Blowup rate control for solution of Jang's equation and its application on Penrose inequality

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We prove that the blowup term of a blowup solution of Jang's equation on an initial data set $(\mathcal{M}, g, k)$ near an arbitrary strictly stable MOTS Σ is exactly $-\frac{1}{\sqrt{\lambda}} \log \tau$, where τ is the distance from Σ and λ is the principal eigenvalue of the MOTS stability operator of Σ. We also prove that the gradient of the solution is of order τ^{-1}. Moreover, we apply these results to get a Penrose-like inequality under additional assumptions.

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2. Preliminary and main results

In this paper, we specify the relationship between the blowup term of any blowup solution of Jang's equation at a strictly stable marginally outer trapped surface (MOTS) and the principal eigenvalue of the stability operator. We also specify the blowup order of the gradient of the solution. We then apply these results to the Jang's slice constructed by J. Metzger (c.f. [2],[3]) to get a Penrose-like inequality.

Jang's equation is a quasilinear elliptic equation on an initial data set and it was proposed by P. Jang in [15]. An important application of this equation is the proof of Positive Mass Theorem on a general initial data set by R. Schoen and S.-T. Yau in [1]. As in [1], we assume $(\mathcal{M}, g, k)$ is an oriented 3-dim initial data set, with g the Riemannian metric, k the second fundamental form, μ the local mass density, J^i the local current density, satisfying the constraint equations

$$\mu = \frac{1}{2}(R - k^{ij}k_{ij} + (tr_g k)^2)$$
$$J^i = \nabla_j(p^{ij} - g^{ij}tr_g k)$$

The initial data set $(\mathcal{M}, g, k)$ is said to be asymptotically flat if the complement of some compact subset of $\mathcal{M}$ consists of finitely many connected components $\mathcal{M}_1, ..., \mathcal{M}_m$, called the ends, each one diffeomorphic to $\mathbb{R}^3 - \overline{B_1(0)}$, and such that in the corresponding coordinate systems the metric tensor g_{ij} converges to the Euclidean metric δ_{ij} and the second fundamental form tensor k_{ij} to zero. More precisely, we require that $|g_{ij} - \delta_{ij}| + |x||\partial_k g_{ij}| = O(|x|^{-q})$ and $k_{ij} = O(|x|^{-q-1})$ as $|x| \to \infty$ for some $q > \frac{1}{2}$. We ask in addition that for some $\beta > 2$, $tr_g(k) = g^{ij}k_{ij} = O(|x|^{-\beta})$ as $x \to \infty$. This last condition is imposed so that certain barriers for the Jang's equation can be constructed far out in the asymptotically flat ends. With each end $\mathcal{M}_k$ we associate a total mass m_k defined by the flux integral:

$$m_k = \frac{1}{16\pi} \int_\infty \sum_{i,j} (g_{ij,j} - g_{jj,i}) d\sigma_i$$

which is the limit of surface integrals taken over large two spheres in $\mathcal{M}_k$. This number m_k is called the ADM mass of $\mathcal{M}_k$, c.f. [37]. In this formulation, the general Positive Mass Theorem states that, if $(\mathcal{M}, g, k)$ is a complete asymptotically flat initial data set and satisfies the dominant energy condition $\mu > |J|$, then the ADM mass of each end is non-negative. In [1], R. Schoen and S.-T. Yau proved this theorem by reducing it to the Riemannian Positive Mass Theorem with the help of Jang's equation, which states that for a 3-dim complete asymptotically flat Riemannain manifold with non-negative scalar curvature, the ADM mass of each end is non-negative.

An important tool applied in this reduction strategy is Jang's equation. Consider the graph of a function $f \in C^2(\mathcal{M})$ as a hypersurface of the product manifold $(\mathcal{M} \times \mathbb{R}, g + dt^2)$. Jang's equation is then:

$$(2.1) \qquad \mathcal{J}[f] = \mathcal{H}[f] - \mathcal{P}[f] = 0,$$

where $\mathcal{H}[f]$ denotes the mean curvature of $graph f \subset \mathcal{M} \times \mathbb{R}$ computed with respect to its downward-pointing unit normal, and $\mathcal{P}[f]$ denotes the trace of k with respect to the induced metric on $graph f$, after extending k trivially along the $\mathbb{R}$-factor. Thus $\mathcal{H}[f], \mathcal{P}[f]$ are given by:

$$\mathcal{H}[f] = (g^{ij} - \frac{f^i f^j}{1 + |\nabla f|^2}) \frac{\nabla_i \nabla_j f}{\sqrt{1 + |\nabla f|^2}}$$

$$\mathcal{P}[f] = (g^{ij} - \frac{f^i f^j}{1 + |\nabla f|^2}) k_{ij}$$

In [1], in order to construct a solution for Jang's equation Eq.(2.1), it was proved that the auxiliary equation:

$$(2.2) \qquad \mathcal{H}[f] - \mathcal{P}[f] = tf$$

has a solution $f_t \in B^{2,\beta}$ when $t > 0$. Here, the weighted Hölder space $B^{2,\beta}$ for $\beta \in (0, 1)$ is defined as the space for all f such that $|f|_{2,\beta}$ is finite, where

$$|f|_{2,\beta} = \sup_{\mathcal{M}}(r^\beta(x)|f(x)| + r^{1+\beta}(x)|\nabla f(x)|$$
$$+ r^{2+\beta}(x)|\nabla^2 f(x)| + r^{2+2\beta}(x)|\nabla^2 f|_{\beta,x})$$
$$|\nabla^2 f|_{\beta,x} = \sup_{x_1, x_2 \in B_{\frac{r(x)}{2}}(x)} \frac{|\nabla^2 f(x_1) - \nabla^2 f(x_2)|}{dist(x_1, x_2)^\beta}$$

where the weight function $r(x)$ satisfies $r(x) \geq 1$ in $\mathcal{M}$, and $r(x) = |x|$ on each end $\mathcal{M}_k$.

The solution f_t for the auxiliary equation Eq.(2.2) satisfies the a priori estimate:

(2.3)
$$\sup_{\mathcal{M}} t|f| \leq \mu_1, \quad \sup_{\mathcal{M}} t|\nabla f| \leq \mu_2$$

where μ_1, μ_2 are constants depending only on the initial data. With this gradient bound and the a priori estimate [1, Proposition 2], a smooth solution f of Jang's equation Eq.(2.1) can be found, as the limit function of the sequence $\{f_{t_i}\}$ for a sequence $\{t_i\}$ converging to zero. This solution of Jang's equation might blow up or down at some apparent horizons Σ. By apparent horizon we mean a closed surface Σ such that one of $\theta^+[\Sigma]$ or $\theta^-[\Sigma]$ vanishes on Σ, where

$$\theta^{\pm} = \text{tr}_g k - k(\nu, \nu) \pm H,$$

is called *inner* and *outer expansions*. Here ν is the unit normal on Σ pointing into $\mathcal{M}$ and H is the mean curvature of Σ in $\mathcal{M}$ with respect to ν. Also by blowup we mean, that outside from an apparent horizon Σ the function f is such that *graphf* is a smooth submanifold of $\mathcal{M} \times \mathbb{R}$ with a cylindrical end converging to $\Sigma \times \mathbb{R}$.

After the solution of Jang's equation has been constructed, it is applied to the proof of the general Positive Mass Theorem. It helps reduce the general Positive Mass Theorem to the Riemannian Positive Mass Theorem. We review the procedure of this reduction strategy here. For each apparent horizon Σ where the solution f blows up or down, denote U a neighborhood of Σ, then in [1] each of these cylindrical end of *graphf* is slightly deformed in $U \times (T, \infty)$ or $U \times (-\infty, -T)$ so that *graphf* coincides with $\Sigma \times \mathbb{R}$ in $U \times (T, \infty)$ or $U \times (-\infty, -T)$. These cylindrical ends are then closed up by a conformal factor ψ which exponentially goes to zero on these ends. After this step, the metric on each of these cylindrical ends is conformally changed to a metric which is uniformly equivalent to the flat metric in a punctured ball. Then, a conformal factor u on $\mathcal{M}$ for this deformed metric $\tilde{g}$ can be found such that $u^4 \tilde{g}$ has zero scalar curvature. The theorem is then proved because of the Riemannian Positive Mass Theorem and the fact that the ADM mass of $u^4 \tilde{g}$ is not greater than that of g.

It is well known that in the time symmetric case when black holes are present, there is a positive lower bound for the mass, in terms of the surface area of the black holes. The Riemannian Penrose inequality may be thought of as a refinement of the Riemannian Positive Mass Theorem in the presence of horizons. It relates the total ADM mass (of a chosen end) m to the area

A of its outermost minimal area enclosure. The inequality states that:

$$(2.4) \qquad m \geq \sqrt{\frac{A}{16\pi}}.$$

for an asymptotically flat Riemannian manifold with nonnegative scalar curvature. It is conjectured that Penrose inequality also holds for general initial data sets, when horizons are replaced by MOTSs. It is natural to consider applying Jang's equation to prove the general Penrose inequality, because until now this is the only equation that can locate the apparent horizons. However, in the proof of the Positive Mass theorem [1], the information of the apparent horizon Σ is lost after the cylindrical end being deformed to coincide with $\Sigma \times \mathbb{R}$ in $U \times (T, \infty)$ or $U \times (-\infty, -T)$. For example, in Schwarzschild spacetime, we can construct a blowup solution for Jang's equation on its time symmetric slice. Then if we applied the same reduction techniques as in [1] to this blowup solution, the cylindrical end will be closed up and the corresponding metric will then be conformally deformed to the flat metric on punctured $\mathbb{R}^3$. If one want to extract some information about Σ, then the original blowup end of the solution is needed. Due to this issue, H. Bray and M. Khuri proposed in [26] a way to prove the Penrose inequality for general initial data sets by using the solution of generalized Jang's equation. Also in the same paper a solution of generalized Jang's equation in spherically symmetrical setting was found and was used to prove Penrose inequality in this case. However, the existence of the solution of generalized Jang's equations is still not clear in the general case. For the original equation proposed by P. Jang, J. Metzger was able to construct in [2] a solution which blows up at the outermost MOTS. In this paper, our intuition is to obtain a sharp estimate of blowup rate of the solution of the original Jang's equation Eq.(2.1) at one blowup end, and thus capture some information of the corresponding horizon Σ in form of a Penrose-like inequality.

There has been a lot of research about Jang's equation and its blowup solution, c.f. [1],[2],[3],[5],[6]. In these papers, various existence theorems of Jang's equation were proved, with different constraints, such as asymptotically decaying and blowup at horizon. Blowup control estimates of these solutions were also given in these papers. The uniqueness of asymptotically decaying solution of Jang's equation which blowup at outermost MOTS in the time symmetric (i.e. $k = 0$) case was proved in [3]. We review these results in Section 3.

In this paper, we study the blowup rate of any possible blowup solution of Jang's equation (2.1) near a strictly stable MOTS Σ on a general initial

data set $(\mathcal{M}, g, k)$. We prove in Section 4 that the blowup term of any such kind of solution is exactly $-\frac{1}{\sqrt{\lambda}} \log \tau$, where τ is the distance from Σ and λ is the principal eigenvalue of the MOTS stability operator of Σ. In Section 5, we prove the order of the gradient of the solution is τ^{-1}. In Section 6, for the solution f_0 constructed by J. Metzger in [2], we prove that the the the coefficients of our estimates for f_0 depend only on the initial data set by using the a priori estimates for f_0. We then prove in Section 7 that under additional assumptions, we can apply these results to the slice $graph f_0$ to get a Penrose-like inequality.

We start from an initial data set $(\mathcal{M}, g, k)$ with boundary, where $\mathcal{M}$ is a 3-manifold equipped with a Riemannian metric g together with a symmetric bilinear form k representing the second fundamental form of the time slice $\mathcal{M}$ in space-time. We assume its boundary $\partial \mathcal{M}$ is consisted of apparent horizons. An apparent horizon is called a *marginally outer trapped surface* (MOTS) if $\theta^+(\Sigma)$ vanishes, and *marginally inner trapped surface* (MITS) if $\theta^-[\Sigma]$ vanishes. If Σ is a MOTS and there is no other MOTS on the outside of it, we call it an *outermost* MOTS. There might be other apparent horizons inside $\mathcal{M}$, but we assume that each connected component of $\partial \mathcal{M}$ has positive distance to the rest of $\partial \mathcal{M}$ and all other apparent horizons inside $\mathcal{M}$.

Now we introduce the definition of MOTS stability operator and its principal eigenvalue. For a more detailed investigation we refer to [14].

Let $\Sigma \subset \mathcal{M}$ be a MOTS and consider a normal variation of Σ in $\mathcal{M}$, that is a map $F : \Sigma \times (-\epsilon, \epsilon) \to \mathcal{M}$ such that $F(\cdot, 0) = id_\Sigma$ and $\frac{\partial}{\partial s}\big|_{s=0} F(p, s) = h\nu$, where h is a function on Σ and ν is the normal of Σ. Then the variation of θ^+ is given by

$$\frac{\partial \theta^+[F(\Sigma, s)]}{\partial s}\bigg|_{s=0} = L_\Sigma h,$$

where L_Σ is a linear elliptic operator of second order along Σ, given by

$$L_\Sigma h = -{}^\Sigma \Delta h + 2S({}^\Sigma \nabla h) + h\big({}^\Sigma div S - |\chi^+|^2 - |S|^2 + \tfrac{1}{2}{}^\Sigma Sc - \mu - J(\nu)\big).$$

In this expression ${}^\Sigma \nabla$, ${}^\Sigma div$ and ${}^\Sigma \Delta$ denote the gradient, divergence and Laplace-Beltrami operator tangential to Σ. The tangential 1-form S is given by $S(\nu) = k(\nu, \nu)$ for any ν tangential to Σ. χ^+ is the bilinear form $\chi^+ = A + k^\Sigma$, where A is the second fundamental form of Σ in M and k^Σ is the projection of k to $T\Sigma \times T\Sigma$. Furthermore, ${}^\Sigma Sc$ denotes the scalar curvature of Σ, $\mu = \frac{1}{2}({}^\Sigma Sc - |k|^2 + (trk)^2)$, and $J = div_M k - dtrk$. It is worth noting that L_Σ is not self-adjoint.

However, it was proved in [14, Lemma 4.1] that when Σ is compact then there is a real eigenvalue λ, called *the principal eigenvalue*, such that the real part of any other eigenvalue of L_Σ is greater or equal to λ. The corresponding eigenfunction β, $L\beta = \lambda\beta$ is unique up to a multiplicative constant and can be chosen to be real and everywhere positive. If λ is positive, Σ is called *strictly stable*. In particular, if Σ is strictly stable as a MOTS, there exists an outward deformation strictly increasing θ^+.

In this paper, we use following ways of foliation near the apparent horizon Σ. We will use Foliation B in most of our proofs.

Foliation A. *Denote ν the normal vector field of Σ pointing into $\mathcal{M}$. Defined the foliation Ψ_A to be the map:*

$$(2.5) \qquad \Psi_A : \Sigma \times [0, \bar{\tau}] \to \mathcal{M} : \; (p, \tau) \mapsto \exp_p^{\mathcal{M}}(\tau\nu)$$

Foliation B. *Suppose Σ is a compact boundary component and a strictly stable MOTS with principal eigenvalue and corresponding eigenfunction $\lambda > 0$. We further scale so that $\min_\Sigma \beta = 1$. Defined the foliation Ψ_B to be the map:*

$(2.6) \qquad \Psi_B : \Sigma \times [0, \bar{s}] \to \mathcal{M}$ *such that*

$\quad (1)\ \Psi_B(p, 0) = p$ *for $p \in \Sigma$*

$\quad (2)\ \dfrac{\partial \Psi_B(p, s)}{\partial s} = \beta(p)\nu_s,$ *where ν_s is the normal to $\Sigma_s := \Psi(\Sigma, s)$*

$\qquad$ *extending the outward pointing normal ν_0 on $\Sigma_0 = \Sigma$.*

Because Σ is compact, β is positive, and $\min_\Sigma \beta = 1$, Ψ_A and Ψ_B are comparable.

Our main results are the following:

Theorem 2.1. *Suppose $(\mathcal{M}, g, k)$ is a smooth initial data set with boundary $\partial\mathcal{M} = \cup_i \Sigma_i$, where each Σ_i is a connected component of the boundary. Let Σ be a boundary component which is a compact and strictly stable MOTS with principal eigenvalue $\lambda > 0$, and f be a solution of Eq.(2.1) in an open neighborhood V of Σ, and blows up at Σ, i.e. $f(x) \to +\infty$ when $x \to \Sigma$. Denote $\tau(x)$ to be the geodesic distance of a point x in a neighborhood of Σ to Σ. Also denote $V_c = \{x \in \mathcal{M} | \tau(x) \le c\}$.*

Then there exists τ_0 depending only on the local geometry near Σ, such that $f(x) + \dfrac{1}{\sqrt{\lambda}} \log \tau(x)$ is a bounded function in V_{τ_0}.

More specifically, under Foliation B, there exist constants a and s_0 depending only on the local geometry near Σ, such that the following barrier control for f holds in U_{s_0} for any $0 < s \leq s_1 \leq s_0$. Here U_{s_0} is the region swept out by Σ_s.

$$(2.7) \qquad -\frac{1}{\sqrt{\lambda}} \log \frac{s}{s_1} + \inf_{\Sigma_{s_1}} f + a(s - s_1)$$

$$\leq f(\cdot, s) \leq -\frac{1}{\sqrt{\lambda}} \log \frac{s}{s_1} + \sup_{\Sigma_{s_1}} f - a(s - s_1)$$

As a consequence, on each Σ_s for $s \in (0, s_0)$, if $p_i \in \Sigma, i = 1, 2$ are the points such that $f(p_1, s) = \sup_{\Sigma_s} f$ and $f(p_2, s) = \inf_{\Sigma_s} f$, then we have the following gradient estimates for f at p_1, p_2:

$$(2.8) \qquad\qquad |\partial_s f(p_1, s)| \leq \frac{1}{\sqrt{\lambda} s} + a$$

$$(2.9) \qquad\qquad |\partial_s f(p_2, s)| \geq \frac{1}{\sqrt{\lambda} s} - a$$

Here

$$\partial_s f(p_i, s) := \lim_{\delta s \to 0} \frac{f(p_i, s + \delta s) - f(p_i, s)}{\delta s}, i = 1, 2.$$

Remark 2.2. *The solution f might blow up at some other boundary components, but as long as that boundary component is a strictly stable MOTS, the blowup rate of f near that component can be described by the principal eigenvalue of that component.*

From the gradient estimate for f under Foliation B at extreme points on each slice Σ_s, we immediately have the following result:

Corollary 2.3. *Assume conditions in Theorem 2.1. Then under Foliation B, if f is constant on Σ_{s_1} for some $s_1 \in (0, s_0)$, then the following gradient estimate holds on on Σ_{s_1}:*

$$\frac{1}{\sqrt{\lambda} s_1} - a \leq |\partial_s f(\cdot, s_1)| \leq \frac{1}{\sqrt{\lambda} s_1} + a$$

Let U be a neighborhood of Σ. We can define a coordinate system on the neighborhood $U \times \mathbb{R}$ of $\Sigma \times \mathbb{R}$ in $\mathcal{M} \times \mathbb{R}$ by taking the fourth coordinate s

to be the parameter in Foliation B. Let $\bar{\Psi} : \Sigma \times (-\epsilon, \epsilon) \to \mathcal{M}$ be the map

$$\bar{\Psi} : \Sigma \times (-\epsilon, \epsilon) \times \mathbb{R} \to \mathcal{M} \times \mathbb{R} : (p, s, z) \mapsto \big(\Psi_B(p, s), z\big).$$

Therefore, under coordinate system $\bar{\Psi}$, following the argument in [1, Proposition 4], the graph of the blowup solution f near Σ on $\mathcal{M}$ can be written as the graph of a function u on the cylindrical end $\Sigma \times \mathbb{R}$. In [2, Section 4], J. Metzger provides a way to extend the C^0 super control to C^2 super control. Therefore, we are able to get some gradient estimates for f:

Theorem 2.4. *Under the same assumptions as Theorem 2.1, suppose f is a blowup solution of Eq.(2.1) in an open neighborhood U near Σ, which is a compact boundary component and a strictly stable MOTS with principal eigenvalue $\lambda > 0$. Denote $N = graphf$. Then, under coordinate system $\bar{\Psi}$,*
(1) there exist positive constants $\bar{z}$ and C_1, C_2, C_3, such that $N \cap (U \times [\bar{z}, \infty))$ can be written as the graph of a function u over $C_{\bar{z}} := \Sigma \times [\bar{z}, \infty)$, and

$$(2.10) \qquad |u(p, z)| + |^{C_{\bar{z}}}\nabla u(p, z)| + |^{C_{\bar{z}}}\nabla^2 u(p, z)| \leq C_1 \exp(-\sqrt{\lambda}z)$$

$$(2.11) \qquad |u(p, z)| \geq C_2 \exp(-\sqrt{\lambda}z)$$

$$(2.12) \qquad |^{C_{\bar{z}}}\nabla u(p, z)| \geq C_3 \exp(-\sqrt{\lambda}z)$$

where $^{C_{\bar{z}}}\nabla$ is the covariant derivative w.r.t. the induced metric on $C_{\bar{z}}$.
(2) Denote $\Sigma_s = \Psi_B(\Sigma, s)$. Then there exists constant s_0, such that the following gradient estimates for f:

$$(2.13) \qquad \frac{C_2}{C_1 s} \leq |\partial_s f(p, s)| \leq \frac{C_1}{C_3 s}$$

$$(2.14) \qquad |\nabla^{\Sigma_s} f(p, s)| \leq \frac{\sqrt{2}C_1^2}{C_2 C_3}$$

hold for $\forall p \in \Sigma$, $\forall s \in (0, s_0]$. Here ∇^{Σ_s} denotes the covariant derivative along Σ_s.

Remark 2.5. *From Eq. (2.7), we can see that the barriers for f depend on $\inf_{\Sigma_{\tau_0}} f$ and $\sup_{\Sigma_{\tau_0}} f$. This is due to the fact that Jang's equation (2.1) is invariant under vertical translation. Therefore, $C_1, C_2, C_3, \bar{z}$ also depend on these quantities. If we hope that $C_1, C_2, C_3, \bar{z}$ can be determined by the initial data, then more constraints are needed, e.g. outer boundary condition or decay in infinity.*

If we put more constraints on $(\mathcal{M}, g, k)$ such that it meets the conditions in [2, Theorem 3.1, Remark 3.3], then a blowup solution f_0 of Jang's equation (2.1) can be constructed at outermost MOTS, with appropriate a priori estimates. On this specific Jang's slice $N_0 = graph f_0$, we can get the same estimates as in Theorem 2.1 and 2.4, with constants $C_1, C_2, C_3, \bar{z}$ only depending on the geometry of the initial data.

Theorem 2.6. *Besides the conditions and notations in Theorem 2.4, we further assume that $\mathcal{M}$ is a 3-dim asymptotically flat manifold with one end, and satisfies the dominant energy condition. Also assume that Σ is the only boundary component, and is a compact outermost MOTS. We further assume that there is no MITS in $\mathcal{M}$. Then a function f_0 on $\mathcal{M}$ can be constructed as in [2, Theorem 3.1], such that $\mathcal{J}[f_0] = 0$, $f_0(x) \to 0$ when $|x| \to \infty$, and it only blows up at Σ. Denote $N_0 = graph f_0$. Then, under coordinate system $\bar{\Psi}$,*

(1) there exist positive constants $\bar{z}$ and C_1, C_2, C_3 only depending on the initial data, such that $N_0 \cap (U \times [\bar{z}, \infty))$ can be written as the graph of a function u_0 over $C_{\bar{z}} := \Sigma \times [\bar{z}, \infty)$, and

$$(2.15) \qquad |u_0(p, z)| + |^{C_{\bar{z}}} \nabla u_0(p, z)| + |^{C_{\bar{z}}} \nabla^2 u_0(p, z)| \leq C_1 \exp(-\sqrt{\lambda} z)$$

$$(2.16) \qquad\qquad\qquad |u_0(p, z)| \geq C_2 \exp(-\sqrt{\lambda} z)$$

$$(2.17) \qquad\qquad\qquad |^{C_{\bar{z}}} \nabla u_0(p, z)| \geq C_3 \exp(-\sqrt{\lambda} z)$$

where $^{C_{\bar{z}}} \nabla$ is the covariant derivative w.r.t. the induced metric on $C_{\bar{z}}$.

(2) Denote $\Sigma_s = \Psi_B(\Sigma, s)$. Then there exists constant s_0, such that the following gradient estimates for f_0:

$$(2.18) \qquad\qquad \frac{C_2}{C_1 s} \leq |\partial_s f_0(p, s)| \leq \frac{C_1}{C_3 s}$$

$$(2.19) \qquad\qquad |\nabla^{\Sigma_s} f_0(p, s)| \leq \frac{\sqrt{2} C_1^2}{C_2 C_3}$$

hold for $\forall p \in \Sigma$, $\forall s \in (0, s_0]$. Here ∇^{Σ_s} denotes the covariant derivative along Σ_s.

Remark 2.7. *Because Foliation A and Foliation B are comparable, Theorem 2.6 also holds under Foliation A, but with constants C_1', C_2', C_3', which are different from C_1, C_2, C_3 by quantities that can be determined by the local geometry near Σ.*

From the construction procedure of f_0, c.f.[2], we can find that if the initial data set $(\mathcal{M}, g, k)$ is spherically symmetric, then f_0 is also spherically symmetric. Then follows Corollary 3 we have:

Corollary 2.8. *Assume conditions and notations in Theorem 2.6 and further assume that $(\mathcal{M}, g, k)$ is spherically symmetric, then f_0 is also spherically symmetric. Furthermore there exist constants a, τ_0 only depending on the local geometry near horizon, such that*

$$\frac{1}{\sqrt{\lambda \tau}} - a \le |\partial_\tau f_0| \le \frac{1}{\sqrt{\lambda \tau}} + a$$

holds for $\tau \in (0, \tau_0]$, where τ is the geodesic distance from Σ.

Theorem 2.6 actually provides another way to foliate the neighborhood of Σ. The level set of f:

$$\Sigma_\gamma = \{f_0 = -\frac{1}{\sqrt{\lambda}} \log \gamma\}$$

will form a foliation near Σ, because it can be proved from Theorem 2.6 that $\partial_\tau f$ is uniformly away from zero. Furthermore, from Theorem 2.6 it can be proved that this foliation is comparable with the Foliation A and B.

Foliation C. *Under the same assumptions and notations as Theorem 2.6. Define the foliation Ψ_C to be the map:*

(2.20)
$$\Psi_C : \Sigma \times [0, \bar{\gamma}] \to \mathcal{M} \ such \ that$$
$$(1) \Psi_C(p, 0) = p \ for \ p \in \Sigma$$
$$(2) \Psi_C(p, \gamma) = u_0(p, -\frac{1}{\sqrt{\lambda}} \log \gamma)$$

Then Theorem 2.6 implies the following:

Corollary 2.9. *Denote τ to be the geodesic distance to Σ, and γ to be the parameter of Foliation C. Then there exist positive constants α_1, α_2 only depending on the initial data, such that:*

$$\alpha_1^{-1} \tau \le \gamma \le \alpha_1 \tau$$
$$\alpha_2^{-1} \le \frac{\partial \gamma}{\partial \tau} \le \alpha_2$$

Recall that on a Jang's slice, the scalar curvature $\overline{R}$ can be written as:

(2.21)
$$\overline{R} = 16\pi(\mu - J(\omega)) + |h - k|_{\overline{g}}^2 + 2|q|_{\overline{g}}^2 - 2div_{\overline{g}}(q)$$

where $\overline{g}$ is the metric on Jang's slice, h is the mean curvature of Jang's slice embedded into $\mathcal{M} \times \mathbb{R}$, and

$$\omega_i = \frac{\nabla_i f}{\sqrt{1 + |\nabla f|^2}}$$

$$q_i = \frac{f^j}{\sqrt{1 + |\nabla f|^2}}(h_{ij} - k_{ij})$$

From the above theorems, if we have one more constraint on the coefficients of the gradient estimates in Theorem 2.6, then we are able to prove a Penrose-like inequality with the help of spinor arguments. Furthermore, from [1, Proposition 2], we know that $|q|_{\overline{g}}$ is bounded near Σ. We are also able to prove the same Penrose-like inequality if we put constraint on this upper bound for $|q|_{\overline{g}}$:

Theorem 2.10. *Assume the same conditions and notations of Theorem 2.6. Suppose Condition 1 and 2 to be the following:*
Condition 1:
$$\lambda \frac{C_1^2}{C_3^2}\left(1 + \frac{2C_1^4}{C_2^2 C_3^2}\right) < 4,$$
where C_1, C_2, C_3 are the constants of the gradient estimates in Theorem 2.6;
Condition 2:
$$|q|_{\overline{g}} < 2\sqrt{\lambda} \text{ near } \Sigma.$$

If one of the above conditions hold, then we have the following Penrose-like inequalities:

(2.22)
$$m \geq \theta\sqrt{\frac{|\Sigma|_g}{16\pi}}$$

where m is the ADM mass, and θ is a positive constant given by Eq.(7.38). θ only depends on the geometry of $(\mathcal{M}, g, k)$, but does not depend on m.

3. Introduction and review of existing results

In this section we review the results of Positive Mass Theorem, the existence of solution of Jang's equation in various settings, and the Riemannian Penrose inequality.

The Riemannian Positive Mass Theorem states:

Theorem 3.1. *Let (M^n, g) be a complete asymptotically flat manifold with nonnegative scalar curvature. If $n < 8$ or if M is spin, then the mass of each end is nonnegative. Moreover, if any of the ends has zero mass, then (M^n, g) is isometric to Euclidean space.*

The $n < 8$ case was first proved by R. Schoen and S.-T. Yau [29], and then by E. Witten for the spin case using a Bochner-Lichnerowicz-Weitzenböck formula [19] and completed later by T. Parker and C. Taubes [30].

It is worth noting that M. Herzlich [16] proved the Riemannian Positive Mass theorem for a 3 dimensional manifold with boundary, providing that the mean curvature of the boundary is not too large:

Theorem 3.2. *Let (M, g) be a $C^{2,\alpha}_\tau$ asymptotically flat manifold of order $\tau > 1/2$ and scalar curvature in L^1. Suppose M has an inner boundary ∂M, homeomorphic to a 2-sphere, whose mean curvature satisfies*

$$H \leq 4\sqrt{\frac{\pi}{Area(\partial M)}}.$$

Then, if the scalar curvature of (M, g) is nonnegative, its mass is nonnegative. Moreover, if its mass is zero, then the manifold is flat.

The Positive Mass Theorem was then extended to the case of general initial data set by R. Schoen and S.-T. Yau in [1] with the help of Jang's equation proposed by P. Jang.

Theorem 3.3. *Let $(\mathcal{M}, g, k)$ be a complete oriented asymptotically flat three dimensional initial data set. Assuming the dominant energy condition, then the ADM mass of each end is non-negative. Moreover, if any of the ends has zero mass, then $(\mathcal{M}, g, k)$ can be isometrically embedded in to four dimensional Minkowski space as a spacelike hypersurface.*

This paper also proved the existence of an asymptotically decaying solution of Jang's equation on the asymptotically flat initial data set $(\mathcal{M}, g, k)$. By asymptotically decaying we mean the solution f satisfies $\partial^j f = O(|x|^{-j-\frac{1}{2}})$, $j = 0, 1, 2, 3$, on each asymptotically flat end. It was also proved that the solution f is well-behaved except that it may blowup at some apparent horizons.

Theorem 3.4. *[1, Proposition 4] Assume the same conditions in Theorem 3.3. There is a sequence $\{t_i\}$ converging to zero and open sets Ω_+, Ω_-, Ω_0, so that if f_i satisfies $\mathcal{J}[f_i] = t_i f_i$, we have:*

1) *The sequence $\{f_i\}$ converges uniformly to $+\infty$ (respectively $-\infty$) on the set Ω_+ (respectively Ω_-), and $\{f_i\}$ converges to a smooth asymptotically decaying solution f of Eq. (2.1) on Ω_0.*

2) *The sets Ω_+ and Ω_- have compact closure, and $\mathcal{M} = \bar{\Omega}_+ \cup \bar{\Omega}_- \cup \bar{\Omega}_0$. Each boundary component Σ of Ω_+ (respectively Ω_-) is a smooth embedded two-sphere satisfying $H_\Sigma - tr_\Sigma(k_{ij}) = 0$ (respectively $H_\Sigma + tr_\Sigma(k_{ij}) = 0$). Moreover, no two connected components of Ω_+ can share a common boundary.*

3) *The graphs N_i of f_i converge smoothly to a properly embedded submanifold N of $\mathcal{M} \times \mathbb{R}$. Each connected component of N is either a component of the graph f, or the cylinder $\Sigma \times \mathbb{R} \subset \mathcal{M} \times \mathbb{R}$ over a boundary component Σ of Ω_+ or Ω_-. Any two connected components of N_0 are separated by a positive distance.*

It was also proved in [1] that if this blowup happens, then N can be written as the graph of a function u on the corresponding cylindrical end, and the derivatives of u up to second order also tend to zero asymptotically on this cylindrical end, c.f.[1, Corollary 2]:

Theorem 3.5. *[1, Corollary 2] Assume the conditions of Theorem 3.4 and let Σ be a connected component of the apparent horizons, on which f tends to $+\infty$ ($-\infty$ respectively). Let U be a neighborhood of Σ with positive distance to any other apparent horizons in $\mathcal{M}$.*

Then for all $\epsilon > 0$ there exists $\bar{z} = \bar{z}(\epsilon)$, depending also on the geometry of $(\mathcal{M}, g, k)$, such that $N \cap (U \times [\bar{z}, \infty))$ can be written as the graph of a function u over $C_{\bar{z}} := \Sigma \times [\bar{z}, \infty)$, so that

$$|u(p, z)| + |^{C_{\bar{z}}}\nabla u(p, z)| + |^{C_{\bar{z}}}\nabla^2 u(p, z)| < \epsilon.$$

for all $(p, z) \in C_{\bar{z}}$. Here, $^{C_{\bar{z}}}\nabla$ denotes covariant differentiation along $C_{\bar{z}}$.

To prove the Penrose inequality, it is necessary to capture the information of the apparent horizons. Therefore, one needs to prescribe the boundary condition for Jang's equation at horizons. One approach is to let the solution blow up near Σ and approximate a cylinder over $\partial\mathcal{M}$. In [2], J. Metzger showed that if Σ is an outermost MOTS, then in fact there *must*

exist a solution to (2.1) which blows up at Σ (and only at Σ, provided there are no MITSs in $\mathcal{M}$).

Theorem 3.6. *[2, Theorem 3.1] If $(\mathcal{M}, g, k)$ be an initial data set with $\partial\mathcal{M} = \partial^-\mathcal{M} \cup \partial^+\mathcal{M}$ such that $\partial^-\mathcal{M}$ is an outermost MOTS, $\theta^+[\partial^+\mathcal{M}] > 0$ and $\theta^-[\partial^+\mathcal{M}] < 0$, then there exists an open set $\Omega_0 \subset \mathcal{M}$ and a function $f_0 : \Omega_0 \to \mathbb{R}$ such that*

1) *$\mathcal{M} \setminus \Omega_0$ does not intersect $\partial\mathcal{M}$,*

2) *$\theta^-[\partial\Omega_0] = 0$ with respect to the normal vector pointing into Ω_0,*

3) *$\mathcal{J}[f_0] = 0$,*

4) *$N^+ = graph f_0 \cap \mathcal{M} \times \mathbb{R}^+$ is asymptotic to the cylinder $\partial^-M \times \mathbb{R}^+$,*

5) *$N^- = graph f_0 \cap \mathcal{M} \times \mathbb{R}^-$ is asymptotic to the cylinder $\partial\Omega_0 \times \mathbb{R}^-$, and*

6) *$f_0|_{\partial^+\mathcal{M}} = 0$.*

This existence theorem still holds if the boundary condition is changed from $f_0|_{\partial^+\mathcal{M}} = 0$ to asymptotic decaying if the initial data set is asymptotically flat (c.f. [2, Remark 3.3]). [3, Proposition 3.1] provided a proof for this case for higher dimension $3 \le n \le 7$. Moreover, if there is no MITS in $\mathcal{M}$, then $\Omega_0 = \mathcal{M}$.

With the blowup solution f_0 constructed in Theorem 3.6 , J. Metzger showed that under the assumption of strict stability, the graph of f_0 can be written as the graph of a function u_0 on the cylindrical end, whose decay rate is exponential with a power directly related to the principal eigenvalue of the MOTS. The general idea is to show the existence of a super-solution with at most logarithmic blowup of the desired rate, c.f.[2, Theorem 4.2, Theorem 4.4]:

Theorem 3.7. *[2, Theorem 4.2] Let $N_0 = graph f_0$ be the manifold constructed in Theorem 3.6 and assume the situation of Theorem 3.5. Then there exists $\bar{z} = \bar{z}(\epsilon)$, depending also on the geometry of $(\mathcal{M}, g, k)$, such that $N_0 \cap (U \times [\bar{z}, \infty))$ can be written as the graph of a function u_0 over $C_{\bar{z}} := \Sigma \times [\bar{z}, \infty)$. If in addition Σ is strictly stable with principal eigenvalue $\lambda > 0$, then for all $\delta < \sqrt{\lambda}$ there exists $c = c(\delta)$ depending only on the data $(\mathcal{M}, g, k)$ and δ such that*

$$|u_0(p, z)| + |^{C_{\bar{z}}}\nabla u_0(p, z)| + |^{C_{\bar{z}}}\nabla^2 u_0(p, z)| \le c \exp(-\delta z).$$

where $^{C_{\bar{z}}}\nabla$ is the covariant derivative w.r.t. the induced metric on $C_{\bar{z}}$.

A non-existence theorems for the blowup solution with higher blowup rate was also proved in [2]:

Theorem 3.8. *[2, Theorem 4.4] Under the assumptions of Theorem 3.7 there are no solutions $h : \Sigma \times [0, \infty) \to \mathbb{R}$ to the equation (2.1) with decay*

$$|h(p, z)| + |^{C_{\bar{z}}}\nabla h(p, z)| + |^{C_{\bar{z}}}\nabla^2 h(p, z)| \leq C \exp(-\delta z)$$

such that $\delta > \sqrt{\lambda}$ and $h > 0$. Here, $^{C_{\bar{z}}}\nabla$ is the covariant derivative w.r.t. the induced metric on $C_{\bar{z}}$.

Remark 3.9. *It is worth noting that Theorem 3.8 can be implied by our theorem.*

In [26], H. Bray and M. Khuri proposed the generalized Jang's equation, in an attempt to prove the Penrose inequality in the setting of general initial data. It is defined as

$$(3.1) \qquad \left(g^{ij} - \frac{\phi^2 f^i f^j}{1 + \phi^2 |\nabla f|^2} \right) \left(\frac{\phi \nabla_{ij} f + \phi_i f_j + \phi_j f_i}{\sqrt{1 + \phi^2 |\nabla f|^2}} - k_{ij} \right) = 0.$$

An appropriate choice of ϕ will provide a proof for Penrose inequality. However, this involves solving coupled equations, and the full existence theorem for the solution remains unsolved, except for spherical symmetric case, c.f. [26]. In [5], Q. Han and M. Khuri proved the existence of blowup solution of generalized Jang's equation, in the case when ϕ is fixed and independent of the solution, on an asymptotically flat initial data set. They also gave the blowup rate estimates for the solution they constructed.

Since this article is not focusing on the generalized version Eq.(3.1), we only include their theorem for the case of $\phi \equiv 1$:

Theorem 3.10. *[5, Theorem 1.1](case $\phi \equiv 1$) Suppose that $(\mathcal{M}, g, k)$ is a smooth, asymptotically flat initial data set, with outermost apparent horizon boundary $\partial \mathcal{M}$ composed of MOTS $\partial^+ \mathcal{M}$ and MITS $\partial^- \mathcal{M}$. Denote by τ the distance function from $\partial \mathcal{M}$ and Σ_τ the level sets of the geodesic flow emanating from $\partial \mathcal{M}$. Assume $c^{-1} \tau \leq \theta^{\pm}(\Sigma_\tau) \leq c\tau$ for some constants $c > 0$, then there exists a smooth asymptotically decaying solution f of the Jang's equation (2.1), such that $f(x) \to \pm\infty$ as $x \to \partial^{\pm}\mathcal{M}$. More precisely, in a*

neighborhood of $\partial^{\pm}\mathcal{M}$:

$$(3.2) \qquad -\alpha^{-1}\log\tau + \beta^{-1} \le \pm f \le -\alpha\log\tau + \beta$$

for some positive constants α and β.

Remark 3.11. *The lower bound blowup control in the above theorem is not only valid for the solution constructed in [5], but also valid for all blowup solutions. This is because the family of sub barriers used to prove the lower bound in Theorem 3.10 are finite at horizon, thus can be applied for all possible blowup solutions. However in this article, a different family of sub barriers are constructed, which provide a better estimate for the blowup solution.*

In [6], C. Williams showed that for large classes of spherically symmetric initial data, there are solutions of the Jang equation which blow up at non-outermost MOTSs, i.e. MOTSs which lie strictly inside of other MOTSs, and even inside of strictly outer trapped surfaces.

Until now there is no result about the uniqueness of the solution of Jang's equation on a general initial data set. For time symmetric case, Jang's equation is reduced to a Jenkins-Serrin type equation. Under some boundary constrain, M. Eichmair and J. Metzger were able to prove the uniqueness for solution of Jang's equation in time symmetric case, c.f. [3, Section 5].

Thus in the time symmetric case, the existence and uniqueness of the asymptotically decaying solution of Jang's equation (2.1) which blows up at outermost MOTS has been proved. We also know by [5, Theorem 1.1] that in this case the blowup rate near horizon is $\log\tau$, where τ is the distance from the horizon. However, we still don't know the coefficient of $\log\tau$ in this case, and whether there are any other lower order blowup terms. In this paper, we will show that even in the non-time-symmetric case, for a blowup solution f of Jang's equation (2.1) at a strictly stable MOTS with principal eigenvalue λ, the only blowup term of f is exactly $-\dfrac{1}{\sqrt{\lambda}}\log\tau$. We are also able to prove that the gradient of f is of order τ^{-1}. For these two parts of the result we don't need extra assumptions on spacetime such as the dominant energy condition or asymptotic flatness. This is because our estimation is local near horizon.

The third part of our result is to apply these estimates to the Jang's slice $N_0 = graphf_0$ constructed by J. Metzger in [2] to prove a Penrose-like inequality.

The Penrose inequality may be thought of as a refinement of the Positive Mass Theorem when black holes are present. It relates the total ADM mass (of a chosen end) m to the area A of its outermost minimal area enclosure. The inequality states that:

$$(3.3) \qquad\qquad\qquad m \geq \sqrt{\frac{A}{16\pi}}.$$

And furthermore it asserts that if equality holds and the outermost minimal area enclosure is the boundary of an open bounded domain $U \subset M$, then $(M - U, g)$ admits an isometric embedding into the Schwarzschild spacetime with second fundamental form given by $k = 0$.

The Riemannian Penrose inequality (time symmetric case) states:

Theorem 3.12 (Riemannian Penrose Inequality). *Let (M^n, g) be a complete asymptotically flat manifold with nonnegative scalar curvature, where $n < 8$. Fix one end. Let m be the mass of that end, and let A be the area of an outer minimizing horizon (with one or more components). Let ω_{n-1} be the area of the standard unit $(n - 1)$-sphere. Then*

$$m \geq \frac{1}{2} \left(\frac{A}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}},$$

with equality if and only if the part of (M, g) outside the horizon is isometric to a Riemannian Schwarzschild manifold outside its unique outer minimizing horizon.

The $n = 3$ case was proved by G. Huisken and T. Ilmanen using inverse mean curvature flow method (c.f. [17]), and also by H. Bray using a conformal flow of metrics (c.f. [27]). Then H. Bray and D. Lee generalized the result to the $n < 8$ case (c.f. [28]). M. Herzlich also gave a Penrose-like inequality in his paper [16].

For the case of general initial data set $(\mathcal{M}, g, k)$ when the dominant energy condition $\mu \geq |J|$ is satisfied, it is possible to obtain a non-sharp Penrose-like inequality with the help of Jang's equation. Recall that on a Jang's slice, the scalar curvature $\overline{R}$ can be written as:

$$(3.4) \qquad \overline{R} = 16\pi(\mu - J(\omega)) + |h - k|_{\overline{g}}^2 + 2|q|_{\overline{g}}^2 - 2div_{\overline{g}}(q)$$

where $\bar{g}$ is the metric on Jang's slice, h is the mean curvature of Jang's slice embedded into $\mathcal{M} \times \mathbb{R}$, and

$$\omega_i = \frac{\nabla_i f}{\sqrt{1 + |\nabla f|^2}}$$

$$q_i = \frac{f^j}{\sqrt{1 + |\nabla f|^2}}(h_{ij} - k_{ij})$$

For the case when vector field q vanishes at a cylindrical end of Jang's slice, M. Khuri proved in [7] a Penrose-like inequality for an initial data set with charge. However, until now the Penrose inequality for the case when q does not vanish at the cylindrical end of Jang's slice still remains open. For example, in Schwarzschild spacetime, we can construct a blowup solution for Jang's equation on its time symmetric slice, and we can calculate that the vector field q does not vanish at the cylindrical end of this Jang's slice. In this article, we will prove a Penrose-like inequality for this case, based on a sharp estimate of the cylindrical end of Jang's slice.

4. Barriers construction

Because Theorem 2.1 is a local estimate of the solution at each strictly stable MOTS boundary component, we can do the estimate separately. It is equivalent to prove our result on one of these components. We denote this compact and strictly stable component as Σ with principal eigenvalue $\lambda > 0$. For convenience we assume $f(x) \to +\infty$ when $x \to \Sigma$.

All of our computation in Section 4 will be under Foliation B. Under this foliation we can choose $\bar{s} > 0$ small enough such that the surfaces $\Sigma_s = \Psi_B(\Sigma, s)$ with $s \in [0, \bar{s}]$ form a local foliation near Σ with lapse β such that

$$\lambda \beta s - \Lambda s^2 \leq \theta^+[\Sigma_s] \leq \lambda \beta s + \Lambda s^2.$$

for some constant Λ.

Denote the region swept out by these Σ_s by $U_{\bar{s}}$. Note that $\partial U_{\bar{s}} = \Sigma \cup \Sigma_{\bar{s}}$ and $dist(\Sigma_{\bar{s}}, \Sigma) \geq \bar{s}$. We can assume that $dist(\Sigma_{\bar{s}}, \partial\mathcal{M}) > 0$. On $U_{\bar{s}}$ we consider a test function of the form $v = \phi(s)$. For such functions Jang's operator can be computed as follows, cf. [2], [4]. Note that $\dfrac{\partial \beta}{\partial s}$ does not

appear because our β is chosen to be a function on Σ.

$$(4.1) \quad \mathcal{J}[\phi] = \frac{\phi'}{\beta\sqrt{1+\beta^{-2}(\phi')^2}}\theta^+ - \left(1 + \frac{\phi'}{\beta\sqrt{1+\beta^{-2}(\phi')^2}}\right) P$$
$$- \frac{k(\nu,\nu)}{1+\beta^{-2}(\phi')^2} + \frac{\phi''}{\beta^2(1+\beta^{-2}(\phi')^2)^{\frac{3}{2}}},$$

where ϕ' denotes the derivative of $v = \phi(s)$ with respect to s. The quantities θ^+, $k(\nu,\nu)$ and $P = trk - k(\nu,\nu)$ are computed on the respective Σ_s.

4.1. Super estimate: order control

The most important part of our proof in this section is the order control of the solution f. If we know nothing about the blowup order of f beforehand, it will be impossible to apply the comparison principle in $U_{\bar{s}}$, even if we have a super solution.

In this section we will prove that for any blowup solution f of Eq.(2.1), there exist constants $c_1 > 0$, $s_1 > 0$ which only depend on the local geometry of the initial data set near the horizon, such that $f < -c_1 \log s$ holds for $s \in (0, s_1]$. In order to prove this inequality, in Proposition 4.3 below, we construct a piecewise smooth function $W(s)$ iteratively, and prove that $f < W$ by induction. Moreover, we observe for $s \to 0$ that the blowup order of $W(s)$ is exactly $-c_1 \log s$ for some positive constant c_1, thus the inequality gets proved. In the next section, we will refine c_1 to $\dfrac{1}{\sqrt{\lambda}}$ by another family of super barriers which will be constructed in Lemma 4.7.

We first construct a family of auxiliary super barriers under the Foliation B. We will use these auxiliary super barriers to construct the upper bound function $W(s)$ in Proposition 4.3.

For $\epsilon > 0$, we set functions w_ϵ to be the following:

Lemma 4.1. *For any $\epsilon > 0$, we construct the following functions w_ϵ on $s \in (\epsilon, \bar{s})$:*

$$(4.2) \qquad\qquad w_\epsilon(s) = -\log(s - \epsilon)$$

These functions have the following properties:

1) *$\lim_{s \to \epsilon} w_\epsilon(s) = +\infty$*

2) *There exist constants $\epsilon_0 > 0$ and $\alpha > 0$ which only depend on $(\mathcal{M}, g, k)$, s.t. $\forall 0 < \epsilon \leq \epsilon_0$, $w_\epsilon(s)$ is a super solution on $s \in (\epsilon, (1+\alpha)\epsilon]$.*

Proof. Property (1) is obvious, thus we only prove Property (2) here. If we plug $\phi = w_\epsilon$ into Eq.(4.1), then for $s \in (\epsilon, (1+\alpha)\epsilon]$, ($\alpha$ is a constant to be determined), each term of Eq.(4.1) will be:

$$\frac{w_\epsilon'}{\beta\sqrt{1+\beta^{-2}(w_\epsilon')^2}} = -\frac{1}{\sqrt{1+\beta^2(s-\epsilon)^2}} = -1 + O((s-\epsilon)^2)$$
$$\theta^+ = \lambda\beta s + O(s^2)$$
$$\frac{1}{1+\beta^{-2}(w_\epsilon')^2} = \frac{\beta^2(s-\epsilon)^2}{1+\beta^2(s-\epsilon)^2} = \beta^2(s-\epsilon)^2 + O((s-\epsilon)^4)$$
$$\frac{w_\epsilon''}{\beta^2(1+\beta^{-2}(w_\epsilon')^2)^{\frac{3}{2}}} = \frac{\beta(s-\epsilon)}{(1+\beta^2(s-\epsilon)^2)^{\frac{3}{2}}} = \beta(s-\epsilon) + O((s-\epsilon)^3)$$

Thus

$$\mathcal{J}[w_\epsilon] = -\lambda\beta s + \beta(s-\epsilon) + O((s-\epsilon)^2)$$
$$= (1-\lambda)\beta s - \beta\epsilon + O((s-\epsilon)^2)$$
$$= (1-\lambda)\beta s - \beta\epsilon + O(\epsilon^2)$$

for $s \in (\epsilon, (1+\alpha)\epsilon]$.

Thus if we set α to be a constant s.t. $(1-\lambda)(1+\alpha) < 1 - \dfrac{\lambda}{2}$, we will have:

$$\mathcal{J}[w_\epsilon] \leq (1 - \frac{\lambda}{2})\beta\epsilon - \beta\epsilon + O(\epsilon^2) = -\frac{\lambda}{2}\beta\epsilon + O(\epsilon^2) < 0$$

holds for $\epsilon \in (0, \epsilon_0]$, where ϵ_0 is a constant only related to the geometry of $(\mathcal{M}, g, k)$ near Σ. $\qquad\square$

Now we focus on a blowup solution f satisfying Eq.(2.1) on $U_{\bar{s}}$. For simplicity we assume that $\bar{s} > (1+\alpha)\epsilon_0$.

By the comparison principle, c.f. [32, Section 10], we get the following result:

Proposition 4.2. *There exist constants ϵ_0 and α, s.t. for any function f that satisfies (2.1) on $U_{\bar{s}}$ and $f \to +\infty$ when $s \to 0$, the following inequality:*

(4.3) $$f(\cdot, s) \leq \sup_{q \in \Sigma} f(q, (1+\alpha)\epsilon) + \log(\frac{\alpha\epsilon}{s - \epsilon})$$

holds for any $\epsilon \in (0, \epsilon_0]$ and $s \in (\epsilon, (1+\alpha)\epsilon]$.

Proof. We assume ϵ_0, α the same as in Lemma 4.1 and $(1+\alpha)\epsilon_0 < \bar{s}$. Then by Property (2) of Lemma 4.1, we know that for $\forall\epsilon \in (0, \epsilon_0)$,

$\overline{w}_\epsilon(s) = w_\epsilon(s) - w_\epsilon((1+\alpha)\epsilon) + \sup_{q\in\Sigma} f(q, (1+\alpha)\epsilon)$ is a super solution on $s \in (\epsilon, (1+\alpha)\epsilon]$. This super solution satisfies $\overline{w}_\epsilon((1+\alpha)\epsilon) \geq f(\cdot, (1+\alpha)\epsilon)$. Moreover, by Property (1) in Lemma 4.1, and the fact that $f(\cdot, \epsilon)$ is finite, we know by the comparison principle that this super solution is actually a super barrier for f for $s \in (\epsilon, (1+\alpha)\epsilon]$. $\qquad\square$

In the above proposition we find super barriers which are only effective in a region which becomes infinitely small when $\epsilon \to 0$. This is not enough for obtaining a upper bound for the blowup rate. In Proposition 4.3 below we use the above w_ϵ to iteratively construct a upper bound function $W(s)$ for f on a fixed region $s \in (0, (1+\alpha)\epsilon_0]$.

For any function f that satisfies (2.1) on $U_{\bar{s}}$ and $f \to +\infty$ when $s \to 0$, we construct the following upper bound $W(s)$ for f iteratively for $s \in (0, (1+\alpha)\epsilon]$:

Proposition 4.3. *For a constant $d \in (0,1)$, and function f that satisfies Eq.(2.1) on $U_{\bar{s}}$ and $f \to +\infty$ when $s \to 0$, we construct $W(s)$ iteratively in the following way:*

1) *For $s \in [(1+d\alpha)\epsilon_0, (1+\alpha)\epsilon_0]$, define*

$$W(s) = \log(\frac{\alpha\epsilon_0}{s - \epsilon_0}) + \sup_{q\in\Sigma} f(q, (1+\alpha)\epsilon_0)$$

2) *For $n \geq 0$, if $W(s)$ is defined in $s \in \left[\dfrac{(1+d\alpha)^{n+1}}{(1+\alpha)^n}\epsilon_0, \dfrac{(1+d\alpha)^n}{(1+\alpha)^{n-1}}\epsilon_0\right)$, then define*

$$W(s) = \log\left(\frac{\alpha\dfrac{(1+d\alpha)^{n+1}}{(1+\alpha)^{n+1}}\epsilon_0}{s - \dfrac{(1+d\alpha)^{n+1}}{(1+\alpha)^{n+1}}\epsilon_0}\right) + W\left(\frac{(1+d\alpha)^{n+1}}{(1+\alpha)^n}\epsilon_0\right)$$

for $s \in \left[\dfrac{(1+d\alpha)^{n+2}}{(1+\alpha)^{n+1}}\epsilon_0, \dfrac{(1+d\alpha)^{n+1}}{(1+\alpha)^n}\epsilon_0\right)$

Then $W(s)$ will be a piecewise smooth function defined on $s \in (0, (1+\alpha)\epsilon_0]$. Furthermore, W will be an upper bound for f, and the following inequality

(4.4) $$f(p, s) \leq W(s)$$

holds for $\forall s \in (0, (1+\alpha)\epsilon_0]$ and $\forall p \in \Sigma$.

Proof. We will prove Ineq.(4.4) on $s \in (0, (1 + \alpha)\epsilon_0]$ by induction.

First, plug $\epsilon = \epsilon_0$ into Ineq.(4.3). By the fact that $(1 + d\alpha)\epsilon_0 > \epsilon_0$, we get Ineq.(4.4) holds on $s \in [(1 + d\alpha)\epsilon_0, (1 + \alpha)\epsilon_0)$.

Next, if Ineq.(4.4) holds for $s \in [\dfrac{(1 + d\alpha)^{n+1}}{(1 + \alpha)^n}\epsilon_0, \dfrac{(1 + d\alpha)^n}{(1 + \alpha)^{n-1}}\epsilon_0)$ for some $n \geq 0$, (especially we have $f(\cdot, \dfrac{(1 + d\alpha)^{n+1}}{(1 + \alpha)^n}\epsilon_0) \leq W(\dfrac{(1 + d\alpha)^{n+1}}{(1 + \alpha)^n}\epsilon_0))$, we apply Ineq.(4.3) for $\epsilon = \dfrac{(1 + d\alpha)^{n+1}}{(1 + \alpha)^{n+1}}\epsilon_0$ and by the fact that $\dfrac{(1 + d\alpha)^{n+2}}{(1 + \alpha)^{n+1}}\epsilon_0 > \epsilon = \dfrac{(1 + d\alpha)^{n+1}}{(1 + \alpha)^{n+1}}\epsilon_0$, we have:

$$f(p, s) \leq \sup_{q \in \Sigma} f(q, \frac{(1 + d\alpha)^{n+1}}{(1 + \alpha)^n}\epsilon_0) + \log \left(\frac{\alpha \dfrac{(1 + d\alpha)^{n+1}}{(1 + \alpha)^{n+1}}\epsilon_0}{s - \dfrac{(1 + d\alpha)^{n+1}}{(1 + \alpha)^{n+1}}\epsilon_0} \right)$$

$$\leq W\left(\frac{(1 + d\alpha)^{n+1}}{(1 + \alpha)^n}\epsilon_0\right) + \log \left(\frac{\alpha \dfrac{(1 + d\alpha)^{n+1}}{(1 + \alpha)^{n+1}}\epsilon_0}{s - \dfrac{(1 + d\alpha)^{n+1}}{(1 + \alpha)^{n+1}}\epsilon_0} \right)$$

$$= W(s)$$

holds for $s \in [\dfrac{(1 + d\alpha)^{n+2}}{(1 + \alpha)^{n+1}}\epsilon_0, \dfrac{(1 + d\alpha)^{n+1}}{(1 + \alpha)^n}\epsilon_0)$, $p \in \Sigma$.

Thus by induction we know that Ineq.(4.4) holds for $s \in [\dfrac{(1 + d\alpha)^{n+1}}{(1 + \alpha)^n}\epsilon_0, \dfrac{(1 + d\alpha)^n}{(1 + \alpha)^{n-1}}\epsilon_0)$, for all interger $n \geq 0$, thus it holds for $s \in (0, (1 + \alpha)\epsilon_0]$. $\square$

In the above construction for W, we can also prove inductively that $W(\dfrac{(1 + d\alpha)^n}{(1 + \alpha)^{n-1}}\epsilon_0) = -n \log d + \sup_{q \in \Sigma} f(q, (1 + \alpha)\epsilon_0)$ for any non-negative interger n.

Thus we know that for for $s \in [\dfrac{(1 + d\alpha)^{N+1}}{(1 + \alpha)^N}\epsilon_0, \dfrac{(1 + d\alpha)^N}{(1 + \alpha)^{N-1}}\epsilon_0)$:

$$f(\cdot, s) \leq W(s)$$
$$\leq W\left(\frac{(1+d\alpha)^{N+1}}{(1+\alpha)^N}\epsilon_0\right)$$
$$= -(N+1)\log d + \sup_{q\in\Sigma} f(q, (1+\alpha)\epsilon_0)$$
$$= (N+1)\log\frac{1}{d} + \sup_{q\in\Sigma} f(q, (1+\alpha)\epsilon_0)$$
$$< (1 - \log_{\frac{1+\alpha}{1+d\alpha}}\left(\frac{s}{(1+\alpha)\epsilon_0}\right))\log\frac{1}{d} + \sup_{q\in\Sigma} f(q, (1+\alpha)\epsilon_0)$$

By setting $d = \dfrac{1}{2}$, we get:

$$(4.5) \qquad f(\cdot, s) \leq -\frac{\log 2}{\log(1+\alpha) - \log(1+\frac{\alpha}{2})}\log s + \sup_{q\in\Sigma} f(q, (1+\alpha)\epsilon_0)$$
$$+ \log 2 + \frac{\log 2 \log((1+\alpha)\epsilon_0)}{\log(1+\alpha) - \log(1+\frac{\alpha}{2})}$$

Remark 4.4. *By the fact that*

$$\lim_{d\to 1^-} \frac{\log d}{\log\frac{1+d\alpha}{1+\alpha}} = \frac{\alpha+1}{\alpha}$$

we know the optimal value of the coefficient of $-\log s$ in the Ineq.(4.5) will be $\frac{\alpha+1}{\alpha}$, and the corresponding inequality will be:

$$f(\cdot, s) \leq -\frac{\alpha+1}{\alpha}\log\frac{s}{(1+\alpha)\epsilon_0} + \sup_{q\in\Sigma} f(q, (1+\alpha)\epsilon_0)$$

Because ϵ_0, α are constants determined by $(\mathcal{M}, g, k)$, thus for any blowup solution f, we have the following upper bound for the blowup order:

Proposition 4.5. *There exist constants s_1, c_1, and C_1 only depend on the local geometry of $(\mathcal{M}, g, k)$ near Σ, s.t. for any function f that satisfies Eq.(2.1) on $U_{\bar{s}}$ and $f \to +\infty$ when $s \to 0$, the following inequality :*

$$(4.6) \qquad f(p, s) \leq \sup_{q\in\Sigma} f(q, s_1) - c_1 \log s + C_1$$

holds for $\forall s \in (0, s_1]$ and $p \in \Sigma$.

4.2. Super estimate: coefficient refinement

In the previous section we prove that any blowup solution of Jang's equation in our case cannot blow up faster than $-c_1 \log s$ for some positive constant c_1 which only depends on $(\mathcal{M}, g, K)$ (c.f. Ineq.(4.6)). In this section we are going to refine this result to $-\dfrac{1}{\sqrt{\lambda}} \log s$.

To achieve this, we need to construct a new family of super barriers. We first compute the expansion of $\mathcal{J}[v_{a,\gamma}]$ for some test functions near the horizon.

Lemma 4.6. *For $\gamma > 0$ and constant a, we construct the following functions $v_{a,\gamma}$:*

$$v_{a,\gamma}(s) = \frac{1}{\sqrt{\lambda}} \int_s^{\bar{s}} \frac{1}{x^\gamma} dx + as$$

for $s \in (0, \bar{s}]$.

These functions have the following properties:

1) *For $\gamma > 1$, $v_{a,\gamma}(s)$ blowup at the rate of $\dfrac{s^{1-\gamma}}{\sqrt{\lambda}(\gamma - 1)}$ when s goes to 0 .*
   For $\gamma < 1$, $v_{a,\gamma}(s)$ is bounded in $U_{\bar{s}}$.

2) $v'_{a,\gamma}(s) = -\dfrac{1}{\sqrt{\lambda} s^\gamma} + a, \; v''_{a,\gamma}(s) = \dfrac{\gamma}{\sqrt{\lambda} s^{\gamma+1}}$

3) *Denote* $\sigma^2 = (a^2 + \beta^2) s^{2\gamma} - \dfrac{2a}{\sqrt{\lambda}} s^\gamma + \dfrac{1}{\lambda}$. *Then for γ close to 1, $\sigma^3 \mathcal{J}[v_{a,\gamma}]$ has the following expansion in $U_{\bar{s}}$:*

$$(4.7) \quad \sigma^3 \mathcal{J}[v_{a,\gamma}] = -\frac{\beta}{\sqrt{\lambda}} s + \frac{\beta\gamma}{\sqrt{\lambda}} s^{2\gamma-1}$$

$$- \frac{1}{\lambda\sqrt{\lambda}}(\theta^+ - \lambda\beta s) + 3a\beta s^{\gamma+1} - \frac{\beta^2}{\sqrt{\lambda}}(k(\nu, \nu) + \frac{P}{2}) s^{2\gamma}$$

$$+ O(s^{3\gamma} + s^{2\gamma+1} + s^{\gamma+2} + s^3)$$

Proof. Properties (1)(2) are obvious, thus we only prove property (3) here. First of all we have the following expansion for the powers of σ near the

horizon:

$$\sigma = \frac{1}{\sqrt{\lambda}} - as^\gamma + \frac{\beta^2\sqrt{\lambda}}{2}s^{2\gamma} + O(s^{3\gamma})$$

$$\sigma^2 = \frac{1}{\lambda} - \frac{2a}{\sqrt{\lambda}}s^\gamma + (a^2 + \beta^2)s^{2\gamma}$$

$$\sigma^3 = \frac{1}{\lambda\sqrt{\lambda}} - \frac{3a}{\lambda}s^\gamma + \frac{3(a^2 + \frac{1}{2}\beta^2)}{\sqrt{\lambda}}s^{2\gamma} + O(s^{3\gamma})$$

If we plug $v_{a,\gamma}$ into Eq.(4.1), then each term of $\sigma^3\mathcal{J}[v_{a,\gamma}]$ will be:

$$\sigma^2 s^\gamma v'_{a,\gamma}(s)\theta^+ = (\frac{1}{\lambda} - \frac{2a}{\sqrt{\lambda}}s^\gamma + (a^2+\beta^2)s^{2\gamma})(-\frac{1}{\sqrt{\lambda}} + as^\gamma)\theta^+$$

$$= (-\frac{1}{\lambda\sqrt{\lambda}} + \frac{3a}{\lambda}s^\gamma + O(s^{2\gamma}))\theta^+$$

$$= -\frac{1}{\sqrt{\lambda}}\beta s + 3a\beta s^{\gamma+1} - \frac{1}{\lambda\sqrt{\lambda}}(\theta^+ - \lambda\beta s)$$
$$+ O(s^{3\gamma} + s^{2\gamma+1} + s^{\gamma+2} + s^3)$$

$$\sigma^3 + \sigma^2 s^\gamma v'_{a,\gamma}(s) = \frac{1}{\lambda\sqrt{\lambda}} - \frac{3a}{\lambda}s^\gamma + \frac{3(a^2+\frac{1}{2}\beta^2)}{\sqrt{\lambda}}s^{2\gamma}$$
$$+ (\frac{1}{\lambda} - \frac{2a}{\sqrt{\lambda}}s^\gamma + (a^2+\beta^2)s^{2\gamma})(-\frac{1}{\sqrt{\lambda}} + as^\gamma)$$
$$+ O(s^{3\gamma} + s^{2\gamma+1} + s^{\gamma+2} + s^3)$$

$$= \frac{\beta^2}{2\sqrt{\lambda}}s^{2\gamma} + O(s^{3\gamma} + s^{2\gamma+1} + s^{\gamma+2} + s^3)$$

$$\sigma s^{2\gamma}\beta^2 = \frac{\beta^2}{\sqrt{\lambda}}s^{2\gamma} + O(s^{3\gamma})$$

$$s^{3\gamma}\beta v''_{a,\gamma}(s) = \frac{\beta\gamma}{\sqrt{\lambda}}s^{2\gamma-1}$$

Put all these together we prove Eq.(4.7). $\square$

By Eq.(4.7), we can construct a family of super barriers (see Lemma 4.7) and sub barriers (see Lemma 4.10 in the next section) for Jang's equation (2.1) on a fixed domain near the horizon.

Proposition 4.7. *There exist constants a, s_2 which only depend on the local geometry of the initial data set near horizon, such that for $\forall 1 < \gamma < \frac{5}{4}$,*

1) $\mathcal{J}[v_{a,\gamma}(s)] \leq 0$ *for $s \in (0, s_2]$.*

2) $\frac{1}{\sqrt{\lambda}} \int_s^{s_2} \frac{1}{x^\gamma} dx + as - as_2 + \sup_{q \in \Sigma} f(q, s_2)$ *is a super barrier for the solution f of Eq.(2.1) on $(0, s_2]$.*

Proof. We first prove Property (1). For the case $\gamma > 1$, rearrange Eq.(4.7) in Lemma 4.6 on $U_{\bar{s}}$ in ascending order. We denote $c_1 = \sup_{U_{\bar{s}}}(-\beta(k(\nu, \nu) + \frac{P}{2}))$ and recall that on $U_{\bar{s}}$ we have $\lambda\beta s - \Lambda s^2 < \theta^+[\Sigma_s] < \lambda\beta s + \Lambda s^2$, then:

$$
\begin{aligned}
\sigma^3 \mathcal{J}[v_{a,\gamma}] = & -\frac{\beta}{\sqrt{\lambda}} s + \frac{\beta\gamma}{\sqrt{\lambda}} s^{2\gamma-1} - \frac{1}{\lambda\sqrt{\lambda}}(\theta^+ - \lambda\beta s) + 3a\beta s^{\gamma+1} \\
& - \frac{\beta^2}{\sqrt{\lambda}}(k(\nu, \nu) + \frac{P}{2}) s^{2\gamma} + O(s^3) \\
\leq & -\frac{\beta}{\sqrt{\lambda}} s + \frac{\beta\gamma}{\sqrt{\lambda}} s^{2\gamma-1} + \frac{\Lambda}{\lambda\sqrt{\lambda}} s^2 + 3a\beta s^{\gamma+1} + c_1 \frac{\beta}{\sqrt{\lambda}} s^{2\gamma} + O(s^3)
\end{aligned}
$$

Thus

$$
\begin{aligned}
-\sqrt{\lambda}\beta^{-1}\sigma^3 \mathcal{J}[v_{a,\gamma}] \geq & \ s - \gamma s^{2\gamma-1} - \frac{\Lambda}{\lambda\beta} s^2 - 3a\sqrt{\lambda} s^{\gamma+1} - c_1 s^{2\gamma} + O(s^3) \\
= & \ s(1 - \gamma s^{2\gamma-2}) - \left(\frac{\Lambda}{\lambda\beta} s^2 + 3a\sqrt{\lambda} s^{\gamma+1} + c_1 s^{2\gamma}\right) + O(s^3) \\
= & \ s(1 + \sqrt{\gamma} s^{\gamma-1})(1 - \sqrt{\gamma} s^{\gamma-1}) \\
& - (1 - \sqrt{\gamma} s^{\gamma-1})(\frac{\Lambda}{\lambda\beta} s^2 + (\frac{\Lambda\sqrt{\gamma}}{\lambda\beta} + 3a\sqrt{\lambda}) s^{\gamma+1}) \\
& - (\frac{\Lambda\gamma}{\lambda\beta} + 3a\sqrt{\lambda\gamma} + c_1) s^{2\gamma} + O(s^3) \\
= & \ (1 - \sqrt{\gamma} s^{\gamma-1})(s + \sqrt{\gamma} s^\gamma - \frac{\Lambda}{\lambda\beta} s^2 - (\frac{\Lambda\sqrt{\gamma}}{\lambda\beta} + 3a\sqrt{\lambda}) s^{\gamma+1}) \\
& + (-3a\sqrt{\lambda\gamma} - \frac{\Lambda\gamma}{\lambda\beta} - c_1) s^{2\gamma} + O(s^3)
\end{aligned}
$$

Let's look at the last line of the above calculation. Notice the following facts:

1) $1 - \sqrt{\gamma} s^{\gamma-1} \geq 0$ for $\forall \gamma > 1$ and $\forall s \in [0, e^{-\frac{1}{2}}]$.

2) Set $a = -\sup_{q \in \Sigma} \dfrac{\frac{5\Lambda}{4\lambda\beta} + |c_1| + 1}{3\sqrt{\lambda}}$, then $-3a\sqrt{\lambda\gamma} - \frac{\Lambda\gamma}{\lambda\beta} - c_1 \geq 1 > 0$ for $\gamma \in (1, \frac{5}{4})$. Notice that a is not related with γ.

3) Because we set a in terms of the local geometry data near the horizon, there exists constant $s_2^* > 0$ which only depends on the same local

geometry (not related with γ), such that

$$s + \sqrt{\gamma}s^\gamma - \frac{\Lambda}{\lambda\beta}s^2 - (\frac{\Lambda\sqrt{\gamma}}{\lambda\beta} + 3a\sqrt{\lambda})s^{\gamma+1}$$

$$\geq s + s^{\frac{5}{4}} - \frac{\Lambda}{\lambda\beta}s^2 - |\frac{\sqrt{5}\Lambda}{2\lambda\beta} + 3a\sqrt{\lambda}|s^2 > 0$$

holds for $s \in (0, s_2^*]$. This is obvious because the coefficients of the leading terms s and $s^{\frac{5}{4}}$ are both positive constants, and their orders are at least $\frac{3}{4}$ away from the orders of the rest terms, whose coefficients are bounded and only depend on the local geometry near the horizon of the initial data set.

Then on $(0, \min(s_2^*, e^{-\frac{1}{2}})]$ we have:

$$-\sqrt{\lambda}\beta^{-1}\sigma^3 \mathcal{J}[v_{a,\gamma}] \geq (-3a\sqrt{\lambda\gamma} - \frac{\Lambda\gamma}{\lambda\beta} - c_1)s^{2\gamma} + O(s^3)$$

$$\geq s^{2\gamma} + O(s^3)$$

$$\geq s^{\frac{5}{2}} + O(s^3)$$

Therefore, there exists constant s_2 only depending on the local geometry of the initial data, such that $-\sqrt{\lambda}\beta^{-1}\sigma^3 \mathcal{J}[v_{a,\gamma}] \geq 0$ for $s \in (0, s_2]$, and thus $\mathcal{J}[v_{a,\gamma}] \leq 0$ because $\lambda > 0$ and β is positive function. Thus $v_{a,\gamma}(s)$ is a supersolution of Jang's equation (2.1) on $(0, s_2]$.

For Property (2) we only need to notice that for $\gamma > 1$, $v_{a,\gamma}$ blows up at the order of $s^{1-\gamma}$, which is much faster than $-c \log s$ for any constant c when $s \to 0$. In the previous section we prove that any solution f cannot blowup faster than $-c \log s$. Thus for $\gamma > 1$, $v_{a,\gamma}$ always blows up faster than any solution f of Eq.(2.1) near the horizon. Therefore a vertical translation of $v_{a,\gamma}$ at $s = s_2$ is enough to make it a super barrier. $\square$

By the comparison principle, c.f. [32, Chapter 10], we have the following result:

Proposition 4.8. *There exist constants a, s_2 which only depend on the local geometry of the initial data set near horizon, such that for $\forall 1 < \gamma < \frac{3}{2}$, and any function f satisfies Jang's equation (2.1) on $U_{\overline{s}}$ and $f \to +\infty$ when $s \to 0$, the following inequality :*

$$(4.8) \qquad f(p, s) \leq \sup_{q \in \Sigma} f(q, s_2) + \frac{1}{\sqrt{\lambda}} \int_s^{s_2} \frac{1}{x^\gamma}dx + as - as_2$$

holds for any $p \in \Sigma$ and $s \in (0, s_2]$.

We let $\gamma \to 1$ in Ineq.(4.8). Then because a, s_2 is not related to γ, we have the following estimate for the upper bound of blowup rate:

Proposition 4.9. *There exist constants a, s_2 which only depend on the local geometry of the initial data set near horizon, such that any function f satisfies Jang's equation 2.1 on $U_{\bar{s}}$ and $f \to +\infty$ when $s \to 0$, the following inequality :*

(4.9) $$f(p, s) \leq \sup_{q \in \Sigma} f(q, s_2) - \frac{1}{\sqrt{\lambda}} \log s + \frac{1}{\sqrt{\lambda}} \log s_2 + as - as_2$$

holds for any $p \in \Sigma$ and $s \in (0, s_2]$.

Thus the first half of Theorem 2.1 is proved.

4.3. Sub estimate

By Eq.(4.7), we can construct a family of sub barriers in a similar way as in the previous section for Jang's equation (2.1) on a fixed domain near the horizon.

Proposition 4.10. *There exist constants a, s_3 which only depend on the local geometry of the initial data set near horizon, such that for $\forall \frac{3}{4} < \gamma < 1$,*

1) $\mathcal{J}[v_{a,\gamma}(s)] \geq 0$ for $s \in (0, s_3]$.

2) $\frac{1}{\sqrt{\lambda}} \int_s^{s_3} \frac{1}{x^\gamma} dx + as - as_3 + \inf_{q \in \Sigma} f(q, s_3)$ is a sub barrier for solution f of Eq.(2.1) on $(0, s_3]$.

Proof. We first prove Property (1). For the case $\gamma < 1$, rearrange the terms of Eq.(4.7) in ascending order on $U_{\bar{s}}$. Denote $c_2 = \inf_{U_{\bar{s}}}(-\beta(k(\nu, \nu) + \frac{P}{2}))$ and recall that on $U_{\bar{s}}$ we have $\lambda \beta s - \Lambda s^2 < \theta^+[\Sigma_s] < \lambda \beta s + \Lambda s^2$, then:

$$\sigma^3 \mathcal{J}[v_{a,\gamma}] = \frac{\beta \gamma}{\sqrt{\lambda}} s^{2\gamma - 1} - \frac{\beta}{\sqrt{\lambda}} s - \frac{\beta^2}{\sqrt{\lambda}}(k(\nu, \nu) + \frac{P}{2}) s^{2\gamma} + 3a\beta s^{\gamma + 1}$$
$$- \frac{1}{\lambda \sqrt{\lambda}}(\theta^+ - \lambda \beta s) + O(s^{3\gamma})$$
$$\geq \frac{\beta \gamma}{\sqrt{\lambda}} s^{2\gamma - 1} - \frac{\beta}{\sqrt{\lambda}} s + c_2 \frac{\beta}{\sqrt{\lambda}} s^{2\gamma} + 3a\beta s^{\gamma + 1} - \frac{\Lambda}{\lambda \sqrt{\lambda}} s^2 + O(s^{3\gamma})$$

Thus

$$\sqrt{\lambda}\beta^{-1}\sigma^3 \mathcal{J}[v_{a,\gamma}] \geq \gamma s^{2\gamma-1} - s + c_2 s^{2\gamma} + 3a\sqrt{\lambda}s^{\gamma+1} - \frac{\Lambda}{\lambda\beta}s^2 + O(s^{3\gamma})$$

$$= \gamma s^{2\gamma-1}(1 - \frac{1}{\gamma}s^{2-2\gamma}) + \left(c_2 s^{2\gamma} + 3a\sqrt{\lambda}s^{\gamma+1} - \frac{\Lambda}{\lambda\beta}s^2\right) + O(s^{3\gamma})$$

$$= \gamma s^{2\gamma-1}(1 - \frac{1}{\sqrt{\gamma}}s^{1-\gamma})(1 + \frac{1}{\sqrt{\gamma}}s^{1-\gamma})$$

$$+ (1 - \frac{1}{\sqrt{\gamma}}s^{1-\gamma})(c_2 s^{2\gamma} + (3a\sqrt{\lambda} + \frac{c_2}{\sqrt{\gamma}})s^{\gamma+1})$$

$$+ (3a\sqrt{\frac{\lambda}{\gamma}} - \frac{\Lambda}{\lambda\beta} + \frac{c_2}{\gamma})s^2 + O(s^{3\gamma})$$

$$= (1 - \frac{1}{\sqrt{\gamma}}s^{1-\gamma})(\gamma s^{2\gamma-1} + \sqrt{\gamma}s^\gamma + c_2 s^{2\gamma} + (3a\sqrt{\lambda} + \frac{c_2}{\sqrt{\gamma}})s^{\gamma+1})$$

$$+ (3a\sqrt{\frac{\lambda}{\gamma}} - \frac{\Lambda}{\lambda\beta} + \frac{c_2}{\gamma})s^2 + O(s^{3\gamma})$$

Let's look at the last line of the above calculation. Notice the following facts:

1) $1 - \frac{1}{\sqrt{\gamma}}s^{1-\gamma} \geq 0$ for $\forall \frac{3}{4} < \gamma < 1$ and $\forall s \in [0, \frac{9}{16}]$.

2) Set $a = \sup_{q \in \Sigma} \frac{2}{3\sqrt{3\lambda}}(1 + \frac{4}{3}|c_2| + \frac{\Lambda}{\lambda\beta})$, then $3a\sqrt{\frac{\lambda}{\gamma}} - \frac{\Lambda}{\lambda\beta} + \frac{c_2}{\gamma} \geq 1 > 0$ for $\gamma \in (\frac{3}{4}, 1)$. Notice that a is not related with γ.

3) Because we set a in terms of the local geometry data near the horizon, there exists constant $s_3^* > 0$ which only depends on the same local geometry (not related with γ), such that

$$\gamma s^{2\gamma-1} + \sqrt{\gamma}s^\gamma + c_2 s^{2\gamma} + (3a\sqrt{\lambda} + \frac{c_2}{\sqrt{\gamma}})s^{\gamma+1}$$

$$\geq \frac{3}{4}s + \frac{\sqrt{3}}{2}s - |c_2|s^{\frac{3}{2}} - |3a\sqrt{\lambda} + \frac{c_2}{\sqrt{\gamma}}|s^{\frac{7}{4}} > 0$$

holds for $s \in (0, s_3^*]$. This is obvious because the coefficients of the leading terms s are both positive constants, and their orders are at least $\frac{1}{2}$ away from the orders of the rest terms, whose coefficients are bounded and only depend on the local geometry near the horizon of the initial data set.

Then on $(0, \min(s_3^*, \frac{9}{16})]$ we have:

$$\sqrt{\lambda}\beta^{-1}\sigma^3 \mathcal{J}[v_{a,\gamma}] \geq (3a\sqrt{\frac{\lambda}{\gamma}} - \frac{\Lambda}{\lambda\beta} + \frac{c_2}{\gamma})s^2 + O(s^{3\gamma})$$
$$\geq s^2 + O(s^{3\gamma})$$
$$\geq s^2 + O(s^{\frac{9}{4}})$$

Therefore, there exists constant s_3 only depending on the local geometry of the initial data, such that $\sqrt{\lambda}\beta^{-1}\sigma^3 \mathcal{J}[v_{a,\gamma}] \geq 0$ for $s \in (0, s_3]$, and thus $\mathcal{J}[v_{a,\gamma}] \geq 0$ because $\lambda > 0$ and β is positive function. Thus $v_{a,\gamma}(s)$ is a supersolution of Jang's equation (2.1) on $(0, s_3]$.

For Property (2) we only need to notice that for $\frac{3}{4} < \gamma < 1$, $v_{a,\gamma}$ is finite at $s = 0$. By the fact that f blows up near the horizon, a vertical translation of $v_{a,\gamma}$ at $s = s_3$ is enough to make it a sub barrier.   □

By the comparison principle, c.f. [32, Chapter 10], we have the following result:

Proposition 4.11. *There exist constants a, s_3 which only depends on the local geometry of the initial data set near horizon, such that for $\forall \frac{3}{4} < \gamma < 1$, and any function f satisfies Jang's equation (2.1) on $U_{\bar{s}}$ and $f \to +\infty$ when $s \to 0$, the following inequality:*

$$(4.10) \qquad f(p, s) \geq \inf_{q \in \Sigma} f(q, s_3) + \frac{1}{\sqrt{\lambda}} \int_s^{s_3} \frac{1}{x^\gamma}dx + as - as_3$$

holds for any $p \in \Sigma$ and $s \in (0, s_3]$.

Let $\gamma \to 1$ in Ineq.(4.10), then because a, s_3 is not related to γ, we have the following estimate for the lower bound of blowup rate:

Proposition 4.12. *There exist constants a, s_3 which only depend on the local geometry of the initial data set near horizon, such that for any function f satisfies Jang's equation 2.1 on $U_{\bar{s}}$ and $f \to +\infty$ when $s \to 0$, the following inequality :*

$$(4.11) \qquad f(p, s) \geq \inf_{q \in \Sigma} f(q, s_3) - \frac{1}{\sqrt{\lambda}} \log s + \frac{1}{\sqrt{\lambda}} \log s_3 + as - as_3$$

holds for any $p \in \Sigma$ and $s \in (0, s_3]$.

Thus the second half of Theorem 2.1 is proved.

Put together our barriers, we prove that in Foliation B, there exist constants a, s_0 which only depend on the local geometry of the initial data set near horizon, such that for any function f satisfies Jang's equation (2.1) on $U_{\bar{s}}$ and $f \to +\infty$ when $s \to 0$, the following inequality :

$$(4.12) \qquad \inf_{q \in \Sigma} f(q, s_0) - \frac{1}{\sqrt{\lambda}} \log \frac{s}{s_0} + a(s - s_0)$$

$$\leq f(p, s) \leq \sup_{q \in \Sigma} f(q, s_0) - \frac{1}{\sqrt{\lambda}} \log \frac{s}{s_0} - a(s - s_0)$$

holds for any $p \in \Sigma$ and $s \in (0, s_0]$. Because s is comparable with the distance to the horizon, thus the above inequality implies the following C^0 lower and upper bound:

Proposition 4.13. *There exist constants a, τ_0 which only depend on the local geometry of the initial data set near horizon, such that for any function f satisfies Jang's equation (2.1) on V_{τ_0} and $f \to +\infty$ when $\tau \to 0$, $|f + \frac{1}{\sqrt{\lambda}} \log \tau|$ is a bounded function in V_{τ_0}. Here τ is the distance function to horizon, and $V_{\tau_0} = \{x \in \mathcal{M} | dist(x, \Sigma) \in (0, \tau_0]\}$*

It is a direct consequence of our barrier arguments that the following inequality holds for $\forall 0 < s \leq s_1 \leq s_0$:

$$(4.13) \qquad \inf_{q \in \Sigma} f(q, s_1) - \frac{1}{\sqrt{\lambda}} \log \frac{s}{s_1} + a(s - s_1)$$

$$\leq f(\cdot, s) \leq \sup_{q \in \Sigma} f(q, s_1) - \frac{1}{\sqrt{\lambda}} \log \frac{s}{s_1} - a(s - s_1)$$

As a consequence of Ineq.(4.13), suppose x_1, x_2 are the points on Σ_{s_1} such that $f(x_1, s_1) = \sup_{q \in \Sigma} f(q, s_1)$ and $f(x_2, s_1) = \inf_{q \in \Sigma} f(q, s_1)$, then:

$$\partial_s f(x_1, s_1) = \lim_{s \to s_1} \frac{f(x_1, s) - f(x_1, s_1)}{s - s_1} = \lim_{s \to s_1^-} \frac{f(x_1, s_1) - f(x_1, s)}{s_1 - s}$$

$$\geq \lim_{s \to s_1^-} \frac{f(x_1, s_1) - \sup_{q \in \Sigma} f(q, s_1) + \frac{1}{\sqrt{\lambda}} \log \frac{s}{s_1} + a(s - s_1)}{s_1 - s}$$

$$= \lim_{s \to s_1^-} \frac{-\frac{1}{\sqrt{\lambda}}(\log s_1 - \log s) - a(s_1 - s)}{s_1 - s}$$

$$= -\frac{1}{\sqrt{\lambda} s_1} - a$$

Similarly we can also get:

$$\partial_s f(x_2, s_1) \leq -\frac{1}{\sqrt{\lambda} s_1} + a$$

Thus Theorem 2.1 gets proved.

5. Gradient estimates

In this section we prove some gradient estimates for the blowup solution f of Jang's equation Eq.(2.1). First we need the following a priori estimate for Jang's equation by R. Schoen and S.-T. Yau:

Theorem 5.1. *[1, Proposition 2] Let $F(x)$ be a given C^2 function on $\mathcal{M}$ and suppose μ_1, μ_2, μ_3 are constants so that*

$$\sup_{\mathcal{M}} |F| \leq \mu_1, \quad \sup_{\mathcal{M}} |\nabla F| \leq \mu_2, \quad \sup_{\mathcal{M}} |\nabla^2 F| \leq \mu_3$$

and f is a C^3 solution of

$$(g^{ij} - \frac{f^i f^j}{1 + |\nabla f|^2})(\frac{\nabla_{ij} f}{\sqrt{1 + |\nabla f|^2}} - k_{ij}) = F$$

Denote $N = graph f$. If $X_0 \in N$, let (y^1, y^2, y^3, y^4) be normal coordinates in $\mathcal{M} \times \mathbb{R}$ centered at X_0 so that the tangent space to N at X_0 is the $y^1 y^2 y^3$-space. Then in a neighborhood of X_0, N is given by the graph of a function $w(y)$, $y = (y^1, y^2, y^3)$. We called this the local defining function w for N.

Then, there is a constant $\rho > 0$ depending only on the initial data and μ_1, μ_2, μ_3 so that for any $X_0 \in N$, the local defining function w for N is defined on $\{|y| \leq \rho\}$, and satisfies for any $\alpha \in (0,1)$,

$$\sup_{|y| \leq \rho} (|w(y)| + |\partial w(y)| + |\partial\partial w(y)| + |\partial\partial\partial w(y)| + |\partial\partial\partial w(y)|_{\alpha,\rho}) \leq c_1(\alpha),$$

where c_1 depends only on α, the initial data, and μ_1, μ_2, μ_3. Moreover, we may require

$$N \cap B_\rho^4(X_0) \subset \{Y : y^4 = w(y)\}$$

We also have the following Harnack-type inequalities:

$$\sup_{N \cap B_\rho^4(X_0)} < \overline{e}_4, \nu > \ \leq c_2 \inf_{N \cap B_\rho^4(X_0)} < \overline{e}_4, \nu >$$

$$\sup_{N \cap B_\rho^4(X_0)} |\overline{\nabla} \log < \overline{e}_4, \nu > | \leq c_3$$

where $\overline{e}_4$ is the downward unit normal to N, ν is the downward unit parallel vector field tangent to the $\mathbb{R}$ factor, $\overline{\nabla}$ is the covariant derivative on $N = graphf$, and c_2, c_3 are constants only depending on the initial data and μ_1, μ_2, μ_3.

Then following the arguments in [1, Proposition 4], we know that $N = graphf$ converges to the cylinder $\Sigma \times \mathbb{R}$. In fact, from the fact that $f - a$ is also a solution of Eq.(2.1), by the estimates of [1, Proposition 2] there is a sequence $\{a_i\}$ tending to infinity such that the graph of $f - a_i$ converges smoothly on compact subsets of $\mathcal{M} \times \mathbb{R}$ to a limiting 3-dim submanifold of $\mathcal{M} \times \mathbb{R}$. Then Harnack inequality in [1, Proposition 2] implies that this limiting manifold is $\Sigma \times \mathbb{R}$. Let U be a neighborhood of Σ. Following the arguments in [1, Proposition 4, Corollary 2], we can define a coordinate system on the neighborhood $U \times \mathbb{R}$ of $\Sigma \times \mathbb{R}$ in $\mathcal{M} \times \mathbb{R}$ by taking the fourth coordinate τ to be the distance function to $\Sigma \times \mathbb{R}$ in $\mathcal{M} \times \mathbb{R}$. Let $\bar{\Psi}' : \Sigma \times (-\epsilon', \epsilon') \to \mathcal{M}$ be the map

$$\bar{\Psi}' : \Sigma \times (-\epsilon', \epsilon') \times \mathbb{R} \to \mathcal{M} \times \mathbb{R} : (p, \tau, z) \mapsto \big(\exp_p(\tau\nu), z\big).$$

Thus $\bar{\Psi}'$ is compatible with Foliation A.

Then for a function h on $C_{\bar{z}}$ we let $graph_{\bar{\Psi}'}h$ be the set

$$graph_{\bar{\Psi}'}h = \{\bar{\Psi}(p, h(p, z), z) : (p, z) \in \Sigma \times \mathbb{R}\}.$$

Therefore, it is a direct consequence of the above reasoning that there exists constant $\bar{z}$, such that $N \cap (U \times [\bar{z}, \infty))$ can be written as a graph of u on $C_{\bar{z}} := \Sigma \times [\bar{z}, \infty)$. Furthermore, by translating the C^0 barriers of f Eq.(2.7) into C^0 barriers of h, we get the following:

Proposition 5.2. *Under the same assumptions as Theorem 2.1, for each solution f of Eq.(2.1) which blows up at Σ, there exist positive constants $\bar{z}$ and C_0', C_2', such that $graphf \cap U \times [\bar{z}, \infty)$ can be written as the graph of a function h over $C_{\bar{z}} := \Sigma \times [\bar{z}, \infty)$ under coordinate $\bar{\Psi}'$, and*

$$C_2' \exp(-\sqrt{\lambda}z) \leq |h(p, z)| \leq C_0' \exp(-\sqrt{\lambda}z)$$

5.1. C^2 bound

In this section, we prove a refinement of [1, Corollary 2] and [2, Theorem 4.2]. [1, Corollary 2] states that $N = graph f$ converges uniformly in C^2 to the cylinder $\Sigma \times \mathbb{R}$ for large values of f, and [2, Theorem 4.2] states that the rate of this convergence is exponential. Recall the statements of [1, Corollary 2] and [2, Theorem 4.2]:

Theorem 5.3. *[1, Corollary 2] Assume the conditions of Theorem 3.4 and let Σ be a connected component of the apparent horizons, on which f tends to $+\infty$ ($-\infty$ respectively). Let U be a neighborhood of Σ with positive distance to any other apparent horizons in $\mathcal{M}$.*

Then for all $\epsilon > 0$ there exists $\bar{z} = \bar{z}(\epsilon)$, depending also on the geometry of $(\mathcal{M}, g, k)$, such that $N \cap (U \times [\bar{z}, \infty))$ can be written as the graph of a function u over $C_{\bar{z}} := \Sigma \times [\bar{z}, \infty)$, so that

$$|u(p, z)| + |^{C_{\bar{z}}}\nabla u(p, z)| + |^{C_{\bar{z}}}\nabla^2 u(p, z)| < \epsilon.$$

for all $(p, z) \in C_{\bar{z}}$. Here, $^{C_{\bar{z}}}\nabla$ denotes covariant differentiation along $C_{\bar{z}}$.

Theorem 5.4. *[2, Theorem 4.2] Let $N_0 = graph f_0$ be the manifold constructed in Theorem 3.6 and assume the situation of Theorem 3.5. Then there exists $\bar{z} = \bar{z}(\epsilon)$, depending also on the geometry of $(\mathcal{M}, g, k)$, such that $N_0 \cap (U \times [\bar{z}, \infty))$ can be written as the graph of a function u_0 over $C_{\bar{z}} := \Sigma \times [\bar{z}, \infty)$. If in addition Σ is strictly stable with principal eigenvalue $\lambda > 0$, then for all $\delta < \sqrt{\lambda}$ there exists $c = c(\delta)$ depending only on the data $(\mathcal{M}, g, k)$ and δ such that*

$$|u_0(p, z)| + |^{C_{\bar{z}}}\nabla u_0(p, z)| + |^{C_{\bar{z}}}\nabla^2 u_0(p, z)| \leq c \exp(-\delta z).$$

where $^{C_{\bar{z}}}\nabla$ is the covariant derivative w.r.t. the induced metric on $C_{\bar{z}}$.

In this section, we are able to prove that the rate of this convergence is $\exp(-\sqrt{\lambda}z)$. In [2, Section 4], J. Metzger provided a way to extend the C^0 super control to C^2 super control. The procedure presented here is the same as in [2, Theorem 4.2]. We briefly outline here for consistence.

In [2, Theorem 4.2], it is computed that the value of Jang's operator for a function h on $\Sigma \times \mathbb{R}$ is the following:

$$\mathcal{J}[h] = \partial_z^2 h + \gamma_{h(p,z)}^{ij} \nabla_{i,j}^2 h - 2\gamma_{h(p,z)}^{ij} \partial_i h \cdot k(\partial_s, \partial_j) - \theta^+[\Sigma_{h(p,z)}]$$
$$+ Q(h, {}^{C_{\bar{z}}}\nabla h, {}^{C_{\bar{z}}}\nabla^2 h)$$

where γ_s is the metric on Σ_s and Q is of the form

$$Q(h, ^{C_{\bar{z}}}\nabla h, ^{C_{\bar{z}}}\nabla^2 h) = h *^{C_{\bar{z}}}\nabla +^{C_{\bar{z}}}\nabla *^{C_{\bar{z}}}\nabla +^{C_{\bar{z}}}\nabla *^{C_{\bar{z}}}\nabla *^{C_{\bar{z}}}\nabla^2 h$$

where $*$ denotes some contraction with a bounded tensor. The vectors ∂_i, $i = 1, 2$ denote directions tangential to Σ and ∂_z the direction along the $\mathbb{R}$-factor in $C_{\bar{z}}$.

By [1, Corollary 2], we know that $Q(h, ^{C_{\bar{z}}}\nabla h, ^{C_{\bar{z}}}\nabla^2 h)$ is low order term. Thus by freezing coefficients, h satisfies a linear, uniformly elliptic equation of the form

$$a^{ij}\partial_i\partial_j h+ < b, ^{C_{\bar{z}}}\nabla h > -\theta^+[\Sigma_{h(p,z)}] = F.$$

Notice that because $\theta^+[\Sigma_\tau] = \tau L_\Sigma 1 + O(\tau^2)$, $\theta^+[\Sigma_{h(p,z)}]$ also decays exponentially with the order $e^{-\sqrt{\lambda}z}$.

Now using standard Schauder interior estimates for linear elliptic equations (c.f. [32, Chapter 6]), we can bound C^2 norm of h by its C^0 norm and the norm of $\theta^+[\Sigma_{h(p,z)}]$. By the fact that they both decay exponentially with the order $e^{-\sqrt{\lambda}z}$, we can conclude that C^2 norm of h also decays exponentially with the order $e^{-\sqrt{\lambda}z}$. Thus we prove the first part of Theorem 2.4.

Theorem 5.5. *Under the same assumptions of Theorem 2.1, for each solution f of Eq.(2.1) which blows up at Σ, there exist positive constants $\bar{z}$ and C'_1, C'_2, such that $graph f \cap U \times [\bar{z}, \infty)$ can be written as the graph of a function h over $C_{\bar{z}} := \Sigma \times [\bar{z}, \infty)$ under coordinate system $\bar{\Psi}'$, and*

$$|h(p, z)| + |^{C_{\bar{z}}}\nabla h(p, z)| + |^{C_{\bar{z}}}\nabla^2 h(p, z)| \le C'_1 \exp(-\sqrt{\lambda}z)$$

and

$$|h(p, z)| \ge C'_2 \exp(-\sqrt{\lambda}z)$$

where $^{C_{\bar{z}}}\nabla$ is the covariant derivative w.r.t. the induced metric on $C_{\bar{z}}$.

For the convenience of the next section, we prefer to rewrite the above theorems under a coordinate system in $U \times \mathbb{R}$ which is compatible with Foliation B.

Let $\bar{\Psi} : \Sigma \times (-\epsilon, \epsilon) \to \mathcal{M}$ be the map

$$\bar{\Psi} : \Sigma \times (-\epsilon, \epsilon) \times \mathbb{R} \to \mathcal{M} \times \mathbb{R} : (p, s, z) \mapsto (\Psi_B(p, s), z).$$

Thus $\bar{\Psi}$ is compatible with Foliation B.

Then for a function u on $C_{\bar{z}}$ we let $graph_{\bar{\Psi}} u$ be the set

$$graph_{\bar{\Psi}} u = \{\bar{\Psi}(p, u(p, z), z) : (p, z) \in \Sigma \times \mathbb{R}\}.$$

For a small neighborhood of Σ, there exist constants $\alpha_i, i = 1, 2, 3$, such that the following inequalities:

$$\frac{s}{\tau} \in (\alpha_1^{-1}, \alpha_1)$$

$$\frac{\partial s}{\partial \tau}, \frac{\partial \tau}{\partial s} \in (\alpha_2^{-1}, \alpha_2)$$

$$\frac{\partial^2 s}{\partial \tau^2}, \frac{\partial^2 \tau}{\partial s^2} \in (\alpha_3^{-1}, \alpha_3)$$

holds in this neighborhood.

Then we can rewrite the above theorem under coordinates defined by Ψ:

Theorem 5.6. *Under the same assumption as Theorem 2.1, for each solution f of Eq.(2.1) which blows up at Σ, there exist positive constants $\bar{z}$ and C_1, C_2, such that $graph f \cap U \times [\bar{z}, \infty)$ can be written as the graph of a function u over $C_{\bar{z}} := \Sigma \times [\bar{z}, \infty)$ under coordinate system $\bar{\Psi}$, and*

$$|u(p, z)| + |^{C_{\bar{z}}} \nabla u(p, z)| + |^{C_{\bar{z}}} \nabla^2 u(p, z)| \le C_1 \exp(-\sqrt{\lambda} z)$$

and

$$|u(p, z)| \ge C_2 \exp(-\sqrt{\lambda} z)$$

where $^{C_{\bar{z}}} \nabla$ is the covariant derivative w.r.t. the induced metric on $C_{\bar{z}}$.

In the above theorem, the constants $C_1, C_2, \bar{z}$ are of course related to the solution f because Jang's equation is invariant under vertical translation. By the fact that super barrier and sub barrier depend on $\sup_{\Sigma_{s_0}} f$ and $\inf_{\Sigma_{s_0}} f$, we know constants $C_1, C_2, \bar{z}$ also depend on these quantities.

5.2. Gradient estimate

Now under the coordinates defined by $\bar{\Psi}$, the hypersurface N can be expressed in two different way in a neighborhood of Σ: either as the graph of f on $\mathcal{M}$, or as the graph of u on cylinder $\Sigma \times \mathbb{R}$:

$$z = f(p, s)$$

$$s = u(p, z)$$

hold for $p \in \Sigma$.

Denote $F(p, z, s) = f(p, s) - z$. Then $F(p, z, u(p, z)) = 0$. Let $e_i, i = 1, 2$ be the orthonormal frame on Σ. Parallel translate them to obtain an orthonormal frame on Σ_s, and still denote them as $e_i, i = 1, 2$. Denote $\partial_i, i = 1, 2$ to be the derivatives taken in the direction of $e_i, i = 1, 2$, respectively, and ∂_s to be the derivative taken w.r.t the parameter s of Foliation B. Then by Implicit Function Theorem,

$$(5.1) \qquad \partial_z u(p, z) = -\big([J_{F,s}(p, z, s)]^{-1} \partial_z F(p, z, s)\big)\big|_{s=u(p,z)}$$

$$(5.2) \qquad \partial_i u(p, z) = -\big([J_{F,s}(p, z, s)]^{-1} \partial_i F(p, z, s)\big)\big|_{s=u(p,z)}$$

Thus we have

$$(5.3) \qquad \partial_s f(p, s)\big|_{s=u(p,z)} = J_{F,s}(p, z, s)\big|_{s=u(p,z)} = \frac{1}{\partial_z u(p, z)}$$

Therefore, by the fact that

$$|\partial_z u(p, z)| \le |^{C_{\bar{z}}} \nabla u(p, z)| \le C_1 e^{-\sqrt{\lambda} z},$$

and

$$s = u(p, z) \ge C_2 e^{-\sqrt{\lambda} z},$$

we have the following lower bound for $|\partial_s f|$:

$$|\partial_s f(p, s)| \ge \frac{1}{C_1 e^{-\sqrt{\lambda} z}} \ge \frac{C_2}{C_1 s}$$

We now prove the upper bound for the gradient estimate for f. As discussed above, it is equivalent to find the lower bound for $|\partial_z u|$. We first prove the following weaker version:

Proposition 5.7. *There exist constants $C > 0$ and $z_1 \ge \bar{z}$ such that for $\forall z > z_1$, there exists a point $p_z \in \Sigma$, such that*

$$\partial_z u(p_z, z) := \lim_{\delta z \to 0} \frac{u(p_z, z + \delta z) - u(p_z, z)}{\delta z} \le -C \exp(-\sqrt{\lambda} z)$$

Proof. For an arbitrary $z_0 \ge \bar{z}$, denote $p_0 \in \Sigma$ to be the point where u achieves its maximum when $z = z_0$, and $s_0 = \max_{p \in \Sigma}(u(p, z_0)) = u(p_0, z_0)$. Then on Σ_{s_0} under Foliation B, f achieves its maximum on Σ_{s_0} at $p_0 \in \Sigma$. By the gradient estimate of f at maximum point on Σ_{s_0} in Theorem 2.1, we

have

$$|\partial_s f(p_0, s_0)| \leq \frac{1}{\sqrt{\lambda}s_0} + a$$

Translate it back to the gradient estimate for u at (p_0, z_0) by Implicit Function Theorem, we have:

$$|\partial_z u(p_0, z_0)| = \frac{1}{|\partial_s f(p_0, s_0)|} \geq \frac{\sqrt{\lambda}s_0}{1 + a\sqrt{\lambda}s_0}$$

$$= \frac{\sqrt{\lambda}u(p_0, z_0)}{1 + a\sqrt{\lambda}u(p_0, z_0)} \geq \frac{\sqrt{\lambda}C_2 e^{-\sqrt{\lambda}z_0}}{1 + aC_0 e^{-\sqrt{\lambda}z_0}} \geq \frac{\sqrt{\lambda}C_2}{2} e^{-\sqrt{\lambda}z_0}$$

The last inequality holds when z_0 is large enough. $\qquad\square$

We denote $\Sigma_z^u := u(\Sigma, z)$. Suppose V is the open domain enclosed by $\Sigma_{\bar{z}}^u$ and Σ. Then Σ_z^u will stay in the tubular area $V \times \mathbb{R}$ if $z > \bar{z}$.

Denote $N = \text{graph}f = \text{graph}u$. The Harnack inequality in [1, Proposition 2] implies that there are constants ρ, c_2, such that for any point X_0 on N,

$$\sup_{N \cap B_\rho^4(X_0)} < \overline{e}_4, \nu > \leq c_2 \inf_{N \cap B_\rho^4(X_0)} < \overline{e}_4, \nu >$$

where

$$< \overline{e}_4, \nu > = \frac{1}{\sqrt{1 + |\nabla f|^2}} = \frac{-\partial_z u}{\sqrt{1 + |C_{\bar{z}} \nabla u|^2}}$$

Now V can be covered by finite balls B_ρ^4, and we suppose constant c_4 is the number of balls that is enough to cover V. Thus each Σ_z^u can be covered by c_4 balls B_ρ^4, for $\forall z > \bar{z}$. This is because each of them stays in a horizontal cut of the tube $V \times \mathbb{R}$, which is exactly V.

Therefore, together with the Harnack inequality, there is a constant c_5, such that

$$\sup_{\Sigma_z^u} < \overline{e}_4, \nu > \leq c_5 \inf_{\Sigma_z^u} < \overline{e}_4, \nu >, \quad \forall z > \bar{z}.$$

Now for any $p \in \Sigma$, we have

$$
\begin{aligned}
|\partial_z u(p,z)| &\geq \frac{|\partial_z u(p,z)|}{\sqrt{1 + |C_{\bar{z}} \nabla u(p,z)|^2}} = \frac{-\partial_z u(p,z)}{\sqrt{1 + |C_{\bar{z}} \nabla u(p,z)|^2}} \\
&= < \bar{e}_4, \nu > \big|_{(p,z)} \geq \inf_{\Sigma_z^u} < \bar{e}_4, \nu > \\
&\geq c_5^{-1} \sup_{\Sigma_z^u} < \bar{e}_4, \nu > \geq c_5^{-1} < \bar{e}_4, \nu > \big|_{(p_0,z)} \\
&= \frac{-\partial_z u(p_z,z)}{c_5 \sqrt{1 + |C_{\bar{z}} \nabla u(p_z,z)|^2}} \geq \frac{C e^{-\sqrt{\lambda}z}}{c_5 \sqrt{1 + C_1^2 e^{-2\sqrt{\lambda}z}}} \\
&\geq \frac{C}{2 c_5 C_1} e^{-\sqrt{\lambda}z}
\end{aligned}
$$

The last line holds if z is big enough.

Set $C_3 = \frac{C}{2 c_5 C_1}$ then we prove the following:

Proposition 5.8. *Assume the condition and notation of Theorem 5.6, then there exist constants $C_3 > 0$, $z_2 \geq \bar{z}$, such that $\forall z \geq z_2, p \in \Sigma$,*

$$
|\partial_z u(p,z)| \geq C_3 e^{-\sqrt{\lambda}z}
$$

Now, by Implicit Function Theorem we have:

$$
|\partial_s f(p,s)| = \frac{1}{|\partial_z u(p,z)|} \leq \frac{1}{C_3 e^{-\sqrt{\lambda}z}} \leq \frac{C_1}{C_3 s}
$$

Moreover,

$$
|\partial_i f(p,s)| = |\partial_i u(p,z)||\partial_s f(p,s)| \leq \frac{C_1^2 e^{-\sqrt{\lambda}z}}{C_3 s} \leq \frac{C_1^2}{C_2 C_3}
$$

Because $\partial_i, i = 1,2$ are the derivatives taken in the direction of the orthonormal frame $e_i, i = 1,2$ on Σ_s, we have:

$$
|\nabla^{\Sigma_s} f(p,s)| \leq \frac{\sqrt{2} C_1^2}{C_2 C_3}
$$

holds for $\forall p \in \Sigma$, $\forall s \in (0, s_0]$. Here ∇^{Σ_s} denotes the covariant derivative along Σ_s.

Put together all the above conclusions, we prove the following:

Proposition 5.9. *Under the same assumptions as Theorem 2.1, suppose f is a blowup solution of Eq.(2.1) in an open neighborhood U near Σ, which*

is a compact boundary component and a strictly stable MOTS with principal eigenvalue $\lambda > 0$. Denote $N = \mathrm{graph} f$. Then, under coordinate system $\bar{\Psi}$,

(1) there exist positive constants $\bar{z}$ and C_1, C_2, C_3, such that $N \cap (U \times [\bar{z}, \infty))$ can be written as the graph of a function u over $C_{\bar{z}} := \Sigma \times [\bar{z}, \infty)$, and

$$(5.4) \qquad |u(p,z)| + |^{C_{\bar{z}}}\nabla u(p,z)| + |^{C_{\bar{z}}}\nabla^2 u(p,z)| \leq C_1 \exp(-\sqrt{\lambda}z)$$

$$(5.5) \qquad\qquad\qquad |u(p,z)| \geq C_2 \exp(-\sqrt{\lambda}z)$$

$$(5.6) \qquad\qquad\qquad |^{C_{\bar{z}}}\nabla u(p,z)| \geq C_3 \exp(-\sqrt{\lambda}z)$$

where $^{C_{\bar{z}}}\nabla$ is the covariant derivative w.r.t. the induced metric on $C_{\bar{z}}$.

(2) Denote $\Sigma_s = \Psi_B(\Sigma, s)$. Then there exists constant s_0, such that the following gradient estimates for f:

$$(5.7) \qquad\qquad \frac{C_2}{C_1 s} \leq |\partial_s f(p,s)| \leq \frac{C_1}{C_3 s}$$

$$(5.8) \qquad\qquad |\nabla^{\Sigma_s} f(p,s)| \leq \frac{\sqrt{2}C_1^2}{C_2 C_3}$$

hold for $\forall p \in \Sigma$, $\forall s \in (0, s_0]$. Here ∇^{Σ_s} denotes the covariant derivative along Σ_s.

Thus Theorem 2.4 gets proved.

6. Dependence of the coefficients

In this section we prove Theorem 2.6. We prove that for the solution f_0 constructed by J. Metzger, c.f. [2], we can have the same estimates as in Theorem 2.4 hold for f_0, while with constants $\bar{z}, C_1, C_2, C_3$ depending only on the geometry of the initial data set.

To remove the dependence of $\bar{z}, C_1, C_2, C_3$ on the vertical translation on the solution f, we need to normalize f such that there is no such kind of freedom. For example, we can normalize f by some outer boundary condition or require that the solution decay at infinity. J. Metzger constructed a blowup solution f_0 in [2] with these constraints, thus in this section we will study this specific example f_0 to see if our $C_1, C_2, C_3, \bar{z}$ can be determined by the geometry of initial data. Therefore, although Theorem 2.1 and Theorem 2.4 work for more general initial data sets, we need to further assume that $\mathcal{M}$ is 3-dim asymptotically flat manifold with one end, satisfies the dominant energy condition. Also assume Σ is the only boundary component, and is a compact strictly stable outermost MOTS.

Then by [14, Lemma 9.7], Σ is topologically 2-sphere. We also assume there is no MITS in $\mathcal{M}$. The existence of asymptotically decaying solution of Jang's equation over $\mathcal{M}$, which only blows up at Σ then follows [2, Theorem 3.1, Remark 3.3] . The detailed construction procedure can be found in [2, Section 3]. We only give a brief introduction about the procedure here for consistence.

First of all, it can be constructed from $(\mathcal{M}, g, k)$ a new data set $(\tilde{M}, \tilde{g}, \tilde{k})$ (c.f. [2], [4]), with following properties:

1) $\mathcal{M} \subset \tilde{\mathcal{M}}$ with $\tilde{g}|_{\mathcal{M}} = g$, $\tilde{k}|_{\mathcal{M}} = k$,

2) $\theta^+[\partial\tilde{\mathcal{M}}] < 0$,

3) $H[\partial\tilde{\mathcal{M}}] > 0$ where H is the mean curvature of $\partial\tilde{\mathcal{M}}$ with respect to the normal pointing out of $\tilde{\mathcal{M}}$,

4) The region $\tilde{\mathcal{M}} \setminus \mathcal{M}$ is foliated by surfaces Σ_s with $\theta^+(\Sigma_s) < 0$.

Then the following Dirichlet boundary value problem can be solved:

(6.1)
$$\begin{cases} \mathcal{J}[f_t] = tf_t & \text{in } \tilde{M} \\ f_t = \frac{\delta}{2t} & \text{on } \partial^-\tilde{M} \\ f_t \to 0 & \text{when } x \to \infty \end{cases}$$

where δ is a lower bound for the mean curvature H on $\partial\tilde{\mathcal{M}}$. The solution f_t of Eq.(6.1) satisfies an estimate of the form

(6.2)
$$\sup_{\tilde{\mathcal{M}}} |f_t| + \sup_{\tilde{\mathcal{M}}} |\nabla f_t| \leq \frac{C}{t},$$

where C is a constant depending only on the data $(\tilde{M}, \tilde{g}, \tilde{k})$ but not on t.

This gradient estimate and the fact that graphs $N_t = graph f_t$ have uniformly bounded curvature in $\tilde{\mathcal{M}} \times \mathbb{R}$ away from the boundary implies that it allows to extract a sequence $t_i \to 0$ such that the N_{t_i} converge smoothly to a manifold N_0 (c.f. [1, Section 4], [4, Proposition 3.8]), which can be proved to be the graph of a function f_0 on $\mathcal{M}$ which satisfies $\mathcal{J}[f_0] = 0$, with the desired asymptotics.

From the local parametric estimate [1, Proposition 2], the a priori bounds at asymptotically flat ends [1, Proposition 3], and the gradient estimate (6.2), it is straightforward to see that for any $\mathcal{M}' \subset \mathcal{M}$ such that $\partial\mathcal{M}'$ does not intersect with Σ, there is a uniform C^0 bound for f_t on $\mathcal{M}'$

$$|\sup_{\mathcal{M}'} f_t| < c_1.$$

where c_1 only depends on the geometry of the initial data. This also bounds the limit function f_0. Thus by setting $\mathcal{M}'$ to be the portion of $\mathcal{M}$ outside Σ_{s_0} in Theorem 2.1, we have $-c_1 < \inf_{q \in \Sigma} f_0(q, s_0) \leq \sup_{q \in \Sigma} f_0(q, s_0) < c_1$ in Eq.(2.7). Thus we have the following:

Proposition 6.1. *Besides the conditions and notations in Theorem 2.4, we further assume that $\mathcal{M}$ is a 3-dim asymptotically flat manifold with one end, and satisfies the dominant energy condition. Also assume that Σ is the only boundary component, and is a compact outermost MOTS. We also assume that there is no MITS in $\mathcal{M}$. Then a function f_0 on $\mathcal{M}$ can be constructed as in [2, Theorem 3.1], such that $\mathcal{J}[f_0] = 0$, $f_0(x) \to 0$ when $|x| \to \infty$, and it only blows up at Σ. Denote $N_0 = graph f_0$. Then,*
(1) there exist positive constants $\bar{z}$ and C_1, C_2, C_3, which only depend on the initial data such that $N_0 \cap (U \times [\bar{z}, \infty))$ can be written as the graph of a function u_0 over $C_{\bar{z}} := \Sigma \times [\bar{z}, \infty)$ under coordinate system $\bar{\Psi}$, and

$$(6.3) \qquad |u_0(p, z)| + |^{C_{\bar{z}}}\nabla u_0(p, z)| + |^{C_{\bar{z}}}\nabla^2 u_0(p, z)| \leq C_1 \exp(-\sqrt{\lambda} z).$$

$$(6.4) \qquad |u_0(p, z)| \geq C_2 \exp(-\sqrt{\lambda} z)$$

$$(6.5) \qquad |^{C_{\bar{z}}}\nabla u_0(p, z)| \geq C_3 \exp(-\sqrt{\lambda} z)$$

where $^{C_{\bar{z}}}\nabla$ is the covariant derivative w.r.t. the induced metric on $C_{\bar{z}}$.

(2) Denote $\Sigma_s = \Psi_B(\Sigma, s)$. Then

$$(6.6) \qquad \frac{C_2}{C_1 s} \leq |\partial_s f_0(p, s)| \leq \frac{C_1}{C_3 s}$$

$$(6.7) \qquad |\nabla^{\Sigma_s} f_0(p, s)| \leq \frac{\sqrt{2} C_1^2}{C_2 C_3}$$

hold for $\forall p \in \Sigma$, $\forall s \in (0, s_0]$. Here ∇^{Σ_s} denotes the covariant derivative along Σ_s.

Thus Theorem 2.6 is proved.

7. Application to Penrose inequality

In previous sections we have obtained a sharp estimate for the blowup rate and gradient of the solution f_0 of Jang's equation, which is contructed by J. Metzger in [2]. In this section we apply this control to the slice $graph f_0$ get a Penrose-like inequality for general initial data sets.

7.1. Basic settings

Assume the same condition as in Theorem 2.6. We will apply our estimates onto the solution f_0 constructed in [2, Theorem 3.1] . Also in this section we still use Foliation B. Recall that in this foliation $\Psi_B : \Sigma \times [0, \bar{s}] \to \mathcal{M}$ the metric g of $\mathcal{M}$ near a neighborhood of Σ can be written as

$$(7.1) \qquad\qquad g = \beta^2 ds^2 + g_{\Sigma_s}$$

and θ^+ satisfies:

$$(7.2) \qquad\qquad \theta^+[\Sigma_s] = \lambda\beta s + O(s^2)$$

Now by Theorem 2.6 we know there are constants C_1, C_2, C_3 which only depend on the initial data, such that:

$$(7.3) \qquad\qquad \frac{C_2}{C_1 s} \leq |\partial_s f_0| \leq \frac{C_1}{C_3 s}$$

$$(7.4) \qquad\qquad |\nabla^{\Sigma_s} f_0(p, s)| \leq \frac{\sqrt{2} C_1^2}{C_2 C_3}$$

Here ∇^{Σ_s} denotes the covariant derivative along Σ_s.

In this Section we will use $\overline{\mathcal{M}}$ to represent $N_0 = graph f_0$. Denote the induced metric on Jang's slice to be $\bar{g}$. Denote $\nabla, \overline{\nabla}$ to be the Levi-Civita connection of $g, \bar{g}$, respectively. Then $\bar{g} = df_0^2 + g$, and the second fundamental form on $\overline{\mathcal{M}}$ embedded into $\mathcal{M} \times \mathbb{R}$ will be $h = \dfrac{\nabla^2 f_0}{\sqrt{1 + |\nabla f_0|^2}}$. Denote $\overline{\Sigma}_s$ to be the lift of Σ_s to $\overline{\mathcal{M}}$. Denote the portion of $\mathcal{M}, \overline{\mathcal{M}}$ outside $\Sigma_s, \overline{\Sigma}_s$ to be $\mathcal{M}_s, \overline{\mathcal{M}}_s$, respectively. We choose orthonormal frames $\{e_1, e_2\}$ and $\{\bar{e}_1, \bar{e}_2\}$ for $T\Sigma_s$ and $T\overline{\Sigma}_s$, respectively. Let e_3 be the normal of Σ_s pointing into $\mathcal{M}_s$ that is tangent to $\mathcal{M}$. We also choose $\{\bar{e}_3, \bar{e}_4\}$ for the normal bundle of Σ_s in $\mathcal{M} \times \mathbb{R}$, such that $\bar{e}_3$ is tangent to the graph $\overline{\mathcal{M}}$ and $\bar{e}_4$ is a downward unit normal vector of $\overline{\mathcal{M}}$ in $\mathcal{M} \times \mathbb{R}$. We denote the induced Levi-Civita connection on Σ_s to be $\overset{\circ}{\nabla}$ and $\phi_s = f_0|_{\Sigma_s}$ to be f's restriction on Σ_s. Denote H_{Σ_s}, $\overline{H}_{\overline{\Sigma}_s}$ as the mean curvature of $\Sigma_s, \overline{\Sigma}_s$ with respect to $e_3, \bar{e}_3$, respectively. Recall that the scalar curvature on the Jang's slice can be written as:

$$(7.5) \qquad \overline{R} = 16\pi(\mu - J(\omega)) + |h - k|_{\bar{g}}^2 + 2|q|_{\bar{g}}^2 - 2div_{\bar{g}}(q)$$

where

$$\omega_i = \frac{\nabla_i f_0}{\sqrt{1 + |\nabla f_0|^2}}$$

$$q_i = \frac{f_0^j}{\sqrt{1 + |\nabla f_0|^2}}(h_{ij} - k_{ij})$$

We will calculate $\overline{H}_{\overline{\Sigma}_s} - q(\overline{e}_3)$ in terms of the geometry quantities on $\mathcal{M}$.

7.2. Calculation of $\overline{H}_{\overline{\Sigma}_s} - q(\overline{e}_3)$

We now calculate the value of $\overline{H}_{\overline{\Sigma}_s} - q(\overline{e}_3)$ in terms of the geometry quantities on $\mathcal{M}$.

From [25, page 10]:

$$(7.6) \quad \overline{H}_{\overline{\Sigma}_s} - q(\overline{e}_3) = (<e_3, \overline{e}_3> + \frac{<e_3, \overline{e}_4>^2}{<e_3, \overline{e}_3>})H_{\Sigma_s} - \frac{<e_3, \overline{e}_4>}{<e_3, \overline{e}_3>}Tr_{\Sigma_s}k$$

Following the calculation of [18, Section 4, Eq.(4.4)], we get:

$$(7.7) \qquad <e_3, \overline{e}_3> = \frac{\sqrt{1 + |\overline{\nabla}\phi_s|^2}}{\sqrt{1 + |\nabla f_0|^2}} \quad \text{and} \quad <e_3, \overline{e}_4> = \frac{\nabla_{e_3} f_0}{\sqrt{1 + |\nabla f_0|^2}}$$

Then the above equations imply:

$$(7.8) \quad \overline{H}_{\overline{\Sigma}_s} - q(\overline{e}_3) = \frac{\sqrt{1 + |\nabla f_0|^2}}{\sqrt{1 + |\overline{\nabla}\phi_s|^2}}\theta^+[\Sigma_s] + \frac{\sqrt{1 + |\overline{\nabla}\phi_s|^2}}{|\nabla_{e_3} f_0| + \sqrt{1 + |\nabla f_0|^2}}Tr_{\Sigma_s}k$$

In the above calculation we use the fact that $\nabla_{e_3} f_0 < 0$ and $|\nabla f_0|^2 = |\nabla_{e_3} f_0|^2 + |\overline{\nabla}\phi_s|^2$.

Therefore, we can have an upper bound for $\overline{H}_{\overline{\Sigma}_s} - q(\overline{e}_3)$ as follows:

$$(7.9) \quad \overline{H}_{\overline{\Sigma}_s} - q(\overline{e}_3) = \frac{\sqrt{1 + |\nabla f_0|^2}}{\sqrt{1 + |\overline{\nabla}\phi_s|^2}}\theta^+[\Sigma_s] + \frac{\sqrt{1 + |\overline{\nabla}\phi_s|^2}}{|\nabla_{e_3} f_0| + \sqrt{1 + |\nabla f_0|^2}}Tr_{\Sigma_s}k$$
$$\leq \sqrt{1 + |\nabla f_0|^2}(\lambda\beta s + O(s^2)) + O(s)$$
$$= \sqrt{\beta^{-2}|\partial_s f_0|^2 + O(1)}(\lambda\beta s + O(s^2)) + O(s)$$
$$\leq \sqrt{\frac{C_1^2}{\beta^2 C_3^2 s^2} + O(1)}\lambda\beta s + O(s)$$
$$= \lambda\frac{C_1}{C_3} + O(s)$$

Remark 7.1. *The standard gradient estimate for the mean curvature type equation actually provides an estimate for $|\nabla f_0|$. From [32, Chapter 16], there exist positive constants C_4, C_5 only depends on $\sup|h|$ and $\sup|\nabla h|$, where h is the mean curvature of $N_0 = graph f_0$ embedded into $\mathcal{M} \times \mathbb{R}$, such that:*

$$|\nabla f_0| \leq C_4 \exp(C_5 f_0/\tau)$$

Then because $f = -\frac{1}{\sqrt{\lambda}} \log \tau + O(1)$ in a neighborhood of Σ, there exists positive constant C_6, s.t.

$$|\nabla f_0| \leq C_4 \exp(C_6 \frac{\log \frac{1}{\tau}}{\tau})$$

This is an upper bound for the gradient estimate of f_0. But the order is not sharp enough for proving a Penrose-like inequality in this section.

7.3. Existence of harmonic spinor

We are now going to find a harmonic spinor on $\overline{\mathcal{M}}$ which is constant at infinity. Because $\overline{\mathcal{M}}$ is 3-dimensional oriented Riemannian manifold, it is also a spin manifold. Denote $\overline{\mathcal{M}}_r$ as the domain enclosed by large sphere S_r, and $\overline{\mathcal{M}}_{s,r} = \overline{\mathcal{M}}_r \cap \overline{\mathcal{M}}_s$. Denote ν_r to be the outer normal of S_r.

Our calculation will follow [16, Section 2] and [18, Section 5]. From [16, Lemma 2.3], we have the following Bochner-Lichnerowicz-Weitzenböck formula:

$$(7.10) \qquad \int_{\overline{\mathcal{M}}_{s,r}} |D\psi|^2 = \int_{\overline{\mathcal{M}}_{s,r}} |\overline{\nabla}\psi|^2 + \frac{1}{4} \int_{\overline{\mathcal{M}}_{s,r}} \overline{R}|\psi|^2$$
$$+ \int_{\partial\overline{\mathcal{M}}_s} <\overline{\nabla}_{\overline{e}_3}\psi + \overline{e}_3 \cdot D\psi, \psi>$$
$$- \int_{S_r} <\overline{\nabla}_{\nu_r}\psi + \nu_r \cdot D\psi, \psi>$$

where ψ is a spinor on $\overline{\mathcal{M}}_{s,r}$, D is the Dirac operator and $\cdot$ is Clifford multiplication on Jang's slice $\overline{\mathcal{M}}_{s,r}$. Here we still use $\overline{\nabla}$ to denote the spin connection on $\overline{\mathcal{M}}_{s,r}$

From [14, Lemma 9.7] we know that

$$\lambda|\partial\mathcal{M}|_g \leq 4\pi,$$

We set

$$C_s = \min_{\partial \mathcal{M}_s} \left(4 - (\overline{H}_{\partial \overline{\mathcal{M}}_s} - q(\overline{e}_3)) \sqrt{\frac{|\partial \mathcal{M}_s|_{\overline{g}}}{\pi}} \right).$$

Then

$$(7.11) \qquad C_s \geq 4 - (\overline{H}_{\partial \overline{\mathcal{M}}_s} - q(\overline{e}_3)) \sqrt{\frac{|\partial \overline{\mathcal{M}}_s|_{\overline{g}}}{\pi}}$$

$$\geq 4 - (\lambda \frac{C_1}{C_3} + O(s)) \sqrt{\frac{|\partial \overline{\mathcal{M}}_s|_{\overline{g}}}{|\partial \mathcal{M}_s|_g}} \sqrt{\frac{|\partial \mathcal{M}_s|_g}{\pi}}$$

$$\geq 4 - (\lambda \frac{C_1}{C_3} + O(s)) \sup_{\Sigma_s} \sqrt{1 + |\overline{\nabla} \phi_s|^2} \sqrt{\frac{4}{\lambda}}$$

$$= 4 - 2\sqrt{\lambda} (\frac{C_1}{C_3} + O(s)) \sup_{\Sigma_s} \sqrt{1 + |\nabla^{\Sigma_s} f|^2}$$

$$\geq 4 - 2\sqrt{\lambda} (\frac{C_1}{C_3} + O(s)) \sqrt{1 + \frac{2C_1^4}{C_2^2 C_3^2}}$$

If we suppose

$$\lambda \frac{C_1^2}{C_3^2} (1 + \frac{2C_1^4}{C_2^2 C_3^2}) < 4,$$

then $C_s > 0$ if s is small enough. Therefore, there exists a positive constant, still denote as s_0, s.t. $\forall s \in (0, s_0]$, we have $C_s > 0$. In all the context below, s_0 always refers to this definition.

We are now going to find a nontrivial solution to the PDE system as in [16]:

$$(7.12) \qquad D\psi = 0 \text{ with boundary condition } P_s^+ \psi = 0$$

where ψ is asymptotically constant and P_s^+ is the L^2 orthogonal projection on the space of eigenvectors of positive eigenvalues of $\slashed{D}$ on $\partial \overline{\mathcal{M}}_s$. Here $\slashed{D}$ is the 2-dimensional Dirac operator on the boundary, defined as:

$$-\slashed{D}\psi = \overline{e}_3 \cdot \overline{e}_1 \cdot \overline{\nabla}_{\overline{e}_1} \psi + \overline{e}_3 \cdot \overline{e}_2 \cdot \overline{\nabla}_{\overline{e}_2} \psi$$

where $\overline{\nabla}$ is the spin connection on the boundary.

Notice that the boundary is a topological 2-sphere when s is small, there is no $\not{D}$- harmonic spinor field on it. We denote the weighted Sobolev space

$$W^{1,2}_{-1}(\overline{\mathcal{M}}_s) = \{v \in W^{1,2}_{loc}(\overline{\mathcal{M}}_s) \mid |x|^{l-1}\overline{\nabla}^l v \in L^2(\overline{\mathcal{M}}_s), \quad l = 0, 1\}.$$

as a notation for both functions and spinors. We then define the space

(7.13) $$\mathcal{H}_s = \{\psi \in W^{1,2}_{-1}(\overline{\mathcal{M}}_s), P_s^+\psi = 0\}$$

This is a Hilbert space w.r.t the $W^{1,2}_{-1}$ norm.

The boundary term for $\partial\overline{\mathcal{M}}_s$ in the Weitzenböck formula can be rearranged as

(7.14) $$\int_{\partial\overline{\mathcal{M}}_s} < \overline{\nabla}_{\overline{e}_3}\psi + \overline{e}_3 \cdot D\psi, \psi >$$

$$= -\int_{\partial\overline{\mathcal{M}}_s} < \psi, \not{D}\psi > -\frac{1}{2}\int_{\partial\overline{\mathcal{M}}_s} \overline{H}_{\partial\overline{\mathcal{M}}_s}|\psi|^2$$

We will follow the steps of calculation in [18, Theorem 5.1] . First for any domain Ω and vector field X, the following equation holds:

$$\int_{\partial\Omega} < X, \nu_{out} > |\psi|^2 = \int_{\Omega} divX|\psi|^2 + \int_{\Omega} X(|\psi|^2)$$

By setting $X = q, \Omega = \overline{\mathcal{M}}_{s,r}$ in the above equation, the Weitzenböck formula can be rearranged as:

(7.15) $$\int_{\overline{\mathcal{M}}_{s,r}} |D\psi|^2 = \int_{\overline{\mathcal{M}}_{s,r}} |\overline{\nabla}\psi|^2 + \frac{1}{4}\int_{\overline{\mathcal{M}}_{s,r}} (\overline{R} + 2div_{\overline{g}}(q))|\psi|^2$$

$$+ \frac{1}{2}\int_{\overline{\mathcal{M}}_{s,r}} q(|\psi|^2) + \frac{1}{2}\int_{\partial\overline{\mathcal{M}}_s} q(\overline{e}_3)|\psi|^2$$

$$- \frac{1}{2}\int_{S_r} q(\nu_r)|\psi|^2 - \int_{\partial\overline{\mathcal{M}}_s} < \psi, \not{D}\psi >$$

$$- \frac{1}{2}\int_{\partial\overline{\mathcal{M}}_s} \overline{H}|\psi|^2 - \int_{S_r} < \overline{\nabla}_{\nu_r}\psi + \nu_r \cdot D\psi, \psi >$$

$$\geq \int_{\overline{\mathcal{M}}_{s,r}} |\overline{\nabla}\psi|^2 + \frac{1}{2}\int_{\overline{\mathcal{M}}_{s,r}} (|q|^2_{\overline{g}}|\psi|^2 + q(|\psi|^2))$$

$$+ \int_{\partial\overline{\mathcal{M}}_s} - < \psi, \not{D}\psi > -\frac{1}{2}\overline{H}|\psi|^2 + \frac{1}{2}q(\overline{e}_3)|\psi|^2$$

$$- \int_{S_r} \frac{1}{2}q(\nu_r)|\psi|^2 + < \overline{\nabla}_{\nu_r}\psi + \nu_r \cdot D\psi, \psi >$$

The second inequality of the above is induced by Eq.(3.4)

$$(7.16) \qquad \overline{R} - 2div_{\overline{g}}(q) \geq 2|q|_{\overline{g}}^2$$

Now if we set $X = q$ in the following inequality (c.f. [18]):

$$(7.17) \qquad |\overline{\nabla}\psi|^2 + |X|^2|\psi|^2 + X(|\psi|^2) \geq 0$$

as

$$X(|\psi|^2) = 2 < \overline{\nabla}_X\psi, \psi >\geq -2|\overline{\nabla}_X\psi||\psi| \geq -2|\overline{\nabla}\psi||X||\psi|$$

Then Ineq.(7.15) will imply:

$$(7.18) \qquad \int_{\overline{\mathcal{M}}_{s,r}} |D\psi|^2 \geq \frac{1}{2}\int_{\overline{\mathcal{M}}_{s,r}} |\overline{\nabla}\psi|^2$$
$$+ \int_{\partial\overline{\mathcal{M}}_s} - <\psi, \slashed{D}\psi> -\frac{1}{2}\overline{H}|\psi|^2 + \frac{1}{2}q(\overline{e}_3)|\psi|^2$$
$$- \int_{S_r} \frac{1}{2}q(\nu_r)|\psi|^2 + <\overline{\nabla}_{\nu_r}\psi + \nu_r \cdot D\psi, \psi>$$

For any spinor $\psi \in W_{-1}^{1,2}$, the Ineq.(7.18) becomes (because C_c^∞ in dense in $W_{-1}^{1,2}$):

$$(7.19) \qquad \int_{\overline{\mathcal{M}}_{s,r}} |D\psi|^2 \geq \frac{1}{2}\int_{\overline{\mathcal{M}}_{s,r}} |\overline{\nabla}\psi|^2$$
$$+ \int_{\partial\overline{\mathcal{M}}_s} - <\psi, \slashed{D}\psi> -\frac{1}{2}(\overline{H} - q(\overline{e}_3))|\psi|^2$$

Now for spinors ψ in $\mathcal{H}_s$, we decompose ψ in the eigenspace of $\slashed{D}$ as: $\psi = \sum_i \psi_i$, where ψ_i is the eigenfunctions w.r.t to α_i, i.e. $\slashed{D}\psi_i = \alpha_i\psi_i$. Under this decomposition, the boundary term of formula Ineq.(7.19) becomes:

$$\int_{\partial\overline{\mathcal{M}}_s} - <\psi, \slashed{D}\psi> -\frac{1}{2}\overline{H}|\psi|^2 + \frac{1}{2}q(\overline{e}_3)|\psi|^2$$
$$= \int_{\partial\overline{\mathcal{M}}_s} -\sum_i \alpha_i|\psi_i|^2 - \frac{1}{2}\overline{H}|\psi|^2 + \frac{1}{2}q(\overline{e}_3)\sum_i |\psi_i|^2$$
$$= \sum_i \int_{\partial\overline{\mathcal{M}}_s} (-\alpha_i - \frac{1}{2}\overline{H} + \frac{1}{2}q(\overline{e}_3))|\psi_i|^2$$

From the lower estimation for the absolute value of eigenvalues of Dirac operator on 2-sphere [23],[24], and together with the fact that our

boundary condition $P_s^+\psi = 0$ forbids any positive eigenvalues, we have $\alpha_i \leq -2\sqrt{\dfrac{\pi}{|\partial\overline{\mathcal{M}}_s|_{\overline{g}}}}$. Now for $s < s_0$, we have:

$$(7.20) \qquad -\alpha_i - \frac{1}{2}\overline{H} + \frac{1}{2}q(\overline{e}_3) \geq 2\sqrt{\frac{\pi}{|\partial\overline{\mathcal{M}}_s|_{\overline{g}}}} - \frac{1}{2}\overline{H} + \frac{1}{2}q(\overline{e}_3)$$

$$\geq \frac{1}{2}\sqrt{\frac{\pi}{|\partial\overline{\mathcal{M}}_s|_{\overline{g}}}}C_s > 0$$

Thus

$$(7.21) \qquad \int_{\partial\overline{M}} - <\psi, \slashed{D}\psi> -\frac{1}{2}\overline{H}|\psi|^2 + \frac{1}{2}q(\overline{e}_3)|\psi|^2$$

$$= \sum_i \int_{\partial\overline{M}_s} (-\alpha_i - \frac{1}{2}\overline{H} + \frac{1}{2}q(\overline{e}_3))|\psi_i|^2$$

$$\geq \sum_i \int_{\partial\overline{M}_s} \frac{1}{2}\sqrt{\frac{\pi}{|\partial\overline{\mathcal{M}}_s|_{\overline{g}}}}C_s|\psi_i|^2$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{|\partial\overline{\mathcal{M}}_s|_{\overline{g}}}}C_s \int_{\partial\overline{M}_s} |\psi|^2$$

$$\geq 0$$

This formula together with formula 7.19 implies the following estimate for spinors in $\mathcal{H}_s$:

$$(7.22) \qquad \int_{\overline{\mathcal{M}}_s} |D\psi|^2 \geq \frac{1}{2}\int_{\overline{\mathcal{M}}_s} |\overline{\nabla}\psi|^2 + \frac{1}{2}\sqrt{\frac{\pi}{|\partial\overline{\mathcal{M}}_s|_{\overline{g}}}}C_s \int_{\partial\overline{M}_s} |\psi|^2$$

$$\geq \frac{1}{2}\int_{\overline{\mathcal{M}}_s} |\overline{\nabla}\psi|^2$$

The right hand side of the above is the Hilbert norm on $W_{-1}^{1,2}$. This implies a coercive estimate for spinors in $\mathcal{H}$, which is enough to establish the existence of harmonic spinors that are asymptotic to constant spinor at infinity. We are then able to get a lower bound for the ADM mass of the initial data set:

Proposition 7.2. *Under Foliation B, we suppose*

$$\lambda\frac{C_1^2}{C_3^2}\left(1 + \frac{2C_1^4}{C_2^2C_3^2}\right) < 4,$$

where C_1, C_2, C_3 are constants appearing in the gradient estimates in Theorem 2.6. Also we suppose ψ_0 is a smooth spinor field which is constant in some chart around infinity and $P_s^+ \psi_0 = 0$. Then for $s < s_0$, there exists a unique $\psi \in \mathcal{H}_s$ s.t.

$$(7.23) \qquad D(\psi_0 + \psi) = 0, P_s^+(\psi_0 + \psi) = 0$$

Furthermore, we have the following lower bound for the ADM mass of the initial data set:

$$(7.24) \qquad 8\pi|\psi_0|^2 m \geq \int_{\overline{\mathcal{M}}_s} |\overline{\nabla}(\psi + \psi_0)|^2 + \sqrt{\frac{\pi}{|\partial\overline{\mathcal{M}}_s|_{\overline{g}}}} C_s \int_{\partial\overline{M}_s} |\psi + \psi_0|^2$$

Proof. We have already establish the existence of the solution of Eq.(7.23). Plugging $\psi + \psi_0$ back into Ineq.(7.18) will imply the following result:

$$(7.25) \quad \int_{S_r} \frac{1}{2} q(\nu_r) |\psi_0 + \psi|^2 + \; < \overline{\nabla}_{\nu_r}(\psi + \psi_0) + \nu_r \cdot D(\psi + \psi_0), \psi + \psi_0 >$$

$$\geq \frac{1}{2} \int_{\overline{\mathcal{M}}_s} |\overline{\nabla}(\psi + \psi_0)|^2 + \frac{1}{2} \sqrt{\frac{\pi}{|\partial\overline{\mathcal{M}}_s|_{\overline{g}}}} C_s \int_{\partial\overline{M}_s} |\psi + \psi_0|^2$$

We take limit $r \to \infty$ in this inequality. The decay rate of q implies that the first term of left hand side disappears under this limiting process. Together with the fact that

$$(7.26) \quad 4\pi|\psi_0|^2 m = \lim_{r \to \infty} \int_{S_r} < \overline{\nabla}_{\nu_r}(\psi + \psi_0) + \nu_r \cdot D(\psi + \psi_0), \psi + \psi_0 >$$

We then prove the following lower bound for the ADM mass:

$$(7.27) \qquad 8\pi|\psi_0|^2 m \geq \int_{\overline{\mathcal{M}}_s} |\overline{\nabla}(\psi + \psi_0)|^2 + \sqrt{\frac{\pi}{|\partial\overline{\mathcal{M}}_s|_{\overline{g}}}} C_s \int_{\partial\overline{M}_s} |\psi + \psi_0|^2$$

$\square$

7.4. Discussion about condition 2

In this section we prove that the similar conclusion still holds under Foliation C if Condition 2 in Theorem 2.10 holds. Under Foliation C, for γ small

enough, we set

$$C'_\gamma = \min_{\partial \mathcal{M}_\gamma} \left(4 - (\overline{H}_{\partial \overline{\mathcal{M}}_\gamma} - q(\bar{e}_3)) \sqrt{\frac{|\partial \overline{\mathcal{M}}_\gamma|_{\overline{g}}}{\pi}} \right)$$

where $\Sigma_\gamma = \Psi_C(\Sigma, \gamma)$ and $\overline{\Sigma}_\gamma$ is the lift of Σ_γ onto $\overline{M}$. $\overline{\Sigma}_\gamma$ can also be treated as the horizontal cuts of the cylindrical end.

Recall that Foliation C is defined as :

(7.28) $\Psi_C : \Sigma \times [0, \bar{\gamma}] \to \mathcal{M}$ such that
$$(1) \Psi_C(p, 0) = p \text{ for } p \in \Sigma$$
$$(2) \Psi_C(p, \gamma) = u_0(p, -\frac{1}{\sqrt{\lambda}} \log \gamma)$$

where u_0 is the graph of $\overline{\mathcal{M}}$ on the cylinder $\Sigma \times \mathbb{R}$.

Then, by Theorem 2.6 we have u_0 and $^{C_{\bar{z}}}\nabla u_0$ both uniformly converging to zero as $z \to \infty$. Thus together with the fact that $\overline{g}$ is the same as g on the level set of f, and $\lambda |\Sigma|_g \leq 4\pi$ (c.f. [14, Lemma 9.7]), we have

$$|\partial \overline{\mathcal{M}}_\gamma|_{\overline{g}} = |\partial \mathcal{M}_\gamma|_g \to |\Sigma|_g \leq \frac{4\pi}{\lambda} \text{ when } \gamma \to 0.$$

Moreover, it is proved in [31] that $\overline{H}_{\partial \overline{\mathcal{M}}_\gamma}$ goes to zero. More specifically, $\overline{H}_{\partial \overline{\mathcal{M}}_\gamma} = < e_3, \bar{e}_3 > H_{\partial \mathcal{M}_\gamma}$. This is because $\overline{g} = df_0^2 + g$ does not change the metric g on the level sets of f_0. It only stretches lengths perpendicular to the level sets of f_0 by a factor of $< e_3, \bar{e}_3 > = \frac{1}{\sqrt{1+|\nabla f_0|^2}}$. The formula $\overline{H}_{\partial \overline{\mathcal{M}}_\gamma} = < e_3, \bar{e}_3 > H_{\partial \mathcal{M}_\gamma}$ is then implied by the first variation formula for area. As we have proved that $|\nabla f_0|$ blows up, $\overline{H}_{\partial \overline{\mathcal{M}}_\gamma}$ goes to zero as $\gamma \to 0$.

From [1, Proposition 2] we know that $|q|_{\overline{g}}$ is bounded near Σ. If furthermore we suppose that the Condition 2 of Theorem 2.10 holds, i.e. $|q|_{\overline{g}} < 2\sqrt{\lambda}$ near Σ, then it is straightforward to see that under this assumption, there exists constant γ_0 s.t. $C'_\gamma > 0$ for $\forall \gamma \in (0, \gamma_0]$. In all the context below, γ_0 always refers to this definition.

Denote P'^+_γ to be the L^2 orthogonal projection on the space of eigenvectors of positive eigenvalues of the corresponding $\slashed{D}$ on $\partial \mathcal{M}_\gamma$, and denote

$$\mathcal{H}'_\gamma = \{\psi \in W^{1,2}_{-1}(\overline{\mathcal{M}}_\gamma), P'^+_\gamma \psi = 0\}.$$

Then similar arguments as in the previous section imply the following inequality for spinors in $\mathcal{H}'_\gamma$:

$$(7.29) \qquad \int_{\overline{\mathcal{M}}_\gamma} |D\psi|^2 \geq \frac{1}{2} \int_{\overline{\mathcal{M}}_\gamma} |\overline{\nabla}\psi|^2 + \frac{1}{2} \sqrt{\frac{\pi}{|\partial\overline{\mathcal{M}}_\gamma|_{\bar{g}}}} C'_\gamma \int_{\partial\overline{M}_\gamma} |\psi|^2$$

$$\geq \frac{1}{2} \int_{\overline{\mathcal{M}}_\gamma} |\overline{\nabla}\psi|^2$$

This implies a coercive estimate for spinors in $\mathcal{H}'_\gamma$. We then reach the same conclusion as in Proposition 7.2:

Proposition 7.3. *We suppose $|q|_{\bar{g}} < 2\sqrt{\lambda}$ near Σ. Also we suppose ψ_0 is a smooth spinor field which is constant in some chart around infinity and $P'^+_\gamma \psi_0 = 0$ under Foliation C. Then, for $\gamma < \gamma_0$, there exists a unique $\psi \in \mathcal{H}'_\gamma$ s.t.*

$$(7.30) \qquad\qquad D(\psi_0 + \psi) = 0, \, P'^+_\gamma(\psi_0 + \psi) = 0$$

Furthermore, we have the following lower bound for the ADM mass of the initial data set:

$$(7.31) \qquad 8\pi|\psi_0|^2 m \geq \int_{\overline{\mathcal{M}}_\gamma} |\overline{\nabla}(\psi + \psi_0)|^2 + \sqrt{\frac{\pi}{|\partial\overline{\mathcal{M}}_\gamma|_{\bar{g}}}} C'_\gamma \int_{\partial\overline{M}_\gamma} |\psi + \psi_0|^2$$

7.5. A Penrose-like inequality

We shall now perform the last step in proving a Penrose-like inequality. We need following lemmas:

Lemma 7.4. *Assume X and Y are two Banach space such that X is continuously embedded into Y. If X is reflexive and $\{x_n\}$ is a bounded sequence in X, then there is a subsequence $\{x_{n_k}\}$ weakly convergent to $x \in X$ both in X and Y.*

Proof. X is a reflexive Banach space, therefore bounded subsets are weakly precompact by Banach-Alaoglu theorem. The Eberlein-Smulian theorem implies that bounded subsets in X are weakly sequentially precompact, and therefore from $\{x_n\}$ we can extract a subsequence $\{x_{n_k}\}$ which is weakly converging to $x \in X$. Let J to be the embedding operator from X to Y. Then for $f \in Y^*$, $f \circ J \in X^*$ because J is continuous. Hence $f(x_n) = (f \circ J)(x_n)$ converges to $(f \circ J)(x) = f(x)$. Therefore, x_n converges weakly to x in Y. $\qquad\square$

Lemma 7.5. *Assume X and Z are two Banach space such that X is compactly embedded into Y. If X is reflexive and $\{x_n\}$ is bounded in X, then there is a subsequence $\{x_{n_k}\}$ convergent to $x \in X$, weakly in X and strongly in Z.*

Proof. For the same reason as above lemma, we can extract a subsequence $\{x_{n_k}\}$ which is weakly converging to $x \in X$. Because X is compactly embedded into Z, thus there is an element $z \in Z$ s.t. there is a subsequence of $\{x_n\}$, still denote as x_{n_k}, converges strongly to z in Z. Thus $\{x_{n_k}\}$ also converges weakly to z in Z. For the same reason as above lemma, x_{n_k} converges weakly to x in Z, thus x and z are equal. $\qquad\square$

Lemma 7.6. *Assume*

$$\sigma_s = \sqrt{\frac{|\overline{\Sigma}_s|_{\overline{g}}}{\pi}} \inf_{f \in C_c^\infty(\overline{\Sigma}_s)} \frac{|df|^2_{L^2(\overline{M}_s)}}{|f|^2_{L^2(\overline{\Sigma}_s)}} = \sqrt{\frac{|\overline{\Sigma}_s|_{\overline{g}}}{\pi}} \inf_{f \in W^{1,2}_{-1}(\overline{\Sigma}_s)} \frac{|df|^2_{L^2(\overline{M}_s)}}{|f|^2_{L^2(\overline{\Sigma}_s)}},$$

then $\sigma_s > 0$ for any $s > 0$. Here f denotes function which is not identical zero.

Proof. Assume $\{f_i\}$ is a sequence of functions minimizing σ_s, and satisfies $|f_i|_{L^2(\overline{\Sigma}_s)} = 1$. Because $W^{1,2}_{-1}(\overline{\mathcal{M}}_s)$ is compactly embedded into $L^2(\overline{\Sigma}_s)$, by above lemma, we can find a subsequence $\{f_{i_k}\}$ and $f \in W^{1,2}_{-1}$, such that $\{f_{i_k}\}$ converge to f weakly in $W^{1,2}_{-1}$, strongly in $L^2(\overline{\Sigma}_s)$. Thus $|f|_{L^2(\overline{\Sigma}_s)} = 1$, and $|df|_{L^2(\overline{\mathcal{M}}_s)} \leq \underline{\lim}_{k \to \infty} |df_{i_k}|_{L^2(\overline{\mathcal{M}}_s)} = \sigma_s$. Suppose that $\sigma_s = 0$, then $|f|_{L^2(\overline{\Sigma}_s)} = 1$, while $|df|_{L^2(\overline{\mathcal{M}}_s)} = 0$. So f must be a constant on $\overline{\mathcal{M}}_s$ while at the same time decay at infinity, which indicates that it can only be zero. This contradicts the fact that $|f|_{L^2(\overline{\Sigma}_s)} = 1$. $\qquad\square$

Theorem 7.7. *Under Foliation B, suppose*

$$\lambda \frac{C_1^2}{C_3^2}\left(1 + \frac{2C_1^4}{C_2^2 C_3^2}\right) < 4,$$

where C_1, C_2, C_3 are constants appearing in the gradient estimates in Theorem 2.6. Then for $s \in (0, s_0]$, we have the following lower bound for the ADM mass of the initial data set $(\mathcal{M}, g, K)$:

$$(7.32) \qquad\qquad m \geq \frac{\sigma_s C_s}{2(C_s + \sigma_s)} \sqrt{\frac{|\overline{\Sigma}_s|_{\overline{g}}}{16\pi}}$$

Proof. From Ineq.(7.24) in Proposition 7.2, and if for simplicity we assume $\lim_{r \to \infty} |\psi_0| = 1$, we will have:

$$(7.33) \qquad 8\pi m \geq \int_{\overline{\mathcal{M}}_s} |\overline{\nabla}\psi_1|^2 + \sqrt{\frac{\pi}{|\partial\overline{\mathcal{M}}_s|_{\overline{g}}}} C_s \int_{\partial\overline{M}_s} |\psi_1|^2$$

where $\psi_1 = \psi + \psi_0$ is a spinor such that $\lim_{r \to \infty} |\psi_1| = 1$.

We now denote functions $h = |\psi_1|$ and $v = h - 1$. Then $v \in W^{1,2}_{-1}$ as a function because of triangular formula. Because $\overline{\nabla}$ is compatible with $<,>$, we have:

$$|\overline{\nabla}|\psi||^2 = \frac{1}{4}|\psi|^{-2}|\overline{\nabla}(|\psi|^2)|^2 = |\psi|^{-2}|<\overline{\nabla}\psi,\psi>|^2 \leq |\overline{\nabla}\psi|^2$$

The above inequality then implies:

$$(7.34) \qquad 8\pi m \geq \int_{\overline{\mathcal{M}}_s} |\overline{\nabla}\psi_1|^2 + \sqrt{\frac{\pi}{|\partial\overline{\mathcal{M}}_s|_{\overline{g}}}} C_s \int_{\partial\overline{M}_s} |\psi_1|^2$$
$$\geq \int_{\overline{\mathcal{M}}_s} |\overline{\nabla}|\psi_1||^2 + \sqrt{\frac{\pi}{|\partial\overline{\mathcal{M}}_s|_{\overline{g}}}} C_s \int_{\partial\overline{M}_s} |\psi_1|^2$$
$$= \int_{\overline{\mathcal{M}}_s} |dv|^2 + \sqrt{\frac{\pi}{|\partial\overline{\mathcal{M}}_s|_{\overline{g}}}} C_s \int_{\partial\overline{M}_s} (1+v)^2$$

By Young's Inequality,

$$(7.35) \qquad (1+v)^2 \geq 1 - \epsilon^{-1} + (1-\epsilon)v^2, \forall \epsilon > 0$$

From definition of σ_s we know that

$$(7.36) \qquad |dv|^2_{L^2(\overline{M}_s)} \geq \sigma_s \sqrt{\frac{\pi}{|\overline{\Sigma}_s|_{\overline{g}_s}}} |v|^2_{L^2(\overline{\Sigma}_s)}$$

Thus put together all the above inequalities and set $\epsilon = 1 + C_s^{-1}\sigma_s$, we get the following Penrose-like inequality for m:

$$8\pi m \geq \int_{\overline{\mathcal{M}}_s} |dv|^2 + \sqrt{\frac{\pi}{|\partial\overline{\mathcal{M}}_s|_{\overline{g}}}} C_s \int_{\partial\overline{M}_s} (1+v)^2$$
$$\geq |dv|^2_{L^2(\overline{M}_s)} + \sqrt{\frac{\pi}{|\partial\overline{\mathcal{M}}_s|_{\overline{g}}}} C_s \left((1-\epsilon^{-1})|\partial\overline{\mathcal{M}}_s|_{\overline{g}} + (1-\epsilon)|v|^2_{L^2(\partial\overline{\mathcal{M}}_s)}\right)$$

$$= \sqrt{\frac{\pi}{|\partial\overline{\mathcal{M}}_s|_{\overline{g}}}} C_s(1-\epsilon^{-1})|\partial\overline{\mathcal{M}}_s|_{\overline{g}} + |dv|^2_{L^2(\overline{M}_s)}$$

$$+ \sqrt{\frac{\pi}{|\partial\overline{\mathcal{M}}_s|_{\overline{g}}}} C_s(1-\epsilon)|v|^2_{L^2(\partial\overline{\mathcal{M}}_s)}$$

$$\geq \sqrt{\frac{\pi}{|\partial\overline{\mathcal{M}}_s|_{\overline{g}}}} \left(C_s(1-\epsilon^{-1})|\partial\overline{\mathcal{M}}_s|_{\overline{g}} + (\sigma_s + C_s(1-\epsilon))|v|^2_{L^2(\partial\overline{\mathcal{M}}_s)} \right)$$

$$= \frac{\sigma_s C_s}{C_s + \sigma_s} \sqrt{\pi|\partial\overline{\mathcal{M}}_s|_{\overline{g}}}$$

□

The similar reasoning holds for Foliation C if Condition 2 of Theorem 2.10 holds, and thus we have a Penrose-like inequality following Proposition 7.3:

Theorem 7.8. *Under Foliation C, denote*

$$\sigma'_\gamma = \sqrt{\frac{|\overline{\Sigma}_\gamma|_{\overline{g}}}{\pi}} \inf_{f\in C_c^\infty(\overline{\Sigma}_\gamma)} \frac{|df|^2_{L^2(\overline{M}_\gamma)}}{|f|^2_{L^2(\overline{\Sigma}_\gamma)}} = \sqrt{\frac{|\overline{\Sigma}_\gamma|_{\overline{g}}}{\pi}} \inf_{f\in W_{-1}^{1,2}(\overline{\Sigma}_\gamma)} \frac{|df|^2_{L^2(\overline{M}_\gamma)}}{|f|^2_{L^2(\overline{\Sigma}_\gamma)}},$$

then $\sigma'_\gamma > 0$ for any $\gamma > 0$. If we suppose $|q|_{\overline{g}} < 2\sqrt{\lambda}$ near Σ, then for $\gamma \in (0, \gamma_0]$, we have the following lower bound for the ADM mass of the initial data set $(\mathcal{M}, g, K)$:

$$(7.37) \qquad m \geq \frac{\sigma'_\gamma C'_\gamma}{2(C'_\gamma + \sigma'_\gamma)} \sqrt{\frac{|\overline{\Sigma}_\gamma|_{\overline{g}}}{16\pi}}$$

We now denote $\theta_s = \dfrac{\sigma_s C_s}{2(C_s + \sigma_s)} \sqrt{\dfrac{|\partial\overline{\mathcal{M}}_s|_{\overline{g}}}{|\partial\mathcal{M}|_g}}$, $\theta'_\gamma = \dfrac{\sigma'_\gamma C'_\gamma}{2(C'_\gamma + \sigma'_\gamma)} \sqrt{\dfrac{|\partial\overline{\mathcal{M}}_\gamma|_{\overline{g}}}{|\partial\mathcal{M}|_g}}$.

Denote

$$(7.38) \qquad \theta = \begin{cases} \sup_{s\in(0,s_0)} \theta_s & \text{if Condition 1 holds,} \\ \sup_{\gamma\in(0,\gamma_0)} \theta'_\gamma & \text{if Condition 2 holds.} \end{cases}$$

Then either Condition 1 or 2 of Theorem 2.10 implies that there is a constant $\theta > 0$ such that

$$(7.39) \qquad m \geq \theta \sqrt{\frac{|\Sigma|_g}{16\pi}}$$

7.6. Schwarzschild time symmetric case

Now we calculate the value of θ in Schwarzschild time symmetric case. Suppose now our initial data set is the Schwarzschild time symmetric slice, and for simplicity we assume $m = 1$. Following J. Metzger's construction procedure in [2, Theorem 3.1], each solution f_t of $\mathcal{J}[f_t] = tf_t$ with Dirichlet boundary condition under spherically symmetric setting will still be spherically symmetric. Thus the Jang's slice $\overline{\mathcal{M}} = graph\,f_0$, which is the limiting manifold of $graph\,f_t$ when $t \to 0$, is also spherically symmetric. It is then straightforward to find the only spherically symmetric solution f_0 of $\mathcal{J}[f_0] = 0$ on $\mathcal{M}$ such that it blows up at $r = 2$, and decays to zero at infinity. By solving the ODE, we know f_0 satisfies:

$$f_0'(r) = -\sqrt{\frac{16r}{(r-2)(r^4-16)}}$$

We use the coordinate system $\{r, \theta, \phi\}$ for both $\mathcal{M}$ and $\overline{\mathcal{M}}$. First of all, $\frac{\partial}{\partial r}\theta^+(S_r)|_{r=2} > 0$, thus the horizon $\{r = 2\}$ is outermost strictly stable MOTS. Also under this coordinate system, the distance function τ to the horizon $\{r = 2\}$ on $\mathcal{M}$ is of order $\sqrt{r-2}$, thus

$$\log \tau = \frac{1}{2}\log(r-2) + O(1).$$

Together with the fact that

$$f_0(r) = -\log(r-2) + O(1) = -2\log \tau + O(1),$$

by Theorem 1 the principal eigenvalue of the horizon $\{r = 2\}$ in $\mathcal{M}$ must be $\lambda = \frac{1}{4}$.

Now the induced metric $\overline{g}$ on $\overline{\mathcal{M}}$ is

$$\overline{g} = g + df_0^2 = \frac{r^5}{(r-2)(r^4-16)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2,$$

and it can be calculated that:

$$|\overline{\Sigma}_r|_{\bar{g}} = 4\pi r^2$$

$$\overline{H}_{\overline{\Sigma}_r} = 2\sqrt{\frac{(r-2)(r^4-16)}{r^7}}$$

$$q(\bar{e}_3) = -\frac{32}{\sqrt{r^7(r^3+2r^2+4r+8)}}$$

Thus

$$C_r = 4 - (\overline{H}_{\partial\overline{\mathcal{M}}_r} - q(\bar{e}_3))\sqrt{\frac{|\partial\overline{\mathcal{M}}_r|_{\bar{g}}}{\pi}}$$

$$= 4 - 4\sqrt{\frac{(r-2)(r^4-16)}{r^5}} - \frac{64}{\sqrt{r^5(r^3+2r^2+4r+8)}}$$

C_r is positive for $\forall r \geq 2$.

The following lemma gives the value of σ_r:

Lemma 7.9. *Suppose (M, g) is a 3-dim spherically symmetric Riemannian manifold equipped with $g = F^2(r)(r)dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2$ and boundary $\partial M = \{r = r_0\}$. Then for any $f \in C_c^\infty(M)$ such that $|f|_{L^2(\partial M)} = 1$, we have*

$$(7.40) \qquad |df|_{L^2(M)}^2 \geq \frac{1}{r_0^2 \int_{r_0}^\infty \frac{F(r)}{r^2}}$$

and the equality holds if and only if f is the spherically symmetric harmonic solution on M.

Proof. The inequality can be proved by Fubini's theorem and Cauchy-Schwarz Inequality:

$$\int_M |df|^2 = \int d\theta \int d\phi \int_{r_0}^\infty |df|^2 \sqrt{detg}dr$$

$$\geq \int d\theta \int d\phi \int_{r_0}^\infty \frac{1}{F^2(r)}(\partial_r f(r,\theta,\phi))^2 \sqrt{detg}dr$$

$$= \int \sin\theta d\theta d\phi \int_{r_0}^\infty (\partial_r f(r,\theta,\phi))^2 \frac{r^2}{F(r)}$$

$$\geq \iint \sin\theta d\theta d\phi \frac{(\int_{r_0}^\infty \partial_r f(r,\theta,\phi))^2}{\int_{r_0}^\infty \frac{F(r)}{r^2}}$$

$$= \frac{1}{\int_{r_0}^{\infty} \frac{F(r)}{r^2}} \iint f(r_0, \theta, \phi)^2 \sin\theta d\theta d\phi$$

$$= \frac{1}{\int_{r_0}^{\infty} \frac{F(r)}{r^2}} \frac{|f|^2_{L^2(\partial M)}}{r_0^2} = \frac{1}{r_0^2 \int_{r_0}^{\infty} \frac{F(r)}{r^2}}$$

The equality holds if and only if $f = f(r)$ and $f'(r) = C\frac{F(r)}{r^2}$. It can be checked that if this happens then $\Delta_M f = 0$.   □

It is then straightforward to calculate the value of σ_r by Lemma 7.9:

$$\sigma_r = \sqrt{\frac{|\overline{\Sigma}_r|_{\bar{g}}}{\pi}} \inf_{f \in C_c^{\infty}(\overline{\Sigma}_r)} \frac{|df|^2_{L^2(\overline{M}_r)}}{|f|^2_{L^2(\overline{\Sigma}_r)}}$$

$$= \frac{2}{r \int_r^{\infty} \sqrt{\frac{x}{(x-2)(x^4-16)}} dx}$$

Then

$$\theta_r = \frac{\sigma_r C_r}{2(C_r + \sigma_r)} \sqrt{\frac{|\partial \overline{\mathcal{M}}_r|_{\bar{g}}}{|\partial \mathcal{M}|_g}}$$

$$= \frac{r}{4(\sigma_r^{-1} + C_r^{-1})}$$

$$= \frac{r}{2r \int_r^{\infty} \sqrt{\frac{x}{(x-2)(x^4-16)}} dx + \frac{1}{1 - \sqrt{\frac{(r-2)(r^4-16)}{r^5}} - \frac{16}{\sqrt{r^5(r^3+2r^2+4r+8)}}}}$$

θ_r is monotonically increasing in r, $\lim_{r\to 2} \theta_r = 0$, $\lim_{r\to\infty} \theta_r = 1$. Although $\lim_{r\to 2} \theta_r = 0$ is not a good property, θ_r quickly becomes significantly non-zero when it leaves away from the horizon $\{r = 2\}$. We list a few numerical results in Table 1.

Thus for example if we look at the domain $U = \{r \in (2, 2.1)\}$, then $\theta = \sup_U \theta_r \approx 0.5$.

However, the property that $\lim_{r\to 2} \theta_r = 0$ is still not a good one. This property means that when we approach the cylindrical end we will lose more and more information, which contradicts our intuition. The following calculation shows that most of the information loss happens at the last step: when we apply Young's Inequality together with capacity.

We first calculate the Green's function of the Dirac operator on $\mathcal{M}$.

Table 1: Numerical Values of θ_r

r	θ_r
2.001	0.3198
2.01	0.3922
2.1	0.5084
2.5	0.6466
3.0	0.7292

Lemma 7.10. *Suppose (M, g) is a 3-dim spherically symmetric Riemannian manifold equipped with $g = F^2(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$ and boundary $\partial M = \{r = r_0\}$. Denote D to be the Dirac operator on M. Then the following:*

$$\psi = c_1 e^{\frac{i\phi}{2}} \begin{bmatrix} \cos \frac{\theta}{2} \\ -\sin \frac{\theta}{2} \end{bmatrix} h(r) + c_2 e^{-\frac{i\phi}{2}} \begin{bmatrix} \sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{bmatrix} h(r)$$

is a solution of $D\psi = 0$. Here c_1, c_2 are complex numbers, and

$$h(r) = Ce^{-\int_r^\infty \frac{F(s)-1}{s} ds}$$

where C is a real constant. Furthermore, we have $|\psi| = (|c_1|^2 + |c_2|^2)h(r)$, and $\not D \psi = -\frac{1}{r_0}\psi$ on the boundary $\partial M = \{r = r_0\}$

Proof. Denote the associated orthonormal frame $\{e_i\}$ and coframe $\{\omega_i\}$ on M by:

$$e_1 = \frac{1}{r} \frac{\partial}{\partial \theta}, \qquad e_2 = \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}, \qquad e_3 = \frac{1}{F(r)} \frac{\partial}{\partial r}.$$

and

$$\omega_1 = rd\theta, \qquad \omega_2 = r \sin \theta d\phi, \qquad \omega_3 = F(r)dr.$$

The connection 1-form $\{\omega_{ij}\}$ is given by $d\omega_i = -\omega_{ij} \wedge \omega_j$. Calculation shows:

$$\omega_{31} = -\frac{1}{rF(r)}\omega_1, \qquad \omega_{32} = -\frac{1}{rF(r)}\omega_2, \qquad \omega_{12} = -\frac{\cot \theta}{r}\omega_2.$$

The spin connection is given by:

$$\nabla = d - \frac{1}{4}\omega_{ij} \otimes e_i \cdot e_j = d - \frac{1}{2}\omega_{12} \otimes e_1 \cdot e_2 - \frac{1}{2}\omega_{13} \otimes e_1 \cdot e_3 \cdot - \frac{1}{2}\omega_{23} \otimes e_2 \cdot e_3$$

Fix the following Pauli matrix throughout this section

$$e_1 \to \begin{bmatrix} & i \\ i & \end{bmatrix}, \quad e_2 \to \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, \quad e_3 \to \begin{bmatrix} i & \\ & -i \end{bmatrix}.$$

Then the Dirac operator is:

$$\begin{aligned}
D &= e_1 \cdot \nabla_{e_1} + e_2 \cdot \nabla_{e_2} + e_3 \cdot \nabla_{e_3} \\
&= e_3 \cdot \frac{1}{F(r)}\frac{\partial}{\partial r} + e_1 \cdot \frac{1}{r}\frac{\partial}{\partial \theta} + e_2 \cdot \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \\
&\quad + \frac{1}{2rF(r)}(e_1 \cdot e_3 \cdot e_1 + e_2 \cdot e_3 \cdot e_2) + \frac{\cot\theta}{2r}e_2 \cdot e_1 \cdot e_2
\end{aligned}$$

Then by separation of variable, we can find a solution of $D\psi = 0$:

$$\psi = c_1 e^{\frac{i\phi}{2}}\begin{bmatrix} \cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} \end{bmatrix} h(r) + c_2 e^{-\frac{i\phi}{2}}\begin{bmatrix} \sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{bmatrix} h(r)$$

where c_1, c_2 are complex numbers, and $h(r)$ is a real-value function satisfies:

$$rh'(r) = (F(r) - 1)h(r)$$

then

$$h(r) = Ce^{-\int_r^\infty \frac{F(s)-1}{s}ds}$$

It can be checked that $|\psi| = (|c_1|^2 + |c_2|^2)h(r)$. Also on the boundary $\partial M = S_{r_0}$, the orthonormal frame will be $\{e_1, e_2\}$, and $\nu = e_3$. It is then straightforward to get the following result on S_{r_0}:

$$\psi = e_3 \cdot e_1 \cdot \nabla_{e_1}^{S_{r_0}}\psi + e_3 \cdot e_2 \cdot \nabla_{e_2}^{S_{r_0}}\psi = \frac{1}{r_0}\psi$$

where $\nabla^{S_{r_0}}$ is the spin connection on the boundary.

Thus $\slashed{D}\psi = -\frac{1}{r_0}\psi$. □

We apply the above lemma to the Jang's slice $\overline{\mathcal{M}}$, then $F(r) = \sqrt{\dfrac{r^5}{(r-2)(r^4-16)}}$. If we require that $h(r) \to 1$ when $r \to \infty$, then

$$h(r) = e^{-\int_r^\infty \frac{F(s)-1}{s} ds}$$

Under this setting ψ is constant at infinity. It can also be checked that $h(r) \to 0$ when $r \to 2$, and $h(r) \to 1$ when $r \to \infty$. Thus if we set $|c_1|^2 + |c_2|^2 = 1$, then $|\psi| \to 1$ when $r \to \infty$.

Then

$$\lim_{r \to 2} \left(\int_{\overline{\mathcal{M}}_r} |dh|^2 + \sqrt{\frac{\pi}{|\partial \overline{\mathcal{M}}_r|_{\bar{g}}}} C_r \int_{\partial \overline{M}_r} h^2 \right)$$

$$= \lim_{r \to 2} \int_{\overline{\mathcal{M}}_r} |dh|^2$$

$$= \iint d\theta d\phi \int_2^\infty \frac{1}{F(r)} (h'(r))^2 \sqrt{det g}\, dr$$

$$= 4\pi \int_2^\infty \frac{r^2}{\sqrt{F(r)}} (h'(r))^2 dr$$

By numerical calculation $\int_2^\infty \frac{r^2}{\sqrt{F(r)}} (h'(r))^2 dr \approx 0.6795$. Although it is not sharp (sharp value should be 2), it is not zero, compared to the fact that $\lim_{r \to 2} \theta_r = 0$.

There are some other information loss. One of them happens when we are applying the Ineq.(7.17):

(7.41) $$|\overline{\nabla}\psi|^2 + |q|^2|\psi|^2 + q(|\psi|^2) \geq 0$$

Another happens when we are using the inequality $|\overline{\nabla}|\psi|| \leq |\overline{\nabla}\psi|$:
One can calculate that:

$$\lim_{r \to 2} \left(\int_{\overline{\mathcal{M}}_r} |\overline{\nabla}\psi|^2 + \sqrt{\frac{\pi}{|\partial \overline{\mathcal{M}}_r|_{\bar{g}}}} C_r \int_{\partial \overline{M}_r} |\psi|^2 \right) \approx 4\pi * 1.0193$$

which is a little sharper than before.

7.7. Conclusion

We prove a Penrose-like inequality $m \geq \theta \sqrt{\dfrac{|\Sigma|_g}{16\pi}}$ under two different conditions on a one-end asymptotically flat 3-dim initial data set $(\mathcal{M}, g, k)$

with boundary $\partial\mathcal{M}$, which is a connected compact strictly stable outermost MOTS. We are able to keep most of the information about the cylindrical end after solving Dirac equation Eq.(7.23). The most serious information loss happens at the last step, when we are using capacity to extract an area term. This is caused by the essential difference between the Green's function for Laplacian and the Green's function for Dirac operator.

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