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Positive energy representations of affine algebras and Stokes matrices of the affine Toda equations

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We give a construction which produces a positive energy representation of the affine Lie algebra $\widehat{\mathfrak{sl}}_{n+1}\mathbb{C}$ from the Stokes data of a solution of the tt*-Toda equations. The construction appears to play a role in conformal field theory. We illustrate this with several examples: the fusion ring, W-algebra minimal models (Argyres-Douglas theory), as well as topological-antitopological fusion itself.

1. Introduction

In this article we give a purely mathematical construction which relates
— integrable p.d.e. (affine Toda equations),
— Stokes data of linear meromorphic o.d.e., and
— representations of infinite-dimensional Lie algebras.

By "purely mathematical" we mean that the construction *a priori* does not depend on concepts from physics. Nevertheless, our project was indeed motivated by physical ideas, originating from topological-antitopological fusion and quantum cohomology. It seems to have a role — as a rather special example, at least — in some mathematical aspects of conformal field theory.

Our principal motivation is the tt*-Toda equations (tt* equations of Toda type). We review these equations and their (global) solutions in section 2. In section 3 we give a Lie-theoretic description of the Stokes data of these solutions — the main technical ingredient here is from [13]. Our construction of positive energy representations of affine Lie algebras from Stokes data is given in section 4.

In section 5 we give several applications of this construction in conformal field theory. We expect that this material could be expanded and developed further. For example, the ingredients of the construction all occur in the ODE/IM Correspondence, and we would expect a relation with that intriguing area.

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2. The tt*-Toda equations

We begin with a brief review of the tt*-Toda equations, in order to motivate our main construction in section 4.

The tt* equations were introduced by Cecotti-Vafa in their study of $N = 2$ supersymmetric field theory (see [2],[3]). They discussed several examples, the most prominent being the tt* equations of "Toda type", or tt*-Toda equations. These are

$$2(w_i)_{t\bar{t}} = -e^{2(w_{i+1}-w_i)} + e^{2(w_i-w_{i-1})}, \ w_i : \mathbb{C}^* \to \mathbb{R}, \ i \in \mathbb{Z} \tag{2.1}$$

where the real functions $w_0, \dots, w_n$ satisfy $w_i = w_{i+n+1}$, $w_i = w_i(|t|)$, $w_i + w_{n-i} = 0$.

From physical considerations, Cecotti-Vafa predicted the existence of solutions of (2.1) with certain properties. As a first step in this direction, the following statement was proved in [18],[14],[15],[16],[22],[23].

Theorem 2.1. *For each $N > 0$, there is a one-to-one correspondence between solutions of (2.1) on $\mathbb{C}^*$ and $\mathfrak{sl}_{n+1}\mathbb{C}$-valued 1-forms $\eta(z)\,dz$ on (the universal cover of) $\mathbb{C}^*$, where*

$$\eta(z) = \left(\begin{array}{c|c|c|c} & & & z^{k_0} \\ \hline z^{k_1} & & & \\ \hline & \ddots & & \\ \hline & & z^{k_n} & \end{array}\right). \tag{2.2}$$

Here the k_i are real numbers satisfying $k_i \in [-1, \infty)$, $n + 1 + \sum_{i=0}^n k_i = N$, and $k_i = k_{n-i+1}$ for $i = 1, \dots, n$. The variable z of (2.2) is related to the variable t of (2.1) by $t = \frac{n+1}{N} z^{\frac{N}{n+1}}$.

A more meaningful correspondence is obtained by introducing real numbers $m_0, \dots, m_n$ with $m_i + m_{n-i} = 0$. The m_i are defined by:

$$m_{i-1} - m_i + 1 = \tfrac{n+1}{N}(k_i + 1). \tag{2.3}$$

(We make the convention that $m_i = m_{i+n+1}$.) In terms of the m_i, we have a one-to-one correspondence between solutions of (2.1) and the convex polytope

$$\{m = \operatorname{diag}(m_0, \ldots, m_n) \mid m_{i-1} - m_i + 1 \geq 0, m_i + m_{n-i} = 0\}.$$

Then the relation between $m_0, \ldots, m_n$ and $w_0, \ldots, w_n$ is simply given by the asymptotics of the solution at $t = 0$, namely

$$w_i \sim -m_i \log |t| \quad (\text{as } t \to 0).$$

Writing $w = \operatorname{diag}(w_0, \ldots, w_n)$, we have $w \sim -m \log |t|$. With this notation (2.3) is equivalent to $z^{\frac{N}{n+1}m}\eta(z)z^{-\frac{N}{n+1}m} = z^{\frac{1}{n+1}\sum_{i=0}^n k_i}\eta(1)$. Thus, the m_i arise simply through "balancing" the k_i.

It is well known that solutions of the Toda equations correspond to certain kinds of harmonic maps. The above relation between solutions w and 1-forms $\eta(z)\,dz$ is, in fact, an example of the generalized Weierstrass representation (or DPW representation) for harmonic maps of surfaces into symmetric spaces [4]. This is based on the loop group Iwasawa factorization [24].

We review this construction very briefly, referring to section 2 of [16] for details. Introducing a loop parameter $\lambda \in S^1$, one can solve the complex o.d.e.

$$L^{-1}L_z = \tfrac{1}{\lambda}\eta, \quad L|_{z=0} = I$$

near $z = 0$ (at least, if all $k_i > -1$, which is the case needed in this article). Then the Toda equations (2.1) are the zero curvature condition for the 1-form[1] $\alpha = L_{\mathbb{R}}^{-1}dL_{\mathbb{R}}$, where $L = L_{\mathbb{R}}L_+$ is a suitable Iwasawa factorization. It can be shown that $L_+ = b + O(\lambda)$ where $b = \operatorname{diag}(b_0, \ldots, b_n)$ and all $b_i > 0$. Then one defines $w_i = \log b_i - m_i \log |t|$. So far this discussion is local (near $z = 0$), and straightforward; the nontrivial aspect of Theorem 2.1 is that the local solutions are in fact globally defined for all $0 < |t| < \infty$.

A summary of results related to Theorem 2.1, with some physical background, can be found in [11].

[1]For simplicity we are now modifying the notation of [16]. In [16], α denotes a gauge equivalent 1-form $\alpha = (L_{\mathbb{R}}G)^{-1}d(L_{\mathbb{R}}G)$, which has the same zero curvature condition. In [16], the nonzero entries of η are $c_i z^{k_i}$ rather than z^{k_i}, and the global solutions are given by certain specific c_i, but we may set all $c_i = 1$ at the expense of modifying the Iwasawa factorization.

Remark 2.2. There are various equivalent forms of (2.1), which depend on the definition of w_i in terms of b_i, and whether t or z is used. In terms of z, for example, $w_i = \log b_i$ gives

$$2(w_i)_{z\bar{z}} = -|z^{k_{i+1}}|^2 e^{2(w_{i+1}-w_i)} + |z^{k_i}|^2 e^{2(w_i - w_{i-1})}.$$

We use the t version (2.1) for consistency with [16]. □

3. Stokes data

The radial condition $w = w(|t|)$ leads to another, quite different, interpretation of equation (2.1): it is the condition that a certain meromorphic connection $\hat{\alpha}$ in the variable $\lambda \in \mathbb{C}^*$ has the property that its monodromy data is independent of z. Details of this formulation can be found in section 2 of [16].

The isomonodromic connection $\hat{\alpha}$ has (single-valued) coefficients which are holomorphic for $\lambda \in \mathbb{C}^*$, but has poles of order 2 at $\lambda = 0$ and $\lambda = \infty$. Its monodromy data consists of Stokes matrices relating solutions on sectors at each pole and "connection matrices" relating solutions at $\lambda = 0$ and $\lambda = \infty$. We refer to chapter 1 of [6] for these concepts from o.d.e. theory.

More generally, it was shown by Dubrovin [5] that the tt* equations always have an isomonodromic formulation with the same pole structure (poles of order 2 at $\lambda = 0$ and $\lambda = \infty$). Monodromy data of such connections can be hard to calculate, but, for the tt*-Toda equations, calculations are facilitated by the close relation between the Toda equations and Lie theory.

This monodromy data was calculated in [15],[16],[12],[13]. The result permits another characterization of the (global) solutions of (2.1), as an alternative to the asymptotic data m, as in the next theorem. Such a characterization had also been predicted by Cecotti-Vafa.

In fact the connection matrices turn out to be the same for all (global) solutions, so we ignore them here. The Stokes data may be specified efficiently as follows:

Theorem 3.1. *For each $N > 0$, there is a one-to-one correspondence between solutions of (2.1) on $\mathbb{C}^*$ and n-tuples of "Stokes parameters"*

$$s = (s_1, \ldots, s_n) \in \mathbb{R}^n$$

with $s_i = s_{n-i+1}$. Explicitly, s_i is the i-th symmetric function of the $n+1$ numbers $e^{(2m_0+n)\frac{\pi\sqrt{-1}}{n+1}}, e^{(2m_1+n-2)\frac{\pi\sqrt{-1}}{n+1}}, \ldots, e^{(2m_n-n)\frac{\pi\sqrt{-1}}{n+1}}$.

The Stokes parameters are (up to sign) the coefficients of the characteristic polynomial of a certain "$(n+1)$-th root of monodromy" matrix $M^{(0)}$, from which the Stokes matrices can be read off. The definition of $M^{(0)}$, and the proof of the theorem, is given in section 6 of [13]. It should be noted that the Stokes parameters are canonical, unlike the Stokes matrices themselves, which depend on various choices. This is important for physical applications.

As it will play a role in the next section, we note the following Lie-theoretic property of $M^{(0)}$, which implies the formula for s_i just given.

Theorem 3.2. *Assume that all $k_i > -1$. Then $M^{(0)}$ is conjugate to the diagonal matrix*

$$M^{(0)}_{\mathrm{diag}} = e^{\frac{2\pi\sqrt{-1}}{n+1}(m+\rho)}$$

where $\rho = \mathrm{diag}(\frac{n}{2}, \frac{n}{2} - 1, \dots, -\frac{n}{2})$.

Proof. When all $k_i > -1$, $M^{(0)}$ is diagonalizable, so the result follows from Proposition 6.9 of [13]. In the notation of [13] and [20], $\rho = x_0$. □

Remark 3.3. In the next section we shall restrict further to $k_i \in \mathbb{Z}_{\geq 0}$. Then $\eta(z)\, dz$ is a holomorphic connection form on $\mathbb{C}$, with a pole at $z = \infty$. The Stokes data turns out to be *equivalent* to the Stokes data $s = (s_1, \dots, s_n)$ of $\hat{\alpha}$. However, the pole does not have semisimple residue, and its order depends on the values of $k_0, \dots, k_n$ so the Stokes data is harder to extract than in the case of $\hat{\alpha}$. These difficulties may be dealt with by classical o.d.e. methods, but it is more efficient to use homogeneity and replace the 1-form $\omega = \frac{1}{\lambda}\eta(z)\, dz$ by a meromorphic 1-form $\hat{\omega}$ in the variable λ, just as we replaced α by $\hat{\alpha}$. The 1-form $\hat{\omega}$ always has a semisimple pole of order 2 at $\lambda = 0$ and a pole of order 1 at $\lambda = \infty$. The Iwasawa factorization shows that the Stokes data of $\hat{\alpha}$ and $\hat{\omega}$ are the same at $\lambda = 0$. The Stokes data of $\hat{\omega}$ is easily calculated. We note that α defines a harmonic bundle, and ω a corresponding Higgs bundle; this is the point of view of [22], [23]. In our notation the harmonic metric is e^{-2w} (cf. section 4.2 of [11]). □

Remark 3.4. As stated in Remark 3.3, the Stokes data of the tt*-Toda connection is the same as the Stokes data of the holomorphic connection, and so the formula in Theorem 3.2 applies also to the holomorphic connection. Although the condition $k_i = k_{n-i+1}$ was imposed in [13], the proof of the formula in Theorem 3.2 makes no use of this condition; it remains valid for arbitrary $k_i \geq -1$. We shall need this fact in sections 4 and 5. □

4. Lie algebra representations from Stokes data

We shall now establish a relation between

— solutions w of (2.1) with all $k_i \in \mathbb{Z}_{\geq 0}$

— positive energy representations of the affine Lie algebra $\widehat{\mathfrak{sl}}_{n+1}\mathbb{C}$

by utilizing the matrices $M^{(0)}$ (i.e. the Stokes data of w).

For the Lie algebra $\mathfrak{g} = \mathfrak{sl}_{n+1}\mathbb{C}$ we choose the standard diagonal Cartan subalgebra $\mathfrak{h} = \{\mathrm{diag}(h_0, \dots, h_n) \mid h_i \in \mathbb{C}, \sum_{i=0}^n h_i = 0\}$, and roots $x_i - x_j$ $(0 \leq i \neq j \leq n)$, where $x_i : \mathrm{diag}(h_0, \dots, h_n) \mapsto h_i$. We take $\alpha_i = x_{i-1} - x_i$ $(1 \leq i \leq n)$ as simple roots, then $\psi = x_0 - x_n$ is the highest root. Here we are using the Lie-theoretic conventions[2] of Examples 2.1, 3.6, 3.11 of [13], which generally follow those of Kostant [20].

We use the bilinear form $B(X, Y) = \mathrm{tr}\, XY$ to identify $\mathfrak{h}^*$ with $\mathfrak{h}$. Then the basic weights are identified with $\epsilon_1, \dots, \epsilon_n \in \mathfrak{h}$, where $\alpha_i(\epsilon_j) = \delta_{ij}$. Explicitly:

$$\epsilon_i = \mathrm{diag}\left(\left(1 - \tfrac{i}{n+1}\right) (\underbrace{1, \dots, 1}_{i}, \underbrace{0, \dots, 0}_{n+1-i}) - \tfrac{i}{n+1} (\underbrace{0, \dots, 0}_{i}, \underbrace{1, \dots, 1}_{n+1-i}) \right).$$

We have $\rho = \epsilon_1 + \cdots + \epsilon_n$. The weight lattice is

$$P = \{\textstyle\sum_{i=1}^n v_i \epsilon_i \mid \text{all } v_i \in \mathbb{Z}\}$$

(see Remark 4.1 below). The (fundamental) Weyl chamber is $C = \{\sum_{i=1}^n v_i \epsilon_i \mid \text{all } v_i \geq 0\}$, and the dominant weights are $P_+ = P \cap C$. The (fundamental) Weyl alcove is

$$A = \{\textstyle\sum_{i=1}^n v_i \epsilon_i \mid \text{all } v_i \geq 0 \text{ and } \sum_{i=1}^n v_i \leq 1\}.$$

Remark 4.1. In section 6 of [13] we put $\mathfrak{h}_\sharp = \{h \in \mathfrak{h} \mid \text{all } \alpha_i(h) \in \mathbb{R}\}$, so that ${\scriptstyle\sqrt{-1}}\,\mathfrak{h}_\sharp$ is the standard Cartan subalgebra of the compact real form $\mathfrak{su}_{n+1}$. Then the integer lattice is $I = {\scriptstyle\sqrt{-1}}\,\mathfrak{h}_\sharp \cap 2\pi{\scriptstyle\sqrt{-1}}\,\mathbb{Z}^{n+1}$. Thus the real roots $\alpha^{\mathrm{real}} = (2\pi{\scriptstyle\sqrt{-1}})^{-1}\alpha$ take integer values on the integer lattice, as do

[2] In this section we drop the requirement $k_i = k_{n-i+1}$, until it is needed (in Corollary 4.5 below) for the relation with solutions w of (2.1).

all real weights. The basic real weights are

$$\Lambda_i^{\mathrm{real}} = \tfrac{1}{2\pi\sqrt{-1}}\left(1 - \tfrac{i}{n+1}\right)(x_0 + \cdots + x_{i-1}) - \tfrac{1}{2\pi\sqrt{-1}}\tfrac{i}{n+1}(x_i + \cdots + x_n)$$

$(1 \le i \le n)$, and the weight lattice is $W = \oplus_{i=1}^n \mathbb{Z}\Lambda_i^{\mathrm{real}}$. With these definitions the Weyl alcove is

$$\mathfrak{A} = \{y \in {\scriptstyle\sqrt{-1}}\,\mathfrak{h}_\sharp \mid \text{all } \alpha_i^{\mathrm{real}} \ge 0 \text{ and } \psi^{\mathrm{real}}(y) \le 1\}.$$

To simplify the presentation in this article we use the convention "without $2\pi_{\sqrt{-1}}$" (which amounts to declaring that the exponential map is $X \mapsto e^{2\pi\sqrt{-1}\,X}$). We obtain P (instead of W) and A (instead of $\mathfrak{A}$). Our integer lattice is the set of integer diagonal matrices in $\mathfrak{sl}_{n+1}\mathbb{C}$. $\square$

We recall (see [19], [24]) that the affine Kac-Moody algebra $\widehat{\mathfrak{sl}}_{n+1}\mathbb{C}$ is an extension of the loop algebra $\Lambda\mathfrak{sl}_{n+1}\mathbb{C}$ by two additive generators, and that the irreducible positive energy representations of $\widehat{\mathfrak{sl}}_{n+1}\mathbb{C}$ of level k are parametrized by dominant weights (Λ, k), where Λ is a dominant weight of $\mathfrak{sl}_{n+1}\mathbb{C}$ of level k.

For nontrivial representations we have $k \in \mathbb{N}$. The dominant weights of $\mathfrak{sl}_{n+1}\mathbb{C}$ of level k are

$$P_k = \{\textstyle\sum_{i=1}^n v_i\epsilon_i \in P_+ \mid \sum_{i=1}^n v_i \le k\}.$$

The following fact is well known, but we give the short proof.

Lemma 4.2. *We have $P_k + \rho = P_+ \cap (k+n+1)\mathring{A}$ where $\mathring{A}$ denotes the interior of the Weyl alcove A.*

Proof. Let $v = \sum_{i=1}^n v_i\epsilon_i$ with all $v_i \in \mathbb{Z}_{\ge 0}$. Then: (i) $v \in P_k + \rho$ iff $v_i \ge 1$ and $\sum_{i=1}^n (v_i - 1) \le k$, i.e. $v_i > 0$ and $\sum_{i=1}^n v_i < k + n + 1$; (ii) $v \in \cap(k + n + 1)A$ iff $v_i \ge 0$ and $\sum_{i=1}^n v_i \le k+n+1$. Hence $v \in \cap(k+n+1)\mathring{A}$ iff $v_i > 0$ and $\sum_{i=1}^n v_i < k + n + 1$. Thus (i) and (ii) are equivalent. $\square$

In view of this, we introduce the following notation:

Definition 4.3. *Let $\mathring{A}_k = \left(\frac{1}{k+n+1}P_+\right) \cap \mathring{A}$. Let $\theta : \mathring{A}_k \to P_k + \rho$ be the identification given by $\theta(v) = (k+n+1)v \in P_+ \cap (k+n+1)\mathring{A} = P_k + \rho$.*

Recall from section 3 that the Stokes data is represented by a certain matrix $M^{(0)}_{\mathrm{diag}} = e^{\frac{2\pi\sqrt{-1}}{n+1}(m+\rho)}$. With the conventions of [13], the corresponding

Lie algebra element $\frac{2\pi\sqrt{-1}}{n+1}(m+\rho)$ is in $\mathfrak{A}$ (see Remark 4.1); with our current conventions we have $\frac{1}{n+1}(m+\rho)$ in A.

We now ask:

(i) for which m does $\frac{1}{n+1}(m+\rho)$ lie in the subset $\mathring{A}_k$, for some k ?

(ii) in that case, what is the corresponding element of $P_k+\rho$?

The answers are:

Theorem 4.4. *Assume that m arises from $k_0,\dots,k_n \in \mathbb{Z}_{\geq 0}$ through formula (2.3) (with $m_0+\cdots+m_n=0$). Then:*
(a) $\frac{1}{n+1}(m+\rho) \in \mathring{A}_k$ for $k=\sum_{i=0}^n k_i$, and
(b) the corresponding element of $P_k+\rho$ is $(\sum_{i=1}^n k_i\epsilon_i)+\rho$.

Proof. First we observe that formula (2.3) is equivalent to

$$N\tfrac{1}{n+1}(m+\rho)=\rho+\textstyle\sum_{i=1}^n k_i\epsilon_i. \tag{4.1}$$

(To verify this, it suffices to apply each simple root α_i to both sides, then use $\alpha_i(m)=m_{i-1}-m_i$ and $\alpha_i(\rho)=1$.) Next we put $k=\sum_{i=0}^n k_i$ (hence $N=n+1+k$). Then (4.1) says that $\theta(\frac{1}{n+1}(m+\rho))=\rho+\sum_{i=1}^n k_i\epsilon_i$, where θ is as in Definition 4.3. This gives both (a) and (b). □

Restricting now to the tt*-Toda situation (i.e. assuming $k_i=k_{n-i+1}$), we obtain:

Corollary 4.5. *Assume that $k_i \in \mathbb{Z}_{\geq 0}$ and $k_i=k_{n-i+1}$. Let $N=n+1+\sum_{i=0}^n k_i$. Then there is a one-to-one correspondence between*
(i) solutions w of the tt-Toda equation given by η (as in Theorem 2.1)*
(ii) Stokes data $M^{(0)}$ given by m (as in Theorem 3.1)
(iii) positive energy representations of $\widehat{\mathfrak{sl}}_{n+1}\mathbb{C}$ with dominant weights $(\Lambda,k)=(\sum_{i=1}^n k_i\epsilon_i, N-(n+1))$.

Example 4.6. Let $k_0=1$ and $k_1=\cdots=k_n=0$. Then $N=n+2$ and $m=-\frac{1}{n+2}\rho$. The corresponding representation has dominant weight $(0,1)$; this is the basic representation of $\widehat{\mathfrak{sl}}_{n+1}\mathbb{C}$. Let us compute the Stokes parameters $s_1,\dots,s_n$ using Theorem 3.1. These are the elementary symmetric functions of $e^{\frac{\pi\sqrt{-1}}{n+2}n}, e^{\frac{\pi\sqrt{-1}}{n+2}(n-2)},\dots,e^{\frac{\pi\sqrt{-1}}{n+2}(-n)}$. When $n+1$ is even, they are the $(n+2)$-th roots of -1, excluding -1 itself. They are the roots of the polynomial

$$\frac{x^{n+2}+1}{x+1} = x^{n+1} - x^n + x^{n-1} - \cdots - x + 1.$$

When $n+1$ is odd, they are the $(n+2)$-th roots of 1, excluding -1, i.e. the roots of the polynomial

$$\frac{x^{n+2}-1}{x+1} = x^{n+1} - x^n + x^{n-1} - \cdots + x - 1.$$

In both cases, all $s_i = 1$. $\square$

To comment on the significance of Theorem 4.4, one might say that it is hardly surprising that a positive energy representation of $\widehat{\mathfrak{sl}}_{n+1}\mathbb{C}$ can be concocted artificially from the element $\frac{1}{n+1}(m+\rho)$ of the Weyl alcove, as the Weyl alcove plays such a fundamental role in the theory of affine Lie algebras. However, Corollary 4.5 says that the (dominant weight of the) representation is given *precisely* by the integers k_i from which the solution w was constructed. Moreover the positive energy representations give *all* the global solutions of (2.1) which are generic (i.e. $m_{i-1} - m_i + 1 > 0$) and rational (i.e. $m_i \in \mathbb{Q}$). These form an open dense subset of all global solutions. Thus the representations are tightly related to the solutions of the tt*-Toda equations through our construction.

Remark 4.7. Positive energy representations of $\widehat{\mathfrak{sl}}_{n+1}\mathbb{C}$ give (projective) representations of the loop group $\Lambda\mathrm{SL}_{n+1}\mathbb{C}$. We have seen (and it is well known) that $\Lambda\mathrm{SL}_{n+1}\mathbb{C}$ plays an important role in solving the Toda equations. Thus one can expect a more direct role for the representation associated to w in Corollary 4.5. Indeed, the solutions are obtained by taking the Iwasawa factorization of the holomorphic $\Lambda\mathrm{SL}_{n+1}\mathbb{C}$-valued function L, and this is equivalent to the Birkhoff factorization of $c(L)^{-1}L$, where c is the real form involution of $\Lambda\mathrm{SL}_{n+1}\mathbb{C}$. This should give a determinant formula for the τ-function of w, as the Birkhoff factorization can be expressed in terms of infinite determinants given by τ-functions. Although we have not pursued this, the existence of such formulae is well known — see section 6 of [15] for a brief explanation of a determinant formula due to Tracy and Widom [27]. $\square$

5. Relations with conformal field theory

In this section we describe three ways in which the construction of section 4 is relevant to conformal field theory. It is written mainly for mathematicians who might not be familiar with physics, but we hope that the ideas sketched here are not too inaccurate and might also be of passing interest to physicists.

1. Topological-antitopological fusion

According to Cecotti and Vafa ([2],[3]) it is the solutions with "integer Stokes data" which represent physically realistic models. With our notation, this means solutions with integer Stokes parameters s_i. Then the s_i can be interpreted as counting Bogomolnyi solitons.

Furthermore, the s_i appear as leading term coefficients in the asymptotics as $t \to \infty$ of the corresponding solution w (see [11] for a precise statement). It follows from this (and Theorem 3.1 above) that the global solutions are characterized equally well by their asymptotics at $t = \infty$ as by their asymptotics at $t = 0$. This is another property predicted by Cecotti and Vafa, on the grounds that w represents the renormalization group flow between the chiral data at $t = 0$ (in our notation, the k_i or m_i) and the soliton data at $t = \infty$ (the s_i).

We have seen a solution of this type already in Example 4.6, where we have $k_0 = 1, k_1 = \cdots = k_n = 0$ and $m = -\frac{1}{n+2}\rho$. All s_i are equal to 1 here. From the tt* point of view, this particular solution corresponds to the supersymmetric A_{n+1} minimal model. Geometrically, it corresponds to an unfolding $\frac{1}{n+2}x^{n+2} - tx$ of the A_{n+1} singularity $\frac{1}{n+2}x^{n+2}$.

Other solutions of the tt* equations with geometric interpretations are those corresponding to the quantum cohomology of Kähler manifolds (or orbifolds). It is implicit in [2] that all such solutions are expected to be globally defined. In the (very special) case of the tt*-Toda equations, the basic example is the quantum cohomology of $\mathbb{C}P^n$, complex projective space. Here we have $k_0 = 0, k_1 = \cdots = k_n = -1$ and $m = -\rho$, and the solution is indeed globally defined. However, the assumption $k_i > -1$ is not satisfied here, and Theorem 3.2 does not apply (in fact, $M^{(0)}$ is not diagonalizable). Nevertheless the formula for the Stokes numbers in Theorem 3.1 does apply, and it gives $s_i = \binom{n+1}{i}$.

Further examples (such as weighted projective spaces and their hypersurfaces) can be found in [17]. In all cases the assumption $k_i > -1$ is violated.

With the prominent exception of the A_{n+1} minimal model, however, the "integer Stokes data" solutions are generally not of the type considered in section 4.

2. The fusion ring

The fusion ring of the WZW model $SU(n+1)_k$ is a certain ring structure on the set of positive energy representations of $\widehat{\mathfrak{sl}}_{n+1}\mathbb{C}$ of level k. We refer to [10] for the background, and [21] for a treatment close to the context of the current article.

The ring can be described succinctly (using the notation of section 4) as follows. For a positive energy representation with dominant weight (Λ, k), where $k \in \mathbb{N}$ and $\Lambda \in P_k$, a "special element" is defined by

$$t_\Lambda = e^{2\pi\sqrt{-1}\,\zeta_\Lambda}, \quad \zeta_\Lambda = \tfrac{\Lambda+\rho}{k+n+1}.$$

Then the level k fusion ideal $I_k(\mathrm{SU}_{n+1})$ of the representation ring $R(\mathrm{SU}_{n+1})$ is defined by

$$I_k(\mathrm{SU}_{n+1}) = \{\text{representations whose characters vanish at all } t_\Lambda, \Lambda \in P_k\}.$$

The level k fusion ring is then $R(\mathrm{SU}_{n+1})/I_k(\mathrm{SU}_{n+1})$.

Our observation concerning this is that

$$\zeta_\Lambda = \tfrac{1}{n+1}(m+\rho),$$

where m corresponds to $\Lambda = \sum_{i=1}^n k_i\epsilon_i$ as in section 4. This follows immediately from (4.1). In other words, the special element t_Λ is precisely our matrix $M^{(0)}_{\mathrm{diag}}$ which represents the Stokes data of the holomorphic 1-form $\eta(z)dz$ (see Remark 3.4).

We do not know a satisfactory explanation of this coincidence. On the one hand, it is well known that fusion arises geometrically from "fusing" moduli spaces of flat SU_{n+1}-connections over Riemann surfaces with a common boundary component, and it is known that such moduli spaces can be described in terms of monodromy data. On the other hand the connections in sections 1-3 are *not* SU_{n+1}-connections.

3. Minimal models

It is well known (e.g. section 9.4 of [24]) that the Virasoro algebra acts by intertwining operators on any $\widehat{\mathfrak{sl}}_{n+1}\mathbb{C}$-module of positive energy. In this way a positive energy representation gives a representation of the Virasoro algebra. Irreducible representations are classified according to their central charge c and conformal dimension h.

Representations of the Virasoro algebra can be used to construct special examples of conformal field theories called minimal models; in a minimal model, the Hilbert space of the theory is a sum of finitely many irreducible representations, and the representations which occur are highly restricted. The theory of these "Virasoro minimal models" is described in [7].

More generally, representations of the W-algebra W_{n+1} (see [9],[1]) can be used to construct "W_{n+1} minimal models". (The W-algebra W_2 is the Virasoro algebra.) The theory of W_{n+1} minimal models is described in [1], from which we quote the following result:

Theorem 5.1. *([1]) Let $p, p' \in \mathbb{N}$ be coprime. Let $\Lambda^{(+)}, \Lambda^{(-)}$ be dominant weights of $\mathfrak{sl}_{n+1}\mathbb{C}$. Then there exists an irreducible representation of W_{n+1} whose central charge is*

$$c = n - n(n+1)(n+2)\tfrac{(p'-p)^2}{pp'} \tag{5.1}$$

and whose conformal dimension h is given by

$$c - 24h = n - 12\left|\alpha_+(\Lambda^{(+)} + \rho) + \alpha_-(\Lambda^{(-)} + \rho)\right|^2, \tag{5.2}$$

where ρ is as in Theorem 3.2, and $\alpha_+ = \sqrt{p'/p}, \alpha_- = -\sqrt{p/p'}$.

Some comments on the notation of [1] are in order before we proceed further. The central charge formula is (6.13) in [1], and the conformal dimension formula is (6.74). The scalars α_+, α_- are given just after (6.13) and in (6.75); we have taken the positive square root for α_+.

For the W_{n+1} minimal model of type (p, p') the dominant weights $\Lambda^{(+)}, \Lambda^{(-)}$ are restricted as follows:

$$\Lambda^{(+)} \in P_{p-(n+1)}, \quad \Lambda^{(-)} \in P_{p'-(n+1)}.$$

This is (6.76) in [1]. However, dominant weights which lie in the same orbit of the centre of SU_{n+1} should be identified. This is (6.77) in [1]. We shall make the action explicit in a moment. By definition, the minimal model of type (p, p') consists of the set of equivalence classes with respect to this action.

The special case $p = n+1, p' = N = n+1+k = n+1+\sum_{i=0}^n k_i$ was considered by Fredrickson and Neitzke in [8], in connection with Argyres-Douglas theories of type (A_n, A_{k-1}). (Our $n+1$ is called K by them, and our $k = \sum_{i=0}^n k_i$ is called N by them.) Here we have

$$\Lambda^{(+)} = 0, \quad \Lambda^{(-)} \in P_k.$$

From now on we shall write $\Lambda = \Lambda^{(-)}$, in keeping with our earlier notation for dominant weights (this should not be confused with the notation $\Lambda = \alpha_+\Lambda^{(+)} + \alpha_-\Lambda^{(-)}$ in (6.73) of [1], which we shall not use).

Formulae (5.1) and (5.2) become, in this case:

$$c = n - \tfrac{1}{N}n(n+2)(N-(n+1))^2 \tag{5.3}$$

$$c - 24h = n - 12\tfrac{n+1}{N}\left|\Lambda - \tfrac{N-(n+1)}{n+1}\rho\right|^2 \tag{5.4}$$

Example 5.2. The values $\Lambda^{(+)} = \Lambda^{(-)} = 0$ are always permitted in the W_{n+1} minimal model, and for these weights we have $h = 0$. However a feature of the special case $p = n + 1$ is that $h \leq 0$ for all weights in the model. This follows from formula (5.5) below. As a consequence, in this special case, the model cannot be unitary. □

The weights Λ in the $(n + 1, n + 1 + k)$ minimal model are restricted to lie in P_k, but (as mentioned above) there is a further reduction given by dividing by the action of the centre of SU_{n+1}, i.e. by the cyclic group $\{I, \iota I, \ldots, \iota^n I\}$ where ι is a primitive $(n + 1)$-th root of unity.

Proposition 5.3. *The action of ι on the weight $\Lambda = \sum_{i=1}^{n} k_i \epsilon_i$ is given by*

$$\iota \cdot \Lambda = \sum_{i=1}^{n} k_{\sigma \cdot i}\, \epsilon_i$$

where σ is the cyclic permutation $\sigma = (01 \cdots n)$. Here we are using the notation $k_0, \ldots, k_n$ as in section 4, i.e. $N = n + 1 + \sum_{i=0}^{n} k_i$ and the subscript i of k_i is interpreted mod $n + 1$.

Proof. The essential ingredient here is the action of the centre on the alcove A, which is explained in detail in [26] and in section 2.4 of [21]. An explicit formula for the case SU_{n+1} can be found in section 4.4 of [26]. Applying this formula to our alcove element $\frac{1}{n+1}(m + \rho)$, we see that the action corresponds to cyclic permutation of the subscripts of the m_i (without changing ρ). From our formula (2.3), this corresponds to cyclic permutation of the subscripts of the k_i. □

Thus the weights $\Lambda = \sum_{i=1}^{n} v_i \epsilon_i$ which occur in the W_{n+1} minimal model of type $(p, p') = (n + 1, N - (n + 1))$ are indexed by "cyclic $(n + 1)$-partitions of $N - (n + 1)$". These may be enumerated as follows.

Proposition 5.4. *Assume that $n + 1$ and N are coprime. Then there are $\frac{1}{N}\binom{N}{n+1}$ equivalence classes of $(n + 1)$-tuples $(k_0, \ldots, k_n) \in \mathbb{Z}_{\geq 0}^{n+1}$ such that $\sum_{i=0}^{n} k_i = N - (n + 1)$, where the equivalence relation is defined by $(k_0, \ldots, k_n) \sim (l_0, \ldots, l_n)$ if $l_i = k_{i-s}$ for some $s \in \mathbb{N}$ (all indices are mod $n + 1$).*

Proof. A reference from the combinatorics literature is Corollary 1 of [25], but we shall give a proof based on the properties of the holomorphic connection $\nabla = d + \eta(z)dz$, which seems appropriate for the present context.

The equation for parallel sections of $\nabla = d - \eta(z)^t dz$ can be written

$$D \begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix} = \eta(z)^t \begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix}, \quad D = \tfrac{d}{dz}.$$

Each y_i satisfies a scalar o.d.e.

$$z^{-k_i} D z^{-k_{i-1}} D \cdots D z^{-k_{i-n}} D\, y_i = y_i$$

where we interpret $k_{i-(n+1)}$ as k_i. As i varies, these scalar equations differ by cyclic permutations, and they are all equivalent in an obvious sense — more formally, they define isomorphic D-modules.

To calculate the number of equivalence classes, we have to calculate the number of (ordered) strings of symbols D ($n+1$ times) and z^{-1} ($N-(n+1) = \sum_{i=0}^n k_i$ times) up to cyclic equivalence.

Now, there are $\binom{N}{n+1}$ ways to choose the positions of the D's. This gives $\frac{1}{N}\binom{N}{n+1}$ cyclic equivalence classes, *assuming that* the orbit of every string has N distinct elements. An orbit has less than N distinct elements if and only if it contains a sub-string of length l, containing m D's say, repeated r times (with $l, r > 1$). In that case we have $lr = N$ and $mr = n+1$, but this is impossible as $n+1$ and N are coprime. □

Now we shall connect this to the construction of section 4. Formula (4.1) can be written

$$N \tfrac{1}{n+1} m = \Lambda - \tfrac{N-(n+1)}{n+1}\rho$$

where $\Lambda = \sum_{i=1}^n k_i \epsilon_i$. Using this, (5.4) becomes

$$c - 24h = n - 12\tfrac{N}{n+1}\,|m|^2. \tag{5.5}$$

Substituting for c from (5.3), we can write the formula for h as

$$24h = -\tfrac{1}{N} n(n+2)(N-(n+1))^2 + 12\tfrac{N}{n+1}|m|^2. \tag{5.6}$$

Alternatively, using the fact that $|\rho|^2 = \frac{1}{12}n(n+1)(n+2)$, we can write

$$h = \tfrac{n+1}{2N}\left(\left|\Lambda - \tfrac{N-(n+1)}{n+1}\rho\right|^2 - \left|\tfrac{N-(n+1)}{n+1}\rho\right|^2 \right). \tag{5.7}$$

Fredrickson and Neitzke arrive at these considerations from a rather different starting point, namely a certain moduli space of Higgs bundles on $\mathbb{C}$.

The moduli space admits an action of $\mathbb{C}^*$, and the fixed points of this action are the forms (or Higgs fields) $\eta(z)dz$ with all $k_i \in \mathbb{Z}_{\geq 0}$. In their notation the quantity $\frac{N}{n+1}|m|^2$ is called μ, and it is identified with a "regulated norm" of $\eta(z)dz$. Our formula (5.5) is then

$$c - 24h = n - 12\mu.$$

Thus we recover Theorem 5.3 of [8]. This relation is the basis for the "somewhat mysterious" bijection ([8], section 5) between Higgs fields and representations of W_{n+1}.

Our construction in section 4 gives a (mathematical) explanation of this bijection. Namely, it shows how the representation arises directly from $\eta(z)dz$ by means of its Stokes data. Thus it is the Stokes data which provides the crucial link.

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