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Operator forms of the nonhomogeneous associative classical Yang-Baxter equation

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This paper studies operator forms of the nonhomogeneous associative classical Yang-Baxter equation (nhacYBe), extending and generalizing such studies for the classical Yang-Baxter equation and the associative Yang-Baxter equation that can be traced back to the works of Semenov-Tian-Shansky and Kupershmidt on Rota-Baxter Lie algebras and $\mathcal{O}$-operators. Solutions of the nhacYBe are characterized in terms of generalized $\mathcal{O}$-operators, and in terms of the classical $\mathcal{O}$-operators precisely when the solutions satisfy an invariant condition. When the invariant condition is compatible with a Frobenius algebra, such solutions have a close relationship with Rota-Baxter operators on the Frobenius algebra. In general, solutions of the nhacYBe can be produced from Rota-Baxter operators, and then from $\mathcal{O}$-operators when the solutions are taken in semi-direct product algebras. In the other direction, Rota-Baxter operators can be obtained from solutions of the nhacYBe in unitizations of algebras. Finally a classification is obtained for solutions of the nhacYBe satisfying the mentioned invariant condition in all unital complex algebras of dimensions two and three. All these solutions are shown to come from Rota-Baxter operators.

1. Introduction

The aim of this paper is to give operator forms of the nonhomogeneous associative classical Yang-Baxter equation in terms of Rota-Baxter operators and the more general $\mathcal{O}$-operators.

1.1. CYBE, AYBE and their operator forms

The classical Yang-Baxter equation (CYBE) was first given in the tensor form

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0,$$

where $r \in \mathfrak{g} \otimes \mathfrak{g}$ and $\mathfrak{g}$ is a Lie algebra (see [16] for details). The CYBE arose from the study of inverse scattering theory in 1980s. Later it was recognized as the "semi-classical limit" of the quantum Yang-Baxter equation which was encountered by C. N. Yang in the computation of the eigenfunctions of a one-dimensional fermion gas with delta function interactions [45] and by R. J. Baxter in the solution of the eight vertex model in statistical mechanics [13]. The study of the CYBE is also related to classical integrable systems and quantum groups (see [16] and the references therein).

An important approach in the study of the CYBE was through the interpretation of its tensor form in various operator forms which proved to be effective in providing solutions of the CYBE, in addition to the well-known work of Belavin and Drinfeld [14]. First Semonov-Tian-Shansky [42] showed that if there exists a nondegenerate symmetric invariant bilinear form on a Lie algebra $\mathfrak{g}$ and if a solution r of the CYBE is antisymmetric, then r can be equivalently expressed as a linear operator $R : \mathfrak{g} \to \mathfrak{g}$ satisfying the operator identity

$$[R(x), R(y)] = R([R(x), y]) + R([x, R(y)]), \quad \forall x, y \in \mathfrak{g}, \tag{1}$$

which is then regarded as an **operator form** of the CYBE. Note that Eq. (1) is exactly the Rota-Baxter relation (of weight zero) in Eq. (4) for Lie algebras.

In order for the approach to work more generally, Kupershmidt revisited operator forms of the CYBE in [28] and noted that, when r is antisymmetric, the tensor form of the CYBE is equivalent to a linear map $r : \mathfrak{g}^* \to \mathfrak{g}$ satisfying

$$[r(x), r(y)] = r(\mathrm{ad}^* r(x)(y) - \mathrm{ad}^* r(y)(x)), \ \forall x, y \in \mathfrak{g}^*,$$

where $\mathfrak{g}^*$ is the dual space of $\mathfrak{g}$ and ad^* is the dual representation of the adjoint representation (coadjoint representation) of the Lie algebra $\mathfrak{g}$. He further generalized the above ad^* to an arbitrary representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ of $\mathfrak{g}$, that is, a linear map $T : V \to \mathfrak{g}$, satisfying

$$[T(u), T(v)] = T(\rho(T(u))v - \rho(T(v))u), \quad \forall u, v \in V,$$

which was regarded as a natural generalization of the CYBE. Such an operator is called an $\mathcal{O}$**-operator** associated to ρ. Note that the operator form (1) of the CYBE given by Semenov-Tian-Shansky is just an $\mathcal{O}$-operator associated to the adjoint representation of $\mathfrak{g}$.

Going in the other direction, any $\mathcal{O}$-operator gives an antisymmetric solution of the CYBE in a semi-direct product Lie algebra, completing the cycle from the tensor form to the operator form and back to the tensor form of the CYBE. Moreover, there is a closely related algebraic structure called the pre-Lie algebra. Any $\mathcal{O}$-operator gives a pre-Lie algebra and conversely, any pre-Lie algebra naturally gives an $\mathcal{O}$-operator of the commutator Lie algebra, and hence naturally gives rise to a solution of the CYBE [5].

An analogue of the CYBE for associative algebras is the **associative Yang-Baxter equation (AYBE)** [2]:

$$r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12} = 0,$$

for $r \in A \otimes A$, where A is an associative algebra (see Definition 2.6 for details). Its form with spectral parameters was given in [38] in connection with the CYBE and the quantum Yang-Baxter equation. The AYBE arose from the study of the (antisymmetric) infinitesimal bialgebras, a notion traced back to Joni and Rota in order to provide an algebraic framework for the calculus of divided differences [24, 25] and, in the antisymmetric case, carrying the same structures under the names of "associative D-bialgebra" in [49] and "balanced infinitesimal bialgebra" in the sense of the opposite algebra in [2]. The AYBE has found applications in various fields in mathematics and mathematical physics such as Poisson geometry, integrable systems, quantum groups and mirror symmetry [4, 27, 29, 33–35, 40, 41, 44].

Motivated by the operator approach to the CYBE and the Rota-Baxter operators with weights, $\mathcal{O}$-operators with weights were introduced to give an operator approach to the AYBE [10], while a method of obtaining Rota-Baxter operators from solutions of the (opposite) AYBE was obtained in [1]. Briefly speaking, under the antisymmetric condition, a solution of the AYBE is an $\mathcal{O}$-operator associated to the dual representation of the adjoint representation, while an $\mathcal{O}$-operator gives an antisymmetric solution of the AYBE in a semi-direct product associative algebra. Furthermore, the role played by pre-Lie algebras in CYBE is similarly played by dendriform algebras introduced by Loday [31], that is, any $\mathcal{O}$-operator induces a dendriform algebra structure on the representation space and conversely, a dendriform algebra gives a natural $\mathcal{O}$-operator and hence there is a construction of (antisymmetric) solutions of the AYBE from dendriform algebras [8, 11]. Moreover, such relationships are generalized to connect the solutions of the AYBE satisfying certain "invariant" conditions and $\mathcal{O}$-operators with weights [8, 10].

In turn, these studies of the AYBE by $\mathcal{O}$-operators with weights led to the introduction of similar $\mathcal{O}$-operators on Lie algebras. These generalizations have found fruitful applications to the CYBE and further to Lax pairs, Lie bialgebras and PostLie algebras [7, 9].

1.2. Nonhomogeneous AYBE and its operator form

The notion of a **nonhomogeneous associative classical Yang-Baxter equation (nhacYBe)** [36] is the equation (detailed in Definition 2.6)

$$r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12} = \mu r_{13}, \tag{2}$$

where μ is a fixed constant. Its opposite form, given in Eq. (8), was called the **associative classical Yang-Baxter equation of weight** μ in [19]. Taking $\mu = 0$ recovers the AYBE.

The nhacYBe arose from the study of the quantum Yang-Baxter equation and Bezout operators. Another motivation for introducing the nhacYBe is the μ**-infinitesimal bialgebras**, that is, a triple $(A, \cdot, \Delta)$ consisting of an algebra $(A, \cdot)$ and a coalgebra (A, Δ) satisfying the compatibility condition

$$\Delta(x \cdot y) = (L(x) \otimes \mathrm{id})\Delta(y) + \Delta(x)(\mathrm{id} \otimes R(y)) - \mu x \otimes y, \ \ \forall x, y \in A,$$

where $L(x), R(x)$ are the left and right multiplication operators of $(A, \cdot)$ respectively. When $\mu = 1$, it was also called a **unital infinitesimal bialgebra** [32] and appeared in several topics such as combinatorics, operads and pre-Lie algebras [21, 22, 47, 48]. A solution of the opposite form of the nhacYBe in a unital algebra gives a μ-infinitesimal bialgebra [19, 36].

Note that while the AYBE has its origin from the CYBE for Lie algebras, when $\mu \neq 0$, the nhacYBe does not have a counterpart for Lie algebras since the term r_{13} on the right hand side of Eq. (2) does not make sense for a Lie algebra.

As in the cases of the CYBE and the AYBE, it is important to study the nhacYBe through its operator forms, to give further understanding on the nature of the equation, and to provide constructions of its solutions. To address the challenge from the nonhomogeneity, $\mathcal{O}$-operators and Rota-Baxter operators are generalized and new approaches are introduced (see Remark 3.14).

1.3. Outline of the paper

We next provide some details of our operator approach of the nhacYBe which also serve as an outline of the paper.

In Section 2, we first generalize the notion of an $\mathcal{O}$-operator whose weight is a scalar to one whose weight is a binary operation. We then interpret solutions of the nhacYBe equivalently in terms of generalized $\mathcal{O}$-operators (Theorem 2.8) and, in the presence of a symmetric Frobenius algebra, in terms of generalized Rota-Baxter algebras (Theorem 2.16). On Frobenius algebras, such an interpretation also gives a correspondence between solutions of the AYBE and Rota-Baxter systems introduced in [15], rather than Rota-Baxter operators by themselves (Corollary 2.18). In order to make a connection with the existing notion of $\mathcal{O}$-operators and Rota-Baxter operators, we explore the additional conditions for solutions of the nhacYBe. As it turns out, a solution r of the nhacYBe can be interpreted in terms of an $\mathcal{O}$-operator precisely when it satisfies the **symmetrized invariant** condition that the extended symmetrizer

$$\mathbf{r} := r + \sigma(r) - \mu(\mathbf{1} \otimes \mathbf{1}) \tag{3}$$

of r is invariant, where σ is the flip map (Theorem 2.22). Note that the parameter μ appears in both the nhacYBe and the invariant condition, especially as the scalar multiple of $\mathbf{1} \otimes \mathbf{1}$ for the latter. In particular, the vanishing of the extended symmetrizer of a solution r means that $(r, -\sigma(r))$ is an associative Yang-Baxter pair in the sense of [15] (Corollary 2.28).

In Section 3, we present a close relationship between the nhacYBe and Rota-Baxter operators including but exceeding the known relationships between the antisymmetric solutions of the AYBE and Rota-Baxter operators of weight zero on Frobenius algebras given in [10]. In unital symmetric Frobenius algebras, when the extended symmetrizer is a multiple of the nondegenerate invariant tensor corresponding to the nondegenerate bilinear form defining the Frobenius algebra structure, that is, the extended symmetrizer is a nondegenerate invariant tensor or zero, there is a characterization of the solutions of the nhacYBe by Rota-Baxter operators (Theorem 3.1). As special cases, taking the matrix algebras gives the correspondence in [36], and taking the trivial extended symmetrizer and $\mu = 0$ yields the correspondence in [10]. Assume that the extended symmetrizer is degenerate. Then in one direction, there is a construction of solutions of the nhacYBe from Rota-Baxter operators satisfying its own invariant conditions (Proposition 3.5). Based on such a construction, we obtain symmetrized invariant solutions of the nhacYBe for $\mu \neq 0$ in semi-direct product algebras from $\mathcal{O}$-operators of weight zero as well as from dendriform algebras of Loday [31]. Note that these constructions are different from the construction of solutions of the AYBE from $\mathcal{O}$-operators given in [10] due to the appearance of the new term $\mu(\mathbf{1} \otimes \mathbf{1})$ in Eq. (3) (Remark 3.14). In the other direction, Rota-Baxter operators can also be obtained from solutions of the nhacYBe in an augmented algebra, that is, the unitization of an associative algebra (Theorem 3.17 and Corollary 3.19).

In Section 4, we give the classification of the symmetrized invariant solutions of the nhacYBe for $\mu \neq 0$ in the unital complex algebras in dimensions two and three. These examples indicate that the symmetrized invariant solutions of the nhacYBe only comprise a small part of all solutions of the nhacYBe. Moreover, we also find that all symmetrized invariant solutions of the nhacYBe for $\mu \neq 0$ in the unital complex algebras in dimensions two and three are obtained from Rota-Baxter operators.

Notations. Throughout the paper, we fix a base field $\mathbf{k}$. Unless otherwise specified, all the vector spaces and algebras are finite-dimensional, although some results and notions remain valid in the infinite-dimensional case. By a $\mathbf{k}$-algebra, we mean an associative algebra over $\mathbf{k}$ not necessarily having a unit.

2. Characterizations of nhacYBe by generalized $\mathcal{O}$-operators

We first recall some basic definitions and facts that will be used in this paper. We then introduce the notion of a generalized $\mathcal{O}$-operator whose weight is a binary operation. Especially, when the binary operation is obtained from an A-bimodule $\mathbf{k}$-algebra, we recover the notion of the $\mathcal{O}$-operator of weight λ. We moreover give a general interpretation of the nhacYBe in terms of generalized $\mathcal{O}$-operators, including a correspondence between solutions of the nhacYBe with $\mu = 0$, that is, the AYBE, and Rota-Baxter systems [15] on Frobenius algebras. Finally under the additional invariant condition, this interpretation gives a correspondence between symmetrized invariant solutions of the nhacYBe and $\mathcal{O}$-operators with weight λ.

2.1. $\mathcal{O}$-operators and Rota-Baxter operators for bimodules

We generalize the notions of $\mathcal{O}$-operators and Rota-Baxter operators from those with scalar weights to the ones with weights given by binary operations. We first briefly recall some background and refer the reader to [6, 10] for further details.

Let $(A, \cdot)$ be a $\mathbf{k}$-algebra. An **A-bimodule** is a $\mathbf{k}$-module V, together with linear maps $\ell, r : A \to \mathrm{End}_{\mathbf{k}}(V)$ satisfying

$$\begin{aligned} \ell(x \cdot y)v = \ell(x)(\ell(y)v), \quad vr(x \cdot y) = (vr(x))r(y), \\ (\ell(x)v)r(y) = \ell(x)(vr(y)), \quad \forall\, x, y \in A, v \in V. \end{aligned}$$

If we want to be more precise, we also denote an A-bimodule V by the triple (V, ℓ, r).

Given a $\mathbf{k}$-algebra $A = (A, \cdot)$ and $x \in A$, define

$$L(x) : A \to A,\ L(x)y = xy; \quad R(x) : A \to A,\ yR(x) = yx,\ \forall\, y \in A$$

to be the left and right actions on A. We further define

$$\begin{aligned} L = L_A : A \to \mathrm{End}_{\mathbf{k}}(A),\ x \mapsto L(x); \\ R = R_A : A \to \mathrm{End}_{\mathbf{k}}(A),\ x \mapsto R(x),\ \forall\, x \in A. \end{aligned}$$

Clearly, (A, L_A, R_A) is an A-bimodule, called the **adjoint A-bimodule**.

There is a natural characterization of semi-direct product extensions of a $\mathbf{k}$-algebra $(A, \cdot)$ by an A-bimodule. Let $\ell, r : A \to \mathrm{End}_{\mathbf{k}}(V)$ be linear maps. Define a multiplication on $A \oplus V$ (still denoted by $\cdot$) by

$$(a + u) \cdot (b + v) := a \cdot b + (\ell(a)v + ur(b)), \quad \forall a, b \in A, u, v \in V.$$

Then as is well known, $A \oplus V$ is a $\mathbf{k}$-algebra, denoted by $A \ltimes_{\ell,r} V$ and called the **semi-direct product** of A by V, if and only if (V, ℓ, r) is an A-bimodule.

For a $\mathbf{k}$-module V and its dual module $V^* := \mathrm{Hom}_{\mathbf{k}}(V, \mathbf{k})$, the usual pairing between them is given by

$$\langle , \rangle : V^* \times V \to \mathbf{k},\ \langle u^*, v \rangle = u^*(v),\ \forall\, u^* \in V^*, v \in V.$$

Identifying V with $(V^*)^*$, we also use $\langle v, u^* \rangle = \langle u^*, v \rangle$.

Let A be a $\mathbf{k}$-algebra and let (V, ℓ, r) be an A-bimodule. Define linear maps $\ell^*, r^* : A \to \mathrm{End}_{\mathbf{k}}(V^*)$ by

$$\langle u^* \ell^*(x), v\rangle = \langle u^*, \ell(x)v\rangle,\ \langle r^*(x)u^*, v\rangle = \langle u^*, vr(x)\rangle,$$
$$\forall x \in A, u^* \in V^*, v \in V,$$

respectively. Then (V^*, r^*, ℓ^*) is also an A-bimodule, called the **dual A-bimodule** of (V, ℓ, r).

To give an operator interpretation of solutions of the nhacYBe, we generalize the notion of $\mathcal{O}$-operators with weights introduced in [10] by dropping the condition that the multiplication $\circ$ on R turns $(R, \circ, \ell, r)$ into an A-bimodule $\mathbf{k}$-algebra.

Definition 2.1. Let $(A, \cdot)$ be a $\mathbf{k}$-algebra. Let (R, ℓ, r) be an A-bimodule and $\circ$ a binary operation on R. A linear map $\alpha : R \to A$ is called an **$\mathcal{O}$-operator of weight $\circ$ associated to** (R, ℓ, r) or simply a **generalized $\mathcal{O}$-operator** if α satisfies

$$\alpha(u) \cdot \alpha(v) = \alpha(\ell(\alpha(u))v) + \alpha(ur(\alpha(v))) + \alpha(u \circ v),\ \ \forall u, v \in R.$$

In particular, if $(R, \ell, r) = (A, L_A, R_A)$ is the adjoint A-bimodule and $\circ$ is a binary operation on A, then an $\mathcal{O}$-operator $\alpha : A \to A$ of weight $\circ$ associated to the A-bimodule (A, L_A, R_A) is called a **Rota-Baxter operator of weight** $\circ$. In this case α satisfies

$$\alpha(x) \cdot \alpha(y) = \alpha(\alpha(x) \cdot y) + \alpha(x \cdot \alpha(y)) + \alpha(x \circ y),\ \ \forall x, y \in A.$$

Example 2.2. In the definition of Rota-Baxter operators with weight $\circ$, when $\circ$ is given by $x \circ y := \lambda x \cdot y$ for a given $\lambda \in \mathbf{k}$, we recover the usual **Rota-Baxter operator of weight** λ, with its defining operator identity

$$P(x) \cdot P(y) = P(x \cdot y) + P(P(x) \cdot y) + \lambda P(x \cdot y),\quad \forall x, y \in A. \tag{4}$$

Here the notion is named after the mathematicians G.-C. Rota [39] and G. Baxter [12] for their early work motivated by fluctuation theory in probability and combinatorics, which again appeared in the work of Connes and Kreimer on renormalization of quantum field theory [18] as a fundamental algebraic structure. See [23] for further details.

We separately define a special case that will be important to us.

Definition 2.3. Let $(A, \cdot)$ be a $\mathbf{k}$-algebra and (R, ℓ, r) be an A-bimodule. Let $s : R \to A$ be a linear map. A linear map $\alpha : R \to A$ is called an **$\mathcal{O}$-operator right twisted by** s associated to (R, ℓ, r) if

$$\alpha(u) \cdot \alpha(v) = \alpha(\ell(\alpha)u))v) + \alpha(ur(\alpha(v))) + \alpha(ur(s(v))), \ \ \forall u, v \in R.$$

Likewise α is called an **$\mathcal{O}$-operator left twisted by** s **associated to** (R, ℓ, r) when the third term one the right hand side of the above equation is replaced by $\alpha(\ell(s(u))v)$.

When the A-bimodule is taken to be (A, L_A, R_A), the operator is called the **Rota-Baxter operator right twisted by** s (resp. **left twisted by** s).

Obviously the operators in Definition 2.3 are the special cases of the operators in Definition 2.1 when the binary operation $\circ$ are defined by

$$u \circ v := ur(s(v)) \text{ (resp. } u \circ v := \ell(s(u))v), \quad \forall u, v \in R.$$

To recover the notion of $\mathcal{O}$-operators with scalar weights introduced in [10], we recall a concept combining A-bimodules with $\mathbf{k}$-algebras [46].

Definition 2.4. Let $(A, \cdot)$ be a $\mathbf{k}$-algebra with multiplication $\cdot$ and let $(R, \circ)$ be a $\mathbf{k}$-algebra with multiplication $\circ$. Let $\ell, r : A \to \mathrm{End}_{\mathbf{k}}(R)$ be linear maps. We call R (or the quadruple $(R, \circ, \ell, r)$) an **A-bimodule k-algebra** if (R, ℓ, r) is an A-bimodule that is compatible with the multiplication $\circ$ on R in the sense that

$$\begin{aligned}&\ell(x)(v \circ w) = (\ell(x)v) \circ w, \ (v \circ w)r(x) = v \circ (wr(x)),\\ &(vr(x)) \circ w = v \circ (\ell(x)w),\end{aligned}$$

for all $x, y \in A, v, w \in R$.

Obviously, $(A, \cdot, L_A, R_A)$ is an A-bimodule $\mathbf{k}$-algebra.

In Definition 2.1, when the A-bimodule (R, ℓ, r) with multiplication $*$ is assumed to be an A-bimodule $\mathbf{k}$-algebra and when $u \circ v = \lambda u * v$ for $\lambda \in \mathbf{k}$, we recover the following notion of an $\mathcal{O}$-operator with weight λ in [10] which is also called a relative Rota-Baxter operator [11].

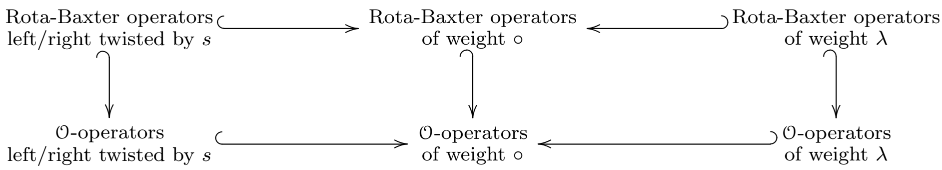
Definition 2.5. Let $(A, \cdot)$ be a $\mathbf{k}$-algebra and let $(R, *, \ell, r)$ be an A-bimodule $\mathbf{k}$-algebra. Let $\lambda \in \mathbf{k}$. A linear map $\alpha : R \to A$ is called an **$\mathcal{O}$-operator of weight** λ associated to $(R, *, \ell, r)$ if α satisfies

$$\alpha(u) \cdot \alpha(v) = \alpha(\ell(\alpha(u))v) + \alpha(ur(\alpha(v))) + \lambda\alpha(u * v), \quad \forall u, v \in R.$$

When $* = 0$, then $\mathcal{O}$ is called an **$\mathcal{O}$-operator** (of weight zero) associate to the A-bimodule (R, ℓ, r).

When R is the A-bimodule **k**-algebra (A, L_A, R_A) with $u \circ v := \lambda u \cdot v$ for $\lambda \in \mathbf{k}$ and the default multiplication $\cdot$ of A, we recover the notion of a Rota-Baxter operator P of weight λ defined in Eq. (4).

These structures can be summarized in the commutative diagram



2.2. Operator forms of solutions of nhacYBe

We recall the notion of the nhacYBe and give an interpretation of solutions of the nhacYBe in terms of the generalized $\mathcal{O}$-operators just introduced.

Let $(A, \cdot, \mathbf{1})$ be a unital **k**-algebra of which the multiplication $\cdot$ is often suppressed. For $r = \sum_i a_i \otimes b_i \in A \otimes A$, denote

$$(5) \quad r_{12} := \sum_i a_i \otimes b_i \otimes \mathbf{1}, \quad r_{13} := \sum_i a_i \otimes \mathbf{1} \otimes b_i, \quad r_{23} := \sum_i \mathbf{1} \otimes a_i \otimes b_i.$$

Then $r_{12}r_{13}, r_{13}r_{23}, r_{23}r_{12}$ are elements in the **k**-algebra $A \otimes A \otimes A$.

Definition 2.6. Let A be a unital **k**-algebra and let $r \in A \otimes A$.

(a) r is a solution of the **associative Yang-Baxter equation (AYBE)**

$$(6) \qquad r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12} = 0$$

in A if the equation holds with the notation in Eq. (5).

(b) Fix a $\mu \in \mathbf{k}$. r is a solution of the **μ-nonhomogeneous associative Yang-Baxter equation (μ-nhacYBe)**

$$(7) \qquad r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12} = \mu r_{13}$$

in A if the equation holds with the notation in Eq. (5).

The opposite form of Eq. (7) is [19]

$$(8) \qquad r_{13}r_{12} + r_{23}r_{13} - r_{12}r_{23} = \mu r_{13}.$$

Definition 2.7. Let A be a unital $\mathbf{k}$-algebra and $\mu \in \mathbf{k}$. Let $r \in A \otimes A$. Define the μ-**extended symmetrizer** of r to be

$$\mathbf{r} := r + \sigma(r) - \mu(\mathbf{1} \otimes \mathbf{1}). \tag{9}$$

The prefix μ in Definitions 2.6 and 2.7 will be suppressed when its meaning is clear from the context.

Let $r \in A \otimes A$. Define linear maps $r^\sharp, r^{t\sharp} : A^* \to A$ by the canonical bijections

$$\begin{aligned}
(_)^\sharp &: A \otimes A \cong \mathrm{Hom}_{\mathbf{k}}(A^*, \mathbf{k}) \otimes A \cong \mathrm{Hom}_{\mathbf{k}}(A^*, A),\\
(_)^{t\sharp} &= (_)^\sharp \sigma : A \otimes A \to \mathrm{Hom}_{\mathbf{k}}(A^*, A).
\end{aligned}$$

Explicitly, $r^\sharp$ and $r^{t\sharp}$ are determined by

$$\langle r^\sharp(a^*), b^*\rangle = \langle r, a^* \otimes b^*\rangle, \quad \langle r^{t\sharp}(a^*), b^*\rangle = \langle r, b^* \otimes a^*\rangle, \quad \forall a^*, b^* \in A^*.$$

With these notations, r is called **nondegenerate** if the linear map $r^\sharp$ or $r^{t\sharp}$ is a linear isomorphism. Otherwise, r is called **degenerate**. Furthermore, r is symmetric if and only if

$$\langle r, a^* \otimes b^*\rangle = \langle r, b^* \otimes a^*\rangle, \text{ that is, } \langle r^\sharp(a^*), b^*\rangle = \langle r^\sharp(b^*), a^*\rangle, \quad \forall a^*, b^* \in A^*.$$

We now give an operator form of solutions of the nhacYBe in terms of the generalized $\mathcal{O}$-operators with weights given by binary operations.

Theorem 2.8. *Let $(A, \cdot, \mathbf{1})$ be a unital $\mathbf{k}$-algebra. For $r \in A \otimes A$, let $\mathbf{r}$ be the extended symmetrizer of r and let $\mathbf{r}^\sharp : A^* \to A$ be the corresponding linear map. Then the following statements are equivalent.*

(a) *The tensor r is a solution of the nhacYBe in A.*

(b) *The following equation holds.*

$$\begin{aligned}
r^\sharp(a^*) \cdot r^\sharp(b^*) &+ r^\sharp(a^* L^*(r^{t\sharp}(b^*)))\\
&- r^\sharp(R^*(r^\sharp(a^*))b^*) - \mu r^\sharp(\langle \mathbf{1}, b^*\rangle a^*) = 0, \quad \forall a^*, b^* \in A^*.
\end{aligned} \tag{10}$$

(c) *The linear map $r^\sharp$ from r is an $\mathcal{O}$-operator right twisted by $-\mathbf{r}^\sharp$ associated to (A^*, R^*, L^*).*

(d) *The following equation holds.*

$$r^{t\sharp}(a^*)\cdot r^{t\sharp}(b^*) - r^{t\sharp}(a^*L^*(r^{t\sharp}(b^*))) + r^{t\sharp}(R^*(r^\sharp(a^*))b^*) - \mu r^{t\sharp}(\langle \mathbf{1}, a^*\rangle b^*) = 0, \forall a^*, b^* \in A^*. \tag{11}$$

(e) *The linear map $r^{t\sharp}$ from $\sigma(r)$ is an $\mathcal{O}$-operator left twisted by $-\mathbf{r}^\sharp$ associated to (A, R^*, L^*).*

Proof. Let $r = \sum_i a_i \otimes b_i$ and $a^*, b^*, c^* \in A^*$.

(a) $\Longleftrightarrow$ (b). We have

$$\begin{aligned}
\langle r_{12}\cdot r_{13}, a^*\otimes b^*\otimes c^*\rangle &= \sum_{i,j}\langle a_i\cdot a_j, a^*\rangle\langle b_i, b^*\rangle\langle b_j, c^*\rangle\\
&= \sum_j \langle r^{t\sharp}(b^*)\cdot a_j, a^*\rangle\langle b_j, c^*\rangle\\
&= \langle r^\sharp(a^*L^*(r^{t\sharp}(b^*))), c^*\rangle,\\
\langle r_{13}\cdot r_{23}, a^*\otimes b^*\otimes c^*\rangle &= \sum_{i,j}\langle a_i, a^*\rangle\langle a_j, b^*\rangle\langle b_i\cdot b_j, c^*\rangle\\
&= \sum_j \langle a_j, b^*\rangle\langle r^\sharp(a^*)\cdot b_j, c^*\rangle\\
&= \langle r^\sharp(a^*)\cdot r^\sharp(b^*), c^*\rangle,\\
\langle -r_{23}\cdot r_{12}, a^*\otimes b^*\otimes c^*\rangle &= -\sum_{i,j}\langle a_i, a^*\rangle\langle a_j\cdot b_i, b^*\rangle\langle b_j, c^*\rangle\\
&= -\sum_j \langle a_j\cdot r^\sharp(a^*), b^*\rangle\langle b_j, c^*\rangle\\
&= \langle -r^\sharp(R^*(r^\sharp(a^*))b^*), c^*\rangle,\\
\langle -\mu r_{13}, a^*\otimes b^*\otimes c^*\rangle &= -\mu\sum_i \langle a_i, a^*\rangle\langle \mathbf{1}, b^*\rangle\langle b_i, c^*\rangle\\
&= \langle -\mu r^\sharp(a^*), c^*\rangle\langle \mathbf{1}, b^*\rangle = \langle -\mu r^\sharp(\langle \mathbf{1}, b^*\rangle a^*), c^*\rangle.
\end{aligned}$$

Hence r satisfies Eq. (7) if and only if Eq. (10) holds.

(b) $\Longleftrightarrow$ (c). From the definition of the extended symmetrizer of r: $\mathbf{r} = r + \sigma(r) - \mu(\mathbf{1}\otimes\mathbf{1})$, we obtain

$$\mathbf{r}^\sharp(b^*) = r^\sharp(b^*) + r^{t\sharp}(b^*) - \mu\langle \mathbf{1}, b^*\rangle\mathbf{1}, \quad \forall b^* \in A^*$$

and hence

$$r^{t\sharp}(b^*) = -r^\sharp(b^*) + \mathbf{r}^\sharp(b^*) + \mu\langle \mathbf{1}, b^*\rangle\mathbf{1}, \quad \forall b^* \in A^*.$$

Further $L^*(\mathbf{1})$ is the identity map on A^*. Thus Eq. (10) is equivalent to

$$\begin{aligned} r^\sharp(a^*)\cdot r^\sharp(b^*) &- r^\sharp(a^*L^*(r^\sharp(b^*))) \\ &- r^\sharp(R^*(r^\sharp(a^*))b^*) + r^\sharp(a^*L^*(\mathbf{r}^\sharp(b^*))) = 0, \ \ \forall a^*, b^*\in A^*, \end{aligned}$$

as needed.

(a) $\Longleftrightarrow$ (d). Similarly, we have

$$\begin{aligned} \langle r_{12}\cdot r_{13}, a^*\otimes b^*\otimes c^*\rangle &= \sum_j \langle r^{t\sharp}(b^*)\cdot a_j, a^*\rangle\langle b_j, c^*\rangle \\ &= \langle r^{t\sharp}(b^*)\cdot r^{t\sharp}(c^*), a^*\rangle, \\ \langle r_{13}\cdot r_{23}, a^*\otimes b^*\otimes c^*\rangle &= \sum_j \langle a_i, a^*\rangle\langle b_i\cdot r^\sharp(b^*), c^*\rangle \\ &= \langle r^{t\sharp}(R^*(r^\sharp(b^*))c^*), a^*\rangle, \\ \langle -r_{23}\cdot r_{12}, a^*\otimes b^*\otimes c^*\rangle &= -\sum_j \langle a_i, a^*\rangle\langle r^{t\sharp}(c^*)\cdot b_i, b^*\rangle \\ &= -\langle r^{t\sharp}(b^*L^*(r^{t\sharp}(c^*))), a^*\rangle, \\ \langle -\mu r_{13}, a^*\otimes b^*\otimes c^*\rangle &= \langle -\mu r^{t\sharp}(c^*), a^*\rangle\langle \mathbf{1}, b^*\rangle \\ &= \langle -\mu r^{t\sharp}(\langle \mathbf{1}, b^*\rangle c^*), a^*\rangle. \end{aligned}$$

Hence r satisfies Eq. (7) if and only if Eq. (11) holds.

(d) $\Longleftrightarrow$ (e). The proof is the same as for (b) $\Longleftrightarrow$ (c). □

We now show that the opposite nhacYBe in Eq. (8) also affords an operator form.

Lemma 2.9. *Let* $(A, \cdot, \mathbf{1})$ *be a unital* $\mathbf{k}$*-algebra. Let* $r\in A\otimes A$*. Then* r *satisfies Eq.* (7) *if and only if* $\sigma(r)$ *satisfies Eq.* (8).

Proof. Let $r=\sum_i a_i\otimes b_i\in A\otimes A$. Then r satisfies Eq. (7) if and only if

$$\begin{aligned} \sum_{i,j}(a_i\cdot a_j\otimes b_i\otimes b_j + a_i\otimes a_j\otimes b_i\cdot b_j \\ - a_j\otimes a_i\cdot b_j\otimes b_i - \mu a_i\otimes \mathbf{1}\otimes b_i) = 0. \end{aligned} \tag{12}$$

On the other hand, $\sigma(r)=\sum_i b_i\otimes a_i$ satisfies Eq. (8) if and only if

$$\begin{aligned} \sum_{i,j}(b_i\cdot b_j\otimes a_j\otimes a_i + b_j\otimes b_i\otimes a_i\cdot a_j \\ - b_i\otimes a_i\cdot b_j\otimes a_j - \mu b_i\otimes \mathbf{1}\otimes a_i) = 0. \end{aligned} \tag{13}$$

Let $\sigma_{13} : A \otimes A \otimes A \to A \otimes A \otimes A$ be the linear map defined by $\sigma(x \otimes y \otimes z) = z \otimes y \otimes x$ for $x, y, z \in A$. It is straightforward to check that the left hand side of Eq. (12) coincides with applying σ_{13} to the left hand side of Eq. (13). This completes the proof. □

Then we have

Corollary 2.10. *Let* $(A, \cdot, \mathbf{1})$ *be a unital* $\mathbf{k}$*-algebra. For* $r \in A \otimes A$*, let* $\mathbf{r}$ *be the extended symmetrizer of* r *and let* $\mathbf{r}^\sharp : A^* \to A$ *be the corresponding linear map. Then* r *satisfies Eq.* (8) *if and only if the linear map* $r^\sharp : A^* \to A$ *from* r *is an* $\mathcal{O}$*-operator left twisted by* $-\mathbf{r}^\sharp$ *associated to* (A^*, R^*, L^*)*.*

Proof. Since $\sigma(r)^\sharp = r^{t\sharp}$, the conclusion follows from Theorem 2.8 and Lemma 2.9. □

2.3. Operator forms of solutions in a Frobenius algebra

We now consider the solutions of the nhacYBe in a Frobenius algebra.

Definition 2.11. Let $(A, \cdot)$ be a $\mathbf{k}$-algebra. A tensor $s \in A \otimes A$ is called **invariant** if

$$(\mathrm{id} \otimes L(x) - R(x) \otimes \mathrm{id})s = 0, \quad \forall x \in A.$$

Lemma 2.12. ([10]) *Let* $(A, \cdot)$ *be a* $\mathbf{k}$*-algebra. Let* $s \in A \otimes A$ *be symmetric. Then the following conditions are equivalent.*

(a) s *is invariant.*

(b) $s^\sharp$ *satisfies*

$$R^*(s^\sharp(a^*))b^* = a^* L^*(s^\sharp(b^*)), \quad \forall a^*, b^* \in A^*.$$

(c) $s^\sharp$ *satisfies*

$$s^\sharp(R^*(x)a^*) = x \cdot s^\sharp(a^*), \quad s^\sharp(a^* L^*(x)) = s^\sharp(a^*) \cdot x, \quad \forall x \in A, a^* \in A^*.$$

Remark 2.13. For a unital $\mathbf{k}$-algebra $(A, \mathbf{1})$, it is obvious that $\mathbf{1} \otimes \mathbf{1}$ is not invariant when $\dim A \geq 2$.

Definition 2.14. A bilinear form $\mathfrak{B} := \mathfrak{B}(\ ,\ )$ on a $\mathbf{k}$-algebra $(A, \cdot)$ is called **invariant** if

$$\mathfrak{B}(a \cdot b, c) = \mathfrak{B}(a, b \cdot c), \quad \forall\ a, b, c \in A.$$

A **Frobenius algebra** $(A, \mathfrak{B})$ is a $\mathbf{k}$-algebra A with a nondegenerate invariant bilinear form $\mathfrak{B}(\ ,\ )$. A Frobenius algebra $(A, \mathfrak{B})$ is called **symmetric** if $\mathfrak{B}(\ ,\ )$ is symmetric.

Let $\mathrm{Iso}_{\mathbf{k}}(M, N)$ denote the set of linear bijections between $\mathbf{k}$-vector spaces M and N of the same dimension. Let $\mathrm{NDHom}(A \otimes A, \mathbf{k})$ and $\mathrm{ND}(A \otimes A)$ denote the sets of nondegenerate bilinear forms on A and nondegenerate tensors in $A \otimes A$ respectively. Then by definition, the linear bijection $\mathrm{Hom}_{\mathbf{k}}(A \otimes A, \mathbf{k}) \cong \mathrm{Hom}_{\mathbf{k}}(A, A^*)$ restricts to a bijection $\mathrm{NDHom}_{\mathbf{k}}(A \otimes A, \mathbf{k}) \cong \mathrm{Iso}_{\mathbf{k}}(A, A^*)$. Similarly, the linear bijection $A \otimes A \cong \mathrm{Hom}_{\mathbf{k}}(A^*, A)$ restricts to a bijection $\mathrm{ND}(A \otimes A) \cong \mathrm{Iso}_{\mathbf{k}}(A^*, A)$. Then thanks to the bijection $\mathrm{Iso}_{\mathbf{k}}(A, A^*) \cong \mathrm{Iso}_{\mathbf{k}}(A^*, A)$ by taking inverse, we obtain a bijection

$$\mathrm{NDHom}_{\mathbf{k}}(A \otimes A, \mathbf{k}) \cong \mathrm{Iso}_{\mathbf{k}}(A, A^*) \cong \mathrm{Iso}_{\mathbf{k}}(A^*, A) \cong \mathrm{ND}(A \otimes A). \tag{14}$$

Explicitly, let $\mathfrak{B}$ be a nondegenerate bilinear form. Let $\phi^\sharp = \phi^\sharp_{\mathfrak{B}} : A^* \to A$ be the linear isomorphism defined by

$$\langle {\phi^\sharp}^{-1}(x), y \rangle = \mathfrak{B}(x, y),\ \forall x, y \in A. \tag{15}$$

The corresponding tensor $\phi \in A \otimes A$ is the one induced from the linear map $\phi^\sharp$.

Lemma 2.15. *Let $(A, \cdot)$ be a $\mathbf{k}$-algebra. A nondegenerate bilinear form $\mathfrak{B}$ is symmetric and invariant* (*and hence gives a symmetric Frobenius algebra* $(A, \cdot, \mathfrak{B})$) *if and only if the corresponding $\phi \in A \otimes A$ via Eq.* (14) *is symmetric and invariant.*

Proof. For $a^*, b^* \in A^*$, let $x = \phi^\sharp(a^*)$ and $y = \phi^\sharp(b^*)$. Then from Eq. (15) we obtain

$$\mathfrak{B}(x, y) = \langle (\phi^\sharp)^{-1}(x), y \rangle = \langle a^*, \phi^\sharp(b^*) \rangle = \langle b^* \otimes a^*, \phi \rangle.$$

Thus $\mathfrak{B}(x, y) - \mathfrak{B}(y, x) = \langle b^* \otimes a^* - a^* \otimes b^*, \phi \rangle$ which shows that $\mathfrak{B}$ is symmetric if and only if ϕ is symmetric.

Then under the symmetric condition of $\mathfrak{B}$ and hence of ϕ, for $z \in A$, we have

$$\begin{aligned}\mathfrak{B}(y \cdot z, x) - \mathfrak{B}(y, z \cdot x) &= \mathfrak{B}(\phi^\sharp(b^*) \cdot z, \phi^\sharp(a^*)) - \mathfrak{B}(\phi^\sharp(b^*), z \cdot \phi^\sharp(a^*)) \\ &= \langle a^*, \phi^\sharp(b^*) \cdot z\rangle - \langle b^*, z \cdot \phi^\sharp(a^*)\rangle \\ &= \langle a^* L^*(\phi^\sharp(b^*)), z\rangle - \langle R^*(\phi^\sharp(a^*))b^*, z\rangle \\ &= \langle a^* L^*(\phi^\sharp(b^*)) - \langle R^*(\phi^\sharp(a^*))b^*, z\rangle.\end{aligned}$$

By Lemma 2.12, this shows that $\mathfrak{B}$ is symmetric and invariant if and only if ϕ is symmetric and invariant. □

Theorem 2.16. *Let $(A, \cdot, \mathbf{1}, \mathfrak{B})$ be a unital symmetric Frobenius algebra. Let $\phi^\sharp : A^* \to A$ be the linear isomorphism defined by Eq. (15). For $r \in A \otimes A$, let the linear maps $P_r, P_r^t : A \to A$ be defined respectively by*

$$P_r(x) := r^\sharp(\phi^\sharp)^{-1}(x), \quad P_r^t(x) := r^{t\sharp}(\phi^\sharp)^{-1}(x), \quad \forall x \in A. \tag{16}$$

Let $\mathbf{r}^\sharp(a^*) := r^\sharp(a^*) + r^{t\sharp}(a^*) - \mu\langle \mathbf{1}, a^*\rangle \mathbf{1}, a^* \in A^*$ be defined by the extended symmetrizer $\mathbf{r}$ of r. Then the following statements are equivalent.

(a) *r is a solution of the nhacYBe in A.*

(b) *The following equation holds.*

$$\begin{aligned}P_r(x) \cdot P_r(y) = P_r(P_r(x) \cdot y) \\ - P_r(x \cdot P_r^t(y)) + \mu\mathfrak{B}(\mathbf{1}, y)P_r(x), \quad \forall x, y \in A.\end{aligned} \tag{17}$$

(c) *The following equation holds.*

$$\begin{aligned}P_r^t(x) \cdot P_r^t(y) = P_r^t(-P_r(x) \cdot y) \\ + P_r^t(x \cdot P_r^t(y)) + \mu\mathfrak{B}(\mathbf{1}, x)P_r^t(y), \quad \forall x, y \in A.\end{aligned} \tag{18}$$

(d) *The operator P_r on A is a Rota-Baxter operator right twisted by $-\mathbf{r}^\sharp(\phi^\sharp)^{-1}$, that is,*

$$\begin{aligned}P_r(x) \cdot P_r(y) = P_r(P_r(x) \cdot y) + P_r(x \cdot P_r(y)) \\ - P_r(x \cdot \mathbf{r}^\sharp(\phi^\sharp)^{-1}(y)), \quad \forall x, y \in A.\end{aligned}$$

(e) *The operator* P_r^t *on* A *is a Rota-Baxter operator left twisted by* $-\mathbf{r}^\sharp(\phi^\sharp)^{-1}$*, that is,*

$$\begin{aligned} P_r^t(x) \cdot P_r^t(y) = P_r^t(P_r^t(x) \cdot y) + P_r^t(x \cdot P_r^t(y)) \\ - P_r^t(\mathbf{r}^\sharp(\phi^\sharp)^{-1}(x) \cdot y), \quad \forall x, y \in A. \end{aligned}$$

Proof. For $x, y \in A$, setting $a^* = {\phi^\sharp}^{-1}(x), b^* = {\phi^\sharp}^{-1}(y)$, we have

$$\begin{aligned} P_r(x) \cdot P_r(y) &= r^\sharp(a^*) \cdot r^\sharp(b^*), \\ P_r(P_r(x) \cdot y) &= r^\sharp {\phi^\sharp}^{-1}(r^\sharp {\phi^\sharp}^{-1}(x) \cdot \phi^\sharp(b^*)) \\ &= r^\sharp {\phi^\sharp}^{-1}(r^\sharp(a^*) \cdot \phi^\sharp(b^*)) = r^\sharp(R^*(r^\sharp(a^*))b^*), \\ P_r(x \cdot P_r^t(y)) &= r^\sharp {\phi^\sharp}^{-1}(\phi^\sharp(a^*) \cdot r^{t\sharp} {\phi^\sharp}^{-1}(y)) \\ &= r^\sharp {\phi^\sharp}^{-1}(\phi^\sharp(a^*) \cdot r^{t\sharp}(b^*)) = r^\sharp(a^* L^*(r^{t\sharp}(b^*))), \\ \mathfrak{B}(\mathbf{1}, y) P_r(x) &= P_r \phi^\sharp(a^*) \mathfrak{B}(\mathbf{1}, y) = r^\sharp(\langle \mathbf{1}, b^* \rangle a^*). \end{aligned}$$

Note that the invariance of ϕ given by Lemma 2.15 is used in deriving Eqs. (17) and (18). By Theorem 2.8, r satisfies Eq. (7) if and only if P_r satisfies Eq. (17). Similarly, we show that r satisfies Eq. (7) if and only if P_r^t satisfies Eq. (18). Hence statements (a) – (c) are equivalent.

Next for any $x \in A$ and $b^* \in A^*$, we have

$$\begin{aligned} \langle P_r(x) + P_r^t(x), b^* \rangle &= \langle r^\sharp({\phi^\sharp}^{-1}(x)) + r^{t\sharp}({\phi^\sharp}^{-1}(x)), b^* \rangle \\ &= \langle \mathbf{r}^\sharp {\phi^\sharp}^{-1}(x) + \mu \langle {\phi^\sharp}^{-1}(x), \mathbf{1} \rangle \mathbf{1}, b^* \rangle \\ &= \langle \mathbf{r}^\sharp {\phi^\sharp}^{-1}(x) + \mu \mathfrak{B}(x, \mathbf{1}) \mathbf{1}, b^* \rangle. \end{aligned}$$

Hence

$$P_r^t(x) = -P_r(x) + \mathbf{r}^\sharp {\phi^\sharp}^{-1}(x) + \mu \mathfrak{B}(x, \mathbf{1}) \mathbf{1}, \quad \forall x \in A.$$

Then the equivalence of the statement (b) (resp. (c)) to the statement (d) (resp. (e)) follows from applying this equation. □

We give an application to Rota-Baxter systems introduced by Brzeziński [15].

Definition 2.17. Let A be a $\mathbf{k}$-algebra. Let $P, S : A \to A$ be two linear maps. The triple (A, P, S) is called a **Rota-Baxter system** if for $x, y \in A$, the following equations hold

$$P(x)P(y) = P(P(x)y + xS(y)), \quad S(x)S(y) = S(P(x)y + xS(y)).$$

Taking $\mu = 0$ in the equivalent statements (a)–(c) in Theorem 2.16 gives

Corollary 2.18. *Let $(A, \cdot, \mathbf{1}, \mathfrak{B})$ be a unital symmetric Frobenius algebra. For $r \in A \otimes A$, let P_r and P_r^t be defined as in Eq. (16). Then r is a solution of the AYBE in Eq. (6) if and only if $(A, P_r, -P_r^t)$ is a Rota-Baxter system.*

2.4. Operator forms of symmetrized invariant solutions of nhacYBe

We now show that, under an invariant condition, solutions of the nhacYBe can be interpreted in terms of the usual $\mathcal{O}$-operators in Definition 2.5.

Definition 2.19. Let $(A, \cdot, \mathbf{1})$ be a unital $\mathbf{k}$-algebra. A tensor $r \in A \otimes A$ is called **symmetrized invariant** if its extended symmetrizer $\mathbf{r}$ defined in Eq. (9) is invariant.

Lemma 2.20. (a) *Let $(A, \cdot, \mathbf{1})$ be a unital $\mathbf{k}$-algebra. Let $s \in A \otimes A$ be symmetric and invariant. Set*

$$a^* \circ b^* := a^* L^*(s^\sharp(b^*)) = R^*(s^\sharp(a^*))b^*, \quad \forall a^*, b^* \in A^*. \tag{19}$$

Then $(A^*, \circ, R^*, L^*)$ is an A-bimodule $\mathbf{k}$-algebra.

(b) *Let $(A^*, \circ, R^*, L^*)$ be an A-bimodule $\mathbf{k}$-algebra. Define a linear map $s^\sharp : A^* \to A$, or equivalently $s \in A \otimes A$, by*

$$\langle s, a^* \otimes b^* \rangle := \langle s^\sharp(a^*), b^* \rangle := \langle b^* \circ a^*, \mathbf{1} \rangle, \quad \forall a^*, b^* \in A^*. \tag{20}$$

Suppose

$$\langle a^* \circ b^*, \mathbf{1} \rangle = \langle b^* \circ a^*, \mathbf{1} \rangle, \quad \forall a^*, b^* \in A^*, \tag{21}$$

and $s^\sharp$ satisfies

$$\langle s^\sharp(a^*) \cdot x, b^* \rangle = \langle b^* \circ a^*, x \rangle, \quad \forall x \in A, a^*, b^* \in A^*. \tag{22}$$

Then s is symmetric and invariant.

Proof. (a). Let $a^*, b^*, c^* \in A^*$ and $x, y \in A$. Then we have

$$\begin{aligned}(a^* \circ b^*) \circ c^* &= a^* L^*(s^\sharp(b^*)) \circ c^* = a^* L^*(s^\sharp(b^*)) L^*(s^\sharp(c^*)),\\ a^* \circ (b^* \circ c^*) &= a^* \circ b^* L^*(s^\sharp(c^*)) = a^* L^*(s^\sharp(b^* L^*(s^\sharp(c^*))))\\ &= a^* L^*(s^\sharp(b^*) * s^\sharp(c^*)).\end{aligned}$$

Hence $(A^*, \circ)$ is a $\mathbf{k}$-algebra. Moreover,

$$\begin{aligned}\langle R^*(x)(a^* \circ b^*), y\rangle &= \langle a^* L^*(s^\sharp(b^*)), y \cdot x\rangle = \langle a^*, s^\sharp(b^*) \cdot y \cdot x\rangle,\\ \langle (R^*(x)a^*) \circ b^*, y\rangle &= \langle R^*(x)a^*, s^\sharp(b^*) \cdot y\rangle = \langle a^*, s^\sharp(b^*) \cdot y \cdot x\rangle.\end{aligned}$$

Hence $R^*(x)(a^* \circ b^*) = (R^*(x)a^*) \circ b^*$. Similarly, we have

$$(a^* \circ b^*)L^*(x) = a^* \circ (b^* L^*(x)), \quad (a^* L^*(x)) \circ b^* = a^* \circ (R^*(x)b^*).$$

Therefore $(A^*, \circ, R^*, L^*)$ is an A-bimodule $\mathbf{k}$-algebra.

(b). Applying Eq. (21) gives

$$\begin{aligned}\langle s, a^* \otimes b^*\rangle &= \langle s^\sharp(a^*), b^*\rangle = \langle b^* \circ a^*, \mathbf{1}\rangle = \langle a^* \circ b^*, \mathbf{1}\rangle\\ &= \langle s^\sharp(b^*), a^*\rangle = \langle s, b^* \otimes a^*\rangle, \quad \forall a^*, b^* \in A^*.\end{aligned}$$

Hence s is symmetric. Since $(A^*, \circ, R^*, L^*)$ is an A-bimodule $\mathbf{k}$-algebra, we have

$$\begin{aligned}\langle x \cdot s^\sharp(b^*), a^*\rangle &= \langle s^\sharp(b^*), a^* L^*(x)\rangle = \langle (a^* L^*(x)) \circ b^*, \mathbf{1}\rangle = \langle a^* \circ (R^*(x)b^*), \mathbf{1}\rangle\\ &= \langle s^\sharp(R^*(x)b^*), a^*\rangle,\\ \langle s^\sharp(b^*) \cdot x, a^*\rangle &= \langle s^\sharp(b^*), R^*(x)a^*\rangle = \langle (R^*(x)a^*) \circ b^*, \mathbf{1}\rangle = \langle b^* \circ (R^*(x)a^*), \mathbf{1}\rangle\\ &= \langle (b^* L^*(x)) \circ a^*, \mathbf{1}\rangle = \langle a^* \circ (b^* L^*(x)), \mathbf{1}\rangle = \langle s^\sharp(b^* L^*(x)), a^*\rangle,\end{aligned}$$

where $x \in A, a^*, b^* \in A^*$. Hence s is invariant. □

Remark 2.21. In fact, under the same conditions as for Lemma 2.20, Eqs. (21) and (22) hold if and only if the following equation holds

$$\langle s^\sharp(a^*) \cdot x, b^*\rangle = \langle b^* \circ a^*, x\rangle = \langle x \cdot s^\sharp(b^*), a^*\rangle, \quad \forall x \in A, a^*, b^* \in A^*.$$

Theorem 2.22. *Let $(A, \cdot, \mathbf{1})$ be a unital $\mathbf{k}$-algebra. Let $r \in A \otimes A$ be symmetrized invariant. Let $\circ$ be the binary operation defined from $\mathbf{r}$ by Eq. (19). Then the following statements are equivalent.*

(a) *The tensor* r *is a solution of the nhacYBe in Eq.* (7).

(b) *When* $\mathbf{r} = 0$, *the map* $r^\sharp$ *is an* $\mathcal{O}$*-operator of weight zero associated to the* A*-bimodule* (A^*, R^*, L^*), *and when* $\mathbf{r} \neq 0$, *the map* $r^\sharp$ *is an* $\mathcal{O}$*-operator of weight* -1 *associated to the* A*-bimodule* $\mathbf{k}$*-algebra* $(A^*, \circ, R^*, L^*)$.

(c) *When* $\mathbf{r} = 0$, *the map* $r^{t\sharp}$ *is an* $\mathcal{O}$*-operator of weight zero associated to the* A*-bimodule* (A^*, R^*, L^*), *and when* $\mathbf{r} \neq 0$, *the map* $r^{t\sharp}$ *is an* $\mathcal{O}$*-operator of weight* -1 *associated to the* A*-bimodule* $\mathbf{k}$*-algebra* $(A^*, \circ, R^*, L^*)$.

Proof. (a) $\Longleftrightarrow$ (b). Since $a^* \circ b^* := a^* L^*(\mathbf{r}^\sharp(b^*))$ and, by Lemma 2.20, $(A^*, \circ, R^*, L^*)$ is an A-bimodule $\mathbf{k}$-algebra, the equivalence follows from Theorem 2.8.

The proof of (a) $\Longleftrightarrow$ (c) follows from the same argument. □

Corollary 2.23. *Let* $(A, \cdot, \mathbf{1})$ *be a unital* $\mathbf{k}$*-algebra. Let* $r \in A \otimes A$ *be symmetrized invariant. Then* r *is a solution of the nhacYBe if and only if* r *satisfies Eq.* (8).

Proof. By Theorem 2.22, the tensor r is a solution the nhacYBe if and only if $\sigma(r)$ is a solution of the nhacYBe, which holds if and only if r is a solution of Eq. (8) by Lemma 2.9. □

Remark 2.24. For a unital $\mathbf{k}$-algebra $(A, \mathbf{1})$, it is obvious that $\mu(\mathbf{1} \otimes \mathbf{1})$ is a solution of the nhacYBe. However, if $\mu \neq 0$ and $\dim A \geq 2$, then the extended symmetrizer of $\mu(\mathbf{1} \otimes \mathbf{1})$ is not invariant (see also Remark 2.13).

Corollary 2.25. *Let* $(A, \cdot, \mathbf{1})$ *be a unital* $\mathbf{k}$*-algebra and* $(A^*, \circ, R^*, L^*)$ *be an* A*-bimodule* $\mathbf{k}$*-algebra satisfying Eq.* (21). *Let* $s^\sharp : A^* \to A$ *be the linear map from* $\circ$ *defined by Eq.* (20) *and satisfying Eq.* (22). *Let* $P : A^* \to A$ *be a linear map satisfying*

$$P(a^*) + P^*(a^*) = s^\sharp(a^*) + \mu\langle a^*, \mathbf{1}\rangle \mathbf{1}, \quad \forall a^* \in A^*, \tag{23}$$

where $P^* : A^* \to A^*$ *is the dual map of* P. *Then the following statements are equivalent.*

(a) *When* $s^\sharp = 0$, P *is an* $\mathcal{O}$*-operator of weight 0 associated to* (A^*, R^*, L^*) *and when* $s^\sharp \neq 0$, P *is an* $\mathcal{O}$*-operator of weight* -1 *associated to* $(A^*, \circ, R^*, L^*)$.

(b) *When* $s^\sharp = 0$, P^* *is an* $\mathcal{O}$*-operator of weight zero associated to* (A^*, R^*, L^*) *and when* $s^\sharp \neq 0$, P^* *is an* $\mathcal{O}$*-operator of weight* -1 *associated to* $(A^*, \circ, R^*, L^*)$.

(c) *The tensor* $r \in A \otimes A$ *defined by* $r^\sharp = P$ *is a symmetrized invariant solution of the nhacYBe.*

(d) *The tensor* $r \in A \otimes A$ *defined by* $r^{t\sharp} = P$ *is a symmetrized invariant solution of the nhacYBe.*

Proof. By Lemma 2.20, the tensor s from $s^\sharp$ is symmetric and invariant. Set $P = r^\sharp$. Then for $a^*, b^* \in A^*$, we have

$$\begin{aligned}\langle P(a^*) + P^*(a^*) + s^\sharp(a^*) - \mu\langle a^*, \mathbf{1}\rangle \mathbf{1}, b^*\rangle \\ = \langle r + \sigma(r) + s - \mu(\mathbf{1} \otimes \mathbf{1}), a^* \otimes b^*\rangle.\end{aligned}$$

Hence P satisfies Eq. (23) if and only if the extended symmetrizer of r is symmetric and invariant. By Theorem 2.22, statement (a) holds if and only if statement (c) holds. Note that in this case, $P^* = r^{t\sharp}$. Therefore by Theorem 2.22, statement (b) holds if and only if statement (a) or statement (c) holds.

Furthermore, by the symmetry of P and P^*, if we set $P = r^{t\sharp}$, then by the above discussion, we can directly show that statement (d) holds if and only if statement (b) holds. This proves that all the statements are equivalent. □

We end this subsection with displaying a relationship between solutions of the nhacYBe with trivial extended symmetrizers and associative Yang-Baxter pairs.

Definition 2.26. ([15]) Let A be a $\mathbf{k}$-algebra. **An associative Yang-Baxter pair** is a pair of elements $r, s \in A \otimes A$ satisfying

$$r_{12}r_{13} - r_{23}r_{12} + r_{13}s_{23} = 0, \quad r_{12}s_{13} - s_{23}s_{12} + s_{13}s_{23} = 0.$$

Proposition 2.27. ([15]) *Let* $(A, \mathbf{1})$ *be a unital* $\mathbf{k}$*-algebra. Let* $r, s \in A \otimes A$. *If* $r - s = \mathbf{1} \otimes \mathbf{1}$, *then the pair* (r, s) *is an associative Yang-Baxter pair if and only if* r *satisfies the nhacYBe with* $\mu = 1$.

Corollary 2.28. *Let $(A, \mathbf{1})$ be a unital $\mathbf{k}$-algebra. Let $r \in A \otimes A$. If*

$$r + \sigma(r) = \mu(\mathbf{1} \otimes \mathbf{1})$$

with $\mu \neq 0$, then r is a solution of the nhacYBe in Eq. (7) if and only if $(r, -\sigma(r))$ is an associative Yang-Baxter pair.

Proof. Let $r \in A \otimes A$ be a solution of the nhacYBe and $r + \sigma(r) = \mu(\mathbf{1} \otimes \mathbf{1})$ with $\mu \neq 0$. Then $r' = \frac{1}{\mu} r$ is a solution of the nhacYBe with $\mu = 1$ and $r' + \sigma(r') = \mathbf{1} \otimes \mathbf{1}$. By Proposition 2.27, $(r', -\sigma(r'))$ is an associative Yang-Baxter pair. Hence $(r, -\sigma(r))$ is an associative Yang-Baxter pair. Similarly, the converse also holds. □

3. NhacYBe and Rota-Baxter operators

In this section, we first give a correspondence between certain Rota-Baxter operators and symmetrized invariant solutions of the nhacYBe with a specific extended symmetrizer $\mathbf{r}$ in unital symmetric Frobenius algebras.

When the tensor $\mathbf{r}$ is degenerate, solutions of the nhacYBe in semi-direct product algebras can still be derived from Rota-Baxter operators, $\mathcal{O}$-operators and dendriform algebras, while Rota-Baxter operators can be derived from solutions of the nhacYBe in unitization algebras.

3.1. NhacYBe and Rota-Baxter operators on Frobenius algebras

Extending the correspondence between solutions of the AYBE and Rota-Baxter systems on Frobenius algebras given in Corollary 2.18 to the nhacYBe, we obtain

Theorem 3.1. *Let $(A, \cdot, \mathbf{1}, \mathfrak{B})$ be a unital symmetric Frobenius algebra. Let $\phi^\sharp : A^* \to A$ be the linear isomorphism from $\mathfrak{B}$ defined by Eq. (15) and let $\phi \in A \otimes A$ be the corresponding invariant symmetric tensor. Suppose $r \in A \otimes A$ has its extended symmetrizer given by*

$$\mathbf{r} := r + \sigma(r) - \mu(\mathbf{1} \otimes \mathbf{1}) = -\lambda\phi. \tag{24}$$

Define linear maps $P_r, P_r^t : A \to A$ respectively by

$$P_r(x) := r^\sharp {\phi^\sharp}^{-1}(x), \quad P_r^t(x) := {r^t}^\sharp {\phi^\sharp}^{-1}(x), \quad \forall x \in A. \tag{25}$$

Then the following conditions are equivalent.

(a) *r is a solution of the nhacYBe in A.*

(b) *P_r is a Rota-Baxter operator of weight λ, that is, Eq.* (4) *holds.*

(c) *P_r^t is a Rota-Baxter operator of weight λ.*

Proof. It follows from Theorem 2.16 by taking $\mathbf{r}^\sharp = -\lambda\phi^\sharp$. □

A different construction of Rota-Baxter operators from solutions of the opposite form of the nhacYBe in Eq. (8) can be found in [19].

Taking $\lambda = \mu = 0$ in Theorem 3.1, we obtain the following result. Note that in this case, $P_r^t = -P_r$.

Corollary 3.2. *[10, Corollary 3.17] An antisymmetric $r \in A \otimes A$ is a solution of the AYBE in Eq.* (6) *if and only if the linear map P_r defined by Eq.* (25) *is a Rota-Baxter operator of weight zero.*

Example 3.3. Let $(A, \cdot) = (\mathrm{End}_{\mathbf{k}}(V), \cdot) = (M_n(\mathbf{k}), \cdot)$ be the matrix algebra, where $n = \dim V$. It is a Frobenius algebra with the invariant bilinear form being the trace form, that is,

$$\mathfrak{B}(x, y) := \mathrm{Tr}(x \cdot y), \quad \forall x, y \in A. \tag{26}$$

Take a basis $\{e_1, \cdots, e_n\}$ of A such that $\mathfrak{B}(e_i, e_j) = \delta_{ij}$. Let

$$\phi = \sum_i e_i \otimes e_i.$$

Therefore Eq. (15) holds. Moreover, since $\mathrm{End}_{\mathbf{k}}(V) \otimes \mathrm{End}_{\mathbf{k}}(V) \cong \mathrm{End}_{\mathbf{k}}(V \otimes V)$, it is known that ϕ is the flip map σ on $V \otimes V$.

Let $r = \sum_i a_i \otimes b_i \in A \otimes A$. Then

$$P_r(x) = r^\sharp {\phi^\sharp}^{-1}(x) = \sum_i \langle {\phi^\sharp}^{-1}(x), a_i \rangle b_i = \sum_i \mathfrak{B}(x, a_i) b_i = \sum_i \mathrm{Tr}(x \cdot a_i) b_i.$$

Similarly, $P_r^t(x) = \sum_i \mathrm{Tr}(x \cdot b_i) a_i$. Suppose that

$$r + \sigma(r) = -\lambda\sigma + \mu(\mathbf{1} \otimes \mathbf{1}) = -\lambda\phi + \mu(\mathbf{1} \otimes \mathbf{1}).$$

If r satisfies Eq. (7), then both P_r and P_r^t are Rota-Baxter operators of weight λ. This is exactly the example given in [36].

Example 3.4. We can be more explicit with Example 3.3 when $n = 2$. Let $E_{ij} \in M_2(\mathbf{k})$, $1 \le i, j \le 2$, be the matrix whose (i, j)-entry is 1 and other entries are zero. Now the matrix algebra $A = M_2(\mathbb{C})$ is a Frobenius algebra with the invariant bilinear form $\mathfrak{B}$ given by Eq. (26). An orthonormal basis with respect to the form is

$$e_1 = \frac{1}{\sqrt{2}}(E_{11} + E_{22}),\ e_2 = \frac{1}{\sqrt{2}}(E_{11} - E_{22}),$$
$$e_3 = \frac{1}{\sqrt{2}}(E_{12} + E_{21}),\ e_4 = \frac{1}{\sqrt{-2}}(E_{12} - E_{21}).$$

Hence the ϕ in Example 3.3 is

$$\begin{aligned}\phi &= e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3 + e_4 \otimes e_4 \\ &= E_{11} \otimes E_{11} + E_{22} \otimes E_{22} + E_{12} \otimes E_{21} + E_{21} \otimes E_{12}.\end{aligned}$$

Note that the unit $\mathbf{1}$ in $M_2(\mathbb{C})$ is $E_{11} + E_{22}$. Then

$$\mathbf{1} \otimes \mathbf{1} = E_{11} \otimes E_{11} + E_{11} \otimes E_{22} + E_{22} \otimes E_{11} + E_{22} \otimes E_{22}.$$

On the other hand, by a direct calculation, we find that $r = E_{12} \otimes E_{21} - E_{11} \otimes E_{22}$ is a solution of the nhacYBe with $\mu = -1$ in $M_2(\mathbb{C})$. Then we have

$$r + \sigma(r) = E_{12} \otimes E_{21} - E_{11} \otimes E_{22} + E_{21} \otimes E_{12} - E_{22} \otimes E_{11} = \phi - \mathbf{1} \otimes \mathbf{1}.$$

Hence by Theorem 3.1, we have a Rota-Baxter operator P_r of weight -1 determined by

$$P_r(E_{11}) = -E_{22}, P_r(E_{21}) = E_{21}, P_r(E_{12}) = P_r(E_{22}) = 0.$$

3.2. From $\mathcal{O}$-operators and dendriform algebras to nhacYBe on semi-direct product algebras

We now show that $\mathcal{O}$-operators of weight zero and dendriform algebras can give rise to solutions of the nhacYBe in some semi-direct product algebras. We first generalize one direction of Theorem 3.1 by relaxing the condition that the extended symmetrizer of r is a multiple of a nondegenerate invariant tensor giving by a symmetric Frobenius algebra.

Proposition 3.5. *Let $(A, \cdot, \mathbf{1})$ be a unital $\mathbf{k}$-algebra. Let $s \in A \otimes A$ be symmetric and invariant. Let $P : A \to A$ be a linear map satisfying*

$$s^\sharp P^*(a^*) + P s^\sharp(a^*) = -\lambda s^\sharp(a^*) + \mu \langle a^*, \mathbf{1} \rangle \mathbf{1}, \quad \forall a^* \in A^*,$$

where P^* is the linear dual of P. Let r_1 and r_2 be defined by $r_1^\sharp = s^\sharp P^*$ and $r_2^\sharp = P s^\sharp$. Explicitly, setting $s = \sum_i a_i \otimes b_i$, then

$$r_1 := \sum_i P(a_i) \otimes b_i, \quad r_2 := \sum_i a_i \otimes P(b_i). \tag{27}$$

If P is a Rota-Baxter operator of weight λ, then r_1 and r_2 are symmetrized invariant solutions of the nhacYBe in A.

Conversely, suppose that s is nondegenerate. Let $r \in A \otimes A$ satisfy

$$r + \sigma(r) = -\lambda s + \mu(\mathbf{1} \otimes \mathbf{1}).$$

Let $P_r, P_r^t : A \to A$ be the linear maps defined respectively by

$$P_r(x) := r^\sharp {s^\sharp}^{-1}(x), \quad P_r^t(x) := {r^t}^\sharp {s^\sharp}^{-1}(x), \quad \forall x \in A.$$

If r is a solution of the nhacYBe, then P_r and P_r^t are Rota-Baxter operators of weight λ.

Proof. In fact, we have $r_2^\sharp = {r_1^t}^\sharp$ since

$$\begin{aligned}\langle {r_1^t}^\sharp(a^*), b^* \rangle &= \langle s^\sharp P^*(b^*), a^* \rangle = \langle s^\sharp(a^*), P^*(b^*) \rangle \\ &= \langle P s^\sharp(a^*), b^* \rangle = \langle r_2^\sharp(a^*), b^* \rangle, \quad \forall a^*, b^* \in A^*.\end{aligned}$$

Hence $r_2 = \sigma(r_1)$. For $a^*, b^* \in A^*$, we have

$$\begin{aligned}&\langle r_1 + \sigma(r_1) + \lambda s - \mu(\mathbf{1} \otimes \mathbf{1}), a^* \otimes b^* \rangle \\ &\quad = \langle s^\sharp P^*(a^*), b^* \rangle + \langle s^\sharp P^*(b^*), a^* \rangle + \lambda \langle s^\sharp(a^*), b^* \rangle - \mu \langle \mathbf{1}, a^* \rangle \langle \mathbf{1}, b^* \rangle \\ &\quad = \langle s^\sharp P^*(a^*) + P s^\sharp(a^*) + \lambda s^\sharp(a^*) - \mu \langle a^*, \mathbf{1} \rangle \mathbf{1}, b^* \rangle = 0.\end{aligned}$$

Hence $r_1+\sigma(r_1)+\lambda s-\mu(\mathbf{1}\otimes\mathbf{1})=0$. For $a^*,b^*,c^*\in A^*$, we have

$$\begin{aligned}
\langle r_1^\sharp(a^*)\cdot r_1^\sharp(b^*),c^*\rangle &= \langle s^\sharp P^*(a^*)\cdot s^\sharp P^*(b^*),c^*\rangle\\
&= \langle s^\sharp P^*(b^*),c^*L^*(s^\sharp P^*(a^*))\rangle\\
&= \langle b^*,P(s^\sharp(c^*)\cdot s^\sharp P^*(a^*))\rangle\\
&= \langle b^*,-P(s^\sharp(c^*)\cdot P(s^\sharp(a^*)))\rangle\\
&\quad+\langle b^*,P(-\lambda s^\sharp(c^*)\cdot s^\sharp(a^*)+\mu\langle\mathbf{1},a^*\rangle s^\sharp(c^*))\rangle,\\
\langle r_1^\sharp(a^*L^*(r_1^\sharp(b^*))),c^*\rangle &= \langle s^\sharp P^*(a^*L^*(s^\sharp P^*(b^*))),c^*\rangle\\
&= \langle a^*,s^\sharp P^*(b^*)\cdot P(s^\sharp(c^*))\rangle\\
&= \langle a^*,s^\sharp(P^*(b^*)L^*P(s^\sharp(c^*)))\rangle\\
&= \langle b^*,P(P(s^\sharp(c^*))\cdot s^\sharp(a^*))\rangle,\\
\langle r_1^\sharp(R^*(r_1^\sharp(a^*))b^*),c^*\rangle &= \langle s^\sharp P^*(R^*(s^\sharp P^*(a^*))b^*),c^*\rangle\\
&= \langle R^*(s^\sharp P^*(a^*))b^*,P(s^\sharp(c^*)\rangle\\
&= \langle b^*,P(s^\sharp(c^*))\cdot s^\sharp P^*(a^*)\rangle\\
&= \langle b^*,-P(s^\sharp(c^*))\cdot P(s^\sharp(a^*))\rangle\\
&\quad+\langle b^*,-\lambda P(s^\sharp(c^*))\cdot s^\sharp(a^*)+\mu\langle\mathbf{1},a^*\rangle P(s^\sharp(c^*))\rangle,\\
\langle\lambda r_1^\sharp(a^*L\cdot(s^\sharp(b^*))),c^*\rangle &= \langle\lambda s^\sharp P^*(a^*L^*(s^\sharp(b^*))),c^*\rangle=\langle a^*,\lambda s^\sharp(b^*)\cdot Ps^\sharp(c^*)\rangle\\
&= \langle a^*,\lambda s^\sharp(b^*L^*(Ps^\sharp(c^*)))\rangle\\
&= \langle b^*,\lambda P(s^\sharp(c^*))\cdot s^\sharp(a^*)\rangle.
\end{aligned}$$

Hence if P is a Rota-Baxter operator of weight λ, then $r_1^\sharp$ is an $\mathcal{O}$-operator associated to the A-bimodule $\mathbf{k}$-algebra $(A^*,\circ,R^*,L^*)$, where $\circ$ is defined from $-\lambda s$. Hence r_1 is a solution of the nhacYBe by Theorem 2.22. By Theorem 2.22 again, r_2 is also a solution of the nhacYBe since $r_2^\sharp=r_1^{t\sharp}=\sigma(r_1)^\sharp$.

If s is nondegenerate, then from the above proof, it is obvious that the converse is true. Alternatively, note that when s is nondegenerate, symmetric and invariant, then it corresponds to a nondegenerate, symmetric and invariant bilinear form $\mathfrak{B}$ by Lemma 2.15 through Eq. (15) such that $(A,\mathfrak{B})$ is a Frobenius algebra. Then the conclusion follows from Theorem 3.1. □

Remark 3.6. When $\mu=0$, the tensor r_1 in Eq. (27) recovers a construction in [19].

In the rest of this subsection, we provide symmetrized invariant solutions of the nhacYBe in semi-direct product algebras from $\mathcal{O}$-operators of weight zero and dendriform algebras by applying Proposition 3.5. We first supply more background.

Let $(A,\cdot)$ be a $\mathbf{k}$-algebra and (V,l,r) be an A-bimodule. Let (V^*,r^*,l^*) be the dual A-bimodule. Denote the semi-direct product algebras

$$\widehat{A} := A \ltimes_{l,r} V, \quad \mathcal{A} := A \ltimes_{r^*,l^*} V^*.$$

Identify a linear map $\beta : V \to A$ with an element in $\mathcal{A}\otimes\mathcal{A}$ by the injective map

$$\mathrm{Hom}_{\mathbf{k}}(V,A) \cong A\otimes V^* \hookrightarrow \mathcal{A}\otimes\mathcal{A}.$$

Proposition 3.7. ([8]) *Let A be a $\mathbf{k}$-algebra and (V,ℓ,r) be an A-bimodule. Let $\alpha : V\to A$ be a linear map. Then α is an $\mathcal{O}$-operator of weight zero if and only if the linear map*

$$\widehat{\alpha}(x,u) := (\alpha(u), -\lambda u),\ \ \forall x\in A, u\in V, \tag{28}$$

is a Rota-Baxter operator of weight λ on the algebra $\widehat{A}$.

Lemma 3.8. ([10]) *Let $(A,\cdot)$ be a $\mathbf{k}$-algebra and (V,l,r) be an A-bimodule. Let $\beta : V\to A$ be a linear map. Then $\widetilde{\beta} = \beta+\sigma(\beta)\in\mathcal{A}\otimes\mathcal{A}$ is invariant if and only if β is a* **balanced A-bimodule homomorphism***, that is,*

$$\begin{aligned} &\beta(l(x)u) = x\cdot\beta(v),\ \ \beta(ur(x)) = \beta(u)\cdot x, \\ &l(\beta(u))v = ur(\beta(v)),\ \ \forall x\in A, u,v\in V. \end{aligned} \tag{29}$$

Theorem 3.9. *Let $(A,\cdot,\mathbf{1})$ be a unital $\mathbf{k}$-algebra and (V,ℓ,r) be an A-bimodule. Assume that $\alpha : V\to A$ is an $\mathcal{O}$-operator of weight zero and $\beta : V^*\to A$ is a balanced A-bimodule homomorphism. Let $\widehat{\alpha}$ be given by Eq.* (28) *and $\widetilde{\beta} := \beta+\sigma(\beta)\in\widehat{A}\otimes\widehat{A}$. Let $r_1, r_2\in\widehat{A}\otimes\widehat{A}$ be defined by*

$$r_1^\sharp := \widetilde{\beta}^\sharp\widehat{\alpha}^*, \quad r_2^\sharp := \widehat{\alpha}\widetilde{\beta}^\sharp.$$

If α and β satisfy

$$\beta\alpha^*(x^*) + \alpha\beta^*(x^*) = \mu\langle x^*, \mathbf{1}\rangle\mathbf{1},\ \ \forall x^*\in A^*,$$

then r_1 and r_2 are symmetrized invariant solutions of the nhacYBe in $\widehat{A}$, with $s=\widetilde{\beta}$.

Proof. By Proposition 3.7, $\widehat{\alpha}$ is a Rota-Baxter operator of weight λ on $\widehat{A}$. By Lemma 3.8, $\widetilde{\beta} \in \widehat{A} \otimes \widehat{A}$ is invariant. Moreover, we have

$$\widehat{\alpha}^*(x^*, u^*) = (0, \alpha^*(x^*) - \lambda u^*), \quad \widetilde{\beta}^\sharp(x^*, u^*) = (\beta(u^*), \beta^*(x^*)),$$
$$\forall x^* \in A^*, u^* \in V^*.$$

Hence for $x^* \in A^*, u^* \in V$, we have

$$\begin{aligned}
&\widetilde{\beta}^\sharp \widehat{\alpha}^*(x^*, u^*) + \widehat{\alpha}\widetilde{\beta}^\sharp(x^*, u^*) + \lambda\widetilde{\beta}^\sharp(x^*, u^*) - \mu\langle (x^*, u^*), (\mathbf{1}, 0)\rangle(\mathbf{1}, 0)\\
&\quad = (\beta\alpha^*(x^*) - \lambda\beta(u^*), 0) + (\alpha\beta^*(x^*), -\lambda\beta^*(x^*))\\
&\qquad + \lambda(\beta(u^*), \beta^*(x^*)) - (\mu\langle x^*, \mathbf{1}\rangle\mathbf{1}, 0)\\
&\quad = (\beta\alpha^*(x^*) + \alpha\beta^*(x^*) - \mu\langle x^*, \mathbf{1}\rangle\mathbf{1}, 0) = 0.
\end{aligned}$$

By Proposition 3.5, the desired result follows. □

Corollary 3.10. *Let $(A, \mathbf{1})$ be a unital $\mathbf{k}$-algebra. Let $s \in A \otimes A$ be symmetric and invariant. Let $P : A \to A$ be a linear map satisfying*

$$s^\sharp P^*(a^*) + P s^\sharp(a^*) = \mu\langle a^*, \mathbf{1}\rangle\mathbf{1}, \quad \forall a^* \in A^*.$$

Suppose that P is a Rota-Baxter operator of weight zero.

(a) *Let $r_1, r_2 \in A \otimes A$ be defined by*

$$r_1^\sharp := s^\sharp P^*, \quad r_2^\sharp := P s^\sharp.$$

Then r_1 and r_2 are symmetrized invariant solutions of the nhacYBe in A whose extended symmetrizers are zero.

(b) *Set $\widehat{A} := A \ltimes_{L,R} A$. Let $\widehat{P}$ be given by Eq. (28) with $\widetilde{s^\sharp} = s^\sharp + \sigma(s^\sharp) \in \widehat{A} \otimes \widehat{A}$. Let $r_3, r_4 \in \widehat{A} \otimes \widehat{A}$ be defined by*

$$r_3^\sharp := \big(\widetilde{s^\sharp}\big)^\sharp \widehat{P}^*, \quad r_4^\sharp := \widehat{P}\big(\widetilde{s^\sharp}\big)^\sharp.$$

Then r_3 and r_4 are symmetrized invariant solutions of the nhacYBe in $\widehat{A}$ with $s = \widetilde{s^\sharp}$.

Proof. (a) follows from Proposition 3.5 with $\lambda = 0$.

(b) follows from Theorem 3.9 where $(V, l, r) = (A, L, R)$ and $P = \alpha$, $\beta = s^\sharp$. Note that in this case, if s is invariant and symmetric, then $s^\sharp$ is a balanced A-module homomorphism, that is, $s^\sharp$ satisfies Eq. (29). □

Corollary 3.11. *Let $(A, \cdot, \mathbf{1})$ be a unital* $\mathbf{k}$*-algebra. Set $\widehat{A} := A \ltimes_{R^*,L^*} A^*$. Assume that $\beta : A \to A$ is a linear map satisfying*

$$\beta(x \cdot y) = \beta(x) \cdot y = x \cdot \beta(y), \ \ \forall x, y \in A. \tag{30}$$

Let $\alpha : A^* \to A$ be an $\mathcal{O}$-operator of weight zero associated to (A^*, R^*, L^*). Let $\widehat{\alpha}$ be given by Eq. (28) *and $\widetilde{\beta} = \beta + \sigma(\beta) \in \widehat{A} \otimes \widehat{A}$. Let $r, r' \in \widehat{A} \otimes \widehat{A}$ be defined by*

$$r^\sharp := \widetilde{\beta}^\sharp \widehat{\alpha}^*, \quad r'^\sharp := \widehat{\alpha}\widetilde{\beta}^\sharp.$$

If α and β satisfy

$$\beta\alpha^*(x^*) + \alpha\beta^*(x^*) = \mu\langle x^*, \mathbf{1}\rangle \mathbf{1}, \ \ \forall x^* \in A^*,$$

then r and r' are symmetrized invariant solutions of the nhacYBe in $\widehat{A}$, when taking $s = \widetilde{\beta}$. In particular, suppose that $\beta = \mathrm{id}$. Then β satisfies Eq. (30)*. Suppose that*

$$\alpha(x^*) + \alpha^*(x^*) = \mu\langle x^*, 1\rangle 1, \ \ \forall x^* \in A^*.$$

(a) *Let $r_1, r_2 \in \widehat{A} \otimes \widehat{A}$ be defined by*

$$r_1^\sharp := \widetilde{\mathrm{id}}^\sharp \widehat{\alpha}^*, \quad r_2^\sharp := \widehat{\alpha}\widetilde{\mathrm{id}}^\sharp.$$

Then r_1 and r_2 are symmetrized invariant solutions of the nhacYBe in $\widehat{A}$ with $s = \widetilde{\mathrm{id}}$.

(b) *Let $r_3, r_4 \in A \otimes A$ be defined by*

$$r_3^\sharp := \alpha, \quad r_4^\sharp := \alpha^*.$$

Then r_3 and r_4 are symmetrized invariant solutions of the nhacYBe in A.

Proof. The first part follows from Theorem 3.9 by taking $(V, l, r) := (A^*, R^*, L^*)$. Note that in this case, Eq. (29) is exactly Eq. (30).

(a) follows from the first part when $\beta = \mathrm{id}$.

(b) follows from Corollary 2.25 in the case that the extended symmetrizer is zero. □

We finally provide solutions of the nhacYBe from dendriform algebras.

Definition 3.12. [31] Let A be a vector space with two binary operations $\prec$ and $\succ$. Then $(A, \prec, \succ)$ is called a **dendriform algebra** if for all $a, b, c \in A$,

$$(a \prec b) \prec c = a \prec (b \prec c + b \succ c),\ (a \succ b) \prec c = a \succ (b \prec c),$$
$$(a \prec b + a \succ b) \succ c = a \succ (b \succ c).$$

Let $(A, \prec, \succ)$ be a dendriform algebra. For $a \in A$, let $L_{\prec}(a)$, $R_{\prec}(a)$ and $L_{\succ}(a)$, $R_{\succ}(a)$ denote the left and right multiplication operators on $(A, \prec)$ and $(A, \succ)$, respectively. Furthermore, define linear maps

$$R_{\prec}, L_{\succ} : A \to \mathrm{End}_{\mathbf{k}}(A), \quad a \mapsto R_{\prec}(a),\ a \mapsto L_{\succ}(a),\ \ \forall a \in A.$$

As is well known, for a dendriform algebra $(A, \prec, \succ)$, the multiplication

$$a \star b := a \prec b + a \succ b, \quad \forall a, b \in A,$$

defines a $\mathbf{k}$-algebra $(A, \star)$, called the **associated algebra** of the dendriform algebra. Moreover, $(A, L_{\succ}, R_{\prec})$ is a bimodule of the algebra $(A, \star)$ [6, 31].

A **unital dendriform algebra** [20] is a $\mathbf{k}$-module $A := \mathbf{k}\mathbf{1} \oplus A^+$ such that $(A^+, \prec, \succ)$ is a dendriform algebra and the operations $\prec$ and $\succ$ are extended (partially) to A by

$$x \prec \mathbf{1} = \mathbf{1} \succ x = x, \quad x \succ \mathbf{1} = \mathbf{1} \prec x = 0, \quad \forall x \in A^+.$$

Note that $\mathbf{1} \prec \mathbf{1}$ and $\mathbf{1} \succ \mathbf{1}$ are not defined. Then $(A, \star, \mathbf{1})$ is a unital $\mathbf{k}$-algebra.

Corollary 3.13. *Let $(A, \prec, \succ, \mathbf{1})$ be a unital dendriform algebra with the unit $\mathbf{1}$. Let $(A, \star)$ be the associated unital $\mathbf{k}$-algebra with the unit $\mathbf{1}$. Suppose that there is a linear map $\beta : A^* \to A$ satisfying*

$$\beta(R_{\prec}^*(x)y^*) = x \star \beta(y^*),\ \beta(y^* L_{\succ}^*(x)) = \beta(y^*) \star x,$$
$$R_{\prec}^*(\beta(y^*))z^* = y^* L_{\succ}^*(\beta(z^*)),$$

for $x \in A, y^*, z^* \in A^*$. Set $\widehat{A} := A \ltimes_{L_{\succ}, R_{\prec}} A$. Let $\widehat{\mathrm{id}}$ be given by Eq. (28), *that is,*

$$\widehat{\mathrm{id}}(x, y) := (y, -\lambda y),\ \ \forall x, y \in A,$$

and $\widetilde{\beta} = \beta + \sigma(\beta) \in \widehat{A} \otimes \widehat{A}$. If in addition, β satisfies

$$\beta(x^*) + \beta^*(x^*) = \mu\langle x^*, \mathbf{1}\rangle \mathbf{1}, \quad \forall x^* \in A^*,$$

then r_1 and r_2 defined by

$$r_1^\sharp := \widetilde{\beta}^\sharp \widehat{\mathrm{id}}^*, \quad r_2^\sharp := \widehat{\mathrm{id}}\widetilde{\beta}^\sharp$$

are symmetrized invariant solutions of the nhacYBe in $\widehat{A}$, with $s = \widetilde{\beta}$.

Proof. Note that the identity map id is an $\mathcal{O}$-operator of the associated algebra $(A, \star)$ associated to the bimodule $(A, L_\succ, R_\prec)$. Hence the conclusion follows from Theorem 3.9. □

Remark 3.14. The above constructions of symmetrized invariant solutions of the nhacYBe are different from the construction of solutions of the AYBE from $\mathcal{O}$-operators given in [10], where the symmetric invariant tensors appearing in the symmetric parts of solutions in the semi-direct product algebras can be "lifted" from linear maps from the bimodules to the k-algebras themselves as Lemma 3.8 illustrates. However, it is not true for the symmetric tensor $\mathbf{1} \otimes \mathbf{1}$ any more, that is, the approach in [10] does not apply here due to the appearance of the new term $\mu(\mathbf{1} \otimes \mathbf{1})$.

3.3. From nhacYBe to Rota-Baxter operators on unitization algebras

We end this section with constructions of Rota-Baxter operators from solutions of the nhacYBe in unitization algebras, or equivalently, augmented algebras.

The **unitization** of a not necessarily unital $\mathbf{k}$-algebra A' is the direct sum $\mathbf{k}$-algebra $A := \mathbf{k} \oplus A'$. An **augmentation map** on a unital $\mathbf{k}$-algebra $(A, \cdot, \mathbf{1})$ is a $\mathbf{k}$-algebra homomorphism $\varepsilon : A \to \mathbf{k}$. An **augmented unital k-algebra** is a unital $\mathbf{k}$-algebra $(A, \cdot, \mathbf{1})$ with an augmentation map ε.

As is well known [17, Theorem 5.1.1], augmented unital $\mathbf{k}$-algebras are precisely the unitizations of (not necessarily unital) algebras given by

$$\mathbf{k} \oplus A' \longleftrightarrow (A, \varepsilon),$$

where $A := \mathbf{k} \oplus A'$, ε is the projection to $\mathbf{k}$, while A' is $\ker \varepsilon$.

Remark 3.15. For an augmented unital $\mathbf{k}$-algebra $(A, \cdot, \mathbf{1}, \varepsilon)$ with augmentation map ε, there is a basis $\{e_1, \cdots, e_n\}$ of A such that $e_1 = \mathbf{1}$ and $\{e_2, \cdots, e_n\}$ is a basis of $\ker \varepsilon = A'$. Let $\{e_1^*, \cdots, e_n^*\}$ be the dual basis. Then $\varepsilon = e_1^*$.

The following conclusion is obvious.

Lemma 3.16. *Let* $(A, \cdot, \mathbf{1})$ *be a unital* $\mathbf{k}$*-algebra and* ε *be an augmentation map. Then* $\varepsilon(\mathbf{1}) = 1_{\mathbf{k}}$*, and*

$$\varepsilon(x \cdot y \cdot z) = \varepsilon(y \cdot z \cdot x) = \varepsilon(z \cdot x \cdot y) = \varepsilon(x)\varepsilon(y)\varepsilon(z), \ \forall x, y, z \in A. \tag{31}$$

Let $(A, \cdot, \mathbf{1}, \varepsilon)$ be an augmented unital $\mathbf{k}$-algebra. Define linear maps

$$\varepsilon_l : A \otimes A \to \mathbf{k} \otimes A, \varepsilon_r : A \otimes A \to A \otimes \mathbf{k}$$

respectively by

$$\varepsilon_l := \varepsilon \otimes \mathrm{id}, \quad \varepsilon_r := \mathrm{id} \otimes \varepsilon.$$

Similarly, define linear maps

$$\begin{aligned}&\varepsilon_{12} : A \otimes A \otimes A \to \mathbf{k} \otimes \mathbf{k} \otimes A, \ \varepsilon_{23} : A \otimes A \otimes A \to A \otimes \mathbf{k} \otimes \mathbf{k}, \\ &\varepsilon_{13} : A \otimes A \otimes A \to \mathbf{k} \otimes A \otimes \mathbf{k}\end{aligned}$$

respectively by

$$\varepsilon_{12} := \varepsilon \otimes \varepsilon \otimes \mathrm{id}, \quad \varepsilon_{23} := \mathrm{id} \otimes \varepsilon \otimes \varepsilon, \quad \varepsilon_{13} := \varepsilon \otimes \mathrm{id} \otimes \varepsilon.$$

Denote the natural isomorphisms of algebras [23]

$$\beta_\ell : \mathbf{k} \otimes A \to A, 1_{\mathbf{k}} \otimes a \mapsto a; \ \ \beta_r : A \otimes \mathbf{k} \to A, x \otimes 1_{\mathbf{k}} \mapsto x, \forall\, x \in A.$$

Similarly, define natural isomorphisms of algebras

$$\begin{aligned}\beta_{12} &: \mathbf{k} \otimes \mathbf{k} \otimes A \to A, \quad 1_{\mathbf{k}} \otimes 1_{\mathbf{k}} \otimes x \mapsto x, \\ \beta_{23} &: A \otimes \mathbf{k} \otimes \mathbf{k} \to A, \quad x \otimes 1_{\mathbf{k}} \otimes 1_{\mathbf{k}} \mapsto x, \\ \beta_{13} &: \mathbf{k} \otimes A \otimes \mathbf{k} \to A, \quad 1_{\mathbf{k}} \otimes x \otimes 1_{\mathbf{k}} \mapsto x, \forall\, x \in A.\end{aligned}$$

For $x \in A$, set

$$\begin{gathered} x_{(l)} := x \otimes \mathbf{1} \in A \otimes A, \ \ x_{(r)} := \mathbf{1} \otimes x \in A \otimes A, \\ x_{(1)} := x \otimes \mathbf{1} \otimes \mathbf{1} \in A \otimes A \otimes A, \ x_{(2)} := \mathbf{1} \otimes x \otimes \mathbf{1} \in A \otimes A \otimes A, \\ x_{(3)} := \mathbf{1} \otimes \mathbf{1} \otimes x \in A \otimes A \otimes A. \end{gathered}$$

Theorem 3.17. *Let* $(A,\cdot,\mathbf{1},\varepsilon)$ *be an augmented unital* $\mathbf{k}$*-algebra. Let* $r=\sum_i a_i\otimes b_i\in A\otimes A$ *be a solution of the nhacYBe and* $\mathbf{r}$ *be the extended symmetrizer of* r*. Define linear maps* $P,P':A\to A$ *by*

$$P(x):=\sum_i \varepsilon(a_i\cdot x)b_i,\quad P'(x):=\sum_i \varepsilon(b_i\cdot x)a_i,\quad \forall x\in A. \tag{32}$$

(a) *If* $\mathbf{r}$ *is nonzero and satisfies*

$$\beta_l(\varepsilon_l(\mathbf{r}\cdot x_{(l)}))=x,\quad \forall x\in A, \tag{33}$$

then P *and* P' *are Rota-Baxter operators of weight* -1*.*

(b) *If* $\mathbf{r}=0$*, then* P *and* P' *are Rota-Baxter operators of weight zero.*

Proof. (a). Let $x,y\in A$. By definition, we have

$$\begin{aligned} P(x)&=\beta_l\varepsilon_l(r\cdot x_{(l)})=\beta_{13}(\varepsilon_{13}(r_{12}\cdot x_{(1)}))\\ &=\beta_{12}(\varepsilon_{12}(r_{13}\cdot x_{(1)}))=\beta_{12}(\varepsilon_{12}(r_{23}\cdot x_{(2)})), \end{aligned} \tag{34}$$

$$\begin{aligned} P'(x)&=\beta_r\varepsilon_r(r\cdot x_{(r)})=\beta_l\varepsilon_l(\sigma(r)\cdot x_{(l)})=\beta_{23}(\varepsilon_{23}(r_{12}\cdot x_{(2)}))\\ &=\beta_{23}(\varepsilon_{23}(r_{13}\cdot x_{(3)}))=\beta_{13}(\varepsilon_{13}(r_{23}\cdot x_{(3)})). \end{aligned} \tag{35}$$

Since r satisfies Eq. (7), we have

$$\begin{aligned} &r_{12}\cdot r_{13}\cdot x_{(1)}\cdot y_{(2)}+r_{13}\cdot r_{23}\cdot x_{(1)}\cdot y_{(2)}-r_{23}\cdot r_{12}\cdot x_{(1)}\cdot y_{(2)}\\ &\quad=\mu r_{13}\cdot x_{(1)}\cdot y_{(2)}. \end{aligned}$$

Applying $\beta_{12}\varepsilon_{12}:A\otimes A\otimes A\to A$ to both sides of the above equation, we get

$$\begin{aligned} &\beta_{12}\varepsilon_{12}\big(r_{12}\cdot r_{13}\cdot x_{(1)}\cdot y_{(2)}+r_{13}\cdot r_{23}\cdot x_{(1)}\cdot y_{(2)}-r_{23}\cdot r_{12}\cdot x_{(1)}\cdot y_{(2)}\big)\\ &\quad=\mu\beta_{12}\big(\varepsilon_{12}(r_{13}\cdot x_{(1)}\cdot y_{(2)})\big). \end{aligned} \tag{36}$$

Furthermore, we have

$$\begin{aligned}(37)\quad \beta_{12}\big(\varepsilon_{12}(r_{12}\cdot r_{13}\cdot x_{(1)}\cdot y_{(2)})\big) &= \beta_{12}(\varepsilon_{12}(\sum_{i,j}(a_i\cdot a_j\cdot x)\otimes(b_i\cdot y)\otimes b_j))\\ &= \beta_{12}(\sum_{i,j}\varepsilon(a_i\cdot a_j\cdot x)\otimes\varepsilon(b_i\cdot y)\otimes b_j)\\ &= \sum_{i,j}\varepsilon(a_i\cdot a_j\cdot x)\varepsilon(b_i\cdot y)b_j\\ &\overset{(32)}{=} \sum_j \varepsilon(P'(y)\cdot a_j\cdot x)b_j\\ &\overset{(31)}{=} \sum_j \varepsilon(a_j\cdot x\cdot P'(y))b_j\\ &\overset{(32)}{=} P(x\cdot P'(y)).\end{aligned}$$

Similarly, we have

$$\beta_{12}\big(\varepsilon_{12}(r_{13}\cdot r_{23}\cdot x_{(1)}\cdot y_{(2)})\big) = P(x)\cdot P(y), \tag{38}$$
$$\beta_{12}\big(\varepsilon_{12}(r_{23}\cdot r_{12}\cdot x_{(1)}\cdot y_{(2)})\big) = P(P(x)\cdot y), \tag{39}$$
$$\beta_{12}\big(\varepsilon_{12}(r_{13}\cdot x_{(1)}\cdot y_{(2)})\big) = \varepsilon(y)P(x). \tag{40}$$

Substituting Eqs. (37)-(40) into Eq. (36) gives

$$P(x)\cdot P(y) + P(x\cdot P'(y)) - P(P(x)\cdot y) = \mu\varepsilon(y)P(x). \tag{41}$$

Since the extended symmetrizer $\mathbf{r}$ of r is nonzero, we have

$$\beta_l\varepsilon_l((r+\sigma(r))\cdot x_{(l)} - \mu x_{(l)}) = \beta_l\varepsilon_l(\mathbf{r}\cdot x_{(l)}).$$

By Eqs. (33), (34) and (35), we obtain

$$P'(x) = x + \mu\varepsilon(x)\mathbf{1} - P(x). \tag{42}$$

Substituting Eq. (42) into Eq. (41) yields

$$\begin{aligned}P(x)\cdot P(y) + P\Big(x\cdot\big(y+\mu\varepsilon(y)\mathbf{1}-P(y)\big)\Big) - P(P(x)\cdot y)\\ = P(x)\cdot P(y) + P(x\cdot y) + \mu\varepsilon(y)P(x) - P(x\cdot P(y)) - P(P(x)\cdot y)\\ = \mu\varepsilon(y)P(x),\end{aligned}$$

that is,

$$P(x) \cdot P(y) = P(P(x) \cdot y) + P(x \cdot P(y)) - P(x \cdot y),$$

as required. Similarly, we prove that P' is also a Rota-Baxter operator of weight -1.

(b). By an argument similar to the proof of Item (a), we also have

$$P(x) \cdot P(y) + P(x \cdot P'(y)) - P(P(x) \cdot y) = \mu\varepsilon(y)P(x). \tag{43}$$

Since the extended symmetrizer of r is zero, we obtain

$$r + \sigma(r) - \mu(\mathbf{1} \otimes \mathbf{1}) = 0,$$

and so

$$\beta_l \varepsilon_l((r + \sigma(r)) \cdot x_{(l)} - \mu x_{(l)}) = 0.$$

By Eqs. (34)-(35), we have

$$P'(x) = \mu\varepsilon(x)\mathbf{1} - P(x). \tag{44}$$

Substituting Eq. (44) into Eq. (43) shows that P is a Rota-Baxter operator of weight zero. A similar argument proves that P' is a Rota-Baxter operator of weight zero. □

Corollary 3.18. *Let $(A, \cdot, \mathbf{1}, \varepsilon)$ be an augmented unital $\mathbf{k}$-algebra. Let $r \in A \otimes A$ be anti-symmetric (i.e. $r + \sigma(r) = 0$). If r satisfies the AYBE, then the operator P defined by Eq.* (32) *is a Rota-Baxter operator of weight zero.*

Proof. It follows from Theorem 3.17 (b) by taking $\mu = 0$. □

Corollary 3.19. *With the conditions in Theorem 3.17, suppose that $\mathbf{r} \in A \otimes A$ is nonzero and invariant, that is, $\mathbf{r} \cdot x_{(l)} = x_{(r)} \cdot \mathbf{r}, \ \forall x \in A$. As in Remark 3.15, let $\{e_1 = \mathbf{1}, e_2, \cdots, e_n\}$ be a basis of A and $\{e_1^*, e_2^*, \cdots, e_n^*\}$ be the dual basis such that $\varepsilon = e_1^*$. Moreover, suppose*

$$\mathbf{r} = \mathbf{1} \otimes \mathbf{1} + \sum_{i,j>1} s_{ij} e_i \otimes e_j.$$

Then the linear maps P and P' defined by Eq. (32) *are Rota-Baxter operators of weight -1.*

Proof. For all $x \in A$, we have

$$\beta_l \varepsilon_l(\mathbf{r} \cdot x_{(l)}) = \beta_l \varepsilon_l(x_{(r)} \cdot \mathbf{r}) = \beta_l(\varepsilon(\mathbf{1}) \otimes x) + \sum_{i,j>1} \beta_l(s_{ij}\varepsilon(e_i) \otimes (x \cdot e_j)) = x,$$

that is, $\mathbf{r}$ satisfies Eq. (33). Hence the conclusion follows from Theorem 3.17. □

Proposition 3.20. *Let $(A, \cdot, \mathbf{1})$ be a unital $\mathbf{k}$-algebra. If $\varepsilon : A \to \mathbf{k}$ is an augmentation map, then the bilinear form $\mathfrak{B}$ on A defined by*

$$\mathfrak{B}(x, y) := \varepsilon(x)\varepsilon(y), \ \ \forall x, y \in A, \tag{45}$$

is symmetric and invariant. Moreover, $\mathfrak{B}$ satisfies

$$\mathfrak{B}(x \cdot y, z) = \mathfrak{B}(y \cdot x, z), \ \ \forall x, y, z \in A.$$

In particular, if $\mathfrak{B}$ is nondegenerate, then A is commutative. Conversely, if $\mathfrak{B}$ is a symmetric invariant bilinear form satisfying

$$\mathfrak{B}(x, y) = \mathfrak{B}(x \cdot y, 1) = \mathfrak{B}(x, 1)\mathfrak{B}(y, 1), \ \ \forall x, y \in A,$$

then the linear map $\varepsilon : A \to \mathbf{k}$ defined by

$$\varepsilon(x) := \mathfrak{B}(x, 1), \ \ \forall x \in A,$$

is an augmentation map.

Proof. All the statements can be verified directly from the definitions. □

Example 3.21. Let $(A, \cdot, \mathbf{1}, \varepsilon)$ be an augmented unital commutative $\mathbf{k}$-algebra. Let $\mathfrak{B}$ be the bilinear form defined by Eq. (45). Suppose that $\mathfrak{B}$ is nondegenerate. Then $(A, \cdot, \mathfrak{B})$ is a symmetric Frobenius algebra. Let $\phi^\sharp : A^* \to A$ be the linear isomorphism defined by Eq. (15). Let $\{e_1 = \mathbf{1}, e_2, \cdots, e_n\}$ be a basis of A satisfying

$$\mathfrak{B}(e_i, e_j) = \delta_{ij}, \ \ \forall i, j = 1, \cdots, n.$$

Then $\phi \in A \otimes A$ is invariant and

$$\phi = \sum_{i=1}^{n} e_i \otimes e_i = \mathbf{1} \otimes \mathbf{1} + \sum_{i=2}^{n} e_i \otimes e_i.$$

By Theorem 3.17 and Corollary 3.19, we find that if r satisfies Eqs. (7) and (24), then the linear maps P and P' defined by Eq. (32) are Rota-Baxter operators of weight λ. Note that this conclusion also follows form Theorem 3.1, since in this case, $P = P_r$ and $P' = P_r^t$, where P_r and P_r^t are defined by Eq. (25).

4. Classification of symmetrized invariant solutions of nhacYBe in low dimensions

In this section, we classify symmetrized invariant solutions of the nhacYBe for $\mu \neq 0$ in the unital complex algebras with dimensions two and three, and find that all of them are obtained from Rota-Baxter operators through Theorem 3.1. It would be interesting to see what happens for algebras in higher dimensions.

4.1. The classification in dimension two

The set of symmetric invariant tensors of a **k**-algebra A is a subspace of $A \otimes A$ and is denoted by $\mathrm{Inv}(A)$.

There are two two-dimensional unital $\mathbb{C}$-algebras whose nonzero products with respect to a basis $\{e_1, e_2\}$ are given by [37]

$$\begin{aligned}&(A1) : e_1e_1 = e_1, e_1e_2 = e_2e_1 = e_2;\\&(A2) : e_1e_1 = e_1, e_2e_2 = e_2.\end{aligned}$$

By [30], for the algebra $(A1)$, there is only one nonzero solution $r = \mu e_1 \otimes e_1$ of the nhacYBe Eq. (7). By Remark 2.24, this solution is not symmetrized invariant.

Consider the solutions of the nhacYBe in the algebra $(A2)$. We find that eight of the nine nonzero solutions are symmetrized invariant, given by

$$\begin{aligned}&r_1 = \mu(e_1 \otimes e_1 + e_2 \otimes e_2 + e_1 \otimes e_2), \quad r_2 = \mu(e_1 \otimes e_1 + e_2 \otimes e_2 + e_2 \otimes e_1),\\&r_3 = \mu e_1 \otimes e_2, \quad r_4 = \mu e_2 \otimes e_1,\\&r_5 = \mu(e_1 \otimes e_1 + e_1 \otimes e_2), \quad r_6 = \mu(e_1 \otimes e_1 + e_2 \otimes e_1),\\&r_7 = \mu(e_2 \otimes e_2 + e_1 \otimes e_2), \quad r_8 = \mu(e_2 \otimes e_2 + e_2 \otimes e_1).\end{aligned}$$

Moreover, all of these solutions are obtained from Rota-Baxter operators by Theorem 3.1.

To see this, note that

$$r_2 = \sigma(r_1), \;\; r_4 = \sigma(r_3), \;\; r_6 = \sigma(r_5), \;\; r_8 = \sigma(r_7),$$

and the unit of the algebra $(A2)$ is $e_1 + e_2$. It is straightforward to show that $\mathrm{Inv}(A2) = \mathrm{span}\{e_1 \otimes e_1, e_2 \otimes e_2\}$. Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be the bilinear forms on $(A2)$ defined by

$$\mathfrak{B}_1(e_1, e_1) = \mathfrak{B}_1(e_2, e_2) = 1, \mathfrak{B}_1(e_1, e_2) = \mathfrak{B}_1(e_2, e_1) = 0;$$
$$\mathfrak{B}_2(e_1, e_1) = 1, \mathfrak{B}_2(e_2, e_2) = -1, \mathfrak{B}_2(e_1, e_2) = \mathfrak{B}_2(e_2, e_1) = 0.$$

Then both $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are symmetric, nondegenerate and invariant. Their corresponding symmetric, invariant tensors from Lemma 2.15 are

$$\phi_1 = e_1 \otimes e_1 + e_2 \otimes e_2, \;\; \phi_2 = e_1 \otimes e_1 - e_2 \otimes e_2,$$

so that $\mathfrak{B}_i(x, y) = \langle \phi_i^{\sharp^{-1}}(x), y\rangle$ for $x, y \in (A2)$ and $i = 1, 2$. Now the 8 symmetrized invariant solutions of the nhacYBe satisfy

$$r_1 + \sigma(r_1) = r_2 + \sigma(r_2) = r_1 + r_2 = \mu\phi_1 + \mu(e_1 + e_2) \otimes (e_1 + e_2);$$
$$r_3 + \sigma(r_3) = r_4 + \sigma(r_4) = r_3 + r_4 = -\mu\phi_1 + \mu(e_1 + e_2) \otimes (e_1 + e_2);$$
$$r_5 + \sigma(r_5) = r_6 + \sigma(r_6) = r_5 + r_6 = \mu\phi_2 + \mu(e_1 + e_2) \otimes (e_1 + e_2);$$
$$r_7 + \sigma(r_7) = r_8 + \sigma(r_8) = r_7 + r_8 = -\mu\phi_2 + \mu(e_1 + e_2) \otimes (e_1 + e_2).$$

By Theorem 3.1, their corresponding linear operators $P_{r_1}, P_{r_2}, P_{r_5}, P_{r_6}$ are Rota-Baxter operators of weight $-\mu$ and $P_{r_3}, P_{r_4}, P_{r_7}, P_{r_8}$ are Rota-Baxter operators of weight μ.

4.2. The classification in dimension three

Any three-dimensional unital $\mathbb{C}$-algebra is isomorphic to one of the following five [26, 43], defined by their nonzero products on a basis $\{e_1, e_2, e_3\}$

$(B1): e_1e_1 = e_1, e_2e_2 = e_2, e_3e_3 = e_3;$
$(B2): e_1e_1 = e_1, e_2e_2 = e_2, e_3e_2 = e_2e_3 = e_3;$
$(B3): e_1e_1 = e_1, e_1e_2 = e_2e_1 = e_2, e_1e_3 = e_3e_1 = e_3, e_2e_2 = e_3;$
$(B4): e_1e_1 = e_1, e_1e_2 = e_2e_1 = e_2, e_1e_3 = e_3e_1 = e_3, e_3e_2 = e_2, e_3e_3 = e_3;$
$(B5): e_1e_1 = e_1, e_1e_2 = e_2e_1 = e_2, e_1e_3 = e_3e_1 = e_3.$

Solutions of the nhacYBe in these algebras were classified in [30]. For the algebras $(B3)$ and $(B5)$, there is exactly one nonzero solution $r = \mu e_1 \otimes e_1$ and it is not symmetrized invariant.

For the algebra $(B4)$, it is straightforward to prove that $\mathrm{Inv}(B4) = 0$. Then by [30], none of the nonzero solutions is symmetrized invariant.

For the algebra $(B2)$, $e_1 + e_2$ is the unit. Moreover, the vector subspace S spanned by e_1 and e_2 is a unital subalgebra of $(B2)$. It is in fact $(A2)$ in Section 4.1. As discussed there, there are 8 symmetrized invariant solutions r_i, $1 \leq i \leq 8$, of the nhacYBe in S, together with the corresponding Rota-Baxter operators $P_{r_i}, 1 \leq i \leq 8$ on $(A2)$. In fact, they are the only nonzero symmetrized invariant solutions of Eq. (7) in $(B2)$. The corresponding Rota-Baxter operators on $(B2)$ are derived from $P_{r_i}, i = 1, \cdots, 8$ by setting $P_{r_i}(e_3) = 0$, as shown in [3].

For the algebra $(B1)$, among the total of 73 nonzero solutions of the nhacYBe given in [30], there are exactly 48 nonzero solutions that are symmetrized invariant. All of these solutions are obtained from Rota-Baxter operators thanks to Theorem 3.1.

Note that the unit $\mathbf{1}$ is $e_1 + e_2 + e_3$ and

$$\mathrm{Inv}(B1) = \mathrm{span}\{e_1 \otimes e_1, e_2 \otimes e_2, e_3 \otimes e_3\}.$$

Set

$$\phi_1 := e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3, \quad \phi_2 := e_1 \otimes e_1 + e_2 \otimes e_2 - e_3 \otimes e_3,$$
$$\phi_3 := e_1 \otimes e_1 - e_2 \otimes e_2 + e_3 \otimes e_3, \quad \phi_4 := -e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3.$$

According to their extended symmetrizers

$$\mathbf{r} := r + \sigma(r) - \mu(\mathbf{1} \otimes \mathbf{1}),$$

these 48 solutions are grouped together as follows.

$$r_1 = \mu(e_2 \otimes e_1 + e_3 \otimes e_1 + e_3 \otimes e_2), \quad r_2 = \mu(e_1 \otimes e_2 + e_1 \otimes e_3 + e_2 \otimes e_3),$$
$$r_3 = \mu(e_2 \otimes e_1 + e_2 \otimes e_3 + e_3 \otimes e_1), \quad r_4 = \mu(e_1 \otimes e_2 + e_3 \otimes e_2 + e_1 \otimes e_3),$$
$$r_5 = \mu(e_1 \otimes e_3 + e_2 \otimes e_1 + e_2 \otimes e_3), \quad r_6 = \mu(e_3 \otimes e_1 + e_1 \otimes e_2 + e_3 \otimes e_2),$$

for which $\mathbf{r} = -\mu\phi_1$ and hence their corresponding linear operators in Theorem 3.1 are Rota-Baxter operators of weight μ. Similarly, we have

$$r_i = r_{i-6} + \mu\phi_1, \quad 7 \leq i \leq 12,$$

for which $\mathbf{r} = \mu\phi_1$ and hence correspond to Rota-Baxter operators of weight $-\mu$;

$$r_i = r_{i-12} + \mu(e_3 \otimes e_3), \quad 13 \leq i \leq 18,$$

for which $\mathbf{r} = -\mu\phi_2$ and hence correspond to Rota-Baxter operators of weight μ;

$$r_i = r_{r-18} + \mu(e_1 \otimes e_1 + e_2 \otimes e_2), \quad 19 \leq i \leq 24,$$

for which $\mathbf{r} = \mu\phi_2$ and hence correspond to Rota-Baxter operators of weight $-\mu$;

$$r_i = r_{i-24} + \mu(e_2 \otimes e_2), \quad 25 \leq i \leq 30,$$

for which $\mathbf{r} = -\mu\phi_3$ and hence correspond to Rota-Baxter operators of weight μ;

$$r_i = r_{i-30} + \mu(e_1 \otimes e_1 + e_3 \otimes e_3), \quad 31 \leq i \leq 36,$$

for which $\mathbf{r} = \mu\phi_3$ and hence correspond to Rota-Baxter operators of weight $-\mu$;

$$r_i = r_{i-36} + \mu(e_1 \otimes e_1), \quad 36 \leq i \leq 42,$$

for which $\mathbf{r} = -\mu\phi_4$ and hence correspond to Rota-Baxter operators of weight μ;

$$r_i = r_{i-42} + \mu(e_2 \otimes e_2 + e_3 \otimes e_3), \quad 43 \leq i \leq 48,$$

for which $\mathbf{r} = \mu\phi_4$ and hence correspond to Rota-Baxter operators of weight $-\mu$.

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