^n$ taken to the limit $\|x\| \rightarrow 0$ leads to

$$(B.6) \quad 0 = \lim_{\|x\| \rightarrow 0} \mathcal{F}_\nu(h, u), \quad \text{for each } \nu > p \quad \text{and all } h \neq 0.$$

Since $\partial_\nu f^{(n)}$ is continuous the limit $\lim_{\|x\| \rightarrow 0} \mathcal{F}_\nu(h, u)$ (at fixed $h \neq 0$) converges uniformly to zero on compact subsets of V^q . Therefore we can take the limit $h \rightarrow 0$ of (B.6) and interchange the limits by the Moore-Osgood theorem [23] to get

$$\begin{aligned} \text{(B.7)} \quad 0 &= \lim_{h \rightarrow 0} \lim_{\|x\| \rightarrow 0} \mathcal{F}_\nu(h, u) = \lim_{\|x\| \rightarrow 0} \lim_{h \rightarrow 0} \mathcal{F}_\nu(h, u) \\ &= \lim_{\|x\| \rightarrow 0} \frac{1}{\|x\|^p} \partial_\nu f^{(n)}(u, w_0), \quad \nu > p, \end{aligned}$$

where in the last equality we simply used the definition of \mathcal{F}_ν . Since the point w_0 in V^q is arbitrary our second claim is established.

We may now return to the proof of the lemma, which follows by simply applying the derivative $\partial^\alpha f$ with $\alpha = (\alpha_1, \dots, \alpha_{p+q})$ and $|\alpha| \leq l$ in the reverse order $\partial_{x_{p+q}}^{\alpha_{p+q}} \dots \partial_{x_1}^{\alpha_1} f$. The derivatives in the first p directions decrease the order to $\mathbf{o}(\|x\|^{l-|\alpha_u|})$, while the remaining derivatives do not change this order at all. This proves the first implication of the lemma. The second implication is trivial because $|\alpha| \geq |\alpha_u|$. \square

This lemma has an immediate application for products of functions. We restrict to the one-dimensional case, since this is all we need in the main text.

Corollary B.4. *Let $l \geq 1$ and $f, g \in C^l([0, a])$, $a > 0$. Assume that $f(x)$ and $g(x)$ are $\mathbf{o}(x^l)$. Then the function $h(x) = f(x)g(x)/x$ extends as a $C^l([0, a])$ function to $x = 0$.*

Proof. It suffices to check that the limits $\lim_{x \rightarrow 0} (\partial_x)^k h(x)$ for $k = 0, \dots, l$ exist. We have

$$(\partial_x)^k h(x) = \sum_{p=0}^k \sum_{r=0}^{k-p} \frac{1}{x^{p+1}} a_{kp} (\partial_x)^{k-p-r} f(x) (\partial_x)^r g(x)$$

where a_{kp} are constants. Lemma B.3 applied to the one-dimensional interval $[0, a]$ ensures that $(\partial_x)^k f(x)$ is C^{l-k} and $\mathbf{o}(x^{l-k})$, and the same for g . Therefore, each term in the sum is $\mathbf{o}(x^{l-k+p+r+l-r-p-1}) = \mathbf{o}(x^{2l-k-1})$ and, in particular, $\mathbf{o}(x^{l-1})$ for $k = 0, \dots, l$. \square

We now introduce an additional structure in the $V^q \subset \mathbb{R}^q$ factor. We split $\mathbb{R}^q = \mathbb{R}^{d_z} \times \mathbb{R}^{d_t}$ (in particular $q = d_z + d_t$) and assume that V^q is of the form $V^q = Z^{d_z} \times T^{d_t}$ where $Z^{d_z} \subset \mathbb{R}^{d_z}$ and $T^{d_t} \subset \mathbb{R}^{d_t}$ are open and connected. Furthermore, we require that Z^{d_z} contains the origin of \mathbb{R}^{d_z} . Thus, we have

now the splitting $\mathbb{R}^{p+q} = \mathbb{R}^p \times \mathbb{R}^{d_z} \times \mathbb{R}^{d_t}$ and we want to concentrate on the first two factors, so we define $\mathbb{R}^d = \mathbb{R}^p \times \mathbb{R}^{d_z}$ (again with the usual identification). Cartesian coordinates in \mathbb{R}^d are denoted by $x_i, i = 1, \dots, d$. As before we introduce the projection π_2 of $\mathbb{R}^{p+q} = \mathbb{R}^d \times \mathbb{R}^{d_t}$ into the first \mathbb{R}^d factor and define the seminorm $|x| = |\pi_2(x)|_{\mathbb{R}^d}$. The set $\mathcal{C}_0 := \{x \in V; |x| = 0\}$ is precisely $\{0_d\} \times T^{d_t}$, where $\{0_d\}$ stands for the origin of \mathbb{R}^d . It is immediate to check that $\mathcal{C}_0 \subset \mathcal{A}$.

We also introduce the usual little-o notation associated to $|x|$ by defining (observe that o is not in boldface now)

$$f \in o(|x|^l) \iff \lim_{|x| \rightarrow 0} \frac{f}{|x|^l} = 0.$$

When $d_z = 0$ we recover the previous setup. Therefore, Lemma B.3 implies in particular that for $l \geq 1$ (irrespectively of the value of d_z)

$$f \in C^l(V) \text{ and } o(|x|^l) \implies \partial^\alpha f \in o(|x|^{l-|\alpha|}).$$

Let us continue by providing two easy but convenient auxiliary results.

Lemma B.5. *Let $f : V \rightarrow \mathbb{R}$. If $f \in \mathbf{o}(\|x\|^l)$ then $f \in o(|x|^l)$.*

Proof. Since $|x|^l \geq \|x\|^l$ for $l \geq 0$ then $|f| |x|^{-l} \leq |f| \|x\|^{-l}$, so $\lim_{|x| \rightarrow 0} |f| |x|^{-l} \leq \lim_{|x| \rightarrow 0} |f| \|x\|^{-l}$. But $|x| \rightarrow 0$ implies $\|x\| \rightarrow 0$, so the limit is zero by the assumption $f \in \mathbf{o}(\|x\|^l)$. \square

Lemma B.6. *Let $f : V \rightarrow \mathbb{R}$ and $k \geq 0$. Then $f \in \mathbf{o}(\|x\|^l) \implies f/|x|^k \in \mathbf{o}(\|x\|^{l-k})$.*

Proof. The result follow directly from $\frac{1}{\|x\|^{l-k}} \frac{|f|}{|x|^k} = \frac{|f|}{\|x\|^l} \frac{\|x\|^k}{|x|^k} \leq \frac{|f|}{\|x\|^l}$, taking the limit $\|x\| \rightarrow 0$. \square

Let us introduce now the vector $\hat{n} := \frac{1}{|x|} (x^i \partial_i)$, which is clearly smooth in $V \setminus \mathcal{C}_0$ and bounded near \mathcal{C}_0 . Define also

$$\hat{n}^i := \hat{n}(x^i) = \frac{x^i}{|x|} = \partial^i |x|.$$

We note the well-known fact that $|x|^{|\alpha|} \partial^\alpha \hat{n}^i$ for $|\alpha| \geq 0$, as well as $|x|^{|\alpha|+1} \partial^\alpha (\frac{1}{|x|})$, are bounded. We will use $v^{(m)}(f)$ for the derivation of f m times along a vector field v . Since \hat{n} does not exist at \mathcal{C}_0 we define $\hat{n}(f) : V \rightarrow \mathbb{R}$ by setting $\hat{n}(f)(\mathcal{C}_0) = 0$, and similarly for higher powers $\hat{n}^{(m)}(f)$.

Lemma B.7. *Let $f : V \rightarrow \mathbb{R}$ be C^l , $l \geq 1$, and assume $f \in \mathbf{o}(\|x\|^l)$. Then, for any $1 \leq k \leq l$*

- (i) $\hat{n}^{(k)}(f) \in \mathbf{o}(\|x\|^{l-k})$ and $C^{l-k}(V)$.
- (ii) $|x| \hat{n}(f) \in \mathbf{o}(\|x\|^l)$.
- (iii) $\frac{1}{|x|^k} f \in \mathbf{o}(\|x\|^{l-k})$ and $C^{l-k}(V)$.

Proof. We start by establishing the following fact: For $n \geq 1$ and $f^{(n)} \in C^n(V)$ satisfying $f^{(n)} \in \mathbf{o}(\|x\|^n)$, it holds

$$(B.8) \quad f^{(n-1)} := \hat{n}(f^{(n)}) \quad \text{is } C^{n-1}(V) \text{ and } \mathbf{o}(\|x\|^{n-1}).$$

By the comment before the lemma, this claim actually involves the function $g : V \rightarrow \mathbb{R}$ defined by $g(\mathcal{C}_0) = 0$ and $g := \hat{n}(f^{(n)}) = \hat{n}^i \partial_i f^{(n)}$ outside \mathcal{C}_0 . Clearly $g \in C^{n-1}(V \setminus \mathcal{C}_0)$. Moreover, given that $\partial_i f^{(n)} \in \mathbf{o}(\|x\|^{n-1})$ (by Lemma B.3), and \hat{n}^i are bounded everywhere, we have $\hat{n}^i \partial_i f^{(n)} \in \mathbf{o}(\|x\|^{n-1})$, which in particular means that $g \in C^0(V)$ and $g \in \mathbf{o}(\|x\|^{n-1})$. Furthermore, the boundedness near the axis of $x^a/\|x\|$ for $1 \leq a \leq p$ and x^j for $p < j \leq d$ combined with Lemma B.3 imply $x^a \partial_a f^{(n)} \in \mathbf{o}(\|x\|^n)$ and $x^j \partial_j f^{(n)} \in \mathbf{o}(\|x\|^n)$, and therefore

$$|x| \hat{n}(f^{(n)}) = |x| \hat{n}^a \partial_a f^{(n)} + |x| \hat{n}^j \partial_j f^{(n)} = x^a \partial_a f^{(n)} + x^j \partial_j f^{(n)} \in \mathbf{o}(\|x\|^n).$$

This already proves point (ii) in the lemma.

It only remains to check that g is $C^{n-1}(V)$. Take α such that $0 \leq |\alpha| \leq n - 1$ and compute

$$\partial^\alpha \left(\hat{n}(f^{(n)}) \right) = \sum_{\beta \leq \alpha} b_\beta \partial^\beta \hat{n}^i \partial^{\alpha-\beta} \partial_i f^{(n)},$$

where $b_\beta \in \mathbb{R}$. The terms in the sum can be written as $|x|^{|\beta|} \partial^\beta \hat{n}^i \frac{1}{|x|^{|\beta|}} \partial^{\alpha-\beta} \partial_i f^{(n)}$, and given that $|x|^{|\beta|} \partial^\beta \hat{n}^i$ are bounded and that $\partial^{\alpha-\beta} \partial_i f^{(n)} \in \mathbf{o}(\|x\|^{n-1-|\alpha|+|\beta|})$ by Lemma B.3 (note that $|\alpha| - |\beta| = |\alpha - \beta|$ because $\beta \leq \alpha$), each term belongs to $\mathbf{o}(\|x\|^{n-1-|\alpha|+|\beta|-|\beta|}) = \mathbf{o}(\|x\|^{n-1-|\alpha|})$. Therefore $\partial^\alpha (\hat{n}(f^{(n)})) \in \mathbf{o}(\|x\|^{n-1-|\alpha|})$ and thus $\partial^\alpha (\hat{n}(f^{(n)})) \in \mathbf{o}(|x|^{n-1-|\alpha|})$ by Lemma B.5. Since $|\alpha| \leq n - 1$ the limits of $\partial^\alpha (\hat{n}(f^{(n)}))$ vanish at $\|x\| = 0$, and thus at $|x| = 0$ in particular. Define the functions

$$g^\alpha := \begin{cases} \partial^\alpha (\hat{n}(f^{(n)})) & x \neq \mathcal{C}_0 \\ 0 & x = \mathcal{C}_0 \end{cases},$$

with $g^0 = g$, which are $C^0(V)$ by construction, and $\mathbf{o}(\|x\|^{n-1-|\alpha|})$ (and thus $\mathbf{o}(|x|^{n-1-|\alpha|})$).

It remains to verify that the differentials (to order $n - 1$) of g on \mathcal{C}_0 exist and vanish for all $c \in \mathcal{C}_0$. We do that at once by showing that Dg_c^β (namely, the differential of g^β at c) vanishes for any $0 \leq |\beta| \leq n - 2$. We compute

$$\lim_{x \rightarrow c} \frac{|g^\beta(x) - g^\beta(c)|}{|x|_{\mathbb{R}^{p+q}}} = \lim_{x \rightarrow c} \frac{|g^\beta(x)|}{|x|_{\mathbb{R}^{p+q}}} \leq \lim_{x \rightarrow c} \frac{|g^\beta(x)|}{|x|} = 0, \text{ for } 0 \leq |\beta| \leq n - 2$$

where in the first equality we insert $g^\beta(c) = 0$ (since $c \in \mathcal{C}_0$), the inequality follows from $|x|_{\mathbb{R}^{p+q}} \geq |x|$, and in the final equality we use that the limit $x \rightarrow c$ is equivalent to $|x| \rightarrow 0$, and that $g^\beta \in \mathbf{o}(|x|^{n-1-|\beta|})$, so that $g^\beta / |x| \in \mathbf{o}(|x|^{n-2-|\beta|})$. By definition of differential, this limit shows that Dg_c^β exists and vanishes for $0 \leq |\beta| \leq n - 2$ and all $c \in \mathcal{C}_0$. For $|\beta| = 0$ this means that g is differentiable at \mathcal{C}_0 with vanishing differential, which establishes that $g \in C^1(V)$. Iterating, and noting that $|\beta| \leq n - 2$ means that we may take up to $n - 1$ derivatives of g , it follows that $g \in C^{n-1}(V)$, and the claim (B.8) is verified.

We now apply this result iteratively to the functions $f^{(l)} := f$ and $f^{(l-s)} := \hat{n}^{(s)}(f^{(l)})$ for $s = 0, \dots, l$. By hypothesis, $f^{(l)}$ is $C^l(V)$ and $\mathbf{o}(\|x\|^l)$, so

$$f^{(n-1)} = \hat{n}(f^{(n)}) \quad \text{is} \quad C^{n-1}(V) \quad \text{and} \quad \mathbf{o}(\|x\|^{n-1})$$

for $1 \leq n \leq l$. Given that $1 \leq l$ and $1 \leq k \leq l$ by assumption, we have $1 \leq l - k + 1 \leq l$, and we can take $n = l - k + 1$ in the preceding statement to conclude $\hat{n}^{(k)}(f^{(l)}) = f^{(l-k)} = \hat{n}(f^{(l-k+1)})$ is $C^{l-k}(V)$ and $\mathbf{o}(\|x\|^{l-k})$. This is item (i) of the Lemma.

The proof of the third point (iii) follows an analogous procedure. First, for $f^{(n)} \in C^n(V)$ and $\mathbf{o}(\|x\|^n)$ we have $\frac{1}{|x|} f^{(n)} \in \mathbf{o}(\|x\|^{n-1})$ by virtue of Lemma B.6. We define a new g , now setting $g(\mathcal{C}_0) = 0$ and $g := \frac{1}{|x|} f^{(n)}$ outside \mathcal{C}_0 and show that $g \in C^{n-1}(V)$. Indeed, for $0 \leq |\alpha| \leq n - 1$,

$$\partial^\alpha \left(\frac{1}{|x|} f^{(n)} \right) = \sum_{\beta \leq \alpha} b_\beta \partial^\beta \frac{1}{|x|} \partial^{\alpha-\beta} f^{(n)}.$$

The terms in the sum can be written as $|x|^{|\beta|+1} \partial^\beta \frac{1}{|x|} \frac{1}{|x|^{|\beta|+1}} \partial^{\alpha-\beta} f^{(n)}$, and given that $|x|^{|\beta|+1} \partial^\beta \frac{1}{|x|}$ are bounded and that $\partial^{\alpha-\beta} f^{(n)} \in \mathbf{o}(\|x\|^{n-|\alpha|+|\beta|})$ by Lemma B.3, we get that each term belongs to $\mathbf{o}(\|x\|^{n-1-|\alpha|})$. Therefore $\partial^\alpha \left(\frac{1}{|x|} f^{(n)} \right) \in \mathbf{o}(\|x\|^{n-1-|\alpha|})$ and thus $\partial^\alpha \left(\frac{1}{|x|} f^{(n)} \right) \in \mathbf{o}(|x|^{n-1-|\alpha|})$ by

Lemma B.5. Since $|\alpha| \leq n - 1$ the limits of $\partial^\alpha \left(\frac{1}{|x|} f^{(n)} \right)$ vanish at $\|x\| = 0$, and thus $|x| = 0$ in particular. The rest of the proof follow the same steps as for the previous g and the same iteration process as for item (i). \square

From here onwards we particularise to $d = 3$, $d_z = 1$ ($\implies p = 2$) and $d_t = 1$. Given Cartesian coordinates in $B^2 \times Z \subset \mathbb{R}^3$, $\{x_i\} = \{x_1, x_2, z\}$, $i = 1, 2, 3$ (indices raised and lowered with δ_{ij}), the seminorms read explicitly $|x| := \sqrt{x^i x_i}$, $\|x\| = \sqrt{x^a x_a} = \sqrt{(x_1)^2 + (x_2)^2}$. Moreover we consider a ball $B^3 \subset \mathbb{R}^3$ centered at the origin $\{0_3\}$ small enough so that $B^3 \subset B^2 \times Z$, and denote by U the corresponding set in V , that is, $U = B^3 \times T \subset V$.

Introduce the additional vectors tangent to the B^3 factor of U

$$(B.9) \quad \hat{\rho} := x^a \partial_a = x_1 \partial_{x_1} + x_2 \partial_{x_2}, \quad \hat{z} := \partial_z, \quad \eta := x_1 \partial_{x_2} - x_2 \partial_{x_1},$$

smooth in U

$$\iota := -\frac{1}{|x|} (z \hat{\rho} - \|x\|^2 \hat{z}),$$

smooth in $U \setminus \mathcal{C}_0$ and extends continuously ($\iota = 0$) to \mathcal{C}_0 .

It is straightforward to check that \hat{n} , ι and η are mutually orthogonal and commute. Their explicit expressions in spherical coordinates in the $B^3 \setminus \{0_3\}$ factor of $U \setminus \mathcal{C}_0$ are given by $\hat{n} = \partial_r$, $\eta = \partial_\phi$ and $\iota = -\sin \theta \partial_\theta$. Regarding the T factor of U , it is now an interval in the real line, and we will denote by t both the points and coordinate within T . Clearly ∂_t commutes with \hat{n} , ι and η .

Let us continue with a result we will also need in the main text.

Lemma B.8. *Let $f : V \rightarrow \mathbb{R}$ be C^l , $l \geq 1$, and assume $f \in \mathbf{o}(\|x\|^l)$. Then $\iota(f) \in \mathbf{o}(\|x\|^l)$. Also, $|x| \iota(f) \in C^{l-1}(V)$.*

Proof. By virtue of (B.9) we have

$$\iota(f) = -\frac{1}{|x|} (z x^a \partial_a f - \|x\|^2 \partial_z f) = -\frac{z}{|x|} x^a \partial_a f + \frac{\|x\|}{|x|} \|x\| \partial_z f.$$

As in the proof above, boundedness near the axis of $x^a/|x|$ combined with Lemma B.3 imply $x^a \partial_a f \in \mathbf{o}(\|x\|^l)$. Boundedness of $z/|x|$ near the axis thus establishes the first term is $\mathbf{o}(\|x\|^l)$. For the second term, Lemma B.3 also ensures that $\partial_z f \in \mathbf{o}(\|x\|^l)$, and hence boundedness of $\|x\|/|x|$ leads to the first result. The final statement follows from the C^∞ smoothness of both factors $z x^a$ and $\|x\|^2$. \square

The main result of this Appendix is the following lemma, which concerns radially symmetric functions in $\{x_1, x_2\}$ (see Appendix A), that is, functions invariant under η . We refer to these functions also as “axially symmetric”. The corresponding trace functions are defined in the domain $U_\eta := \{\rho \in \mathbb{R}_{\geq 0}; (\rho, 0, z, t) \in U\} \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^2$. We will also use the notation introduced in Section 5.1 regarding the spheres $S_r := \{x \in U; t = \text{const.}, |x| = r > 0\}$. Observe that we suppress the label t in S_r . As in the main text, given any vector field V in \mathbb{R}^3 , we write V_\parallel for its tangential projection to the spheres S_r . The operator \bar{d} acting on a scalar β gives the differential of its pull-back on each of the spheres. We also let $\langle \cdot, \cdot \rangle$ denote the euclidean metric on \mathbb{R}^3 and observe that $\langle V, \cdot \rangle = |x|^2 g_{S^2}(V_\parallel, \cdot)$, where g_{S^2} is the standard round unit metric. Note also that η and ι coincide with η_\parallel and ι_\parallel respectively, and $\langle \eta, \eta \rangle = \langle \iota, \iota \rangle = \|x\|^2$.

We have the ingredients to state the key result of the Appendix. In the first item of the lemma we use the big-O notation with its usual meaning.

Lemma B.9. *Let k, l be non-negative integers.*

- 1) *Let $P(z, t)$ be $C^m(U)$ with $m \geq 0$. Then the equation on U*

$$(B.10) \quad |x| \iota(Z_P) = \|x\|^{2k} z^l P(z, t)$$

with $k \geq 1$ admits an axially symmetric solution $Z_P \in C^m(U)$ and $O(|x|^{2k+l-1})$. Moreover, if $m \geq 1$, $\hat{n}(Z_P) \in O(|x|^{2k+l-2})$.

- 2) *Let $\Phi^{(m)}$ be radially symmetric in $\{x_1, x_2\}$, $C^m(U)$ and $o(\|x\|^m)$ with $m \geq 1$. Then the equation on U*

$$(B.11) \quad |x| \iota(Z_\Phi) = \|x\|^{2k} z^l \Phi^{(m)}$$

with $2k + l \geq 1$ admits an axially symmetric solution $Z_\Phi \in C^m(U \setminus \mathcal{C}_0) \cap C^0(U)$ and $o(|x|^{2k+l+m-1})$. Moreover, $\hat{n}(Z_\Phi) \in o(|x|^{2k+l+m-2})$.

- 3) *Let $\Gamma^{(m)}$ be radially symmetric in $\{x_1, x_2\}$, $C^m(U \setminus \mathcal{C}_0) \cap C^0(U)$ and $O(|x|)$ with $m \geq 0$. Then the equation on U*

$$(B.12) \quad |x| \iota(Z_\Gamma) = \|x\|^2 \Gamma^{(m)}$$

admits an axially symmetric solution $Z_\Gamma \in C^m(U \setminus \mathcal{C}_0) \cap C^0(U)$ and $O(|x|^2)$ which is also C^{m+1} on each sphere S_r . Moreover, $\hat{n}(Z_\Gamma)$ and $\partial_t(Z_\Gamma)$ are C^m on each sphere S_r .

Proof. Since the inhomogeneous (source) terms of all the equations are invariant under η , and we only care about existence, it will suffice to consider solutions which are radially symmetric in $\{x_1, x_2\}$. We concentrate first on the problem for Z_P . Clearly $\hat{\varrho}$ (see (B.9)) acts on radial functions in $\{x_1, x_2\}$ as $\hat{\varrho}(g(\|x\|)) = \rho \partial_\rho g(\rho)|_{\rho=\|x\|}$. Note that the trace of the function $|x|$ is $\sqrt{\rho^2 + z^2}$. Equation (B.10) for Z_P is cast in terms of the traces in U_η as

$$(B.13) \quad -\rho(z\partial_\rho - \rho\partial_z)\mathfrak{Z}_P(\rho, z, t) = \rho^{2k} z^l P(z, t).$$

It is direct to check that

$$\mathfrak{Z}_P(\rho, z, t) = \int_0^z (\rho^2 + z^2 - s^2)^{k-1} s^l P(s, t) ds$$

satisfies the equation. The assumption $k \geq 1$ ensures that the integrand is a polynomial in even powers of ρ , which by Lemma A.1 ensures, in turn, that Z_P is $C^m(U)$ (i.e. the differentiability of P). In terms of functions on U

$$Z_P = \int_0^z (|x|^2 - s^2)^{k-1} s^l P(s, t) ds$$

satisfies (B.10). In spherical coordinates, and using the change of variable $s = \lambda r$ in the integral, we can reexpress Z_P as

$$(B.14) \quad Z_P(r, \theta, \phi) = r^{2k+l-1} \int_0^{\cos \theta} (1 - \lambda^2)^{k-1} \lambda^l P(\lambda r, t) d\lambda.$$

Boundedness of the integral for $k \geq 1$ establishes that $Z_P \in O(|x|^{2k+l-1})$ as claimed. We compute now a radial derivative using (B.14), taking into account that in spherical coordinates $\hat{n} = \partial_r$, to obtain

$$\begin{aligned} \hat{n}(Z_P)(r, \theta, \phi) &= (2k + l - 1) \frac{Z_P}{r} \\ &\quad + r^{2k+l-1} \int_0^{\cos \theta} (1 - \lambda^2)^{k-1} \lambda^{l+1} \partial_z P(\lambda r, t) d\lambda. \end{aligned}$$

This time boundedness of the integral for $k \geq 1$ implies $\hat{n}(Z_P) \in O(|x|^{2k+l-2})$.

We proceed with the second point. The proof rests on an iteration based on the following claim.

Claim. Fix integers $m \geq 1$ and $0 \leq s \leq m$ and let $f^{(s)} \in C^s(U)$ be a radially symmetric function (in $\{x_1, x_2\}$) satisfying $|x| f^{(s)} \in \mathfrak{o}(\|x\|^{s+1})$ for

$0 \leq s \leq m - 1$ and $f^{(m)} \in \mathbf{o}(\|x\|^m)$. Let l, k be non-negative integers satisfying $2k + l \geq 1$ and define $g^{(s)} = \|x\|^{2k} z^l f^{(s)}$. Then the equation

$$(B.15) \quad |x| \iota(\beta) = g^{(s)}$$

admits a unique axially symmetric solution $\beta = \widehat{\beta}_s$ such that its restriction to S_r is C^s and satisfies the boundary condition $\widehat{\beta}_s|_{\theta=0} = 0$. Moreover, this function is also C^0 with respect to r and t for $r > 0$ and satisfies $\widehat{\beta}_s \in o(\|x\|^{2k+l+s-1})$, so that in particular it extends continuously to \mathcal{C}_0 as $\widehat{\beta}_s|_{\mathcal{C}_0} = 0$.

Proof of the claim. We observe that the assumptions imply (since $\|x\|/|x|$ is bounded)

$$(B.16) \quad \begin{aligned} f^{(s)} &\in \mathbf{o}(\|x\|^s), & |x| f^{(s)} &\in \mathbf{o}(\|x\|^{\min\{s+1, m\}}), & 0 \leq s \leq m, \\ f^{(m)}/\|x\| &\in o(|x|^{m-1}), & |x| f^{(s)}/\|x\| &\in \mathbf{o}(|x|^s), & 0 \leq s \leq m - 1, \end{aligned}$$

the second line following from the first combined with Lemma B.5 (and $m \geq 1$). Clearly $g^{(s)} \in C^s(U)$ and, since $\|x\|^{2k} z^l / |x|$ is also bounded, $g^{(s)} \in \mathbf{o}(\|x\|^{\min\{s+1, m\}})$. We start by writing (B.15) in spherical coordinates in $U \setminus \mathcal{C}_0$ recalling that $\iota = -\sin \theta \partial_\theta$ and $g^{(s)}$ is invariant under $\eta = \partial_\phi$, as

$$(B.17) \quad \partial_\theta \beta = -\frac{1}{r \sin \theta} g^{(s)}(\theta, r, t) \quad \text{for } r > 0.$$

We may write the solution $\widehat{\beta}_s$ that satisfies $\widehat{\beta}_s|_{\theta=0} = 0$ as

$$(B.18) \quad \widehat{\beta}_s(\theta; r, t) = -\int_0^\theta \frac{g^{(s)}(\lambda, r, t)}{r \sin \lambda} d\lambda.$$

Since in particular $g^{(s)} \in \mathbf{o}(\|x\|^1)$ the integrand is bounded in all the domain of integration and thus the integral exists and $\widehat{\beta}_s(\theta; r, t)$ is continuous in θ up to the boundary. Clearly, $\widehat{\beta}_s$ thus constructed is C^0 on each S_r . On the other hand, since $g^{(s)} \in C^s(U)$ then $g^{(s)}$ is C^s in r and t for $r > 0$, and therefore $\widehat{\beta}_s(\theta; r, t)$ is also C^0 in r and t for $r > 0$. Moreover, at each fixed t we have

$$(B.19) \quad \begin{aligned} \left| \widehat{\beta}_s \Big|_{\theta; r, t} \right|_{r^{2k+l+s-1}} &\leq \frac{1}{r^{2k+l+s-1}} \int_0^\theta \left| \frac{g^{(s)}}{r \sin \lambda} \right| d\lambda \leq \frac{\theta}{r^{2k+l+s-1}} \sup_{|x|=r} \left| \frac{g^{(s)}}{\|x\|} \right| \\ &\leq \frac{\theta}{r^{s-1}} \sup_{|x|=r} \left| \frac{f^{(s)}}{\|x\|} \right| = \frac{\theta}{r^s} \sup_{|x|=r} \left| \frac{|x| f^{(s)}}{\|x\|} \right|, \end{aligned}$$

where we have used the explicit form of $g^{(s)} = \|x\|^{2k} z^l f^{(s)}$ and $\sup_{|x|=r} (\|x\|^{2k} |z^l|) \leq r^{2k+l}$ in the last inequality. Properties (B.16) imply in particular that $|x| f^{(s)} / \|x\| \in o(|x|^s)$ for all $s \in \{0, \dots, m\}$, so, the last term is $o(1)$ and we conclude

$$(B.20) \quad \lim_{|x| \rightarrow 0} \frac{|\widehat{\beta}_s|}{|x|^{2k+l+s-1}} = 0, \quad 0 \leq s \leq m.$$

The claim for $s = 0$ has been shown. In the following we deal with $s \geq 1$. In order to obtain the differentiability of the solutions $\widehat{\beta}_s$ along the spheres S_r one could differentiate (B.18) repeatedly and take control over the behaviour of the terms around the axis. That is trivial for the first derivative, but for $s \geq 2$ we follow a more straightforward strategy. Note also that with the integral expression only we cannot extract any sort of differentiability (nor continuity) of $\partial_r \widehat{\beta}_s(\theta; r, t)$ on S_r because the fact that $g^{(s)}$ is $\mathbf{o}(\|x\|^s)$ does not translate to $\partial_r g^{(s)}$ in any way in general.

A convenient form of expressing equation (B.15) is

$$(B.21) \quad |x| d\beta(\iota) = \frac{1}{\langle \eta, \eta \rangle} g^{(s)} \langle \iota, \iota \rangle,$$

where at the right hand side we are just using $\langle \iota, \iota \rangle = \langle \eta, \eta \rangle$. Clearly $d\beta(\iota) = \bar{d}\beta(\iota_{\parallel})$ outside the origin. On the other hand, since we are constructing solutions invariant under η , it is enough to restrict β so that $d\beta(\eta) = \bar{d}\beta(\eta_{\parallel}) = \eta(\beta) = 0$. Therefore, since $\langle \iota, \eta \rangle = 0$, equation (B.21) is equivalent to

$$(B.22) \quad \bar{d}\beta(\cdot) = \frac{1}{|x|} \frac{1}{\eta^2} g^{(s)} \langle \iota, \cdot \rangle = \frac{|x|}{\|x\|^2} g^{(s)} g_{\mathbb{S}^2}(\iota_{\parallel}, \cdot) =: g_{\mathbb{S}^2}(V, \cdot)$$

on each sphere S_r , where we have defined the vector

$$V := \frac{|x|}{\|x\|^2} g^{(s)} \iota_{\parallel} = -\|x\|^{-2} g^{(s)} (zx_1 \partial_{x_1} + zx_2 \partial_{x_2} - \|x\|^2 \partial_z),$$

and (B.9) has been used in the second equality. We deal first with the regularity and behaviour around the axis of the vector V . The applications of V to the Cartesian coordinate functions (observe $V(t) = 0$) provide

$$V(x_1) = -z \frac{x_1}{\|x\|} g^{(s)} \frac{1}{\|x\|}, \quad V(x_2) = -z \frac{x_2}{\|x\|} g^{(s)} \frac{1}{\|x\|}, \quad V(z) = g^{(s)}.$$

Since $g^{(s)}$ is $\mathfrak{o}(\|x\|^s)$ in particular, Lemma A.5 ensures that $\frac{x_1}{\|x\|}g^{(s)}$ and $\frac{x_2}{\|x\|}g^{(s)}$ are C^s and $\mathfrak{o}(\|x\|^s)$. We apply next Lemma A.3 to these two functions to find that $\frac{1}{\|x\|}g^{(s)}$ is C^{s-1} . Given that, also, $\frac{1}{\|x\|}g^{(s)} \in \mathfrak{o}(\|x\|^{s-1})$ we finally apply Lemma A.5 to $\frac{1}{\|x\|}g^{(s)}$ to conclude that $\frac{x_2}{\|x\|}\frac{1}{\|x\|}g^{(s)}$ and $\frac{x_1}{\|x\|}\frac{1}{\|x\|}g^{(s)}$ are C^{s-1} and $\mathfrak{o}(\|x\|^{s-1})$. Therefore $V(x_1)$ and $V(x_2)$ are C^{s-1} and $\mathfrak{o}(\|x\|^{s-1})$ in U , while $V(z)$ is clearly C^s . As a result, V is a C^{s-1} vector field on U , thus on each S_r .

For $s = 1$ it suffices now to trivially use the equation (B.22), which applied to $\partial_{x^i} = \{\partial_{x_1}, \partial_{x_2}, \partial_z\}$ gives

$$\bar{d}\beta(\partial_{x^i}) = V(x^i), \quad \bar{d}\beta(\partial_t) = 0.$$

Given the above, any solution β of (B.22) invariant under η satisfies $\bar{d}\beta \in C^0(S_r)$. Therefore we have that $\widehat{\beta}_1 \in C^1(S_r)$ in particular.

We deal now with the case $s \geq 2$. The application of the divergence $(\star_{\mathbb{S}^2}\bar{d}\star_{\mathbb{S}^2})$ at both sides of (B.22) yields

$$(B.23) \quad \Delta_{\mathbb{S}^2}\beta = \operatorname{div}_{\mathbb{S}^2}V$$

on each S_r . Since we are dealing with a one-dimensional problem (because $\eta(\beta) = 0$) and V is a C^{s-1} vector field on each S_r , the solutions β of the equation (B.23) for $s \geq 2$ must be C^s on each S_r .⁸ Observe that a solution β of (B.23) is unique up to an additive constant at each S_r , hence up to a radially symmetric (and t -dependent) function. Therefore, the solution $\widehat{\beta}_s(r, t)$ (for $r > 0$) for each s constructed above is the unique solution at each S_r and t , fixed by the condition $\widehat{\beta}_s|_{\theta=0}(r, t) = 0$, i.e. vanishing at the north poles of each S_r , and satisfies $\widehat{\beta}_s \in C^s(S_r)$. This finishes the proof of the claim. □

We deal now with the original equation (B.11). Applying the claim to $s = m \geq 1$ in (B.15) and $g^{(m)} = \|x\|^{2k}z^l\Phi^{(m)}$ and setting $Z_\Phi(r, t) = \widehat{\beta}_m(r, t)$, we conclude that this function is C^m on each S_r and $\mathfrak{o}(\|x\|^{2k+l+m-1})$. It remains to show that Z_Φ is also differentiable in r and t , which will then imply $Z_\Phi \in C^m(U \setminus \mathcal{C}_0)$.

⁸Without axial symmetry we would need that the inhomogeneous term is Hölder $C^{s,\alpha}$.

We first consider the radial derivatives. Let us set $f^{(m)} := \Phi^{(m)}$ and define $f^{(s)}$ by the iteration

$$(B.24) \quad f^{(s-1)} := (2k + l - 1) \frac{1}{|x|} f^{(s)} + \hat{n}(f^{(s)}) \quad \text{for } 1 \leq s \leq m.$$

Points (i) and (iii) in Lemma B.7 show, after a trivial iteration starting at $f^{(m)} \in C^m(U)$ and $\mathbf{o}(\|x\|^m)$, that $f^{(s)}$ is $C^s(U)$ and $\mathbf{o}(\|x\|^s)$ for $0 \leq s \leq m$. In addition, since $|x| \hat{n}(f^{(s)}) \in \mathbf{o}(\|x\|^{(s)})$ (by point (ii) in Lemma B.7), (B.24) implies $|x| f^{(s)} \in \mathbf{o}(\|x\|^{s+1})$ for all $0 \leq s \leq m - 1$. The point of introducing these functions is that $\hat{n}^{(m-s)}(Z_\Phi)$ with $s \in \{0, \dots, m\}$ satisfies the equation

$$(B.25) \quad |x| \iota(\hat{n}^{(m-s)}(Z_\Phi)) = \|x\|^{2k} z^l f^{(s)}.$$

We show this by iteration. The statement is clearly true for $s = m$ (the original equation). Assume it is true for a given $s \in \{1, \dots, m\}$ and apply \hat{n} to (B.25). Using $\hat{n}(|x|) = 1$, $\hat{n}(\|x\|) = |x|^{-1} \|x\|$ and $\hat{n}(z) = |x|^{-1} z$, together with the fact that \hat{n} and ι commute yields

$$\begin{aligned} \iota(\hat{n}^{(m-s)}(Z_\Phi)) + |x| \iota(\hat{n}^{(m-s+1)}(Z_\Phi)) &= \|x\|^{2k} z^l \left((2k + l) \frac{f^{(s)}}{|x|} + \hat{n}(f^{(s)}) \right) \\ \implies |x| \iota(\hat{n}^{(m-(s-1))}(Z_\Phi)) &= \|x\|^{2k} z^l \left((2k + l - 1) \frac{f^{(s)}}{|x|} + \hat{n}(f^{(s)}) \right) \\ &= \|x\|^{2k} z^l f^{(s-1)}, \end{aligned}$$

where in the second line we inserted the equation for $\iota(\hat{n}^{(m-s)}(Z_\Phi))$ and the definition of $f^{(s-1)}$. Thus, equation (B.25) is also true for $s - 1$, and hence for all $s \in \{0, \dots, m\}$.

Moreover, the function $\hat{n}^{(m-s)}(Z_\Phi)$ vanishes on the line $\theta = 0$. Thus, all the conditions of the claim are satisfied for equation (B.25) and we conclude that $\hat{n}^{(m-s)}(Z_\Phi)$ is C^s on each sphere and $\mathbf{o}(\|x\|^{2k+l+s-1})$. This proves that all radial derivatives $\partial_r^{(m-s)} Z_\Phi$ for $s \in \{0, \dots, m\}$ are $C^s(U \setminus \mathcal{C}_0)$ and, moreover, extend continuously to $r = 0$ with the value zero.

Regarding the derivatives with respect to t , a similar (in fact much easier) argument applies. Indeed, $\partial_t^{(m-s)} Z_\Phi$, $s \in \{0, \dots, m\}$ satisfies the equation

$$|x| \iota \left(\partial_t^{(m-s)} Z_\Phi \right) = \|x\|^{2k} z^l \partial_t^{(m-s)} \Phi^{(m)}.$$

Lemma B.3 ensures that $f^{(s)} := \partial_t^{(m-s)} \Phi^{(m)}$ is $C^s(U)$ and $\mathbf{o}(\|x\|^m)$, and this also implies in particular $|x| f^{(s)} \in \mathbf{o}(\|x\|^{s+1})$, $s \in \{0, \dots, m - 1\}$. We may

apply our main claim to $g^{(s)} = \|x\|^{2k} z^l f^{(s)}$ to conclude that $\partial_t^{(m-s)} Z_\Phi$ is C^s on each S_r and $o(\|x\|^{2k+l+s-1})$ for $s \in \{0, \dots, m\}$.

To sum up, $Z_\Phi(r, t)$ is C^m on each S_r and it is also C^m with respect to the parameters $\{r, t\}$ for $r > 0$ and $t \in T$. We may coordinate the sphere with two charts so that, together with r and t , we also cover $U \setminus \mathcal{C}_0$ with two coordinate charts. The coordinate change to cartesian coordinates is smooth on $U \setminus \mathcal{C}_0$, so we have proved that (B.15) on U admits an axially symmetric solution $Z_\Phi \in C^m(U \setminus \mathcal{C}_0)$ (as function of $\{x_1, x_2, z, t\}$), and $o(\|x\|^{2k+l+m-1})$.

The proof of point 3 is more direct than the previous one because the right-hand side of the equation has better behaviour near the axis. We first consider $m \geq 1$, for which equation (B.12) has the form of (B.15) with a $C^1(U)$ right-hand side. We therefore recover (B.23) in the form

$$\Delta_{\mathbb{S}^2} Z_\Gamma = \operatorname{div}_{\mathbb{S}^2} V,$$

where the vector V is defined as in (B.22) with $g^{(s)}$ replaced by $\|x\|^2 \Gamma^{(m)}$, i.e. $V = \|x\| \Gamma^{(m)} \iota_{\parallel} = -\Gamma^{(m)}(zx_1 \partial_{x_1} + zx_2 \partial_{x_2} - \|x\|^2 \partial_z)$. Clearly, V has the same differentiability of $\Gamma^{(m)}$, that is, $C^m(U \setminus \mathcal{C}_0)$, and therefore V is C^m on each S_r . As a result (see above), the solution Z_Γ is a C^{m+1} function on each S_r . We denote by $\widehat{Z}_\Gamma(r, t)$ the solution for $r > 0$ that vanishes at the north poles of each S_r . Using spherical coordinates, as done previously, we can express that solution by

$$(B.26) \quad \widehat{Z}_\Gamma(\theta; r, t) = - \int_0^\theta r \sin \lambda \Gamma^{(m)}(\lambda, r, t) d\lambda.$$

This integral expression directly shows that $\widehat{Z}_\Gamma(\theta; r, t)$ is C^m with respect to $\{r, t\}$ by construction, which together with $\widehat{Z}_\Gamma \in C^{m+1}(S_r)$, implies $\widehat{Z}_\Gamma \in C^m(U \setminus \mathcal{C}_0)$. To obtain the differentiability on each S_r of both $\hat{n}(Z_\Gamma)$ and $\partial_t(Z_\Gamma)$ we differentiate (B.12) accordingly, to get

$$\|x\| \iota(\hat{n}(Z_\Gamma)) = \|x\|^2 \left(\frac{\Gamma^{(m)}}{\|x\|} + \hat{n}(\Gamma^{(m)}) \right), \quad \|x\| \iota(\partial_t Z_\Gamma) = \|x\|^2 \left(\partial_t \Gamma^{(m)} \right).$$

The terms within brackets at the right hand sides in both equations are $C^{m-1}(U \setminus \mathcal{C}_0)$, and the same argument as above involving the Laplace equation shows that $\hat{n}(Z_\Gamma)$ and $\partial_t(Z_\Gamma)$ are $C^{(m-1)}(S_r)$.

It only remains to consider the case $m = 0$. It suffices to use the integral expression (B.26) which is obviously still valid. It is immediate that the integral is continuous in $\{\theta, r, t\}$ for $r > 0$, from where it follows easily that $\widehat{Z}_\Gamma \in C^0(U \setminus \mathcal{C}_0)$. Clearly this function can be differentiated once

with respect to θ on $\theta \in [0, \pi]$, with continuous derivative. Again, we easily conclude that $\widehat{Z}_\Gamma \in C^1(S_r)$ on each S_r .

It only remains to show that, irrespectively of the value of $m \geq 0$, $\widehat{Z}_\Gamma \in (|x|^2)$, as this already implies that this function extends continuously to \mathcal{C}_0 . We compute

$$\frac{|\widehat{Z}_\Gamma|(\theta; r, t)}{r^2} \leq \int_0^\theta \left| \sin \lambda \frac{\Gamma^{(m)}}{r} \right| d\lambda \leq \theta \sup_{|x|=r} \left| \frac{\Gamma^{(m)}}{|x|} \right|.$$

By the assumption $\Gamma^{(m)} \in O(|x|)$, the last term is bounded and the property follows. \square

Corollary B.10. *Let $\mathcal{V}, \mathcal{V}'$ be finite subsets of $\mathbb{N} \times \mathbb{N}$ with the restrictions $\mathcal{V} \subset \{k \geq 1\} \times \{l \geq 0\}$ and $\mathcal{V}' \subset \{2k' + l' \geq 1\}$ and define $b := \min_{\mathcal{V}} \{2k + l\}, c := \min_{\mathcal{V}'} \{2k' + l'\}$. Consider the equation on U*

$$(B.27) \quad \iota(\gamma) = \sum_{(k,l) \in \mathcal{V}} \|x\|^{2k} z^l P_k(z, t) + \sum_{(k',l') \in \mathcal{V}'} \|x\|^{2k'} z^{l'} \Phi_{l'k'}^{(m)}(x),$$

where P_k are C^m in their arguments and $\Phi_{l'k'}^{(m)}$ are $C^m(U)$ and $o(\|x\|^m)$. If $m \geq 1$ the equation admits an axially symmetric solution

$$\gamma = |x| (Z_P + Z_\Phi)$$

where $Z_P + Z_\Phi \in C^m(U \setminus \mathcal{C}_0)$ with $Z_P \in O(|x|^{b-1})$ and $Z_\Phi \in o(|x|^{c+m-1})$. Moreover, $\hat{n}(Z_P) \in O(|x|^{b-2})$ and $\hat{n}(Z_\Phi) \in o(|x|^{c+m-2})$. In particular,

$$\gamma \in O(|x|^b), \quad \hat{n}(\gamma) \in O(|x|^{b-1}) \quad \text{provided } b \leq c + m,$$

$$\gamma \in o(|x|^{c+m}), \quad \hat{n}(\gamma) \in o(|x|^{c+m-1}) \quad \text{for } b > c + m.$$

Proof. Write $\gamma = |x| \widehat{Z}$ so that \widehat{Z} solves

$$|x| \iota(\widehat{Z}) = \sum_{(k,l) \in \mathcal{V}} \|x\|^{2k} z^l P_k(z, t) + \sum_{(k',l') \in \mathcal{V}'} \|x\|^{2k'} z^{l'} \Phi_{l'k'}^{(m)}(x),$$

Since the equation is linear the solution decomposes into a sum and we may apply Lemma B.9 to each term. The rest is immediate. \square

Corollary B.11. *With the above definitions (in particular $\Gamma^{(m-1)}$ is axially symmetric, $C^{m-1}(U \setminus \mathcal{C}_0) \cap C^0(U)$ and $O(|x|)$, cf. point 3 in Lemma B.9) the equation on U*

$$(B.28) \quad \iota(\gamma_0) = \sum_{(k,l) \in \mathcal{V}} \|x\|^{2k} z^l P_{lk}(z, t) \\ + \sum_{(k',l') \in \mathcal{V}'} \|x\|^{2k'} z^{l'} \Phi_{l'k'}^{(m)}(x) + \|x\|^2 \Gamma^{(m-1)},$$

with $m \geq 1$ admits an axially symmetric solution $\gamma_0 \in C^{m-1}(U \setminus \mathcal{C}_0) \cap C^0(U)$, which is also C^m on each sphere S_r , and moreover $\hat{n}(\gamma_0)$ and $\partial_t \gamma_0$ are C^{m-1} on each sphere S_r . In addition, $\gamma_0 \in O(|x|^3)$ provided $3 \leq b \leq c + m$.

Proof. We use the previous corollary, which ensures the existence of the solution γ , followed by the addition $\gamma_0 = \gamma + |x| Z_\Gamma$ of the solution Z_Γ for (B.12) given by the third point of the lemma. \square

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