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Existence and uniqueness of compact rotating configurations in GR in second order perturbation theory

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Existence and uniqueness of rotating fluid bodies in equilibrium is still poorly understood in General Relativity (GR). Apart from the limiting case of infinitely thin disks, the only known global results in the stationary rotating case (Heilig [14] and Makino [21]) show existence in GR nearby a Newtonian configuration (under suitable additional restrictions). In this work we prove existence and uniqueness of rigidly (slowly) rotating fluid bodies in equilibrium to second order in perturbation theory in GR. The most widely used perturbation framework to describe slowly rigidly rotating stars in the strong field regime is the Hartle-Thorne model. The model involves a number of hypotheses, some explicit, like equatorial symmetry or that the perturbation parameter is proportional to the rotation, but some implicit, particularly on the structure and regularity of the perturbation tensors and the conditions of their matching at the surface. In this work, with basis on the gauge results obtained in [25], the Hartle-Thorne model is fully derived from first principles and only assuming that the perturbations describe a rigidly rotating finite perfect fluid ball (with no layer at the surface) with the same barotropic equation of state as the static ball. Rigidly rotating fluid balls are analyzed consistently in second order perturbation theory by imposing only basic differentiability requirements and boundedness. Our results prove in particular that, at this level of approximation, the spacetime must be indeed equatorially symmetric and is fully determined by two parameters, namely the central pressure and the uniform angular velocity of the fluid.

1. Introduction

Equilibrium configurations of self-gravitating rotating fluid bodies is an important and difficult subject in Einstein's theory of general relativity. From a physical perspective, they model astrophysical objects of finite size with strong gravitational fields, such as compact stars. One aspect of the problem is to construct and study physically realistic examples. Here the main tools are numerical methods and perturbation approaches. Another important aspect is to understand structural issues such as existence and uniqueness properties of the model. Several approaches have considered the exterior and interior problems separately. Existence results for the Dirichlet problem for both the interior and the exterior problems on fixed boundaries have been estalished in [36, 37], while the geometric uniqueness of the exterior problem given an interior metric has been proved in [26, 41] (see [42] for the Einstein-Maxwell case). In the perturbative setting (to second order) the constraints on the Cauchy data coming from the interior problem that need to be imposed to guarantee existence and uniqueness of the exterior are known [20].

Concernig the global problem, static (non rotating) and spherically symmetric perfect fluid bodies in General Relativity (GR) are known to exist and be unique given an equation of state satisfying some mild conditions and the value of the central pressure [35] (see also [30]). The solution is either of infinite extent (and then the energy density vanishes at infinity), or of finite extent so that it can be matched to Schwarzschild. Much less is known in the rotating case, for which we do not even have a single explicit solution describing a rotating finite object with its corresponding asymptotically flat exterior, except in the limiting case of infinitely thin disks [18, 28]. In the rotating global problem we only have results on existence of solutions sufficiently close to Newtonian configurations [14, 21]. In fact, even in the simpler Newtonian context the problem is highly non-trivial and a subject of active current research [15, 16, 39, 40]. Existence results for rotating configurations of other matter models have been established for Vlasov matter [3] and elastic bodies [1].

As a step forward towards establishing an existence and uniqueness proof of rotating configurations in GR far away from Newtonian regimes we analyze the problem for "slowly" rotating perfect fluid bodies in the context of second order perturbation theory in GR. In informal terms our main result is (see Theorem 7.3 for a precise version)

Theorem 1.1. *Given a static and spherically symmetric perfect fluid body of finite extent in General Relativity with central pressuce P_c, there exists a solution of the second order perturbed field equations in General Relatity satisfying:*

(a) *The perturbation is stationary and axially symmetric.*

(b) *The interior is a rigidly rotating perfect fluid with central pressure P_c and with the same barotropic equation of state as the spherical body.*

(c) *The exterior is vacuum (without cosmological constant) and bounded at infinity.*

(d) *The matching conditions (with absence of surface layers) are fulfilled at the boundary of the body.*

Moreover, the solution is uniquely determined by the angular velocity of the fluid, the configuration is equatorially symmetric and the boundary of the *body is stationary and axially symmetric. In addition, if the first order perturbation is non zero, then the perturbation parameter can be taken to be proportional to the angular velocity.*

This result is interesting in two respects. Firstly, we hope it can pave the way for applying an implicit function method to show existence of rotating configurations near any static spherical model in the fully non-linear theory and in the strong field regime. Secondly, as already mentioned, perturbation methods are widely used to study slowly rotating fluids. The literature in the subject is vast and, to a large extent, is based on the perturbation framework put forward by Hartle [11] and Hartle and Thorne [13] in the 60's. Under suitable extra assumptions (some explicit and some implicit) these works provided plausibility arguments towards the validity of the theorem above and, in fact, this validity has been taken for granted in the literature since then (not only in the Hartle-Thorne approach, but also in related or other perturbation methods, e.g. [6, 8]). Given the importance of the Hartle-Thorne approach we prove our theorem in their setup and therefore provide a rigorous proof for its validity, once the relevant correction found in [33] is incorporated. Thus, our theorem provides a rigorous and firm basis for all the results based on perturbations à la Hartle-Thorne where either the correction in [33] is irrelevant, or has already been taken into account. This applies in particular to the well known scalability property of the perturbative models widely used in astrophysics (see e.g. [5]).

We have just mentioned explicit and implicit extra assumptions, as well as plausibility arguments, in the Hartle-Thorne approach. Let us be more specific on this. By extra explicit assumptions we mean equatorial symmetry and that the perturbation parameter is proportional to the angular velocity. To discuss the extra implicit assumptions, let us review some basic facts about perturbation theory in GR. Perturbations to second order around a background spacetime (M, g) are described by two (symmetric and 2-covariant) tensors on M, K_1 and K_2, for the first and second order respectively. The perturbed metric corresponds to the one-parameter family g_ε given by

$$g_\varepsilon = g + \varepsilon K_1 + \frac{1}{2}\varepsilon^2 K_2 + O(\varepsilon^3),$$

where ε is a small parameter, called "perturbation parameter". Given a static and spherically symmetric background configuration

$$g = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

in static and spherical coordinates, the first part of the classical studies consisted in restricting a priori the form of the perturbation tensors. Besides on the condition of stationarity and axial symmetry, this step was based primarily on physical arguments relying on how different metric components should be excited at different orders in perturbation theory. This, combined with a convenient gauge choice (based on a suitable form of stationary and axisymmetric metrics [12]), was used to write perturbation tensors (see e.g. [11], [6]) as[1]

$$\begin{aligned} K_1^H &= 2\omega(r,\theta) r^2 \sin^2\theta dt d\phi, \\ K_2^H &= \left(-4e^{\nu(r)}h(r,\theta) + 2r^2\sin^2\theta\omega^2(r,\theta)\right) dt^2 + 4e^{\lambda(r)}m(r,\theta)dr^2 \\ &\quad + 4k(r,\theta)r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \end{aligned}$$

in terms of four functions $\{\omega, h, m, k\}$. Moreover, these four "perturbation functions" were assumed to cover both the interior and the exterior of the fluid ball and to be *continuous (and ω also with continuous first derivatives)* across the boundary of the fluid, located at $r = a, a \in \mathbb{R}$. This implicit assumption was combined with some plausibility arguments based on the field equations for a rigidly rotating perfect fluid and vacuum in order to achieve an important simplification of the *angular structure* of the functions, namely, that $\omega = \omega(r)$ and that the expansion of the other functions in terms of Legendre polynomials contains only the $\ell = 0, 1, 2$ components. The $\ell = 1$ components were made to vanish by assuming equatorial symmetry.

Given this setting, the (perturbed) field equations for a rigidly rotating perfect fluid with the barotropic equation of state of the background in the interior and vacuum in the exterior were studied assuming (again implicitly) that (i) the functions in K_1^H and K_2^H are bounded at the origin $r = 0$, (ii) satisfy the aforementioned continuity conditions at $r = a$, and (iii) are zero at infinity. Under these assumptions, plus a fixed value of the central pressure, it is argued that the field equations yield unique solutions depending on a single scaling parameter that can thus be absorbed in the perturbation parameter ε.

While point (iii) is justified quite directly by demanding an asymptotically flat metric g_ε, the rest of implicit assumptions were not rigorously established as necessary consequences of the problem under consideration. In addition, the physical and plausibility arguments must be replaced by rigorous arguments. These were the tasks we set up ourselves to do. Actually, our results hold under weaker requirements, since *the only global assumption*

[1] The function m here corresponds to $e^\lambda m/r$ in [11, 13].

we shall need is that the perturbation tensors stay bounded. Asymptotic flatness turns out to be a consequence of boundedness and the field equations.

In a first approach to this problem [33] two of us dropped assumption (ii) regarding the "matching" of the functions at $r = a$, by resorting to the general perturbed matching theory to second order developed in [22]. It was found that the point (ii) is inconsistent with the rest of the setting (this is the correction alluded to above). More precisely, there is a gauge in which ω is indeed C^1 and h, k are C^0 at the surface, but the function m presents a jump proportional to the value of the energy density at the surface. This fact has consequences in the computation of the mass in terms of the radius (see [31, 32]).[2]

Point (i) above or, more specifically, the issue of existence of a suitable gauge that transforms a general stationary, axially symmetric and orthogonally transtive (see below for definitions) first and second order perturbation tensor into a suitable canonical form, while keeping under control their differentiability and boundedness properties, turned out to be a much harder task than originally expected. This problem has been solved in [25], where we prove that the canonical form can be achieved with the loss of only one derivative and keeping all the relevant quantities bounded near the origin. This is the content of Theorem 6.3 in [25] and its Corollary 6.4, which here we collect together as Theorem 2.2. This result is of a purely geometric nature (i.e. independent of any field equations) and yields a "canonical form" that is still more general than the form of K_1^H and K_2^H above.

This "canonical form" carries an associated gauge freedom, which is identified in Proposition 6.9 in [25] and recovered here as Proposition 2.5

The results in [25] are the starting point of the present paper, where we derive rigorously the Hartle-Thorne model without not only any ad-hoc or implicit assumption, but also without assuming *equatorial symmetry* nor any *a priori* relationship whatsoever between the perturbation parameter and the angular velocity.

The proof starts with three basic steps, to be carried out both at first and second order: (1) obtain the field equations in terms of a set of convenient functions that encode all the necessary information to solve the interior and exterior problems, (2) solve the perturbed matching conditions for the perturbation tensors to first and second order in terms of those functions together with the functions that describe the deformation of the surface of the star, and (3) join the interior and exterior problems at the common

[2]It is worth mentioning that this correction is present, although it was somehow forgotten, in the original Newtonian approach by Chandrasekhar [9], see [34].

boundary Σ. Then, using elliptic methods that exploit the regularity and boundedness properties of the "canonical form", the analysis of the interior and exterior problems at each order follows by (a) proving that a number of relevant homogeneous problems only accept the trivial solution, (b) using the remaining gauge freedom left (at each stage) to get rid of spurious solutions, and finally, (c) proving existence and uniqueness of the remaining problems. We stress that the perturbed matching problem in step (b) is solved without imposing any a priori condition. In particular, we allow the deformation of the surface to be non-axially symmetric and time-dependent. It is the global problem itself that, a posteriori, forces the deformation of the body to be stationary and axially symmetric.

Although this procedure needs to be applied firstly to the first order problem and then to the second order, we follow a strategy that allows to treat both cases at once. This strategy is based on a bootstrap-type argument based on the fact that a second order perturbation problem with identically vanishing first order perturbation tensor is formally equivalent to a first order problem. We thus set up without a priori justification a very specific form for the first order perturbation tensor (which in fact corresponds to K_1^H with $\omega(r)$) and solve the second order problem under this assumption. We call this the *base global perturbation scheme*. The bootstrap argument closes by showing that this problem, when restricted to an identically vanishing K_1^H implies that the second order perturbation tensor must necessarily take the form assumed in the base perturbation scheme. In other words, the first order global problem is a particular case of the bootstrap argument, applied with a vanishing first order tensor, while the second order global problem becomes then the bootstrap argument itself.

1.1. Plan of the paper

The paper is structured as follows. In Section 2 we set up the stage by recalling the definition of static and spherically symmetric spacetime and establishing our basic set of global and differentiability assumptions. Next, we state the two main results of our previous paper [25] concerning the structure of stationary and axially symmetric perturbations. Theorem 2.2, establishes the regularity and differentiability properties of the functions when the perturbation tensors are cast in "canonical form", while Proposition 2.5 provides the full class of gauge transformations that preserve the form of the perturbation tensors in the later *base perturbation scheme*.

In Section 3 we establish the background spacetime; a static and spherically symmetric spacetime containing two regions matched across a hypersurface that preserves the symmetries. One of the regions solves the field equations for a perfect fluid with a barotropic equation of state and non-negative energy-density and pressure, and the other one is just Schwarzschild. Such a background is called *perfect fluid ball configuration* (Definition 3.1).

Section 4 is devoted to writing down the second order perturbed field equations, derived from the Einstein field equations, first for a general fluid and then particularizing to the rigidly rotating case. We also recall a well-known result on the relationship between rigid rotation and orthogonal transtivity of the group action, which is needed to make contact with the geometric results in [25]. Finally we find the consequences of the imposition of a barotropic equation of state (the same as for the background) at the level of the perturbed field equations.

In Section 5 we set up and ellaborate the global *base perturbation scheme*, which lies at the basis of the boostrap-type argument described above. The section starts with a detailed description of the a priori assumptions that define the base scheme. All these assumptions are justified later as part of the bootstrap argument. We split the assumptions into five blocks, B1 to B5, because several intermediate results only require a subset thereof. The next step, developed in Subsection 5.1, is to write down the explicit form of the field equations, both in the interior and in the exterior domains, for the perturbation tensors of the base scheme. Part of the computation, which may be of independent interest, is postponed to Appendix A where a fully covariant expression for the first order perturbations of the Ricci tensor is obtained (actually for more general background spacetimes). A key step in this subsection is the introduction of functions $\hat{h}$, $\hat{q}$ and $\hat{v}$ which are nearly gauge invariant (Lemma 5.2) and in terms of which the field equations simplify. In particular, this allows us to prove that part of the gauge freedom can be used to eliminate the $\ell = 1$ Legendre sector of the functions (Proposition 5.9). At this point, we will have isolated a set of functions and the corresponding equations that fully characterise the base scheme in the interior and the exterior regions: $\{\mathcal{W}(r,\theta), \widehat{q}_0(r), \widehat{v}_0(r), \sigma(r), \widehat{v}_2(r), \widehat{v}_\perp(r,\theta), f(r,\theta)\}$, where $\widehat{q}_0(r)$ and $\sigma(r)$ are free. The equation of state of the background is imposed in Subsection 5.1.1 to provide an algebraic expression for $\widehat{q}_0$ in terms of the rest. So far, no connection between the interior and exterior problems has been made. Subsection 5.2 is devoted to do this. The geometric matching problem, which is technically rather involved, is left to Appendix B. The results of this Appendix are independent of any field equations and hence may find applications in other situations. In particular no symmetry assumptions are made on how the matching surface gets perturbed, so they generalise the matching conditions obtained in [33], where axial symmetry was imposed. The geometric matching results of the appendix are specialized to our specific fluid problem in Proposition 5.12.

In Section 6 we tackle the global problem of existence of uniqueness of the base scheme. The section starts with a core result (Proposition 6.1) that provides existence of a decomposition in terms of Legendre polynomials of functions satisfying a sufficiently general elliptic global problem. This, together with the existence and uniqueness results shown in Appendix D, are the basic ingredients for this section. Subsection 6.1 is devoted to showing existence and uniqueness of the angular component $\mathcal{W}$. In Subsection 6.2 we prove that $\widehat{v}_\perp(r,\theta)$ must vanish everywhere and that $\widehat{v}_2(r)$ is unique and vanishes if the first order perturbation of the base scheme is zero. The existence of a barotropic equation of state together with the use of (most of) the remaining gauge freedom is finally used in Subsection 6.3 to settle the $\ell = 0$ sector and find the existence and uniqueness result of the base global scheme (Proposition 6.10). This result is the basis of the bootstrap argument.

In Section 7 we use the bootstrap argument in terms of the base global scheme to obtain the main result of the paper. After discussing the gauge behaviour and physical meaning of the integration parameter ($P_c^{(2)}$) introduced in the previous sections, a first use of the bootstrap argument provides Proposition 7.2, which states the result for the first order problem for the first order perturbation in "canonical form". A second use of the bootstrap argument for the second order problem provides the final and main result of the paper, Theorem 7.3. In Remark 7.4 we provide the explicit procedure for the calculation of the global unique solution in a fully fixed gauge. We stress that when the energy density of the star does not vanish at the boundary, these expressions correct [33] the standard formulae used in the literature. In addition, our gauge fixing respects the condition that the perturbation tensors stay bounded at infinity, something which has been often overlooked in applications of the Hartle-Thorne model. Finally we exploit the freedom of re-defining the perturbation parameter in order to write down the one parameter family of metrics in the familiar form used in the literature, and discuss the physical interpretation of the only free parameter in the model.

1.2. Notation

Most of the notation used in this paper will be specified along the way. Here we only fix the basic objects.

A C^{n+1} spacetime (M, g) is a four-dimensional (we never consider other dimensions in this paper) orientable C^{n+2} manifold M endowed with a time-oriented Lorentzian metric g of class C^{n+1} and signature $+2$. Our sign conventions for the Riemann and Ricci tensor follow e.g [43]. Scalar products of two vector fields X, Y with the metric g will be denoted by $\langle X, Y\rangle$. The covector metrically related to a vector X is denoted with boldface, $\boldsymbol{X} := g(X, \cdot)$. Throughout the paper, for functions of one argument, a prime means derivative with respect to the argument. 0_m denotes the point at the origin in $\mathbb{R}^m$.

2. Stationary and axisymmetric perturbation scheme

In this Section we summarize the results in Paper 1 needed below. Specifically we quote a theorem on the existence of a canonical form for the perturbation metric tensors to first and second order and the regularity of the corresponding coefficient functions, as well as the most general gauge transformation that respects this form (for a specific form of the first order tensor, since this is all we shall need).

The background is spherically symmetric and satisfying appropriate global conditions. The definitions are as in [25].

Definition 2.1. A spacetime (M, g) is static and spherically symmetric if it admits an $SO(3)$ group of isometries acting transitively on spacelike surfaces (which may degenerate to points), and a Killing vector ξ which is timelike everywhere, commutes with the generators of $SO(3)$ and is orthogonal to the $SO(3)$ orbits.

Our global and differentiability requirements on the spacetime are as follows:

Assumption H_1**:** $M \simeq U^3 \times I$ where $I \subset \mathbb{R}$ is an open interval and U^3 is a radially symmetric domain of $\mathbb{R}^3$ with the orbits of the Killing ξ along the I factor and $SO(3)$ acting in the standard way on U^3. Moreover, in the cartesian coordinates $\{x, y, z, t\}$ of $U^3 \times I$, the metric g is

$$g = -e^{\nu} dt^2 + \upsilon (x_i dx^i)^2 + \chi \delta_{ij} dx^i dx^j$$

with ν, υ, χ being C^{n+1} radially symmetric functions of x, y, z.

We note that the Killing vector $\xi = \partial_t$ is hypersurface orthogonal, hence the name "static and spherically symmetric". The centre of symmetry $\mathcal{C}_0 \subset M$ is by definition the set of points invariant under $SO(3)$. By the global diffeomorphism $\simeq$ in assumption H_1 we have $\mathcal{C}_0 \simeq \{0_3 \cap U^3\} \times I$, so $\mathcal{C}_0$ is non-empty if and only if U^3 is a ball.

All geometric objects in M will be identified with their image by $\simeq$ and viceversa. This applies for instance to $\mathcal{C}_0$, or to the function $|x| := \sqrt{x^2+y^2+z^2}$ on U^3, which also defines a function on M. The orbits of the $SO(3)$ action are the spheres $S_r := \{|x| = r\}$, which we view again as subsets of $U^3 \times I$ or of M depending on the context.

Define two functions $\lambda, \mathcal{R} : M \to \mathbb{R}$ by

$$e^{\lambda} := \chi + \upsilon |x|^2, \qquad \mathcal{R}^2 := \chi |x|^2, \qquad \mathcal{R} \geq 0.$$

Both are well defined because χ and $\chi + \upsilon|x|^2$ are positive everywhere (otherwise g is not a Lorentzian metric). It is clear that $\lambda \in C^{n+1}(M)$ and $\mathcal{R} \in C^{n+1}(M \setminus \mathcal{C}_0) \cap C^0(M)$, and that both are radially symmetric when expressed in $\{x, y, z, t\}$ coordinates.

We shall mostly work in spherical coordinates $\{r, \theta, \phi, t\}$ defined from $\{x, y, z, t\}$ in the standard way. This coordinate system covers $M \setminus \mathcal{A}$, where $\mathcal{A} = \{x = 0, y = 0\}$ is the axis of the Killing vector $\eta = \partial_\phi$. On this domain the metric g takes the form

$$g = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + \mathcal{R}^2(r)\left(d\theta^2 + \sin^2\theta d\phi^2\right), \qquad \xi = \partial_t. \tag{2.1}$$

We make the usual abuse of notation of writing functions in different coordinate systems with the same symbol (the meaning should be clear from the context). Nevertheless we write explicitly the arguments when we want to make clear which representation is being used (we have already followed this convention in (2.1) when writing $\nu(r)$ etc.)

We can now quote the main theorem in [25]. To fix the basic notation, we recall that perturbation tensors are defined through a family of C^{n+1} spacetimes $(M_\varepsilon, \hat{g}_\varepsilon)$, that includes the background (M, g) for $\varepsilon = 0$, diffeomorphically identified through some gauge ψ_ε (C^{n+2} for each ε). To first and to second order, the respective perturbation tensors K_1 and K_2 are defined as

$$K_1 = \left.\frac{dg_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0}, \quad K_2 = \left.\frac{d^2 g_\varepsilon}{d\varepsilon^2}\right|_{\varepsilon=0}, \tag{2.2}$$

where $g_\varepsilon := \psi_\varepsilon^*(\hat{g}_\varepsilon)$ on (M, g). For the precise notion of "perturbation scheme", "inheritance of an orthogonally transitive isometry group action", as well as "gauge transformations" and our notation for gauge vectors we refer to [25]. For completeness, though, we recall here that a perturbation scheme is said to be of class C^{n+1} when the family $\hat{g}_\varepsilon$ is C^{n+1} and the perturbation tensors K_1, K_2 are, respectively, C^n and C^{n-1}. We also recall that a two-dimensional group of isometries, generated by say $\{\xi, \eta\}$, acts orthogonally transitively when the 2-planes orthogonal to the group orbits generate surfaces. In four dimensions, this happens if and only if the two scalars $\star(\boldsymbol{\xi} \wedge \boldsymbol{\eta} \wedge d\boldsymbol{\xi})$ and $\star(\boldsymbol{\xi} \wedge \boldsymbol{\eta} \wedge d\boldsymbol{\eta})$, where we use $\star$ for the Hodge dual operation, vanish identically.

Theorem 2.2 (Canonical form [25]). *Let (M, g) be a static and spherically symmetric background satisfying assumption* H_1*, with g of class C^{n+1} with $n \geq 2$, given in spherical coordinates by* (2.1)*. Let us be given a C^{n+1} maximal perturbation scheme $(M_\varepsilon, \hat{g}_\varepsilon, \{\psi_\varepsilon\})$ inheriting the orthogonal transitive stationary and axisymmetric action generated by $\{\xi = \partial_t, \eta = \partial_\phi\}$. Then, there exists gauge vectors V_1 and V_2, that commute with η, are tangent to S_r as well as orthogonal to η, and extend continuously to zero at $\mathcal{C}_0$, such that the gauge transformed tensors K_1^Ψ and K_2^Ψ are of class $C^{n-1}(M \setminus \mathcal{C}_0)$ and $C^{n-2}(M \setminus \mathcal{C}_0)$ respectively, and such that the functions defined on $M \setminus \mathcal{C}_0$ by*

$$h^{(1)} := -\frac{1}{4} e^{-\nu} K_1^\Psi(\partial_t, \partial_t) \tag{2.3}$$

$$k^{(1)} := \frac{1}{4\eta^2} K_1^\Psi(\eta, \eta) \tag{2.4}$$

$$-x\chi\omega = K_1^\Psi(\partial_t, \partial_y), \qquad y\chi\omega = K_1^\Psi(\partial_t, \partial_x), \tag{2.5}$$

$$\begin{aligned} m^{(1)} &:= \frac{1}{4}\left\{ K_1^{\Psi}{}^{\alpha}_{\ \alpha} + e^{-\nu} K_1^\Psi(\partial_t, \partial_t) - 8k^{(1)} \right\} \\ &= \frac{1}{4}\left\{ K_1^{\Psi}{}^{\alpha}_{\ \alpha} - 4h^{(1)} - 8k^{(1)} \right\} \end{aligned} \tag{2.6}$$

$$h := -\frac{1}{4} e^{-\nu} \left(K_2^\Psi(\partial_t, \partial_t) - 2\eta^2\omega^2 \right) \tag{2.7}$$

$$k := \frac{1}{4\eta^2} K_2^\Psi(\eta, \eta), \tag{2.8}$$

$$-x\chi\mathcal{W} = K_2^\Psi(\partial_t, \partial_y), \qquad y\chi\mathcal{W} = K_2^\Psi(\partial_t, \partial_x), \tag{2.9}$$

$$m := \frac{1}{4}\left\{ K_2^{\Psi}{}^{\alpha}_{\ \alpha} + e^{-\nu} K_2^\Psi(\partial_t, \partial_t) - 8k \right\} \tag{2.10}$$

have the following properties:

(a.1) $h^{(1)}$ *extends to a* $C^n(M)$ *function.*

(a.2) ω *extends to a* $C^{n-1}(M)$ *function.*

(a.3) *The vector field* $\omega\eta$ *is* $C^n(M \setminus \mathcal{C}_0)$.

(a.4) $m^{(1)}$ *and* $k^{(1)}$ *are* $C^n(M \setminus \mathcal{C}_0)$ *and bounded near* $\mathcal{C}_0$.

(b.1) h *is* $C^{n-1}(M \setminus \mathcal{C}_0)$ *and bounded near* $\mathcal{C}_0$.

(b.2) $\mathcal{W}$ *is* $C^{n-2}(M \setminus \mathcal{C}_0)$ *and bounded near* $\mathcal{C}_0$.

(b.3) *The vector field* $\mathcal{W}\eta$ *is* $C^{n-1}(M \setminus \mathcal{C}_0)$.

(b.4) m *and* k *are* $C^{n-1}(M \setminus \mathcal{C}_0)$ *and bounded near* $\mathcal{C}_0$.

Moreover, there exist two functions $f^{(1)}$ *and* f *defined on* $M \setminus \mathcal{C}_0$*, invariant under* ξ *and* η *and satisfying*

(a.5) $f^{(1)}$ *is* $C^{n-1}(M \setminus \mathcal{C}_0)$*, bounded near* $\mathcal{C}_0$, $C^n(S_r)$ *on all spheres* S_r, $\partial_\theta f^{(1)}$ *is* C^{n-1} *outside the axis and extends continuously to* $\mathcal{A} \setminus \mathcal{C}_0$*, where it vanishes, and both* $\partial_r f^{(1)}$ *and* $\partial_t f^{(1)}$ *are* $C^{n-1}(S_r)$ *on all spheres* S_r,

(b.5) f *is* $C^{n-2}(M \setminus \mathcal{C}_0)$*, bounded near the origin,* $C^{n-1}(S_r)$ *on all spheres* S_r, $\partial_\theta f$ *is* C^{n-2} *outside the axis and extends continuously to* $\mathcal{A} \setminus \mathcal{C}_0$*, where it vanishes, and both* $\partial_r f$ *and* $\partial_t f$ *are* $C^{n-2}(S_r)$ *on all spheres* S_r,

so that K_1^Ψ *and* K_2^Ψ *take the following form on* $M \setminus \mathcal{A}$

$$
\begin{aligned}
(2.11) \qquad K_1^\Psi = &-4e^{\nu(r)}h^{(1)}(r,\theta)dt^2 - 2\omega(r,\theta)\mathcal{R}^2(r)\sin^2\theta dtd\phi \\
&+ 4e^{\lambda(r)}m^{(1)}(r,\theta)dr^2 + 4k^{(1)}(r,\theta)\mathcal{R}^2(r)(d\theta^2 + \sin^2\theta d\phi^2) \\
&+ 4e^{\lambda(r)}\partial_\theta f^{(1)}(r,\theta)\mathcal{R}(r)drd\theta,
\end{aligned}
$$

$$
\begin{aligned}
(2.12) \qquad K_2^\Psi = &\left(-4e^{\nu(r)}h(r,\theta) + 2\omega^2(r,\theta)\mathcal{R}^2(r)\sin^2\theta\right)dt^2 \\
&+ 4e^{\lambda(r)}m(r,\theta)dr^2 + 4k(r,\theta)\mathcal{R}^2(r)(d\theta^2 + \sin^2\theta d\phi^2) \\
&+ 4e^{\lambda(r)}\partial_\theta f(r,\theta)\mathcal{R}(r)drd\theta - 2\mathcal{W}(r,\theta)\mathcal{R}^2(r)\sin^2\theta dtd\phi.
\end{aligned}
$$

Since the form of the perturbation tensors in Theorem 2.2 is used repeatedly in the paper, we put forward the following definition:

Definition 2.3. First and second order perturbation tensors on a static and spherically symmetric background that have the structure and regularity properties given in Theorem 2.2 are said to be in *canonical form*.

Remark 2.4 ([25]). In the setup of Theorem 2.2, let K_1 and K_2 be perturbation tensors defined by the perturbation scheme $(M_\varepsilon, \hat{g}_\varepsilon, \{\psi_\varepsilon\})$ and K_1^Ψ, K_2^Ψ be the corresponding tensors in canonical form. If the background admits no further local isometries and the perturbation scheme is restricted so that the inherited axial Killing vector $\hat{\eta}_\varepsilon = d\psi_\varepsilon(\eta)$ is independent of the choice $\psi_\varepsilon \in \{\psi_\varepsilon\}$, then the gauge vectors V_1 and V_2 transforming K_1 and K_2 into *fixed* K_1^Ψ and K_2^Ψ are unique up to the addition of a Killing vector of the background that commutes with η. We emphasize that the condition on $\hat{\eta}_\varepsilon$ is no restriction at all if $\hat{g}_\varepsilon$, $\varepsilon \neq 0$, admits only one axial symmetry.

As we shall see, the field equations for perturbed fluid balls restrict strongly the first order metric perturbation tensor. It is an essential ingredient of this paper to understand the full gauge freedom that respects this restricted form. The following result, proved in [25], achieves this.

Proposition 2.5 (Gauge freedom [25]). *Let (M, g) be a static and spherically symmetric spacetime as in Theorem 2.2. Assume that $\mathcal{R}'(r)$ and $\nu'(r)$ do not vanish identically on open sets and consider the following first and second order perturbation tensors*

$$K_1 = -2\omega(r,\theta)\mathcal{R}^2(r)\sin^2\theta dt d\phi, \tag{2.13}$$

$$\begin{aligned} K_2 = {} & \left(-4e^{\nu(r)}h(r,\theta) + 2\omega^2(r,\theta)\mathcal{R}^2(r)\sin^2\theta\right) dt^2 + 4e^{\lambda(r)}m(r,\theta)dr^2 \\ & + 4k(r,\theta)\mathcal{R}^2(r)\left(d\theta^2 + \sin^2\theta d\phi^2\right) + 4e^{\lambda(r)}\partial_\theta f(r,\theta)\mathcal{R}(r) dr d\theta \\ & - 2\mathcal{W}(r,\theta)\mathcal{R}^2(r)\sin^2\theta dt d\phi. \end{aligned} \tag{2.14}$$

Then a first order gauge vector V_1 preserves the form of K_1 (i.e. there is ω^g such that $K_1^g := K_1 + \mathcal{L}_{V_1} g$ is given by (2.13) with $\omega \longrightarrow \omega^g$) if and only if, up to the addition of a Killing vector of the background,

$$V_1 = Ct\partial_\phi, \qquad C \in \mathbb{R}, \qquad \textit{and then} \qquad \omega^g = \omega - C. \tag{2.15}$$

For V_1 as in (2.15), the second order gauge vector V_2 preserves the form of K_2 if and only if

$$\begin{aligned} & V_2 = At\partial_t + Bt\partial_\phi + 2\mathcal{Y}(r,\theta)\partial_r + 2\alpha(r)\sin\theta\partial_\theta + \zeta, \\ & A, B \in \mathbb{R}, \quad \zeta \textit{ Killing vector of } g, \end{aligned} \tag{2.16}$$

and $K_2^g := K_2 + \mathcal{L}_{V_2} g + 2\mathcal{L}_{V_1} K_1^g - \mathcal{L}_{V_1}\mathcal{L}_{V_1} g$ *takes the form (2.14) with the coefficients* h, m, k, f *transformed to*

$$h^g = h + \frac{1}{2}A + \frac{1}{2}\mathcal{Y}\nu', \tag{2.17}$$

$$k^g = k + \mathcal{Y}\frac{\mathcal{R}'}{\mathcal{R}} + \alpha(r)\cos\theta, \tag{2.18}$$

$$m^g = m + \mathcal{Y}_{,r} + \frac{1}{2}\mathcal{Y}\lambda', \tag{2.19}$$

$$f^g = f + \frac{\mathcal{Y}}{\mathcal{R}} - \mathcal{R}e^{-\lambda}\alpha'\cos\theta + \beta(r), \tag{2.20}$$

$$\mathcal{W}^g = \mathcal{W} - B, \tag{2.21}$$

where the arbitrary function $\beta(r)$ *arises because* K_2^g *only involves* $\partial_\theta f^g$*.*

Remark 2.6. It is important to stress that this proposition includes in particular the *full* gauge freedom that preserves the first order metric perturbation tensor in canonical form. Indeed, by setting $K_1 = 0$ and $V_1 = 0$, second order metric perturbation tensors transform under a gauge change in exactly the same way as the first order perturbation tensors do. Since the tensor K_2 in (2.14) is fully general (in the canonical form) it follows that the most general transformation vector that respects a general K_1 in canonical form is given by $V_1 = V_2$, with V_2 as given in (2.16) and $\{h^{(1)}, m^{(1)}, k^{(1)}, f^{(1)}, \omega\}$ transform exactly as the corresponding (2.17)–(2.21).

Exploiting the gauge freedom to first and second order in Proposition 2.5 will be an important tool to prove the results of this paper. We will use the following notation for it.

Notation 2.7. We will denote by $\{\Psi(C; A, B, \mathcal{Y}, \alpha)\}$ the family of gauges described to second order by the gauge vectors (2.15) and (2.16) and such that the gauged functions satisfy the regularity properties of the corresponding functions in Theorem 2.2.. When e.g. $\alpha(r)$ has already been fixed, so that the gauge vectors are restricted to the form (2.15)–(2.16) with $\alpha(r) = 0$, the corresponding family will be denoted by $\{\Psi(C; A, B, \mathcal{Y})\} \subset \{\Psi(C; A, B, \mathcal{Y}, \alpha)\}$. This notation extends naturally to any subset of gauge parameters in the family.

3. Background spherically symmetric global model

In this section we recall the basic construction of a spherically symmetric spacetime consisting of two regions matched across a hypersurface that preserves the symmetries. We distinguish the two regions as "interior" and "exterior", but at this point this is merely a convention. We use $(+)$ to label objects in the interior, and a $(-)$ for the exterior. We denote by (M, g) the static and spherically symmetric spacetime resulting from the matching $M = M^+ \cup M^-$ of two C^{n+1} $(n \geq 4)$ static and spherically symmetric spacetimes $(M^\pm, g^\pm)$ with boundaries $\Sigma^\pm$. The matching hypersurface is $M^- \cap M^+ \simeq \Sigma^+ \simeq \Sigma^-$. We will use coordinates $\{t_\pm, r_\pm, \theta_\pm, \phi_\pm\}$ on $(M^\pm, g^\pm)$ covering a neighbourhood of the boundaries $\Sigma^\pm$, such that the metrics read

$$g^\pm = -e^{\nu_\pm(r_\pm)} dt_\pm^2 + e^{\lambda_\pm(r_\pm)} dr_\pm^2 + \mathcal{R}_\pm{}^2(r_\pm) \left(d\theta_\pm^2 + \sin^2\theta_\pm d\phi_\pm^2 \right).$$

By spherical symmetry and staticity, the hypersurfaces $\Sigma^\pm$ can be described by embeddings from an abstract manifold Σ (called *the boundary*), coordinated by $\{\tau, \vartheta, \varphi\}$, by means of

$$\Sigma^+ = \{t_+ = \tau,\ r_+ = a_+,\ \theta_+ = \vartheta,\ \phi_+ = \varphi\}, \tag{3.1}$$

$$\Sigma^- = \{t_- = \tau,\ r_- = a_-,\ \theta_- = \vartheta,\ \phi_- = \varphi\}, \tag{3.2}$$

where $a_\pm$ are constants. We may choose $r_\pm$ so that r_+ takes values to the left of a_+ in the real line and r_- to the right of a_-. Clearly, $\mathcal{R}_\pm(a_\pm) > 0$ (the boundary is a hypersurface). We fix uniquely the unit normals $\mathrm{n}^\pm$ so that n^+ points M^+ inwards and n^- points M^- outwards. Thus

$$\mathrm{n}^+ = -e^{-\frac{\lambda_+(a_+)}{2}} \partial_{r_+}|_{\Sigma^+}, \qquad \mathrm{n}^- = -e^{-\frac{\lambda_-(a_-)}{2}} \partial_{r_-}|_{\Sigma^-}. \tag{3.3}$$

$\Sigma^\pm$ are obviousy timelike everywhere and their first and second fundamental forms read

$$h_{ij}^\pm dx^i dx^j = -e^{\nu_\pm(a_\pm)} d\tau^2 + \mathcal{R}_\pm^2(a_\pm)(d\vartheta^2 + \sin^2\vartheta d\varphi^2), \tag{3.4}$$

$$\kappa_{ij}^\pm dx^i dx^j = e^{-\frac{\lambda_\pm(a_\pm)}{2}} \left(\frac{1}{2} e^{\nu_\pm(a_\pm)} \nu_\pm'(a_\pm) d\tau^2 - \mathcal{R}_\pm(a_\pm)\mathcal{R}'(a_\pm)(d\vartheta^2 + \sin^2\vartheta d\varphi^2) \right). \tag{3.5}$$

The matching conditions across Σ require that the first and second fundamental forms on both sides agree i.e. $[h] = [\kappa] = 0$, where for any object $[f] := f^+ - f^-$. When a quantity f satisfies $[f] = 0$ we write $f^+ = f^- := f$

on Σ. From (3.4)–(3.5) the matching conditions are equivalent to

$$[\mathcal{R}] = 0, \qquad [\nu] = 0, \qquad [e^{-\lambda/2}\nu'] = 0, \qquad [e^{-\lambda/2}\mathcal{R}'] = 0. \tag{3.6}$$

The last two can also be written as $[\mathrm{n}(\nu)] = [\mathrm{n}(\mathcal{R})] = 0$. So far, no field equations have been imposed. We summarize the construction with the following definition.

Definition 3.1. A spacetime (M, g) si called **static and spherically symmetric with two regions** if it is composed by $(M^{\pm}, g^{\pm}, \Sigma^{\pm})$ as described in this section and satisfies the matching conditions (3.6).

3.1. Background field equations

Our background spacetime describes a non-rotating self-gravitating fluid of finite extent. Thus, it consists of two regions, one solving the gravitational field equations for a perfect fluid and the other for vacuum. In the context of General Relativity without cosmological constant, which we assume from now on, the field equations are $\mathrm{Ein}_g = \varkappa T$, where Ein_g is the Einstein tensor of g, $\varkappa$ is the gravitational coupling constant and T is the energy-momentum tensor of the matter. For a perfect fluid

$$T_{\mu\nu} = (E + P)u_\mu u_\nu + P g_{\mu\nu},$$

where P is the pressure, E the density and u is the (unit timelike) four-velocity of the fluid. For the metric (2.1) the perfect-fluid Einstein field equations hold if and only if, in addition to $u = e^{-\frac{\nu}{2}}\xi$ and

$$\varkappa P = e^{-\lambda}\frac{\mathcal{R}'}{\mathcal{R}}\left(\frac{\mathcal{R}'}{\mathcal{R}} + \nu'\right) - \frac{1}{\mathcal{R}^2}, \tag{3.7}$$

$$\varkappa E = e^{-\lambda}\left(-2\frac{\mathcal{R}''}{\mathcal{R}} - \frac{\mathcal{R}'^2}{\mathcal{R}^2} + \frac{\mathcal{R}'}{\mathcal{R}}\lambda'\right) + \frac{1}{\mathcal{R}^2}, \tag{3.8}$$

the following ODE is satisfied

$$\nu'' = -2\frac{\mathcal{R}''}{\mathcal{R}} + \frac{\mathcal{R}'}{\mathcal{R}}\left(2\frac{\mathcal{R}'}{\mathcal{R}} + \lambda' + \nu'\right) + \frac{1}{2}\nu'\left(\lambda' - \nu'\right) - \frac{2e^{\lambda}}{\mathcal{R}^2} \tag{3.9}$$
$$\Longrightarrow \quad P' + \frac{\nu'}{2}(E + P) = 0.$$

The implication is in fact an equivalence wherever $\mathcal{R}' \neq 0$. From (3.7), any critical value r_{crit} of $\mathcal{R}(r)$ outside the centre(s) of symmetry (i.e. satisfying

$\mathcal{R}'(r_{crit}) = 0$, $\mathcal{R}(r_{crit}) \neq 0$) must have $P|_{r_{crit}} < 0$. The boundary of the fluid ball (with vacuum exterior) is located at $P = 0$ (the fact that $P|_\Sigma = 0$ is a general consequence of the Israel conditions and in our setup it follows immediately from (3.7) and (3.6)). Thus, either $P > 0$ or $P < 0$ in the interior of the body, and the physical case is $P > 0$. Also on physical grounds it must be that the energy density of the fluid is non-negative and positive somewhere. We make this assumption explicit:

Assumption H_2**:** The background spacetime has two non-empty regions, one vacuum and one covered by a self-gravitating fluid satisfying $P \geq 0$ and $E \geq 0$. Moreover, there is at least one point in the fluid where $E > 0$.

The condition $P \geq 0$ implies that $\mathcal{R}(r)$ is strictly monotonic and we can set $\mathcal{R}(r) = r$, which we assume from now on. The field equations (3.7)–(3.9) become

$$\lambda' = \frac{1}{r}(1 - e^\lambda) + re^\lambda \varkappa E, \tag{3.10}$$

$$\nu' = \frac{1}{r}(e^\lambda - 1) + re^\lambda \varkappa P, \tag{3.11}$$

$$\nu'' = \frac{1}{r}\left(\frac{2}{r} + \lambda' + \nu'\right) + \frac{1}{2}\nu'\left(\lambda' - \nu'\right) - \frac{2e^\lambda}{r^2} \tag{3.12}$$

$$\iff \qquad P' = -\frac{\nu'}{2}(E + P).$$

Consider the convenient and standard background quantities

$$j(r) := e^{-(\lambda+\nu)/2}, \tag{3.13}$$

$$1 - \frac{\varkappa M(r)}{4\pi r} := e^{-\lambda}. \tag{3.14}$$

The former satisfies

$$\frac{j'}{j} = -\frac{1}{2}\left(\lambda' + \nu'\right) = -\frac{1}{2} re^\lambda \varkappa (E + P), \tag{3.15}$$

while the latter allows one to replace the variables $\{\lambda, \nu\}$ by $\{M, P\}$ as follows: (3.14) and (3.11) give

$$\nu' = \frac{\varkappa}{r(4\pi r - \varkappa M)}(M + 4\pi r^3 P), \tag{3.16}$$

and the system (3.10)–(3.12) takes the standard form (see (9) and (10) in [29])

$$M' = 4\pi r^2 E, \tag{3.17}$$

$$P' = -\frac{\varkappa(E+P)(M+4\pi r^3 P)}{8\pi r^2(1-\frac{\varkappa M}{4\pi r})}. \tag{3.18}$$

These are the well-known TOV equations [29]. These equations are usually suplemented with a barotropic equation of state (EOS) $E(P)$ which closes the system. A substantial portion of the paper does not rely on the existence of a barotropic EOS. We will make the assumption explicit when needed (in Section 5.1.1).

The vacuum case is obviously Schwarzschild, for which

$$M = M_{\text{T}} \quad \in \mathbb{R} \qquad e^{-\lambda(r)} = e^{\nu(r)} = 1 - \frac{\varkappa M_{\text{T}}}{4\pi r}.$$

The matching conditions (3.6) read, after setting $\mathcal{R}_\pm(r_\pm) = r_\pm$,

$$a_+ = a_-(=a), \quad [\nu] = 0, \quad [\lambda] = 0, \quad [\nu'] = 0, \tag{3.19}$$

and are interpreted as follows: $[\lambda] = 0$ is equivalent to the continuity of the mass $M_{\text{T}} = M(a)$, $[\nu] = 0$ fixes uniquely the additive integration constant that arises when solving (3.16) and $[\nu'] = 0$ corresponds to $[P] = 0$, which, in principle, determines a. Note that $[\nu'] = 0$ also provides

$$\nu'_\pm(a) = \frac{1}{a}(e^{\lambda(a)} - 1), \tag{3.20}$$

where the equality follows directly from (3.11).

Finally, the field equations combined with the matching conditions (3.19) allow us to express the jumps of higher order derivatives in terms of the fluid variables (A_0 is a constant whose explicit form is not needed)

$$[\lambda'] = a e^{\lambda(a)} \varkappa [E], \tag{3.21}$$

$$[\nu''] = \frac{1}{a}\left(1 + \frac{a\nu'(a)}{2}\right)[\lambda'], \tag{3.22}$$

$$[\lambda''] = a e^{\lambda(a)} \varkappa [E'] + [\lambda'^2], \tag{3.23}$$

$$[\nu'''] = \frac{1}{a}\left(1 + \frac{a\nu'(a)}{2}\right)[\lambda''] + A_0[\lambda']. \tag{3.24}$$

Note that the jumps of $[\nu'']$ and $[\lambda']$ are proportional. All these expressions are valid also when two perfect fluids are matched.

Everything we have said so far in this section holds locally near the boundaries. We now make a global assumption similar in spirit to assumption H_1. Since the spacetime (M, g) is now composed of two regions $(M^{\pm}, g^{\pm})$ with boundaries $\Sigma^{\pm}$, we modify the assumption as follows

Assumption H_1'**:** The interiors $(\mathrm{Int}(M^{\pm}), g^{\pm})$ satisfy assumption H_1 with corresponding diffeomorphisms

$$\mathrm{Int}(M^+) \simeq B^+ \times I, \qquad \mathrm{Int}(M^-) \simeq (\mathbb{R}^3 \setminus \overline{B^+}) \times I,$$

where B^+ is an open ball centered at the origin. Moreover $\Sigma^{\pm} \simeq (\partial B^+) \times I$.

Under assumptions H_1' and H_2, it must be the case that the fluid lies in the interior M^+. Indeed, if M^+ were vacuum then $M_{\mathrm{T}} = 0$ and one easily concludes from (3.17)–(3.18) together with $E \geq 0$ and $P(a) = 0$ that $P \leq 0$ in the fluid region, which is a contradiction. Consequently the coordinate r takes values in $r \in (0, a]$ in the interior (fluid) region and $r \in [a, \infty)$ in the exterior (vacuum) domain. The spacetime is C^{n+1} (with $n \geq 4$) everywhere except at Σ, in particular in a neighbourhood of the centre $r = 0$. Since the vacuum region is the exterior M^- can now write

$$e^{-\lambda_-(r)} = e^{\nu_-(r)} = 1 - \frac{\varkappa M_{\mathrm{T}}}{4\pi r} \qquad \Longrightarrow \qquad j_-(r) = 1, \tag{3.25}$$

where an additive integration constant in ν has been adjusted to zero. This choice fixes the (otherwise arbitrary) freedom in scaling the static Killing ξ by a positive constant. Moreover, one has, in addition, $[E] = E_+(a)$ and $[E'] = E_+'(a)$.

We now make use of the following result on the differentiability of radially symmetric functions (see e.g. [25] or Lemma 3.1 in [2])

Lemma 3.2. *Let $q : B^+ \to \mathbb{R}$ be radially symmetric, i.e. such that there exists ${}^{tr}q : [0, a_0) \to \mathbb{R}$ (the trace of q) with $q(x) = {}^{tr}q(|x|)$. Then $q \in C^n(B^+)$ $(n \geq 0)$ if and only if ${}^{tr}q$ is $C^n([0, a_0))$ (i.e. up to the inner boundary) and all its odd derivatives up to order n vanish at zero. Equivalently, if and only if*

$${}^{tr}q(r) = \mathcal{P}_n(r^2) + \Phi^{(n)}(r), \tag{3.26}$$

where $\mathcal{P}_n$ is a polynomial of degree $[\frac{n}{2}]$ and $\Phi^{(n)}$ is $C^n([0, a_0))$ and satisfies $\Phi^{(n)}(r) = o(r^n)$.

This Lemma implies that $\lambda(r)$ and $\nu(r)$ in the + region (as functions of one variable r) are C^{n+1} up to boundary, and admit an expansion

$$\begin{aligned} \lambda(r) &= \lambda_0 + \lambda_2 r^2 + \lambda_4 r^4 + \Phi^{(5)}_\lambda(r), \\ \nu(r) &= \nu_0 + \nu_2 r^2 + \nu_4 r^4 + \Phi^{(5)}_\nu(r), \end{aligned} \tag{3.27}$$

with $\lambda_0, \lambda_2, \lambda_4, \nu_0, \nu_2, \nu_4 \in \mathbb{R}$, and $\Phi^{(5)}_\lambda(r)$, $\Phi^{(5)}_\nu(r)$ are $C^{n+1}([0,a])$ and vanish, together with their derivatives up to order five, at $r=0$. Combining this with the field equations (3.10)–(3.11) near $r=0$ one finds, in particular,

$$\lambda_0 = 0, \quad \lambda_2 = \frac{\varkappa}{3} E_c, \quad \nu_2 = \frac{\varkappa}{6}\left(E_c + 3P_c\right), \tag{3.28}$$

where $E_c = E(0)$ and $P_c = P(0)$ are the values of the energy density and pressure at the origin, while ν_0 will be determined by the matching condition $[\nu] = 0$. Expressions (3.27)–(3.28) give

$$\begin{aligned} e^{\lambda(r)} &= 1 + \frac{\varkappa}{3} E_c r^2 + O(r^4), \\ e^{\nu(r)} &= e^{\nu_0}\left(1 + \frac{\varkappa}{6}(E_c + 3P_c) r^2 + O(r^4)\right). \end{aligned} \tag{3.29}$$

These expansions together with (3.14) imply that $M(r) \in O(r^3)$.

Another consequence of assumptions H_1' and H_2 is that $\nu(r)$ is free of critical values outside the origin. First of all, equation (3.17) together with $M \in O(r^3)$ and $E(r) \geq 0$ implies $M(r) \geq 0$. Furthermore, the quantity $4\pi r - \varkappa M(r)$ is positive for r sufficiently close to zero, so regularity of the spacetime imposes (by (3.18)) that $r > \varkappa M(r)/4\pi$ for all $r \leq a$ and $r > \varkappa M_{\mathrm{T}}/4\pi$ for $r \geq a$ (in fact this property holds in much more general circumstances [23]). With these properties it is clear from (3.16) that $\nu' > 0$ away from the origin.

The setup described in this section is summarized in the following definition.

Definition 3.3. A C^{n+1} **perfect fluid ball configuration** is a static and spherically symmetric spacetime with two regions, c.f. Definition 3.1, satisfying assumptions H_1' and H_2.

Whenever this definition is invoked, all the results and notation introduced in this section will be understood.

4. Perturbed Einstein's field equations to second order

We review in this section the perturbations of the Ricci tensor in terms of K_1 and K_2, and the perturbations of the perfect fluid under the assumption of rigid rotation. Recall that this means that the fluid 3-velocity, as observed by the stationary observer, is uniform in both space and time and only has one component along the axial direction. This precludes, in particular, the presence of convective motions inside the fluid. We then write down the first and second order perturbed Einstein field equations under these conditions. This part, just like the previous one, is a reminder of known things and it is included to make the paper as self-contained as possible and to fix some notation.

4.1. First and second order perturbations of the Ricci tensor

Given two metrics g and g_ε, the respective Riemann tensors, denoted by $R^{\mu}{}_{\alpha\nu\beta}$ and $R_\varepsilon{}^{\mu}{}_{\alpha\nu\beta}$, are related by (e.g. [43])

$$(4.1)\quad R_\varepsilon{}^{\mu}{}_{\alpha\nu\beta} = R^{\mu}{}_{\alpha\nu\beta} + \nabla_\nu S_\varepsilon{}^{\mu}{}_{\alpha\beta} - \nabla_\beta S_\varepsilon{}^{\mu}{}_{\alpha\nu} + S_\varepsilon{}^{\mu}{}_{\nu\rho} S_\varepsilon{}^{\rho}{}_{\alpha\beta} - S_\varepsilon{}^{\mu}{}_{\beta\rho} S_\varepsilon{}^{\rho}{}_{\alpha\nu}$$

where ∇ is the Levi-Civita derivative of g and the tensor S_ε is the difference of the respective connections of g_ε and g, explicitly

$$S_\varepsilon{}^{\mu}{}_{\alpha\beta} = \frac{1}{2} g_\varepsilon{}^{\sharp\mu\nu} \left(\nabla_\alpha g_{\varepsilon\nu\beta} + \nabla_\beta g_{\varepsilon\nu\alpha} - \nabla_\nu g_{\varepsilon\alpha\beta} \right) =: g_\varepsilon^{\sharp\mu\nu} H_{\varepsilon\nu\alpha\beta}$$

where the last equality defines H_ε and the tensor $g_\varepsilon^{\sharp\mu\nu}$ is the contravariant metric associated to g_ε. Recalling that g_ε depends differentiably on ε, that $g_{\varepsilon=0} = g$, and the definitions (2.2), it follows directly from $\frac{dg_\varepsilon^{\sharp\alpha\beta}}{d\varepsilon} = -g_\varepsilon^{\sharp\alpha\mu} g_\varepsilon^{\sharp\beta\nu} \frac{dg_{\varepsilon\mu\nu}}{d\varepsilon}$ that

$$\left.\frac{dg_\varepsilon^{\sharp\alpha\beta}}{d\varepsilon}\right|_{\varepsilon=0} = -K_1^{\alpha\beta}, \qquad \left.\frac{d^2 g_\varepsilon^{\sharp\alpha\beta}}{d\varepsilon^2}\right|_{\varepsilon=0} = -K_2^{\alpha\beta} + K_1^{\alpha\mu} K_{1\mu}{}^{\beta}.$$

We emphasize that all objects are defined in (M, g) and that all indices are raised and lowered with the background metric g. Define also

$$(4.2)\qquad S^{(1)}{}_{\mu\alpha\beta} := \left.\frac{dH_{\varepsilon\mu\alpha\beta}}{d\varepsilon}\right|_{\varepsilon=0} = \frac{1}{2} \left(\nabla_\alpha K_{1\mu\beta} + \nabla_\beta K_{1\mu\alpha} - \nabla_\mu K_{1\alpha\beta} \right),$$

$$(4.3)\qquad S^{(2)}{}_{\mu\alpha\beta} := \left.\frac{d^2 H_{\varepsilon\mu\alpha\beta}}{d\varepsilon^2}\right|_{\varepsilon=0} = \frac{1}{2} \left(\nabla_\alpha K_{2\mu\beta} + \nabla_\beta K_{2\mu\alpha} - \nabla_\mu K_{2\alpha\beta} \right),$$

from which it follows directly

$$\left.\frac{dS_{\varepsilon}{}^{\mu}{}_{\alpha\beta}}{d\varepsilon}\right|_{\varepsilon=0} = S^{(1)\mu}{}_{\alpha\beta}, \qquad \left.\frac{d^2 S_{\varepsilon}{}^{\mu}{}_{\alpha\beta}}{d\varepsilon^2}\right|_{\varepsilon=0} = S^{(2)\mu}{}_{\alpha\beta} - 2K_1^{\mu\nu} S^{(1)}{}_{\nu\alpha\beta}.$$

Taking the first and second derivative of (4.1) with respect to ε at $\varepsilon = 0$, and using that $S_{\varepsilon}|_{\varepsilon=0} = 0$, the following expressions are directly obtained

$$\left.\frac{dR_{\varepsilon}{}^{\mu}{}_{\alpha\nu\beta}}{d\varepsilon}\right|_{\varepsilon=0} = \nabla_{\nu} S^{(1)\mu}{}_{\alpha\beta} - \nabla_{\beta} S^{(1)\mu}{}_{\alpha\nu}, \tag{4.4}$$

$$\begin{aligned}\left.\frac{d^2 R_{\varepsilon}{}^{\mu}{}_{\alpha\nu\beta}}{d\varepsilon^2}\right|_{\varepsilon=0} &= \nabla_{\nu}\left(S^{(2)\mu}{}_{\alpha\beta} - 2K_1^{\mu\rho} S^{(1)}{}_{\rho\alpha\beta}\right) \\ &\quad - \nabla_{\beta}\left(S^{(2)\mu}{}_{\alpha\nu} - 2K_1^{\mu\rho} S^{(1)}{}_{\rho\nu\alpha}\right) \\ &\quad + 2S^{(1)\mu}{}_{\nu\rho} S^{(1)\rho}{}_{\alpha\beta} - 2S^{(1)\mu}{}_{\beta\rho} S^{(1)\rho}{}_{\alpha\nu}.\end{aligned} \tag{4.5}$$

We can elaborate (4.5) by expanding the second terms in the parentheses and inserting $\nabla_{\mu} K_{1\alpha\beta} = S^{(1)}{}_{\alpha\beta\mu} + S^{(1)}{}_{\beta\alpha\mu}$. The result is

$$\begin{aligned}\left.\frac{d^2 R_{\varepsilon}{}^{\mu}{}_{\alpha\nu\beta}}{d\varepsilon^2}\right|_{\varepsilon=0} &= \nabla_{\nu} S^{(2)\mu}{}_{\alpha\beta} - \nabla_{\beta} S^{(2)\mu}{}_{\alpha\nu} \\ &\quad + 2K_1^{\mu\rho}\left(\nabla_{\beta} S^{(1)}{}_{\rho\nu\alpha} - \nabla_{\nu} S^{(1)}{}_{\rho\alpha\beta}\right) \\ &\quad + 2S^{(1)\rho}{}_{\nu\alpha} S^{(1)}{}_{\rho}{}^{\mu}{}_{\nu} - 2S^{(1)\rho}{}_{\beta\alpha} S^{(1)}{}_{\rho}{}^{\mu}{}_{\nu}.\end{aligned} \tag{4.6}$$

From (4.4) and (4.6), the first and second order perturbations of the Ricci tensor are obtained by simply contracting the μ and ν indices, namely

$$R^{(1)}_{\alpha\beta} := \left.\frac{dR_{\varepsilon\alpha\beta}}{d\varepsilon}\right|_{\varepsilon=0} = \nabla_{\mu} S^{(1)\mu}{}_{\alpha\beta} - \nabla_{\beta} S^{(1)\mu}{}_{\alpha\mu} \tag{4.7}$$

$$= \frac{1}{2}\left(\nabla_{\mu}\nabla_{\alpha} K_1{}^{\mu}{}_{\beta} + \nabla_{\mu}\nabla_{\beta} K_1{}^{\mu}{}_{\alpha} - \nabla_{\mu}\nabla^{\mu} K_{1\alpha\beta} - \nabla_{\alpha}\nabla_{\beta} K_1{}^{\mu}{}_{\mu}\right), \tag{4.8}$$

$$
\begin{aligned}
(4.9) \quad R^{(2)}_{\alpha\beta} &:= \left.\frac{d^2 R_{\varepsilon\alpha\beta}}{d\varepsilon^2}\right|_{\varepsilon=0} \\
&= \frac{1}{2}\left(\nabla_\mu\nabla_\alpha K_2{}^\mu{}_\beta + \nabla_\mu\nabla_\beta K_2{}^\mu{}_\alpha - \nabla_\mu\nabla^\mu K_{2\alpha\beta} - \nabla_\beta\nabla_\alpha K_2{}^\mu{}_\mu\right) \\
&\quad + \frac{1}{2}\nabla_\beta\nabla_\alpha\left(K_1{}^{\mu\rho}K_{1\mu\rho}\right) - \left(\nabla_\beta K_1{}^{\mu\rho}\right)\left(\nabla_\alpha K_{1\mu\rho}\right) \\
&\quad + K_1{}^{\mu\rho}\left(\nabla_\mu\nabla_\rho K_{1\alpha\beta} - \nabla_\mu\nabla_\alpha K_{1\rho\beta} - \nabla_\mu\nabla_\beta K_{1\rho\alpha}\right) \\
&\quad + 2S^{(1)\rho}{}_{\mu\alpha}S^{(1)}{}_\rho{}^\mu{}_\beta - 2S^{(1)\rho}{}_{\beta\alpha}S^{(1)}{}_\rho{}^\mu{}_\mu,
\end{aligned}
$$

where we have inserted (4.2)–(4.3) and in the second expression we have also used

$$
\begin{aligned}
2K_1{}^{\mu\rho}\nabla_\beta S^{(1)}{}_{\rho\mu\alpha} &= K_1{}^{\mu\rho}\nabla_\beta\left(S^{(1)}{}_{\rho\mu\alpha} + S^{(1)}{}_{\mu\rho\alpha}\right) \\
&= K_1{}^{\mu\rho}\nabla_\beta\nabla_\alpha K_{1\mu\rho} \\
&= \frac{1}{2}\nabla_\beta\nabla_\alpha\left(K_1{}^{\mu\rho}K_{1\mu\rho}\right) - \left(\nabla_\beta K_1{}^{\mu\rho}\right)\left(\nabla_\alpha K_{1\mu\rho}\right).
\end{aligned}
$$

Expression (4.9) is advantageous over alternative forms because it is manifestly symmetric in α, β.

4.2. Perfect fluid source

Let us now assume that the matter content of the perturbed scheme is a perfect fluid, that is, the energy momentum tensor $\widehat{T}_\varepsilon$ at each $(M_\varepsilon, \hat{g}_\varepsilon)$ has the form

$$
(4.10) \qquad \widehat{T}_\varepsilon - \frac{1}{2}(\mathrm{tr}_{\hat{g}_\varepsilon}\widehat{T}_\varepsilon)\hat{g}_\varepsilon = (\hat{E}_\varepsilon + \hat{P}_\varepsilon)\hat{\boldsymbol{u}}_\varepsilon \otimes \hat{\boldsymbol{u}}_\varepsilon + \frac{1}{2}(\hat{E}_\varepsilon - \hat{P}_\varepsilon)\hat{g}_\varepsilon,
$$

where $\hat{\boldsymbol{u}}_\varepsilon$ is the ($\hat{g}_\varepsilon$-unit) one-form fluid flow, and $\hat{E}_\varepsilon$ and $\hat{P}_\varepsilon$ the mass-energy density and pressure. These expressions are pullbacked onto (M, g) as

$$
(4.11) \qquad T_\varepsilon - \frac{1}{2}(\mathrm{tr}_{g_\varepsilon}T_\varepsilon)g_\varepsilon = (E_\varepsilon + P_\varepsilon)\boldsymbol{U}_\varepsilon \otimes \boldsymbol{U}_\varepsilon + \frac{1}{2}(E_\varepsilon - P_\varepsilon)g_\varepsilon,
$$

where, in particular, $\boldsymbol{U}_\varepsilon := \psi^*_\varepsilon(\hat{\boldsymbol{u}}_\varepsilon)$. The vectors (in contravariant form) $\hat{u}_\varepsilon$ are pushforwarded through ψ_ε^{-1} to a family of fluid vectors $u_\varepsilon := d\psi_\varepsilon^{-1}(\hat{u}_\varepsilon)$. It is immediate that $\boldsymbol{U}_\varepsilon = g_\varepsilon(u_\varepsilon, \cdot)$, $\boldsymbol{U}_\varepsilon(u_\varepsilon) = -1$ hold.

The field equations of the perturbed scheme are $\widehat{\mathrm{Ein}}_{\hat{g}_\varepsilon} = \varkappa \widehat{T}_\varepsilon$, and are pullbacked onto (M, g), and rearranged, as

$$\mathrm{Ric}_\varepsilon = \varkappa(T_\varepsilon - \frac{1}{2}(\mathrm{tr}_{g_\varepsilon} T_\varepsilon) g_\varepsilon). \tag{4.12}$$

Define

$$E^{(1)} := \left.\frac{dE_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0}, \quad P^{(1)} := \left.\frac{dP_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0}, \quad u^{(1)} := \left.\frac{du_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0},$$
$$E^{(2)} := \left.\frac{d^2E_\varepsilon}{d\varepsilon^2}\right|_{\varepsilon=0}, \quad P^{(2)} := \left.\frac{d^2P_\varepsilon}{d\varepsilon^2}\right|_{\varepsilon=0}, \quad u^{(2)} := \left.\frac{d^2u_\varepsilon}{d\varepsilon^2}\right|_{\varepsilon=0}.$$

From $\boldsymbol{U}_\varepsilon = g_\varepsilon(u_\varepsilon, \cdot)$ the perturbations of the fluid velocity one-forms are

$$\left.\frac{d\boldsymbol{U}_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0} = K_1(u, \cdot) + \boldsymbol{u^{(1)}}, \tag{4.13}$$
$$\left.\frac{d^2\boldsymbol{U}_\varepsilon}{d\varepsilon^2}\right|_{\varepsilon=0} = K_2(u, \cdot) + 2K_1(u^{(1)}, \cdot) + \boldsymbol{u^{(2)}}, \tag{4.14}$$

where $u := u_\varepsilon|_{\varepsilon=0}$ is the background fluid velocity vector. The normalisation condition $\boldsymbol{U}_\varepsilon(u_\varepsilon) = -1$ implies, upon taking successive ε derivatives at $\varepsilon = 0$, the two algebraic constraints

$$2\boldsymbol{u}(u^{(1)}) + K_1(u, u) = 0, \tag{4.15}$$
$$K_2(u, u) + 4K_1(u^{(1)}, u) + 2\boldsymbol{u^{(1)}}(u^{(1)}) + 2\boldsymbol{u}(u^{(2)}) = 0, \tag{4.16}$$

which determine the components of $u^{(1)}$ and $u^{(2)}$ along u. The perturbed Einstein field equations arise from the ε derivatives of (4.12) with (4.11), and yield

$$\begin{aligned} R^{(1)}_{\alpha\beta} = {} & \varkappa \left(E^{(1)} + P^{(1)}\right) u_\alpha u_\beta \\ & + \varkappa(E + P) \left((K_{1\alpha\mu} u^\mu + u^{(1)}{}_\alpha) u_\beta + (K_{1\beta\mu} u^\mu + u^{(1)}{}_\beta) u_\alpha \right) \\ & + \frac{1}{2}\varkappa \left(E^{(1)} - P^{(1)}\right) g_{\alpha\beta} + \frac{1}{2}\varkappa (E - P) K_{1\alpha\beta}, \end{aligned} \tag{4.17}$$

after using (4.13) and (4.15). The second order equations are similarly obtained from the second derivative of (4.12) and using (4.13)–(4.16),

(4.18)
$$\begin{aligned}
R^{(2)}_{\alpha\beta} &= \varkappa \left(E^{(2)} + P^{(2)}\right) u_\alpha u_\beta \\
&\quad + 2\varkappa(E^{(1)} + P^{(1)}) \Big(\big(K_{1\alpha\mu}u^\mu + u^{(1)}{}_\alpha\big)u_\beta + \big(K_{1\beta\mu}u^\mu + u^{(1)}{}_\beta\big)u_\alpha \Big) \\
&\quad + \varkappa(E+P) \Big(\big(K_{2\alpha\mu}u^\mu + 2K_{1\alpha\mu}u^{(1)\mu} + u^{(2)}{}_\alpha\big)u_\beta \\
&\qquad + \big(K_{2\beta\mu}u^\mu + 2K_{1\beta\mu}u^{(1)\mu} + u^{(2)}{}_\beta\big)u_\alpha \\
&\qquad +2\big(K_{1\alpha\mu}u^\mu + u^{(1)}{}_\alpha\big)\big(K_{1\beta\mu}u^\mu + u^{(1)}{}_\beta\big)\Big) \\
&\quad + \frac{1}{2}\varkappa\left(E^{(2)} - P^{(2)}\right) g_{\alpha\beta} + \varkappa\left(E^{(1)} - P^{(1)}\right) K_{1\alpha\beta} + \frac{1}{2}\varkappa\,(E-P)\, K_{2\alpha\beta}.
\end{aligned}$$

Let us now assume that the spacetime (M, g) admits a hypersurface orthogonal timelike Killing vector ξ and an axial Killing vector η. We assume further that the perturbation scheme inherits the local symmetry generated by ξ and η and that for each ε, the spacetime $(M_\varepsilon, \hat{g}_\varepsilon)$ is a solution of the Einstein's field equations for a rigidly rotating perfect fluid, i.e. that there exists a constant (on each ε) Ω_ε and a positive function $\hat{N}_\varepsilon \in C^{n+1}(M_\varepsilon)$ such that

$$\hat{u}_\varepsilon = \hat{N}_\varepsilon(\hat{\xi}_\varepsilon + \Omega_\varepsilon \hat{\eta}_\varepsilon), \tag{4.19}$$

where $\hat{\xi}_\varepsilon := d\psi_\varepsilon(\xi)$ and $\hat{\eta}_\varepsilon := d\psi_\varepsilon(\eta)$. The pullback of the field equations of the perturbed scheme and the relations (4.19) on M translate into the spacetime (M, g_ε) being a solution of the Einstein's field equations (4.12) with (4.11) and

$$u_\varepsilon = N_\varepsilon \left(\xi + \Omega_\varepsilon \eta\right)$$

for some positive function $N_\varepsilon \in C^{n+1}(M)$.

Staticity of the background imposes that u is parallel to ξ and then (2.1) implies

$$u = e^{-\frac{\nu}{2}}\xi, \qquad \Longleftrightarrow \qquad N_\varepsilon|_{\varepsilon=0} = e^{-\frac{\nu}{2}}, \qquad \Omega_\varepsilon|_{\varepsilon=0} = 0. \tag{4.20}$$

In terms of the following quantities

$$\Omega^{(1)} := \left.\frac{d\Omega_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0}, \qquad \Omega^{(2)} := \left.\frac{d^2\Omega_\varepsilon}{d\varepsilon^2}\right|_{\varepsilon=0},$$
$$u^{(1)0} := \left.\frac{dN_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0}, \qquad u^{(2)0} := \left.\frac{d^2N_\varepsilon}{d\varepsilon^2}\right|_{\varepsilon=0},$$

the first and second order pertubation fluid velocity vectors are, using (4.20),

$$u^{(1)} = u^{(1)0}\xi + e^{-\frac{\nu}{2}}\Omega^{(1)}\eta,$$
$$u^{(2)} = u^{(2)0}\xi + \left(2u^{(1)0}\Omega^{(1)} + e^{-\frac{\nu}{2}}\Omega^{(2)}\right)\eta.$$

Recall that the components $u^{(1)0}$ and $u^{(2)0}$ are determined by the algebraic constraints (4.15)–(4.16), so we can write $u^{(1)}$ and $u^{(2)}$ in terms of the metric perturbation tensors and the constants $\Omega^{(1)}$ and $\Omega^{(2)}$ as

(4.21)
$$u^{(1)} = \frac{1}{2}e^{-\frac{3\nu}{2}}K_1(\xi,\xi)\xi + e^{-\frac{\nu}{2}}\Omega^{(1)}\eta,$$
(4.22)
$$u^{(2)} = e^{-\frac{3\nu}{2}}\left(\frac{1}{2}K_2(\xi,\xi) + \frac{3}{4}e^{-\nu}K_1(\xi,\xi)^2 + 2\Omega^{(1)}K_1(\xi,\eta) + \Omega^{(1)2}\langle\eta,\eta\rangle\right)\xi$$
$$+ e^{-\frac{\nu}{2}}\left(e^{-\nu}K_1(\xi,\xi)\Omega^{(1)} + \Omega^{(2)}\right)\eta.$$

We now exploit a well-known relation between orthogonal transitivity of the Abelian group action and rigid rotation of the self-gravitating fluid, forced upon by the Einstein field equations (see e.g. [38, Chapter 19.2]). In our present set up, the specific result we need is as follows.

Proposition 4.1 (Rigid rotation and orthogonal transitivity). *Let (M,g) be a spacetime with C^{n+1} ($n \geq 3$) metric that admits an Abelian G_2 group of isometries generated by $\{\xi,\eta\}$. Assume also that M is simply connected and that η is an axial symmetry with a non-empty set of fixed points. Let $(M_\varepsilon, \hat{g}_\varepsilon, \{\psi_\varepsilon\})$ be a C^{n+1} maximal perturbation scheme inheriting this group (c.f. Definition 2.1 in [25]). If the matter content of the perturbation scheme is that of a rigidly rotating perfect fluid (or vacuum), then the background is orthogonally transitive and the perturbation scheme inherits this property.*

Proof. By assumption, for each ε the spacetime $(M_\varepsilon, \hat{g}_\varepsilon)$ is a solution of the Einstein's field equations with (4.10) and (4.19). Therefore it is straightforward to check that the one-forms $\widehat{\mathrm{Ric}_\varepsilon}(\hat{\xi}_\varepsilon, \cdot)$ and $\widehat{\mathrm{Ric}_\varepsilon}(\hat{\eta}_\varepsilon, \cdot)$ lie in $span\{\hat{\boldsymbol{\xi}}_\varepsilon, \hat{\boldsymbol{\eta}}_\varepsilon\}$, where $\hat{\boldsymbol{\eta}}_\varepsilon = \hat{g}_\varepsilon(\hat{\eta}_\varepsilon, \cdot)$ and $\hat{\boldsymbol{\xi}}_\varepsilon = \hat{g}_\varepsilon(\hat{\xi}_\varepsilon, \cdot)$. Standard curvature identities (see e.g. [38, Chapter 19.2]) imply that this fact, in addition to the commutation of ξ and η, implies that the functions $O_{1\varepsilon} := \star(\hat{\boldsymbol{\eta}}_\varepsilon \wedge \hat{\boldsymbol{\xi}}_\varepsilon \wedge d\hat{\boldsymbol{\xi}}_\varepsilon)$ and $O_{2\varepsilon} := \star(\hat{\boldsymbol{\eta}}_\varepsilon \wedge \hat{\boldsymbol{\xi}}_\varepsilon \wedge d\hat{\boldsymbol{\eta}}_\varepsilon)$ satisfy [19]

$$dO_{1\varepsilon} = 0, \qquad dO_{2\varepsilon} = 0.$$

Simply connectedness implies that $O_{1\varepsilon}$ and $O_{2\varepsilon}$ are constant (for each ε), while the existence of points where η vanishes (the axis) implies they must, in fact, be zero. Therefore the group generated by $\{\hat{\xi}_\varepsilon, \hat{\eta}_\varepsilon\}$ on $(M_\varepsilon, \hat{g}_\varepsilon)$ for each ε is orthogonal transitive. The result follows, in particular for $\varepsilon = 0$. □

Observe that the fluid quantities are gauge dependent in general. In particular, the perturbed pressure at first and second order transforms as (see e.g. [7])

$$P^{(1)g} = P^{(1)} + V_1(P), \quad P^{(2)g} = P^{(2)} + V_2(P) + V_1(P^{(1)} + P^{(1)g}). \tag{4.23}$$

Eventually, uniqueness of the solutions will rely on one free integration constant associated to the value of the perturbed pressure at the origin. In order to assign a clear physical meaning to this parameter it is necesary to show that the perturbed central pressure is a gauge invariant quantity. The next lemma proves this fact, to second order, within the perturbation scheme of Theorem 2.2 provided the configuration is equatorially symmetric. Later on we will show that this symmetry is a necessary consequence of the field equations.

Lemma 4.2. *Within the C^{n+1} ($n \geq 3$) perturbation scheme introduced in Theorem 2.2 and with the definitions above, (i) the value $P^{(1)}(0) := P^{(1)}|_{r=0}$ is gauge invariant and (ii) if the configuration has equatorial symmetry then $P^{(2)}(0) := P^{(2)}|_{r=0}$ is also gauge invariant.*

Proof. The first order V_1 and second order V_2 gauge vectors within the perturbation scheme are $C^{n+1}(M)$ and $C^n(M)$ respectively (see [25]). On the other hand, P is $C^{n-1}(M)$. Note $P^{(1)}$ and $P^{(2)}$ are $C^{n-2}(M)$ and $C^{n-3}(M)$ respectively and therefore both functions are continuous, in particular, at the origin. First, Lemma 3.2 implies dP vanishes at $r = 0$. Therefore, the term $V_1(P) \equiv dP(V_1)$ vanishes at $r = 0$, and the claim (i) follows from (4.23).

The same argument applies to $V_2(P)$, so in order to show gauge invariance of $P^{(2)}$ (4.23) it suffices to prove that $V_1(P^{(1)} + P^{(1)g}) = 0$ at the origin. Under the stationary and axisymmetric perturbation scheme of Theorem 2.2, the functions $P^{(1)}$ and $P^{(1)g}$ are time independent and axially symmetric. Moreover, by assumption they are equatorially symmetric, i.e. invariant under $z \to -z$. Let $f \in C^1(M)$ be any function with these properties and consider the equatorially invariant hypersurface $\mathcal{E} := \{z = 0\}$. The restriction $f|_{\mathcal{E}}$ is independent of t and radially symmetric in $\{x, y\}$. Hence Lemma 3.2 implies that $d(f|_{\mathcal{E}})$ vanishes at the origin. We can decompose uniquely $V_1 = V_1^z + V_1^{\mathcal{E}}$, with V_1^z along ∂_z and $V_1^{\mathcal{E}}$ tangent to $\mathcal{E}$. By equatorial symmetry $V_1^z(f)|_{\mathcal{E}} = 0$. Therefore $V_1(f)|_{\mathcal{E}} = V_1^{\mathcal{E}}(f)|_{\mathcal{E}} = V_1^{\mathcal{E}}(f|_{\mathcal{E}})$, and thus $V_1(f)|_{\mathcal{E}} = d(f|_{\mathcal{E}})(V_1^{\mathcal{E}})$ vanishes at the origin. Applying this fact to $f = P^{(1)} + P^{(1)g}$, the claim (ii) follows. □

So far no equation of state for the perfect fluid has been imposed. We shall say that the perturbation scheme satisfies a barotropic equation of state if there exists a C^2 function of one variable $P(E)$ such that, for each value of ε, the pressure and density of the fluid are related by $P_\varepsilon = P(E_\varepsilon)$. Note that we do not allow dependence on ε in the equation of state itself, and thus the barotropic EOS is that of the background. Taking ε-derivatives at $\varepsilon = 0$, the perturbed pressures are written in terms of the perturbed densitities as

$$\begin{aligned} &P^{(1)} - \frac{dP}{dE} E^{(1)} = 0, \\ &P^{(2)} - \frac{dP}{dE} E^{(2)} - \frac{d^2P}{dE^2} {E^{(1)}}^2 = 0. \end{aligned} \tag{4.24}$$

where the derivatives $\frac{dP}{dE}$ etc. are evaluated at the background density.

5. "Base" global perturbation scheme

In order to tackle the first and second order problems we have to deal with the first and second order perturbation tensors as given in (2.11) and (2.12). It is obvious that the problem involves two steps, namely addressing the first order problem first and dealing with the second order one afterwards. However, as mentioned in the Introduction, there is a strategy that allows one to treat both cases at the same time, and this entails a considerable simplification of the proof.

The underlying idea is that a second order perturbation problem under the assumption that the first order perturbation tensor vanishes identically is completely equivalent to a first order problem. This fact is both physically and geometrically clear, and can be checked explicitly: all the equations, matching conditions, etc. are identical for the first order perturbation tensor and for the second order perturbation tensor after setting the first order tensor to zero (the only difference is in the differentiability class, which to second order is one order lower than to first order, but this poses no problem as the differentiability assumed at second order suffices for the argument).

The idea is then to use a bootstrap type of argument. We assume a specific form for the first order perturbation tensor which includes zero as a particular case, and leave the second order tensor completely free (within our perturbation scheme, naturally). We then analyze the second order problem in full detail. After this has been done, we set the first order perturbation tensor to zero and all the conclusions that we find apply immediately to the first order problem. This will allow us to show that our assumption on the first order perturbation tensor is in fact a consequence of the first order problem. This will close the bootstrap and hence all the results obtained under the scheme will be fully general. We call the restricted second order problem the "base" perturbation scheme and is built as follows:

The base perturbation scheme: The global manifold consists of an interior region (M^+) and an exterior region (M^-) separated by a hypersurface Σ. We construct a second order perturbation for each region around a background configuration (we drop $\pm$ indices) satisfying items B1-B3 below. We then construct the global model by solving the most general perturbed matching problem (item B4). We finally consider a *barotropic base scheme* (and will explicitly specify *barotropic*) when the barotropic EOS of the background is assumed in the interior, i.e. items B1-B5 below hold.

B1: *The background corresponds to a finite perfect fluid ball configuration according to Definition 3.3 with $E_c + P_c \neq 0$.*

Remark: In particular, the metric, which we take to be C^{n+1} on each region with $n \geq 4$, is given by (2.1) with $\mathcal{R}(r) = r$, and $\xi := \partial_t$, $\eta := \partial_\phi$. Define $\mathfrak{S}_{\alpha\beta} := \xi_\alpha \eta_\beta + \xi_\beta \eta_\alpha$, so that $\mathfrak{S} = -2e^\nu r^2 \sin^2\theta dt d\phi$ in those coordinates.

B2: *The first order metric perturbation tensors $K_1^\pm$ are bounded and*

B2.1: *read (dropping $\pm$ indices)*

$$K_1 = \varpi e^{-\nu}\mathfrak{S}, \tag{5.1}$$

for given functions $\varpi_+ \in C^{n+2}(M^+ \setminus \mathcal{C}_0) \cap C^2(M^+)$ and $\varpi_- \in C^\infty(M^-)$, both radially symmetric and bounded.

Remark: In spherical coordinates, this assumption translates onto

$$K_1 = -2\varpi(r) r^2 \sin^2\theta dt d\phi, \tag{5.2}$$

where $\varpi_-(r)$ is $C^\infty([a,\infty))$, and, by virtue of Lemma 3.2, $\varpi_+(r)$ is $C^2([0,a]) \cap C^{n+2}((0,a])$, and admits the decomposition

$$\varpi_+(r) = \varpi_0 + \varpi_2 r^2 + \Phi^{(2)}_\varpi(r), \tag{5.3}$$

where $\varpi_0, \varpi_2 \in \mathbb{R}$ and $\Phi^{(2)}_\varpi(r) \in C^2([0,a])$ and $o(r^2)$.

B2.2: *The functions $\varpi_\pm(r)$ satisfy the equation (we drop the $\pm$ signs)*

$$\begin{aligned} &\frac{1}{r^3}\frac{d}{dr}\left(r^4 j \frac{d\varpi}{dr}\right) + 4j'(\varpi - \Pi^{(1)}) = 0 \quad \Longleftrightarrow \\ &\varpi'' = (\lambda' + \nu')\left(\frac{1}{2}\varpi' + \frac{2}{r}(\varpi - \Pi^{(1)})\right) - \frac{4}{r}\varpi' \end{aligned} \tag{5.4}$$

where $\Pi^{(1)}_+ \in \mathbb{R}$ in M^+ and $\Pi^{(1)}_- = 0$ in M^-.

Remark: In M^- we have $j' = 0$ and thus the value of $\Pi^{(1)}_-$ is irrelevant, so we fix it to zero for definiteness. The general solution on M^- is given by

$$\varpi_-(r) = \frac{2J_\varpi}{r^3} + \varpi_\infty, \qquad \text{with} \qquad J_\varpi, \varpi_\infty \in \mathbb{R}.$$

The vector fields $r^{-1}\eta$ and ξ are smooth and bounded in M^-. Thus, boundedness of K_1 demands that $K_1(\xi, r^{-1}\eta) = -\varpi(r)e^{-\nu} r \sin^2\theta$ is also bounded. This condition clearly requires $\varpi_\infty = 0$ and the function ϖ is

$$\varpi_-(r) = \frac{2J_\varpi}{r^3}, \qquad J_\varpi \in \mathbb{R}. \tag{5.5}$$

By Proposition 2.5, a first order gauge transformation with $V_1^- = C^- t\partial_\phi$ changes $\varpi_- \to \varpi_- - C^-$. Only $C^- = 0$ respects the condition $\varpi_\infty = 0$ and we conclude that boundedness of K_1 fixes the first order gauge freedom in the exterior completely.

In the interior region M^+, equation (5.4) combined with (3.27) determines

$$\varpi_2 = \frac{2}{5}(\varpi_0 - \Pi^{(1)}_+)(\lambda_2 + \nu_2). \tag{5.6}$$

Clearly, if $\varpi_+ = 0$ then (5.4) implies $\Pi^{(1)}_+ = 0$.

We introduce an auxiliary number[3] m, restricted to $n \geq m \geq 2$, that prescribes the differentiability of K_2, and assume the outcome of Theorem 2.2.

B3: *The second order metric perturbation tensors* $K_2^{\pm}$ *satisfy:*

B3.1: $K_2^{\pm}$ *are* C^m *(in* $M^+ \setminus \mathcal{C}_0$ *and* M^-*, respectively) and bounded everywhere, and in spherical coordinates (dropping* $\pm$ *indices) have the form*

$$K_2 = \left(-4e^{\nu(r)}h(r,\theta) + 2\varpi^2(r)r^2\sin^2\theta\right)dt^2 + 4e^{\lambda(r)}m(r,\theta)dr^2 + 4k(r,\theta)r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) + 4e^{\lambda(r)}\partial_\theta f(r,\theta)rdrd\theta - 2\mathcal{W}(r,\theta)r^2\sin^2\theta dtd\phi \tag{5.7}$$

$$=: K_2^H + \mathcal{W}e^{-\nu}\mathfrak{S}, \tag{5.8}$$

where the functions in (5.7) *correspond to the traces (in* $\{x,y\}$*) of the axially symmetric functions with same name, which satisfy*

* $h_+, m_+, k_+ \in C^{m+1}(M^+ \setminus \mathcal{C}_0)$ *and bounded near* $\mathcal{C}_0$, $h_-, m_-, k_- \in C^{m+1}(M^-)$
* $\mathcal{W}_+ \in C^m(M^+ \setminus \mathcal{C}_0)$ *and bounded near* $\mathcal{C}_0$, $\mathcal{W}_- \in C^m(M^-)$,
* *the vectors* $\mathcal{W}\eta_\pm$ *are* $C^{m+1}(M^+ \setminus \mathcal{C}_0)$ *and* $C^{m+1}(M^-)$ *respectively.*
* $f_+ \in C^m(M^+ \setminus \mathcal{C}_0)$ *and bounded near* $\mathcal{C}_0$, $f_- \in C^m(M^-)$*, both* $f_\pm$ *are* $C^{m+1}(S_r)$ *on all spheres* S_r*, all* $\partial_r f_\pm$ *and* $\partial_t f_\pm$ *are* $C^m(S_r)$ *on all spheres* S_r*, and finally* $\partial_\theta f_\pm$ *are* C^m *outside the axis* $\mathcal{A}$ *and extend continuously to* $\mathcal{A} \setminus \mathcal{C}_0$ *where they vanish.*

B3.2: K_2^- *solves the second order field equations* (4.18) *for vacuum and* K_2^+ *solves the second order field equations* (4.18) *for a rigidly rotating perfect fluid* (4.22) *with* $\Omega^{(1)} = \Pi_+^{(1)}$ *and* $\Omega^{(2)} = \Pi_+^{(2)} \in \mathbb{R}$.

B4: *The first and second order perturbed matching conditions (c.f. Appendix B) hold on* Σ*. Moreover we assume* $[\varpi] = 0$.

Remark: The first order perturbed matching conditions demand $[\varpi] = b_1 \in \mathbb{R}$ (see Proposition B.1). By Proposition 2.5 a first order gauge transformation with $V_1^+ = C^+ t\partial_\phi$ changes $\varpi_+ \to \varpi_+ - C^+$. Given that C^- has already been fixed to zero, the condition $[\varpi] = 0$ is fulfilled if and only if C^+ is chosen to be $C^+ = b_1$. We conclude that (i) the assumption $[\varpi] = 0$ entails no loss of generality, and (ii) that

[3] This parameter is not to be confused with the function m. The context will clarify the intended meaning.

this condition together with boundedness at infinity fixes completely the gauge freedom at first order, c.f. Proposition 2.5.

B5: *The perfect fluid satisfies the background barotropic equation of state.*

Some additional remarks are in order. We first stress that we *do not assume equatorial symmetry.* The assumption of boundedness on K_2 will not play any role until we tackle the global problems in Section 6. Finally, proving that B2.1 and B2.2 hold necessarily, and that there exist indeed solutions of (5.4) in $C^2([0,a)) \cap C^{n+2}((0,a))$, will be part of the bootstrap argument.

Observe that the full set of gauge transformations compatible with the base scheme is given by Proposition 2.5, restricted to $C^+ = C^- = 0$, or, in the Notation 2.7, the class $\{\Psi(A, B, \mathcal{Y}, \alpha)\}$.

We introduce at this point some relevant definitions and notation that will be useful for both the interior (+) and the exterior (−) problems. Let $D = \mathbb{R}^3 \setminus \{0\}$, $D^+ = \overline{B_a} \setminus \{0\}$ and $D^- = D \setminus B_a$, where B_a is the ball of radius $a > 0$ centered at the origin. Since we deal with interior and exterior functions that take different values at the boundary we also introduce the disjoint union $\widehat{D} := D^+ \sqcup D^-$ endowed with the disjoint union topology. Let $\{r, \theta, \phi\}$ be standard spherical coordinates on $\widehat{D}$. For each $r > 0$, we let $S_r := \{|x| = r\}$. Observe both $D^\pm$ contain S_a and that $\widehat{D}$ constains two copies thereof. We shall use the following notation for the geometry of S_r.

Notation 5.1 (Notation in S_r). We endow S_r with the standard metric of radius one $g_{\mathbb{S}^2} = d\theta^2 + \sin^2\theta d\phi^2$. We fix the orientation of S_r so that $\{\partial_\theta, \partial_\phi\}$ is positively oriented and denote by $\boldsymbol{\eta}_{\mathbb{S}^2}$ the corresponding volume form. The Hodge dual on p-forms of $(S_r, g_{\mathbb{S}^2})$ is denoted by $\star_{\mathbb{S}^2}$ and we define $\bar{\boldsymbol{\eta}} := g_{\mathbb{S}^2}(\eta, \cdot)$ i.e. the metrically related one-form of $\eta = \partial_\phi$. The covariant derivative associated to $g_{\mathbb{S}^2}$ is $\overline{D}$, the corresponding Laplacian is $\Delta_{\mathbb{S}^2}$ and tensors on $(S_r, g_{\mathbb{S}^2})$ carry capital Latin indices $A, B, \cdots$.

We define P_ℓ to be the Legendre polynomial of order ℓ on $(S_r, g_{\mathbb{S}^2})$. More precisely, P_ℓ is the only solution invariant under $\eta = \partial_\phi$ of the eigenvalue problem $(\Delta_{\mathbb{S}^2} + \ell(\ell+1))P_\ell = 0$, $\ell \in \mathbb{N}$, with the normalization choice $\int_{\mathbb{S}^2} P_\ell P_{\ell'} \boldsymbol{\eta}_{\mathbb{S}^2} = \frac{4\pi}{2\ell+1}\delta_{\ell\ell'}$ and satisfying $P_\ell > 0$ on the north pole (defined by $\theta = 0$). The first three Legendre polynomials are $P_0 = 1$, $P_1 = \cos\theta$, $P_2 = \frac{1}{2}(3\cos^2\theta - 1)$. Given any function $f : \widehat{D} \to \mathbb{R}$ we define the following 'components'

$$f_\ell(r) := \frac{2\ell+1}{4\pi} \int_{S_r} f P_\ell \boldsymbol{\eta}_{\mathbb{S}^2}. \tag{5.9}$$

We emphasize that the integration is on S_r with the volume form of the standard round metric of radius one (in particular $\int_{S_r} \boldsymbol{\eta}_{\mathbb{S}^2} = 4\pi$). For functions f independent of ϕ, an alternative equivalent definition is $f_\ell(r) := \frac{2\ell+1}{2}\int_0^\pi f(r,\theta)P_\ell(\cos\theta)\sin\theta d\theta$. Whenever the ℓ subindices can lead to confusion, we will also use $f_{(0)}$, $f_{(1)}, f_{(2)}$ etc.

We will use the same name for objects $F_\pm$ defined respectively on $M^\pm$ invariant under ξ and the corresponding objects $F_\pm$ defined on D^+ and D^-. Any object F defined on $\widehat{D}$ is said to be composed by $\{F_+, F_-\}$ when $F|_{D^\pm} = F_\pm$. Viceversa, given $\{F_+, F_-\}$ defined on $D^\pm$ we will use F to refer to the object defined on $\widehat{D}$ whose restriction to $D^\pm$ is $F_\pm$. For scalar functions, whenever $F_+ \in C^{m_1}(D^+)$ and $F_- \in C^{m_2}(D^-)$ we will equivalently write $F \in C^{m_1}(D^+) \cap C^{m_2}(D^-)$. Moreover, if $[F] = 0$ then F is also well defined on D, and if $F \in C^0(D^+) \cap C^0(D^-)$ then $F \in C^0(D)$. Observe this extends to any function and any of its (partial) derivatives iteratively. This notation will also translate to the corresponding intervals on the real line for the coordinate r.

5.1. Field equations for the base perturbation scheme

In this section we write down explicitly the field equations for the second order of the base perturbation scheme. More specifically, we develop the point B3.2 of the base perturbation scheme under assumptions B1 and B2. We use the results introduced in Section 4 combined with Appendix A, where we derive with covariant methods the first order perturbed Ricci tensor for a first order perturbation of the form $K_1 = w\mathfrak{S}$, with w depending on r, θ.

We start with the first order problem. Expression (4.21) and $K_1(\xi,\xi) = 0$ impose $u^{(1)} = \Pi^{(1)} e^{-\frac{\nu}{2}}\eta$, where the redefinition of constants $\Omega^{(1)} \to \Pi^{(1)}$ has been made. Thus

$$K_{1\alpha\mu}u^\mu + u^{(1)}{}_\alpha = e^{-\nu/2}(\Pi^{(1)} - \varpi)\eta_\alpha. \tag{5.10}$$

By Proposition A.1 and the notation introduced in Remark A.2, the first order perturbed Ricci tensor of (5.2) (defined in (4.8)) has the form $R^{(1)}_{\alpha\beta} = \mathfrak{R}(\varpi e^{-\nu})\mathfrak{S}_{\alpha\beta}$. Inserting into (4.17) shows that the first order Einstein field equations require

$$E^{(1)} = P^{(1)} = 0. \tag{5.11}$$

For the second order problem, it is advantageous to split the second order Ricci tensor $R^{(2)}_{\alpha\beta}$ of (5.2) and (5.7) into two terms. Define $R^{(2)H}_{\alpha\beta}$ as

$R^{(2)}_{\alpha\beta}$ computed with K_2^H in (5.8), i.e. $R^{(2)H}_{\alpha\beta} := R^{(2)}_{\alpha\beta}(\mathcal{W}=0)$. Then, by virtue of (4.9) and (4.8) together with Proposition A.1 we have

$$R^{(2)}_{\alpha\beta} = R^{(2)H}_{\alpha\beta} + \mathfrak{R}(\mathcal{W}e^{-\nu})\mathfrak{S}_{\alpha\beta}. \tag{5.12}$$

As for the right hand side of (4.18), we start by computing the second order perturbation vector $u^{(2)}$. We use $K_1(\xi,\eta) = -\varpi\langle\eta,\eta\rangle$ and $K_2(\xi,\xi) = -4e^{\nu}h + 2\varpi^2 r^2\sin^2\theta$, so that (4.22) yields, after the redefinition of constants $\Omega^{(2)} \to \Pi^{(2)}$,

$$u^{(2)} = e^{-\frac{3\nu}{2}}\left(-2e^{\nu}h + (\varpi - \Pi^{(1)})^2 r^2 \sin^2\theta\right)\xi + e^{-\frac{\nu}{2}}\Pi^{(2)}\eta. \tag{5.13}$$

One immediately finds

$$\begin{aligned} &{K_2^H}_{\alpha\mu}u^{\mu} + 2K_{1\alpha\mu}u^{(1)\mu} + {u^{(2)}}_{\alpha} \\ &\qquad = \left(2e^{\frac{-\nu}{2}}h - e^{-\frac{3\nu}{2}}\left(\varpi^2 - \Pi^{(1)2}\right)r^2\sin^2\theta\right)\xi_\alpha + e^{-\frac{\nu}{2}}\Pi^{(2)}\eta_\alpha. \end{aligned} \tag{5.14}$$

Inserting (5.10), (5.14) and (5.12) into (4.18) and using $\mathfrak{S}_{\alpha\mu}u^{\mu} = -\mathcal{W}e^{-\nu/2}\eta_\alpha$ the second order field equations take the form

$$\begin{aligned} R^{(2)H}_{\alpha\beta} &+ \mathfrak{R}(\mathcal{W}e^{-\nu})\mathfrak{S}_{\alpha\beta} = e^{-\nu}\varkappa\left(E^{(2)} + P^{(2)}\right)\xi_\alpha\xi_\beta \\ &+ \varkappa(E+P)\left\{\left(4e^{-\nu}h - 2e^{-2\nu}\left(\varpi^2 - \Pi^{(1)2}\right)r^2\sin^2\theta\right)\xi_\alpha\xi_\beta\right. \\ &\left.+ e^{-\nu}\Pi^{(2)}\mathfrak{S}_{\alpha\beta} + 2e^{-\nu}(\varpi - \Pi^{(1)})^2\eta_\alpha\eta_\beta\right\} + \frac{1}{2}\varkappa\left(E^{(2)} - P^{(2)}\right)g_{\alpha\beta} \\ &+ \frac{1}{2}\varkappa\,(E-P)\,{K_2^H}_{\alpha\beta} - \frac{1}{2}\varkappa(E+3P)\mathcal{W}e^{-\nu}\mathfrak{S}_{\alpha\beta}. \end{aligned} \tag{5.15}$$

Now, an explicit computation shows that $R^{(2)H}_{\alpha\beta}$ has vanishing $\xi_{(\alpha}\eta_{\beta)}$ component so equation (5.15) splits into two, namely the component along $\mathfrak{S}$, which is

$$-\mathfrak{R}(\mathcal{W}e^{-\nu}) + \varkappa(E+P)e^{-\nu}\Pi^{(2)} - \frac{1}{2}\varkappa(E+3P)\mathcal{W}e^{-\nu} = 0, \tag{5.16}$$

and the rest

$$
\begin{aligned}
0 = (\mathrm{Eq})_{\alpha\beta} := &- R^{(2)H}_{\alpha\beta} + e^{-\nu}\varkappa\left(E^{(2)} + P^{(2)}\right)\xi_\alpha\xi_\beta \\
&+ \varkappa(E+P)\left\{\left(4e^{-\nu}h - 2e^{-2\nu}(\varpi^2 - \Pi^{(1)2})r^2\sin^2\theta\right)\xi_\alpha\xi_\beta \right.\\
&\quad \left. +2e^{-\nu}(\varpi - \Pi^{(1)})^2\eta_\alpha\eta_\beta\right\} \\
&+ \frac{1}{2}\varkappa\left(E^{(2)} - P^{(2)}\right)g_{\alpha\beta} + \frac{1}{2}\varkappa\,(E-P)\,K^H_{2\ \alpha\beta}.
\end{aligned} \tag{5.17}
$$

In principle, these are nine equations (note $(\mathrm{Eq})_{t\phi} \equiv 0$ holds identically, by construction). Two of them, (Eq_{tt}) and (Eq_{rr}), determine $E^{(2)}$ and $P^{(2)}$ algebraically. For the moment we are interested in studying a subset of seven independent linear combinations which do not involve $E^{(2)}$ nor $P^{(2)}$. Introducing the notation $\mathfrak{a}, \mathfrak{b}, \cdots := \{r, \theta\}$ and $\mathfrak{i}, \mathfrak{j}, \cdots := \{t, \phi\}$, one convenient such subset is

$$
\begin{aligned}
&(\mathrm{Eq})_{\mathfrak{a}\,\mathfrak{i}} = 0, \quad (\mathrm{Eq})_{r\theta} = 0, \\
&\frac{(\mathrm{Eq})_{\phi\phi}}{g_{\phi\phi}} - \frac{(\mathrm{Eq})_{\theta\theta}}{g_{\theta\theta}} = 0, \quad \frac{(\mathrm{Eq})_{\theta\theta}}{g_{\theta\theta}} - \frac{(\mathrm{Eq})_{rr}}{g_{rr}} = 0.
\end{aligned} \tag{5.18}
$$

Before writing them down explicitly, let us introduce the following scalar functions and discuss their properties,

$$
\begin{aligned}
&\widehat{h} := h - \frac{1}{2}r\nu' f, \\
&\widehat{v} := k + \widehat{h} - f = k + h - f\left(\frac{1}{2}r\nu' + 1\right), \\
&\widehat{q} := m + \widehat{h} - e^{-\lambda/2}\left(e^{\lambda/2} r f\right)_{,r} = m + h - \frac{1}{2}rf\left(\lambda' + \nu'\right) - (rf)_{,r}.
\end{aligned} \tag{5.19}
$$

Given the differentiability and boundedness properties of the original set $\{h, m, k, f\}$ (point B3), and that $n \geq m$, we have $\widehat{h}, \widehat{v} \in C^m(D^+) \cap C^m(D^-)$, $C^{m+1}(S_r)$ on all spheres S_r, and bounded near $\mathcal{C}_0$, and $\widehat{q} \in C^{m-1}(D^+) \cap C^{m-1}(D^-)$ is also $C^m(S_r)$ on all spheres S_r.

The motivation behind these definitions is their very special gauge behaviour, as described in the following lemma. Its proof is by explicit calculation using the results of Proposition 2.5.

Lemma 5.2. *Under the gauge vector V_2 given by (2.16) with $\mathcal{R} = r$ the functions $\widehat{h}$, $\widehat{v}$, $\widehat{q}$ transform as*

$$\widehat{h}^g = \widehat{h} + \frac{1}{2}A + \frac{1}{2}r\nu'\left(re^{-\lambda}\alpha'\cos\theta - \beta\right), \tag{5.20}$$

$$\widehat{v}^g = \widehat{v} + \frac{1}{2}A + \alpha\cos\theta + \left(re^{-\lambda}\alpha'\cos\theta - \beta\right)\left(1 + \frac{1}{2}r\nu'\right), \tag{5.21}$$

$$\begin{aligned}\widehat{q}^g = \widehat{q} + \frac{1}{2}A + \cos\theta\left(\frac{1}{2}r^2e^{-\lambda}\alpha'\left(\lambda' + \nu'\right) + (r^2e^{-\lambda}\alpha')'\right)\\ - \left(\frac{1}{2}r\beta(\lambda' + \nu') + (r\beta)_{,r}\right),\end{aligned} \tag{5.22}$$

where $\beta(r)$ is the arbitrary function that enters f in (2.20).

Note that the gauge function $\mathcal{Y}$ has disappeared from these transformations so that $\widehat{h}$, $\widehat{v}$ and $\widehat{q}$ go a long way towards being gauge invariant. The remaining gauge transformation is fully explicit in the variable θ. This will be important in the following.

We can now write down explicitly equations (5.16) and (5.18). The first set involves a long computation which has been carried out with the aid of computer algebra systems. As for the second, its explicit form can be obtained directly from (A.7) after taking into account that λ_ξ and λ_η, as defined in Proposition A.1 take the form

$$\begin{aligned} R_{\alpha\beta}\xi^\beta = -\frac{\varkappa}{2}\left(E + 3P\right)\xi_\alpha, \quad R_{\alpha\beta}\eta^\beta = \frac{\varkappa}{2}\left(E - P\right)\eta_\alpha \\ \Longrightarrow \lambda_\xi = -\frac{\varkappa}{2}\left(E + 3P\right), \quad \lambda_\eta = \frac{\varkappa}{2}\left(E - P\right) \\ \Longrightarrow \lambda_\xi + \lambda_\eta = -2\varkappa P. \end{aligned}$$

The result is

Lemma 5.3. *In the setup described above, the field equations (5.16) and (5.18) take, respectively, the following explicit form*

$$\frac{\partial}{\partial r}\left(r^4 j\frac{\partial\mathcal{W}}{\partial r}\right) + \frac{r^2je^\lambda}{\sin^3\theta}\frac{\partial}{\partial\theta}\left(\sin^3\theta\frac{\partial\mathcal{W}}{\partial\theta}\right) + 4r^3j'(\mathcal{W} - \Pi^{(2)}) = 0, \tag{5.23}$$

and

$$0 = (Eq)_{\mathfrak{ai}} \equiv 0, \tag{5.24}$$

$$0 = (Eq)_{r\theta} \equiv \frac{\partial}{\partial\theta}\left(\frac{2}{r}\widehat{q} - 2\widehat{v}_{,r} + (\widehat{q} - 2\widehat{h})\nu'\right), \tag{5.25}$$

$$\begin{aligned} 0 &= \frac{(Eq)_{\phi\phi}}{g_{\phi\phi}} - \frac{(Eq)_{\theta\theta}}{g_{\theta\theta}} \\ &\equiv -e^{-(\lambda+\nu)} r \sin^2\theta \left(r\varpi'^2 + 2\left(\lambda' + \nu'\right)(\varpi - \Pi^{(1)})^2\right) \\ &\quad + \frac{2}{r^2}\left(\widehat{q}_{,\theta\theta} - \frac{\cos\theta}{\sin\theta}\widehat{q}_{,\theta}\right), \end{aligned} \tag{5.26}$$

$$\begin{aligned} 0 &= \frac{(Eq)_{\theta\theta}}{g_{\theta\theta}} - \frac{(Eq)_{rr}}{g_{rr}} \\ &\equiv 2e^{-\lambda}\widehat{v}_{,rr} - \frac{2}{r^2}\Delta_{\mathbb{S}^2}\widehat{v} - e^{-\lambda}(\nu' + \lambda')\widehat{v}_{,r} - \frac{4}{r^2}\widehat{v} \\ &\quad + 4e^{-\lambda}\nu'\widehat{h}_{,r} - e^{-\lambda}\left(\nu' + \frac{2}{r}\right)\widehat{q}_{,r} + \frac{2}{r^2}\left(\frac{\cos\theta}{\sin\theta}\widehat{q}_{,\theta} + 2\widehat{q}\right) \\ &\quad - e^{-(\lambda+\nu)} r^2 \sin^2\theta\, \varpi'^2. \end{aligned} \tag{5.27}$$

Remark 5.4. Equation (5.23) includes vacuum as a particular case. As in the remark after B2.2, the constant $\Pi^{(2)}$ is irrelevant in the vacuum region M^- (where $j' = 0$). Without loss of generality, we shall set $\Pi^{(2)}_- = 0$ in M^-.

Remark 5.5. The fundamental underlying reason that will allow us to close the bootstrap argument below is the decoupling of equation (5.23), which only involves $\mathcal{W}$, and the system (5.24)–(5.27), which only involves the rest of terms in K_2, that is K_2^H.

Remark 5.6. Since the second order Einstein field equations reduce to the first order equations when $K_1 = 0$, it follows from this lemma that the field equations (4.17) for (5.2) are equivalent to (5.4) plus (5.11).

Remark 5.7. The requirement $m \geq 2$ in the bootstrap hypothesis B3.1 allows us to work with classical solutions of (5.24)–(5.27) because, as discussed after (5.19), $\widehat{h}_\pm, \widehat{v}_\pm$ are $C^2(D^\pm)$, $\widehat{q}_\pm$ is $C^1(D^\pm)$ and all of them are $C^2(S_r)$ on each sphere S_r.

Remark 5.8. For later use we introduce the notation

$$f_\omega(r) := \frac{1}{6}e^{-(\lambda+\nu)} r^3 \left(r\varpi'^2 + 2\left(\lambda' + \nu'\right)(\varpi - \Pi^{(1)})^2\right), \tag{5.28}$$

which arises in the inhomogeneous term in (5.26).

Equation (5.26) is a second order linear ODE in θ for the function $\widehat{q}$ and can be explicitly integrated. Its general solution is

$$\widehat{q}(r,\theta) = \widehat{q}_0(r) + \widehat{q}_1(r)P_1(\cos\theta) + f_\omega(r)P_2(\cos\theta),$$

where $\widehat{q}_0(r)$ and $\widehat{q}_1(r)$ are free functions of r. We show next that $\widehat{q}_1(r)$ is pure gauge, i.e. that it can be set to zero by a suitable choice of gauge transformation (5.22). In terms of the functions

$$b(r) := r^2 e^{-\lambda}\alpha', \qquad c(r) := r\beta, \tag{5.29}$$

the gauge transformation law of $\widehat{q}$ (5.22) takes the form

$$\widehat{q}^g = \widehat{q} + \frac{1}{2}A + P_1(\cos\theta)\left(\frac{b}{2}\left(\lambda' + \nu'\right) + b'\right) - \left(\frac{c}{2}(\lambda' + \nu') + c'\right).$$

We impose that $b(r)$ solves the ODE

$$\frac{b}{2}(\lambda' + \nu') + b' = -\widehat{q}_1, \tag{5.30}$$

so that $\widehat{q}^g$ becomes $\widehat{q}^g = \widehat{q}_0 + f_\omega P_2(\cos\theta)$. Dropping the superindex ^{g} one has

$$\widehat{q} = \widehat{q}_0(r) + f_\omega(r)P_2(\cos\theta), \tag{5.31}$$

and we have proved that $\widehat{q}_1(r)$ can be gauged away, as claimed. The remaining gauge freedom is given by the general solution of the homogeneous part of (5.30), i.e.

$$b = b_0 e^{-\frac{1}{2}(\lambda+\nu)}, \qquad b_0 \in \mathbb{R}.$$

which, in terms of $\alpha(r)$ is (by the definition of $b(r)$ in (5.29))

$$\alpha' = \frac{b_0}{r^2}e^{\frac{1}{2}(\lambda-\nu)}. \tag{5.32}$$

This residual gauge will be used later to simplify $\widehat{v}$.

We now insert $\widehat{q}$ from (5.31) into equations (5.25) and (5.27) and perform a trivial integration in θ in the first one, which introduces an arbitrary function $\sigma(r)$,

$$0 = \left(\frac{2}{r} + \nu'\right)(\widehat{q}_0 + f_\omega P_2(\cos\theta)) - 2\widehat{v}_{,r} - 2\widehat{h}\nu' + \sigma(r)\nu', \tag{5.33}$$

$$\begin{aligned} 0 = {} & 2e^{-\lambda}\widehat{v}_{,rr} - \frac{2}{r^2}\Delta_{\mathbb{S}^2}\widehat{v} - e^{-\lambda}(\nu' + \lambda')\widehat{v}_{,r} - \frac{4}{r^2}\widehat{v} + 4e^{-\lambda}\nu'\widehat{h}_{,r} \\ & - e^{-\lambda}\left(\nu' + \frac{2}{r}\right)\left(\widehat{q}_0{}' + f_\omega{}' P_2(\cos\theta)\right) + \frac{2}{r^2}(2\widehat{q}_0 - f_\omega) \\ & - \frac{2}{3}e^{-(\lambda+\nu)}r^2\varpi'^2(1 - P_2(\cos\theta)). \end{aligned} \tag{5.34}$$

Equation (5.33) determines $\widehat{h}$ algebraically (recall that ν' is nowhere zero outside the centre). Inserting the result into (5.34) yields

$$\begin{aligned} & r^2\widehat{v}_{,rr} + \frac{1}{2}r^2\left(\lambda' + \nu' - 4\frac{\nu''}{\nu'}\right)\widehat{v}_{,r} + e^{\lambda}\Delta_{\mathbb{S}^2}\widehat{v} + 2e^{\lambda}\widehat{v} \\ & \quad = \mathcal{F}_0(r) + \mathcal{F}_2(r)P_2(\cos\theta), \end{aligned} \tag{5.35}$$

where we have introduced

$$\begin{aligned} \mathcal{F}_0(r) := {} & -2\widehat{q}_0\left(\frac{r\nu''}{\nu'} + 1 - e^{\lambda}\right) + \widehat{q}_0'\left(r + \frac{1}{2}r^2\nu'\right) \\ & - \frac{1}{3}e^{-\nu}r^4\varpi'^2 - e^{\lambda}f_\omega + r^2\sigma'\nu', \end{aligned} \tag{5.36}$$

$$\mathcal{F}_2(r) := f_\omega{}'\left(r + \frac{1}{2}r^2\nu'\right) - 2f_\omega\frac{r\nu''}{\nu'} + \frac{1}{3}e^{-\nu}r^4\varpi'^2 - 2f_\omega, \tag{5.37}$$

and recall f_ω is defined in (5.28).

For later use, let us find the most general solution of the homogenous part of (5.35) (i.e. with $\mathcal{F}_0 = \mathcal{F}_2 = 0$) with the form $\widehat{v} = W(r)P_1(\cos\theta)$. It is immediate that this will be a solution iff

$$2W'' + \left(\lambda' + \nu' - \frac{4\nu''}{\nu'}\right)W' = 0. \tag{5.38}$$

This is a second order ODE that can be trivially integrated once. However, finding the general solution is a harder problem, which we address by exploiting the gauge behaviour of $\widehat{v}$ described in (5.21). Consistency of the whole construction requires that the gauge transformation (5.21) restricted to $A = \beta(r) = 0$ and $\alpha(r)$ satisfying (5.32) must transform solutions of (5.35) into solutions. The $\ell = 1$ component of any such solution must solve the homogeneous PDE. The gauge transformation above preserves the $\ell = 1$ character of the function, so it produces another solution of the same homogeneous PDE. In other words, a solution $W(r)$ of (5.38) transforms under this gauge into another solution $W^g(r)$. This is true in particular for $W(r) = 0$, which is an obvious solution. Summing up, for any function $\gamma(r)$ satisfying

$$\gamma' = \frac{b_0}{r^2} e^{\frac{1}{2}(\lambda - \nu)} \tag{5.39}$$

it must be the case that the function

$$W(r) = \left(\gamma(r) + \frac{b_0}{r} e^{-\frac{1}{2}(\lambda+\nu)} \left(1 + \frac{1}{2} r\nu' \right) \right) \tag{5.40}$$

solves (5.38). It is a matter of direct computation to confirm that this is indeed the case. We still need to show that (5.40) is the general solution. Given that the expression involves two arbitrary constants, namely an additive intregration constant γ_0 in (5.39), and b_0, we need to make sure that it contains two linearly independent solutions. One solution is $\gamma(r) = \gamma_0 \in \mathbb{R}$, $b_0 = 0$, so it suffices to check that the solution with $b_0 \neq 0$ is not constant. Computing the derivative of (5.40) and using the background field equation (3.12) yields

$$W' = -\frac{b_0}{2} e^{-\frac{1}{2}(\lambda+\nu)} (\nu')^2,$$

which is not identically zero when $b_0 \neq 0$. We conclude that indeed (5.40) is the general solution of (5.38), and moreover, that it is regular everywhere.

Take now an arbitrary solution $\widehat{v}$ of (5.35). We define $\widehat{v}_\perp$ by means of

$$\widehat{v}(r,\theta) := \widehat{v}_0(r) + \widehat{v}_1(r) P_1(\cos\theta) + \widehat{v}_2(r) P_2(\cos\theta) + \widehat{v}_\perp(r,\theta),$$

where $\widehat{v}_0(r), \widehat{v}_1(r), \widehat{v}_2(r)$ are the components defined as in (5.9). As mentioned, $\widehat{v}_1(r) P_1(\cos\theta)$ necessarily satisfies the homogeneous part of equation (5.35), so there must exist $\gamma(r)$ solving (5.39) such that $\widehat{v}_1(r) = W(r)$ as given in (5.40). Apply now the gauge transformation (5.21) with $\alpha(r) = -\gamma(r)$. The transformed function $\widehat{v}^g$ reads

$$\begin{aligned} \widehat{v}^g(r,\theta) &= \widehat{v}_0(r) + \widehat{v}_1(r)\cos\theta + \widehat{v}_\perp(r,\theta) + \frac{1}{2}A - W(r) - \beta\left(1 + \frac{1}{2} r\nu'\right) \\ &= \widehat{v}_0(r) + \frac{1}{2}A - \beta\left(1 + \frac{1}{2} r\nu'\right) + \widehat{v}_2(r) P_2(\cos\theta) + \widehat{v}_\perp(r,\theta). \end{aligned} \tag{5.41}$$

To sum up, we have found a gauge transformation of the form (2.16) that gets rid of the $\ell = 1$ terms of both $\widehat{q}(r,\theta)$ and $\widehat{v}(r,\theta)$. This choice fixes completely the function $\alpha(r)$, so the remaining gauge freedom is encoded in the constants A, B and the functions $\beta(r), \mathcal{Y}(r)$.

So far, we have ignored the perturbed field equations involving $P^{(2)}$ and $E^{(2)}$. The following proposition summarizes the previous results and incorporates the information on $P^{(2)}$ and $E^{(2)}$ needed later.

Proposition 5.9. *Assume* B1-B3 *in the base perturbation scheme. Then, $u^{(2)}$ is given by (5.13) and equation $(Eq)_{t\phi} = 0$ is equivalent to (5.23). In addition, there is a class of gauges $\{\Psi(A, B, \mathcal{Y})\}$, c.f. Notation 2.7, for which the remaining Einstein field equations for a perfect fluid (including vacuum) to second order are satisfied if only if, in terms of the functions defined in (5.19),*

$$\widehat{q}(r,\theta) = \widehat{q}_0(r) + f_\omega(r) P_2(\cos\theta), \tag{5.42}$$

$$\widehat{v}(r,\theta) = \widehat{v}_0(r) + \widehat{v}_2(r) P_2(\cos\theta) + \widehat{v}_\perp(r,\theta), \tag{5.43}$$

$$\widehat{h}(r,\theta) = \frac{1}{2}\sigma(r) - \frac{\widehat{v}_{,r}}{\nu'} + \left(\frac{1}{r\nu'} + \frac{1}{2}\right)\left(\widehat{q}_0 + f_\omega P_2(\cos\theta)\right), \tag{5.44}$$

with $f_\omega(r)$ given explicitly by (5.28), where $\widehat{q}_0(r)$, $\sigma(r)$ are free functions and $\widehat{v}_0$, $\widehat{v}_2$, $\widehat{v}_\perp$ satisfy

$$r^2 \widehat{v}_0{}'' + \frac{1}{2} r^2 \left(\lambda' + \nu' - 4\frac{\nu''}{\nu'}\right) \widehat{v}_0{}' + 2e^{\lambda} \widehat{v}_0 = \mathcal{F}_0(r), \tag{5.45}$$

$$r^2 \widehat{v}_2{}'' + \frac{1}{2} r^2 \left(\lambda' + \nu' - 4\frac{\nu''}{\nu'}\right) \widehat{v}_2{}' - 4e^{\lambda} \widehat{v}_2 = \mathcal{F}_2(r), \tag{5.46}$$

$$r^2 \widehat{v}_{\perp,rr} + \frac{1}{2} r^2 \left(\lambda' + \nu' - 4\frac{\nu''}{\nu'}\right) \widehat{v}_{\perp,r} + e^{\lambda} \Delta_{\mathbb{S}^2} \widehat{v}_\perp + 2e^{\lambda} \widehat{v}_\perp = 0, \tag{5.47}$$

with $\mathcal{F}_0$ and $\mathcal{F}_2$ explicitly given by (5.36)–(5.37).

Moreover, the second order perturbed pressure $P^{(2)}$ and energy-density $E^{(2)}$ are determined algebraically from the previous functions and have the explicit forms

$$P^{(2)} = F_P^{(2)}(r) + \frac{4}{\nu'} P' \left(\frac{1}{2} f r \nu' + \widehat{h} + \frac{1}{3} e^{-\nu} r^2 (\varpi - \Pi^{(1)})^2 P_2(\cos\theta)\right), \tag{5.48}$$

$$E^{(2)} = F_E^{(2)}(r) + \frac{4}{\nu'} E' \left(\frac{1}{2} f r \nu' + \widehat{h} + \frac{1}{3} e^{-\nu} r^2 (\varpi - \Pi^{(1)})^2 P_2(\cos\theta)\right), \tag{5.49}$$

where

$$F_P^{(2)} := 2(E+P)\left(\mathcal{I} + \frac{1}{3}e^{-\nu}r^2(\varpi - \Pi^{(1)})^2\right), \tag{5.50}$$

$$F_E^{(2)} := -\frac{4}{\nu'}\left\{(E+P)\mathcal{I}' + E'\left(\mathcal{I} + \frac{1}{3}e^{-\nu}r^2(\varpi - \Pi^{(1)})^2\right)\right\}, \tag{5.51}$$

$$2\varkappa(E+P)\mathcal{I} := \frac{2}{r^2}\widehat{q}_0 + \varkappa(E+3P)\sigma + e^{-\lambda}\sigma'\nu' - \frac{4}{r^2}f_\omega. \tag{5.52}$$

Remark 5.10. Expressions (5.48)–(5.51) also hold in vacuum (i.e. when $E = P = E^{(2)} = P^{(2)} = 0$). In this case, (5.48) with (5.50) and (5.52) imply

$$\frac{2}{r^2}\widehat{q}_0 + e^{-\lambda}\sigma'\nu' - \frac{4}{r^2}f_\omega = 0. \tag{5.53}$$

Conversely, one easily checks that if $P = E = 0$ and (5.53) holds then $P^{(2)} = E^{(2)} = 0$.

Proof. All the statements not involving $P^{(2)}$ or $E^{(2)}$ have been already established except for (5.44), which is a direct consequence of (5.33), and the equivalence of the PDE (5.35) with (5.45)–(5.47), which is a direct consequence of the splitting (5.43).

The two remaining field equations in (5.17), namely $(\mathrm{Eq})_{rr}$ and $(\mathrm{Eq})_{tt}$ provide explicit algebraic expressions for $P^{(2)}$ and $E^{(2)}$. The resulting expressions can be rewritten in the form given in (5.48)–(5.49) with the definitions below them. □

Remark 5.11. For this proposition the perturbation of the fluid has not been assumed to satisfy any barotropic equation of state.

5.1.1. Barotropic equation of state. In this subsection we analyse assumption B5, namely that the perfect fluid satisfies the equation of state of the background. This assumption yields an additional constraint affecting only the $\ell = 0$ sector in the interior region.

The existence of a barotropic equation of state is equivalent to demand (4.24). Since $P^{(1)} = E^{(1)} = 0$ in the "base" scheme, this is simply

$$P^{(2)}E' - E^{(2)}P' = 0. \tag{5.54}$$

Given the expressions (5.48)–(5.49), this, in turn, is equivalent to $F_P^{(2)}E' - F_E^{(2)}P' = 0$. From the explicit forms (5.50)–(5.51) and recalling that $\nu' =$

$-2P'/(E+P)$, see (3.12), it follows that

$$F_P^{(2)}E' - F_E^{(2)}P' = -2(E+P)^2\mathcal{I}'.$$

Thus, the barotropic equation of state yields a first integral $\mathcal{I} = \mathcal{I}_0 \in \mathbb{R}$, or explicitly

$$\frac{2}{r^2}\widehat{q}_0 + \varkappa(E+3P)\sigma + e^{-\lambda}\sigma'\nu' - \frac{4}{r^2}f_\omega = 2\mathcal{I}_0\varkappa(E+P), \tag{5.55}$$

which provides an algebraic equation for $\widehat{q}_0$ in terms of background and first order quantities as well as the free function $\sigma(r)$. The derivation has been done in the interior domain D^+. However, by Remark 5.10 this equation also holds in D^- for any constant $\mathcal{I}_0^-$. Following our convention, we set $\mathcal{I}_0 := \{\mathcal{I}_0^+, \mathcal{I}_0^-\}$ and work with both domains at the same time.

In terms of $\mathcal{I}_0$, expressions (5.50)–(5.51) simplify to

$$F_P^{(2)} = 2(E+P)\left(\mathcal{I}_0 + \frac{1}{3}e^{-\nu}r^2(\varpi - \Pi^{(1)})^2\right), \tag{5.56}$$

$$F_E^{(2)} = -\frac{4}{\nu'}E'\left(\mathcal{I}_0 + \frac{1}{3}e^{-\nu}r^2(\varpi - \Pi^{(1)})^2\right). \tag{5.57}$$

Observe that these expressions only involve background and first order quantities. Replacing back into (5.50) and (5.51), we can also simplify $P^{(2)}$ and $E^{(2)}$. It is convenient to write them in terms of the original (non-hatted) function h (see (5.19)). The result is

$$P^{(2)} = -2(E+P)\left(h - \mathcal{I}_0 + \frac{1}{3}e^{-\nu}r^2(\varpi - \Pi^{(1)})^2\left(P_2(\cos\theta) - 1\right)\right), \tag{5.58}$$

$$E^{(2)} = \frac{4}{\nu'}E'\left(h - \mathcal{I}_0 + \frac{1}{3}e^{-\nu}r^2(\varpi - \Pi^{(1)})^2\left(P_2(\cos\theta) - 1\right)\right). \tag{5.59}$$

Under a change of gauge in $\{\Psi(C; A, B, \mathcal{Y}, \alpha)\}$, i.e. (2.15) and (2.16), h changes as (2.17), while $P^{(2)}$, taking into account that $P^{(1)} = 0$, does as $P^{(2)g} = P^{(2)} - \mathcal{Y}\nu'(E+P)$, c.f. (4.23). Substracting equation (5.58) and its gauged counterpart we thus have

$$\mathcal{Y}\nu'(E+P) = P^{(2)} - P^{(2)g} = 2(E+P)\left(h^g - h - \mathcal{I}_0^g + \mathcal{I}_0\right)$$
$$= 2(E+P)\left(\frac{1}{2}A + \frac{1}{2}\mathcal{Y}\nu' - \mathcal{I}_0^g + \mathcal{I}_0\right).$$

Therefore, because $E_c + P_c \neq 0$ by assumption, the first integral in the interior $\mathcal{I}_0^+$ is gauged transformed by

$$\mathcal{I}_0^{+g} = \mathcal{I}_0^+ + \frac{1}{2}A^+. \tag{5.60}$$

Since $\varpi_+(r) \in C^2([0,a])$ and $h_+ \in C^{m+1}(M^+ \setminus \mathcal{C}_0)$ and bounded near $\mathcal{C}_0$, it follows that $P^{(2)}$ is also bounded near the centre and, *as long as the limit of* h *at* $r = 0$ *exists* it holds

$$\lim_{r\to 0} P^{(2)} = 2(E_c + P_c)\left(\mathcal{I}_0^+ - \lim_{r\to 0} h\right) \qquad \text{if } \lim_{r\to 0} h \text{ exists.} \tag{5.61}$$

Let us advance here the limit of h will exist as a consequence of the field equations. This will be discussed in Section 6.3.

Tackling the problem for the $\ell = 0$ sector means taking care of the functions $\widehat{v}_0(r)$, $\widehat{q}_0(r)$ and $\sigma(r)$. From Proposition 5.9, $\widehat{v}_0, \sigma$ must satisfy (5.45) with (5.36) and the barotropic EOS forces $\widehat{q}_0$ to satisfy (5.55) in both domains $D^{\pm}$. The key is to introduce a change of unknowns and replace the pair $\{\widehat{v}_0(r), \sigma(r)\}$ in terms of two functions $\{\delta(r), \varsigma(r)\}$ by means of

$$\widehat{v}_0 = \frac{1}{2}\delta\left(2 + r\nu'\right) + \mathcal{I}_0, \tag{5.62}$$

$$\nu' r\sigma + 2e^{\lambda}\left(\delta + \varsigma\frac{1}{2 + r\nu'}\right) = \mathcal{I}_0(r\nu' - 2). \tag{5.63}$$

This change of functions is invertible because $2 + r\nu' = 1 + e^{\lambda}(1 + r^2\varkappa P)$ by (3.11) and the right-hand side is everywhere positive.

The function $\widehat{q}_0$ is obtained from the barotropic EOS condition (5.55). In terms of the new variables and replacing also E, P from the background equations (3.10)–(3.11) this gives

$$\widehat{q}_0 = \frac{1}{2}r\delta(\lambda' + \nu') + (r\delta)' + \mathcal{I}_0 + \varsigma\frac{\nu'^2 r^2 + 2e^{\lambda}}{(2 + r\nu')^2} + \varsigma'\frac{r}{2 + r\nu'} + 2f_{\omega}. \tag{5.64}$$

We now insert this in (5.45) and apply the change (5.62) and (5.63). A long but straightforward calculation that uses (3.12) and (5.28) gives

$$\begin{aligned} r^2\varsigma'' + \frac{1}{2}r^2\left(\nu' + \lambda' - 4\frac{\nu''}{\nu'}\right)\varsigma' + 2e^{\lambda}\varsigma \\ = -r^3 e^{-\nu}\left(2(\lambda' + \nu')(\varpi - \Pi^{(1)})^2 - \varpi'^2 r\right) - 4\mathcal{F}_2, \end{aligned} \tag{5.65}$$

where $\mathcal{F}_2$ was given in (5.37).

Equation (5.65) is remarkable for several reasons. First of all it involves only the unknown function $\varsigma(r)$ (this means that the function $\delta(r)$ is completely unrestricted). Moreover, the homogeneous part of the equation is identical to the one in (5.45) in terms of $\widehat{v}_0$. This is quite unexpected, given the rather involved change of functions (5.62)–(5.63). Moreover, the inhomogeneous term in (5.65) involves *only* background and first order quantities, unlike (5.45) which also involves unknowns. It is also interesting that this inhomogeneous term is directly related to $\mathcal{F}_2(r)$, which appeared in the $\ell = 2$ sector of the equations.

This equation for ς is the key object that will allow us in Section 6.3 to obtain existence and uniqueness of the barotropic base scheme.

5.2. Matching conditions for the base perturbation scheme

In this section we find the necessary and sufficient conditions that the base perturbation scheme must satisfy so that the second order perturbed matching conditions at the boundary of the fluid ball are satisfied. The perturbed matching conditions derived in [4, 27] (first order) and [22] (second order) are summarized and explained in Appendix B, where we also determine the most general matching conditions in a spherically symmetric static background with two regions, as defined in 3.1, for a first perturbation tensor $K_1 = -2\mathcal{R}^2\omega(r,\theta)dtd\phi$ and K_2 of the general form (5.7). The results are purely geometric and do not rely on any field equations. Moreover, they extend the matching conditions obtained in [33] in that the matching hypersurface Σ is not assumed to be axially symmetric, and are in turn generalised to a still unfixed function $\mathcal{R}(r)$. Throughout this section we use the notation of Appendix B.

The base perturbation scheme fits into the setup of Appendix B as the particular case where $\omega(r,\theta) = \varpi(r)$, $\mathcal{R}(r) = r$ and matching hypersurface Σ located at $r_+ = r_- = a$. Proposition B.1 states that ϖ must satisfy

$$[\varpi] = b_1, \quad b_1 \in \mathbb{R}, \qquad [\varpi'] = 0. \tag{5.66}$$

Let us recall that in the base scheme we have further fixed the first order gauges so that $b_1 = 0$.

In addition, the first order deformation functions $Q_1^{\pm}(\tau, \vartheta, \varphi)$ on either side satisfy the conditions listed in (B.12). The explicit forms of Λ_1 and Λ_2, defined in (B.11), are

$$\Lambda_1 = \frac{1}{2}e^{\nu-\lambda}\left(\nu'' + \frac{1}{2}\nu'\left(\nu' - \lambda'\right)\right), \qquad \Lambda_2 = \frac{1}{2}re^{-\lambda}\lambda'.$$

Therefore the conditions (B.12) in our present setup become

$$[Q_1] = 0, \qquad Q_1[\lambda'] = 0 \qquad Q_1[\nu''] = 0, \tag{5.67}$$

after using that $\mathrm{n} = -e^{-\frac{\lambda}{2}}\partial_r$ and that the background matching conditions are $[r] = [\nu] = [\lambda] = [\nu'] = 0$ (see (3.19)). Moreover, by (3.21)–(3.22)) these equations are equivalent to

$$[Q_1] = 0, \qquad Q_1[E] = 0. \tag{5.68}$$

The second order matching conditions for the base perturbation scheme are obtained from Proposition B.7 with $\omega(r,\theta) = \varpi(r)$ and $\mathcal{R}(r) = r$. It follows

$$Q_1^2[\mathrm{n}(\Lambda_2)] = Q_1^2\left[\frac{1}{2}e^{-3\lambda/2}\left(-r\lambda'' + \lambda'(r\lambda' - 1)\right)\right] = -\frac{a}{2}e^{-3\lambda(a)/2}Q_1^2[\lambda''],$$

where we used (5.67) (and the obvious fact that $Q_1^2[a] = Q_1^2[b] = 0 \Longrightarrow Q_1^2[ab] = 0$). By a slightly longer, but analogous, calculation one finds

$$Q_1^2\left[\frac{\nu'}{r}\mathrm{n}(\Lambda_2) + \frac{2}{e^{\nu}}\mathrm{n}(\Lambda_1)\right] = -e^{-3\lambda(a)/2}Q_1^2[\nu'''].$$

Using this and rewriting $\{h, k, m\}$ in terms of $\{\widehat{v}, \widehat{h}, \widehat{q}\}$ as defined in (5.19), all the terms involving f in the matching conditions (B.42)–(B.49) drop out.

The result is

$$[\Xi] = ae^{\lambda(a)/2}\left(2c_0 + (2c_1 + H_1)\cos\vartheta\right), \tag{5.69}$$
$$[\mathcal{W}] = D_3, \tag{5.70}$$
$$[\mathcal{W}_{,r}] = 2e^{-\lambda(a)/2}Q_1[\varpi''], \tag{5.71}$$
$$[\widehat{v} - \widehat{h}] = c_0 + c_1\cos\vartheta, \tag{5.72}$$
$$[\widehat{h}] = \frac{1}{2}\left(H_0 + a\nu'(a)c_0\right) + \frac{1}{4}a\nu'(a)\left(H_1 + 2c_1\right)\cos\vartheta, \tag{5.73}$$
$$\begin{aligned}&\left[\widehat{q} + \widehat{h} - 2\widehat{v} - r(\widehat{v} - \widehat{h})_{,r}\right] \\ &\quad = \left(H_1 - \frac{1}{2}e^{\lambda(a)}\left(2c_1 + H_1\right)\right)\cos\vartheta \\ &\qquad + \frac{1}{2}\left[\Xi e^{-\lambda/2}\left(\frac{\lambda'}{2} - \frac{1}{r}\right)\right] - \frac{1}{4}e^{-\lambda(a)}Q_1^2\left[\lambda''\right],\end{aligned} \tag{5.74}$$
$$\begin{aligned}&-[\widehat{h}_{,r}] + \frac{a\nu'(a)}{2}[\widehat{v}_{,r} - \widehat{h}_{,r}] + \nu'(a)\left(1 - \frac{a\nu'(a)}{2}\right)[\widehat{v} - \widehat{h}] \\ &\quad = -\frac{1}{2}\nu'(a)\left(\left(1 - \frac{a\nu'(a)}{2}\right)H_1 - \frac{1}{2}e^{\lambda(a)}\left(2c_1 + H_1\right)\right)\cos\vartheta \\ &\qquad - \frac{1}{4}\left[\Xi e^{-\lambda/2}\left(\nu'' + \nu'^2 - \frac{\nu'}{r}\right)\right] + \frac{1}{4}e^{-\lambda(a)}Q_1^2\left[\nu'''\right].\end{aligned} \tag{5.75}$$

Specifically, Proposition B.7 states that the matching conditions are satisfied if and only if there exist constants c_0, c_1, H_0, H_1 and D_3 and functions $\Xi^\pm$ on Σ such that (5.69)–(5.75) hold. So far no field equations have been used. In the next proposition we determine the matching conditions when the field equations hold.

Proposition 5.12 (Perturbed matching). *Assume the setup of the base perturbation scheme (*B1-B3*). Restrict to the class of gauges $\{\Psi(A, B, \mathcal{Y})\}$, c.f. Notation 2.7, at both sides M^+ and M^- so that the results of Proposition 5.9 hold.*

Then the second order matching conditions across $\Sigma = \{r = a\}$ are satisfied if and only if there exists contants D_3, c_0, H_0 such that

$$[\mathcal{W}] = D_3, \tag{5.76}$$
$$[\mathcal{W}_{,r}] = 0, \tag{5.77}$$
$$[\widehat{v}] = \frac{H_0}{2} + \left(1 + \frac{a\nu'(a)}{2}\right)c_0, \tag{5.78}$$

(5.79)
$$[\widehat{v}_{,r}] = \left(\frac{1}{a} + \frac{\nu'(a)}{2}\right) \times \left([\widehat{q}_0] - \frac{H_0}{2} + \frac{1}{3}e^{-\nu(a)}a^4\varkappa E_+(a)(\varpi_+(a) - \Pi^{(1)})^2 P_2(\cos\vartheta)\right) - \left(\frac{e^{\lambda(a)}}{a} + \frac{1}{2}a\nu'(a)^2\right)c_0,$$

(5.80)
$$[\sigma] = \left(\frac{1}{2} - \frac{1}{a\nu'(a)}\right)H_0 - \frac{2e^{\lambda(a)}}{a\nu'(a)}c_0,$$

(5.81)
$$[\sigma'] = \frac{1}{2}\varkappa Q_1^2 E_+'(a) - \frac{2e^{\lambda(a)}}{a^2\nu'(a)}[\widehat{q}_0] \qquad \underline{\text{provided}} \quad E_+(a) = 0.$$

Remark 5.13. Note that in the case $E_+(a) = 0$ and $E_+'(a) \neq 0$, the matching condition (5.81) forces the first order deformation function Q_1 to be a constant on Σ.

Proof. We start by computing the linear combination $2a(5.75) + (2 + a\nu'(a))(5.74)$. This is advantageous because the factor involving Ξ becomes, after inserting ν'' from the background equation (3.12),

$$\frac{a}{2}\left[\Xi e^{-\lambda/2}\left(-\nu'\left(\frac{1}{r} + \frac{\nu'}{2}\right) + \frac{2e^\lambda - 4}{r^2}\right)\right] = e^{-\lambda/2}\left(-\frac{\nu'}{2}\left(1 + \frac{r\nu'}{2}\right) + \frac{e^\lambda - 2}{r}\right)\Bigg|_{r=a}[\Xi],$$

the equality being true because the term in parenthesis is continuous across Σ. Another simplification occurs with the terms involving Q_1^2, which become

$$\frac{a}{2}e^{-\lambda(a)}Q_1^2\left[\nu''' - \left(\frac{1}{r} + \frac{\nu'}{2}\right)\lambda''\right],$$

and this is zero as a consequence of (3.24). The explicit form of the linear combination is

(5.82)
$$(2 + a\nu'(a))[\widehat{q} - \widehat{v}] + \left(2 - a\nu'(a) + a^2\nu'^2(a)\right)[\widehat{h} - \widehat{v}] - 2a[\widehat{v}_{,r}] = \left(\left(2 + \frac{r^2\nu'^2}{2}\right)H_1 - e^\lambda(2c_1 + H_1)\right)\Bigg|_{r=a}\cos\vartheta + e^{-\lambda/2}\left(-\frac{\nu'}{2}\left(1 + \frac{r\nu'}{2}\right) + \frac{e^\lambda - 2}{r}\right)\Bigg|_{r=a}[\Xi].$$

Thus, the matching conditions to be satisfied are (5.69)–(5.74) and (5.82). In all of them $\widehat{h}$ and $\widehat{q}$ are to be understood as short-hands of the explicit expressions given in (5.42) and (5.44).

We next show the necessity of (5.76)–(5.78). There is nothing to prove in (5.76). For (5.77) we need to determine $[\varpi'']$. Since $\lambda'_- + \nu'_- = 0$ and $[\varpi'] = 0$ we compute from (5.4)

$$[\varpi''] = (\lambda'_+ + \nu'_+)\left(\frac{1}{2}\varpi'_+ + \frac{2}{a}(\varpi_+ - \Pi^{(1)})\right) = [\lambda']\left(\frac{1}{2}\varpi'_+ + \frac{2}{a}(\varpi_+ - \Pi^{(1)})\right).$$

Since $Q_1[\lambda'] = 0$, (5.77) follows at once from (5.71). For the rest of expressions, we first observe that neither $\widehat{v} - \widehat{h}$ nor $\widehat{h}$ have terms $\ell = 1$ in the decompositions (5.43) and (5.44). Thus (5.72) and (5.73) force $c_1 = H_1 = 0$. Expression (5.78) is then an immediate consequence of (5.72) and (5.73). Concerning (5.79), we substitute (5.69), (5.72), (5.73) into (5.82) to find

$$[\widehat{v}_{,r}] = \left(\frac{1}{r} + \frac{\nu'}{2}\right)\Big|_{r=a}\left([\widehat{q}] - \frac{H_0}{2}\right) - \left(\frac{e^\lambda}{r} + \frac{1}{2}r\nu'^2\right)\Big|_{r=a} c_0,$$

and this transforms into (5.78) after inserting $\widehat{q} = \widehat{q}_0 + f_\omega P_2(\cos\theta))$ and computing the jump of f_ω from its explicit expression in (5.28) as

$$\begin{aligned}[f_\omega] &= \frac{1}{6}e^{-(\lambda(a)+\nu(a))}a^3\left(a[\varpi'^2] + 2\left[(\lambda' + \nu')(\varpi - \Pi^{(1)})^2\right]\right)\\ &= \frac{1}{3}e^{\lambda(a)}a^4\varkappa E_+(a)(\varpi_+(a) - \Pi^{(1)})^2,\end{aligned} \tag{5.83}$$

where in the second equality we used $[\varpi'] = 0$ as well as the general identity $[AB] = A^+[B] + B^-[A]$ applied to $A = (\varpi - \Pi^{(1)})^2$ and $B = \lambda' + \nu'$, together with (3.21) and $\lambda'_- + \nu'_- = 0$. Expression (5.80) is obtained directly from (5.73) after taking into account that $\widehat{h}$ is given by (5.44) and $[\widehat{v}_{,r}]$ has already been computed.

Finally, we establish (5.81). First of all we take the radial derivative of $\widehat{h}$ defined in (5.44) and replace $\widehat{v}_{,rr}$ from (5.34) to obtain

$$\begin{aligned}\widehat{h}_{,r} = &-\frac{\sigma'}{2} + \frac{e^\lambda}{r^2\nu'}\left(\Delta_{\mathbb{S}^2}\widehat{v} + 2\widehat{v}\right) - \frac{\widehat{v}_{,r}}{2\nu'}\left(\frac{2\nu''}{\nu'} - (\lambda' + \nu')\right)\\ &+ \frac{1}{r\nu'}\left(\frac{1}{r} + \frac{\nu''}{\nu'}\right)(\widehat{q}_0 + f_\omega P_2) - \frac{e^\lambda}{r^2\nu'}(2q_0 - f_\omega)\\ &+ \frac{e^{-\nu}}{3\nu'}r^2\varpi_r^2(1 - P_2).\end{aligned} \tag{5.84}$$

Under the assumption $[E]=0$, we have $[\nu'']=0$ and $[\lambda']=0$ (cf. (3.21) and (3.22)). This implies that no term in (5.74) involves jumps of products with two or more discontinuous factors. For instance, the term involving Ξ is

$$\frac{1}{2}\left[\Xi e^{-\lambda/2}\left(\frac{\lambda'}{2}-\frac{1}{r}\right)\right]=\left(\frac{a\lambda'(a)}{2}-1\right)c_0.$$

Inserting (5.84) into (5.74) one can solve for $[\sigma']$. A straightforward, if somewhat long calculation, yields (5.81) after using (5.78)–(5.80), as well as $[\lambda'']=ae^{\lambda(a)}\varkappa E'_+(a)$, cf. (3.23), and

$$\lambda'_\pm(a)=-\nu'_\pm(a)=a^{-1}(1-e^{\lambda(a)}),$$

which is a consequence of the background field equations (3.10)–(3.11) under $E_+(a)=0$.

This proves the "only if" part of the proposition. To show sufficiency we only need to care about (5.74) when $[E]\neq 0$, as this is the only equation left out. We now have $[\lambda']\neq 0$ and hence the term involving Ξ in (5.74) becomes, after applying again the identity $[AB]=A^+[B]+B^-[A]$,

$$\frac{1}{2}\left[\Xi e^{-\lambda/2}\left(\frac{\lambda'}{2}-\frac{1}{r}\right)\right]=\frac{1}{2}[\Xi]e^{-\lambda(a)/2}\left(\frac{\lambda'_+(a)}{2}-\frac{1}{a}\right)+\frac{e^{-\lambda(a)/2}}{4}[\lambda']\Xi_-.$$

Thus, equation (5.74) can be solved for Ξ_- and hence imposes no additional restrictions on the matching. This concludes the "if" part of the proposition. □

6. Existence and uniqueness results of the "base" second order global problem

We start by proving the following global decomposition result, for which we use the analytic results discussed in Appendices C and D. This proposition will be used later in several circumstances.

Proposition 6.1. *Let* $D=\mathbb{R}^3\setminus\{0\}$, $D^+=\overline{B}_a\setminus\{0\}$ *and* $D^-=D\setminus B_a$, *where* B_a *is the ball of radius* $a>0$ *centered at the origin. Let* $\{r,\theta,\phi\}$ *be standard spherical coordinates on* D. *Consider* $\widehat{u}\in C^2(D^+)\cap C^2(D^-)\cap C^1(D)$, *invariant under* $\eta=\partial_\phi$ *and satisfying the PDE*

$$r^2\widehat{u}_{,rr}+r\mathcal{A}(r)\widehat{u}_{,r}+V(r)\left(\Delta_{\mathbb{S}^2}\widehat{u}+\gamma(r)\widehat{u}\right)=0 \tag{6.1}$$

on D^+ *and* D^-. *Assume that the functions* $V(r)$, $\gamma(r)$, $\mathcal{A}(r)$ *satisfy*

(i) *$V(r) \geq 0$, $\gamma(r)$ is bounded from above,*

(ii) *the parts $V^-(r)$, $\gamma^-(r)$, $\mathcal{A}^-(r)$ are $C^1([a,\infty))$ functions and $V^+(r)$, $\gamma^+(r)$, $\mathcal{A}^+(r)$ extend to the origin as $C^1([0,a])$ functions.*

(iii) *the following limits exist and are finite*

$$\begin{aligned} &\lim_{r\to 0} V^+(r) = V_0, && \lim_{r\to\infty} V^-(r) = V_\infty, \\ &\lim_{r\to 0} \mathcal{A}^+(r) = a_0, && \lim_{r\to\infty} \mathcal{A}^-(r) = a_\infty, \\ &\lim_{r\to 0} \gamma^+(r) = \gamma_0, && \lim_{r\to\infty} \gamma^-(r) = \gamma_\infty, \end{aligned} \tag{6.2}$$

with $V_0 > 0$ and $V_\infty > 0$.

Suppose, in addition, that $\widehat{u}$ is bounded in D. Define $\gamma_{max} := \sup_D \gamma$. Then the following holds:

- *If $\gamma_{max} < 0$ then $\widehat{u} = 0$.*
- *If $\gamma_{max} \geq 0$ define ℓ_{max} as the largest natural number satisfying $\ell(\ell+1) \leq \gamma_{max}$. Then there exist functions $\widehat{u}_\ell(r) \in C^2((0,a]) \cap C^2([a,\infty)) \cap C^1(0,\infty) \cap L^\infty(0,\infty)$ with $\ell \in \{0, \cdots, \ell_{max}\}$ such that*

$$\widehat{u}(r,\theta) = \sum_{\ell=0}^{\ell_{max}} \widehat{u}_\ell(r) P_\ell(\cos\theta). \tag{6.3}$$

Proof. For all $\ell \in \mathbb{N} \cup \{0\}$ define $\widehat{u}_\ell(r) := \frac{2\ell+1}{4\pi} \int_{\mathbb{S}^2} \widehat{u} P_\ell \boldsymbol{\eta}_{\mathbb{S}^2}$. It is clear that this function is C^2 on $I_a^+ := (0,a]$ as well as on $I_a^- := [a,\infty)$. It is also C^1 on $(0,\infty)$. On $I_a^\pm$ we compute

$$\begin{aligned} r^2 \frac{d^2\widehat{u}_\ell}{dr^2} &+ r\mathcal{A}(r)\frac{d\widehat{u}_\ell}{dr} + V(r)\left(\gamma(r) - \ell(\ell+1)\right)\widehat{u}_\ell \\ &= \frac{2\ell+1}{4\pi}\int_{\mathbb{S}^2} \left(r^2 \widehat{u}_{,rr} + r\mathcal{A}(r)\widehat{u}_{,r} + V(r)(\gamma(r) - \ell(\ell+1))\widehat{u}\right) P_\ell \boldsymbol{\eta}_{\mathbb{S}^2} \\ &= \frac{2\ell+1}{4\pi}\int_{\mathbb{S}^2} -V(r)\left(\Delta_{\mathbb{S}^2}\widehat{u} + \ell(\ell+1)\widehat{u}\right) P_\ell \boldsymbol{\eta}_{\mathbb{S}^2} = 0, \end{aligned}$$

where in the second equality we used the PDE (6.1) and in the last one we integrated by parts twice and used $\Delta_{\mathbb{S}^2} P_\ell = -\ell(\ell+1)P_\ell$. Thus, $\widehat{u}_\ell$ satisfies the following ODE on $I_a^\pm$

$$r^2 \frac{d^2\widehat{u}_\ell}{dr^2} + r\mathcal{A}(r)\frac{d\widehat{u}_\ell}{dr} + V(r)\left(\gamma(r) - \ell(\ell+1)\right)\widehat{u}_\ell = 0. \tag{6.4}$$

Boundedness of $\widehat{u}$ on D implies that $\widehat{u}_\ell(r)$ is bounded on $(0,\infty)$. It suffices to apply Theorem D.3 to the problem for $\{\widehat{u}^+_\ell, \widehat{u}^-_\ell\}$ with

$$b_0 = V_0(\gamma_0 - \ell(\ell+1), \qquad b_\infty = V_\infty(\gamma_\infty - \ell(\ell+1)),$$

$\mathcal{F}^\pm = 0$ and $d_0 = d_1 = 0$ to ensure that if $b_0, b_\infty < 0$ then the only bounded solution is the trivial one. Restrict ℓ to satisfy $\ell > \ell_{\max}$. Since, by definition of supremum, $\gamma_{\max} \geq \gamma_0$ and $\gamma_{\max} \geq \gamma_\infty$ it follows $\ell(\ell+1) > \gamma_0$ and $\ell(\ell+1) > \gamma_\infty$ and thus $b_0, b_\infty < 0$ because $V_0, V_\infty > 0$ by assumption. To sum up, if $\ell > \ell_{\max}$ then $\widehat{u}^\pm_\ell = 0$.

At any $r > 0$, the function $\widehat{u}|_{S_r}$ is C^2 and invariant under ∂_ϕ. Thus, it can uniquely decomposed as

$$\widehat{u}|_{S_r} = \sum_{\ell=0}^{\infty} \widehat{u}_\ell(r) P_\ell$$

where convergence is in L^2. All terms after $\ell_{\max}$ are zero, so convergence is also pointwise and we conclude that $\widehat{u}$ takes the form

$$\widehat{u}(r,\theta) := \sum_{\ell=0}^{\ell_{\max}} \widehat{u}_\ell(r) P_\ell(\cos\theta)$$

as claimed in the Proposition. □

Remark 6.2. It is clear from the proof that the condition that $\widehat{u}$ is independent of ϕ can be dropped, at the expense that the decomposition in this case is in terms of all spherical harmonics of order $\ell \leq \ell_{\max}$ and not just the Legendre polynomials.

6.1. Global problem for $\mathcal{W}$: existence and uniqueness

In this subsection we study the existence, uniqueness and structural properties of the function $\mathcal{W}$, which is restricted to be bounded and satisfy the PDE (5.23) on each side $M^\pm$ together with the matching conditions (5.76) and (5.77).

By item B3.1 of the base perturbation scheme the one-form $\mathcal{W}\bar{\boldsymbol{\eta}} \in C^{m+1}(D^+) \cap C^{m+1}(D^-)$ with $m \geq 2$ and hence on each S_r. We use the terminology introduced in Notation 5.1. By the Hodge decomposition on the sphere, there exist two functions $\tau, \mathcal{G}$ on each S_r satisfying (recall that

$\Pi^{(2)} = 0$ on D^-)

$$(\mathcal{W} - \Pi^{(2)})\bar{\boldsymbol{\eta}} = d\mathcal{H} + \star_{\mathbb{S}^2} d\mathcal{G}. \tag{6.5}$$

Since $\overline{D}_A((\mathcal{W} - \Pi^{(2)})\eta^A) = \eta(\mathcal{W}) = 0$, the function $\mathcal{H}$ is constant on each S_r, and can be set to zero without loss of generality. The potential function $\mathcal{G}$ solves

$$\Delta_{g_{\mathbb{S}^2}} \mathcal{G} = -\mathrm{div}_{\mathbb{S}^2}\left((\mathcal{W} - \Pi^{(2)})\bar{\boldsymbol{\eta}}\right), \tag{6.6}$$

where, as usual, the divergence of a one-form $\boldsymbol{w}$ is $\mathrm{div}_{\mathbb{S}^2}\boldsymbol{w} = \overline{D}_A w^A$ with indices raised with $g_{\mathbb{S}^2}$. The right-hand side of (6.6) is C^m on each S_r, so $\mathcal{G}$ is C^{m+2} as a function on S_r.[4] The solution is unique up to an additive constant on each S_r, hence up to a radially symmetric function. In the spherical coordinates $\{\theta, \phi\}$, the Hodge decomposition (6.5) takes the explicit form

$$(\mathcal{W} - \Pi^{(2)}) \sin\theta = \partial_\theta \mathcal{G}. \tag{6.7}$$

Let $\widetilde{\mathcal{G}}$ be the unique solution of this PDE satisfying the boundary condition $\widetilde{\mathcal{G}}|_{\theta=0} = 0$, i.e. vanishing at the north pole of each sphere S_r. The right-hand side of (6.6) is C^m as a function of r both on $r \in (0, a]$ and on $r \in [a, \infty)$. Since the boundary condition is differentiable in r, the solution $\widetilde{\mathcal{G}}$ is $C^m(D^+) \cap C^m(D^-)$. Moreover, $\mathcal{W}$ is bounded on $\widehat{D}$, so the same holds for $\widetilde{\mathcal{G}}$. It turns out to be convenient to extract the $\ell = 0$ component of $\widetilde{\mathcal{G}}$ and define

$$\mathcal{G} := \widetilde{\mathcal{G}} - \frac{1}{4\pi}\int_{S_r} \widetilde{\mathcal{G}}\boldsymbol{\eta}_{\mathbb{S}^2}.$$

It is clear from this definition that $\mathcal{G} \in C^m(D^+) \cap C^m(D^-)$, bounded on $\widehat{D}$ and satisfies $\int_{S_r} \mathcal{G}\boldsymbol{\eta}_{\mathbb{S}^2} = 0$. Next we obtain the PDE that $\mathcal{G}$ must satisfy. We insert $\mathcal{W} - \Pi^{(2)} = (\sin\theta)^{-1}\partial_\theta\mathcal{G}$ into (5.23) and find, after a direct calculation,

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial r}\left(r^4 j \frac{\partial \mathcal{G}}{\partial r}\right) + r^2 j e^{\lambda}\left(\Delta_{\mathbb{S}^2}\mathcal{G} + 2\mathcal{G}\right) + 4r^3 j' \mathcal{G}\right) = 0.$$

[4]The problem is one-dimensional and therefore no Hölder requirement is needed.

Integrating in θ there appears an arbitrary integration function of r which is then uniquely fixed by the condition $\int_{S_r} \mathcal{G}\boldsymbol{\eta}_{\mathbb{S}^2} = 0$. Thus,

$$\frac{\partial}{\partial r}\left(r^4 j \frac{\partial \mathcal{G}}{\partial r}\right) + r^2 j e^{\lambda}\left(\Delta_{\mathbb{S}^2}\mathcal{G} + 2\mathcal{G}\right) + 4r^3 j' \mathcal{G} = 0 \tag{6.8}$$

on $I_a^+ = (0, a]$ and $I_a^- = [a, \infty)$. We also need to determine the matching conditions for $\mathcal{G}$. From Proposition 5.12 (specifically from $[\mathcal{W}] = D_3$, $[\mathcal{W}_{,r}] = 0$) and (6.7), the jump of $\mathcal{G}$ and $\partial_r \mathcal{G}$ satisfy

$$\begin{aligned} &\left(D_3 - \Pi_+^{(2)}\right)\sin\theta = \partial_\theta[\mathcal{G}], \quad 0 = \partial_\theta[\partial_r\mathcal{G}] \\ \iff \quad &[\mathcal{G}] = (\Pi_+^{(2)} - D_3)\cos\theta, \quad [\partial_r \mathcal{G}] = 0 \end{aligned} \tag{6.9}$$

where in the integration we have imposed that neither $[\mathcal{G}]$ nor $[\partial_r\mathcal{G}]$ have $\ell = 0$ term.

We start with a lemma on existence and uniqueness of bounded solutions of (6.8).

Lemma 6.3. *Let $\mathcal{G} \in C^m(D^+) \cap C^m(D^-)$ $(m \geq 2)$ be bounded and satisfy $\int_{S_r} \mathcal{G}\boldsymbol{\eta}_{\mathbb{S}^2} = 0$ together with* (6.8) *on D^+ and D^- and the jumps*

$$[\mathcal{G}] = l_0 \cos\theta, \qquad [\partial_r \mathcal{G}] = 0, \qquad l_0 \in \mathbb{R}. \tag{6.10}$$

Assume that $E_c + P_c \neq 0$. Then, there exists a unique radially symmetric bounded function $G \in C^2(\overline{D}^+) \cap C^{n+2}(D^+) \cap C^\infty(D^-) \cap C^1(D)$ satisfying $G(0) = 1$ and a constant $\mathcal{W}_c \in \mathbb{R}$ such that

$$\mathcal{G}(r,\theta) = \begin{cases} -\mathcal{W}_c G(r) P_1(\cos\theta) & \text{on } D^+ \\ -(\mathcal{W}_c G(r) + l_0) P_1(\cos\theta) & \text{on } D^-. \end{cases} \tag{6.11}$$

Moreover, the function $G(r)$ on D^- is given by

$$G^-(r) = -\frac{G'(a)a^4}{3r^3} + G_\infty, \qquad G_\infty := G(a) + aG'(a)/3 \tag{6.12}$$

and $G'(a) > 0$, $G_\infty > 1$. Clearly also $\lim_{r\to\infty} G = G_\infty$.

Proof. Let us define $\mathcal{G}_\perp := \mathcal{G} - \frac{3}{4\pi}\int_{S_r} \mathcal{G} P_1 \boldsymbol{\eta}_{\mathbb{S}^2}$. We want to apply Proposition 6.1, so we check that all hypotheses are satisfied. By construction $\mathcal{G}_\perp$ is $C^m(D^+) \cap C^m(D^-)$, bounded in $\widehat{D}$, and has no $\ell = 0$, or $\ell = 1$ components.

By (6.10), it satisfies $[\mathcal{G}_\perp] = [\partial_r \mathcal{G}_\perp] = 0$, so that $\mathcal{G}_\perp \in C^1(D)$. The PDE (6.8) in expanded form is

$$\mathcal{G}_{\perp,rr} + \left(\frac{4}{r} + \frac{j'}{j}\right)\mathcal{G}_{\perp,r} + \frac{e^\lambda}{r^2}\left(\Delta_{\mathbb{S}^2}\mathcal{G}_\perp + \left(2 + \frac{4re^{-\lambda}j'}{j}\right)\mathcal{G}_\perp\right) = 0, \tag{6.13}$$

so it fits into the general form (6.1) with

$$\begin{aligned} V(r) = e^\lambda, \quad \gamma(r) = 2 + \frac{4re^{-\lambda}j'}{j} = 2 - 2r^2\varkappa(E+P), \\ \mathcal{A}(r) = 4 + r\frac{j'}{j} = 4 - \frac{1}{2}r^2 e^\lambda \varkappa(E+P), \end{aligned} \tag{6.14}$$

where (3.15) has been substituted in the last two expressions. These functions are all $C^n([0,a]) \cap C^n([a,\infty))$. By assumption H_2 we have $\sup_{\widehat{D}} \gamma(r) = 2$, and the limit conditions (6.2) are all fullfilled (c.f. (3.29)) with

$$\lim_{r\to 0} V^+(r) = 1, \quad \lim_{r\to\infty} V^-(r) = 1, \quad \lim_{r\to 0}\mathcal{A}^+(r) = 4, \quad \lim_{r\to\infty}\mathcal{A}^-(r) = 4,$$
$$\lim_{r\to 0}\gamma^+(r) = 2, \quad \lim_{r\to\infty}\gamma^-(r) = 2.$$

All the conditions of Proposition 6.1 are satisfied and $\ell_{\max} = 1$ so we conclude that $\mathcal{G}_\perp$ must be of the form $\mathcal{G}_\perp = \mathcal{G}^0_\perp(r) + \mathcal{G}^1_\perp(r)P_1(\cos\theta)$. However, by construction $\mathcal{G}_\perp$ has no such components, hence it vanishes identically. Consequently, $\mathcal{G}$ has only $\ell = 1$ component, i.e. takes the form

$$\mathcal{G}(r,\theta) = \mathcal{G}_1(r)P_1(\cos\theta) \tag{6.15}$$

for some radially symmetric function $\mathcal{G}_1$ at either $D^\pm$. From (6.13) and (6.14), this function satisfies the ODE

$$\frac{1}{r^3}\frac{d}{dr}\left(r^4 j \frac{d\mathcal{G}_1}{dr}\right) + 4j'\mathcal{G}_1 = 0 \tag{6.16}$$

or, in expanded form,

$$r^2\mathcal{G}_1'' + r\left(4 - \frac{1}{2}r^2 e^\lambda\varkappa(E+P)\right)\mathcal{G}_1' - 2r^2 e^\lambda \varkappa(E+P)\mathcal{G}_1 = 0, \tag{6.17}$$

on $I_a^+ = (0,a]$ and $I_a^- = [a,\infty)$ together with the jumps (from (6.10))

$$[\mathcal{G}_1] = l_0, \qquad [\mathcal{G}_1'] = 0. \tag{6.18}$$

We now show that (6.17) admits a unique solution, up to scale, which is $C^1(a,\infty)$ and bounded. We start with the interior domain I_a^+. Equation (6.17) satisfies the requirements of items *(ii)–(iii)* of Lemma D.2 with

$$\mathcal{A}(r) = 4 - \frac{1}{2} r^2 e^{\lambda} \varkappa (E+P), \qquad \mathcal{B}(r) = r^2 \mathcal{Q}(r)$$
$$\text{with} \quad \mathcal{Q}(r) = -2 e^{\lambda_+} \varkappa (E+P),$$

so that $a_0 = 4$, and $\mathcal{Q}(0) = -2\varkappa(E_c + P_c) \neq 0$ (by assumption of the lemma). Therefore, Lemma D.2 ensures that there exists a unique up to scaling function $g^+(r)$ that stays bounded in $(0,a)$, and extends to a function in $C^2([0,a])$ satisfying $g^+(0) \neq 0$ and $g^{+\prime}(0) = 0$. It is clear that $g^+(r) \in C^{n+2}((0,a])$ also. We fix the scale by imposing $g^+(0) = 1$. In the exterior part I_a^-, equation (6.17) can be solved explicity. The solution is

$$g^-_{l_1,l_2}(r) = -\frac{2l_1}{r^3} - l_2 \qquad l_1, l_2 \in \mathbb{R}. \tag{6.19}$$

Consider the function $\{g^+(r), g^-_{l_1,l_2}(r)\}$. This corresponds to a function $g(r) \in C^1(0,\infty)$ if and only if

$$g^+(a) = g^-_{l_1,l_2}(a) = -\frac{2l_1}{a^3} - l_2, \qquad g^{+\prime}(a) = {g^-_{l_1,l_2}}'(a) = \frac{6l_1}{a^4}.$$

It is clear that this system admits a unique solution $\{l_1, l_2\}$ with corresponding $g^-(r) := g^-_{l_1,l_2}(r)$ given by

$$g^-(r) := g^+(a) + \frac{a}{3} g^{+\prime}(a) \left(1 - \frac{a^3}{r^3}\right). \tag{6.20}$$

This establishes the existence of a unique bounded function $g(r) \in C^2([0,a]) \cap C^{n+2}((0,a]) \cap C^1(0,\infty) \cap C^\infty([a,\infty)$ satisfying $g(0) = 1$ and solving (6.17) on $I_a^\pm$. This $g(r)$ is the trace of a bounded radially symmetric function $G : \overline{D} \longrightarrow \mathbb{R}$. It is immediate from the properties of $g(r)$ that $G \in C^{n+2}(D^+) \cap C^\infty(D^-) \cap C^1(D)$. Moreover, since $g(r) \in C^2([0,a])$ and satisfies $g'(0) = 0$, Taylor's theorem gives $g(r) = 1 + g_2 r^2 + \Phi_g^{(2)}(r)$ where $g_2 \in \mathbb{R}$ and $\Phi_g^{(2)}(r)$ is $C^2([0,a])$ and $o(r^2)$. Using Lemma 3.2 it follows that $G \in C^2(\overline{D}^+)$. This proves the first claim of the Lemma.

The explicit form (6.12) follows at once from (6.20). Given that $0 \not\equiv E + P \geq 0$ and since $g^+(0) = 1$ and $g^{+\prime}(0) = 0$, Lemma C.4 establishes that $g^+(a) > 1$ and $g^{+\prime}(a) > 0$. Therefore $G(a) > 1$, $G'(a) > 0$ and the claim $G_\infty > 1$ also follows.

Concerning the function $\mathcal{G}_1(r)$, by the uniqueness up to scale of bounded solutions of (6.17) on I_a^+, there exists a constant $\mathcal{W}_c$ such that $\mathcal{G}_1(r) = -\mathcal{W}_c g(r)$ on I_a^+ (the introduction of a minus sign will be convenient later). On I_a^-, $\mathcal{G}_1^-(r)$ has the form (6.19) for some constants l_1 and l_2. Imposing the jumps (6.10) it is immediate that $\mathcal{G}_1^-(r) = -(\mathcal{W}_c g(r) + l_0)$ on I_a^-. Combining with (6.15) concludes the proof. □

Remark 6.4. In the proof of this lemma it has been useful to distinguish the function G from its trace $g(r)$. For the rest of the paper, this is no longer necessary, so we use the same symbol G for both. This follows the general convention used throughout the paper.

We can now prove the following result on existence and uniqueness of $\mathcal{W}$.

Proposition 6.5 (Existence and uniqueness of $\mathcal{W}$). *Assume the setup of the base perturbation scheme (*B1-B4*). Then*

1) *$\mathcal{W}$ is radially symmetric on $\overline{D}$, i.e. it is a function $\mathcal{W}(r)$.*
2) *There exists a (unique) choice of constants $B^{\pm}$ in the gauge freedom* (2.16) *such that the transformed function (still denoted by $\mathcal{W}$) is continuous across $r = a$ and fulfils the property that $K_2(\xi, r^{-1}\eta)$ is bounded at infinity. Moreover, this function is given by*

$$\mathcal{W} = \mathcal{W}_c(G(r) - G_\infty), \qquad \mathcal{W}_c \in \mathbb{R}$$

where the function $G(r)$ and constant G_∞ are defined in Lemma 6.3. In particular $\mathcal{W} \in C^{n+2}(M^+ \setminus \mathcal{C}_0) \cap C^2(M^+) \cap C^\infty(M^-) \cap C^1(M \setminus \mathcal{C}_0)$,

$$\mathcal{W}_- = \frac{2J}{r^3}, \qquad J := -\mathcal{W}_c G'(a)\frac{a^4}{6}$$

and J vanishes if and only if $\mathcal{W}_c = 0$. The parameter $\Pi_+^{(2)}$ corresponding to this function is $\Pi_+^{(2)} = -\mathcal{W}_c G_\infty$

Proof. Let $\mathcal{G}$ be the unique function related to $\mathcal{W}$ by the Hodge decomposition (6.7) and satisfying $\int_{S_r} \mathcal{G}\boldsymbol{\eta}_{\mathbb{S}^2} = 0$. This function satisfies the jumps (6.10), so all the hypothesis of Lemma 6.3 hold and we conclude that $\mathcal{G}$ is given by (6.11) with $l_0 = \Pi_+^{(2)} - D_3$. Inserting back into (6.7) and using that

$\Pi_-^{(2)} = 0$ and (6.12), $\mathcal{W}$ can be written as

(6.21)
$$\mathcal{W} = \begin{cases} \mathcal{W}_c G(r) + \Pi_+^{(2)} & \text{on } D^+ \\ \mathcal{W}_c G(r) + \Pi_+^{(2)} - D_3 = \frac{2J}{r^3} + \mathcal{W}_c G_\infty + \Pi_+^{(2)} - D_3 & \text{on } D^-. \end{cases}$$

with $J := -\mathcal{W}_c G'(a) \frac{a^4}{6}$. This proves item *1*. From (5.7), boundedness of the component $K_2(\xi, r^{-1}\eta)$ on M^- is equivalent to $\lim_{r\to\infty} \mathcal{W} = 0$ (compare item B2.2 in the base scheme). In addition, given that $G(r)$ is C^1 on D, the function $\mathcal{W}$ is continuous on D if and only if the constant D_3 can be transformed away. To achieve both properties, and given the transformation law (2.21), we use the gauge transformation with vectors $V_2^- = (\mathcal{W}_c G_\infty + \Pi_+^{(2)} - D_3)t\partial_\phi$ on M^- and $V_2^+ = (\mathcal{W}_c G_\infty + \Pi_+^{(2)})t\partial_\phi$ on M^+. It is clear that no other possible choice of the constants $B^\pm$ in (2.16) can accomplish this. The gauge transformed $\mathcal{W}$ (which we still call $\mathcal{W}$) is now given by $\mathcal{W} = \mathcal{W}_c(G(r) - G_\infty)$ everywhere and the corresponding parameter $\Pi_+^{(2)} = -\mathcal{W}_c G_\infty$, as follows directly from (6.21) on D^+. All the properties claimed in the proposition are immediate consequences of the corresponding properties for $G(r)$ obtained in Lemma 6.3. In particular J vanishes if and only if $\mathcal{W}_c$ does because $G'(a) > 0$. □

6.2. Global problem for $\widehat{v}$: existence and uniqueness of the $\ell \geq 2$ sector

In this subsection we apply Proposition 6.1 to deal with the global problem for $\widehat{v}$, consisting of the PDE (5.35) at either side $D^\pm$ plus the matching conditions (5.78)–(5.81) in Proposition 5.12. In contrast to the problem for $\mathcal{W}$, however, we cannot prove uniqueness of $\widehat{v}$ yet, since the radially symmetric ($\ell = 0$) part still contains one free function (the integrating factor $\sigma(r)$). Adding the requirement of a barotropic equation of state (in the next subsection) will allow us to tackle the existence and uniqueness of the $\ell = 0$ sector of $\widehat{v}$.

Proposition 6.6 (Global problem for $\widehat{v}$). *Assume the setup of the base perturbation scheme (*B1-B4*) and restrict to the class of gauges $\{\Psi(A, B, \mathcal{Y})\}$ constructed in Proposition 5.9 in both $M^\pm$. Then, $\widehat{v}(r,\theta)$ must have the form*

(6.22)
$$\widehat{v}(r,\theta) = \widehat{v}_0(r) + \widehat{v}_2(r) P_2(\cos\theta).$$

Moreover, the field equations and matching conditions for $\widehat{v}_2(r)$ admit a unique bounded solution. This solution satisfies

- $\widehat{v}_2^+(r) \in C^{n+1}((0,a])$ *and* $\widehat{v}_2^+(r)$ *is* $O(r^4)$ *and extends as a* $C^1([0,a])$ *function, and* $\widehat{v}_2^{+\prime}(r)$ *is* $O(r^3)$ *near* $r=0$,
- $\widehat{v}_2^-(r) \in C^\infty([a,\infty))$, $\widehat{v}_2^-(r)$ *is* $O(r^{-4})$ *and* $\widehat{v}_2^{-\prime}(r)$ *is* $O(r^{-5})$ *near* $r = \infty$.

In particular if $\varpi_\pm(r) = 0$ *then* $\widehat{v}_2(r) = 0$. *These results are independent of the function* $\beta(r)$.

Proof. By Proposition 5.9 the decomposition (5.43) holds on both regions, and the function $\widehat{v}_\perp$ satisfies, c.f. (5.47),

$$r^2 \widehat{v}_{\perp,rr} + r\mathcal{A}(r)\widehat{v}_{\perp\, r} + V(r)\left(\Delta_{\mathbb{S}^2}\widehat{v}_\perp + 2\widehat{v}_\perp\right) = 0 \tag{6.23}$$

with

$$V(r) = e^\lambda, \qquad \mathcal{A}(r) = \frac{1}{2} r\left(\lambda' + \nu' - 4\frac{\nu''}{\nu'}\right).$$

Equation (6.23) is of the form (6.1) with $\gamma(r) = 2$. Recall that $\lambda(r), \nu(r) \in C^{n+1}(M^+) \cap C^{n+1}(M^-)$, so the same holds for $V(r)$. The values of $V(r)$ at the origin and infinity are, respectively, $V_0 = 1$ (by (3.29)) and $V_\infty = 1$. Concerning $\mathcal{A}(r)$, we use the expansion at the origin for $\nu(r)$ in (3.27) together with $\nu_2 \neq 0$, which follows from (3.28) because of assumption H_2 and the base perturbation scheme condition $E_c + P_c \neq 0$. With that,

$$\frac{r}{\nu'_+} = \frac{1}{2\nu_2} + \Phi^{(2)}, \qquad \Phi^{(2)} \in C^{n-1}(M^+) \text{ and } O(r^2). \tag{6.24}$$

Thus, $\mathcal{A}^+ \in C^{n-1}(M^+)$ and $\mathcal{A}^+(0) = -2$. Therefore (6.23) satisfies the requirements of Proposition 6.1 with $V_0 = 1$, $V_\infty = 1$, $a_0 = -2$, $a_\infty = 4$. Since $\gamma(r) = 2$ we have $\ell_{\max} = 1$. We conclude that $\widehat{v}_\perp$ must be of the form $\widehat{v}^0_\perp(r)P_0(\cos\theta) + \widehat{v}^1_\perp(r)P_1(\cos\theta)$ and hence identically zero since by construction $\widehat{v}_\perp$ does not have such components. The decomposition (5.43) gives (6.22) at once. Furthermore, since the class of gauges is restricted to $\alpha(r) = 0$, Lemma 5.2 ensures that (6.22) holds in the class of gauges $\{\Psi(A, B, \mathcal{Y})\}$, i.e. for arbitrary parameters A, B and free function $\mathcal{Y}(r)$ in (2.16), as well as for any choice of $\beta(r)$.

It remains to show that $\widehat{v}_2(r)$ exists and is unique, and obtain its behavior around $r = 0$ and $r \to \infty$. The problem for $\widehat{v}_2$ is given by equation

(5.46) at both $\pm$ sides, together with the matching conditions obtained from (5.78)–(5.79) of Proposition 5.12, which explicitly read

$$(6.25)\quad [\widehat{v}_2] = 0, \qquad [\widehat{v}_2'] = \frac{1}{6}\left(2 + a\nu'(a)\right) e^{-\nu(a)} a^3 \varkappa E_+(a)(\varpi_+(a) - \Pi^{(1)}_+)^2.$$

We want to apply Theorem D.3 with $\mathcal{F} = \mathcal{F}_2$ given in (5.37), and

$$\mathcal{A}^{\pm}(r) = \frac{1}{2} r \left(\nu'_{\pm} + \lambda'_{\pm} - 4\frac{\nu''_{\pm}}{\nu'_{\pm}} \right), \qquad \mathcal{B}^{\pm}(r) = -4e^{\lambda_{\pm}}.$$

Let us check that all the hypotheses are satisfied. We have already seen that $\mathcal{A}^+(r) \in C^{n-1}([0,a])$ and $a_0 = \mathcal{A}^+(0) = -2$, while we have $b_0 = \mathcal{B}^+(0) = -4$ (c.f. (3.29)). In the exterior, we may write (by the background field equations)

$$(6.26)\qquad \mathcal{A}^-(r) = 2(1 + e^{\lambda_-}), \qquad \mathcal{B}^-(r) = -4e^{\lambda_-},$$

where $\lambda_-(r)$ is given explicitly in (3.25). Consequently,

$$a_\infty = \lim_{r\to+\infty} \mathcal{A}^-(r) = 4, \qquad b_\infty = \lim_{r\to+\infty} \mathcal{B}^-(r) = -4,$$
$$\lim_{r\to+\infty} r^2 \frac{d\mathcal{A}^-(r)}{dr} = -\frac{\varkappa M_{\mathrm{T}}}{2\pi}, \qquad \lim_{r\to+\infty} r^2 \frac{d\mathcal{B}^-(r)}{dr} = \frac{\varkappa M_{\mathrm{T}}}{\pi}.$$

The function $\mathcal{F}_2^+(r)$ is $C^{n-1}((0,a])$ and extends continuously to the centre, where it vanishes. The structure of $\mathcal{F}_2^+(r)$ around $r = 0$ is obtained using (3.27), (5.3) and (5.6), and it is found to be of the form $\mathcal{F}_2^+ = r^6(\sigma_6 + \Phi^{(1)}_{\mathcal{F}})$ where $\sigma_6 \in \mathbb{R}$ and $\Phi^{(1)}_{\mathcal{F}}$ is $o(1)$. Concerning $\mathcal{F}_2^-$, inserting the background vacuum field equations in (5.37) gives

$$(6.27)\qquad \mathcal{F}_2^- = \frac{1}{3} r^4 (e^{\lambda} - 1) {\varpi'_-}^2 = \frac{3\varkappa M_{\mathrm{T}} J_\varpi^2}{\pi r^5}\left(1 + O\left(\frac{1}{r}\right)\right)$$

where the second equality follows from the explicit form (5.5) of ϖ_-. Hence, $\mathcal{F}_2^+$ and $\mathcal{F}_2^-$ satisfy the requirements of Theorem D.3 with $\alpha_0 = 6$ and $\alpha_\infty = -5$. The quantities λ^0_- and λ^∞_- defined in Theorem D.3 take the values $\lambda^0_- = -4$ and $\lambda^\infty_- = -4$. All the hypothesis of Theorem D.3 are satisfied, including (D.13), as well as $\lambda^0_- + 1 \leq 0$ and $\alpha_0 - 1 \geq 0$. Consequently, there exists a unique solution $\{\widehat{v}_2^+(r), \widehat{v}_2^-(r)\}$ that stays bounded on $(0,\infty)$, and, moreover, $\widehat{v}_2^+(r)$ is $O(r^4)$, extends as a $C^1([0,a])$ function, and $\widehat{v}_2^{+\prime}(r)$ is $O(r^3)$ because $\lambda^0_- = -4$ and $\alpha_0 = 6$. The behaviour of the solution $\widehat{v}^-$ and its derivative $\widehat{v}^{-\prime}$ near $r=\infty$ is obtained from Theorem D.3 with the values $\min\{|\lambda_-^\infty|, |\alpha_\infty|\} = 4$ and $\min\{|\lambda_-^\infty - 1|, |\alpha_\infty - 1|\} = 5$ respectively. The differentiability of the solutions in D^+ and D^- follow from the fact that the coefficients $\mathcal{A}^+$, $\mathcal{B}^+$ and $\mathcal{F}_2^+$ are C^{n-1}, C^n and C^{n-1} on $(0,a)$ respectively, while $\mathcal{A}^-$, $\mathcal{B}^-$ and $\mathcal{F}_2^-$ are $C^\infty([a,\infty))$.

As above, Lemma 5.2 ensures that the class of gauges $\{\Psi(A,B,\mathcal{Y})\}$ given by (2.16) with arbitrary parameters A, B, free function $\mathcal{Y}(r)$ and $\alpha(r)=0$, and also free choice of $\beta(r)$, keeps $\widehat{v}_2(r)$ invariant.

The final statement concerning the case $\varpi_\pm = 0$ is immediate since $\widehat{v}_2(r) = 0$ solves the ODE (5.46) with $\mathcal{F}_2(r) = 0$. □

6.3. Barotropic equation of state: existence and uniqueness of $\widehat{v}$

Let us recapitulate. Propositions 5.9 and 6.6 have shown the existence of a class of gauges $\{\Psi(A,B,\mathcal{Y})\}$ and free $\beta(r)$ where $\widehat{v}$ only has $\ell=0$ and $\ell=2$ components. Inverting the definitions (5.19), the original functions $\{h,m,k\}$ take the form (on either side $D^\pm$)

$$h = \widehat{h} + \frac{1}{2}r\nu' f$$
$$= \frac{1}{2}\sigma - \frac{1}{\nu'}(\widehat{v}_0' + \widehat{v}_2' P_2(\cos\theta))$$
$$+ \left(\frac{1}{r\nu'} + \frac{1}{2}\right)(\widehat{q}_0 + f_\omega P_2(\cos\theta)) + \frac{1}{2}r\nu' f, \tag{6.28}$$

$$k = \widehat{v} - \widehat{h} + f$$
$$= \widehat{v}_0 + \widehat{v}_2 P_2(\cos\theta) - \frac{1}{2}\sigma + \frac{1}{\nu'}(\widehat{v}_0' + \widehat{v}_2' P_2(\cos\theta))$$
$$- \left(\frac{1}{r\nu'} + \frac{1}{2}\right)(\widehat{q}_0 + f_\omega P_2(\cos\theta)) + f \tag{6.29}$$

$$m = \widehat{q} - h + \frac{1}{2}rf\left(\lambda' + \nu'\right) + (rf)_{,r}$$
$$= \left(1 - \frac{1}{r\nu'} - \frac{1}{2}\right)(\widehat{q}_0 + f_\omega P_2(\cos\theta))$$
$$- \frac{1}{2}\sigma + \frac{1}{\nu'}(\widehat{v}_0' + \widehat{v}_2' P_2(\cos\theta)) + \frac{1}{2}r\lambda' f + (rf)_{,r}, \tag{6.30}$$

where $\sigma, \widehat{q}_0, \widehat{q}_2, \widehat{v}_0, \widehat{v}_2, f_\omega$ are functions of r, while f is still a free function depending on r, θ.

From the previous subsections, and leaving aside $f(r,\theta)$ (to be discussed later), the only part of the solution where existence and uniqueness has not yet been established is the $\ell = 0$ sector, where the unknowns are $\{\widehat{v}_0, \widehat{q}_0, \sigma\}$. In this section we accomplish this by imposing the background barotropic EOS. As discussed in subsection 5.1.1, $\widehat{q}_0(r)$ is then given explicitly by (5.55) and it is useful to replace the unknowns $\{\widehat{v}_0(r), \sigma(r)\}$ by $\{\delta(r), \varsigma(r)\}$. The main advantage is that $\varsigma(r)$ decouples from $\delta(r)$ and satisfies a global problem for which we can show existence and uniqueness, while $\delta(r)$ will be shown later to be pure gauge. Let us first focus on the problem for $\varsigma(r)$.

We already have the equations that ς^+ and ς^- satisfy in their respective domains, i.e. (5.65). Let us determine the jumps of ς^+ and ς^- across $r = a$, as well as the regularity conditions of ς^+ around $r = 0$, both following from assumptions B2 and B3 of the base scheme. Incidentally, no conditions at $r = \infty$ will be needed, since the field equations will provide bounded solutions only. We start with the regularity and observe, first of all, that (5.62) already implies $\delta(r)$ is $C^m(0, a)$ and bounded near $r = 0$. Since we have a priori information on $\{h, m, k\}$, let us rewrite (6.28)–(6.30) in terms of $\{\delta, \varsigma\}$. After a straightforward calculation and introducing the auxiliary function

$$\Gamma(r) := \varsigma' \frac{1}{\nu'} + \varsigma \frac{r\nu'}{2 + r\nu'} + \frac{2 + r\nu'}{r\nu'} 2f_\omega \tag{6.31}$$

as a shorthand, we have

$$\begin{aligned} h &= \frac{1}{2} r\nu'(\delta + f) + \mathcal{I}_0 + \frac{1}{2}\Gamma + \left(\frac{2 + r\nu'}{2r\nu'} f_\omega - \frac{1}{\nu'} \widehat{v}_2' \right) P_2(\cos\theta), \\ k &= \delta + f - \frac{1}{2}\Gamma + \left(\widehat{v}_2 + \frac{1}{\nu'}\widehat{v}_2' - \frac{2 + r\nu'}{2r\nu'} f_\omega \right) P_2(\cos\theta), \\ m &= \frac{1}{2} r\lambda'(\delta + f) + (r(\delta + f))_{,r} + \frac{r\nu' - 2}{2(2 + r\nu')}\Gamma \\ &\quad + \varsigma \frac{4e^\lambda}{2(2 + r\nu')^2} + \left(\frac{1}{\nu'}\widehat{v}_2' - \frac{2 - r\nu'}{2r\nu'} f_\omega \right) P_2(\cos\theta). \end{aligned} \tag{6.32}$$

By Proposition 6.6, $\widehat{v}_2(r)$ is $C^{n+1}((0, a])$ and $O(r^4)$, extends C^1 at the origin and $\widehat{v}_2'(r)$ is $O(r^3)$. Moreover, $\delta(r)$ is $C^m(0, a)$ and bounded near $r = 0$ and $f_\omega(r)$ is $O(r^4)$ as follows from its defining expression (5.28) together with (3.27) and (5.3). Consequently, the expression for h (or that for k) forces $\Gamma(r)$ to be of class $C^m((0, a])$ and bounded near $r = 0$. This implies that $r\Gamma(r)$ must vanish as $r \to 0$. From (6.31), this limit is

$$0 = \lim_{r \to 0} r\Gamma(r) = \lim_{r \to 0} \left(\frac{1}{2\nu_2} \varsigma'(r) + r^3 \nu_2 \varsigma(r) \right), \tag{6.33}$$

where in the second equality we used (6.24) and (3.27). On the other hand, *if the limit of* $\Gamma(r)$ *as* $r \to 0$ *exists* then so does the limit of h, and therefore the expression of h in (6.32) inserted in (5.61) provides

$$\lim_{r\to 0} P^{(2)} = -(E_c + P_c) \lim_{r\to 0} \Gamma(r) \qquad \text{if } \lim_{r\to 0} \Gamma(r) \text{ exists.} \tag{6.34}$$

We next obtain the jumps that $\varsigma(r)$ must satisfy on $r = a$. The matching conditions (5.78)–(5.79) imply, restricting to the $\ell = 0$ sector,

$$[\widehat{v}_0] = \frac{H_0}{2} + \left(1 + \frac{a\nu'(a)}{2}\right) c_0, \tag{6.35}$$

$$[\widehat{v}_0'] = \left(\frac{1}{a} + \frac{\nu'(a)}{2}\right)\left([\widehat{q}_0] - \frac{H_0}{2}\right) - \left(\frac{e^{\lambda(a)}}{a} + \frac{1}{2}a\nu'(a)^2\right) c_0. \tag{6.36}$$

Eliminating δ from (5.62) into (5.63) gives an expression relating $\widehat{v}_0$ and ς. Taking the diference at both sides and inserting (6.35) gives

$$[\varsigma] = \frac{1}{4} e^{-\lambda(a)}(e^{\lambda(a)} + 3)(1 - e^{\lambda(a)})(H_0 - 2[\mathcal{I}_0]), \tag{6.37}$$

where $\nu'(a)$ is substituted from (3.20). To obtain $[\varsigma']$ we make use of (5.64), after eliminating δ with (5.62), at both $\pm$ sides. The expression contains $[\widehat{v}_0]$ and $[\widehat{v}_0']$, which we substitute by their expressions in (6.35) and (6.36). The terms containing $[\widehat{q}_0]$ cancel. Inserting (6.37) and using (3.20) we obtain

$$\begin{aligned}[\varsigma'] = {} & \frac{1}{4a} e^{-\lambda(a)}(e^{2\lambda(a)} + 3)(e^{\lambda(a)} - 1)(H_0 - 2[\mathcal{I}_0]) \\ & - \frac{2}{a}(e^{\lambda(a)} + 1)[f_\omega],\end{aligned} \tag{6.38}$$

keeping in mind that the explicit expression of $[f_\omega]$ is given by (5.83).

Later in the paper we will face the issue of fixing the gauge completely. To do that it will be determinant to understand the role of the parameter $H_0 - 2[\mathcal{I}_0]$. In preparation for that, let us introduce $\varsigma_*(r)$ as the function that satisfies the same equation as ς and shares its behaviour around $r = 0$, namely

$$\lim_{r\to 0}\left(\frac{1}{2\nu_2}\varsigma_*'(r) + r^3\nu_2\varsigma_*(r)\right) = 0, \tag{6.39}$$

but with jumps given by

$$[\varsigma_*] = 0, \quad [\varsigma_*'] = -\frac{2}{a}(e^{\lambda(a)} + 1)[f_\omega]. \tag{6.40}$$

We also introduce the corresponding function $\Gamma_\star$

$$\Gamma_* := \varsigma_*' \frac{1}{\nu'} + \varsigma_* \frac{r\nu'}{2 + r\nu'} + \frac{2 + r\nu'}{r\nu'} 2 f_\omega. \tag{6.41}$$

and require

$$\text{if} \quad \lim_{r\to 0} \Gamma_\star \quad \text{and} \quad \lim_{r\to 0} \Gamma \quad \text{exist} \qquad \Longrightarrow \qquad \lim_{r\to 0} \Gamma_\star = \lim_{r\to 0} \Gamma. \tag{6.42}$$

In the next proposition we establish existence and uniqueness of ς_*. The corresponding result for the original ς is obtained as a corollary.

Proposition 6.7 (Existence and uniqueness of ς_*). *The problem for $\varsigma_*(r)$, namely equation* (5.65) *on D^+ and D^- with matching conditions on $r = a$ given by* (6.40) *and such that the restriction around the origin* (6.39) *holds, admits a one-parameter family of solutions. In addition, the limits $\lim_{r\to 0} \Gamma(r)$ and $\lim_{r\to 0} \Gamma_\star(r)$ exist and the function $\varsigma_\star$ is uniquely determined by the value $P_c^{(2)} := \lim_{r\to 0} P^{(2)}$ by means of*

$$\lim_{r\to 0} \Gamma_\star = -\frac{P^{(2)}}{E_c + P_c}. \tag{6.43}$$

This solution has the following properties:

1) *$\varsigma_*^+(r)$ is of class $C^{n+1}((0,a])$, extends to a $C^1([0,a])$ function, and has the form*

$$\varsigma_*^+(r) = -\varkappa P_c^{(2)} \frac{E_c + 3P_c}{6(E_c + P_c)} \varsigma_-^+(r) + \varsigma_P^+(r), \tag{6.44}$$

where $\varsigma_-^+(r)$ and $\varsigma_P^+(r)$ are unique: $\varsigma_-^+(r)$ solves the homogeneous part of (5.65) *and satisfies* (6.48), *while $\varsigma_P^+(r)$ is the only particular solution of* (5.65) *that is $O(r^4)$.*

2) *$\varsigma_*^-(r)$ reads*

$$\varsigma_*^-(r) = e^{\lambda_-(r)} \left(\frac{\varsigma_A}{r^2} + \frac{\varsigma_B}{r} \right) + 2 J_\varpi^2 \frac{1}{r^4} (2 + e^{\lambda_-(r)}), \tag{6.45}$$

where $\lambda_-(r)$ is given by (3.25) *and ς_A, ς_B are constants fully determined by the matching conditions in terms of quantities of the background configuration, plus $J_\varpi, \Pi^{(1)}$ and $\{P_c^{(2)}, \varsigma_-^+(a), \varsigma_P^+(a), \varsigma_-^{+\prime}(a), \varsigma_P^{+\prime}(a)\}$.*

3) *If* $\varpi_\pm = 0$ *then*

$$\varsigma_*^+(r) = -\varkappa P_c^{(2)} \frac{E_c + 3P_c}{6(E_c+P_c)} \varsigma_-^+(r), \tag{6.46}$$

$$\varsigma_*^-(r) = \varkappa P_c^{(2)} \varsigma_{**}^-(r), \tag{6.47}$$

where

$$\varsigma_{**}^-(r) := \frac{E_c + 3P_c}{6(E_c+P_c)} e^{\lambda_-(r)} \left(\left(\frac{a^2}{r^2} - \frac{a}{r} \right) e^{-\lambda(a)} a {\varsigma_-^+}'(a) + \left(\frac{a^2}{r^2} - \frac{a}{r}(1 + e^{-\lambda(a)}) \right) \varsigma_-^+(a) \right).$$

Proof. We first analize the equation for ς_*^+ in D^+. To do that we make use of Lemma D.2 for the homogeneous part of (5.65). Lemma D.2 applies with (changing t for r)

$$\mathcal{A}^+(r) = \frac{1}{2} r \left(\nu_+' + \lambda_+' - 4 \frac{\nu_+''}{\nu_+'} \right), \qquad \mathcal{B}^+(r) = 2e^{\lambda_+},$$

which have been already analised (except for a different constant factor in $\mathcal{B}^+$) in the proof of Proposition 6.6. We showed $\mathcal{A}^+(r) \in C^{n-1}([0,a])$ and $a_0 = \mathcal{A}^+(0) = -2$, while $\mathcal{B}^+(r) \in C^{n+1}([0,a])$ with $b_0 = 2$ (observe that $b_0 \geq 0$, which prevents us from using Theorem D.3). As a result $\lambda_+ = -1$ and $\lambda_- = -2$, and therefore point *(i)* of Lemma D.2 ensures there exist two linearly independent solutions $\varsigma_\pm^+(r)$, which necessarily are of class $C^{n+1}(0,a)$, with

$$\begin{aligned} &\varsigma_+^+(r) = r(1+o(1)), \qquad {\varsigma_+^+}'(r) = 1 + o(1), \\ &\varsigma_-^+(r) = r^2(1+o(1)), \qquad {\varsigma_-^+}'(r) = r(2 + o(1)). \end{aligned} \tag{6.48}$$

The inhomogeneous term of equation (5.65) reads

$$\widehat{\mathcal{F}}_0^+ := -r^3 e^{-\nu_+} \left(2(\lambda_+' + \nu_+')(\varpi_+ - \Pi^{(1)})^2 - {\varpi_+'}^2 r \right) - 4\mathcal{F}_2^+. \tag{6.49}$$

Although Theorem D.3 cannot be applied directly, we may still use several constructions introduced in its proof, specifically regarding the properties of the particular solution U_p introduced there. By direct inspection, the function $\widehat{\mathcal{F}}_0^+$ is $C^{n-1}([0,a])$, just like $\mathcal{F}_2^+$ (c.f. (5.37)). Its structure around

$r = 0$ is obtained from (3.27), (5.3), (5.6), plus the fact that $\mathcal{F}_2^+$ is $O(r^6)$, and turns out to be $\widehat{\mathcal{F}}_0^+ = -r^4(4e^{-\nu_0}(\varpi_0 - \Pi^{(1)})^2(\lambda_2 + \nu_2) + o(1))$. In the notation of Theorem D.3 with $\mathcal{F}^+ = \widehat{\mathcal{F}}_0^+$ we have $\alpha_0 = 4$, so that $\alpha_0 + \lambda_+ = 3$ and $\alpha_0 + \lambda_- = 2$ and therefore the general solution of the equation for $\varsigma^+(r)$ has the form

$$\varsigma_*^+(r) = c_+\varsigma_+^+(r) + c_-\varsigma_-^+(r) + \varsigma_P^+(r),$$

where the particular solution satisfies, see Remark D.5,

$$\varsigma_P^+(r) = r^4(\varsigma_P^0 + o(1)), \qquad \varsigma_P^{+\prime}(r) = r^3(4\varsigma_P^0 + o(1)) \tag{6.50}$$

with ς_P^0 a fixed number (see (D.19)). From (6.48) and (6.50), the requirement (6.39) forces $c_+ = 0$. This implies, taking into account (6.24) and $f_\omega \in O(r^4)$, that $\lim_{r\to 0}\Gamma_* = c_-/\nu_2$.

So far we only used equation (5.65) and (6.39). Both are satisfied by the original function ς^+, so it must also be that $\varsigma^+(r) = \hat{c}_-\varsigma_-^+(r) + \varsigma_P^+(r)$ for an, a priori, different integration constant $\hat{c}_-$. However, since $\lim_{r\to 0}\Gamma = \hat{c}_-/\nu_2$, condition (6.42) implies $\hat{c}_- = c_-$ and we conclude that $\varsigma^+(r) = \varsigma_\star^+(r)$. Using $\nu_2 = \varkappa(E_c + 3P_c)/6$, and that $E_c + P_c \neq 0$, the relation (6.34) fixes c_- as

$$c_- = -\frac{\varkappa(E_c + 3P_c)}{6(E_c + P_c)}P_c^{(2)}.$$

This proves (6.43) and item *1.*

Equation (5.65) for ς_* in the exterior region D^- reads

$$r^2\varsigma_*^{-\prime\prime}(r) + 2r(e^{\lambda_-} + 1)\varsigma_*^{-\prime}(r) + 2e^{\lambda_-}\varsigma_*^-(r) = -12J_\varpi^2\frac{1}{r^4}(e^{\lambda_-} - 4),$$

after inserting (6.26) for $\mathcal{A}^-(r)$, (6.27) and (5.5). We do not replace $\lambda_-(r)$ by its explicit form (3.25) for conciseness. The general solution is given by

$$\varsigma_*^-(r) = e^{\lambda_-(r)}\left(\frac{\varsigma_A}{r^2} + \frac{\varsigma_B}{r}\right) + 2J_\varpi^2\frac{1}{r^4}(2 + e^{\lambda_-(r)}), \qquad \varsigma_A, \varsigma_B \in \mathbb{R}.$$

The integration constants ς_A, ς_B are restricted to satisfy the jumps (6.40), which can be arranged in the form

$$\begin{pmatrix} 1 & a \\ 2 & a \end{pmatrix}\begin{pmatrix} \varsigma_A \\ \varsigma_B \end{pmatrix} = \begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix}, \tag{6.51}$$

where κ_1, κ_2 depend on $\{P_c^{(2)}, \varsigma_-^+(a), \varsigma_P^+(a), J_\varpi, a, M_{\mathrm{T}}, \varsigma_-^{+\prime}(a), \varsigma_P^{+\prime}(a), [f_\omega]\}$, with $[f_\omega]$ given by (5.83). The 2×2 matrix has determinant $-a \neq 0$ and therefore there exist unique values of ς_A, ς_B that fulfill these conditions. This proves item *2*, as well as the global existence and uniqueness claim.

Assume now $\varpi_\pm = 0$. The inhomogeneous term vanishes $\widehat{\mathcal{F}}_0^+ = 0$ (see (6.49) and (5.37)) and therefore $\varsigma_P^+(r) = 0$, so that (6.46) follows. In addition, $J_\varpi = \Pi^{(1)} = [f_\omega] = 0$, and we can solve (6.51), to obtain

$$\varsigma_A = \frac{\varkappa(E_c + 3P_c)}{6(E_c + P_c)} P_c^{(2)} a^2 \left(a e^{-\lambda(a)} \varsigma_-^{+\prime}(a) + \varsigma_-^{+}(a) \right), \tag{6.52}$$

$$\varsigma_B = -\frac{\varkappa(E_c + 3P_c)}{6(E_c + P_c)} P_c^{(2)} a \left(a \varsigma_-^{+\prime}(a) + (e^{-\lambda(a)} + 1) \varsigma_-^{+}(a) \right). \tag{6.53}$$

Inserting into (6.45) gives (6.47) after using the explicit form (3.25) of $\lambda_-(r)$. □

Corollary 6.8. *The function ς is given by*

$$\varsigma^+(r) = \varsigma_*^+(r), \tag{6.54}$$

$$\varsigma^-(r) = (H_0 - 2[\mathcal{I}_0])\, e^{\lambda_-(r)} \times \left(-\frac{3}{4} \left(\frac{\varkappa M_{\mathrm{T}}}{4\pi} \right)^2 \frac{1}{r^2} + \left(\frac{\varkappa M_{\mathrm{T}}}{4\pi} \right) \frac{1}{r} \right) + \varsigma_*^-(r). \tag{6.55}$$

Proof. As shown above, only (6.55) needs attention. The function $\tilde{\varsigma} := \varsigma - \varsigma_*$ satisfies the homogeneous part of equation (5.65). The general solution in the exterior D^- is thus given by

$$\tilde{\varsigma}^-(r) = e^{\lambda_-(r)} \left(\frac{\tilde{\varsigma}_A}{r^2} + \frac{\tilde{\varsigma}_B}{r} \right).$$

It suffices to obtain the constants $\tilde{\varsigma}_A$ and $\tilde{\varsigma}_B$ from the matching conditions

$$\tilde{\varsigma}^-(a) = -[\tilde{\varsigma}] = -\frac{1}{4} e^{-\lambda(a)} (e^{\lambda(a)} + 3)(1 - e^{\lambda(a)})(H_0 - 2[\mathcal{I}_0]),$$

$$\tilde{\varsigma}^{-\prime}(a) = -[\tilde{\varsigma}'] = -\frac{1}{4a} e^{-\lambda(a)} (e^{2\lambda(a)} + 3)(e^{\lambda(a)} - 1)(H_0 - 2[\mathcal{I}_0]),$$

that follow from (6.37)–(6.38) and (6.40). Using the explicit form of $\lambda(a)$, c.f. (3.25), we obtain (6.55). □

Remark 6.9. As a consequence of the above results, the function $\Gamma^+ = \Gamma_*^+$ is determined in terms of quantities of the background configuration plus

$J_\varpi, \Pi^{(1)}$ and $P_c^{(2)}$, and near $r = 0$ it has the form

$$\Gamma^+(r) = -\frac{1}{E_c + P_c} P_c^{(2)} + o(1). \tag{6.56}$$

The function Γ in D^- takes the form

$$\Gamma^-(r) = -2\frac{4\pi r - \varkappa M_{\mathrm{T}}}{8\pi r - \varkappa M_{\mathrm{T}}}(H_0 - 2[\mathcal{I}_0]) + \Gamma_*^-(r), \tag{6.57}$$

with Γ_*^- being fully determined in terms of quantities of the background configuration plus $J_\varpi, \Pi^{(1)}$ and $\{P_c^{(2)}, \varsigma_-^+(a), \varsigma_P^+(a), \varsigma_-^{+\prime}(a), \varsigma_P^{+\prime}(a)\}$.

Having established the existence result for ς, we show that the functions that remain undetermined, namely $\delta(r)$ and $f(r,\theta)$, are pure gauge. Since we want to stay in the context where Propositions 6.5 (point *3*) and Proposition 6.6 can be applied, the available gauge freedom has already been restricted to the subset $\{\Psi(A,\mathcal{Y})\}$, namely a function $\mathcal{Y}(r,\theta)$ and a constant A, on each side $M^\pm$. There is also the free integration function $\beta(r)$ on each side. Our next result fixes this freedom and removes the functions f and δ altogether. This concludes our analysis of the base perturbation scheme.

Proposition 6.10 (Existence and uniqueness of the barotropic base scheme). *Assume the setup of the barotropic base perturbation scheme (*B1-B5*). Then, there exists a gauge on each region given by $\varphi^+ := \{\Psi()^+\}$ and $\varphi^- := \{\Psi()^-\}$ (no arguments left), c.f. Notation 2.7, in which*

1. *the two items of Proposition 6.5 hold,*
2. *$\beta^\pm$ can be fixed such that $f_+^\varphi = 0$ and $f_-^\varphi = 0$,*
3. *The solutions of the field equations, denoted by $\{h_+^\varphi, k_+^\varphi, m_+^\varphi\}$ in D^+ and $\{h_-^\varphi, k_-^\varphi, m_-^\varphi\}$ in D^-, exist and for given $P_c^{(2)} \in \mathbb{R}$ are unique. Moreover, the corresponding composed functions in $\widehat{D}$ take the form*

$$\begin{aligned} h^\varphi(r,\theta) &= h_0^\varphi(r) + h_2^\varphi(r)P_2(\cos\theta),\\ m^\varphi(r,\theta) &= m_0^\varphi(r) + m_2^\varphi(r)P_2(\cos\theta),\\ k^\varphi(r,\theta) &= k_2^\varphi(r)P_2(\cos\theta), \end{aligned}$$

with $h_0^\varphi, h_2^\varphi, m_0^\varphi, m_2^\varphi, k_2^\varphi \in C^n((0,a]) \cap C^\infty([a,\infty))$, extend continuously to $r = 0$ and are bounded.

Furthermore, if $\varpi^{\pm}=0$ *then* $k^{\varphi}_{+}=0$ *and*

$$h^{\varphi}_{+}=-\varkappa P_c^{(2)}\frac{E_c+3P_c}{24(E_c+P_c)}\left(\frac{2+r\nu'_{+}}{\nu'_{+}}\varsigma^{+\prime}_{-}+r\nu'_{+}\varsigma^{+}_{-}\right), \tag{6.58}$$

$$\begin{aligned} m^{\varphi}_{+}=-\varkappa P_c^{(2)}\frac{E_c+3P_c}{24(E_c+P_c)}\frac{1}{r\nu'^2_{+}} \\ \times\left(\left((2+r\nu'_{+})(2+r\lambda'_{+})-4e^{\lambda_{+}}\right)\varsigma^{+\prime}_{-}\right. \\ \left.+\left((2+r\lambda'_{+})r\nu'_{+}-4e^{\lambda_{+}}\right)\varsigma^{+}_{-}\right),\end{aligned} \tag{6.59}$$

while $k^{\varphi}_{-}=0$ *and*

$$h^{\varphi}_{-}=\varkappa P_c^{(2)}\left(\frac{8\pi r-\varkappa M_{\rm T}}{4\varkappa M_{\rm T}}r\varsigma^{-\prime}_{**}+\frac{\varkappa M_{\rm T}}{4(4\pi r-\varkappa M_{\rm T})}\varsigma^{-}_{**}\right), \tag{6.60}$$

$$\begin{aligned} m^{\varphi}_{-}=\varkappa P_c^{(2)}\left(\frac{16\pi r-3\varkappa M_{\rm T}}{4\varkappa M_{\rm T}}r\varsigma^{-\prime}_{**}\right. \\ \left.-\left(8\pi r+\frac{(16\pi r-3\varkappa M_{\rm T})\varkappa^2M_{\rm T}^2}{4(4\pi r-\varkappa M_{\rm T})^2}\right)\frac{4\pi r-\varkappa M_{\rm T}}{\varkappa M_{\rm T}(8\pi r-\varkappa M_{\rm T})}\varsigma^{-}_{**}\right),\end{aligned} \tag{6.61}$$

where $\varsigma^{+}_{-}(r)$ *and* $\varsigma^{-}_{**}(r)$ *are defined in Proposition 6.7.*

Proof. We start considering the classes $\{\Psi(A,B,\mathcal{Y})^{\pm}\}$ in which Proposition 6.6 holds, both on M^{+} and M^{-}. We apply a change of gauge (2.17)–(2.20) in Proposition 2.5 with $\mathcal{R}=r$, $\alpha(r)=0$ and

$$\mathcal{Y}=-r\left(\delta+f-\frac{\Gamma}{2}\right). \tag{6.62}$$

We also fix $\beta(r)$ by

$$\beta(r)=\delta-\frac{\Gamma}{2}. \tag{6.63}$$

Applying this to (6.32) one obtains

$$h^{g}=\frac{1}{2}\left(A+2\mathcal{I}_0+\frac{\Gamma}{2}\left(2+r\nu'\right)\right)+\left(\frac{2+r\nu'}{2r\nu'}f_{\omega}-\frac{1}{\nu'}\widehat{v}'_2\right)P_2(\cos\theta), \tag{6.64}$$

$$k^{g}=\left(\widehat{v}_2+\frac{1}{\nu'}\widehat{v}'_2-\frac{2+r\nu'}{2r\nu'}f_{\omega}\right)P_2(\cos\theta), \tag{6.65}$$

$$m^g = \frac{1}{r\nu'}\left(\frac{\Gamma}{4}\left((2+r\nu')(2+r\lambda') - 4e^{\lambda}\right) - \frac{2e^{\lambda}}{2+r\nu'}\varsigma - \frac{1}{6}r^3 e^{-\nu}\left(2(\lambda'+\nu')(\varpi - \Pi^{(1)})^2 - r\varpi'^2\right)\right) + \left(\frac{1}{\nu'}\widehat{v}_2' - \frac{2-r\nu'}{2r\nu'}f_\omega\right)P_2(\cos\theta), \tag{6.66}$$

$$f^g = 0. \tag{6.67}$$

This already proves item *2*. Observe that this partial gauge fixing still leaves arbitrary the constants A, B on each side. We now choose (uniquely) B^+ and B^- so that the second point in Proposition 6.5 holds.

At the interior we choose $A^+ = -2\mathcal{I}_0^+$, so that (6.64)–(6.66) become

$$h_+^\varphi = \frac{\Gamma^+}{4}\left(2 + r\nu'_+\right) + \left(\frac{2+r\nu'_+}{2r\nu'_+}f_\omega^+ - \frac{1}{\nu'_+}\widehat{v}_2^{+\prime}\right)P_2(\cos\theta), \tag{6.68}$$

$$k_+^\varphi = \left(\widehat{v}_2^+ + \frac{1}{\nu'_+}\widehat{v}_2^{+\prime} - \frac{2+r\nu'_+}{2r\nu'_+}f_\omega^+\right)P_2(\cos\theta), \tag{6.69}$$

$$m_+^\varphi = \frac{1}{r\nu'_+}\left(\frac{\Gamma^+}{4}\left((2+r\nu'_+)(2+r\lambda'_+) - 4e^{\lambda_+}\right) - \frac{2e^{\lambda_+}}{2+r\nu'_+}\varsigma^+ - \frac{1}{6}r^3 e^{-\nu_+}\left(2(\lambda'_+ + \nu'_+)(\varpi_+ - \Pi^{(1)})^2 - r{\varpi'_+}^2\right)\right) + \left(\frac{1}{\nu'_+}\widehat{v}_2^{+\prime} - \frac{2-r\nu'_+}{2r\nu'_+}f_\omega^+\right)P_2(\cos\theta). \tag{6.70}$$

The behaviour of these expressions as $r \to 0$, using (3.27)–(3.29), (6.24), (6.56), that $f_\omega \in O(r^4)$, and Propositions 6.6 and 6.7 as well as Corollary 6.8, is given by

$$\lim_{r\to 0} h_+^\varphi = -\frac{1}{2(E_c + P_c)}P_c^{(2)}, \qquad \lim_{r\to 0} k_+^\varphi = 0, \qquad \lim_{r\to 0} m_+^\varphi = 0,$$

i.e. the limits exist. This shows in particular that the change of gauge defined by (6.62) lies within the class of gauges described in Notation 2.7 and therefore that (6.68)–(6.70) are written in an admissible and fully fixed gauge $\{\Psi()^+\}$ (no arguments left), which we have denoted simply by φ^+. These expressions involve only functions whose existence, uniqueness and regularity properties have already been established in Propositions 6.6 and 6.7 together with Corollary 6.8. Specifically $\widehat{v}_2^+, \varsigma^+$ are $C^{n+1}((0,a])$, $\widehat{v}_2^+$ is unique and ς^+, and thus also Γ^+ by Remark 6.9, is unique up to the constant $P_c^{(2)}$. The claim for $\{h_+^\varphi, k_+^\varphi, m_+^\varphi\}$ follows.

Regarding $\{h^g_-, k^g_-, m^g_-\}$ in D^- we choose $A^- = -2\mathcal{I}_0^+ + H_0$. Then (6.64)–(6.66) become, in the fixed gauge $\varphi^- = \{\Psi()^-\}$ (no arguments left),

$$
\begin{aligned}
(6.71)\quad h^\varphi_- &= \frac{1}{4}(2 + r\nu'_-)\Gamma^-_* \\
&\quad + \frac{1}{\varkappa M_{\rm T}}\left(3(8\pi r - \varkappa M_{\rm T})\frac{J^2_\varpi}{r^4} - (4\pi r - \varkappa M_{\rm T})r\widehat{v_2^-}'\right)P_2(\cos\theta),\\
(6.72)\quad k^\varphi_- &= \left(\widehat{v_2^-} + \frac{4\pi r - \varkappa M_{\rm T}}{\varkappa M_{\rm T}}r\widehat{v_2^-}' - \frac{8\pi r - \varkappa M_{\rm T}}{\varkappa M_{\rm T}}3\frac{J^2_\varpi}{r^4}\right)P_2(\cos\theta),\\
(6.73)\quad m^\varphi_- &= \left(8\pi r - \frac{(16\pi r - 3\varkappa M_{\rm T})(8\pi r - \varkappa M_{\rm T})}{4\pi r - \varkappa M_{\rm T}}\right)\frac{3J^2_\varpi}{\varkappa M_{\rm T} r^4}\\
&\quad - \left(8\pi r + \frac{(16\pi r - 3\varkappa M_{\rm T})\varkappa^2 M^2_{\rm T}}{4(4\pi r - \varkappa M_{\rm T})^2}\right)\frac{4\pi r - \varkappa M_{\rm T}}{\varkappa M_{\rm T}(8\pi r - \varkappa M_{\rm T})}\varsigma^-_*\\
&\quad - \frac{16\pi r - 3\varkappa M_{\rm T}}{4\varkappa M_{\rm T}}r\varsigma^{-\prime}_*\\
&\quad + \left(\frac{4\pi r - \varkappa M_{\rm T}}{\varkappa M_{\rm T}}r\widehat{v_2^-}' - \frac{8\pi r - 3\varkappa M_{\rm T}}{\varkappa M_{\rm T}}3\frac{J^2_\varpi}{r^4}\right)P_2(\cos\theta)
\end{aligned}
$$

after using (3.25), (5.5), (5.28), (6.55) and (6.57). It is straigforward to check first that given (6.45), ς^-_* is $O(1/r)$ and $\varsigma^{-\prime}_*$ is $O(1/r^2)$ and therefore Γ^-_*, c.f. (6.41), is bounded near $r = \infty$. Then, since $\widehat{v}_2$ is $O(1/r^4)$ and $\widehat{v}'_2$ is $O(1/r^5)$, c.f. Proposition 6.6, the three functions in (6.71)–(6.73) are bounded in D^-, which justifies the fact that the change of gauge was indeed within the class $\{\Psi\}$ from Notation 2.7. Again, Propositions 6.6 and 6.7 together with Corollary 6.8 have established all the required existence, uniqueness and regularity properties: $\widehat{v_2^-}, \varsigma^-$ are both $\in C^\infty([a,\infty))$, $\widehat{v_2^-}$ is uniquely determined and ς_* is explicitly given in (6.45) with the constants ς_A and ς_B fully determined once $P_c^{(2)}$ is fixed, and Remark 6.9 establishes the same for Γ_*. The claim for $\{h^\varphi_-, k^\varphi_-, m^\varphi_-\}$ follows.

We now consider the particular case $\varpi^\pm = 0$, so that, in particular, $\Pi^{(1)}_+ = J_\varpi = 0$ and $f_\omega(r) = 0$. First, Proposition 6.6 gives $\widehat{v_2^\pm}(r) = 0$ and therefore all terms in the $\ell = 2$ sector in (6.68)–(6.73) vanish. Thence we already have that $k^\varphi_+ = k^\varphi_- = 0$. On the other hand, Proposition 6.7 item *3*, and Corollary 6.8, provide the form of ς^+ and the explicit expression of ς^-, which inserted in (6.68), (6.70), (6.71), using (6.31), and (6.73) yield (6.58) and (6.59) in D^+, and (6.60) and (6.61) in D^-. □

Remark 6.11. Combining $A^+ = -2\mathcal{I}_0^+$ with (5.60) it follows that, in the gauge φ, the constant $\mathcal{I}_0^\varphi$ vanishes. Hence, the perturbed pressure and energy density (5.58)–(5.59) in this gauge are

$$P^{(2)\varphi} = -2(E+P)\left(h_+^{\varphi} + \frac{1}{3}e^{-\nu}r^2(\varpi - \Pi^{(1)})^2\left(P_2(\cos\theta) - 1\right)\right),$$
$$E^{(2)\varphi} = \frac{4}{\nu'}E'\left(h_+^{\varphi} + \frac{1}{3}e^{-\nu}r^2(\varpi - \Pi^{(1)})^2\left(P_2(\cos\theta) - 1\right)\right).$$

7. Existence and uniqueness of the general set up

We are now ready to apply the results obtained for the "base" perturbation scheme to solve, using a bootstrap argument, the general first order and second order problems for the perturbation scheme in the canonical form over a background configuration for a rigidly rotating perfect fluid interior and vacuum exterior following Definition 2.3.

Before stating the main results of this paper, it is necessary to discuss the physical meaning of the constant $P_c^{(2)}$ that has been introduced in the previous section. We already know that $P^{(2)} = \lim_{r\to 0} P^{(2)}$ (Proposition 6.7), so one might think that this parameter already has a clear meaning. However, the point is more subtle than one may think, as we discuss next.

7.1. The perturbed central pressure

In this work, three different sets of gauge vectors play a role. Theorem 2.2 assumes a C^{n+1} perturbation scheme. So, in particular it assumes perturbation tensors K_1^o and K_2^o that are $C^n(M^\pm)$ and $C^{n-1}(M^\pm)$ respectively. The first set of gauge vectors is the standard one, namely vector fields that respect this differentiability class everywhere. They correspond to gauge vectors $V_1 \in C^{n+1}(M^\pm)$ at first order and $V_2 \in C^n(M^\pm)$ at second order. The second set of gauge vectors is the one that transforms K_1^o, K_2^o into K_1^Ψ, K_2^Ψ as given in Theorem 2.2. These gauge vectors are no longer differentiable everywhere. However, it is part of the content of Theorem 2.2 that they can be chosen to have no radial component and to extend continuously at the origin. Moreover, under a very mild extra condition discussed in Remark 2.4, when the target tensors K_1^Ψ and K_2^Ψ are taken as fixed, these vectors are uniquely defined up to a linear combination of the background Killings ξ and η. So, all such vectors extend continuously to the origin and have no radial component. We call this "canonical gauge transformation". The third class is defined in Proposition 2.5 and has been called $\{\Psi(C; A, B, \mathcal{Y}, \alpha)\}$ in Notation 2.7. This class has been extensively used in the analysis of the base perturbation scheme.

Concerning the perturbed pressures $P^{(1)}$ amd $P^{(2)}$, the field equations imply that these functions are of class $C^{n-2}(M^+)$ and $C^{n-3}(M^+)$ in the starting gauge K_1^o and K_2^o. We already know (Lemma 4.2) that, under this first set of gauge transformations, $P_c^{(2)}$ is invariant provided the configuration has equatorial symmetry (and this property is true in the present setup, see below).

In the base perturbation scheme we have only assumed the *outcome* of Theorem 2.2. While $P^{(1)} = 0$ followed directly from the field equations, at this level of generality we did not know a priori that $P^{(2)}$ is well-behaved at the centre (not even bounded). We prefered this route (instead of assuming the *hypotheses* of Theorem 2.2, which would of course would have been justified) in order to emphasize that, even with this generality, imposing that the fluid satisfies a barotropic equation of state (independent of the perturbation parameter ε) already forces the continuity of $P^{(2)}$ at the centre and hence the existence of the parameter $P_c^{(2)}$.

We now discuss the gauge invariance of $P_c^{(2)}$ under the canonical gauge transformation and under $\{\Psi(C; A, B, \mathcal{Y}, \alpha)\}$ when the *hypotheses* of Theorem 2.2 are assumed (instead of only its conclusions). We are only interested in the case when $P^{(1)} = 0$.

Lemma 7.1. *Let K_1^o and K_2^o be perturbation tensors defined by the C^{n+1} ($n \geq 3$) perturbation scheme $(M_\varepsilon, \hat{g}_\varepsilon, \{\psi_\varepsilon\})$ assumed in Theorem 2.2. Assume further that the perturbed field equations for a rigidly rotating perfect fluid hold on M^+ with $P^{(1)} = 0$ and set $P_c^{(2)} := P^{(2)}(0)$.*

(i) *If $\hat{g}_\varepsilon$, $\varepsilon \neq 0$, only admits one axial Killing vector, then $P_c^{(2)}$ is gauge invariant under the canonical gauge transformation.*

(ii) *$P_c^{(2)}$ is gauge invariant under $\{\Psi(C; A, B, \mathcal{Y}, \alpha)\}$.*

Proof. For the gauge vectors V_1 and V_2 that transform K_1^o and K_2^o into the form K_1^Ψ and K_2^Ψ given in Theorem 2.2, we know that V_1 and V_2 have no radial component (because the condition described in Remark 2.4 is satisfied). Since the background pressure P is radially symmetric we have $V_1(P) = V_2(P) = 0$ outside the centre, and in fact everywhere because V_1, V_2 extend continuosly to the centre. It is now immediate from (4.23) that $P^{(1)g} = P^{(1)} = 0$ and $P^{(2)g} = P^{(2)}$. In particular, the value at the centre $P_c^{(2)} = P^{(2)}(0)$ is gauge invariant. This proves item *(i)*.

For item *(ii)*, let V_1, V_2 be gauge vectors given by (2.15) and (2.16) respectively. Since V_1 has again no radial component, the property $P = P^{(1)} = 0$ follows as before. Now, (4.23) gives $P^{(2)g} = P^{(2)} + V_2(P) = P^{(2)} +$

$2\mathcal{Y}(r,\theta)\partial_r(P)$. Boundedness of $\mathcal{Y}$ and the vanishing of dP at the centre, ensured by Lemma 3.2, proves that $P_c^{(2)}$ is also gauge invariant in this case. □

This result allows us to call $P_c^{(2)}$ "the perturbed central pressure" (to second order) in an unambiguous way.

7.2. Existence and uniqueness of the first order problem

We focus first on the first order problem, i.e. that for (2.11). To do that we only need to consider the *base perturbation scheme* with $\varpi(r) = \Pi^{(1)} = 0$ and identify K_1^{Ψ} with K_2 in (5.7) thanks to the substitutions

$$\mathcal{W}(r,\theta) \to \omega(r,\theta), \quad h \to h^{(1)}, \quad m \to m^{(1)}, \quad k \to k^{(1)}, \quad f \to f^{(1)}, \tag{7.1}$$

at both sides $\pm$, while the perturbed density and pressure in the exterior M^- get substituted by

$$P^{(2)} \to P^{(1)}, \qquad E^{(2)} \to E^{(1)}. \tag{7.2}$$

Obviously, also

$$\mathcal{W}_c \to \omega_c, \qquad \Pi_+^{(2)} \to \Pi_+^{(1)}, \qquad P_c^{(2)} \to P_c^{(1)}, \tag{7.3}$$

and the latter is called simply "perturbed pressure at the origin". We make the full argument precise in the proof of the following proposition.

Proposition 7.2 (Rotating stars to 1st order). *Consider a C^{n+1} perfect fluid ball configuration according to Definition 3.3 with $n \geq 4$ and $E_c + P_c \neq 0$. Let us be given a C^{n+1} maximal perturbation scheme $(M_\varepsilon, \hat{g}_\varepsilon, \{\psi_\varepsilon\})$ inheriting the Abelian G_2 generated by $\{\xi = \partial_t, \eta = \partial_\phi\}$. Assume that the corresponding first order perturbation tensor K_1*

- *solves the 1st order perturbed equations for a rigidly rotating perfect fluid with the barotropic equation of state of the background in M^+ and for vacuum in M^-,*
- *satisfies the linearized matching conditions across the boundary of the fluid ball.*
- *it is bounded,*
- *the perturbed pressure vanishes at the origin, $P_c^{(1)} = 0$.*

Then there exist first order gauge vectors (at each interior and exterior regions $M^{\pm}$) such that the gauge transformed tensor K_1^{φ} takes the form

$$K_1^{\varphi} = -2\omega_c(G(r) - G_{\infty})r^2 \sin^2\theta dt d\phi, \tag{7.4}$$

where $\omega_c \in \mathbb{R}$, $G(r)$ is the unique $C^1(0,\infty)$ solution of (6.17) *in D with $G(0) = 1$, and $G_{\infty} = \lim_{r\to\infty} G(r) = G(a) + aG'(a)/3 > 1$. Moreover, G^+ extends to a $C^2(M^+) \cap C^{n+2}(M^+ \setminus \mathcal{C}_0)$ function, while G^- is $C^{\infty}([a,\infty))$ and given explicitly by* (6.20)*.*

Furthermore, the parameter $\Omega^{(1)} = -\omega_c G_{\infty}$, and the first order perturbed pressure and density $P^{(1)}$ and $E^{(1)}$ vanish identically.

Proof. We start by setting the problem under the frame of the *base perturbation scheme*. Point B1 of the *base perturbation scheme* is satisfied by assumption. We set $K_1^{\pm} = 0$, i.e. $\varpi_{\pm} = 0$, and the whole point B2 is trivially satisfied with $\Pi^{(1)} = J_{\varpi} = 0$, while the matching conditions (5.66) and (5.67) are also satisfied for $b_1 = Q_1^{\pm} = 0$. By Proposition 4.1 the perturbation scheme inherits the orthogonal transitivity of the group generated by $\{\xi,\eta\}$ and therefore Theorem 2.2 applies to both the interior and exterior regions $M^{\pm}$ ensuring that there exists a gauge transformation at each region $M^{\pm}$ for which the first order perturbation tensors take the form (2.11) on each $M^{\pm}$. With the identifications in (7.1) and setting $\mathcal{R}(r) = r$ (which is allowed by the assumptions in Definition 3.3), the properties of the functions in (2.11) imply that point B3.1 of the *base perturbation scheme* is satisfied for $K_2 = K_1^{\Psi}$ with $m = n-1$ (on both M^+ and M^-). Finally, the points B3.2, B4 and B5 are incorporated as assumptions in the Proposition.

It suffices now to apply Proposition 6.10 for the case $\varpi^{\pm} = 0$ and impose $P_c^{(2)} = 0$, which is the translation of $P_c^{(1)} = 0$ under (7.2). Applying again the translation (7.2) to the outcome of this Proposition gives (7.4) as well as all the listed properties of $G(r)$.

The parameter $\Omega^{(1)}$ was called $\Pi_+^{(1)}$ in the *base scheme* and takes the value (by Proposition 6.5 after appyling the translation (7.3)) $\Pi_+^{(1)} = -\omega_c G_{\infty}$. Moreover, Remark 6.11, applied to $\varpi = 0$ and $h^{\varphi} = 0$ provides $E^{(2)} = P^{(2)} = 0$. The translation (7.2) provides the final claim. Calling $P_c^{(2)}$ the "perturbed pressure at the origin" is justified by Lemma 7.1 □

7.3. Existence and uniqueness to second order

Given the previous result for the first order problem, the second order problem just follows the *base perturbation scheme*, and we just need to make direct use of Proposition 6.10. We make the result and the argument precise in the following.

Theorem 7.3 (Rotating stars to second order). *Consider a C^{n+1} perfect fluid ball configuration according to Definition 3.3 with $n \geq 4$ and $E_c + P_c \neq 0$. Let us be given a C^{n+1} maximal perturbation scheme $(M_\varepsilon, \hat{g}_\varepsilon, \{\psi_\varepsilon\})$ inheriting the Abelian G_2 generated by $\{\xi = \partial_t, \eta = \partial_\phi\}$. Assume that the corresponding first and second order perturbation tensors K_1 and K_2*

- *solve the perturbed equations to second order for a rigidly rotating perfect fluid with the barotropic equation of state of the background in M^+ and for vacuum in M^-,*
- *satisfy the matching conditions to second order across the boundary of the fluid ball,*
- *are bounded,*
- *the perturbed pressure at the origin vanishes, $P_c^{(1)} = P_c^{(2)} = 0$.*

Then there exist first and second order gauge vectors (at each interior and exterior regions $M^\pm$) such that the gauge transformed tensors K_1^φ and K_2^φ are of class $C^2(M^+) \cap C^{n+2}(M^+ \setminus \mathcal{C}_0) \cap C^\infty(M^-) \cap C^1(M)$ and $C^0(M^+) \cap C^n(M^+ \setminus \mathcal{C}_0) \cap C^\infty(M^-)$ respectively, and take the form

$$K_1^\varphi = -2\omega_c(G(r) - G_\infty) r^2 \sin^2\theta dt d\phi, \tag{7.5}$$

$$\begin{aligned} K_2^\varphi = & \left(-4e^{\nu(r)}h(r,\theta) + 2\omega_c^2(G(r) - G_\infty)^2 r^2 \sin^2\theta\right) dt^2 \\ & + 4e^{\lambda(r)} m(r,\theta) dr^2 + 4k(r,\theta) r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \\ & - 2\mathcal{W}_c(G(r) - G_\infty) r^2 \sin^2\theta dt d\phi, \end{aligned} \tag{7.6}$$

with

$$\begin{aligned} h(r,\theta) &= h_0(r) + h_2(r) P_2(\cos\theta), \\ m(r,\theta) &= m_0(r) + m_2(r) P_2(\cos\theta), \\ k(r,\theta) &= k_2(r) P_2(\cos\theta), \end{aligned} \tag{7.7}$$

where $\omega_c, \mathcal{W}_c \in \mathbb{R}$ are free parameters and

(i) *$G(r)$ is the unique $C^1(0,\infty)$ solution of (6.17) in D with $G(0) = 1$. Moreover, $G_\infty = \lim_{r\to\infty} G(r) = G(a) + aG'(a)/3 > 1$, $G'(a) > 0$ and $G(r)$ in D^+ extends to a $C^2([0,a]) \cap C^{n+2}((0,a])$ function.*

(ii) *The functions* $h_0(r), h_2(r), m_0(r), m_2(r), k_2(r)$ *are of class* $C^n((0,a]) \cap C^\infty([a,\infty))$, *extend continuously to* $r = 0$, *are bounded, are uniquely determined by* ω_c, *and all vanish if* $\omega_c = 0$.

(iii) *The rotation parameters* $\Omega^{(1)} = -\omega_c G_\infty$ *and* $\Omega^{(2)} = -\mathcal{W}_c G_\infty$.

Remark 7.4. In all the expressions referred to in this remark the replacement $\varpi_+ - \Pi_+^{(1)} \to \omega_c G$ is to be made.

The global solution in the gauge φ is obtained, apart from $G(r)$, in terms of two fully determined functions $\varsigma_*(r)$ and $\widehat{v}_2(r)$. The function $\widehat{v}_2(r)$ is the unique bounded solution of the equation (5.46) with (5.37) and (5.28) that satisfies the matching conditions

$$[\widehat{v}_2] = 0, \qquad [\widehat{v}_2'] = \frac{1}{6} e^{\lambda(a)}(1 + e^{\lambda(a)}) a^3 \varkappa E_+(a) G^2(a),$$

c.f. Proposition 6.6. The function $\varsigma_*(r)$ is the unique bounded solution of the equation (5.65) with (5.37) and (5.28) that satisfies (6.39) and the matching conditions

$$[\varsigma_*] = 0, \qquad [\varsigma_*'] = -\frac{2}{3} e^{\lambda(a)}(1 + e^{\lambda(a)}) a^3 \varkappa E_+(a) G^2(a),$$

and is determined by the value $P_c^{(2)} = \lim_{r\to 0} P^{(2)}$ through the relation (6.43), c.f. Proposition 6.7. Although in Theorem 7.3 we have set $P_c^{(2)} = 0$, the solution has been obtained for general values of the second order perturbed pressure $P_c^{(2)}$. This may be of independent interest.

- In the interior region we have $E^{(1)} = P^{(1)} = 0$, and the functions $\{h, m, k\}$ and $\{E^{(2)}, P^{(2)}\}$ are given by the right-hand sides of (6.68)–(6.70) and Remark 6.11, respectively, with $\Gamma(r)$ as defined in (6.31), f_ω as in (5.28) and ς_*^+, $\widehat{v}_2^+$ being the interior parts of ς_*, $\widehat{v}_2$.
- In the exterior region the functions $\{h, m, k\}$ are given by the right-hand sides of (6.71)–(6.73) with $\Gamma_\star$ as defined in (6.41) and $J_\varpi \to -\omega_c G'(a) a^4/6$, with ς_*^- and $\widehat{v}_2^-$ being the exterior parts of ς_* and $\widehat{v}_2$.

Proof. All the hypotheses of Proposition 7.2 are satisfied, so (7.5) as well as item *(i)* follow readily. We now use the *base perturbation scheme*. The first point B1 is satisfied by assumption. Point B2 holds with $K_1 = K_1^\varphi$ and $\varpi = \omega_c(G - G_\infty)$, $\Pi_+^{(1)} = -\omega_c G_\infty$. Proposition 4.1 and Theorem 2.2 imply the existence of a second order gauge vector that transforms the second order tensor onto the form $K_2^{\Psi\pm}$ as given in (2.12) on each $M^\pm$. By the same Theorem 2.2, point B3.1 of the *base perturbation scheme* is satisfied with $m = n - 2$. Since $n \geq 4$ by assumption, the condition $m \geq 2$ holds. Finally, B3.2, B4 and B5 are satisfied by assumption.

It suffices now to apply Proposition 6.10 with $\varpi = \omega_c(G - G_\infty)$ to conclude (7.6), (7.7), as well as point *(ii)*. As before, calling $P_c^{(2)}$ the "perturbed pressure at the origin" (at second order) is justified by Lemma 7.1. Item *(iii)* follows because $\Omega^{(2)}$ corresponds to $\Pi_+^{(2)}$ in the base scheme and its value was given in Proposition 6.5 (the value of $\Omega^{(1)}$ already appears in Proposition 7.2).

The matching conditions for ς_* and $\widehat{v}_2$ in the Remark follow from (6.40) and (6.25) respectively, for $\varpi = \omega_c(G - G_\infty)$ and $\Pi_+^{(1)} = -\omega_c G_\infty$, so that $\varpi_+(a) - \Pi_+^{(1)} = \omega_c G(a)$ and therefore $[f_\omega] = \frac{1}{3} e^{\lambda(a)} a^4 \varkappa E_+(a) G^2(a)$, c.f. (5.83). The expressions are simplified using (3.20) and that $\nu(a) = -\lambda(a)$, c.f. (3.25) and (3.19). □

We finish the paper by writing down the family of metrics g_ε to second order for the gauge φ obtained in this theorem. The final step will be to exploit the freedom in redefining the perturbation parameter $\varepsilon \to \tilde{\varepsilon} = \tilde{\varepsilon}(\varepsilon)$, inherent to any perturbation theory, as well as the scalability of the perturbations, to obtain the clasical form of the stationary and axially symmetric perturbations around static balls.

From Theorem 7.3 we have

$$
\begin{aligned}
(7.8) \qquad g_\varepsilon = g &- \varepsilon(2\omega_c + \varepsilon \mathcal{W}_c)(G(r) - G_\infty) r^2 \sin^2\theta dt d\phi \\
&+ \varepsilon^2 \Bigg(\left(-2e^{\nu(r)} h(r,\theta) + \omega_c^2 (G(r) - G_\infty)^2 r^2 \sin^2\theta \right) dt^2 \\
&+ 2e^{\lambda(r)} m(r,\theta) dr^2 + 2k(r,\theta) r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \Bigg) + O(\varepsilon^3).
\end{aligned}
$$

If $\omega_c = 0$ then (7.8) reduces to $g_\varepsilon = g - \varepsilon^2 \mathcal{W}_c (G(r) - G_\infty) r^2 \sin^2\theta dt d\phi + O(\varepsilon^3)$, which simply means that the perturbation is set to start at second order, which then becomes the first non-trivial order and takes exacly the same form as the first order with $\omega_c = \mathcal{W}_c/2$. We can thus assume $\omega_c \neq 0$ without loss of generality. The change $\tilde{\varepsilon} = \varepsilon(\omega_c + \varepsilon \mathcal{W}_c/2)$, after a suitable rescaling

$$\{h, m, k\} \to \{h/\omega_c^2, m/\omega_c^2, k/\omega_c^2\}$$

yields

$$g_{\tilde{\varepsilon}} = g - 2\tilde{\varepsilon}(G(r) - G_\infty)r^2 \sin^2\theta dt d\phi + \tilde{\varepsilon}^2\Big(\left(-2e^{\nu(r)}h(r,\theta) + (G(r) - G_\infty)^2 r^2 \sin^2\theta\right) dt^2 + 2e^{\lambda(r)}m(r,\theta)dr^2 + 2k(r,\theta)r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\Big) + O(\tilde{\varepsilon}^3). \quad (7.9)$$

This redefinition amounts to setting $\omega_c = 1$ and $\mathcal{W}_c = 0$. Let us stress the fact that all functions in this last expression are unique, and that the only free paramenter that enters the first and second order perturbations, which we have taken to be ω_c, is now integrated into $\tilde{\varepsilon}$. This corresponds to the property, widely used in the literature, that stationary and axially symmetric perturbations of fluid balls to second order depend on a single free parameter, that can be encoded in the perturbation parameter and which physically is related to the rotation of the fluid. As discussed in the introduction, establishing this fact rigorously was one of the aims of this paper.

Let us finally stress that the relation of the parameters and functions in (7.9) with the rotation of the star, as measured by the static observer ξ at infinity, requires the full control of the gauges and the jump at the surface of the relevant functions, and is of global nature and gauge invariant, as discussed in [33]. One detail that is missing in [33] is that boundedness of the perturbation forces the gauges (at first order) to be fixed so that ξ remains to be the stationary observer for g_ε at infinity. This fixing of gauges, as done in the present paper, yields the following fluid velocity

$$u = F_\varepsilon(\xi - \tilde{\varepsilon}G_\infty\eta) + O(\tilde{\varepsilon}^3), \quad F_\varepsilon := e^{-\frac{\nu}{2}} + \frac{1}{2}\tilde{\varepsilon}^2 e^{-\frac{3\nu}{2}}\left(-2e^\nu h + G^2 r^2 \sin^2\theta\right). \quad (7.10)$$

Indeed, the redefinition $\{\omega_c \to 1, \mathcal{W}_c \to 0\}$ gives (by Theorem 7.3, item (ii)) $\{\Omega^{(1)} = -G_\infty = -(G(a) + aG'(a)/3) < -1, \Omega^{(2)} = 0\}$ and (7.10) follows from (4.21) and (4.22) applied to (7.9). The velocity of rotation of u_ε along η as measured with respect to ξ at infinity is thus given by $-\tilde{\varepsilon}G_\infty$, and vanishes iff $\tilde{\varepsilon} = 0$.

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Appendix A. First order perturbed Ricci tensor in covariant form

In this appendix we derive, in a fully covariant manner, the first order perturbed Ricci tensor in backgrounds admitting two Killing vectors ξ and η satisfying suitable conditions (see Proposition A.1 below) and for metric perturbation tensors K_1 with a single component along the ξ, η direction. It turns out that the perturbed Ricci tensor preserves this structure, namely its only non-zero component is again along the ξ, η direction. In practice this implies a decoupling of the perturbed field equations.

In the main text we apply these results in the context of static and spherically symmetric backgrounds. However, the decoupling holds in much more generality. In view of their potential interest for other problems we present the general result.

We start by writing down an expression for the perturbed Ricci tensor in an arbitrary background when the metric perturbation tensor K_1 is splitted as $K_{1\alpha\beta} = w\mathfrak{S}_{\alpha\beta}$, where w and $\mathfrak{S}_{\alpha\beta}$ are for the moment any C^2 scalar and symetric $(0,2)$-tensor, respectively. Directly from the definition (4.2)

$$\text{(A.1)} \qquad \begin{aligned} S^{(1)}{}_{\mu\alpha\beta} &= \frac{1}{2}\left(\nabla_\alpha w\, \mathfrak{S}_{\mu\beta} + \nabla_\beta w\, \mathfrak{S}_{\mu\alpha} - \nabla_\mu w\, \mathfrak{S}_{\alpha\beta}\right) + w\mathcal{P}_{\mu\alpha\beta}, \\ \mathcal{P}_{\mu\alpha\beta} &:= \frac{1}{2}\left(\nabla_\alpha \mathfrak{S}_{\mu\beta} + \nabla_\beta \mathfrak{S}_{\mu\alpha} - \nabla_\mu \mathfrak{S}_{\alpha\beta}\right). \end{aligned}$$

Taking the trace in $\mu\alpha$ in (4.2) yields immediately

$$\text{(A.2)} \qquad S^{(1)\mu}{}_{\mu\beta} = \frac{1}{2}\nabla_\beta K_1{}^\mu{}_\mu = \frac{1}{2}\nabla_\beta\left(w\mathfrak{S}^\mu{}_\mu\right).$$

We also need to compute

$$\begin{aligned}
\text{(A.3)}\quad \nabla_\mu S^{(1)\mu}{}_{\alpha\beta} &= \frac{1}{2}\left(\nabla^\mu\nabla_\alpha w\,\mathfrak{S}_{\mu\beta} + \nabla^\mu\nabla_\beta w\,\mathfrak{S}_{\mu\alpha} - \nabla^\mu\nabla_\mu w\,\mathfrak{S}_{\alpha\beta}\right)\\
&\quad + \frac{1}{2}\big(\nabla_\alpha w\,\nabla^\mu\mathfrak{S}_{\mu\beta} + \nabla_\beta w\,\nabla^\mu\mathfrak{S}_{\mu\alpha} - 2\nabla^\mu w\,\nabla_\mu\mathfrak{S}_{\alpha\beta}\\
&\qquad + \nabla^\mu w\,\nabla_\alpha\mathfrak{S}_{\mu\beta} + \nabla^\mu w\,\nabla_\beta\mathfrak{S}_{\mu\alpha}\big) + w\nabla_\mu\mathcal{P}^\mu{}_{\alpha\beta}\\
&= \frac{1}{2}\Big(\nabla_\alpha\left(\nabla^\mu w\mathfrak{S}_{\mu\beta}\right) + \nabla_\beta\left(\nabla^\mu w\mathfrak{S}_{\mu\alpha}\right) - \nabla^\mu\nabla_\mu w\,\mathfrak{S}_{\alpha\beta}\Big)\\
&\quad - \nabla_\mu w\nabla^\mu\mathfrak{S}_{\alpha\beta} + \frac{1}{2}\left(\nabla_\alpha w\nabla^\mu\mathfrak{S}_{\mu\beta} + \nabla_\beta w\nabla^\mu\mathfrak{S}_{\mu\alpha}\right)\\
&\quad + w\nabla_\mu\mathcal{P}^\mu{}_{\alpha\beta}.
\end{aligned}$$

Inserting (A.2)–(A.3) into (4.7), yields the following (fully general) identity:

$$\begin{aligned}
R^{(1)}_{\alpha\beta} &= \frac{1}{2}\Big(\nabla_\alpha\left(\nabla^\mu w\,\mathfrak{S}_{\mu\beta}\right) + \nabla_\beta\left(\nabla^\mu w\,\mathfrak{S}_{\mu\alpha}\right) - \nabla^\mu\nabla_\mu w\,\mathfrak{S}_{\alpha\beta}\Big) - \nabla_\mu w\,\nabla^\mu\mathfrak{S}_{\alpha\beta}\\
&\quad + \frac{1}{2}\Big(\nabla_\alpha w\,\nabla^\mu\mathfrak{S}_{\mu\beta} + \nabla_\beta w\,\nabla^\mu\mathfrak{S}_{\mu\alpha}\Big) + w\,\nabla_\mu\mathcal{P}^\mu{}_{\alpha\beta} - \frac{1}{2}\nabla_\alpha\nabla_\beta\left(w\,\mathfrak{S}^\mu{}_\mu\right).
\end{aligned}$$

We now assume that (M,g) admits two Killing vectors ξ and η and that $\mathfrak{S}_{\alpha\beta} = \xi_\alpha\eta_\beta + \xi_\beta\eta_\alpha$. Then $\nabla^\mu w\,\mathfrak{S}_{\mu\alpha} = \xi(w)\eta_\alpha + \eta(w)\xi_\alpha$, $\mathfrak{S}^\mu{}_\mu = 2\langle\xi,\eta\rangle$ and

$$\nabla^\mu\mathfrak{S}_{\mu\alpha} = \xi_\mu\nabla^\mu\eta_\alpha + \eta_\mu\nabla^\mu\xi_\alpha = -\nabla_\alpha\langle\xi,\eta\rangle.$$

Moreover, the Killing equations also imply

$$\text{(A.4)}\qquad \mathcal{P}_{\mu\alpha\beta} = \nabla_\alpha\xi_\mu\,\eta_\beta + \nabla_\beta\xi_\mu\,\eta_\alpha + \nabla_\alpha\eta_\mu\,\xi_\beta + \nabla_\beta\eta_\mu\,\xi_\alpha,$$

whose divergence is, after using the standard identity $\nabla_\mu\nabla_\alpha\xi^\mu = R_{\alpha\mu}\xi^\mu$ (and similarly for η),

$$\begin{aligned}
\nabla_\mu\mathcal{P}^\mu{}_{\alpha\beta} &= R_{\alpha\mu}\xi^\mu\eta_\beta + R_{\beta\mu}\xi^\mu\eta_\alpha + R_{\alpha\mu}\eta^\mu\xi_\beta + R_{\beta\mu}\eta^\mu\xi_\alpha\\
&\quad - 2\nabla_\alpha\xi^\mu\nabla_\beta\eta_\mu - 2\nabla_\beta\xi^\mu\nabla_\alpha\eta_\mu.
\end{aligned}$$

Putting everything together, it follows that

$$\begin{aligned}
\text{(A.5)}\qquad R^{(1)}_{\alpha\beta} &= \nabla_{(\alpha}\left(\xi(w)\eta_{\beta)}\right) + \nabla_{(\alpha}\left(\eta(w)\xi_{\beta)}\right)\\
&\quad - \frac{1}{2}\left(\nabla_\mu\nabla^\mu w\right)\mathfrak{S}_{\alpha\beta} - \nabla^\mu w\,\nabla_\mu\mathfrak{S}_{\alpha\beta} - \nabla_{(\alpha}w\nabla_{\beta)}\langle\xi,\eta\rangle\\
&\quad + 2w\left(R_{\mu(\alpha}\xi^\mu\eta_{\beta)} + R_{\mu(\alpha}\eta^\mu\xi_{\beta)}\right)\\
&\quad - 4w\nabla_{(\alpha}\xi^\mu\nabla_{\beta)}\eta_\mu - \nabla_\alpha\nabla_\beta\left(w\langle\xi,\eta\rangle\right).
\end{aligned}$$

where brackets denote symmetrization. This is a general identity valid for a perturbation tensor $K_{1\alpha\beta}$ of the form $K_{1\alpha\beta} = 2w\xi_{(\alpha}\eta_{\beta)}$ with ξ and η Killing vectors of the background.

This general identity may have applications in several contexts. For the purposes of this paper we need the following particular case:

Proposition A.1. *Let (M, g) be a spacetime admitting two Killing vectors ξ and η satisfying the following three conditions:*

(i) *ξ and η are perpendicular, i.e. $\langle \xi, \eta \rangle = 0$,*

(ii) *$[\xi, \eta] = 0$,*

(iii) *both ξ and η are hypersurface orthogonal and non-null on an open set U.*

Consider a first order perturbation tensor $K_{1\alpha\beta} = w\mathfrak{S}_{\alpha\beta}$ with $\mathfrak{S}_{\alpha\beta} := 2\xi_{(\alpha}\eta_{\beta)}$ and $w \in C^2(U)$ satisfying $\xi(w) = \eta(w) = 0$. Then, on U, the first order perturbation of the Ricci tensor is

$$R^{(1)}_{\alpha\beta} = \mathfrak{S}_{\alpha\beta}\Bigg((\lambda_\xi + \lambda_\eta) w - \frac{1}{2\langle \xi, \xi \rangle \langle \eta, \eta \rangle} \nabla_\mu \Big(\langle \xi, \xi \rangle \langle \eta, \eta \rangle \nabla^\mu w \Big) - \frac{w}{2\langle \xi, \xi \rangle \langle \eta, \eta \rangle} \nabla_\mu \langle \xi, \xi \rangle \nabla^\mu \langle \eta, \eta \rangle \Bigg), \tag{A.6}$$

where λ_ξ and λ_η are defined by $\lambda_\xi = \frac{1}{\langle \xi, \xi \rangle} Ric(\xi, \xi)$ and $\lambda_\eta = \frac{1}{\langle \eta, \eta \rangle} Ric(\eta, \eta)$.

Remark A.2. In the main text we use this result several times. For notational simplicity, it is convenient to define the second order differential operator $\mathfrak{R}(f)$

$$\mathfrak{R}(f) := \Bigg((\lambda_\xi + \lambda_\eta) f - \frac{1}{2\langle \xi, \xi \rangle \langle \eta, \eta \rangle} \nabla_\mu \Big(\langle \xi, \xi \rangle \langle \eta, \eta \rangle \nabla^\mu f \Big) - \frac{f}{2\langle \xi, \xi \rangle \langle \eta, \eta \rangle} \nabla_\mu \langle \xi, \xi \rangle \nabla^\mu \langle \eta, \eta \rangle \Bigg) \tag{A.7}$$

with λ_ξ and λ_η as above, so that (A.6) is simply $R^{(1)}_{\alpha\beta} = \mathfrak{R}(w)\mathfrak{S}_{\alpha\beta}$.

Proof. We work on U. Being hypersurface orthogonal and non-null, the derivatives of ξ and η are necessarily of the form

$$\nabla_\alpha \xi_\beta = \xi_\alpha H_\beta - \xi_\beta H_\alpha \qquad H_\alpha := -\frac{1}{2\langle \xi, \xi\rangle} \nabla_\alpha \langle \xi, \xi\rangle \tag{A.8}$$

$$\nabla_\alpha \eta_\beta = \eta_\alpha M_\beta - \eta_\beta M_\alpha \qquad M_\alpha := -\frac{1}{2\langle \eta, \eta\rangle} \nabla_\alpha \langle \eta, \eta\rangle . \tag{A.9}$$

For any Killing field ξ and vector X it holds ($\mathcal{L}$ denotes Lie derivative)

$$\xi^\mu \nabla_\mu \langle X, X\rangle = \mathcal{L}_\xi \langle X, X\rangle = 2\langle \mathcal{L}_\xi X, X\rangle .$$

As a consequence, the commutation property $[\xi, \eta] = 0$ implies $\xi^\alpha M_\alpha = \eta^\alpha H_\alpha = 0$. We can now compute

$$\begin{aligned} &\nabla_\alpha \xi^\mu \nabla_\beta \eta_\mu = \left(\xi_\alpha H^\mu - \xi^\mu H_\alpha\right)\left(\eta_\beta M_\mu - \eta_\mu M_\beta\right) = \xi_\alpha \eta_\beta H_\mu M^\mu \qquad \Longrightarrow \\ &2\nabla_{(\alpha} \xi^\mu \nabla_{\beta)} \eta_\mu = \mathfrak{S}_{\alpha\beta}\left(H_\mu M^\mu\right) = \mathfrak{S}_{\alpha\beta}\left(\frac{1}{4\langle \xi, \xi\rangle\langle \eta, \eta\rangle} \nabla_\mu \langle \xi, \xi\rangle \nabla^\mu \langle \eta, \eta\rangle\right) \end{aligned}$$

and also

$$\begin{aligned} \nabla_\mu \mathfrak{S}_{\alpha\beta} &= 2\nabla_\mu\left(\xi_{(\alpha}\eta_{\beta)}\right) = 2\left(\nabla_\mu \xi_{(\alpha}\right)\eta_{\beta)} + 2\left(\nabla_\mu \eta_{(\alpha}\right)\xi_{\beta)} \\ &= 2\xi_\mu H_{(\alpha}\eta_{\beta)} + 2\eta_\mu M_{(\alpha}\xi_{\beta)} - \left(H_\mu + M_\mu\right)\mathfrak{S}_{\alpha\beta} \\ &= 2\xi_\mu H_{(\alpha}\eta_{\beta)} + 2\eta_\mu M_{(\alpha}\xi_{\beta)} + \frac{1}{2\langle \xi, \xi\rangle\langle \eta, \eta\rangle} \nabla_\mu\Big(\langle \xi, \xi\rangle\langle \eta, \eta\rangle\Big)\mathfrak{S}_{\alpha\beta}. \end{aligned}$$

Hypersurface orthogonality implies that both ξ and η are eigenvectors of the Ricci tensor, so that

$$R_{\alpha\mu}\xi^\mu = \lambda_\xi \xi_\alpha, \qquad R_{\alpha\mu}\eta^\mu = \lambda_\eta \eta_\alpha,$$

with λ_ξ and λ_η as defined in the Proposition. Thus, under assumptions (i),(ii) and (iii), the first order perturbation of the Ricci tensor (A.5) simplifies to

$$\begin{aligned} R^{(1)}_{\alpha\beta} = &\nabla_{(\alpha}\left(\xi(w)\eta_{\beta)}\right) + \nabla_{(\alpha}\left(\eta(w)\xi_{\beta)}\right) \\ &+ \xi(w)\frac{1}{\langle \xi, \xi\rangle}\nabla_{(\alpha}\langle \xi, \xi\rangle\eta_{\beta)} + \eta(w)\frac{1}{\langle \eta, \eta\rangle}\nabla_{(\alpha}\langle \eta, \eta\rangle\xi_{\beta)} \\ &+ \mathfrak{S}_{\alpha\beta}\Bigg((\lambda_\xi + \lambda_\eta)w - \frac{1}{2\langle \xi, \xi\rangle\langle \eta, \eta\rangle}\nabla_\mu\Big(\langle \xi, \xi\rangle\langle \eta, \eta\rangle\nabla^\mu w\Big) \\ &\quad - \frac{w}{2\langle \xi, \xi\rangle\langle \eta, \eta\rangle}\nabla_\mu\langle \xi, \xi\rangle\nabla^\mu\langle \eta, \eta\rangle\Bigg). \end{aligned}$$

When, in addition, w is invariant under ξ and η, the first four terms vanish and the perturbed Ricci tensor is proportional to the metric perturbation tensor K_1, with explicit expression given in (A.6). □

Remark A.3. We note that under the assumptions of this proposition, we may insert (A.8) and (A.9) into the expression for $\mathcal{P}$ in (A.4) to get

$$\mathcal{P}_{\mu\alpha\beta} = -\frac{1}{2}\frac{\nabla_\mu\left(\langle\xi,\xi\rangle\langle\eta,\eta\rangle\right)}{\langle\xi,\xi\rangle\langle\eta,\eta\rangle}\mathfrak{S}_{\alpha\beta} + \frac{1}{\langle\xi,\xi\rangle}\xi_\mu\eta_{(\alpha}\nabla_{\beta)}\langle\xi,\xi\rangle + \frac{1}{\langle\eta,\eta\rangle}\eta_\mu\xi_{(\alpha}\nabla_{\beta)}\langle\eta,\eta\rangle$$

and the tensor $S^{(1)}$ (A.1) takes the following form

$$\begin{aligned} S^{(1)}{}_{\mu\alpha\beta} = {} & \frac{1}{2}\left(\nabla_\alpha w\,\mathfrak{S}_{\mu\beta} + \nabla_\beta w\,\mathfrak{S}_{\mu\alpha} - \nabla_\mu w\,\mathfrak{S}_{\alpha\beta}\right) \\ & - \frac{1}{2}\frac{\nabla_\mu\left(\langle\xi,\xi\rangle\langle\eta,\eta\rangle\right)}{\langle\xi,\xi\rangle\langle\eta,\eta\rangle}K_{1\alpha\beta} + \frac{w}{\langle\xi,\xi\rangle}\xi_\mu\eta_{(\alpha}\nabla_{\beta)}\langle\xi,\xi\rangle \\ & + \frac{w}{\langle\eta,\eta\rangle}\eta_\mu\xi_{(\alpha}\nabla_{\beta)}\langle\eta,\eta\rangle. \end{aligned} \tag{A.10}$$

Appendix B. Geometrical stationary and axisymmetric perturbed matching to second order

The perturbed matching to second order for the Hartle setup presented in [33] assumes that the perturbed matching hypersurface is axially symmetric, so that the interior and exterior regions are stationary and axially symmetric both in structure and in shape. In this appendix we revisit that framework by dropping any assumption on the perturbation of the matching hypersurface, thus considering the general case. Furthemore, for the sake of generality, we will also include the radial functions $\mathcal{R}_\pm$ at either side in the background configuration. To be more precise, in Propositions 1 and 2 in [33], apart from having set $\mathcal{R}(r) = r$, all four functions Q_1, T_1, Q_2, T_2 on Σ are assumed not to depend on φ. We present in the following the corresponding general results.

We start by recalling the perturbed matching theory to second order, as developed in [22] (see [4, 27] for the first order). We do this for completeness and also because, following [33], it allows us to introduce a quantity with better gauge behaviour that simplifies the expressions to some extent.

The first order matching conditions require the equality of two pairs of symmetric tensors $h^{(1)}_\pm$, $\kappa^{(1)}_\pm$ defined on the background matching hypersurface Σ. Geometrically, these tensors correspond, respectively, to the linear pertubations of the first and second fundamental forms of the matching hypersurfaces Σ_ε in the one-paramenter family of spacetimes $(M, g_\varepsilon^\pm)$ defining the perturbation. They take explicit forms in terms of background quantities, the metric perturbation tensor $K_1^\pm$ and a vector field $Z_1^\pm$ along Σ which encodes the first order variation of the matching hypersurface with ε. Its decomposition into normal and tangential components $Z_1^\pm = Q_1^\pm \mathrm{n}^\pm + T_1{}^\pm$, where $\mathrm{n}^\pm$ is the unit normal to $\Sigma^\pm$, introduces two scalars $Q_1^\pm$ which describe the deformation of $\Sigma^\pm$ as a set of points, and two tangential vectors $T_1{}^\pm$ which determine how the different points within the sets are identified. The construction to second order is analogous and involves tensors $h_\pm^{(2)}$, $\kappa_\pm^{(2)}$ and vector fields $Z_2^\pm = Q_2^\pm \mathrm{n}^\pm + T_2{}^\pm$ along Σ.

We drop the $\pm$ indexes for simplicity. The matching problem involves two independent gauges, the usual spacetime gauge and a hypersurface gauge. The former involves two vectors V_1 and V_2 called (spacetime) gauge vectors and affect Z_1 and Z_2 as [22]

$$Z_1^g = Z_1 - V_1|_\Sigma, \qquad Z_2^g = Z_2 - V_2 - 2\nabla_{Z_1} V_1 + \nabla_{V_1} V_1|_\Sigma. \tag{B.1}$$

The hypersurface gauge involves two vector fields U_1 (first order) and U_2 (second order) both tangential to Σ and transform Z_1 and Z_2 as [22]

$$Z_1^h = Z_1 + U_1, \qquad Z_2^h = Z_2 + U_2 + 2\nabla_{U_1} Z_1 - \sigma\kappa(U_1, U_1)\mathrm{n}, \tag{B.2}$$

where $\sigma = +1$ when Σ is timelike and $\sigma = -1$ when Σ is spacelike[5]. One possible use of the hypersurface gauge is setting to zero the tangential parts at one side of T_1 and T_2 (either side but not both sides simultaneously). Concerning the effect of (B.2) on the normal components, we observe that the scalar Q_1 is not affected at all. This just reflects the fact that the hypersurface gauge does not modify the matching hypersurfaces as sets of points and only affects how they are identified pointwise. This is no longer true at second order. The underlying reason is that Z_2 measures "accelerations" (in the sense of second order changes) and this has the not so obvious consequence that Q_2 is affected by U_1. From the second in (B.2) it follows

$$Q_2^h = Q_2 + 2U_1(Q_1) - \sigma\kappa(U_1, U_1 + 2T_1).$$

[5]In this paper we only deal with $\sigma = +1$, but here we present the general expressions in terms of the new variables.

This suggests the construction of the *hypersurface gauge invariant* quantity (cf. [33])

$$\widehat{Q}_2 := Q_2 + \sigma\kappa(T_1, T_1) - 2T_1(Q_1). \tag{B.3}$$

We therefore rewrite the explicit expressions of $h^{(1)}$, $\kappa^{(1)}$, $h^{(2)}$, $\kappa^{(2)}$ given in Propositions 2 and 3 in [22] in terms of this gauge invariant quantity $\widehat{Q}_2$. The first order objects $h^{(1)}$, $\kappa^{(1)}$ are independent of Q_2, so we simply reproduce from [22]:

$$h^{(1)}{}_{ij} = \pounds_{T_1} h_{ij} + 2Q_1\kappa_{ij} + K_{1\alpha\beta} e^\alpha_i e^\beta_j, \tag{B.4}$$

$$\begin{aligned} \kappa^{(1)}{}_{ij} = {} & \pounds_{T_1}\kappa_{ij} - \sigma D_i D_j Q_1 + Q_1\left(-\mathrm{n}^\mu \mathrm{n}^\nu R_{\alpha\mu\beta\nu} e^\alpha_i e^\beta_j + \kappa_{il}\kappa_j{}^l\right) \\ & + \frac{\sigma}{2} K_1^\perp \kappa_{ij} - \mathrm{n}_\mu S^{(1)\mu}{}_{\alpha\beta} e^\alpha_i e^\beta_j, \end{aligned} \tag{B.5}$$

where D is the Levi-Civita covariant derivative of the (background) induced metric h on Σ, $S^{(1)}$ is defined in (4.2), e^α_i are tangent vectors to Σ and $K_1^\perp := K_1(\mathrm{n}, \mathrm{n})$.

For second order quantities, we replace Q_2 in terms of $\widehat{Q}_2$ in the expressions in [22, Proposition 3]. The result is

$$\begin{aligned} h^{(2)}{}_{ij} = {} & \mathcal{L}_{T_2} h_{ij} + 2\widehat{Q}_2\kappa_{ij} + K_{2\alpha\beta} e^\alpha_i e^\beta_j + 2\mathcal{L}_{T_1} h^{(1)}{}_{ij} - \mathcal{L}_{T_1}\mathcal{L}_{T_1} h_{ij} \\ & + \mathcal{L}_{2Q_1\tau^{(1)} - 2Q_1\kappa(T_1) - D_{T_1}T_1} h_{ij} + 4\sigma Q_1 K_1^\perp \kappa_{ij} \\ & + 2Q_1^2\left(-\mathrm{n}^\mu \mathrm{n}^\nu R_{\alpha\mu\beta\nu} e^\alpha_i e^\beta_j + \kappa_{il}\kappa^l_j\right) + 2\sigma D_i Q_1 D_j Q_1 \\ & - 4Q_1 \mathrm{n}_\mu S^{(1)\mu}{}_{\alpha\beta} e^\alpha_i e^\beta_j, \end{aligned} \tag{B.6}$$

$$\begin{aligned} \kappa^{(2)}{}_{ij} = {} & \mathcal{L}_{T_2}\kappa_{ij} - \sigma D_i D_j \widehat{Q}_2 \\ & + \left(\widehat{Q}_2 + \sigma Q_1 K_1^\perp\right)\left(-\mathrm{n}^\mu \mathrm{n}^\nu R_{\alpha\mu\beta\nu} e^\alpha_i e^\beta_j + \kappa_{il}\kappa^l_j\right) \\ & - \mathrm{n}_\mu S^{(2)\mu}{}_{\alpha\beta} e^\alpha_i e^\beta_j + 2\mathcal{L}_{T_1}\kappa^{(1)}{}_{ij} \\ & + \kappa_{ij}\Bigg(\frac{\sigma}{2} K_2^\perp - \frac{1}{4}(K_1^\perp)^2 - \sigma\left(\tau^{(1)}{}_l + \sigma D_l Q_1\right)\left(\tau^{(1)l} + \sigma D^l Q_1\right) \\ & \quad + 2\sigma Q_1 \mathrm{n}_\mu \mathrm{n}^\rho \mathrm{n}^\delta S^{(1)\mu}{}_{\rho\delta}\Bigg) \\ & + \left(\sigma K_1^\perp \mathrm{n}_\mu + 2\tau^{(1)}{}_\mu + 2\sigma D_\mu Q_1\right) S^{(1)\mu}{}_{\alpha\beta} e^\alpha_i e^\beta_j \\ & - 2Q_1 \mathrm{n}_\mu \mathrm{n}^\nu (\nabla_\nu S^{(1)\mu}{}_{\alpha\beta}) e^\alpha_i e^\beta_j \\ & - 2\mathrm{n}_\mu \mathrm{n}^\nu S^{(1)\mu}{}_{\alpha\nu}\left(e^\alpha_i D_j Q_1 + e^\alpha_j D_i Q_1\right) \end{aligned} \tag{B.7}$$

$$
\begin{aligned}
&- 2Q_1 \mathrm{n}_\mu S^{(1)\mu}{}_{\alpha\beta} e_l^\beta \left(e_i^\alpha \kappa^l_j + e_j^\alpha \kappa^l_i \right) \\
&+ \mathcal{L}_{-\frac{1}{2}K_1^\perp \mathrm{grad}(Q_1) + 2\sigma Q_1 \kappa(\mathrm{grad}(Q_1))} h_{ij} \\
&+ \frac{1}{2} \left(D_i Q_1 D_j K_1^\perp + D_j Q_1 D_i K_1^\perp \right) \\
&- \mathcal{L}_{T_1} \mathcal{L}_{T_1} \kappa_{ij} - \mathcal{L}_{2Q_1\kappa(T_1) + D_{T_1} T_1} \kappa_{ij} - 2\sigma Q_1 \mathcal{L}_{\mathrm{grad}(Q_1)} \kappa_{ij} \\
&- Q_1^2 \Big(\mathrm{n}^\mu \mathrm{n}^\nu \mathrm{n}^\delta (\nabla_\delta R_{\alpha\mu\beta\nu}) e_i^\alpha e_j^\beta + 2 \mathrm{n}^\mu \mathrm{n}^\nu R_{\delta\mu\alpha\nu} e_l^\delta e_j^\alpha \kappa^l_i \\
&\quad + 2 \mathrm{n}^\mu \mathrm{n}^\nu R_{\delta\mu\alpha\nu} e_l^\delta e_i^\alpha \kappa^l_j \Big),
\end{aligned}
$$

where $S^{(2)}$ is given in (4.3), $K_2^\perp := K_2(\mathrm{n}, \mathrm{n})$, $\tau^{(1)}$ is the tangent vector defined by $h(\tau^{(1)}, e_i) = K_1(\mathrm{n}, e_i)$, $D_\mu Q$ is defined by $\{\mathrm{n}^\mu D_\mu Q_1 = 0,\ e_i^\mu D_\mu Q_1 = D_i Q_1\}$ and, for any tangent vector V, $\kappa(V)$ is the tangent vector with components $\kappa^i_j V^j$.

The perturbed matching conditions at first order [4, 22, 27] demand the existence of $Q_1^\pm$ and $T_1{}^\pm$ such that $[h^{(1)}] = [\kappa^{(1)}] = 0$. At second order [22] the perturbed matching conditions hold iff there exist $\widehat{Q}_2^\pm$ and $T_2{}^\pm$ such that $[h^{(2)}] = [\kappa^{(2)}] = 0$.

We may now apply the perturbed matching theory to our specific setting. As in the main text, for any pair of quantities $F^\pm$ (we use $+$ and $-$ as super or subindexes indistinctly) on Σ satisfying $[F] = 0$ we simply write $F^+ = F^- =: F$.

Proposition B.1. *Let (M, g) be a static and spherically symmetric spacetime with two regions as in Definition 3.1. Consider the metric perturbation tensors $K_1^\pm$ of the form*

$$K_1 = -2\omega(r, \theta) \mathcal{R}^2(r) \sin^2\theta dt d\phi \tag{B.8}$$

at either side $M^\pm$. Let us assume that

$$(i) \quad \mathrm{n}(\mathcal{R})|_\Sigma \neq 0, \quad (ii) \quad \left. \left(\frac{1}{2}\mathrm{n}(\nu) - \frac{\mathrm{n}(\mathcal{R})}{\mathcal{R}} + \frac{1}{\mathcal{R}\mathrm{n}(\mathcal{R})} \right) \right|_\Sigma \neq 0, \quad (iii) \quad \mathrm{n}(\nu)|_\Sigma \neq 0, \tag{B.9}$$

where $\mathrm{n} := -e^{-\lambda/2}\partial_r$. The perturbations $K_1^\pm$ satisfy the first order matching conditions if and only if there exists a constant b_1 such that

$$[\omega] = b_1 \in \mathbb{R}, \qquad [\mathrm{n}(\omega)] = 0. \tag{B.10}$$

Moreover, introducing the quantities

$$\Lambda_1 = \frac{1}{2}e^{\nu}\left(\mathrm{n}(\mathrm{n}(\nu)) + \frac{1}{2}(\mathrm{n}(\nu))^2\right), \qquad \Lambda_2 = -\mathcal{R}\mathrm{n}(\mathrm{n}(\mathcal{R})), \tag{B.11}$$

the deformation vectors $Z_1^{\pm} = Q_1^{\pm}\mathrm{n}^{\pm} + T_1{}^{\pm}$ *must satisfy*

$$[T_1] = b_1\tau\eta + \zeta, \qquad [Q_1] = 0, \qquad [\Lambda_1]\, Q_1 = 0, \qquad [\Lambda_2]\, Q_1 = 0, \tag{B.12}$$

where $\eta = \partial_\varphi$ *and* ζ *is an arbitrary background Killing vector.*

Remark B.2. This proposition holds in full generality, i.e. no a priori restriction (such as e.g. axial symmetry) is assumed on how the matching hypersurface is deformed to first order.

Remark B.3. Conditions (B.9) are well-defined because the expresions they involve agree when computed from either side of the matching hypersurface. This is a consequence of the background matching conditions (3.6).

Proof. Let $g_{\mathbb{S}^2}$ be the standard round sphere on $\mathbb{S}^2$ and denote by $\overline{D}$ its associated derivative. We use coordinates $\{\tau, \vartheta, \varphi\} := \{\tau, x^A\}$ on Σ. For any tangent vector $V = V^\tau\partial_\tau + V^A\partial_{x^A}$, and any symmetric tensor $S = a_1 d\tau^2 + a_2 g_{\mathbb{S}^2}$, with a_1, a_2 constants, the Lie derivative $\mathcal{L}_V S$ can be expressed as

$$\begin{aligned}\mathcal{L}_V S = 2a_1\dot{V}^\tau d\tau^2 &+ 2\left(a_1\overline{D}_A V^\tau + a_2\partial_\tau V_A\right) d\tau dx^A \\ &+ a_2\left(\overline{D}_A V_B + \overline{D}_B V_A\right) dx^A dx^B\end{aligned} \tag{B.13}$$

where the dot denotes derivative with respect to τ and Latin indices $A, B, \cdots$ are raised and lowered with $g_{\mathbb{S}^2}$. By (3.4)–(3.5), the tensors h_{ij}, κ_{ij} are both of this form. For notational convenience we write them as

$$h = \alpha_1 d\tau^2 + \alpha_2 g_{\mathbb{S}^2}, \qquad \kappa = \beta_1 d\tau^2 + \beta_2 g_{\mathbb{S}^2}, \tag{B.14}$$

$$\begin{aligned}&\alpha_1 := -e^{\nu}|_\Sigma, \qquad \alpha_2 := \mathcal{R}^2|_\Sigma, \\ &\beta_1 := -\frac{1}{2}e^{\nu}\mathrm{n}(\nu)|_\Sigma, \qquad \beta_2 := \mathcal{R}\mathrm{n}(\mathcal{R})|_\Sigma.\end{aligned} \tag{B.15}$$

Note that α_1, α_2 are both non-zero. The first set of matching conditions (B.4) are

$$\mathcal{L}_{[T_1]}h + 2[Q_1]\kappa + [\Phi^\star(K_1)] = 0, \tag{B.16}$$

where $\Phi^\star$ is the pull-back to Σ. Note that the last term in (B.16) has components only in $d\tau d\varphi$. Applying (B.13), the $\{A, B\}$ components of (B.16) read

$$\alpha_2 \left(\overline{D}_A[T_1]_B + \overline{D}_B[T_1]_A\right) + 2\beta_2[Q_1]g_{\mathbb{S}^2\,AB} = 0. \tag{B.17}$$

Thus $[T_1]_A(\tau)$ is a conformal Killing vector of $\mathbb{S}^2$. Let Y^a $(a = 1, 2, 3)$ be the spherical harmonics with $\ell = 1$ on the sphere. More specifically, Y^a is defined as the restriction of the Cartesian coordinate x^a to the unit sphere, and the labels are chosen so that the rotation generated by η has axis along x^3. The spherical harmonics Y^a satisfy $\overline{D}_A\overline{D}_B Y^a = -Y^a g_{\mathbb{S}^2\,AB}$ and the six dimensional algebra of conformal Killing vectors on $\mathbb{S}^2$ is spanned by $\{\overline{D}_A Y^a\}$ (proper conformal Killings) and $\{\epsilon_{AB}\overline{D}^B Y^a\}$ (Killing vectors) where ϵ_{AB} is the volume form of $(\mathbb{S}^2, g_{\mathbb{S}^2})$ with $\{\partial_\vartheta, \partial_\varphi\}$ positively oriented. The axial Killing vector η is tangent to the foliation of Σ by spheres, so in particular it defines an axial Killing vector on the unit sphere and we can write $\eta = \eta^A\partial_A$. By definition we have $\eta_A := g_{\mathbb{S}^2\,AB}\eta^B = \epsilon_{AB}\overline{D}^B Y^3$. In expressions without indexes, we will use $\bar{\boldsymbol{\eta}} := g_{\mathbb{S}^2}(\eta, \cdot)$ to distinguish η_A from η_α. Note that $\boldsymbol{\eta} = \alpha_2\bar{\boldsymbol{\eta}}$, where $\boldsymbol{\eta}$ is defined by lowering indices with the induced metric on Σ.

Consequently, (B.17) is equivalent to the existence of functions $f_a(\tau), q_a(\tau)$ such that $[T_1]_A = f_a(\tau)\overline{D}_A Y^a + q_a(\tau)\epsilon_{AB}\overline{D}^B Y^a$ and

$$[Q_1] = \frac{\alpha_2}{\beta_2} f_a(\tau) Y^a, \tag{B.18}$$

where we have used assumption (i) in (B.9), i.e. $\beta_2 \neq 0$. Now, the tensor $[\Phi^\star(K_1)]$ is $[\Phi^\star(K_1)] = -2[\omega]\alpha_2\eta_A d\tau dx^A$ and the $\{\tau, A\}$ component of (B.16) becomes

$$\begin{aligned} &\alpha_1\overline{D}_A[T_1]^\tau + \alpha_2\partial_\tau[T_1]_A - [\omega]\alpha_2\eta_A = 0 \qquad \Longleftrightarrow \\ &\overline{D}_A\left(\alpha_1[T_1]^\tau + \alpha_2\dot{f}_a Y^a\right) + \epsilon_{AB}\overline{D}^B\left(\alpha_2\dot{q}_a Y^a\right) - [\omega]\alpha_2\eta_A = 0. \end{aligned} \tag{B.19}$$

Taking $\overline{D}^A$ of (B.19) and using that $\overline{D}^A([\omega]\eta_A) = 0$ it follows

$$\Delta_{g_{\mathbb{S}^2}}\left(\alpha_1[T_1]^\tau + \alpha_2\dot{f}_a Y^a\right) = 0,$$

hence the term in parenthesis depends only on τ, i.e.

$$[T_1]^\tau = -\frac{\alpha_2}{\alpha_1}\dot{f}_a Y^a + C_0^{(1)}(\tau). \tag{B.20}$$

Substituting back into (B.19) yields

$$\epsilon_{AB}\overline{D}^B\left(\alpha_2\dot{q}_aY^a\right)=[\omega]\alpha_2\eta_A. \tag{B.21}$$

At each value of τ, the left hand side is a Killing vector of $\mathbb{S}^2$. Since η_A is also a Killing vector of the sphere, this imples that $[\omega]$ can at most depend on τ. However, since $\omega(r,\theta)$ we conclude that $[\omega]$ is constant, and we write $[\omega]=b_1$. Recalling that $\eta_A=\epsilon_{AB}\overline{D}^BY^3$, equation (B.21) can be written as

$$\alpha_2\epsilon_{AB}\overline{D}^B\left(\dot{q}_aY^a-b_1Y^3\right)=0,$$

from which it imediatelly follows that $\dot{q}_1=\dot{q}_2=0$ and $\dot{q}_3=b_1\iff q_3=b_1\tau+c_3$ with c_3 constant. Finally, the $\{\tau,\tau\}$ component of (B.16) is, using (B.13),

$$\alpha_1\frac{d[T_1]^\tau}{d\tau}+\beta_1[Q_1]=0 \qquad\iff\qquad \alpha_1\dot{C}_0^{(1)}-\alpha_2\ddot{f}_aY^a+\frac{\beta_1\alpha_2}{\beta_2}f_aY^a=0$$

where in the last equality we inserted (B.18) and (B.20). This implies that $C_0^{(1)}$ is constant and that $\ddot{f}_a=\frac{\beta_1}{\beta_2}f_a$. Sumarizing, the linearized matching conditions $[h^{(1)}{}_{ij}]=0$ are fullfilled iff

$$[T_1]=\left(-\frac{\alpha_2}{\alpha_1}\dot{f}_aY^a+C_0^{(1)}\right)\partial_\tau+f_a(\tau)\overline{D}^AY^a\partial_A+b_1\tau\eta+\zeta_0, \tag{B.22}$$

$$[Q_1]=\frac{\alpha_2}{\beta_2}f_a(\tau)Y^a,$$

$$[\omega]=b_1,\qquad \ddot{f}_a=\frac{\beta_1}{\beta_2}f_a \tag{B.23}$$

where ζ_0 is any Killing vector on the sphere. In particular, we have established the first in (B.10).

We next impose the second set of linearized matching conditions. The last term in (B.5) (we drop the $\pm$ indexes here) is, using Remark A.3 with $w=\omega e^{-\nu}$,

$$\begin{aligned}\Phi^\star\left(\mathrm{n}_\mu S^{(1)\mu}{}_{\alpha\beta}\right)&=-\frac{1}{2}\left(\mathrm{n}\left(\omega e^{-\nu}\right)+\frac{\mathrm{n}\left(\langle\xi,\xi\rangle\langle\eta,\eta\rangle\right)}{\langle\xi,\xi\rangle\langle\eta,\eta\rangle}\omega e^{-\nu}\right)\Big|_\Sigma\Phi^\star(\mathfrak{S}_{\alpha\beta})\\&=\frac{1}{2\alpha_1}\left(\mathrm{n}(\omega)+\frac{2\omega}{\mathcal{R}}\mathrm{n}(\mathcal{R})\right)\Big|_\Sigma\Phi^\star(\mathfrak{S}_{\alpha\beta})\\&=\left(\alpha_2\mathrm{n}(\omega)|_\Sigma+2\beta_2\omega|_\Sigma\right)d\tau\otimes_s\bar{\boldsymbol{\eta}}\end{aligned} \tag{B.24}$$

after using $\langle \xi, \xi \rangle = -e^{\nu} \stackrel{\Sigma}{=} \alpha_1$, $\langle \eta, \eta \rangle = \mathcal{R}^2 \sin^2 \theta$ and the fact that

$$\Phi^{\star}(\mathfrak{S}) = 2\alpha_1 \alpha_2 d\tau \otimes_s \bar{\boldsymbol{\eta}}, \tag{B.25}$$

where $\otimes_s$ stands for symmetrized tensor product, $\boldsymbol{\alpha} \otimes_s \boldsymbol{\beta} = \frac{1}{2}(\boldsymbol{\alpha} \otimes \boldsymbol{\beta} + \boldsymbol{\beta} \otimes \boldsymbol{\alpha})$. The Hessian $D_i D_j Q$ of any function Q on Σ has the following components

$$D_\tau D_\tau Q = \partial_\tau \partial_\tau Q, \quad D_\tau D_A Q = \partial_\tau \partial_A Q, \quad D_A D_B Q = \overline{D}_A \overline{D}_B Q. \tag{B.26}$$

Equations (B.5) are therefore

$$\begin{aligned} \mathcal{L}_{[T_1]} \kappa_{ij} - D_i D_j [Q_1] - [Q_1 \mathrm{n}^\mu \mathrm{n}^\nu R_{\alpha\mu\beta\nu} e_i^\alpha e_j^\beta] \\ + [Q_1] \kappa_{il} \kappa^l_j - (\alpha_2 [\mathrm{n}(\omega)] + 2\beta_2 [\omega]) \, (d\tau \otimes_s \bar{\boldsymbol{\eta}})_{ij} = 0, \end{aligned} \tag{B.27}$$

where we have used $K_1^{\perp} = 0$ and have inserted (B.24). We first consider the $\{\tau, A\}$ component. The third and fourth terms are spherically symmetric, hence their $\{\tau, A\}$ component vanishes. The first two are computed using (B.13) and (B.26) as well as (B.22). The result is

$$\left(-\frac{\beta_1 \alpha_2}{\alpha_1} + \beta_2 - \frac{\alpha_2}{\beta_2} \right) \dot{f}_a \overline{D}_A Y^a - \frac{1}{2} \alpha_2 [\mathrm{n}(\omega)] \eta_A = 0. \tag{B.28}$$

The first factor in parenthesis is (ii) in (B.9), hence non-zero by assumption. Since the vector fields $\overline{D}_A Y^a$, η_A are linearly independent and $\alpha_2 \neq 0$, (B.28) is equivalent to $\dot{f}_a = 0$ and $[\mathrm{n}(\omega)] = 0$. The former combined with (B.23) and $\beta_1 = 0$ forces $f_a = 0$ and (B.22) simplifies to

$$[T_1] = b_1 \tau \eta + C_0^{(1)} \partial_\tau + \zeta_0, \qquad [Q_1] = 0.$$

This proves the first two in (B.12) with $\zeta := C_0^{(1)} \partial_\tau + \zeta_0$ any Killing vector on Σ. Equations (B.27) have been reduced to $Q_1 [\mathrm{n}^\mu \mathrm{n}^\nu R_{\alpha\mu\beta\nu} e_i^\alpha e_j^\beta] = 0$. It is straightforward to check that

$$\mathrm{n}^\mu \mathrm{n}^\nu R_{\alpha\mu\beta\nu} e_i^\alpha e_j^\beta = \Lambda_1 \delta_i^\tau \delta_j^\tau + \Lambda_2 g_{\mathbb{S}^2 \, AB} \delta_i^A \delta_j^B, \tag{B.29}$$

which proves the last two statements of the Proposition. □

Remark B.4. The presence of a Killing vector ζ in (B.12) is a consequence of the isometries present in the background configuration, and can never be determined [24].

Remark B.5. Whenever $\mathcal{R}(r) = r$ we have $\Lambda_2 = re^{-\lambda}\lambda'/2$ and $2e^{\lambda-\nu}\Lambda_1 = \nu'' - \nu'\lambda'/4 + \nu'^2/2$, and this lemma extends to the general case (without axial symmetry on Q_1 and T_1) the consequences of Proposition 1 in [33], and in particular $Q_1[\lambda'] = Q_1[\nu''] = 0$ from (B.12). We note that the condition $\nu' \neq 0$ is (wrongly) missing in Proposition 1 in [33].

Before going into the second order matching problem we state and prove a lemma that will simplify the computations.

Lemma B.6. *Let (M, g) be a static and spherically symmetric spacetime with two regions as in Definition 3.1. Assume that the hypotheses in Proposition B.1 hold and that the corresponding first order matching conditions are satisfied.*

Consider second order metric perturbation tensors $K_2^\pm$ of the form

$$K_2 = \left(-4e^{\nu(r)}h(r,\theta) + 2\omega^2(r,\theta)\mathcal{R}^2(r)\sin^2\theta\right)dt^2 - 2\mathcal{W}(r,\theta)\mathcal{R}^2(r)\sin^2\theta dt d\phi + 4e^{\lambda(r)}m(r,\theta)dr^2 + 4k(r,\theta)\mathcal{R}^2(r)(d\theta^2 + \sin^2\theta d\phi^2) + 4e^{\lambda(r)}\partial_\theta f(r,\theta)\mathcal{R}(r) dr d\theta. \tag{B.30}$$

Apply first a hypersurface gauge defined by $U_1 = -T_1{}^- - b_1\tau\partial_\varphi$, $U_2 = 0$ and then a spacetime gauge on each side defined by $V_1^- = -b_1 t\partial_\phi$, $V_1^+ = 0$ and $V_2^\pm = 0$. Using superstript ^{g} to denote spacetime quantities in the final gauge and ^{hg} to denote hypersurface quantities in the final hypersurface and spacetime gauges, the following identitites hold

$$\begin{aligned} &\omega_-^g = \omega_- + b_1, \qquad \omega_+^g = \omega_+, \\ &[\mathrm{n}(\omega^g)] = [\mathrm{n}(\omega)] = 0, \quad [\mathrm{n}(\mathrm{n}(\omega^g))] = [\mathrm{n}(\mathrm{n}(\omega))] \\ &f_+^g = f_+ + \beta_+(r_+), \quad f_-^g = f_- + \beta_-(r_-) \\ &\qquad \Longrightarrow \quad [f^g] = [f] + [\beta], \quad [\beta] \in \mathbb{R} \end{aligned} \tag{B.31}$$

$$[\omega^g] = 0, \quad [h^g] = [h], \quad [k^g] = [k], \quad [m^g] = [m], \quad [\mathcal{W}^g] = [\mathcal{W}], \tag{B.32}$$

$$[\mathrm{n}(h^g)] = [\mathrm{n}(h)], \quad [\mathrm{n}(k^g)] = [\mathrm{n}(k)], \quad [\mathrm{n}(\mathcal{W}^g)] = [\mathrm{n}(\mathcal{W})], \tag{B.33}$$

$$[T_1{}^{hg}] = \zeta, \qquad Q_1^{hg\pm} = Q_1^\pm \quad (\Longrightarrow \quad [Q_1^{hg}] = [Q_1] = 0), \tag{B.34}$$

$$\begin{aligned} [T_2{}^{hg}] = [T_2] &- 2b_1\tau D_{T_1{}^-}\eta - 2D_{T_1{}^-}\zeta - b_1^2\tau^2 D_\eta \eta \\ &- 2b_1\tau D_\eta\zeta - 2b_1 Q_1\tau\kappa(\eta), \end{aligned} \tag{B.35}$$

$$\begin{aligned} &\widehat{Q}_2^{hg+} = \widehat{Q}_2^+, \quad \widehat{Q}_2^{hg-} = \widehat{Q}_2^- - 2b_1\tau\eta(Q_1) \\ &\qquad \Longrightarrow \quad [\widehat{Q}_2^{hg}] = [\widehat{Q}_2] + 2b_1\tau\eta(Q_1^-). \end{aligned} \tag{B.36}$$

Proof. By Proposition 2.5 with $C = -b_1$ (resp. $C = 0$) it follows $\omega^g_- = \omega_- + b_1$ (resp. $\omega^g_+ = \omega_+$), so that, in particular, $\mathrm{n}(\omega^g_\pm) = \mathrm{n}(\omega_\pm)$, $\mathrm{n}(\mathrm{n}(\omega^g_\pm)) = \mathrm{n}(\mathrm{n}(\omega_\pm))$. As a result,

$$[\omega^g] = [\omega] - b_1 \overset{(B.10)}{=} 0, \qquad [\mathrm{n}(\omega^g)] = [\mathrm{n}(\omega)] \overset{(B.10)}{=} 0.$$

The same proposition with $A = B = \mathcal{Y} = \alpha = 0$ (the second order gauge vector V_2 vanishes on both sides) gives (B.31)–(B.33). Concerning the deformation vectors, we apply the hypersurface gauge transformation law (B.2), followed by the spacetime gauge transformation (B.1) and insert $V_2 = U_2 = 0$. The result is

$$Z_1^{hg} = Z_1^h - V_1|_\Sigma = Z_1 + U_1 - V_1|_\Sigma, \tag{B.37}$$

$$\begin{aligned} Z_2^{hg} &= Z_2^h - \nabla_{V_1} V_1|_\Sigma - 2\nabla_{Z_1^h} V_1 + 2\nabla_{V_1} V_1|_\Sigma \\ &= Z_2 + 2\nabla_{U_1} Z_1 - \kappa(U_1, U_1) n - 2\nabla_{Z_1^h} V_1 + \nabla_{V_1} V_1|_\Sigma \\ &= Z_2 + 2\nabla_{U_1} Z_1 - \kappa(U_1, U_1) n - 2\nabla_{Z_1^{hg}} V_1|_\Sigma - \nabla_{V_1} V_1|_\Sigma. \end{aligned} \tag{B.38}$$

Inserting in (B.37) the explicit forms of U_1 and V_1 in the Lemma and using (B.12) yields

$$\left.\begin{aligned} {T_1}^{hg+} &= {T_1}^+ - {T_1}^- - b_1\tau\eta = [T_1] - b_1\tau\eta = \zeta \\ {T_1}^{hg-} &= {T_1}^- - {T_1}^- - b_1\tau\eta + b_1\tau\eta = 0 \end{aligned}\right\} \implies [{T_1}^{hg}] = \zeta, \tag{B.39}$$

$$Q_1^{hg+} = Q_1^+, \quad Q_1^{hg-} = Q_1^-. \tag{B.40}$$

This proves (B.34). For $Z_2^\pm$ we first compute

$$\begin{aligned} \nabla_{U_1} Z_1^\pm &= \nabla_{U_1}({T_1}^\pm + Q_1^\pm \mathrm{n}) \\ &= D_{U_1} {T_1}^\pm - \kappa(U_1, {T_1}^\pm)\mathrm{n} + U_1(Q_1^\pm)\mathrm{n} + Q_1^\pm \kappa(U_1). \end{aligned}$$

Inserting into (B.38), together with $[Q_1] = 0$, $U_1 = -{T_1}^-$, $V_1^- = -b_1 t\eta$, $V_1^+ = 0$ and the first order hg quantity $Z_1^{hg-} = Q_1\mathrm{n} + b_1\tau\eta$, leads to

$$\begin{aligned} Z_2^{hg+} &= Q_2^+\mathrm{n} + {T_2}^+ + 2D_{U_1}{T_1}^+ - 2\kappa(U_1, {T_1}^+)\mathrm{n} + 2U_1(Q_1)\mathrm{n} \\ &\quad + 2Q_1\kappa(U_1) - \kappa(U_1, U_1)\mathrm{n}, \\ Z_2^{hg-} &= Z_2^- + 2D_{U_1}{T_1}^- - 2\kappa(U_1, {T_1}^-)\mathrm{n} + 2U_1(Q_1)\mathrm{n} + 2Q_1\kappa(U_1) \\ &\quad - \kappa(U_1, U_1)\mathrm{n} + 2\nabla_{Q_1\mathrm{n}}(b_1 t\eta)|_\Sigma - \nabla_{b_1 t\eta}(b_1 t\eta)|_\Sigma \\ &= Q_2^-\mathrm{n} + {T_2}^- + 2D_{U_1}{T_1}^- - 2\kappa(U_1, {T_1}^-)\mathrm{n} + 2U_1(Q_1)\mathrm{n} + 2Q_1\kappa(U_1) \\ &\quad - \kappa(U_1, U_1)\mathrm{n} + 2b_1 Q_1 \tau\kappa(\eta) - b_1^2\tau^2\left(D_\eta\eta - \kappa(\eta, \eta)\mathrm{n}\right), \end{aligned}$$

To do this computation it is useful to introduce the spacelike unit vector field $\mathrm{n} := -e^{-\lambda/2}\partial_r$. This field restricts to Σ as the unit normal before and commutes with η, so that $\nabla_{\mathrm{n}}\eta = [\mathrm{n},\eta] + \nabla_\eta \mathrm{n} \overset{\Sigma}{=} \kappa(\eta)$. It must stressed that the extension of the normal n does not change the outcome of the computation. Extracting the tangential to Σ and using $U_1 = -T_1{}^- - b_1\tau\eta$ we obtain the diference

$$
\begin{aligned}
[T_2{}^{hg}] &= T_2{}^+ + 2D_{U_1}T_1{}^+ + 2Q_1\kappa(U_1) \\
&\quad - \left(T_2{}^- + 2D_{U_1}T_1{}^- + 2Q_1\kappa(U_1) + 2b_1Q_1\tau\kappa(\eta) - b_1^2\tau^2 D_\eta\eta\right) \\
&= [T_2] - 2D_{(T_1{}^- + b_1\tau\eta)}[T_1] - 2b_1Q_1\tau\kappa(\eta) + b_1^2\tau^2 D_\eta\eta \\
&= [T_2] - 2D_{T_1{}^-}(b_1\tau\eta + \zeta) - 2b_1\tau D_\eta(b_1\tau\eta + \zeta) \\
&\quad - 2b_1Q_1\tau\kappa(\eta) + b_1^2\tau^2 D_\eta\eta \\
&= [T_2] - 2b_1\tau D_{T_1{}^-}\eta - 2D_{T_1^-}\zeta - b_1^2\tau^2 D_\eta\eta - 2b_1\tau D_\eta\zeta - 2b_1Q_1\tau\kappa(\eta).
\end{aligned}
$$

This proves (B.35). Regarding the normal parts, we have

$$
\begin{aligned}
Q_2^{hg+} &= Q_2^+ - 2\kappa(U_1, T_1{}^+) + 2U_1(Q_1) - \kappa(U_1,U_1), \\
Q_2^{hg-} &= Q_2^- - 2\kappa(U_1, T_1{}^-) + 2U_1(Q_1) - \kappa(U_1,U_1) + b_1^2\tau^2\kappa(\eta,\eta),
\end{aligned}
$$

and the corresponding gauge invariant quantities (B.3) are

$$
\begin{aligned}
\widehat{Q}_2^{hg+} &\overset{(B.3)}{=} Q_2^{hg+} + \kappa(T_1{}^{hg+}, T_1{}^{hg+}) - 2T_1{}^{hg+}(Q_1) \\
&= Q_2^+ - 2\kappa(U_1, T_1{}^+) + 2U_1(Q_1) - \kappa(U_1,U_1) \\
&\quad + \kappa(T_1{}^+ + U_1, T_1{}^+ + U_1) - 2(T_1{}^+ + U_1)(Q_1) \\
&= Q_2^+ + \kappa(T_1{}^+, T_1{}^+) - 2T_1{}^+(Q_1) \overset{(B.3)}{=} \widehat{Q}_2^+, \\
\widehat{Q}_2^{hg-} &\overset{(B.3)}{=} Q_2^{hg-} + \kappa(T_1{}^{hg-}, T_1{}^{hg-}) - 2T_1{}^{hg-}(Q_1) \overset{(B.39)}{=} Q_2^{hg-} \\
&= Q_2^- - 2\kappa(U_1, T_1{}^-) + 2U_1(Q_1) - \kappa(U_1,U_1) + b_1^2\tau^2\kappa(\eta,\eta) \\
&= Q_2^- + 2\kappa(T_1{}^-, T_1{}^-) + 2b_1\tau\kappa(\eta, T_1{}^-) - 2T_1{}^-(Q_1) - 2b_1\tau\eta(Q_1) \\
&\quad - \kappa(T_1{}^-, T_1{}^-) - b_1^2\tau^2\kappa(\eta,\eta) - 2b_1\tau\kappa(\eta, T_1{}^-) + b_1^2\tau^2\kappa(\eta,\eta) \\
&= \widehat{Q}_2^- - 2b_1\tau\eta(Q_1),
\end{aligned}
$$

which establishes (B.36). □

We can now solve the second order matching problem.

Proposition B.7. *Let (M, g) be a static and spherically symmetric spacetime with two regions as in Definition 3.1. Assume that the hypotheses in*

Proposition B.1 hold and that the corresponding first order matching conditions are satisfied.

Consider second order metric perturbation tensors $K_2^{\pm}$ of the form

$$K_2 = \left(-4e^{\nu(r)}h(r,\theta) + 2\omega^2(r,\theta)\mathcal{R}^2(r)\sin^2\theta\right)dt^2 - 2\mathcal{W}(r,\theta)\mathcal{R}^2(r)\sin^2\theta dt d\phi + 4e^{\lambda(r)}m(r,\theta)dr^2 + 4k(r,\theta)\mathcal{R}^2(r)(d\theta^2 + \sin^2\theta d\phi^2) + 4e^{\lambda(r)}\partial_\theta f(r,\theta)\mathcal{R}(r)drd\theta. \tag{B.41}$$

Then the second order matching conditions are satisfied if and only if there exist functions $\widehat{Q}_2^{\pm}$ on Σ such that, in terms of

$$\Xi^{\pm} := \widehat{Q}_2^{\pm} - 2\mathcal{R}(e^{\lambda/2}f)|_{\Sigma^{\pm}} + 2T_1{}^{\pm}(Q_1), \tag{B.42}$$

the following expressions hold

$$[\Xi] = -\left.\frac{\mathcal{R}}{\mathrm{n}(\mathcal{R})}\right|_{\Sigma}(2c_0 + (2c_1 + H_1)\cos\vartheta), \tag{B.43}$$

$$[\mathcal{W}] = D_3 - 2\zeta_0(\omega^+|_{\Sigma}), \tag{B.44}$$

$$[\mathrm{n}(\mathcal{W})] = -2\zeta_0(\mathrm{n}(\omega)|_{\Sigma}) - 2Q_1[\mathrm{n}(\mathrm{n}(\omega))], \tag{B.45}$$

$$[k] = -\,\mathrm{n}(\mathcal{R})|_{\Sigma}\,[e^{\lambda/2}f] + c_0 + c_1\cos\vartheta, \tag{B.46}$$

$$[h] = \frac{1}{2}H_0 + \left.\frac{\mathcal{R}\mathrm{n}(\nu)}{4\mathrm{n}(\mathcal{R})}\right|_{\Sigma}(2[k] + H_1\cos\vartheta), \tag{B.47}$$

$$[m] = 2[k] + \left.\frac{\mathcal{R}}{\mathrm{n}(\mathcal{R})}\right|_{\Sigma}[\mathrm{n}(k)] + \left(H_1 - \frac{(2c_1 + H_1)}{2\mathrm{n}(\mathcal{R})^2|_{\Sigma}}\right)\cos\vartheta + \frac{1}{2}\left[(\Xi + 2\mathcal{R}e^{\lambda/2}f)\left(-\frac{1}{\mathcal{R}\mathrm{n}(\mathcal{R})}\Lambda_2 + \frac{\mathrm{n}(\mathcal{R})}{\mathcal{R}}\right)\right] - \left.\frac{1}{2\mathcal{R}\mathrm{n}(\mathcal{R})}\right|_{\Sigma}(Q_1)^2[\mathrm{n}(\Lambda_2)], \tag{B.48}$$

$$[\mathrm{n}(h)] = \left.\frac{\mathcal{R}\mathrm{n}(\nu)}{2\mathrm{n}(\mathcal{R})}\right|_{\Sigma}[\mathrm{n}(k)] + \frac{\mathrm{n}(\nu)}{2}\left.\left(1-\frac{\mathcal{R}\mathrm{n}(\nu)}{2\mathrm{n}(\mathcal{R})}\right)\right|_{\Sigma}(2[k]+H_1\cos\vartheta) - \left.\frac{\mathrm{n}(\nu)}{4\mathrm{n}(\mathcal{R})^2}\right|_{\Sigma}(H_1+2c_1)\cos\vartheta + \frac{\mathrm{n}(\nu)}{4}\left[\left(\Xi+2\mathcal{R}e^{\lambda/2}f\right)\times\left(-\frac{1}{\mathcal{R}\mathrm{n}(\mathcal{R})}\Lambda_2-\frac{2}{e^{\nu}\mathrm{n}(\nu)}\Lambda_1+\frac{\mathrm{n}(\mathcal{R})}{\mathcal{R}}-\frac{\mathrm{n}(\nu)}{2}\right)\right] - \frac{1}{4}(Q_1)^2\left[\frac{\mathrm{n}(\nu)}{\mathcal{R}\mathrm{n}(\mathcal{R})}\mathrm{n}(\Lambda_2)+\frac{2}{e^{\nu}}\mathrm{n}(\Lambda_1)\right], \tag{B.49}$$

where D_3, H_1, H_0, c_0, c_1 are arbitrary constants.

Remark B.8. As in Remark B.5, setting $\mathcal{R}(r) = r$ the results of this proposition extend to the general case (without assuming axial symmetry on Q_1, T_1, Q_2 and T_2) the outcome of Proposition 2 in [33].

Proof. We exploit Lemma B.6 to simplify the proof. Specifically, we solve the problem in the gauge g and hg and then we translate into the original gauge. For the sake of notational simplicity we drop the superindexes g and hg along the proof and we only restore them at the end.

In the gauge of Lemma B.6 we have ${T_1}^- = 0$ (by (B.39)) and $[\omega] = 0$ (so that we may simply write ω). Thus, Proposition B.1 gives ${T_1}^+ = \zeta = C_0^{(1)}\partial_\tau + \zeta_0$ with ζ_0 a Killing vector on the sphere. It is useful to introduce the tangent vector to Σ given by

$$\mathcal{J}_2 := [T_2] - 2Q_1\kappa(\zeta) - D_\zeta\,\zeta.$$

The second order matching conditions $[{h^{(2)}}_{ij}] = 0$ obtained from (B.6) become, after using Proposition B.1

$$0 = [{h^{(2)}}_{ij}] = \mathcal{L}_{\mathcal{J}_2}h_{ij} + 2[\widehat{Q}_2]\kappa_{ij} + [\Phi^\star(K_2)]_{ij} + 2\mathcal{L}_\zeta\,{h^{(1)}}_{ij} - 4Q_1[\mathrm{n}_\nu {S^{(1)\mu}}_{\alpha\beta}e_i^\alpha e_j^\beta] = \mathcal{L}_{\mathcal{J}_2}h_{ij} + 2[\widehat{Q}_2]\kappa_{ij} + [\Phi^\star(K_2)]_{ij} + 4\zeta(Q_1)\kappa_{ij} + 2\mathcal{L}_\zeta\,\Phi^\star(K_1)_{ij}, \tag{B.50}$$

where in the last equality we inserted the explicit expression of $h^{(1)}$ from (B.4) and used the facts that $\mathcal{L}_\zeta\, h_{ij} = \mathcal{L}_\zeta\, \kappa_{ij} = 0$ and, in the present gauge, also $[\mathrm{n}_\nu S^{(1)\mu}{}_{\alpha\beta} e_i^\alpha e_j^\beta] = 0$. To ellaborate (B.50) further, we use $\Phi^\star(K_1) = -2\alpha_2 \omega|_\Sigma d\tau \otimes_s \bar{\boldsymbol{\eta}}$, (B.25), so that

$$\mathcal{L}_\zeta\, \Phi^\star(K_1) = -2\alpha_2 d\tau \otimes_s \left(\zeta_0(\omega|_\Sigma)\bar{\boldsymbol{\eta}} + \omega|_\Sigma g_{\mathbb{S}^2}([\zeta_0, \eta], \cdot)\right).$$

Inserting this and the pull-back of (B.41) on Σ transforms (B.50) into

$$\begin{aligned}\text{(B.51)}\quad & \mathcal{L}_{\mathcal{J}_2} h + 2[\widehat{Q}_2]\kappa + 4\alpha_1[h]d\tau^2 \\ & \quad - 2\alpha_2[\mathcal{W}]d\tau \otimes_s \bar{\boldsymbol{\eta}} + 4\alpha_2[k]g_{\mathbb{S}^2} + 4\zeta(Q_1)\kappa \\ & \qquad - 4\alpha_2 d\tau \otimes_s \left(\zeta_0(\omega|_\Sigma)\bar{\boldsymbol{\eta}} + \omega|_\Sigma g_{\mathbb{S}^2}([\zeta_0, \eta], \cdot)\right) = 0.\end{aligned}$$

We consider first the A, B components of this expression. Decomposing $\mathcal{J}_2 = \mathcal{J}_2^\tau \partial_\tau + \mathcal{J}_2^A \partial_A$ and computing $\mathcal{L}_{\mathcal{J}_2} h$ with the general identity (B.13), we find that these components give

$$\alpha_2(\overline{D}_A \mathcal{J}_{2B} + \overline{D}_B \mathcal{J}_{2A}) + 2\beta_2[\widehat{Q}_2]g_{\mathbb{S}^2 AB} + 4\alpha_2[k]g_{\mathbb{S}^2 AB} + 4\beta_2\zeta(Q_1)g_{\mathbb{S}^2 AB} = 0.$$

As in the proof of Proposition B.1, this is equivalent to the existence of six functions $f_a^{(2)}(\tau)$, $q_a^{(2)}(\tau)$ such that

$$\text{(B.52)}\qquad \mathcal{J}_{2A} = f_a^{(2)}(\tau)\overline{D}_A Y^a + q_a^{(2)}(\tau)\epsilon_{AB}\overline{D}^B Y^a,$$

$$\text{(B.53)}\qquad [\widehat{Q}_2] + 2\zeta(Q_1) = \frac{\alpha_2}{\beta_2}\left(f_a^{(2)}(\tau)Y^a - 2[k]\right).$$

We next consider the $\{\tau, A\}$ component of (B.51). Another application of (B.13) gives

$$\begin{aligned}\text{(B.54)}\quad & \overline{D}_A\left(\alpha_1 \mathcal{J}_2^\tau + \alpha_2 \dot{f}_a^{(2)} Y^a\right) + \alpha_2 \dot{q}_a^{(2)} \epsilon_{AB}\overline{D}^B Y^a \\ & \qquad - \alpha_2\left([\mathcal{W}]\eta_A + 2\zeta_0(\omega|_\Sigma)\eta_A + 2\omega|_\Sigma[\zeta_0, \eta]_A\right) = 0.\end{aligned}$$

The divergence $\overline{D}^A$ of the second term is identically zero. The divergence of the last term is also zero because $\eta(\mathcal{W}) = 0$ and η_A, $[\zeta_0, \eta]_A$ are Killing vectors (hence divergence-free) and, in addition,

$$\begin{aligned}&\eta^A \overline{D}_A\left(\zeta_0^B \overline{D}_B(\omega|_{\Sigma_0})\right) + [\zeta_0, \eta]^A \overline{D}_A(\omega|_\Sigma) \\ &\quad = \eta^A \zeta_0^B \overline{D}_A \overline{D}_B(\omega|_\Sigma) + (\zeta_0^B \overline{D}_B \eta^A)\overline{D}_A(\omega|_\Sigma) \\ &\quad = \zeta_0^B \overline{D}_B\left(\eta^A \overline{D}_A(\omega|_\Sigma)\right) = 0,\end{aligned}$$

where in the first equality we expanded the Lie bracket and in the last equality we used $\eta(\omega|_\Sigma) = 0$. Thus, the divergence of (B.54) is equivalent to

$$\text{(B.55)} \quad \Delta_{g_{\mathbb{S}^2}} \left(\alpha_1 \mathcal{J}_2^\tau + \alpha_2 \dot{f}_a^{(2)} Y^a \right) = 0 \quad \Longleftrightarrow \quad \mathcal{J}_2^\tau = C_0^{(2)}(\tau) - \frac{\alpha_2}{\alpha_1} \dot{f}_a^{(2)} Y^a,$$

with $C_0^{(2)}(\tau)$ an integration function depending only on τ. Substituting back into (B.54) yields

$$\text{(B.56)} \qquad \dot{q}_a^{(2)} \epsilon_{AB} \overline{D}^B Y^a - [\mathcal{W}]\eta_A - 2\zeta_0(\omega|_\Sigma)\eta_A - 2\omega|_\Sigma[\zeta_0, \eta]_A = 0.$$

We now decompose $\zeta_{0A} = \zeta_{0a}\epsilon_{AB}\overline{D}^B Y^a$, $\zeta_{0a} \in \mathbb{R}$ and define $\eta_A^a := \epsilon_{AB}\overline{D}^B Y^a$ so that in particular $\eta = \eta^3$. The commutation relations are, $a, b, \dots = 1, 2, 3$,

$$[\eta^a, \eta^b] = -\epsilon^{ab}{}_c \eta^c$$

with ϵ^{abc} the Levi-Civita totally antisymmetric symbol, so (B.56) takes the form

$$\dot{q}_a^{(2)} \eta_A^a - [\mathcal{W}]\eta_A - 2\zeta_0(\omega|_\Sigma)\eta_A + 2\omega|_\Sigma \zeta_{0b}\epsilon^{b3}{}_a \eta_A^a = 0.$$

By linear independence of $\{\eta^a\}$, this is equivalent to

$$\text{(B.57)} \qquad \dot{q}_3^{(2)} = [\mathcal{W}] + 2\zeta_0(\omega|_\Sigma),$$

$$\text{(B.58)} \qquad \dot{q}_a^{(2)} = -2\omega|_\Sigma \zeta_{0b}\epsilon^{b3}{}_a, \quad a = 1, 2.$$

Since $[\mathcal{W}]$ and $\omega|_\Sigma$ are constant along τ it follows that $\ddot{q}_a^{(2)} = 0$, i.e. there exist six constants $b_a^{(2)}$ and $d_a^{(2)}$ such that

$$\text{(B.59)} \qquad \begin{aligned} & q_a^{(2)} = b_a^{(2)}\tau + d_a^{(2)}, \qquad [\mathcal{W}] = b_3^{(2)} - 2\zeta_0(\omega|_\Sigma), \\ & b_a^{(2)} + 2\omega|_\Sigma \zeta_{0b}\epsilon^{b3}{}_a = 0, \quad a = 1, 2. \end{aligned}$$

It only remains to impose the $\{\tau, \tau\}$ component of (B.51), which is

$$\alpha_1 \dot{\mathcal{J}}_2{}^\tau + \beta_1[\widehat{Q}_2] + 2\alpha_1[h] + 2\beta_1\zeta(Q_1) = 0.$$

Upon inserting (B.53) and (B.55) this is equivalent to

$$\alpha_1 \dot{C}_0^{(2)} + 2\alpha_1[h] + \alpha_2 \left[\left(-\ddot{f}_a^{(2)} + \frac{\beta_1}{\beta_2} f_a^{(2)} \right) Y^a - \frac{2\beta_1}{\beta_2}[k] \right] = 0.$$

The fact that $[h]$ and $[k]$ are τ-independent and φ-independent imposes that $\dot{C}_0^{(2)}$ is constant and that the term in parentheses is constant for $a = 3$ and zero for $a = 1, 2$. In other words, there exist $c_0^{(2)}, c_1^{(2)}, f_0^{(2)} \in \mathbb{R}$ such that

$$
\text{(B.60)} \qquad \begin{aligned} & C_0^{(2)}(\tau) = c_0^{(2)} + c_1^{(2)}\tau, \quad \ddot{f}_a^{(2)} - \frac{\beta_1}{\beta_2} f_a^{(2)} = f_0^{(2)} \delta_a^3, \\ & [h] = \frac{\beta_1 \alpha_2}{\alpha_1 \beta_2}[k] - \frac{1}{2} c_1^{(2)} + \frac{\alpha_2 f_0^{(2)}}{2\alpha_1} Y^3. \end{aligned}
$$

Summarizing, the second order matching conditions $[h^{(2)}{}_{ij}] = 0$ are equivalent to (B.52), (B.53), (B.55), (B.59) and (B.60). We next deal with $[\kappa^{(2)}{}_{ij}] = 0$. We note the following facts:

(a) $\mathrm{n}_\mu \mathrm{n}^\nu S^{(1)\mu}{}_{\nu\alpha} = 0,$ (see $(A.10)$)
(b) $[S^{(1)}{}_{\mu\alpha\beta} e_l^\mu e_i^\alpha e_j^\beta] = 0,$ as a consequence of $[\omega] = 0$,
(c) $[\mathrm{n}_\mu S^{(1)\mu}{}_{\nu\alpha} e_i^\alpha e_j^\beta] = 0,$ as consequence of $[\omega] = 0, [\mathrm{n}(\omega)] = 0$ (see (B.24)).

Additional facts that we will use are

(d) $\mathcal{L}_{[T_1]} \kappa_{ij} = \mathcal{L}_\zeta \kappa_{ij} = 0,$
(e) $\mathcal{L}_{[T_2]} \kappa_{ij} - \mathcal{L}_{2Q_1 \kappa([T_1]) + [D_{T_1} T_1]} \kappa_{ij} = \mathcal{L}_{\mathcal{J}_2} \kappa_{ij},$
(f)
$$
\begin{aligned}
[\mathcal{L}_{T_1} \kappa^{(1)}{}_{ij}] &= \mathcal{L}_{[T_1]} \kappa^{(1)}{}_{ij} = \mathcal{L}_{\vec{\zeta}} \kappa^{(1)}{}_{ij} \\
&= \mathcal{L}_\zeta \Big(-D_i D_j Q_1 + Q_1 \left(- \mathrm{n}^\mu \mathrm{n}^\nu R_{\alpha\mu\beta\nu} e_i^\alpha e_j^\beta + \kappa_{il} \kappa_j{}^l \right) \Big|_{\Sigma^-} \\
&\qquad - \mathrm{n}_\mu S^{(1)\mu}{}_{\alpha\beta} e_i^\alpha e_j^\beta \Big|_\Sigma \Big) \\
&= -D_i D_j \, \zeta(Q_1) + \zeta(Q_1) \left(- \mathrm{n}^\mu \mathrm{n}^\nu R_{\alpha\mu\beta\nu} e_i^\alpha e_j^\beta + \kappa_{il} \kappa_j{}^l \right) \Big|_{\Sigma^-} \\
&\quad - \mathcal{L}_\zeta \left((\alpha_2 \mathrm{n}(\omega)|_\Sigma + 2\beta_2 \omega|_\Sigma) \, d\tau \otimes_s \bar{\boldsymbol{\eta}} \right)_{ij},
\end{aligned}
$$

where in (f) we used that ζ is the restriction to Σ of an ambient Killing vector tangential to Σ, which has the consequence that $\mathcal{L}_\zeta$ commutes with the Hessian of (Σ, h) and that it anhilates $(-\mathrm{n}^\mu \mathrm{n}^\nu R_{\alpha\mu\beta\nu} e_i^\alpha e_j^\beta + \kappa_{il} \kappa_j{}^l)|_{\Sigma^-}$. Note that $\mathrm{n}^\mu \mathrm{n}^\nu R_{\alpha\mu\beta\nu} e_i^\alpha e_j^\beta$ may be discontinuous on Σ, but since it is multiplied by $\zeta(Q_1)$, it does not matter (by (B.29) and (B.12)) whether we evaluate it on Σ^- (as we have chosen), or on Σ^+. We have also inserted (B.24) in the last equality.

Using (a)–(f) in (B.7) together with $K_1^\perp = 0$, $\tau^{(1)} = 0$ and $[Q_1] = 0$, the equations $[\kappa^{(2)}{}_{ij}] = 0$ become

$$
\begin{aligned}
\text{(B.61)} \quad 0 = [\kappa^{(2)}{}_{ij}] = {} & \mathcal{L}_{\mathcal{J}_2}\kappa_{ij} - D_i D_j \left([\widehat{Q}_2] + 2\,\zeta(Q_1) \right) \\
& + \left[\widehat{Q}_2 \left(-\mathrm{n}^\mu \mathrm{n}^\nu R_{\alpha\mu\beta\nu} e_i^\alpha e_j^\beta + \kappa_{il}\kappa_j^l \right) \right] \\
& + 2\,\zeta(Q_1) \left(-\mathrm{n}^\mu \mathrm{n}^\nu R_{\alpha\mu\beta\nu} e_i^\alpha e_j^\beta + \kappa_{il}\kappa_j^l \right) \Big|_{\Sigma^-} \\
& - 2\mathcal{L}_\zeta \left((\alpha_2 \mathrm{n}(\omega)|_\Sigma + 2\beta_2 \omega|_\Sigma)\, d\tau \otimes_s \bar{\boldsymbol{\eta}} \right)_{ij} \\
& - \left[\mathrm{n}_\mu S^{(2)\mu}{}_{\alpha\beta} e_i^\alpha e_j^\beta \right] + \frac{1}{2}[K_2^\perp]\kappa_{ij} \\
& - 2Q_1 \left[\mathrm{n}_\mu \mathrm{n}^\nu (\nabla_\nu S^{(1)\mu}{}_{\alpha\beta}) e_i^\alpha e_j^\beta \right] \\
& - Q_1^2 \Big[\mathrm{n}^\mu \mathrm{n}^\nu \mathrm{n}^\delta (\nabla_\delta R_{\alpha\mu\beta\nu}) e_i^\alpha e_j^\beta + 2\mathrm{n}^\mu \mathrm{n}^\nu R_{\delta\mu\alpha\nu} e_l^\delta e_j^\alpha \kappa_i^l \\
& \quad + 2\mathrm{n}^\mu \mathrm{n}^\nu R_{\delta\mu\alpha\nu} e_l^\delta e_i^\alpha \kappa_j^l \Big].
\end{aligned}
$$

We proceed with the first and third terms in the third line. Consider, as before, the extension $\mathrm{n} := -e^{-\lambda/2}\partial_r$ of the normal vector off the matching hypersurface (the result is independent of how we extend). Directly from the definition of $S^{(2)\mu}{}_{\alpha\beta}$ (4.3) we get

$$
\text{(B.62)} \qquad \mathrm{n}_\mu S^{(2)\mu}{}_{\alpha\beta} = \frac{1}{2} \left(\nabla_\alpha \left(K_{2\mu\beta}\mathrm{n}^\mu \right) + \nabla_\beta \left(K_{2\mu\alpha}\mathrm{n}^\mu \right) - \mathcal{L}_\mathrm{n} K_{2\alpha\beta} \right).
$$

For any one-form P_α one has the following identity, easy to prove,

$$
\Phi^\star(\nabla P)_{ij} = D_i P_j + \kappa_{ij}(P_\alpha \mathrm{n}^\alpha|_\Sigma)
$$

where $P_i = \Phi^\star(P)_i$. Applying this to (B.62) it follows

$$
\text{(B.63)} \quad \mathrm{n}_\mu S^{(2)\mu}{}_{\alpha\beta} e_i^\alpha e_j^\beta = \frac{1}{2} \left(D_i \tau^{(2)}{}_j + D_j \tau^{(2)}{}_i - \Phi^\star(\mathcal{L}_\mathrm{n} K_2)_{ij} \right) + K_2^\perp \kappa_{ij}
$$

where we have defined $\tau^{(2)}{}_i := K_{2\alpha\beta}\mathrm{n}^\alpha|_\Sigma e_i^\beta$ and recall that $K_2^\perp := K_{2\alpha\beta}\mathrm{n}^\alpha \mathrm{n}^\beta|_\Sigma$. For the $\nabla S^{(1)}$-term, we first observe that by the background symmetries (or by direct computation) the vector field n is geodesic, i.e.

$\nabla_{\mathrm{n}}\mathrm{n} = 0$. Thus, (A.10) (c.f. (B.24)) gives

$$
\begin{aligned}
\mathrm{n}_\mu \mathrm{n}^\nu (\nabla_\nu S^{(1)\mu}{}_{\alpha\beta}) &= \mathrm{n}^\nu \nabla_\nu \left(\mathrm{n}_\mu S^{(1)\mu}{}_{\alpha\beta} \right) \\
&= -\frac{1}{2} \mathrm{n}^\nu \nabla_\nu \left[e^{-\nu} \left(\mathrm{n}(\omega) + \frac{2\omega \mathrm{n}(\mathcal{R})}{\mathcal{R}} \right) \mathfrak{S}_{\alpha\beta} \right] \\
&= \mathrm{n}^\nu \nabla_\nu \left[\sin^2\theta \left(\mathcal{R}^2 \mathrm{n}(\omega) + 2\omega \mathcal{R} \mathrm{n}(\mathcal{R}) \right) (dt \otimes_s d\phi)_{\alpha\beta} \right] \\
&= \sin^2\theta \Big[\mathrm{n} \left(\mathcal{R}^2 \mathrm{n}(\omega) + 2\omega \mathcal{R} \mathrm{n}(\mathcal{R}) \right) \\
&\qquad - \left(\mathcal{R}^2 \mathrm{n}(\omega) + 2\omega \mathcal{R} \mathrm{n}(\mathcal{R}) \right) \left(\frac{1}{2} \mathrm{n}(\nu) + \frac{\mathrm{n}(\mathcal{R})}{\mathcal{R}} \right) \Big] (dt \otimes_s d\phi)_{\alpha\beta}
\end{aligned}
$$

where we replaced $w = \omega e^{-\nu}$ and in the last equality we inserted

$$
\mathrm{n}^\nu \nabla_\nu \nabla_\alpha t = -\frac{1}{2} \mathrm{n}(\nu) dt_\alpha, \qquad \mathrm{n}^\nu \nabla_\nu \nabla_\alpha \phi = -\frac{1}{\mathcal{R}} \mathrm{n}(\mathcal{R}) d\phi_\alpha,
$$

which follows by a simple computation. Consequently,

$$
\begin{aligned}
\left[\mathrm{n}_\mu \mathrm{n}^\nu (\nabla_\nu S^{(1)\mu}{}_{\alpha\beta}) \right] &= \left[\mathrm{n} \left(\mathcal{R}^2 \mathrm{n}(\omega) + 2\omega \mathcal{R} \mathrm{n}(\mathcal{R}) \right) \right] (d\tau \otimes_s \bar{\boldsymbol{\eta}})_{ij} \\
&= \alpha_2 [\mathrm{n}(\mathrm{n}(\omega))] + \omega|_\Sigma [\mathrm{n}(\mathrm{n}(\mathcal{R}^2))] \, (d\tau \otimes_s \bar{\boldsymbol{\eta}})_{ij} \, .
\end{aligned}
$$

With this and (B.63), equation (B.61) is rewritten as

$$
\begin{aligned}
\text{(B.64)} \qquad & \mathcal{L}_{\mathcal{J}_2} \kappa_{ij} - D_i D_j \left([\widehat{Q}_2] + 2\,\zeta(Q_1) \right) \\
& + \left[\widehat{Q}_2 \left(-\mathrm{n}^\mu \mathrm{n}^\nu R_{\alpha\mu\beta\nu} e_i^\alpha e_j^\beta + \kappa_{il} \kappa^l_j \right) \right] \\
& + 2\,\zeta(Q_1) \left(-\mathrm{n}^\mu \mathrm{n}^\nu R_{\alpha\mu\beta\nu} e_i^\alpha e_j^\beta + \kappa_{il} \kappa^l_j \right) \Big|_{\Sigma^-} \\
& - 2\mathcal{L}_\zeta \left((\alpha_2 \mathrm{n}(\omega)|_\Sigma + 2\beta_2 \omega|_\Sigma) \, d\tau \otimes_s \bar{\boldsymbol{\eta}} \right)_{ij} \\
& - \frac{1}{2} \left(D_i [\tau^{(2)}{}_j] + D_j [\tau^{(2)}{}_i] \right) + \frac{1}{2} \left[(\mathcal{L}_{\mathrm{n}} K_2)_{\alpha\beta} e_i^\alpha e_j^\beta \right] \\
& - \frac{1}{2} [K_2^\perp] \kappa_{ij} - 2 Q_1 \alpha_2 [\mathrm{n}(\mathrm{n}(\omega))] \, (d\tau \otimes_s \bar{\boldsymbol{\eta}})_{ij} \\
& - Q_1^2 \Big[\mathrm{n}^\mu \mathrm{n}^\nu \mathrm{n}^\delta (\nabla_\delta R_{\alpha\mu\beta\nu}) e_i^\alpha e_j^\beta \\
& \qquad + 2\mathrm{n}^\mu \mathrm{n}^\nu R_{\delta\mu\alpha\nu} e_l^\delta e_j^\alpha \kappa^l_i + 2\mathrm{n}^\mu \mathrm{n}^\nu R_{\delta\mu\alpha\nu} e_l^\delta e_i^\alpha \kappa^l_j \Big] = 0.
\end{aligned}
$$

where we have also used the first order matching condition $Q_1[\mathrm{n}(\mathrm{n}(\mathcal{R}^2))] = 0$. So far we imposed no restriction on K_2. We now use (B.41), which implies

$$
[K_2^\perp] = 4[m], \qquad [\tau^{(2)}{}_i] = -2D_i[\mathcal{R} e^{\lambda/2} f]
$$

so that, in particular,

$$-\frac{1}{2}\left(D_i[\tau^{(2)}{}_j] + D_j[\tau^{(2)}{}_i]\right) = 2D_iD_j[\mathcal{R}e^{\lambda/2}f]. \tag{B.65}$$

We start by analyzing the $\{A, B\}$ component of (B.64). The backgroung spherical symmetry and the fact that $(\Phi^\star(K_2))_{AB}$ is proportional to $g_{\mathbb{S}^2 AB}$ implies

$$\overline{D}_A\overline{D}_B\left(2[\mathcal{R}e^{\lambda/2}f] - [\widehat{Q}_2] - 2\,\zeta(Q_1)\right) + \Theta g_{\mathbb{S}^2 AB} = 0 \tag{B.66}$$

for some function Θ that will be determined later. This equation states that $\overline{D}_A(2\mathcal{R}[e^{\lambda/2}f] - [\widehat{Q}_2] - 2\,\zeta(Q_1))$ is a conformal Killing vector on the sphere. The most general conformal Killing vector which, in addition, is a gradient is a linear combination (with coefficients that may depend of τ) of $\overline{D}_AY^a$. Hence, there exist three functions $s_a(\tau)$ such that

$$\begin{aligned}\overline{D}_A\left(2\mathcal{R}[e^{\lambda/2}f] - [\widehat{Q}_2] - 2\,\zeta(Q_1)\right) &= s_a(\tau)\overline{D}_AY^a\\ \Longleftrightarrow \qquad 2\mathcal{R}[e^{\lambda/2}f] - [\widehat{Q}_2] - 2\,\zeta(Q_1) &= s_a(\tau)Y^a + s_0(\tau),\end{aligned}$$

where $s_0(\tau)$ is a further integration "constant". Inserting (B.53) we finally arrive at

$$\mathcal{R}[e^{\lambda/2}f] = -\frac{\alpha_2}{\beta_2}[k] + \frac{1}{2}s_0(\tau) + \frac{1}{2}\left(s_a(\tau) + \frac{\alpha_2}{\beta_2}f_a^{(2)}(\tau)\right)Y^a. \tag{B.67}$$

This equation already provides relations between s_a and $f_a^{(2)}$, but we will come to that later. With this information, (B.66) reduces to

$$\Theta = s_a(\tau)Y^a. \tag{B.68}$$

To find the explicit form of Θ (as well as for the rest of equations) we need $[(\mathcal{L}_{\rm n}K_2)_{\alpha\beta}e_i^\alpha e_j^\beta]$. It is convenient to extend also e_i^α to a spacetime neighbourhood of Σ (the result being again independent of how the extension is made). We make the natural choice $e_i^\alpha\partial_\alpha = \partial_{x^i}$. The structure $\mathrm{n}^\mu = \mathrm{n}^r(r)\delta_r^\mu$ implies

$$(\mathcal{L}_{\rm n}K_2)_{\alpha\beta}e_i^\alpha e_j^\beta = \mathrm{n}\left(K_{2ij}\right).$$

Note that K_{2ij} are spacetime scalars so their directional derivative is well-defined. We can now analyze the $\{\tau, A\}$ component of (B.64). The

$i = \tau, j = A$ component of (B.65) is zero because $\mathcal{R}e^{\lambda/2}f|_\Sigma$ does not depend on τ. Inserting the general identity (B.13) and using the forms of $\mathcal{J}_2$, $[Q_2] + 2\,\zeta(Q_1)$ and $(K_2)_{\tau A}$, one finds

$$\begin{aligned}
&\left(\beta_2 - \frac{\beta_1\alpha_2}{\alpha_1} - \frac{\alpha_2}{\beta_2}\right) \dot{f}^{(2)}_a \overline{D}_A Y^a + \beta_2 \dot{q}^{(2)}_a \epsilon_{AB} \overline{D}^B Y^a \\
&\quad + \left(\alpha_2 \mathrm{n}(\omega)|_\Sigma + 2\beta_2 \omega|_\Sigma\right) \zeta_{0b} \epsilon^{b3}{}_a \epsilon_{AB} \overline{D}^B Y^a \\
&\quad - \Bigg(\mathcal{L}_{\zeta_0}\left(\alpha_2 \mathrm{n}(\omega)|_\Sigma + 2\beta_2\omega|_\Sigma\right) \\
&\qquad + \left(\beta_2[\mathcal{W}] + \frac{1}{2}\alpha_2[\mathrm{n}(\mathcal{W})]\right) + Q_1\alpha_2[\mathrm{n}(\mathrm{n}(\omega))|_\Sigma]\Bigg)\eta_A = 0,
\end{aligned}$$

where we used $\mathcal{L}_{\zeta_0}\boldsymbol{\eta}_A = -\zeta_{0b}\epsilon^{b3}{}_a\epsilon_{AB}\overline{D}^B Y^a$. By the hypotheses of the Proposition, the first factor in parenthesis is non-zero, so linear independence of $\overline{D}_A Y^a$ and $\epsilon_{AB}\overline{D}^B Y^a$ implies firstly that $\dot{f}^{(2)}_a = 0$, which combined with the second in (B.60) gives

$$f^{(2)}_a = 0 \quad (a = 1, 2), \qquad f^{(2)}_0 = -\frac{\beta_1}{\beta_2} f^{(2)}_3, \qquad f^{(2)}_3 \text{constant},$$

and secondly that, for $a = 1, 2$,

$$\beta_2 \dot{q}^{(2)}_a + \alpha_2 \mathrm{n}(\omega)|_\Sigma \zeta_{0b}\epsilon^{b3}{}_a + 2\beta_2\omega|_\Sigma\zeta_{0b}\epsilon^{b3}{}_a = 0 \quad \overset{(B.58)}{\Longleftrightarrow} \quad \mathrm{n}(\omega)|_\Sigma\zeta_{0b}\epsilon^{b3}{}_a = 0,$$

and for $a = 3$, after inserting (B.57),

$$[\mathrm{n}(\mathcal{W})] = -2\mathcal{L}_{\zeta_0}(\mathrm{n}(\omega)|_\Sigma) - 2Q_1[\mathrm{n}(\mathrm{n}(\omega))].$$

Observe that the constancy of $f^{(2)}_3$ and the vanishing of $f^{(2)}_1$, $f^{(2)}_2$, together with the fact that f and $[k]$ are τ- and φ-independent imply, via (B.67), that $s_a = 0$ for $a = 1, 2$, and s_0, s_3 are both constant. With these restrictions, $\mathcal{J}_2$ (from (B.52) and (B.55)), $[\widehat{Q}_2]$ (from (B.53)) and (B.67) become

$$\begin{aligned}
\mathcal{J}_2 &= \left(c^{(2)}_0 + c^{(2)}_1\tau\right)\partial_\tau \\
&\quad + \left(f^{(2)}_3 \overline{D}^A Y^3 + (b^{(2)}_a \tau + d^{(2)}_a)\epsilon^{AB}\overline{D}_B Y^a\right)\partial_A, \qquad \text{(B.69)}
\end{aligned}$$

$$[\widehat{Q}_2] = 2\mathcal{R}[e^{\frac{\lambda}{2}}f] - 2\,\zeta(Q_1) - s_3 Y^3 - s_0, \qquad \text{(B.70)}$$

$$[k] = -\frac{\beta_2}{\alpha_2}\mathcal{R}[e^{\lambda/2}f] + \frac{\beta_2}{2\alpha_2}s_0 + \frac{1}{2}\left(\frac{\beta_2}{\alpha_2}s_3 + f^{(2)}_3\right)Y^3. \qquad \text{(B.71)}$$

The remaining equations involve the curvature terms in the last line of (B.64). With our extension $e_i = \partial_{x^i}$, it holds $[n, e_i] = 0$, and then

$$\mathrm{n}^\alpha \nabla_\alpha e_i^\beta |_\Sigma = \kappa_i^j e_j^\beta .$$

Using that the normal field n is geodesic, as well as (B.29) and (B.14), we compute

$$\begin{aligned}
&\mathrm{n}^\mu \mathrm{n}^\nu \mathrm{n}^\delta (\nabla_\delta R_{\alpha\mu\beta\nu}) e_i^\alpha e_j^\beta + 2\mathrm{n}^\mu \mathrm{n}^\nu R_{\delta\mu\alpha\nu} e_l^\delta e_j^\alpha \kappa_i^l + 2\mathrm{n}^\mu \mathrm{n}^\nu R_{\delta\mu\alpha\nu} e_l^\delta e_i^\alpha \kappa_j^l \\
&\quad = \mathrm{n}^\delta \nabla_\delta \left(\mathrm{n}^\mu \mathrm{n}^\nu R_{\alpha\mu\beta\nu} e_i^\alpha e_j^\beta \right) + \mathrm{n}^\mu \mathrm{n}^\nu R_{\delta\mu\alpha\nu} e_l^\delta e_j^\alpha \kappa_i^l + \mathrm{n}^\mu \mathrm{n}^\nu R_{\delta\mu\alpha\nu} e_l^\delta e_i^\alpha \kappa_j^l \\
&\quad = \left(\mathrm{n}(\Lambda_1) + \frac{2\beta_1}{\alpha_1} \Lambda_1 \right) \delta_i^\tau \delta_j^\tau + \left(\mathrm{n}(\Lambda_2) + \frac{2\beta_2}{\alpha_2} \Lambda_2 \right) g_{\mathbb{S}^2\, AB} \delta_i^A \delta_j^B ,
\end{aligned}$$

and therefore, given that $Q_1[\Lambda_2] = 0$ and $Q_1[\Lambda_1] = 0$, the last line in (B.64) simplifies to

$$\begin{aligned}
&Q_1^2 \left[\mathrm{n}^\mu \mathrm{n}^\nu \mathrm{n}^\delta (\nabla_\delta R_{\alpha\mu\beta\nu}) e_i^\alpha e_j^\beta + 2\mathrm{n}^\mu \mathrm{n}^\nu R_{\delta\mu\alpha\nu} e_l^\delta e_j^\alpha \kappa_i^l + 2\mathrm{n}^\mu \mathrm{n}^\nu R_{\delta\mu\alpha\nu} e_l^\delta e_i^\alpha \kappa_j^l \right] \\
&\quad = Q_1^2[\mathrm{n}(\Lambda_1)] \delta_i^\tau \delta_j^\tau + Q_1^2[\mathrm{n}(\Lambda_2)] \delta_i^A \delta_j^B g_{\mathbb{S}^2\, AB} .
\end{aligned}$$

We also need $\kappa_{il} \kappa^l{}_j$, namely

$$\kappa_{il} \kappa^l{}_j = \frac{\beta_1^2}{\alpha_1} \delta_i^\tau \delta_j^\tau + \frac{\beta_2^2}{\alpha_2} \delta_i^A \delta_j^B g_{\mathbb{S}^2\, AB},$$

and we can finally obtain the explicit form of Θ by collecting the appropriate A, B terms in (B.64) (all except for the two Hessians):

$$\begin{aligned}
\Theta = &- 2\beta_2 f_a^{(2)} Y^a + \left[\widehat{Q}_2 \left(-\Lambda_2 + \frac{\beta_2^2}{\alpha_2} \right) \right] + 2\,\zeta(Q_1) \left(-\Lambda_2^- + \frac{\beta_2^2}{\alpha_2} \right) \\
&+ 2\alpha_2 [\mathrm{n}(k)] + 4[k]\beta_2 - 2[m]\beta_2 - Q_1^2[\mathrm{n}(\Lambda_2)],
\end{aligned}$$

where for any quantity a we set $a^- := a|_{\Sigma^-}$. Hence (B.68), and the properties we have found for $f_a^{(2)}$ and s_a yield

(B.72)
$$\begin{aligned}
[m] = &\, 2[k] - \frac{1}{2\beta_2} Q_1^2[\mathrm{n}(\Lambda_2)] + \frac{\alpha_2}{\beta_2}[\mathrm{n}(k)] \\
&- \left(f_3^{(2)} + \frac{s_3}{2\beta_2} \right) Y^3 + \frac{1}{2} \left[\widehat{Q}_2 \left(-\frac{\Lambda_2}{\beta_2} + \frac{\beta_2}{\alpha_2} \right) \right] \\
&+ \zeta(Q_1) \left(-\frac{\Lambda_2^-}{\beta_2} + \frac{\beta_2}{\alpha_2} \right),
\end{aligned}$$

The last step is to impose the $\{\tau,\tau\}$ component of (B.64). Using $\mathcal{J}_2^\tau = c_0^{(2)} + c_1^{(2)}\tau$ (see (B.69)) and the fact that $[\widehat{Q}_2] + 2\,\zeta(Q_1)$ is τ-independent (see (B.70)), this $\{\tau,\tau\}$ component is

$$
\begin{aligned}
2\beta_1 c_1^{(2)} + \left[\widehat{Q}_2\left(-\Lambda_1 + \frac{\beta_1^2}{\alpha_1}\right)\right] + 2\zeta\left(Q_1\right)\left(-\Lambda_1^- + \frac{\beta_1^2}{\alpha_1}\right)& \\
+ 2\alpha_1[\mathrm{n}(h)] + 4\beta_1[h] - 2[m]\beta_1 - Q_1^2[\mathrm{n}(\Lambda_1)] = 0&,
\end{aligned}
$$

where we also used $[\mathrm{n}(\mathcal{R}^2\omega^2)] = 0$. Solving for $[\mathrm{n}(h)]$ and inserting $[m]$ from (B.72) and $[h]$ from (B.60) one finds

$$
\begin{aligned}
[\mathrm{n}(h)] = {}& \frac{\beta_1\alpha_2}{\alpha_1\beta_2}[\mathrm{n}(k)] + \frac{\beta_1}{\alpha_1}\left(1 - \frac{\beta_1\alpha_2}{\alpha_1\beta_2}\right)\left(2[k] - f_3^{(2)}Y^3\right) - \frac{\beta_1}{2\alpha_1\beta_2}s_3Y^3 \\
&+ \frac{\beta_1}{2\alpha_1}\left(\left[\widehat{Q}_2\left(-\frac{\Lambda_2}{\beta_2} + \frac{\Lambda_1}{\beta_1} + \frac{\beta_2}{\alpha_2} - \frac{\beta_1}{\alpha_1}\right)\right]\right. \\
&+ 2\,\zeta(Q_1)\left(-\frac{\Lambda_2^-}{\beta_2} + \frac{\Lambda_1^-}{\beta_1} + \frac{\beta_2}{\alpha_2} - \frac{\beta_1}{\alpha_1}\right) \\
&\left.+ Q_1^2\left[-\frac{\mathrm{n}(\Lambda_2)}{\beta_2} + \frac{\mathrm{n}(\Lambda_1)}{\beta_1}\right]\right).
\end{aligned}
$$

This concludes the process of solving the second order matching conditions in the g-gauge. We put together the results and restore the $g's$ and $hg's$:

$$
\begin{aligned}
[\mathcal{W}^g] &= b_3^{(2)} - 2\zeta_0(\omega^g|_\Sigma), \\
[\mathrm{n}(\mathcal{W})^g] &= -2\zeta_0(\mathrm{n}(\omega^g)|_\Sigma) - 2Q_1^{hg}[\mathrm{n}(\mathrm{n}(\omega^g))], \\
[k^g] &= -\frac{\beta_2}{\alpha_2}\mathcal{R}[e^{\lambda/2}f^g] + \frac{\beta_2}{2\alpha_2}s_0 + \frac{1}{2}\left(\frac{\beta_2}{\alpha_2}s_3 + f_3^{(2)}\right)Y^3, \qquad \text{(B.73)} \\
[h^g] &= -\frac{1}{2}c_1^{(2)} + \frac{\beta_1\alpha_2}{2\alpha_1\beta_2}\left(2[k^g] - f_3^{(2)}Y^3\right), \\
[m^g] &= 2[k^g] - \frac{1}{2\beta_2}(Q_1^{hg})^2[\mathrm{n}(\Lambda_2)] + \frac{\alpha_2}{\beta_2}[\mathrm{n}(k)] \qquad \text{(B.74)} \\
&\quad - \left(f_3^{(2)} + \frac{s_3}{2\beta_2}\right)Y^3 + \frac{1}{2}\left[\widehat{Q}_2^{hg}\left(-\frac{\Lambda_2}{\beta_2} + \frac{\beta_2}{\alpha_2}\right)\right] \\
&\quad + \zeta(Q_1^{hg})\left(-\frac{\Lambda_2^-}{\beta_2} + \frac{\beta_2}{\alpha_2}\right),
\end{aligned}
$$

$$\begin{aligned}
\text{(B.75)}\quad [\mathrm{n}(h^g)] &= \frac{\beta_1\alpha_2}{\alpha_1\beta_2}[\mathrm{n}(k^g)] + \frac{\beta_1}{\alpha_1}\left(1-\frac{\beta_1\alpha_2}{\alpha_1\beta_2}\right)\left(2[k^g] - f_3^{(2)}Y^3\right)\\
&\quad - \frac{\beta_1}{2\alpha_1\beta_2}s_3Y^3 + \frac{\beta_1}{2\alpha_1}\left(\left[\widehat{Q}_2^{hg}\left(-\frac{\Lambda_2}{\beta_2}+\frac{\Lambda_1}{\beta_1}+\frac{\beta_2}{\alpha_2}-\frac{\beta_1}{\alpha_1}\right)\right]\right.\\
&\quad + 2\zeta(Q_1^{hg})\left(-\frac{\Lambda_2^-}{\beta_2}+\frac{\Lambda_1^-}{\beta_1}+\frac{\beta_2}{\alpha_2}-\frac{\beta_1}{\alpha_1}\right)\\
&\quad \left.+ (Q_1^{hg})^2\left[-\frac{\mathrm{n}(\Lambda_2)}{\beta_2}+\frac{\mathrm{n}(\Lambda_1)}{\beta_1}\right]\right),\\
\text{(B.76)}\quad [{T_2}^{hg}] &= \left(c_0^{(2)} + c_1^{(2)}\tau\right)\partial_\tau\\
&\quad + \left(f_3^{(2)}\overline{D}^AY^3 + (b_m^{(2)}\tau + d_m^{(2)})\epsilon^{AB}\overline{D}_BY^m\right)\partial_A\\
&\quad + 2Q_1^{hg}\kappa(\zeta) + D_\zeta\,\zeta,\\
\text{(B.77)}\quad [\widehat{Q}_2^{hg}] &= 2\mathcal{R}[e^{\lambda/2}f^g] - 2\zeta(Q_1^{hg}) - s_3Y^3 - s_0,
\end{aligned}$$

where $f_3^{(2)}, c_0^{(2)}, c_1^{(2)}, b_1^{(2)}, b_2^{(2)}, b_3^{(2)}, d_1^{(2)}, d_2^{(2)}, d_3^{(2)}, s_0, s_3$ are constants, the spherical Killing ζ_0 decomposes as $\zeta_0 = \zeta_{0a}\eta^a$ and $\{\zeta_{0a}\}$ and $\{b_a^{(2)}\}$ are related by

$$b_a^{(2)} + 2\omega^g|_\Sigma \zeta_{0b}{\epsilon^{b3}}_a = 0 \qquad (a=1,2).$$

Observe that the constant s_0 appears only in (B.73) and (B.77), accompanying the term $[e^{\lambda/2}f^g]$. This reflects the fact that f is defined up to an arbitrary additive function that can be different on both sides of Σ. The arbitrary difference $[e^{\lambda/2}(f^g - f)]$ can thus be combined with s_0 to produce a single constant. This will be used in the redefinition of s_0 below. The constants $c_0^{(2)}$, $d_a^{(2)}$ appear only in $[{T_2}^{hg}]$ and state that $[{T_2}^{hg}]$ is defined up to an additive Killing vector $\zeta^{(2)} := c_0^{(2)}\partial_\tau + d_a^{(2)}\epsilon^{AB}\overline{D}_BY^a\partial_A$ of (Σ, h_{ij}).

We can now apply the gauge relations described in Lemma B.6 to rewrite these conditions in the original gauge. We introduce the redefinitions of constants (see Remark B.9 below)

$$\begin{aligned}
&f_3^{(2)} \to -H_1, \quad b_3^{(2)} \to D_3, \quad c_1^{(2)} \to -H_0,\\
\text{(B.78)}\qquad &s_0 \to 2\frac{\alpha_2}{\beta_2}c_0 + 2\mathcal{R}[e^{\lambda/2}(f^g - f)], \quad s_3 \to \frac{\alpha_2}{\beta_2}(2c_1 + H_1)
\end{aligned}$$

and use the explicit expression (B.15) for $\alpha_1, \alpha_2, \beta_1, \beta_2$ which imply

$$\frac{\beta_1 \alpha_2}{\alpha_1 \beta_2} = \frac{\mathcal{R}\mathrm{n}(\nu)}{2\mathrm{n}(\mathcal{R})}, \qquad \frac{\beta_2}{\alpha_2} = \frac{\mathrm{n}(\mathcal{R})}{\mathcal{R}}, \qquad \frac{\beta_1}{\alpha_1} = \frac{1}{2}\mathrm{n}(\nu),$$

and the first four equations yield (B.44)–(B.47) immediately. From Lemma B.6 we have $Q_1^{hg\pm} = Q_1^{\pm} = Q_1$, $\widehat{Q}_2^{hg+} = \widehat{Q}_2^{+}$ and $\widehat{Q}_2^{hg-} = \widehat{Q}_2^{-} - 2b_1\tau\eta(Q_1)$, while Proposition B.1 states $[T_1] = b_1\tau\eta + \zeta$. Recalling the definition (B.42) it is immediate that (B.77) is equivalent to (B.43). Next, we use the identity $[ab] = a^+[b] + [a]b^-$ (valid for any a, b) to compute, for an arbitrary quantity $\mathfrak{P}$,

$$\begin{aligned}
[\widehat{Q}_2^{hg}\mathfrak{P}] + 2\zeta(Q_1^{hg})\mathfrak{P}^- &= [\widehat{Q}_2\mathfrak{P}] + 2b_1\tau\eta(Q_1)\mathfrak{P}^- + 2\zeta(Q_1)\mathfrak{P}^- \\
&= [(\Xi + 2\mathcal{R}e^{\lambda/2}f)\,\mathfrak{P}] - 2[T_1(Q_1)\,\mathfrak{P}] + 2(\zeta(Q_1) + b_1\tau\eta(Q_1))\mathfrak{P}^- \\
&= [(\Xi + 2\mathcal{R}e^{\lambda/2}f)\,\mathfrak{P}] - 2{T_1}^+(Q_1)[\mathfrak{P}].
\end{aligned}$$

This identity applied respectively to

$$\mathfrak{P} = -\frac{\Lambda_2}{\beta_2} + \frac{\beta_2}{\alpha_2} \qquad \text{and} \qquad \mathfrak{P} = -\frac{\Lambda_2}{\beta_2} + \frac{\Lambda_1}{\beta_1} + \frac{\beta_2}{\alpha_2} - \frac{\beta_1}{\alpha_1}$$

transforms (B.74) into into (B.48) and (B.75) into (B.49), after using that ${T_1}^+(Q_1)[\mathfrak{P}] = {T_1}^+(Q_1[\mathfrak{P}]) = 0$, which follows form the constancy of $[\mathfrak{P}]$ on Σ and (B.12). To conclude the proof, note that (B.76) simply determines ${T_2}^+$ in terms of ${T_2}^-$. Neither term appears in the rest of expressions, so this condition poses no additional restriction to the matching. □

Remark B.9. The redefinition of constants (B.78) at the end of the proof has been done so that the result matches the expressions found and used in [33].

Appendix C. Basic analytic lemmas

We use the notation, conventions and definitions of elliptic operators in [10]. Specifically, U denotes a domain of $\mathbb{R}^n$ (i.e. a connected open subset). As usual ∂U denotes its topological boundary and $\overline{U}$ its closure. A second order operator $L = a^{ij}(x)\frac{\partial^2}{\partial x_i x_j} + b^i(x)\frac{\partial}{\partial x_i} + c(x)$, $a^{ij}(x) = a^{ji}(x)$, defined on U is uniformly elliptic if the lowest eigenvalue $\lambda(x)$ and largest eigenvalue $\Lambda(x)$ satisfy that λ is positive and Λ/λ is bounded on U. At points $x \in \partial U$ where the outer normal exists, this will be denoted by ∂_ν.

We need the following version of the boundary point lemma and maximum principle.

Lemma C.1 (Boundary point lemma). *Suppose that L is uniformly elliptic, $u \in C^2(U)$ and $Lu \geq 0$ in U. Let $x_0 \in \partial U$ be such that*

1) *u is continuous at x_0 and $u(x_0) \geq 0$.*
2) *$u(x_0) > u(x)$ for all $x \in U$,*
3) *∂U satisfies an interior sphere condition at x_0 (i.e. there exists a ball $B \subset U$ with $x_0 \in \partial B$).*
4) *$c \leq 0$ and $|c|/\lambda$, $|b^i|/\lambda$ are bounded in B.*

Then the outer normal derivative of u at x_0, if it exists, satisfies the strict inequality

$$\partial_\nu u(x_0) > 0. \tag{C.1}$$

Although not stated in this form in [10], the proof of Lemma 3.4 in [10] also establishes this version. Concerning the next result, its validity is explicitly stated in a remark after Theorem 3.5 in [10].

Theorem C.2 (Strong maximum principle). *Let L be uniformly elliptic on a domain U and $u \in C^2(U)$ satisfy $Lu \geq 0$ (≤ 0). Assume $c \leq 0$ and $|c|/\lambda$, $|b|/\lambda$ are locally bounded in U. Then u cannot achieve a non-negative maximum (non-positive minimum) in the interior of U unless it is constant.*

We shall use these results in a very simple context, namely for second order ODE operators. We consider two types of intervals $I^+ = (0, a)$ $(a > 0)$ and $I^- = (a, \infty)$. In both cases the interior sphere condition is obviously satisfied. We use r to denote the real coordinate of I^+ and I^-. The outer normal derivative at $r = a$ is obviously ∂_r for I^+ and ∂_r for I^-.

The first result we need is the following (the proof is an essentially trivial consequence of the previous results, but we include it for completeness)

Lemma C.3. *On $I^+ = (0, a)$, let L^+ be*

$$L^+ := \frac{d^2}{dr^2} + b^+(r)\frac{d}{dr} + c^+(r), \tag{C.2}$$

where $|b^+(r)|$ and $|c^+(r)|$ are locally bounded in $(0, a]$. Let $f \in C^2(I^+) \cap C^0(\overline{I^+}) \cap C^1((0, a])$ satisfy $L^+ f = 0$. Assume $c^+(r) \leq 0$. Then,

(i) $f(a) > f(0) \geq 0 \implies \partial_r f(a) > 0,$

(ii) $f(a) < f(0) \leq 0 \implies \partial_r f(a) < 0,$

(iii) $f(a) = f(0) = 0 \implies f(r) = 0 \quad \forall r \in \overline{I^+}.$

Proof. L^+ is obviously uniformly elliptic with $\lambda = 1$. Note, in particular that $|c^+|/\lambda$ and $|b^+|/\lambda$ are locally bounded in I^+. Consider first the case $f(a) > f(0) \geq 0$. Since f is non-constant, the strong maximum principle implies that the supremum of f in $\overline{I^+}$ is $f(a)$ and it is achieved only at $r = a$. By the boundedness of c and $|b^i|$ on $(0, a]$ we may apply the boundary point lemma at $r = a$ to conclude $\partial_r f(a) > 0$. The case (ii) follows from (i) when applied to $-f$. Finally, when $f(a) = f(0) = 0$ the strong maximum principle implies $f(r) = 0$ immediately. □

Lemma C.4. *In the setting of Lemma C.3 assume further that $c^+(r)$ is not identically zero, $f(0) \geq 0$ and $\partial_r f(0) = 0$. Then $f(a) > f(0)$ and $\partial_r f(a) > 0$.*

Proof. The supremum of f in $\overline{I^+}$ is clearly non-negative and f cannot be constant because $c^+(r)$ is not identically zero. Thus, the strong maximum principle implies that the supremum can only be achieved at the boundary. This supremum cannot be $f(0)$ because $\partial_r f(0) = 0$ would contradict the boundary point lemma. Thus, the supremum is at $f(a) > f(0)$ and the boundary point lemma implies $\partial_r f(a) > 0$, as claimed. □

An analogous result to Lemma C.3 holds for the unbounded domain I^-.

Lemma C.5. *On $I^- = (a, \infty)$, let L^- be*

$$L^- := \frac{d^2}{dr^2} + b^-(r)\frac{d}{dr} + c^-(r), \tag{C.3}$$

where $|b^-(r)|$ and $|c^-(r)|$ are bounded in $\overline{I^+}$. Let $f \in C^2(I^-) \cap C^1(\overline{I^-})$ satisfy $L^- f = 0$ and

$$\lim_{r \to \infty} f(r) = f_\infty < \infty \tag{C.4}$$

Assume that $c^-(r) \leq 0$ in I^-. Then,

(i) $f(a) > f_\infty \geq 0 \implies \partial_r f(a) < 0,$

(ii) $f(a) < f_\infty \leq 0 \implies \partial_r f(a) > 0,$

(iii) $f(a) = f_\infty = 0 \implies f(r) = 0 \quad \forall r \in \overline{I^-}.$

Proof. The first two statements are immediate consequences of the strong maximum principle and boundary point lemma. For the third one, assume by contradiction that there is $r_0 > a$ with $f(r_0) \neq 0$. By replacing $f \to -f$ we may assume without loss of generality that $f(r_0) > 0$. By the limit assumption (C.4) with $f_\infty = 0$ there exists r_1 sufficiently large (in particular satisfying $r_1 > r_0$) with $f(r_1) < f(r_0)$. The strong maximum principle applied to $(0, r_1)$ gives a contradition, because the function is not constant but its supremum (which is at least $f(r_0)$ and hence positive) is achieved necessarily at an interior point. Thus, it must be that $f(r) = 0$ as claimed. □

Appendix D. Existence and uniqueness of bounded global solutions of a class of ODE

We use the following result (Corollary 6.2 in [17]). We will use $f \in C^0((a,\infty])$ to indicate $f \in C^0(a,\infty)$ and that the limit of $f(s)$ as $s \to \infty$ exists and is finite.

Lemma D.1. *Consider the second order homonegous ODE*

$$\ddot{z} + \alpha(s)\dot{z} + \beta(s)z = 0 \tag{D.1}$$

defined on the interval (s_0, ∞)*. Assume that* $\alpha, \beta \in C^0((s_0,\infty])$ *and let* $\alpha_0 := \lim_{s\to\infty} \alpha(s)$*,* $\beta_0 := \lim_{s\to\infty} \beta(s)$*. Assume further that*

$$\int_{s_0}^{\infty} |\alpha(s) - \alpha_0| \, ds < \infty, \qquad \int_{s_0}^{\infty} |\beta(s) - \beta_0| \, ds < \infty. \tag{D.2}$$

Define

$$\mu_\pm := \frac{-\alpha_0 \pm \sqrt{\alpha_0^2 - 4\beta_0}}{2}.$$

If $\mu_\pm$ *are real and distinct, then (D.1) admits two lineary independent real solutions* $z_\pm(s)$ *satisfying the following asymptotic behaviour at* $s \to \infty$

$$z_\pm(s) = e^{\mu_\pm s}\left(1 + o(1)\right), \qquad \dot{z}_\pm(s) = e^{\mu_\pm s}\left(\mu_\pm + o(1)\right). \tag{D.3}$$

We want to apply this result to analyze the behaviour of solutions to ODE with certain type of singularities at $t = 0$. Specifically, in the main text we need the following lemma.

Lemma D.2. *Consider the second order homogeneous ODE*

$$t^2 x'' + t\mathcal{A}(t)x' + \mathcal{B}(t)x = 0 \tag{D.4}$$

defined in the interval $(0, t_0)$*. Assume that* $\mathcal{A}(t), \mathcal{B}(t) \in C^1([0, t_0))$ *and let* $a_0 := \mathcal{A}(0)$ *and* $b_0 := \mathcal{B}(0)$*. Define*

$$\lambda_\pm := \frac{a_0 - 1 \pm \sqrt{(a_0-1)^2 - 4b_0}}{2}.$$

(i) *If* $4b_0 < (a_0 - 1)^2$ *there exist two real linearly independent solutions* $x_\pm(t)$ *of (D.4), and have the following behaviour near* $t = 0$*:*

$$x_\pm(t) = t^{-\lambda_\pm}\left(1 + o(1)\right), \qquad x'_\pm(t) = -t^{-(1+\lambda_\pm)}\left(\lambda_\pm + o(1)\right). \tag{D.5}$$

(ii) *If either* $b_0 < 0$ *or* $(b_0 = 0, a_0 > 1)$ *then there exists a unique up to scaling solution* $x(t)$ *of (D.4) that stays bounded in* $(0, t_0)$*.* $x(t)$ *extends continuously at* $t = 0$ *with* $x(0) = 0$ *if* $b_0 < 0$ *and* $x(0) \neq 0$ *if* $(b_0 = 0, a_0 > 1)$*.*

(iii) *When* $(b_0 = 0, a_0 > 1)$ *assume further that* $\mathcal{B}(t) = t^2\mathcal{Q}(t)$ *where* $\mathcal{Q}(t) \in C^1([0, t_0))$ *and satisfying* $\mathcal{Q}(0) \neq 0$*. Then, the bounded solution* $x(t)$ *in item (ii) extends to a* C^2 *function in* $[0, t_0)$ *satisfiying* $x'(0) = 0$*.*

Proof. Consider the change of variables $t(s) = e^{-s}$ which sends $(0, t_0)$ to $(s_0 := -\ln t_0, \infty)$. Define $z(s) := x(t(s))$, $\alpha(s) := 1 - \mathcal{A}(t(s))$, $\beta(s) := \mathcal{B}(t(s))$. The ODE (D.4) takes the form (D.1). For any function $\gamma(s)$ we have the equality

$$\int_a^\infty \gamma(s)ds = \int_0^{e^{-a}} \frac{\gamma(s(t))}{t} dt.$$

Since the functions $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are C^1 up to $t = 0$, $|\mathcal{A}(t) - a_0|/t$ and $|\mathcal{B}(t) - b_0|/t$ are bounded, so the hypotheses of Lemma D.1 are satisfied. In addition $\alpha_0 = 1 - a_0$, $\beta_0 = b_0$ so that, in particular $\mu_\pm = \lambda_\pm$. When $4b_0 < (a_0 - 1)^2$ we have that λ_+ and λ_- are real and distinct. The linearly independent solutions $z_\pm(s)$ whose existence is guaranteed by Lemma D.1 show the existence of two solutions $x_\pm(t)$ with the behaviour claimed in (D.5). This proves item (i).

For item (ii), in either case $b_0 < 0$ or $(b_0 = 0, a_0 > 1)$ we have $\lambda_+ > 0$ and $\lambda_- \leq 0$. The solution $x_+(t)$ of item (i) is unbounded near zero, while $x_-(t)$ is bounded. Since the general solution is a linear combination of both, the first statement follows. The continuous extension at $t = 0$ is direct from (D.5) given that $\lambda_- < 0$ when $b_0 < 0$ and $\lambda_- = 0$ when $(b_0 = 0, a_0 > 1)$.

Finally, for item (iii) we already know by item (ii) that $x_-(t)$ (which is the only one up to scaling that remains bounded) admits a continuous extension to $t = 0$. Furthermore the corresponding $z_-(s)$ satisfies

(D.6) $$\lim_{s\to\infty} \dot{z}_-(s) = 0$$

as a consequence of (D.3) and $\mu_- = \lambda_- = 0$. We next prove that $x'_-(t)$ satisties $\lim_{t\to 0^+} x'_-(t) = 0$. First observe that $\mathcal{Q}(0) \neq 0$ implies that there is $t_1 > 0$ sufficiently small such that $\mathcal{B}(t) = t^2\mathcal{Q}(t)$ has a constant sign in $(0, t_1)$. We restrict to this domain and to the equivalent in the s-variable $(s_1 := -\ln t_1, \infty)$, where $\beta(s)$ is guaranteed to vanish nowhere. Since both $\alpha(s)$ and $\beta(s)$ are $C^1((s_1,\infty))$ we may take a derivative of (D.1) and replace $z(s)$ obtained algebraically from (D.1) itself. The result is the following ODE for $\zeta(s) := \dot{z}(s)$

(D.7) $$\ddot{\zeta} + \left(\alpha(s) - \frac{\dot{\beta}(s)}{\beta(s)}\right)\dot{\zeta} + \left(\beta(s) + \dot{\alpha}(s) - \frac{\alpha(s)\dot{\beta}(s)}{\beta(s)}\right)\zeta = 0.$$

It is immediate to compute

$$\dot{\alpha}(s) = t\mathcal{A}'(t)\big|_{t=e^{-s}}, \qquad \frac{\dot{\beta}(s)}{\beta(s)} = -2 - t\frac{\mathcal{Q}'(t)}{\mathcal{Q}(t)}\bigg|_{t=e^{-s}}$$

so the coefficients of the ODE (D.7) are continuous in $(s_1, \infty]$. One checks easily that the integral conditions (D.2) are also satisfied. The corresponding constants of Lemma D.1 are

$$\mu_\pm = \frac{a_0 - 3 \pm |a_0 + 1|}{2}.$$

Using now $a_0 > 1$ we find $\mu_+ = a_0 - 1$ and $\mu_- = -2$, so by Lemma D.1, the function ζ must have the asymptotic behaviour

$$\zeta(s) = e^{-2s}\left(a_1 + o(1)\right) + e^{(a_0-1)s}\left(a_2 + o(1)\right)$$

with constants a_1, a_2. But $\zeta(s) = \dot{z}(s)$ is forced to approach zero at infinity (see (D.6)), so $a_2 = 0$. We conclude that

(D.8) $$x'(t) = -\left(e^s\zeta(s)\right)\big|_{s=-\ln t} = -t\left(a_1 + o(1)\right)$$

and we have shown that $x'(t)$ extends continuously to $t = 0$ with the value zero. It only remains to show that $x''(t)$ also extends continuously to $t = 0$, but this follows at once from the ODE itself

$$x''(t) + t^{-1}\mathcal{A}(t)x'(t) + \mathcal{Q}(t)x(t) = 0$$

since we already know that the second and third terms extend continuously to $t = 0$ (the second term by (D.8)). $\square$

The following theorem is the main result of the appendix.

Theorem D.3. *Let a be a positive constant. Assume that $\mathcal{A}^+, \mathcal{B}^+ : [0,a] \to \mathbb{R}$ and $\mathcal{A}^-, \mathcal{B}^- : [a,\infty) \to \mathbb{R}$ are C^1 on their respective domains and that the limits*

$$\text{(D.9)} \quad \lim_{r\to+\infty} \mathcal{A}^-(r), \qquad \lim_{r\to+\infty} \mathcal{B}^-(r), \qquad \lim_{r\to+\infty} r^2 \frac{d\mathcal{A}^-(r)}{dr}, \qquad \lim_{r\to+\infty} r^2 \frac{d\mathcal{B}^-(r)}{dr}$$

exist and are finite. Define the constants

$$a_0 = \mathcal{A}^+(0), \quad b_0 = \mathcal{B}^+(0), \quad a_\infty = \lim_{r\to+\infty} \mathcal{A}^-(r), \quad b_\infty = \lim_{r\to+\infty} \mathcal{B}^-(r)$$

and assume that $b_0, b_\infty < 0$. Let $\mathcal{F}^+ \in C^0([0,a])$, $\mathcal{F}^- \in C^0([a,\infty))$, $d_0, d_1 \in \mathbb{R}$ and consider the ODE problem $(\star)$ defined by

$$\text{(D.10)} \quad r^2 \frac{d^2u^+(r)}{dr^2} + r\mathcal{A}^+(r)\frac{du^+(r)}{dr} + \mathcal{B}^+(r)u^+(r) = \mathcal{F}^+(r) \qquad \text{on } (0,a],$$

$$\text{(D.11)} \quad r^2 \frac{d^2u^-(r)}{dr^2} + r\mathcal{A}^-(r)\frac{du^-(r)}{dr} + \mathcal{B}^-(r)u^-(r) = \mathcal{F}^-(r) \qquad \text{on } [a,\infty),$$

$$\text{(D.12)} \quad u^+(a) - u^-(a) = d_0, \qquad \left.\frac{du^+}{dr}\right|_{r=a} - \left.\frac{du^-}{dr}\right|_{r=a} = d_1.$$

Assume that the inhomogeneous terms satisfy $\mathcal{F}^+(r) = r^{\alpha_0}(\sigma_+ + o(1))$ near $r = 0$ and $\mathcal{F}^-(r) = r^{\alpha_\infty}(\sigma_- + o(1))$ near infinity, with constants α_0, α_∞ and $\sigma_\pm$. Define

$$\lambda^0_\pm := \frac{a_0 - 1 \pm \sqrt{(a_0-1)^2 - 4b_0}}{2}, \qquad \lambda^\infty_\pm := \frac{1 - a_\infty \pm \sqrt{(a_\infty-1)^2 - 4b_\infty}}{2}.$$

If the constants satisfy

$$\text{(D.13)} \qquad \alpha_0 \geq 0, \quad \alpha_0 + \lambda^0_- \neq 0, \quad \alpha_\infty \leq 0, \quad \alpha_\infty - \lambda^\infty_- \neq 0,$$

then $(\star)$ has a unique bounded in $(0,\infty)$ solution $\{u^+(r), u^-(r)\}$ and moreover

- *$u^+(r)$ can be extended as a $C^0([0,a])$ function of order $O(r^{\min\{-\lambda^0_-,\alpha_0\}})$, and if $1+\lambda^0_- \le 0$ and $\alpha_0 - 1 \ge 0$, then $u^+(r)$ can be also extended as a $C^1([0,a])$ function and $u^{+\prime}(r)$ is $O(r^{\min\{-(1+\lambda^0_-),\alpha_0-1\}})$.*
- *$u^-(r)$ is of order $O(r^{-\min\{|\lambda^\infty_-|,|\alpha_\infty|\}})$ and $u^{-\prime}(r)$ is $O(r^{-\min\{|\lambda^\infty_- -1|,|\alpha_\infty-1|\}})$ near $r=\infty$.*

Remark D.4. If $\mathcal{F}^\pm = 0$ and $d_0 = d_1 = 0$ the unique solution is the trivial $u(r) = 0$, which also extends to the origin.

Proof. We first analyse the homogeneous problem. By Lemma D.2, item (i), the homogeneous equation (D.10) with $\mathcal{F}^+ = 0$ admits two linearly independent solutions $u^+_+(r)$ and $u^+_-(r)$, both of class $C^2((0,a])$ (we may include $r=a$ because $\mathcal{A}^+,\mathcal{B}^+$ are C^1 up to this boundary), with behaviour near $r=0$ given by

$$u^+_\pm(r) = r^{-\lambda^0_\pm}\left(1+o(1)\right), \quad \frac{du^+_\pm(r)}{dr} = -r^{-(1+\lambda^0_\pm)}\left(\lambda^0_\pm + o(1)\right). \tag{D.14}$$

Since $\lambda^0_- < 0$, because $b_0 < 0$ by assumption, $u^+_-(r)$ extends to a $C^0([0,a])$ function with $u^+_-(0) = 0$. In the domain $[a,\infty)$ we consider the change of coordinate $r = t^{-1}$ which transforms the homogeneous ODE (D.11) with $\mathcal{F}^- = 0$ into the form

$$t^2\frac{d^2\widehat{u}^-(t)}{dt^2} + t(2-\widehat{\mathcal{A}}(t))\frac{d\widehat{u}^-(t)}{dt} + \widehat{\mathcal{B}}(t)\widehat{u}^-(t) = 0$$

where for any function $f(r)$ we denote by $\widehat{f}(t) := f(t^{-1}) : (0,a^{-1}] \to \mathbb{R}$. Conditions (D.9) imply inmediately that $\widehat{\mathcal{A}},\widehat{\mathcal{B}}$ extend to $t=0$ as $C^1([0,a^{-1}])$ functions, and we may apply item (i) in Lemma D.2 to conclude that there exist two independent solutions $u^-_\pm(t) \in C^2([a,\infty))$ satisfying

$$\widehat{u}^-_\pm(t) = t^{-\lambda^\infty_\pm}\left(1+o(1)\right), \quad \frac{d\widehat{u}^-_\pm(t)}{dt} = -t^{-(1+\lambda^\infty_\pm)}\left(\lambda^\infty_\pm + o(1)\right).$$

In terms of the original function, this behaviour translates onto

$$u^-_\pm(r) = r^{\lambda^\infty_\pm}\left(1+o(1)\right), \quad \frac{du^-_\pm(r)}{dr} = r^{\lambda^\infty_\pm - 1}\left(\lambda^\infty_\pm + o(1)\right). \tag{D.15}$$

Since $\lambda_-^\infty < 0$, because $b_\infty < 0$ by assumption, $u_-^-(r)$ vanishes at $r \to \infty$. We let $W^\pm(r)$ be the Wronskian of the functions $u_-^\pm(r)$, $u_+^\pm(r)$, i.e.

$$W^\pm(r) := u_-^\pm(r)\frac{du_+^\pm(r)}{dr} - u_+^\pm(r)\frac{du_-^\pm(r)}{dr}.$$

It is inmediate from the previous considerations that

$$W^+(r) = r^{-a_0}\left(-\sqrt{(a_0-1)^2-4b_0} + o(1)\right) \qquad \text{near } r=0,$$
$$W^-(r) = r^{-a_\infty}\left(\sqrt{(a_\infty-1)^2-4b_\infty} + o(1)\right) \qquad \text{near } r=+\infty.$$

We may now include the inhomogeneus term. The general solution of the inhomogeneous problem on each domain in given by the general formula

$$u^\pm(r) = C_+^\pm u_+^\pm(r) + C_-^\pm u_-^\pm(r) + u_P^\pm(r), \tag{D.16}$$

where $C_+^\pm$, $C_-^\pm$ are arbitrary constants and a particular solution $u_P^\pm(r)$ on each domain is given by

$$u_P^\pm(r) = u_+^\pm(r)\int_{r_\pm}^r \frac{u_-^\pm(s)\mathcal{F}^\pm(s)}{s^2W^\pm(s)}ds - u_-^\pm(r)\int_{r_\pm}^r \frac{u_+^\pm(s)\mathcal{F}^\pm(s)}{s^2W^\pm(s)}ds, \tag{D.17}$$

where $r_\pm$ are arbitrary values subject to $r_- \in (0,a]$ and $r_+ \in [a,\infty)$. The behaviour of the integrands near zero and near infinity are, respectively,

$$\frac{u_\mp^+(s)\mathcal{F}^+(s)}{s^2W^+(s)} = s^{\alpha_0+\lambda_\pm^0-1}(\mu^+ + o(1)),$$
$$\frac{u_\mp^-(s)\mathcal{F}^-(s)}{s^2W^-(s)} = s^{\alpha_\infty-\lambda_\pm^\infty-1}(\mu^- + o(1)),$$

for suitable constants μ^+, μ^-. Since $\lambda_+^0, \lambda_+^\infty$ are positive, assumption (D.13) implies $\alpha_0 + \lambda_\pm^0 \neq 0$, $\alpha_\infty - \lambda_\pm^\infty \neq 0$. It is then straighforward to check, using l'Hôpital's rule, that

$$\int_{r^+}^r \frac{u_\mp^+(s)\mathcal{F}^+(s)}{s^2W^+(s)}ds = r^{\alpha_0+\lambda_\pm^0}\left(\frac{\mu^+}{\alpha_0+\lambda_\pm^0} + o(1)\right) + \mathcal{Q}_\pm^+ \qquad \text{near} \quad r=0,$$
$$\int_{r^-}^r \frac{u_\mp^-(s)\mathcal{F}^-(s)}{s^2W^-(s)}ds = r^{\alpha_\infty-\lambda_\pm^\infty}\left(\frac{\mu^-}{\alpha_\infty-\lambda_\pm^\infty} + o(1)\right) + \mathcal{Q}_\pm^- \qquad \text{near} \quad r=\infty,$$

where $\mathcal{Q}^+_\pm$ and $\mathcal{Q}^-_\pm$ are constants. Consequently

$$\text{(D.18)} \quad U^+_P(r) := u^+_P(r) - \mathcal{Q}^+_+ u^+_+(r) + \mathcal{Q}^+_- u^+_-(r)$$

$$\text{(D.19)} \quad = r^{\alpha_0}\left(\frac{\mu^+(\lambda^0_- - \lambda^0_+)}{(\alpha_0+\lambda^0_+)(\alpha_0+\lambda^0_-)} + o(1)\right) \qquad \text{near } r=0,$$

$$\text{(D.20)} \quad U^-_P(r) := u^-_P(r) - \mathcal{Q}^-_+ u^-_+(r) + \mathcal{Q}^-_- u^-_-(r)$$

$$= r^{\alpha_\infty}\left(\frac{\mu^-(\lambda^\infty_+ - \lambda^\infty_-)}{(\alpha_\infty-\lambda^\infty_+)(\alpha_\infty-\lambda^\infty_-)} + o(1)\right) \qquad \text{near } r=+\infty.$$

Absorbing the constants $\mathcal{Q}^\pm_\pm$ into $C^\pm_\pm$, the general solution (D.16) has the form

$$u^\pm(r) = C^\pm_+ u^\pm_+(r) + C^\pm_- u^\pm_-(r) + U^\pm_P(r).$$

Note that $U^+_P(r)$ is bounded near $r=0$ while $U^-_P(r)$ is bounded at infinity. We now impose that the solution $\{u^+(r), u^-(r)\}$ is bounded everywhere. Since $\lambda^0_-, \lambda^\infty_- < 0$ and $\lambda^0_+, \lambda^\infty_+ > 0$ this is equivalent to setting $C^+_+ = C^-_+ = 0$ and we are left with two constants to determine. Thus, the general solution (D.16) reads

$$\text{(D.21)} \qquad u^\pm(r) = C^\pm_- u^\pm_-(r) + U^\pm_P(r).$$

Imposing the matching conditions (D.12) yields a system of two equations of the form

(D.22)

$$\begin{pmatrix} u^+_-(a) & -u^-_-(a) \\ \frac{du^+_-}{dr}\big|_{r=a} & -\frac{du^-_-}{dr}\big|_{r=a} \end{pmatrix}\begin{pmatrix} C^+_- \\ C^-_- \end{pmatrix} = \begin{pmatrix} d_0 + U^-_P(a) - U^+_P(a) \\ d_1 + \frac{dU^-_P}{dr}\big|_{r=a} - \frac{dU^+_P}{dr}\big|_{r=a} \end{pmatrix}.$$

We apply now Lemma C.3 to $u^+_-(r)$ and Lemma C.5 to $u^-_-(r)$ to conclude that $u^+_-(r)$ and $u^-_-(r)$ and their derivatives are all non-zero at a and, moreover, $u^+_-(a)$ has the same sign as it derivative at a, while $u^-_-(a)$ has opposite sign than its derivative. It follows that the 2×2 matrix in (D.22) is invertible, and hence there exits a unique pair of constants $\{C^+_-, C^-_-\}$ satifying the transition conditions (D.12). This concludes the proof of existence and uniqueness of a bounded solution of problem $(\star)$.

We conclude with the the behaviour of the first derivative of the solutions (D.21), i.e.

$$\frac{du^\pm(r)}{dr} = C^\pm_- \frac{du^\pm_-(r)}{dr} + \frac{dU^\pm_P(r)}{dr},$$

at the origin and at infinity correspondingly. Firstly, the terms $dU_P^{\pm}/dr$ are obtained by direct differentiation of their definitions (D.18) and (D.20) and introducing (D.14)–(D.15) together with the differentiation of (D.17), which provides

$$\frac{du_P^{\pm}(r)}{dr} = \frac{du_+^{\pm}(r)}{dr}\int_{r_\pm}^{r}\frac{u_-^{\pm}(s)\mathcal{F}^{\pm}(s)}{s^2W^{\pm}(s)}ds - \frac{du_-^{\pm}(r)}{dr}\int_{r_\pm}^{r}\frac{u_+^{\pm}(s)\mathcal{F}^{\pm}(s)}{s^2W^{\pm}(s)}ds.$$

The results are

(D.23)
$$\frac{dU_P^{+}(r)}{dr} = r^{\alpha_0-1}\left(\frac{\alpha_0\mu^+(\lambda_-^0-\lambda_+^0)}{(\alpha_0+\lambda_+^0)(\alpha_0+\lambda_-^0)}+o(1)\right) \qquad \text{near } r=0,$$

(D.24)
$$\frac{dU_P^{-}(r)}{dr} = r^{\alpha_\infty-1}\left(\frac{\alpha_\infty\mu^-(\lambda_+^\infty-\lambda_-^\infty)}{(\alpha_\infty-\lambda_+^\infty)(\alpha_\infty-\lambda_-^\infty)}+o(1)\right) \qquad \text{near } r=\infty.$$

Regarding $u^+(r)$, the assumption $1+\lambda_-^0\leq 0$ ensures, c.f. (D.14), that du_-^+/dr has a limit at $r\to 0$ and is, in fact, $O(r^{-(1+\lambda_-^0)})$. The expression (D.23) implies that if $\alpha_0-1\geq 0$ then dU_P^+/dr also has a limit at $r\to 0$ and is $O(r^{(\alpha_0-1)})$. The claim for $u^+(r)$ follows. As for $u^-(r)$, since $\alpha_\infty\leq 0$ and $\lambda_-^\infty<0$ by assumption, the claim follows analogously from (D.15) and (D.24). □

Remark D.5. The behaviour of the first derivative of the particular solution $U_p^+(r)$ near the origin is given by (D.23), and therefore, if $\alpha_0-1\geq 0$ then $U_P^+(r)$ extends to a $C^1([0,a])$ function as $U_P^+(r)=r^{\alpha_0}(U_P^0+o(1))$ and $U_P^{+\prime}(r)=r^{\alpha_0-1}(\alpha_0U_P^0+o(1))$ with $U_P^0\in\mathbb{R}$.

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