ADV. THEOR. MATH. PHYS.
Volume 26, Number 9, 2909–2961, 2022

On the equivalence of the KMS condition and the variational principle for quantum lattice systems with mean-field interactions

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We extend Araki's well-known results on the equivalence of the KMS condition and the variational principle for equilibrium states of quantum lattice systems with short-range interactions, to a large class of models possibly containing mean-field interactions (representing an extreme form of long-range interactions). This result is reminiscent of van Hemmen's work on equilibrium states for mean-field models. The extension was made possible by our recent outcomes on states minimizing the free energy density of mean-field models on the lattice, as well as on the infinite volume dynamics for such models.

1. Introduction

As is widely accepted in theoretical and mathematical physics, the equilibrium state of a given *finite* quantum system is the Gibbs state associated with its Hamiltonian. Among other things, this state is stationary with respect to the dynamics generated by the Hamiltonian. In general, Gibbs states are ill-defined for infinite systems and, thus, several notions of thermodynamic equilibrium were proposed for these systems, based on properties of finite volume Gibbs states, like free energy minimization, complete passivity, Kubo-Martin-Schwinger (KMS) and Gibbs property, to mention the most commonly used. All the listed properties uniquely define Gibbs states of any finite quantum system. In particular, they are all equivalent[1] to each other for finite systems, but that is not the case for infinite systems. It is therefore important, from both the mathematical and the physical viewpoints, to understand the relation between these mathematical definitions of equilibrium states of infinite quantum systems. In this scope, here we focus on fermions and quantum spins on infinite lattices and analyze the relation between the minimization of the free energy density and the KMS property of space-homogeneous states, in presence of mean-field, or long-range[2], interactions.

Because of their elegant definition and good mathematical properties (see, e.g., [1, Sections 5.3-5.4]), KMS states certainly constitute the most popular notion of equilibrium states for infinite systems. For instance, they provide the Kelvin-Planck statement of the 2nd law of thermodynamics with a precise mathematical meaning, the complete passivity of these states [2]. The KMS property refers to a *dynamical* notion of equilibrium. By contrast, a *static* notion of thermodynamic equilibrium can also be given by defining equilibrium states via the minimization of the free energy density of the quantum system under consideration. In opposition to the dynamical viewpoint, this approach requires some space homogeneity of states. On the other hand, the static approach has the big advantage of getting around the problem of the existence of quantum dynamics in the infinite volume limit. Moreover, the existence of equilibrium states in the static sense can, in general, be easily ensured (by standard compactness and lower semi-continuity arguments), whereas, in many cases, the mere existence of KMS states is an issue, even if the infinite volume dynamics exists and is continuous in the appropriate sense.

Giving a precise and convenient mathematical meaning to the infinite volume dynamics of quantum systems can be highly non-trivial: On the

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[1] More precisely, in contrast to the other conditions, the complete passivity does not fix the temperature of the system, i.e., this condition only implies that the state is Gibbs for some temperature.

[2] We frequently prefer the term “long-range” instead of “mean-field”, because the latter can refer to different scalings and has thus some ambiguity, whereas the former is precisely defined later on.

one hand, the Green-function method (see, e.g., [1, Section 6.3.4]) allows one to implement, under very general conditions, the infinite volume limit of quantum dynamics as a strongly continuous group of unitaries acting in some abstract Hilbert space. Note however that this approach needs (initial) states to be fixed. On the other hand, the KMS approach requires a strongly continuous group of $*$-automorphisms of the (original) C^*-algebra of the given quantum system. Such a group of $*$-automorphisms should, in principle, not depend on the particular choices of states for the system. In fact, for lattice fermion and quantum spin systems with *short-range* interactions, such a continuous group of $*$-automorphisms is well-defined in the infinite volume limit, independently of some choice of states. See [3, 4] for the fermionic case and [1, 5] for the quantum spin one. For short-range interactions, the notion of KMS states naturally applies and the equivalence between the KMS condition and the minimization of the infinite volume free energy density (at the same inverse temperature) can be proven for space-homogeneous states: Inspired by works of Dobrushin and Lanford-Ruelle on classical spins, Araki contributed in 1974 [6] such an equivalence proof for quantum spins with short-range interactions on a (infinite square) lattice of arbitrary dimension. The case of lattice fermions (with short-range interactions) was only studied in detail much later, in 2003, by Araki and Moriya [4].

Quantum many-body systems in the continuum are expected to have, in general, no such dynamics in the thermodynamic limit, but only a *state-dependent* Heisenberg dynamics (see, e.g., [1, Section 6.3]). To get around this issue in the continuum, in presence of interactions, one may use a UV cut-off of some kind, as it is recently discussed and proven in [7]. The same issue is also expected for interactions slowly decaying in space, even in the lattice case. For instance, as already remarked in the seminal work [8] on the mathematics of the BCS theory, in presence of mean-field interactions, the finite volume quantum dynamics does *not* generally converge within the (original) C^*-algebra of the given system, when the volume diverges. See also [9, Section 4.3].

From the sixties to the nineties, there were a lot of studies on infinite volume dynamics of lattice fermion and quantum spin systems with mean-field interactions, by Bóna, Duffield, Duffner, Haag, Hepp, Lieb, van Hemmen, Rieckers, Thirring, Unnerstall, Wehrl, Werner, and others. Mathematically rigorous results were obtained for models that are permutation-invariant, almost exclusively for quantum spins (except, e.g., in [8, 10]). In this case, the infinite volume dynamics refers to an effective time-evolution on a finite-dimensional Hilbert space[3], whereas the KMS theory were proposed to deal with infinite systems, the finite (lattice) case being trivial from the point of view of this theory. During the last two decades, many studies on the dynamics of fermion systems in the continuum with mean-field interactions were also performed, by Benedikter, Elgart, Erdös, Jakšić, Petrat, Pickl, Porta, Schlein, Yau, and others, but (positive temperature) thermal states were not considered. In this context, the KMS condition is irrelevant. For a detailed discussion on this subject, see [9, Section 1] and references therein.

In 2013, [11] contributed the complete characterization of space-homogeneous free-energy-density-minimizing states of quantum lattice systems with mean-field interactions, in terms of free energy density minimizers of their so-called Bogoliubov (short-range) approximations. In fact, in contrast to the short-range case [1, 4, 5], note that strict minimizers of the free energy density may not exist, but the (weak*) limits of minimizing sequences of states offer a good, more general, notion of equilibrium states, in the long-range case.

In 2021, [9, 12] derived the infinite volume dynamics of space homogeneous quantum lattice systems with mean-field, or long-range, interactions in great generality. In fact, it was shown that the infinite volume dynamics of such systems is always equivalent to an intricate combination of classical and short-range quantum dynamics. A simple illustration of this general result is given in [13, 14]. [12] shows in particular that, within the cyclic representation associated with the initial state, the infinite volume limit of dynamics is well-defined in the σ-weak operator topology and corresponds to a *state-dependent* Heisenberg dynamics.

Combined with [11], the outcomes of the recent papers [9, 12] pave the way to a study of the relations between static and dynamical notions of equilibrium, respectively characterized by the minimization of the free energy density and the KMS conditions, for very general infinite quantum lattice systems with mean-field interactions. This is the main objective of this paper. In particular, the precise status of the KMS condition in presence of mean-field, or long-range, interactions will be examined.

Our main results are summarized as follows: To simplify discussions, we take a translation-invariant[4] model $\mathfrak{m}$ for fermions on the lattice with mean-field interactions, within a very general class of models. Fix once and

[3] The one-site Hilbert space for quantum spins and the fermionic Fock space associated with the one-site Hilbert space for lattice fermions.

[4] See Remark 1.5.

for all the (non-zero) temperature of the system. By [11, Theorem 2.12], the infinite volume (grand-canonical) pressure P of the model $\mathfrak{m}$ is given by the following variational problem

$$\mathrm{P} = -\inf f(E_1) \in \mathbb{R} \ , \tag{1}$$

where $f : E_1 \to \mathbb{R}$ is the free energy density functional associated with $\mathfrak{m}$, defined on the (metrizable weak*-compact) convex set E_1 of all translation-invariant states of the system. Define the (metrizable weak*-compact) convex set $\mathit{\Omega} \subseteq E_1$ of so-called generalized equilibrium states as being the set of all weak*-limit points of minimizing sequences of this variational problem. We then have two observations at this point:

Observation 1: As explained above, there is no state-independent Heisenberg dynamics in infinite volume, but a state-dependent one actually exists, in relation with the ergodic decomposition of translation-invariant states. See [12, Theorem 4.3] or Equation (59) below.

Observation 2: By [11, Theorem 2.39], the set $\mathcal{E}(\mathit{\Omega})$ of extreme points of the weak*-compact convex set $\mathit{\Omega}$ belongs to the set

$$\mathbf{M} \doteq \underset{d \in \mathcal{C}}{\cup} M_{\Phi(d)}$$

of strict minimizers of the free energy densities associated with effective translation-invariant *short-range* interactions $\Phi(d)$, $d \in \mathcal{C}$, which refer to Bogoliubov approximations of the full interaction of the model $\mathfrak{m}$ [11, Section 2.10.1]. In other words, $\mathcal{E}(\mathit{\Omega}) \subseteq \mathbf{M} \subseteq E_1$. The set $\mathcal{C}$ (of parameters d) and the corresponding Bogoliubov approximations strongly depend on the particular choice of the model $\mathfrak{m}$ and will be precisely defined in the sequel.

By [6, Theorem 1] (quantum spins) and [4, Corollary 6.7 and Theorem 12.11] (lattice fermions), the elements of $M_{\Phi(d)}$ are KMS states with respect to a well-defined dynamics on the underlying C^*-algebra $\mathcal{U}$, since the interactions $\Phi(d)$, $d \in \mathcal{C}$, are short-range. By Observations 1 and 2, this suggests that one should decompose arbitrary generalized equilibrium states of $\mathfrak{m}$ in terms of extreme ones, in order to understand the status of the KMS condition in presence of mean-field interactions. In fact, one can apply the Choquet theorem (see, for instance, [11, Theorem 10.18]) to the metrizable and weak*-compact convex set $\mathit{\Omega} \subseteq E_1$: Any generalized equilibrium state

$\omega \in \varOmega$ is the barycenter of a unique probability measure ν_ω which is supported on the set $\mathcal{E}(\varOmega)$, i.e.,

$$\omega(\cdot) = \int_{\mathcal{E}(\varOmega)} \hat{\omega}(\cdot)\, \nu_\omega(\mathrm{d}\hat{\omega}) \ . \tag{2}$$

There are two caveats in relation to this strategy:

Problem (a): By [11, Lemma 9.8], there are uncountably many translation-invariant lattice fermion or quantum spin systems with mean-field interactions such that $\mathcal{E}(\varOmega) \nsubseteq \mathcal{E}(E_1)$, where $\mathcal{E}(E_1)$ is the set of extreme points of E_1. This is problematic because it prevents us from using the ergodicity of extreme states of E_1. See [11, Theorem 1.16].

Problem (b): We prove in Theorem 4.6 the existence of uncountably many models with mean-field interactions having generalized equilibrium states $\omega \in \varOmega$, whose Choquet measure ν_ω is non-orthogonal. This is problematic because it prevents us from using the Effros Theorem [15, Theorem 4.4.9], in the scope of the theory of direct integrals of measurable families of Hilbert spaces, operators, von Neumann algebras, and C^*-algebra representations. See [12, Sections 5-6] for more details.

However, one can also apply the Choquet theorem [11, Theorem 10.18] to the metrizable and weak*-compact convex set $E_1 \supseteq \varOmega$ of all translation-invriant states of the system: Each generalized equilibrium state $\omega \in \varOmega$ is again the barycenter of a unique probability measure μ_ω which is supported on the set $\mathcal{E}(E_1)$, i.e.,

$$\omega(\cdot) = \int_{\mathcal{E}(E_1)} \hat{\omega}(\cdot)\, \mu_\omega(\mathrm{d}\hat{\omega}) \ . \tag{3}$$

This time, μ_ω is always an orthogonal measure, as stated, for instance, in [12, Theorem 5.1]. Compare with Problem (b). There is however an accessory problem in decomposing generalized equilibrium states in the set $\mathcal{E}(E_1)$ of extreme points of E_1, instead of decomposing them in $\mathcal{E}(\varOmega)$, i.e., in terms of extreme generalized equilibrium states of $\mathfrak{m}$:

Problem (c): Being solution to the variational problem (1), extreme generalized equilibrium states satisfy a sort of Euler-Lagrange equation in relation with the approximating interactions $\Phi(d)$, $d \in \mathcal{C}$, also called gap equations in the Physics literature. See [11, Theorem 2.39]. If $\mathcal{E}(\varOmega) \nsubseteq \mathcal{E}(E_1)$ (cf. [11, Lemma 9.8]) then the above probability measure μ_ω is supported on a set of states that do not a priori satisfy these equations, which turn out to be essential in the present study.

Problems (a)–(c) are direct consequences of the presence of mean-field interactions. To properly deal with all of them, Theorem 4.7 is a key technical result showing that the unique orthogonal probability measure μ_ω representing a generalized equilibrium state $\omega \in \mathit{\Omega} \subseteq E_1$ in terms of ergodic states, via Equation (3), is supported on the set $\mathbf{M}$ (of minimizers of the free energy density functional of short-range Bogoliubov approximations), i.e.,

$$\mu_\omega \left(\mathcal{E}(E_1) \cap \mathbf{M} \right) = 1. \tag{4}$$

See Observation 2.

Recall now Observation 1, i.e., the fact that, in presence of mean-field interactions, there is generally no well-defined infinite volume dynamics on the corresponding C^*-algebra, but rather a state-dependent one. A first very natural way to get around this problem is to weaken the KMS property by imposing it only "fiberwise", in the ergodic decomposition (3). By doing so, Problems (a)–(b) are solved. In order to solve Problem (c), we consider the set B of all Bogoliubov states of the given long-range model, which are defined here as being states $\rho \in E_1$ satisfying the above mentioned Euler-Lagrange equations "fiberwise", i.e., for any $\hat{\rho} \in E_1$ these equations are μ_ρ-almost surely satisfied for some $d_{\hat{\rho}} \in \mathcal{C}$. In this way we get an extension of [6, Theorem 1] (quantum spins) and [4, Corollary 6.7, Proposition 12.1, Theorems 7.5 and 12.11] (lattice fermions) to a very general class of quantum lattice systems with mean-field, or long-range, interactions:

Theorem 1.1. $\mathit{\Omega} \cap \mathrm{B} = \mathrm{K} \cap \mathrm{B}$, *where* $\mathrm{K} \subseteq E_1$ *is the set of states* $\rho \in E_1$ *such that* $\hat{\rho} \in E_1$ *is* μ_ρ*-almost surely a KMS state for the strongly continuous group of* $*$*-automorphisms generated by the approximating interaction* $\Phi(d_{\hat{\rho}})$ *for some* $d_{\hat{\rho}} \in \mathcal{C}$.

This assertion corresponds to Theorem 5.4. Meanwhile, Theorem 4.7, i.e., Equation (4), allows us to identify a very large subclass of quantum lattice systems with mean-field interactions for which the Choquet decompositions of generalized equilibrium states ω in $\mathit{\Omega}$ and E_1 are identical, i.e., $\nu_\omega = \mu_\omega$ and $\mathcal{E}(\mathit{\Omega}) \subseteq \mathcal{E}(E_1)$ in Equations (2) and (3):

$$\omega(\cdot) = \int_{\mathcal{E}(E_1)} \hat{\omega}(\cdot)\, \mu_\omega(\mathrm{d}\hat{\omega}) = \int_{\mathcal{E}(\mathit{\Omega})} \hat{\omega}(\cdot)\, \mu_\omega(\mathrm{d}\hat{\omega}) \ . \tag{5}$$

In this case, Problems (a)–(c) *disappear* and one then obtains the following:

Corollary 1.2. $\mathit{\Omega} = \mathrm{K} \cap \mathrm{B}$.

This assertion refers to Corollary 5.5 and shows that generalized equilibrium states are *"fiberwise"* KMS states. The subclass of quantum lattice systems used in this corollary basically includes all lattice fermion and quantum spin systems of condensed matter physics with mean-field interactions – at least to our knowledge.

In order to get the usual (global) KMS property for generalized equilibrium states one shall first use the cyclic representations associated with these states, to make sense of the infinite volume dynamics, by [12, Theorem 4.3]. We apply this procedure to the above subclass of quantum lattice systems. In this case, for any generalized equilibrium state $\omega \in \mathit{\Omega}$, whose associated cyclic representation of the C^*-algebra $\mathcal{U}$ is denoted by $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$, we show in Proposition 5.7 the existence of a σ-weakly continuous group $\mathbf{\Lambda}^\omega \equiv (\mathbf{\Lambda}_t^\omega)_{t\in\mathbb{R}}$ of $*$-automorphisms of the von Neumann algebra $\pi_\omega(\mathcal{U})''$, which results from the infinite volume limit of the full (long-range) dynamics in this representation. Then, we get the following statement:

Theorem 1.3. *The normal extension $\tilde{\omega}$ of the generalized equilibrium state $\omega \in \mathit{\Omega}$ to the von Neumann algebra $\pi_\omega(\mathcal{U})''$ is a KMS state for the σ-weakly continuous group $\mathbf{\Lambda}^\omega$.*

This assertion refers to Theorem 5.8. It represents another natural extension of [6, Theorem 1] (quantum spins) and [4, Corollary 6.7 and Theorem 12.11] (lattice fermions) to quantum lattice systems with mean-field, or long-range, interactions.

In our opinion, these results improve the mathematical status of mean-field quantum models, like the BCS model of (conventional) superconductivity theory, as they allow to use well-known powerful machineries, like the Tomita-Takesaki modular theory [15, Section 2.5] and the KMS theory [1, Sections 5.3-5.4]. For instance, by Corollary 5.9, in the cyclic representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ of any generalized equilibrium state $\omega \in \mathit{\Omega}$, the infinite volume limit of the full dynamics is nothing else than the (time-rescaled) modular $*$-automorphism group associated with (the cyclic and separating vector) Ω_ω and the von Neumann algebra $\pi_\omega(\mathcal{U})''$, just as in the short-range case.

To conclude, the paper is organized as follows: Section 2 presents the general mathematical framework. Observe that we focus on lattice fermion systems. See Remark 1.4. In Section 3, we give a brief account on the theory of fermion systems on the lattice with short-range interactions, which is used as a springboard to introduce the theory of mean-field, or long-range, models in Section 4. Note that the mathematical setting used in the current paper is – up to minor modifications – the one of [9, 11], including the notation.

We thus provide it in a concise way. The main results, i.e., Corollary 5.5, Theorems 5.4 and 5.8, are finally given in Section 5.

Remark 1.4 (Quantum spin systems).
Our study focuses on lattice fermion systems, which are, from a technical point of view, slightly more difficult than quantum spin systems, because of a non-commutativity issue at different lattice sites. However, all the results presented here hold true for quantum spin systems, via obvious modifications.

Remark 1.5 (Periodic quantum lattice systems).
Our study focuses on lattice fermion systems that are translation invariant (in space). However, all the results presented here hold true for space-periodic lattice fermion systems, by appropriately redefining the spin set[5]*. A similar argument holds true for quantum spin systems.*

Remark 1.6 (Inverse temperature of quantum lattice systems).
In all the paper, we fix once and for all the inverse temperature of quantum systems via the strictly positive parameter $\beta \in \mathbb{R}^+$. This parameter is usually not referred to in the notation, unless it is important for the reader's comprehension.

2. Algebraic formulation of lattice fermion systems

2.1. CAR algebra for lattice fermions

2.1.1. Background lattice. For some dimension $d \in \mathbb{N}$, which is fixed once and for all in the sequel, let $\mathfrak{L} \doteq \mathbb{Z}^d$ and $\mathcal{P}_f \subseteq 2^{\mathfrak{L}}$ be the set of all non-empty finite subsets of $\mathfrak{L}$. In order to define the thermodynamic limit, we use the cubic boxes

$$\Lambda_L \doteq \{(x_1, \ldots, x_d) \in \mathfrak{L} : |x_1|, \ldots, |x_d| \leq L\} \in \mathcal{P}_f \ , \qquad L \in \mathbb{N}, \tag{6}$$

as a so-called van Hove sequence. We also fix, once and for all, a positive-valued symmetric function $\mathbf{F} : \mathfrak{L}^2 \to (0, 1]$ satisfying $\mathbf{F}(x, x) = 1$ for all $x \in$

[5]In fact, one can see the lattice points in a (space) period as a single point in an equivalent lattice on which particles have an enlarged spin set.

$\mathfrak{L}$,

$$\|\mathbf{F}\|_{1,\mathfrak{L}} \doteq \sup_{y\in\mathfrak{L}} \sum_{x\in\mathfrak{L}} \mathbf{F}(x,y) \in [1,\infty) \ , \tag{7}$$

as well as

$$\mathbf{D} \doteq \sup_{x,y\in\mathfrak{L}} \sum_{z\in\mathfrak{L}} \frac{\mathbf{F}(x,z)\,\mathbf{F}(z,y)}{\mathbf{F}(x,y)} < \infty \ . \tag{8}$$

Examples of functions satisfying all these conditions, are given by

$$\mathbf{F}(x,y) = \mathrm{e}^{-\varsigma|x-y|}(1+|x-y|)^{-(d+\epsilon)} \ , \qquad x,y \in \mathfrak{L}^2,$$

for every positive parameter $\varsigma \in \mathbb{R}_0^+$ and $\epsilon \in \mathbb{R}^+$.

2.1.2. The CAR C^*-algebra. For any subset $\Lambda \subseteq \mathfrak{L}$, $\mathcal{U}_\Lambda$ denotes the universal unital C^*-algebra generated by elements $\{a_{x,\mathrm{s}}\}_{x\in\Lambda,\mathrm{s}\in\mathrm{S}}$ satisfying the canonical anti-commutation relations (CAR):

$$\left\{ \begin{array}{lll} a_{x,\mathrm{s}}a_{y,\mathrm{t}} + a_{y,\mathrm{t}}a_{x,\mathrm{s}} & = & 0 \\ a_{x,\mathrm{s}}^*a_{y,\mathrm{t}} + a_{y,\mathrm{t}}a_{x,\mathrm{s}}^* & = & \delta_{x,y}\delta_{\mathrm{s},\mathrm{t}}\mathfrak{1} \end{array} \right. , \qquad x,y \in \Lambda,\ \mathrm{s},\mathrm{t} \in \mathrm{S},$$

where $\mathfrak{1}$ stands for the unit of the algebra, $\delta_{\cdot,\cdot}$ is the Kronecker delta and S is some finite set (representing an orthonormal basis of spin modes), which is fixed once and for all. We use the notation

$$|A|^2 \doteq A^*A, \qquad A \in \mathcal{U}_\Lambda,\ \Lambda \subseteq \mathfrak{L}, \tag{9}$$

to shorten the equations, in particular in the context of long-range models.

By identifying the generators $\{a_{x,\mathrm{s}}\}_{x\in\Lambda\cap\Lambda',\mathrm{s}\in\mathrm{S}}$ of both C^*-algebras $\mathcal{U}_\Lambda$ and $\mathcal{U}_{\Lambda'}$, $\{\mathcal{U}_\Lambda\}_{\Lambda\in 2^{\mathfrak{L}}}$ is a net of C^*-algebras with respect to inclusion: For all subsets $\Lambda, \Lambda' \subseteq \mathfrak{L}$ so that $\Lambda \subseteq \Lambda'$, one has $\mathcal{U}_\Lambda \subseteq \mathcal{U}_{\Lambda'}$. For $\Lambda = \mathfrak{L}$ we use the notation $\mathcal{U} \equiv \mathcal{U}_{\mathfrak{L}}$. Observe additionally that the subspace

$$\mathcal{U}_0 \doteq \bigcup_{\Lambda\in\mathcal{P}_f} \mathcal{U}_\Lambda \subseteq \mathcal{U} \equiv \mathcal{U}_{\mathfrak{L}} \tag{10}$$

is a dense $*$-algebra of the CAR C^*-algebra $\mathcal{U}$ of the infinite lattice. In particular, $\mathcal{U}$ is separable, because $\mathcal{U}_\Lambda$ has finite dimension for all (finite subsets) $\Lambda \in \mathcal{P}_f$ and the set $\mathcal{P}_f$ is countable. Elements of $\mathcal{U}_0$ are called local elements of $\mathcal{U}$. The (real) Banach subspace of all self-adjoint elements of $\mathcal{U}$ is denoted by $\mathcal{U}^{\mathbb{R}} \subsetneq \mathcal{U}$.

The local causality of quantum field theory is broken in CAR algebras and physical quantities are therefore defined from the subalgebra of even elements, which are defined as follows: Given a fixed parameter $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$, the condition

$$\mathrm{g}_\theta(a_{x,\mathrm{s}}) = \mathrm{e}^{-i\theta} a_{x,\mathrm{s}} , \qquad x \in \mathbb{Z}^d, \ \mathrm{s} \in \mathrm{S}, \tag{11}$$

defines a unique $*$-automorphism g_θ of the C^*-algebra $\mathcal{U}$. Note that, for any $\Lambda \subseteq \mathfrak{L}$, $\mathrm{g}_\theta(\mathcal{U}_\Lambda) \subseteq \mathcal{U}_\Lambda$ and thus g_θ canonically defines a $*$-automorphism of the subalgebra $\mathcal{U}_\Lambda$. A special role is played by g_π. Elements $A, B \in \mathcal{U}_\Lambda$, $\Lambda \subseteq \mathfrak{L}$, satisfying $\mathrm{g}_\pi(A) = A$ and $\mathrm{g}_\pi(B) = -B$ are respectively called even and odd. (Elements $A \in \mathcal{U}_\Lambda$ satisfying $\mathrm{g}_\theta(A) = A$ for any $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$ are called gauge invariant.) The space of even elements of $\mathcal{U}$ is denoted by

$$\mathcal{U}^+ \doteq \{A \in \mathcal{U} : A = \mathrm{g}_\pi(A)\} \subseteq \mathcal{U} . \tag{12}$$

In fact, it is a C^*-subalgebra of the C^*-algebra $\mathcal{U}$. In Physics, $\mathcal{U}^+$ is seen as more fundamental than $\mathcal{U}$, because of the local causality in quantum field theory, which holds in the first C^*-algebra, but not in the second one. See, e.g., discussions in [9, Section 2.3].

2.2. States of lattice fermion systems

2.2.1. Even states. States on the C^*-algebra $\mathcal{U}$ are, by definition, linear functionals $\rho : \mathcal{U} \to \mathbb{C}$ which are positive, i.e., for all elements $A \in \mathcal{U}$, $\rho(|A|^2) \geq 0$, and normalized, i.e., $\rho(\mathfrak{1}) = 1$. Equivalently, the linear functional ρ is a state iff $\rho(\mathfrak{1}) = 1$ and $\|\rho\|_{\mathcal{U}^*} = 1$. The set of all states on $\mathcal{U}$ is denoted

$$E \doteq \bigcap_{A \in \mathcal{U}} \{\rho \in \mathcal{U}^* : \rho(\mathfrak{1}) = 1, \ \rho(|A|^2) \geq 0\} . \tag{13}$$

This convex set is metrizable and compact with respect to the weak* topology. Mutatis mutandis, for every $\Lambda \subseteq \mathfrak{L}$, we define the set E_Λ of all states on the sub-algebra $\mathcal{U}_\Lambda \subseteq \mathcal{U}$. For any $\Lambda \subseteq \mathfrak{L}$, we use the symbol ρ_Λ to denote the restriction of any $\rho \in E$ to the sub-algebra $\mathcal{U}_\Lambda$. This restriction is clearly a state on $\mathcal{U}_\Lambda$.

Even states on $\mathcal{U}_\Lambda$, $\Lambda \subseteq \mathfrak{L}$, are, by definition, the states $\rho \in E_\Lambda$ satisfying $\rho \circ \mathrm{g}_\pi = \rho$. In other words, the even states on $\mathcal{U}_\Lambda$ are exactly those vanishing on all odd elements of $\mathcal{U}_\Lambda$. The set of even states on $\mathcal{U}$ can be canonically identified with the set of states on the C^*-subalgebra $\mathcal{U}^+$ of even elements, by [9, Proof of Proposition 2.1]. As a consequence, physically relevant states on $\mathcal{U}$ are even.

2.2.2. Translation-invariant states. Lattice translations refer to the group homomorphism $x \mapsto \alpha_x$ from $(\mathbb{Z}^d, +)$ to the group of $*$-automorphisms of the CAR C^*-algebra $\mathcal{U}$ of the (infinite) lattice $\mathfrak{L}$, which is uniquely defined by the condition

$$\alpha_x(a_{y,\mathrm{s}}) = a_{y+x,\mathrm{s}} \ , \quad y \in \mathfrak{L},\ \mathrm{s} \in \mathrm{S} \ . \tag{14}$$

This group homomorphism is used to define the translation invariance of states and interactions of lattice fermion systems.

The state $\rho \in E$ is said to be translation-invariant iff it satisfies $\rho \circ \alpha_x = \rho$ for all $x \in \mathbb{Z}^d$. The space of translation-invariant states on $\mathcal{U}$ is the convex set

$$E_1 \doteq \bigcap_{x\in\mathbb{Z}^d,\ A\in\mathcal{U}} \{\rho \in \mathcal{U}^* : \rho(\mathfrak{1}) = 1,\ \rho(|A|^2) \geq 0,\ \rho = \rho \circ \alpha_x\} \ , \tag{15}$$

which is again metrizable and compact with respect to the weak* topology. Any translation-invariant state is even, by [11, Lemma 1.8]. Thanks to the Krein-Milman theorem [16, Theorem 3.23], E_1 is the weak*-closure of the convex hull of the (non-empty) set $\mathcal{E}(E_1)$ of its extreme points, which turns out to be a weak*-dense (G_δ) subset [9, 17]:

$$E_1 = \overline{\mathrm{co}}\mathcal{E}(E_1) = \overline{\mathcal{E}(E_1)} \ , \tag{16}$$

where $\overline{\mathrm{co}}(K)$ denotes the weak*-closed convex hull of a set K. This fact is well-known and is also true for quantum spin systems on lattices [15, Example 4.3.26 and discussions p. 464].

Since E_1 is metrizable (by separability of $\mathcal{U}$), the Choquet theorem applies: We denote by Σ_E the (Borel) σ-algebra generated by weak*-open subsets of the set E of all states. $\mathcal{E}(E_1)$ is of course a Borel set for it is a G_δ. Recall meanwhile that any positive functional $\rho \in \mathcal{U}_+^*$ is associated with a unique (up to unitary equivalence) cyclic representation $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$ of $\mathcal{U}$. Two positive linear functionals $\rho_1, \rho_2 \in \mathcal{U}^*$ are said to be orthogonal whenever

$$(\mathcal{H}_{\rho_1} \oplus \mathcal{H}_{\rho_2}, \pi_{\rho_1} \oplus \pi_{\rho_2}, \Omega_{\rho_1} \oplus \Omega_{\rho_2})$$

is the cyclic representation of $\mathcal{U}$ associated with the positive functional $\rho_1 + \rho_2 \in \mathcal{U}^*$. A probability measure μ on E is called orthogonal whenever $\rho_{\mu_{\mathfrak{B}}}$

and $\rho_{\mu_{E\backslash\mathfrak{B}}}$ are orthogonal for any $\mathfrak{B}\in\Sigma_E$, where

$$\rho_{\mu_{\mathfrak{B}_0}}(A)\doteq\int_{\mathfrak{B}_0}\rho(A)\,\mu(\mathrm{d}\rho)\ ,\qquad A\in\mathcal{U},\ \mathfrak{B}_0\in\Sigma_E\ .$$

See [15, Lemma 4.1.19 and Definition 4.1.20]. With these definitions we are in a position to state [12, Theorem 5.1] for translation-invariant states, which provides a (stronger) version of the Choquet theorem (in this particular case):

Theorem 2.1 (Ergodic orthogonal decomposition).
For any $\rho\in E_1$, there is a unique probability measure[6] *μ_ρ on E such that $\mu_\rho(\mathcal{E}(E_1))=1$ and*

$$\rho(A)=\int_{E_1}\hat{\rho}(A)\,\mu_\rho(\mathrm{d}\hat{\rho})\ ,\qquad A\in\mathcal{U}\ .$$

Moreover, μ_ρ is an orthogonal measure on (E,Σ_E).

In particular, E_1 is a Choquet simplex. In fact, up to an affine homeomorphism, E_1 is the so-called Poulsen simplex [11, Theorem 1.12].

The unique decomposition of a translation-invariant state $\rho\in E_1$ in terms of extreme translation-invariant states $\hat{\rho}\in\mathcal{E}(E_1)$, given in Theorem 2.1, is also called the *ergodic* decomposition of ρ because of the following fact: Define the space-averages of any element $A\in\mathcal{U}$ by

$$A_L\doteq\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L}\alpha_x(A)\ ,\qquad L\in\mathbb{N}\ .\tag{17}$$

Then, by definition, a translation-invariant state $\hat{\rho}\in E_1$ is *ergodic* iff

$$\lim_{L\to\infty}\hat{\rho}(|A_L|^2)=|\hat{\rho}(A)|^2\ ,\qquad A\in\mathcal{U}\ .\tag{18}$$

(Recall Equation (9).) By [11, Theorem 1.16], any extreme translation-invariant state is ergodic and conversely. In other words, the set of extreme translation-invariant states is equal to

$$\mathcal{E}(E_1)=\{\hat{\rho}\in E_1:\hat{\rho}\text{ is ergodic}\}=\bigcap_{A\in\mathcal{U}}\left\{\hat{\rho}\in E_1:\lim_{L\to\infty}\hat{\rho}(|A_L|^2)=|\hat{\rho}(A)|^2\right\}\ .\tag{19}$$

[6] For E is a metrizable compact space, any finite measure is regular and tight. Thus, here, probabilities measures are just the same as normalized Borel measures.

Remark 2.2 (Periodic states).
For any given $\vec{\ell} \doteq (\ell_1, \ldots, \ell_d) \in \mathbb{N}^d$*, let* $(\mathbb{Z}^d_{\vec{\ell}}, +) \subseteq (\mathbb{Z}^d, +)$ *be the subgroup defined by* $\mathbb{Z}^d_{\vec{\ell}} \doteq \ell_1\mathbb{Z} \times \cdots \times \ell_d\mathbb{Z}$*. Any state* $\rho \in E$ *satisfying* $\rho \circ \alpha_x = \rho$ *for all* $x \in \mathbb{Z}^d_{\vec{\ell}}$ *is called* $\vec{\ell}$*-periodic. All properties given above on translation-invariant states hold true for* $\vec{\ell}$*-periodic states, via obvious modifications. In particular, the weak***-compact convex set of* $\vec{\ell}$*-periodic states is again the so-called Poulsen simplex [11, Theorem 1.12]. By [9, Proposition 2.3], the union of all sets of* $\vec{\ell}$*-periodic states,* $\vec{\ell} \in \mathbb{N}^d$*, is a weak***-dense subset of the set of all even states, the physically relevant ones.*

3. Infinite volume short-range models

3.1. Short-range interactions

A (complex) *interaction* is, by definition, any mapping $\Phi : \mathcal{P}_f \to \mathcal{U}^+$ from the set $\mathcal{P}_f \subseteq 2^{\mathfrak{L}}$ of all non-empty finite subsets of $\mathfrak{L}$ to the C^*-subalgebra $\mathcal{U}^+$ (12) of even elements such that $\Phi_\Lambda \in \mathcal{U}_\Lambda$ for all $\Lambda \in \mathcal{P}_f$. The set $\mathcal{V}$ of all interactions can be naturally endowed with the structure of a complex vector space (via the usual point-wise vector space operations), as well as with the antilinear involution

$$\Phi \mapsto \Phi^* \doteq (\Phi^*_\Lambda)_{\Lambda \in \mathcal{P}_f} \ . \tag{20}$$

An interaction Φ is said to be self-adjoint iff $\Phi = \Phi^*$. The set $\mathcal{V}^{\mathbb{R}}$ of all self-adjoint interactions forms a real subspace of the space of all (complex) interactions.

The (normed) space of short-range interactions is defined by

$$\mathcal{W} \doteq \{\Phi \in \mathcal{V} : \|\Phi\|_{\mathcal{W}} < \infty\} \ , \tag{21}$$

its norm being defined by

$$\|\Phi\|_{\mathcal{W}} \doteq \sup_{x,y \in \mathfrak{L}} \sum_{\Lambda \in \mathcal{P}_f,\, \Lambda \supseteq \{x,y\}} \frac{\|\Phi_\Lambda\|_{\mathcal{U}}}{\mathbf{F}(x,y)} \ , \qquad \Phi \in \mathcal{V} \ , \tag{22}$$

where $\mathbf{F}$ is the positive-valued symmetric function introduced in Section 2.1. $(\mathcal{W}, \|\cdot\|_{\mathcal{W}})$ is a separable Banach space. The (real) Banach subspace of self-adjoint short-range interactions is denoted

$$\mathcal{W}^{\mathbb{R}} \doteq \mathcal{V}^{\mathbb{R}} \cap \mathcal{W} \ ,$$

similar to $\mathcal{U}^{\mathbb{R}} \subsetneq \mathcal{U}$ and $\mathcal{V}^{\mathbb{R}} \subsetneq \mathcal{V}$.

By definition, the interaction $\Phi \in \mathcal{V}$ is translation-invariant iff

$$\Phi_{\Lambda+x} = \alpha_x(\Phi_\Lambda) \ , \qquad x \in \mathbb{Z}^d, \ \Lambda \in \mathcal{P}_f \ , \tag{23}$$

where $\{\alpha_x\}_{x\in\mathbb{Z}^d}$ is the family of (translation) $*$-automorphisms of $\mathcal{U}$ defined by (14), while

$$\Lambda + x \doteq \{y + x \in \mathfrak{L} : y \in \Lambda\} \ , \qquad x \in \mathbb{Z}^d, \ \Lambda \in \mathcal{P}_f \ . \tag{24}$$

The (separable) Banach subspace of translation-invariant and short-range interactions of $\mathcal{V}$ is denoted

$$\mathcal{W}_1 \doteq \bigcap_{x\in\mathbb{Z}^d, \ \Lambda\in\mathcal{P}_f} \{\Phi \in \mathcal{W} : \Phi_{\Lambda+x} = \alpha_x(\Phi_\Lambda)\} \subsetneq \mathcal{W} \subsetneq \mathcal{V} \ . \tag{25}$$

Finally, the (real) Banach subspace of interactions that are simultaneously self-adjoint, translation-invariant and short-range, is denoted

$$\mathcal{W}_1^{\mathbb{R}} \doteq \mathcal{V}^{\mathbb{R}} \cap \mathcal{W}_1 \subseteq \mathcal{W}^{\mathbb{R}} \ , \tag{26}$$

similar to $\mathcal{U}^{\mathbb{R}} \subsetneq \mathcal{U}$, $\mathcal{V}^{\mathbb{R}} \subsetneq \mathcal{V}$ and $\mathcal{W}^{\mathbb{R}} \subsetneq \mathcal{W}$.

3.2. Dynamics generated by short-range interactions

3.2.1. Limit derivations. Local energy elements associated with a given complex interaction $\Phi \in \mathcal{V}$ correspond to the following sequence within the C^*-subalgebra $\mathcal{U}^+$ (12) of even elements:

$$U_L^\Phi \doteq \sum_{\Lambda\subseteq\Lambda_L} \Phi_\Lambda \in \mathcal{U}_{\Lambda_L} \cap \mathcal{U}^+ \ , \qquad L \in \mathbb{N} \ , \tag{27}$$

where we recall that Λ_L, $L \in \mathbb{N}$, are the cubic boxes (6) used to define the thermodynamic limit. If $\Phi \in \mathcal{V}^{\mathbb{R}}$ is self-adjoint then $(U_L^\Phi)_{L\in\mathbb{N}}$ is a sequence (of local Hamiltonians) in $\mathcal{U}^{\mathbb{R}}$. By straightforward estimates using Equations

(7) and (22), note that, for arbitrary short-range interactions $\Phi, \Psi \in \mathcal{W}$,

$$\left\|U_L^{\Phi} - U_L^{\Psi}\right\|_{\mathcal{U}} = \left\|U_L^{\Phi-\Psi}\right\|_{\mathcal{U}} \leq |\Lambda_L| \, \|\mathbf{F}\|_{1,\mathfrak{L}} \, \|\Phi - \Psi\|_{\mathcal{W}} \ , \qquad L \in \mathbb{N} \ . \tag{28}$$

The sequence $(\delta_L^{\Phi})_{L\in\mathbb{N}} \subseteq \mathcal{B}(\mathcal{U})$ of local derivations associated with any given interaction $\Phi \in \mathcal{V}$ is then defined by

$$\delta_L^{\Phi}(A) \doteq i\left[U_L^{\Phi}, A\right] \doteq i\left(U_L^{\Phi} A - A U_L^{\Phi}\right) \ , \qquad A \in \mathcal{U}, \ L \in \mathbb{N} \ .$$

If $\Phi \in \mathcal{V}^{\mathbb{R}}$ is self-adjoint then $(\delta_L^{\Phi})_{L\in\mathbb{N}}$ is a sequence of symmetric derivations (or $*$-derivations). By [9, Corollary 3.5], for any short-range interaction $\Phi \in \mathcal{W}$ and every local element $A \in \mathcal{U}_0$, the limit

$$\delta^{\Phi}(A) \doteq \lim_{L\to\infty} \delta_L^{\Phi}(A) \doteq \sum_{\Lambda \in \mathcal{P}_f} [\Phi_\Lambda, A] \tag{29}$$

exists and defines a (densely defined) derivation δ^{Φ} on the C^* algebra $\mathcal{U}$, whose domain includes the dense $*$-algebra $\mathcal{U}_0 \subseteq \mathcal{U}$ of local elements defined by (10). Additionally, δ^{Φ} is symmetric (or a $*$-derivation) whenever Φ is self-adjoint, i.e., $\Phi \in \mathcal{W}^{\mathbb{R}}$.

3.2.2. Limit short-range dynamics in the Heisenberg picture. We now consider time-dependent interactions. Let $\Psi \in C(\mathbb{R};\mathcal{W})$ be a continuous function from $\mathbb{R}$ to the Banach space $\mathcal{W}$ of short-range interactions. Then, for any $L \in \mathbb{N}$, there is a unique (fundamental) solution $(\tau_{t,s}^{(L,\Psi)})_{s,t\in\mathbb{R}}$ in $\mathcal{B}(\mathcal{U})$ to the (finite volume) non-autonomous evolution equations

$$\forall s,t \in \mathbb{R}: \qquad \partial_s \tau_{t,s}^{(L,\Psi)} = -\delta_L^{\Psi(s)} \circ \tau_{t,s}^{(L,\Psi)} \ , \qquad \tau_{t,t}^{(L,\Psi)} = \mathbf{1}_{\mathcal{U}} \ , \tag{30}$$

and

$$\forall s,t \in \mathbb{R}: \qquad \partial_t \tau_{t,s}^{(L,\Psi)} = \tau_{t,s}^{(L,\Psi)} \circ \delta_L^{\Psi(t)} \ , \qquad \tau_{s,s}^{(L,\Psi)} = \mathbf{1}_{\mathcal{U}} \ . \tag{31}$$

In these equations, $\mathbf{1}_{\mathcal{U}}$ refers to the identity mapping from $\mathcal{U}$ to itself. Note also that, for any $L \in \mathbb{N}$ and $\Psi \in C(\mathbb{R};\mathcal{W})$, $(\tau_{t,s}^{(L,\Psi)})_{s,t\in\mathbb{R}}$ is a continuous two-parameter family of bounded operators that satisfies the (reverse) cocycle property

$$\tau_{t,s}^{(L,\Psi)} = \tau_{r,s}^{(L,\Psi)} \tau_{t,r}^{(L,\Psi)} \ , \qquad s,r,t \in \mathbb{R} \ .$$

If $\Psi \in C(\mathbb{R};\mathcal{W}^{\mathbb{R}})$ then $\delta_L^{\Psi(t)}$ is always a symmetric derivation and thus, in this case, $\tau_{t,s}^{(L,\Psi)}$ is a $*$-automorphism of $\mathcal{U}$ for all lengths $L \in \mathbb{N}$ and times $s,t \in \mathbb{R}$.

By [9, Proposition 3.7], in the thermodynamic limit $L \to \infty$, for any fixed $\Psi \in C(\mathbb{R}; \mathcal{W}^{\mathbb{R}})$, $(\tau_{t,s}^{(L,\Psi)})_{s,t\in\mathbb{R}}$ converges strongly, uniformly for s,t on compacta, to a strongly continuous two-parameter family $(\tau_{t,s}^{\Psi})_{s,t\in\mathbb{R}}$ of $*$-automorphisms of $\mathcal{U}$, which is the unique solution in $\mathcal{B}(\mathcal{U})$ to the non-autonomous evolutions equation

$$\text{(32)} \qquad \forall s,t \in \mathbb{R}: \qquad \partial_t \tau_{t,s}^{\Psi} = \tau_{t,s}^{\Psi} \circ \delta^{\Psi(t)} , \qquad \tau_{s,s}^{\Psi} = \mathbf{1}_{\mathcal{U}} ,$$

in the strong sense on the dense $*$-algebra $\mathcal{U}_0 \subseteq \mathcal{U}$ of local elements defined by (10). In particular, it satisfies the reverse cocycle property:

$$\text{(33)} \qquad \tau_{t,s}^{\Psi} = \tau_{r,s}^{\Psi} \tau_{t,r}^{\Psi} , \qquad s,r,t \in \mathbb{R} .$$

This refers to a non-autonomous limit dynamics in the Heisenberg picture of quantum mechanics. Taking a constant self-adjoint short-range interaction

$$\Phi \in \mathcal{W}^{\mathbb{R}} \subseteq C(\mathbb{R}; \mathcal{W}^{\mathbb{R}})$$

and fixing $s = 0$, we obtain a C_0-group $\tau^{\Phi} \equiv (\tau_t^{\Phi})_{t\in\mathbb{R}}$ of $*$-automorphisms of $\mathcal{U}$. This refers now to an autonomous limit dynamics, again in the Heisenberg picture of quantum mechanics.

3.3. Equilibrium states of short-range interactions

3.3.1. Energy density functional on translation-invariant states. The energy density of a state $\rho \in E$ with respect to a given interaction $\Phi \in \mathcal{V}$ is defined by

$$e_{\Phi}(\rho) \doteq \limsup_{L\to\infty} \frac{\mathrm{Re}\{\rho\left(U_L^{\Phi}\right)\}}{|\Lambda_L|} + i \limsup_{L\to\infty} \frac{\mathrm{Im}\{\rho\left(U_L^{\Phi}\right)\}}{|\Lambda_L|} \in (-\infty,\infty] + i(-\infty,\infty] ,$$

Λ_L, $L \in \mathbb{N}$, being the cubic boxes (6). Observe from Inequality (28) that

$$\text{(34)} \qquad |e_{\Phi}(\rho) - e_{\Psi}(\rho)| \leq \|\mathbf{F}\|_{1,\mathfrak{L}} \|\Phi - \Psi\|_{\mathcal{W}} , \qquad \Phi, \Psi \in \mathcal{W} .$$

By [9, Proposition 3.2], for any translation-invariant state $\rho \in E_1$ (15) and each translation-invariant short-range interaction $\Phi \in \mathcal{W}_1$,

$$\text{(35)} \qquad e_{\Phi}(\rho) = \lim_{L\to\infty} \frac{\rho\left(U_L^{\Phi}\right)}{|\Lambda_L|} = \rho(\mathfrak{e}_{\Phi}) ,$$

where $\mathfrak{e}_{(\cdot)} : \mathcal{W} \rightarrow \mathcal{U}$ is the continuous mapping from the Banach space $\mathcal{W}$ to the CAR algebra $\mathcal{U}$, defined by

$$\mathfrak{e}_{\Phi} \doteq \sum_{\mathcal{Z}\in\mathcal{P}_f,\ \mathcal{Z}\ni 0} \frac{\Phi_{\mathcal{Z}}}{|\mathcal{Z}|} \in \mathcal{U}\ , \qquad \Phi \in \mathcal{W}\ . \tag{36}$$

In particular, for any fixed $\Phi \in \mathcal{W}_1$, the mapping $\rho \mapsto e_{\Phi}(\rho)$ from E_1 to $\mathbb{C}$ is a weak*-continuous affine functional. Given any fixed translation-invariant state $\rho \in E_1$, the linear mapping $\Phi \mapsto e_{\Phi}(\rho)$ from $\mathcal{W}_1$ to $\mathbb{C}$ is continuous, by (34). Note also that, for all $\rho \in E_1$, $e_{\Phi}(\rho)$ is a real number, whenever the interaction $\Phi \in \mathcal{V}$ is short-range, translation-invariant and self-adjoint, i.e., $\Phi \in \mathcal{W}_1^{\mathbb{R}}$.

3.3.2. Entropy density functional on translation-invariant states. The entropy density functional $s : E_1 \rightarrow \mathbb{R}_0^+$ is the von Neumann entropy per unit volume in the thermodynamic limit, that is,

$$s(\rho) \doteq -\lim_{L\rightarrow\infty} \left\{ \frac{1}{|\Lambda_L|} \mathrm{Trace}\ \left(\mathrm{d}_{\rho_{\Lambda_L}} \ln \mathrm{d}_{\rho_{\Lambda_L}} \right) \right\}\ , \qquad \rho \in E_1\ ,$$

where we recall that ρ_{Λ_L} is the restriction of the translation-invariant state $\rho \in E_1$ to the finite-dimensional CAR algebra $\mathcal{U}_{\Lambda_L}$ of the cubic box Λ_L defined by (6). Here, $\mathrm{d}_{\rho_{\Lambda_L}} \in \mathcal{U}_{\Lambda_L}$ is the (uniquely defined) density matrix representing the state ρ_{Λ_L} via a trace[7]:

$$\rho_{\Lambda_L}(\cdot) = \mathrm{Trace}\ \left(\cdot\ \mathrm{d}_{\rho_{\Lambda_L}} \right)\ .$$

By [11, Lemma 4.15], the functional s is well-defined on the set E_1 of translation-invariant states. See also [4, Section 10.2]. By [11, Lemma 1.29], the entropy density functional s is a weak*-upper semi-continuous affine functional.

3.3.3. Equilibrium states as minimizers of the free energy density. Equilibrium states of lattice fermion systems are always defined in relation to a fixed self-adjoint interaction, which determines the energy density of states as well as the microscopic dynamics. Here, we define equilibrium states as minimizers of the free energy density functional, in direct relation with the notion of (grand-canonical) pressure: For a fixed $\beta \in \mathbb{R}^+$, the infinite volume pressure P is the real-valued function on the real Banach subspace $\mathcal{W}_1^{\mathbb{R}}$ (26)

[7] For $\Lambda \in \mathcal{P}_f$, the trace on the finite-dimensional C^*-algebra $\mathcal{U}_\Lambda$ refers to the usual trace on the fermionic Fock space representation.

of interactions that are self-adjoint, translation-invariant and short-range, defined by

$$\Phi \mapsto \mathrm{P}_\Phi \doteq \lim_{L\to\infty} \frac{1}{\beta|\Lambda_L|} \ln \mathrm{Trace}(\mathrm{e}^{-\beta U_L^\Phi}) \ .$$

Recall that the parameter $\beta \in \mathbb{R}^+$ is the inverse temperature of the system. It is fixed once and for all and, therefore, it is often omitted in our discussions or notation, see Remark 1.6. By [11, Theorem 2.12], the above pressure is well-defined and, for any $\Phi \in \mathcal{W}_1^{\mathbb{R}}$,

$$\mathrm{P}_\Phi = -\inf f_\Phi(E_1) < \infty \ , \tag{37}$$

the mapping $f_\Phi : E_1 \to \mathbb{R}$ being the free energy density functional defined on the set E_1 of translation-invariant states by

$$f_\Phi \doteq e_\Phi - \beta^{-1} s \ . \tag{38}$$

Recall that $e_\Phi : E_1 \to \mathbb{R}$ is the energy density functional defined in Section 3.3.1 for any $\Phi \in \mathcal{W}_1$, while $s : E_1 \to \mathbb{R}_0^+$ is the entropy density functional presented in Section 3.3.2.

As explained in Sections 3.3.1–3.3.2, the functionals e_Φ, $\Phi \in \mathcal{W}_1^{\mathbb{R}}$, and $-\beta^{-1}s$, $\beta \in \mathbb{R}^+$, are weak*-lower semi-continuous and affine. In particular, the functional f_Φ (38) is weak*-lower semi-continuous and affine. Therefore, for any $\Phi \in \mathcal{W}_1^{\mathbb{R}}$, this functional has minimizers in the weak*-compact set E_1 of translation-invariant states. Similarly to what is done for translation-invariant quantum spin systems (see, e.g., [1, 18]), for any $\Phi \in \mathcal{W}_1^{\mathbb{R}}$, the set M_Φ of translation-invariant equilibrium states of fermions on the lattice is, by definition, the (non-empty) set

$$M_\Phi \doteq \{\omega \in E_1 : f_\Phi(\omega) = \inf \ f_\Phi(E_1) = -\mathrm{P}_\Phi\} \tag{39}$$

of all minimizers of the free energy density functional f_Φ over the set E_1. By affineness and weak*-lower semi-continuity of f_Φ, M_Φ is a (non-empty) weak*-closed face of E_1 for any $\Phi \in \mathcal{W}_1^{\mathbb{R}}$.

Recall that this is not the only reasonable way of defining equilibrium states. For fixed interactions, they can also be defined as tangent functionals to the corresponding pressure functional or via other conditions like the local stability, the Gibbs condition or the Kubo-Martin-Schwinger (KMS) condition. All these definitions are generally not completely equivalent to each other. For instance, the free energy density minimizing property and the tangent property assume translation invariance (or, at least, periodicity), whereas other conditions like the KMS one are independent of the space invariance of states. We discuss a well-known partial result on this question in the following section. For more details, we recommend the paper [4].

3.3.4. Equilibrium states as KMS states. The KMS condition was introduced in 1957 by Kubo and Martin, and by Schwinger in 1959, in the context of thermodynamic Green's functions. It has led to the notion of *KMS states*. There are various equivalent definitions of KMS states, see, e.g., [1, Sections 5.3-5.4]. Here we use the following one: Given a C_0-group $\tau \equiv (\tau_t)_{t\in\mathbb{R}}$ of $*$-automorphisms of $\mathcal{U}$ and an inverse temperature $\beta \in \mathbb{R}^+$, a state $\rho \in E$ is a (τ, β)-KMS state iff

$$\int_{\mathbb{R}} f(t - i\beta)\rho\left(A\tau_t(B)\right) \mathrm{d}t = \int_{\mathbb{R}} f(t)\rho\left(\tau_t(B)A\right) \mathrm{d}t \tag{40}$$

for all $A, B \in \mathcal{U}$ and any function f being the (holomorphic) Fourier transform of a smooth function with compact support. See, for instance, [5, Equation (13) and Lemma III.3.1 in Chapter III].

Given any fixed $\beta \in \mathbb{R}^+$ and self-adjoint short-range interaction $\Phi \in \mathcal{W}^{\mathbb{R}}$, the set of translation-invariant (τ^{Φ}, β)-KMS states associated with the C_0-group $\tau^{\Phi} \equiv (\tau_t^{\Phi})_{t\in\mathbb{R}}$ of $*$-automorphisms of $\mathcal{U}$ defined in Section 3.2.2 is denoted by

$$K_{\Phi} \doteq \left\{\omega \in E_1 : \omega \text{ is a } (\tau^{\Phi}, \beta)\text{-KMS state}\right\} . \tag{41}$$

One big advantage of the KMS property, as compared to other notions of equilibrium states like the one presented in Section 3.3.3, is that the translation-invariance of states and interactions is not needed. However, as we are dealing with translation-invariant equilibrium states only, this generalization is not further considered here.

The set K_{Φ} is non-empty, weak*-compact and convex, for any $\Phi \in \mathcal{W}^{\mathbb{R}}$. In fact, following [4], we show below that equilibrium states, as minimizers of the free energy density functional f_{Φ}, are exactly the translation-invariant (τ^{Φ}, β)-KMS states.

Theorem 3.1 (Equilibrium states as KMS states).
For any $\Phi \in \mathcal{W}_1^{\mathbb{R}}$, $M_{\Phi} = K_{\Phi}$, see Equations (39) and (41).

Proof. Fix $\Phi \in \mathcal{W}_1^{\mathbb{R}}$. From Equation (22),

$$\sum_{\Lambda\in\mathcal{P}_f,\, \Lambda\supseteq\{0\}} \|\Phi_{\Lambda}\|_{\mathcal{U}} \leq \|\Phi\|_{\mathcal{W}} < \infty$$

and it follows from the translation invariance of the interaction Φ that

$$H_L^\Phi \doteq \sum_{\Lambda\in\mathcal{P}_f,\ \Lambda_L\cap\Lambda\neq\emptyset} \Phi_\Lambda\ , \qquad L\in\mathbb{N}\ ,$$

is well-defined and

$$\delta^\Phi(A) = i[H_L^\Phi, A]\ , \qquad A\in\mathcal{U}_{\Lambda_L}\ ,$$

thanks to Equation (29), keeping in mind that $\Phi_\Lambda \in \mathcal{U}_\Lambda \cap \mathcal{U}^+$ for $\Lambda\in\mathcal{P}_f$. Then, Φ is a (translation covariant) general potential in the sense of [4, Section 5.5], which comes from the dynamics (i.e., C_0-group) τ^Φ. From [4, Theorem 5.13], there is a unique standard potential $\tilde{\Phi}$, in the sense of [4, Definition 5.10], which is associated with the same dynamics τ^Φ. Additionally, since

$$\frac{1}{|\Lambda_L|}\left\|U_L^\Phi - H_L^\Phi\right\|_{\mathcal{U}} \leq \frac{1}{|\Lambda_L|}\sum_{\Lambda\in\mathcal{P}_f:\Lambda\nsubseteq\Lambda_L,\Lambda\cap\Lambda_L\neq\emptyset}\|\Phi_\Lambda\|_{\mathcal{U}}\ ,$$

with the right-hand side of this inequality going to zero as $L\to\infty$, thanks to Equation (22), we use now [4, Theorem 9.5] to deduce that this standard potential $\tilde{\Phi}$ defines the same free energy density functional (38) and, consequently, the same equilibrium states (defined as minimizers of the free energy density functional). We can then directly apply, on the one hand, [4, Corollary 6.7 and Theorem 12.11] to conclude that $M_\Phi\subseteq K_\Phi$, and, on the other hand, [4, Theorem 7.5 and Proposition 12.1] to conclude that $K_\Phi\subseteq M_\Phi$. □

Recall that a state $\rho\in E$ on $\mathcal{U}$ is called *faithful* iff $\rho\left(|A|^2\right)>0$ for all nonzero elements $A\in\mathcal{U}$. It is, by definition, *modular* iff Ω_ρ is separating for the von Neumann algebra $\pi_\rho(\mathcal{U})''\subseteq\mathcal{B}(\mathcal{H}_\rho)$, where $(\mathcal{H}_\rho,\pi_\rho,\Omega_\rho)$ is the cyclic representation of $\mathcal{U}$ associated with ρ. KMS states on simple C^*-algebras like $\mathcal{U}$ are modular and faithful. We thus deduce the following from Theorem 3.1:

Corollary 3.2 (Equilibrium states as faithful states).
For any $\Phi\in\mathcal{W}_1$, translation-invariant equilibrium states $\omega\in M_\Phi$ are faithful and modular.

Proof. The modular property of translation-invariant equilibrium states is a direct consequence of Theorem 3.1 and [1, Corollary 5.3.9]. Given $\Phi\in\mathcal{W}_1$, it follows that a translation-invariant equilibrium state $\omega\in M_\Phi$ with associated cyclic representation $(\mathcal{H}_\omega,\pi_\omega,\Omega_\omega)$ is faithful whenever π_ω is a $*$-isomorphism between $\mathcal{U}$ and $\pi_\omega(\mathcal{U})$. The C^*-algebra $\mathcal{U}$ is a UHF (uniformly hyperfinite) algebra and is thus simple, i.e., the only closed two-sided ideals of $\mathcal{U}$ are the trivial one, $\{0\}$, and $\mathcal{U}$ itself. See, e.g., [17, Section 8] or [15, Corollary 2.6.19]. As a consequence, π_ω must be a $*$-isomorphism between $\mathcal{U}$ and $\pi_\omega(\mathcal{U})$, for, clearly, $\pi_\omega(\mathfrak{1}) \neq 0$ (being the identity operator on $\mathcal{H}_\omega$). □

4. Infinite volume long-range models

4.1. Long-range models

Let

$$\mathbb{S} \doteq \{\Phi \in \mathcal{W}_1 : \|\Phi\|_{\mathcal{U}} = 1\}$$

be the unit sphere of the Banach space $\mathcal{W}_1$ (25) of translation-invariant short-range interactions. Denote by $\mathcal{S}_1$ the space of signed Borel measures of bounded variation on $\mathbb{S}$, which is a real Banach space whose norm is the total variation of measures

$$\|\mathfrak{a}\|_{\mathcal{S}_1} \doteq |\mathfrak{a}|(\mathbb{S}) \ , \qquad \mathfrak{a} \in \mathcal{S}_1 \ .$$

We are now in a position to define the space of long-range models. It is the separable (real) Banach space

$$\mathcal{M} \doteq \left\{\mathfrak{m} \in \mathcal{W}^{\mathbb{R}} \times \mathcal{S}_1 : \|\mathfrak{m}\|_{\mathcal{M}} < \infty\right\} \ , \tag{42}$$

whose norm is

$$\|\mathfrak{m}\|_{\mathcal{M}} \doteq \|\Phi\|_{\mathcal{W}} + \|\mathfrak{a}\|_{\mathcal{S}_1} \ , \qquad \mathfrak{m} \doteq (\Phi, \mathfrak{a}) \in \mathcal{M} \ . \tag{43}$$

The spaces $\mathcal{W}^{\mathbb{R}}$ and $\mathcal{S}_1$ are canonically seen as subspaces of $\mathcal{M}$, i.e.,

$$\mathcal{W}^{\mathbb{R}} \subseteq \mathcal{M} \qquad \text{and} \qquad \mathcal{S}_1 \subseteq \mathcal{M} \ . \tag{44}$$

In particular, $\Phi \equiv (\Phi, 0) \in \mathcal{M}$ for $\Phi \in \mathcal{W}^{\mathbb{R}}$ and $\mathfrak{a} \equiv (0, \mathfrak{a}) \in \mathcal{M}$ for $\mathfrak{a} \in \mathcal{S}_1$.

We define the dense subspace

$$\mathcal{M}_0 \doteq \bigcup_{L \in \mathbb{N}} \mathcal{M}_{\Lambda_L} \tag{45}$$

of the space $\mathcal{M}$ of long-range models, where, for any finite subset $\Lambda \in \mathcal{P}_f$,

$$\mathcal{M}_\Lambda \doteq \left\{(\Phi, \mathfrak{a}) \in \mathcal{M} : \|\mathfrak{a}\|_{\mathcal{S}_1} \doteq |\mathfrak{a}|(\mathbb{S}) = |\mathfrak{a}|(\mathbb{S} \cap \mathcal{W}_\Lambda)\right\} \ , \tag{46}$$

$\mathcal{W}_\Lambda$ being the closed subspace of finite-range translation-invariant interactions defined by

$$\mathcal{W}_\Lambda \doteq \{\Phi \in \mathcal{W}_1 : \Phi_{\mathcal{Z}} = 0 \text{ whenever } \mathcal{Z} \nsubseteq \Lambda,\ \mathcal{Z} \ni 0\} \ . \tag{47}$$

(If $0 \notin \Lambda \in \mathcal{P}_f$ then $\mathcal{W}_\Lambda = \{0\}$, but this is of course not the case of interest here.)

Long-range models $\mathfrak{m} \doteq (\Phi, \mathfrak{a})$ are not necessarily translation-invariant, because their short-range component Φ is not required to be translation-invariant. We thus define

$$\mathcal{M}_1 \doteq \mathcal{W}_1^{\mathbb{R}} \times \mathcal{S}_1 \subsetneq \mathcal{M} \tag{48}$$

as being the (real) Banach space of translation-invariant long-range models.

Remark 4.1 (Equivalent definition of long-range models).
[11] and [9, 12] have different definitions of long-range models. We use here the formalism introduced in [9, 12]. Compare (48) with [11, Definition 2.1]. However, [9, Section 8] shows that the results of [11] apply equally to all translation-invariant long-range models $\mathfrak{m} \in \mathcal{M}_1$.

4.2. Purely repulsive and purely attractive long-range models

By the Hahn decomposition theorem, any signed measure $\mathfrak{a}$ of bounded variation on the unit sphere $\mathbb{S}$ of the Banach space $\mathcal{W}_1$ has a unique decomposition

$$\mathfrak{a} = \underbrace{\mathfrak{a}_+}_{\text{long-range repulsion}} - \underbrace{\mathfrak{a}_-}_{\text{long-range attraction}} \tag{49}$$

$\mathfrak{a}_\pm$ being two positive finite measures vanishing on disjoint Borel sets, respectively denoted by $\mathbb{S}_\mp \subseteq \mathbb{S}$. Recall that such a decomposition is called the Jordan decomposition of the measure of bounded variation $\mathfrak{a}$ and $|\mathfrak{a}| = \mathfrak{a}_+ + \mathfrak{a}_-$. Long-range attractions are represented by the measure $\mathfrak{a}_-$, whereas $\mathfrak{a}_+$ refers to long-range repulsions. A long-range model $\mathfrak{m} \doteq (\Phi, \mathfrak{a}) \in \mathcal{M}$ is said to be *purely attractive* iff $\mathfrak{a}_+ = 0$, while it is *purely repulsive* iff $\mathfrak{a}_- = 0$.

Distinguishing between these two special types of models is important because the effects of long-range attractions and repulsions on the structure of corresponding sets of (generalized) equilibrium states can be very different: By [11, Theorem 2.25], long-range attractions have no particular effect on the structure of the set of (generalized, translation-invariant) equilibrium states, which is still a (non-empty) weak*-closed face of the set E_1 of translation-invariant states, like for short-range interactions. By contrast, long-range repulsions have generally a geometrical effect by possibly breaking the face structure of the set of (generalized) equilibrium states (see [11, Lemma 9.8]). This feature of long-range repulsions leads us to introduce the notion of *simple* long-range models, in Definition 4.4 below.

4.3. Dynamics generated by long-range models

4.3.1. Local derivations and long-range dynamics. The local Hamiltonians associated with any long-range model $\mathfrak{m} \doteq (\Phi, \mathfrak{a}) \in \mathcal{M}$ are the (well-defined) self-adjoint elements

$$U_L^{\mathfrak{m}} \doteq U_L^{\Phi} + \frac{1}{|\Lambda_L|} \int_{\mathbb{S}} \left| U_L^{\Psi} \right|^2 \mathfrak{a}\left(\mathrm{d}\Psi\right) , \qquad L \in \mathbb{N} , \tag{50}$$

where we recall that $|A|^2 \doteq A^* A$ for any $A \in \mathcal{U}$, see (9). Note that $U_L^{(\Phi,0)} = U_L^{\Phi}$ for any self-adjoint short-range interaction $\Phi \in \mathcal{W}^{\mathbb{R}}$ (cf. (44)) and straightforward estimates yield the bound

$$\|U_L^{\mathfrak{m}}\|_{\mathcal{U}} \leq |\Lambda_L| \, \|\mathbf{F}\|_{1,\mathfrak{L}} \, \|\mathfrak{m}\|_{\mathcal{M}} , \qquad L \in \mathbb{N} , \ \mathfrak{m} \in \mathcal{M} , \tag{51}$$

by Equations (28) and (43).

The sequence $(\delta_L^{\mathfrak{m}})_{L\in\mathbb{N}}$ of local (symmetric) derivations of the C^*-algebra $\mathcal{U}$, associated with any fixed long-range model $\mathfrak{m} \in \mathcal{M}$, is defined by

$$\delta_L^{\mathfrak{m}}(A) \doteq i \left[U_L^{\mathfrak{m}}, A\right] \doteq i \left(U_L^{\mathfrak{m}} A - A U_L^{\mathfrak{m}}\right) , \qquad A \in \mathcal{U} , \ L \in \mathbb{N} . \tag{52}$$

Note that $\delta_L^{(\Phi,0)} = \delta_L^{\Phi}$ for every length $L \in \mathbb{N}$ and any self-adjoint short-range interaction $\Phi \in \mathcal{W}^{\mathbb{R}}$ (cf. (44)). For any $\mathfrak{m} \in \mathcal{M}$ and $L \in \mathbb{N}$, the local long-range dynamics is defined to be the continuous group $(\tau_t^{(L,\mathfrak{m})})_{t\in\mathbb{R}}$ of $*$-automorphisms of $\mathcal{U}$ generated by the bounded derivation $\delta_L^{\mathfrak{m}}$. Equivalently,

$$\tau_t^{(L,\mathfrak{m})}(A) \doteq \mathrm{e}^{itU_L^{\mathfrak{m}}} A \mathrm{e}^{-itU_L^{\mathfrak{m}}} , \qquad A \in \mathcal{U} , \ t \in \mathbb{R} . \tag{53}$$

Note that $\tau_t^{(L,(\Phi,0))} = \tau_t^{(L,\Phi)}$ for any self-adjoint short-range interaction $\Phi \in \mathcal{W}^{\mathbb{R}}$, every length $L \in \mathbb{N}$ and all times $t \in \mathbb{R}$. See Section 3.2.2.

4.3.2. Dynamical self-consistency equations. Generically, long-range dynamics in infinite volume are equivalent to intricate combinations of a classical and short-range (infinite volume) quantum dynamics. This fact results from the existence of a solution to a (dynamical) *self-consistency equation*. In order to present this equation, we need some preliminary definitions: For any long-range model $\mathfrak{m} = (\Phi, \mathfrak{a}) \in \mathcal{M}$ and every function $c = (c_\Psi)_{\Psi \in \mathbb{S}} \in L^2(\mathbb{S}; \mathbb{C}; |\mathfrak{a}|)$, we define a so-called approximating (self-adjoint, short-range) interaction by

$$\Phi_{\mathfrak{m}}(c) \doteq \Phi + 2 \int_{\mathbb{S}} \mathrm{Re}\,\{\overline{c_\Psi}\Psi\}\, \mathfrak{a}\,(\mathrm{d}\Psi) \in \mathcal{W}^{\mathbb{R}} \ . \tag{54}$$

The integral in the last definition, which refers to a self-adjoint interaction, i.e., an element of the space $\mathcal{W}^{\mathbb{R}}$, has to be understood as follows:

$$\left(\int_{\mathbb{S}} \mathrm{Re}\,\{\overline{c_\Psi}\Psi\}\, \mathfrak{a}\,(\mathrm{d}\Psi) \right)_{\Lambda} \doteq \int_{\mathbb{S}} \mathrm{Re}\,\{\overline{c_\Psi}\Psi_{\Lambda}\}\, \mathfrak{a}\,(\mathrm{d}\Psi) \ , \qquad \Lambda \in \mathcal{P}_f \ . \tag{55}$$

Note that the integral in the definiens is well-defined because, for each $\Lambda \in \mathcal{P}_f$, the integrand is an absolutely integrable (measurable) function taking values in a finite-dimensional normed space, which is $\mathcal{U}_\Lambda$.

Then, by [9, Theorem 6.5], if $\mathfrak{m} = (\Phi, \mathfrak{a}) \in \mathcal{M}_0$ (see (45)–(46) for the definition of the dense subspace $\mathcal{M}_0 \subseteq \mathcal{M}$) there is a unique continuous[8] mapping $\varpi^{\mathfrak{m}}$ from $\mathbb{R}$ to the space of automorphisms[9] (or self-homeomorphisms) of E such that

$$\varpi^{\mathfrak{m}}\,(t;\rho) = \rho \circ \tau_{t,0}^{\Phi^{(\mathfrak{m},\rho)}} \ , \qquad t \in \mathbb{R} \ , \ \rho \in E \ , \tag{56}$$

where $\Phi^{(\mathfrak{m},\rho)} \in C(\mathbb{R}; \mathcal{W}^{\mathbb{R}})$ is defined for any $\mathfrak{m} \in \mathcal{M}_0$ and $\rho \in E$ by

$$\Phi^{(\mathfrak{m},\rho)}(t) \doteq \Phi_{\mathfrak{m}}(\varpi^{\mathfrak{m}}\,(t;\rho)\,(\mathfrak{e}_{(\cdot)})) \ , \qquad t \in \mathbb{R} \ , \tag{57}$$

the mapping $\mathfrak{e}_{(\cdot)} : \mathbb{S} \to \mathcal{U}$ being defined by (36), while the strongly continuous two-parameter family $(\tau_{t,s}^{\Phi^{(\mathfrak{m},\rho)}})_{s,t \in \mathbb{R}}$ is the unique solution to (32) for $\Psi = \Phi^{(\mathfrak{m},\rho)}$. For any fixed $t \in \mathbb{R}$ and $\rho \in E$, note that $\varpi^{\mathfrak{m}}\,(t;\rho)\,(\mathfrak{e}_{(\cdot)})$ is a continuous bounded function on $\mathbb{S}$. In particular, it belongs to $L^2(\mathbb{S}; \mathbb{C}; |\mathfrak{a}|)$ and thus, at fixed $t \in \mathbb{R}$ and $\rho \in E$, the right-hand side of (57) is an approximating (short-range) interaction, as defined by Equation (54). The continuity

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[8]We endow the set $C\,(E;E)$ of continuous functions from E to itself with the topology of uniform convergence. See [9, Equation (100)] for more details.

[9]I.e., elements of $C\,(E;E)$ with inverse. Note the inverse (in the sense of functions) of an element of $C\,(E;E)$ is again an element of this space, i.e., it is continuous, by (weak*) compactness of E.

of $\Phi^{(\mathfrak{m},\rho)}$ is a consequence of the continuity of the mappings

$$\varpi^{\mathfrak{m}} : \mathbb{R} \to C\left(E; E\right) \qquad \text{and} \qquad \Phi_{\mathfrak{m}}(\cdot) : L^2(\mathbb{S}; \mathbb{C}; |\mathfrak{a}|) \to \mathcal{W}^{\mathbb{R}}.$$

Equation (56) is named here the (dynamical) self-consistency equation.

4.3.3. Limit long-range dynamics in the Schrödinger picture. Any long-range model $\mathfrak{m} \in \mathcal{M}$ leads to a sequence of finite volume dynamics $(\tau_t^{(L,\mathfrak{m})})_{t\in\mathbb{R}}$, $L \in \mathbb{N}$, defined by (53). At length $L \in \mathbb{N}$, the time-evolution $(\rho_t^{(L)})_{t\in\mathbb{R}}$ of any state $\rho \in E$ is given by

$$\rho_t^{(L)} \doteq \rho \circ \tau_t^{(L,\mathfrak{m})} \ . \tag{58}$$

Equation (58) refers to the Schrödinger picture of quantum mechanics.

At fixed $A \in \mathcal{U}$ and $t \in \mathbb{R}$, the thermodynamic limit $L \to \infty$ of $\tau_t^{(L,\mathfrak{m})}(A)$ does not necessarily exist in $\mathcal{U}$, but the limit $L \to \infty$ of $\rho_t^{(L)}$ can still make sense: Fix once and for all a translation-invariant long-range model $\mathfrak{m} \in \mathcal{M}_1 \cap \mathcal{M}_0$. Recall that E_1 denotes the set (15) of translation-invariant states, with set $\mathcal{E}(E_1)$ of extreme points, and that, for any $\rho \in E_1$, there is an orthogonal (unique) probability measure μ_ρ on E_1 with support in $\mathcal{E}(E_1)$ such that

$$\rho\left(A\right) = \int_{\mathcal{E}(E_1)} \hat{\rho}\left(A\right) \ \mathrm{d}\mu_\rho\left(\hat{\rho}\right) \ , \qquad A \in \mathcal{U} \ ,$$

by Theorem 2.1. By the ergodicity property of extreme translation-invariant states (see (19)), one can prove that, for any time $t \in \mathbb{R}$ and every element $A \in \mathcal{U}$,

$$\begin{aligned} \lim_{L\to\infty} \rho_t^{(L)}\left(A\right) &= \int_{\mathcal{E}(E_1)} \varpi^{\mathfrak{m}}\left(t; \hat{\rho}\right)\left(A\right) \ \mathrm{d}\mu_\rho\left(\hat{\rho}\right) \\ &= \int_{\mathcal{E}(E_1)} \hat{\rho} \circ \tau_{t,0}^{\Psi^{(\mathfrak{m},\hat{\rho})}}\left(A\right) \ \mathrm{d}\mu_\rho\left(\hat{\rho}\right) \ , \end{aligned} \tag{59}$$

$\varpi^{\mathfrak{m}}$ being the solution to the self-consistency equation (56). See [12, Theorem 5.8]. This result is not restricted to translation-invariant states but it can be extended to all periodic states, which form a weak*-dense subset of the set of all even states, the physically relevant ones. See Remark 2.2 and [9, Proposition 2.3].

4.4. Equilibrium states of long-range models

4.4.1. The space-averaging functional. In addition to the energy density and entropy density functionals, respectively defined in Sections 3.3.1–3.3.2, we need the so-called space-averaging functional in order to study the thermodynamic properties of long-range models. This new density functional is defined on the set E_1 of translation-invariant states as follows: For any $A \in \mathcal{U}$, the mapping $\Delta_A : E_1 \to \mathbb{R}$ is (well-)defined by

$$\rho \mapsto \Delta_A(\rho) \doteq \lim_{L\to\infty} \rho\left(|A_L|^2\right) \in \left[|\rho(A)|^2, \|A\|_{\mathcal{U}}^2\right] ,$$

where $|A_L|^2 \doteq A_L^* A_L$ (see (9)) and A_L is defined by (17) for any $L \in \mathbb{N}$. Compare with Equation (18). See also [11, Section 1.3]. By [11, Theorem 1.18], the functional Δ_A is affine and weak*-upper semi-continuous. Thanks again to [11, Theorem 1.18], note additionally that, at any fixed (translation-invariant state) $\rho \in E_1$,

$$|\Delta_A(\rho) - \Delta_B(\rho)| \leq (\|A\|_{\mathcal{U}} + \|B\|_{\mathcal{U}})\|A - B\|_{\mathcal{U}} , \qquad A, B \in \mathcal{U} .$$

For any signed Borel measure $\mathfrak{a}$ of bounded variation on $\mathbb{S}$, we define the space-averaging functional $\Delta_{\mathfrak{a}} : E_1 \to \mathbb{R}$ on translation-invariant states by

$$\rho \mapsto \Delta_{\mathfrak{a}}(\rho) \doteq \int_{\mathbb{S}} \Delta_{\mathfrak{e}_\Psi}(\rho)\, \mathfrak{a}(\mathrm{d}\Psi) , \tag{60}$$

the continuous mapping $\mathfrak{e}_{(\cdot)} : \mathcal{W} \to \mathcal{U}$ being defined by Equation (36). By [11, Theorem 1.18], $\Delta_{\mathfrak{a}}$ is a well-defined, affine and weak*-upper semi-continuous functional on the set E_1 of translation-invariant states.

4.4.2. Generalized equilibrium states. We give here the extension of the notion of equilibrium states of Section 3.3.3, to general long-range models, by using again the variational principle associated with the infinite volume pressure. An important issue appears in this more general situation, because of the lack of weak*-continuity of the free energy density functional in presence of long-range repulsions (Section 4.2), as explained below in more detail.

We start by giving the (grand-canonical) pressure in the thermodynamic limit: At any given inverse temperature $\beta \in \mathbb{R}^+$, the infinite volume pressure P for translation-invariant long-range models is, by definition, the real-valued function on the Banach space $\mathcal{M}_1$ (48) of translation-invariant long-range models, defined by

$$\mathfrak{m} \mapsto \mathrm{P}_{\mathfrak{m}} \doteq \lim_{L\to\infty} \frac{1}{\beta|\Lambda_L|} \ln \mathrm{Trace}(\mathrm{e}^{-\beta U_L^{\mathfrak{m}}}) \ .$$

By [11, Theorem 2.12], this mapping is well-defined and, for any $\mathfrak{m} = (\Phi, \mathfrak{a}) \in \mathcal{M}_1$,

$$\mathrm{P}_{\mathfrak{m}} = -\inf f_{\mathfrak{m}}(E_1) \in \mathbb{R} \ , \tag{61}$$

where $f_{\mathfrak{m}} : E_1 \to \mathbb{R}$ is the free energy density functional defined by

$$f_{\mathfrak{m}} \doteq \Delta_{\mathfrak{a}} + f_{\Phi} = \Delta_{\mathfrak{a}} + e_{\Phi} - \beta^{-1} s \ . \tag{62}$$

See Equation (38), defining the free energy density functional f_Φ for any self-adjoint translation-invariant and short-range interaction $\Phi \in \mathcal{W}_1^{\mathbb{R}}$. Observe that Equation (61) is an extension of (37) – which refers the space $\mathcal{W}_1^{\mathbb{R}}$ of short-range models only – to the space $\mathcal{M}_1 \supseteq \mathcal{W}_1^{\mathbb{R}}$ of long-range models.

Similar to (39), for any translation-invariant long-range model $\mathfrak{m} \in \mathcal{M}_1$, one might define the set of equilibrium states by

$$M_{\mathfrak{m}} \doteq \{\omega \in E_1 : f_{\mathfrak{m}}(\omega) = \inf\ f_{\mathfrak{m}}(E_1) = -\mathrm{P}_{\mathfrak{m}}\} \ . \tag{63}$$

Note however that the free energy density functional $f_{\mathfrak{m}}$ is in general not weak*-lower semi-continuous on E_1 and it is thus a priori not clear whether $M_{\mathfrak{m}}$ is empty or not. In fact, by Equation (49), for any translation-invariant long-range model $\mathfrak{m} = (\Phi, \mathfrak{a}) \in \mathcal{M}_1$,

$$f_{\mathfrak{m}} = \underbrace{\Delta_{\mathfrak{a}_+}}_{\text{weak*-upper semi-cont.}} + \underbrace{\left(-\Delta_{\mathfrak{a}_-} + f_{\Phi}\right)}_{\text{weak*-lower semi-cont.}} \ .$$

Therefore, instead of considering $M_{\mathfrak{m}}$, we define

$$\Omega_{\mathfrak{m}} \doteq \Big\{\omega \in E_1 : \exists \{\rho_n\}_{n=1}^{\infty} \subseteq E_1 \text{ weak}^* \text{ converging to } \omega \text{ such that } \lim_{n\to\infty} f_{\mathfrak{m}}(\rho_n) = \inf\ f_{\mathfrak{m}}(E_1)\Big\} \tag{64}$$

as being the set of *generalized* equilibrium states of any fixed translation-invariant long-range model $\mathfrak{m} \in \mathcal{M}_1$ (at inverse temperature $\beta \in \mathbb{R}^+$). Observe for instance that, under periodic boundary conditions, the accumulation points of (finite-volume) Gibbs states associated with any long-range model $\mathfrak{m} \in \mathcal{M}_1$ and $\beta \in \mathbb{R}^+$ always belong to $\Omega_{\mathfrak{m}}$, but not necessarily to $M_{\mathfrak{m}}$, by [11, Theorem 3.13].

Obviously, by weak*-compactness of E_1, the set $\Omega_{\mathfrak{m}}$ is non-empty and $\Omega_{\mathfrak{m}} \supseteq M_{\mathfrak{m}}$. This definition can be expressed in terms of the graph of $f_{\mathfrak{m}}$:

$$\Omega_{\mathfrak{m}} \times \{\inf f_{\mathfrak{m}}(E_1)\} = (E_1 \times \{\inf f_{\mathfrak{m}}(E_1)\}) \cap \overline{\mathrm{Graph}(f_{\mathfrak{m}})} \ ,$$

where the closure of the graph of $f_{\mathfrak{m}}$ refers to the product topology of the weak* topology on E_1 and the usual topology on $\mathbb{R}$. It follows that $\Omega_{\mathfrak{m}}$ is weak*-closed and convex, by affineness of $f_{\mathfrak{m}}$. Thus, $\Omega_{\mathfrak{m}}$ is a weak*-compact convex subset of E_1. See [11, Lemma 2.16]. If $\mathfrak{a}_+ = 0$ then $\Omega_{\mathfrak{m}} = M_{\mathfrak{m}}$ is a (non-empty) weak*-closed face of the Poulsen simplex E_1. By contrast, as already mentioned above, a long-range repulsion $\mathfrak{a}_+$ has generally a *geometrical* effect on the set $\Omega_{\mathfrak{m}}$, by possibly breaking its face structure in E_1. This effect can lead to long-range order of generalized equilibrium states. See [11, Section 2.9].

4.4.3. Thermodynamic game. Through a version of the approximating Hamiltonian method [11, Section 2.10], [11, Theorem 2.36] shows that, for any long-range model $\mathfrak{m} = (\Phi, \mathfrak{a}) \in \mathcal{M}_1$, the pressure $\mathrm{P}_{\mathfrak{m}}$ is given by a (Bogoliubov) min-max variational problem on the Hilbert space $L^2(\mathbb{S}; \mathbb{C}; |\mathfrak{a}|)$ of square integrable functions on the sphere $\mathbb{S}$, which is interpreted as the result of a two-person zero-sum game, as it is explained in this section.

For any translation-invariant long-range model $\mathfrak{m} = (\Phi, \mathfrak{a}) \in \mathcal{M}_1$, recall that functions $c = (c_\Psi)_{\Psi \in \mathbb{S}} \in L^2(\mathbb{S}; \mathbb{C}; |\mathfrak{a}|)$ are parameters of approximating interactions $\Phi_{\mathfrak{m}}(c) \in \mathcal{W}_1^{\mathbb{R}}$, which are defined by (54). By Equation (27), the energy observables associated with $\Phi_{\mathfrak{m}}(c)$ equal

$$U_L^{\Phi_{\mathfrak{m}}(c)} = U_L^{\Phi} + \int_{\mathbb{S}} 2\mathrm{Re}\left\{\overline{c_\Psi} U_L^{\Psi}\right\} \mathfrak{a}\,(\mathrm{d}\Psi) \ , \qquad L \in \mathbb{N} \ .$$

One then deduces from Equations (37)–(38) that

$$\mathrm{P}_{\Phi_{\mathfrak{m}}(c)} = -\inf f_{\Phi_{\mathfrak{m}}(c)}\,(E_1) \ , \qquad c \in L^2(\mathbb{S}; \mathbb{C}; |\mathfrak{a}|) \ , \tag{65}$$

where, for any translation-invariant state $\rho \in E_1$,

$$f_{\Phi_{\mathfrak{m}}(c)}(\rho) = \int_{\mathbb{S}} 2\mathrm{Re}\left\{\overline{c_\Psi} e_\Psi(\rho)\right\} \mathfrak{a}\,(\mathrm{d}\Psi) + e_\Phi(\rho) - \beta^{-1} s(\rho) \ .$$

As compared to the pressure $\mathrm{P}_{\mathfrak{m}}$ for translation-invariant long-range models $\mathfrak{m} \in \mathcal{M}_1$, $\mathrm{P}_{\Phi_{\mathfrak{m}}(c)}$ is, in principle, easier to analyze, because it comes from a purely short-range interaction $\Phi_{\mathfrak{m}}(c) \in \mathcal{W}_1^{\mathbb{R}}$.

Recall Equation (49): $\mathfrak{a} = \mathfrak{a}_+ - \mathfrak{a}_-$ with $\mathfrak{a}_\pm$ being two positive finite measures vanishing on any subset of $\mathbb{S}_\mp$, respectively, where $\mathbb{S}_\pm$ are Borel sets referring to the Jordan decomposition of $\mathfrak{a}$. Then, we define two Hilbert spaces corresponding respectively to the long-range repulsive and attractive components, $\mathfrak{a}_+$ and $\mathfrak{a}_-$, of any translation-invariant long-range model $\mathfrak{m} \in \mathcal{M}_1$:

$$L^2_\pm(\mathbb{S};\mathbb{C}) \doteq L^2(\mathbb{S};\mathbb{C};\mathfrak{a}_\pm) \ . \tag{66}$$

Note that we canonically have the equality

$$L^2(\mathbb{S};\mathbb{C};|\mathfrak{a}|) = L^2_+(\mathbb{S};\mathbb{C}) \oplus L^2_-(\mathbb{S};\mathbb{C}) \ .$$

The approximating free energy density functional

$$\mathfrak{f}_{\mathfrak{m}} : L^2_-(\mathbb{S};\mathbb{C}) \times L^2_+(\mathbb{S};\mathbb{C}) \to \mathbb{R}$$

is defined by

$$\mathfrak{f}_{\mathfrak{m}}(c_-, c_+) \doteq -\|c_+\|_2^2 + \|c_-\|_2^2 - \mathrm{P}_{\Phi_{\mathfrak{m}}(c_- + c_+)} \ , \qquad c_\pm \in L^2_\pm(\mathbb{S};\mathbb{C}) \ .$$

The *thermodynamic game* is the two-person zero-sum game defined from $\mathfrak{f}_{\mathfrak{m}}$, with one of its conservative values being equal (up to a minus sign) to the pressure $\mathrm{P}_{\mathfrak{m}}$ (see [11, Theorem 2.36 (♯)]):

$$\mathrm{P}_{\mathfrak{m}} = -\inf_{c_- \in L^2_-(\mathbb{S};\mathbb{C})} \sup_{c_+ \in L^2_+(\mathbb{S};\mathbb{C})} \mathfrak{f}_{\mathfrak{m}}(c_-, c_+) \ , \qquad \mathfrak{m} \in \mathcal{M}_1 \ .$$

Compare this equality with Equations (61)-(62). The sup and inf in the above optimization problem are attained, i.e., they are respectively a max and a min and the set

$$\mathcal{C}_{\mathfrak{m}} \doteq \left\{ d_- \in L^2_-(\mathbb{S};\mathbb{C}) : \max_{c_+ \in L^2_+(\mathbb{S};\mathbb{C})} \mathfrak{f}_{\mathfrak{m}}(d_-, c_+) = -\mathrm{P}_{\mathfrak{m}} \right\} \tag{67}$$

(of conservative strategies of the "attractive player") is non-empty, norm-bounded and weakly compact, by [11, Lemma 8.4 (♯)]. In the particular case of purely repulsive long-range models, i.e., when $\mathfrak{a}_- = 0$, $\mathcal{C}_{\mathfrak{m}} = \{0\} = L^2_-(\mathbb{S};\mathbb{C})$, which is in this case the unique equivalent class of all complex-valued functions on $\mathbb{S}$, as $\mathfrak{f}_{\mathfrak{m}}$ is independent of c_-.

Note that, in general, there is no saddle point, since the sup and inf do generally not commute. See [11, p. 42]. In [11, Lemma 8.3 (♯)] it is proven that, when $\mathfrak{a}_+ \neq 0$, for all functions $c_- \in L^2_-(\mathbb{S};\mathbb{C})$, the set

$$\left\{ d_+ \in L^2_+(\mathbb{S};\mathbb{C}) : \max_{c_+ \in L^2_+(\mathbb{S};\mathbb{C})} \mathfrak{f}_{\mathfrak{m}}(c_-, c_+) = \mathfrak{f}_{\mathfrak{m}}(c_-, d_+) \right\} \tag{68}$$

has exactly one element, which we denote by $\mathrm{r}_+(c_-)$. By [11, Lemma 8.8], if $\mathfrak{a}_+ \neq 0$ then the mapping

$$\mathrm{r}_+ : c_- \mapsto \mathrm{r}_+(c_-) \tag{69}$$

defines a continuous functional from $L^2_-(\mathbb{S};\mathbb{C})$ to $L^2_+(\mathbb{S};\mathbb{C})$, where $L^2_-(\mathbb{S};\mathbb{C})$ and $L^2_+(\mathbb{S};\mathbb{C})$ are endowed with the weak and norm topologies, respectively. This mapping is called the thermodynamic decision rule of the translation-invariant long-range model $\mathfrak{m} \in \mathcal{M}_1$. In the particular case of purely attractive long-range models, i.e., when $\mathfrak{a}_+ = 0$, $\mathfrak{f}_{\mathfrak{m}}$ is independent of c_+ and one trivially has $\mathrm{r}_+ = 0$, since $L^2_+(\mathbb{S};\mathbb{C}) = \{0\}$ in this case.

4.4.4. Self-consistency of generalized equilibrium states. The structure of the set $\Omega_{\mathfrak{m}}$ (64) of generalized (translation-invariant) equilibrium states can be now discussed in detail, with respect to the thermodynamic game.

For any translation-invariant long-range model $\mathfrak{m} = (\Phi, \mathfrak{a}) \in \mathcal{M}_1$ and every function $c \in L^2(\mathbb{S};\mathbb{C};|\mathfrak{a}|)$, we define the (possibly empty) set

$$\Omega_{\mathfrak{m}}(c) \doteq \left\{ \omega \in M_{\Phi_{\mathfrak{m}}(c)} : e_{(\cdot)}(\omega) = c \right\} \subseteq E_1 , \tag{70}$$

where, for any fixed translation-invariant state $\rho \in E_1$, the continuous and bounded mapping $e_{(\cdot)}(\rho) : \mathbb{S} \to \mathbb{C}$ is defined from (35)–(36) by

$$e_{\Psi}(\rho) \doteq \rho(\mathfrak{e}_{\Psi}) , \qquad \Psi \in \mathbb{S} , \tag{71}$$

while $M_{\Phi_{\mathfrak{m}}(c)}$ is the set (39) of equilibrium states associated with the approximating interaction $\Phi = \Phi_{\mathfrak{m}}(c) \in \mathcal{W}_1^{\mathbb{R}}$ defined by (54). Recall that $M_{\Phi_{\mathfrak{m}}(c)}$ is a weak*-closed face of E_1. Then, we obtain a (static) self-consistency condition for generalized equilibrium states, which says that any extreme point of $\Omega_{\mathfrak{m}}$ must belong to the set

$$\Omega_{\mathfrak{m}}(d_- + \mathrm{r}_+(d_-)) \tag{72}$$

for some $d_- \in \mathcal{C}_{\mathfrak{m}}$, where r_+ is defined by (69), and $\mathcal{C}_{\mathfrak{m}}$ is the non-empty, norm-bounded, weakly compact set defined by (67). This self-consistency condition refers, in a sense, to Euler-Lagrange equations for the variational problem defining the thermodynamic game. More precisely, we have the following statements:

Theorem 4.2 (Self-consistency of equilibrium states [11]). *Let $\mathfrak{m} \in \mathcal{M}_1$ be any translation-invariant long-range model.*
(i)

$$\Omega_{\mathfrak{m}} = \overline{\mathrm{co}} \left(\bigcup_{d_- \in \mathcal{C}_{\mathfrak{m}}} \Omega_{\mathfrak{m}} \left(d_- + \mathrm{r}_+(d_-) \right) \right) \ .$$

(ii) *The set $\mathcal{E}(\Omega_{\mathfrak{m}})$ of extreme points of the weak*-compact convex set $\Omega_{\mathfrak{m}}$ is included in the union of the sets*

$$\mathcal{E} \left(\Omega_{\mathfrak{m}} \left(d_- + \mathrm{r}_+(d_-) \right) \right) \ , \qquad d_- \in \mathcal{C}_{\mathfrak{m}} \ ,$$

of all extreme points of $\Omega_{\mathfrak{m}} \left(d_- + \mathrm{r}_+(d_-) \right)$, $d_- \in \mathcal{C}_{\mathfrak{m}}$, which are non-empty, convex, mutually disjoint, weak-closed subsets of E_1.*

Assertion (i) results from [11, Theorem 2.21 (i)] and [11, Theorem 2.39 (i)], while (ii) corresponds to [11, Theorem 2.39 (ii)].

Theorem 4.2 implies in particular that, for any extreme state $\hat{\omega} \in \mathcal{E}(\Omega_{\mathfrak{m}})$ of $\Omega_{\mathfrak{m}}$, there is a unique $d_- \in \mathcal{C}_{\mathfrak{m}}$ such that

$$d \doteq d_- + \mathrm{r}_+(d_-) = e_{(\cdot)}(\hat{\omega}) \ . \tag{73}$$

In the Physics literature on superconductors, the above equality refers to the so-called gap equations. Conversely, for any $d_- \in \mathcal{C}_{\mathfrak{m}}$, there is some generalized equilibrium state ω satisfying the condition above, but ω is not necessarily an extreme point of $\Omega_{\mathfrak{m}}$. In the case of purely attractive long-range models $\mathfrak{m} = (\Phi, \mathfrak{a}) \in \mathcal{M}_1$, i.e., if $\mathfrak{a}_+ = 0$ (see Section 4.2, in particular (49)), we get a stronger version of Theorem 4.2 (ii) as a direct consequence of (its previous version and) the following proposition:

Proposition 4.3 (Self-consistency of equilibrium states [11]). *If $\mathfrak{m} \in \mathcal{M}_1$ is purely attractive then, for all $d_- \in \mathcal{C}_{\mathfrak{m}}$, one has*

$$\Omega_{\mathfrak{m}} \left(d_- + \mathrm{r}_+(d_-) \right) = M_{\Phi_{\mathfrak{m}}(d_- + \mathrm{r}_+(d_-))} \ .$$

In particular, the sets $\Omega_{\mathfrak{m}} \left(d_- + \mathrm{r}_+(d_-) \right)$ are weak-closed faces of E_1.*

See [11, Proposition 7.4]. By [11, Remark 2.40] and Theorem 2.1, if $\mathfrak{m} \in \mathcal{M}_1$ is purely attractive then $\Omega_{\mathfrak{m}}$ is a Choquet simplex with

$$\mathcal{E}(\Omega_{\mathfrak{m}}) = \bigcup_{d_- \in \mathcal{C}_{\mathfrak{m}}} \mathcal{E}\left(\Omega_{\mathfrak{m}}\left(d_- + \mathrm{r}_+(d_-)\right)\right) \subseteq \mathcal{E}(E_1)\ . \tag{74}$$

Compare with Theorem 4.2 (ii) and see Lemma 4.9 below. This result also holds true for long-range models which are not necessarily purely attractive, but have instead the following property:

Definition 4.4 (Simple long-range models).
We say that the long-range model $\mathfrak{m} \in \mathcal{M}_1$ is simple iff, for all $d_- \in \mathcal{C}_{\mathfrak{m}}$, the set $M_{\Phi_{\mathfrak{m}}(d_- + \mathrm{r}_+(d_-))}$ consists of one single point.

This definition means that the effective interactions describing a simple long-range model, via the so-called Bogoliubov approximation, refer to fermion systems without (first-order) phase transitions, i.e., with a unique equilibrium state. Remark that this property is always true for long-range models leading to approximating (short-range) interactions that are quasi-free, like in the BCS theory. Such a property is relevant here because it prevents the long-range repulsion from breaking the face structure of the set of generalized equilibrium states (see also [11, Lemma 9.8]). This is a consequence of the following assertion, which is similar to Proposition 4.3 for purely attractive models:

Proposition 4.5 (Self-consistency – simple models).
If $\mathfrak{m} \in \mathcal{M}_1$ is simple then, for all $d_- \in \mathcal{C}_{\mathfrak{m}}$, one has

$$\Omega_{\mathfrak{m}}\left(d_- + \mathrm{r}_+(d_-)\right) = M_{\Phi_{\mathfrak{m}}(d_- + \mathrm{r}_+(d_-))}\ .$$

In particular, the sets $\Omega_{\mathfrak{m}}\left(d_- + \mathrm{r}_+(d_-)\right)$ are (trivially) weak-closed faces of E_1.*

Proof. For any model $\mathfrak{m} \in \mathcal{M}_1$,

$$\emptyset \neq \Omega_{\mathfrak{m}}\left(d_- + \mathrm{r}_+(d_-)\right) \subseteq M_{\Phi_{\mathfrak{m}}(d_- + \mathrm{r}_+(d_-))}\ , \qquad d_- \in \mathcal{C}_{\mathfrak{m}}\ ,$$

by Theorem 4.2 (ii). Hence, if $\mathfrak{m}$ is simple, i.e., $M_{\Phi_{\mathfrak{m}}(d_- + \mathrm{r}_+(d_-))}$ consists of one single point for every $d_- \in \mathcal{C}_{\mathfrak{m}}$, then the equality stated in the proposition must be satisfied. □

By Proposition 4.5, similar to purely attractive long-range models, $\Omega_{\mathfrak{m}}$ is a Choquet simplex and Equation (74) also holds true for all simple models $\mathfrak{m} \in \mathcal{M}_1$. See also Lemma 4.9 below. This is another improvement of Theorem 4.2 (ii) in the case of simple long-range models.

4.4.5. Extreme decompositions of generalized equilibrium states. By [11, Lemma 2.16], for any long-range model $\mathfrak{m} = (\Phi, \mathfrak{a}) \in \mathcal{M}_1$, the non-empty set $\Omega_{\mathfrak{m}} \subseteq E_1$ is weak*-compact and convex. If the model is purely attractive, i.e., $\mathfrak{a}_+ = 0$, or simple then $\Omega_{\mathfrak{m}}$ is a face of E_1, by Equation (74). Nevertheless, in general, $\Omega_{\mathfrak{m}}$ may not be a face (see [11, Lemma 9.8]) and we would like to know whether, despite of this fact, the Choquet measures representing elements of $\Omega_{\mathfrak{m}}$ are *orthogonal* measures. This is important in order to be able to use the theory of direct integrals of measurable families of Hilbert spaces, operators, von Neumann algebras, and C^*-algebra representations, as described in [12, Sections 5-6], together with the Effros Theorem [15, Theorem 4.4.9]. Unfortunately, as soon as we have long-range repulsions, the orthogonality property can be lost:

Theorem 4.6 (Non-orthogonality of extremal decomp. in $\Omega_{\mathfrak{m}}$). *Assume that $|\mathrm{S}| \geq 4$. Then, there are uncountably many models $\mathfrak{m} = (\Phi, \mathfrak{a}) \in \mathcal{M}_1$, with $\mathfrak{a} = \mathfrak{a}_+$ (i.e., the long-range model is purely repulsive), having a generalized equilibrium state $\omega \in \Omega_{\mathfrak{m}}$, whose (Choquet) decomposition on the set $\mathcal{E}(\Omega_{\mathfrak{m}})$ of extreme points of $\Omega_{\mathfrak{m}}$ is non-orthogonal.*

Proof. Given any fixed $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$, recall that g_θ is the unique $*$-automorphism of the C^*-algebra $\mathcal{U}$ defined by Equation (11). Pick a (non self-adjoint) local element $A \in \mathcal{U}_0$ satisfying $\|A\|_{\mathcal{U}} = 1$ and

$$A = -\mathrm{g}_{\theta_1}(A) = -\mathrm{g}_{\theta_2}(A) \tag{75}$$

for some $\theta_1, \theta_2 \in \mathbb{R}/(2\pi\mathbb{Z})$, $\theta_1 \neq \theta_2$. Assume also that, for some translation-invariant state $\hat{\rho}_0 \in E_1$, one has $\hat{\rho}_0(A) \neq 0$ as well as

$$\hat{\rho}_0 \circ \mathrm{g}_{\theta_1} \neq \hat{\rho}_0 \circ \mathrm{g}_{\theta_2} \ . \tag{76}$$

For instance, for all $\lambda \in \mathbb{C}$ and any $(x_1, \mathrm{s}_1), \ldots, (x_4, \mathrm{s}_4) \in \mathbb{Z}^d \times \mathrm{S}$,

$$\begin{aligned}\mathrm{g}_{-\pi/4}\left(\lambda a_{x_1,\mathrm{s}_1} a_{x_2,\mathrm{s}_2} a_{x_3,\mathrm{s}_3} a_{x_4,\mathrm{s}_4}\right) &= -\lambda a_{x_1,\mathrm{s}_1} a_{x_2,\mathrm{s}_2} a_{x_3,\mathrm{s}_3} a_{x_4,\mathrm{s}_4} \\ &= \mathrm{g}_{\pi/4}\left(\lambda a_{x_1,\mathrm{s}_1} a_{x_2,\mathrm{s}_2} a_{x_3,\mathrm{s}_3} a_{x_4,\mathrm{s}_4}\right) \ .\end{aligned}$$

Note that

$$A \doteq \lambda a_{x_1,\mathrm{s}_1} a_{x_2,\mathrm{s}_2} a_{x_3,\mathrm{s}_3} a_{x_4,\mathrm{s}_4} \neq 0$$

if $\lambda \neq 0$ and $(x_1, \mathrm{s}_1), (x_2, \mathrm{s}_2), (x_3, \mathrm{s}_3), (x_4, \mathrm{s}_4)$ are different from each other. If $|\mathrm{S}| \geq 4$ and $\mathrm{s}_1, \mathrm{s}_2, \mathrm{s}_3, \mathrm{s}_4 \in \mathrm{S}$ are different from each other, then there is a product state $\hat{\rho}_0 \in E_1$ such that, for all $x \in \mathbb{Z}^d$,

$$\hat{\rho}_0(a_{x,\mathrm{s}_1} a_{x,\mathrm{s}_2} a_{x,\mathrm{s}_3} a_{x,\mathrm{s}_4}) = \hat{\rho}_0(a_{x,\mathrm{s}_1} a_{x,\mathrm{s}_2}) \hat{\rho}_0(a_{x,\mathrm{s}_3} a_{x,\mathrm{s}_4}) \neq 0 ,$$

because on-site states separate the elements of the on-site C^*-algebra $\mathcal{U}_{\{0\}}$ and $a_{0,\mathrm{s}_1} a_{0,\mathrm{s}_2} a_{0,\mathrm{s}_3} a_{0,\mathrm{s}_4} \in \mathcal{U}_{\{0\}}$ is a non-vanishing even[10] element. Observe that, in this case, Equation (76) holds true for $\theta_1 = -\pi/4$ and $\theta_2 = \pi/4$, i.e., $\hat{\rho}_0 \circ \mathrm{g}_{-\pi/4} \neq \hat{\rho}_0 \circ \mathrm{g}_{\pi/4}$, because

$$\begin{aligned} \hat{\rho}_0 \circ \mathrm{g}_{-\pi/4}(a_{0,\mathrm{s}_1} a_{0,\mathrm{s}_2}) &= \mathrm{e}^{-i\pi/2} \hat{\rho}_0(a_{0,\mathrm{s}_1} a_{0,\mathrm{s}_2}) \neq \mathrm{e}^{i\pi/2} \hat{\rho}_0(a_{0,\mathrm{s}_1} a_{0,\mathrm{s}_2}) \\ &= \hat{\rho}_0 \circ \mathrm{g}_{\pi/4}(a_{0,\mathrm{s}_1} a_{0,\mathrm{s}_2}) , \end{aligned}$$

since $\hat{\rho}_0(a_{0,\mathrm{s}_1} a_{0,\mathrm{s}_2}) \neq 0$. Note also that

$$\{\lambda a_{0,\mathrm{s}_1} a_{0,\mathrm{s}_2} a_{0,\mathrm{s}_3} a_{0,\mathrm{s}_4} : \lambda \in \mathbb{C}, \|\lambda a_{0,\mathrm{s}_1} \cdots a_{0,\mathrm{s}_4}\|_{\mathcal{U}} = 1\} \subseteq \mathcal{U}$$

is an uncountable set.

We can also assume that $\hat{\rho}_0 \in \mathcal{E}(E_1)$, i.e., $\hat{\rho}_0$ is ergodic. In fact, note that the above example already corresponds to this special case, for product states are always ergodic. By Equation (75),

$$\hat{\rho}_0(A) = -\hat{\rho}_1(A) = -\hat{\rho}_2(A) \neq 0 , \tag{77}$$

where $\hat{\rho}_1 \doteq \hat{\rho}_0 \circ \mathrm{g}_{\theta_1}$ and $\hat{\rho}_2 \doteq \hat{\rho}_0 \circ \mathrm{g}_{\theta_2}$. Since g_θ is a $*$-automorphism of $\mathcal{U}$ and $\mathrm{g}_{\theta_1} \neq \mathrm{g}_{\theta_2}$, $\hat{\rho}_1 \neq \hat{\rho}_0$ and $\hat{\rho}_2 \neq \hat{\rho}_0$ are two different states, see (76). As $\hat{\rho}_0 \in \mathcal{E}(E_1)$, by using the relations $\alpha_x \circ \mathrm{g}_\theta = \mathrm{g}_\theta \circ \alpha_x$ for all $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$ and $x \in \mathbb{Z}^d$, we infer from Equations (17)–(19) that the states $\hat{\rho}_1$ and $\hat{\rho}_2$ are also extreme states of E_1, i.e., $\hat{\rho}_1 \in \mathcal{E}(E_1)$ and $\hat{\rho}_2 \in \mathcal{E}(E_1)$.

As explained in the proof of [11, Lemma 4.18], for any local element $A \in \mathcal{U}_0$, there exists a finite-range translation-invariant interaction $\Phi^A \in \mathcal{W}_1$ such that

$$\|\Phi^A\|_{\mathcal{W}_1} = \|A\|_{\mathcal{U}} \qquad \text{and} \qquad \rho(A) = e_{\Phi^A}(\rho) , \qquad \rho \in E_1, \tag{78}$$

with $e_{\Phi^A} : E_1 \to \mathbb{C}$ being defined by Equation (35) for $\Phi = \Phi^A$. For instance, assuming that $|\mathrm{S}| \geq 4$, if $A = \lambda a_{0,\mathrm{s}_1} a_{0,\mathrm{s}_2} a_{0,\mathrm{s}_3} a_{0,\mathrm{s}_4}$, where $\mathrm{s}_1, \mathrm{s}_2, \mathrm{s}_3, \mathrm{s}_4 \in \mathrm{S}$ are

[10]On the one hand, a product state is constructed from an even on-site state, by [4, Theorem 11.2]. On the other hand, there is a on-site state separating 0 and $a_{0,\mathrm{s}_1} a_{0,\mathrm{s}_2} a_{0,\mathrm{s}_3} a_{0,\mathrm{s}_4}$ and, since $a_{0,\mathrm{s}_1} a_{0,\mathrm{s}_2} a_{0,\mathrm{s}_3} a_{0,\mathrm{s}_4}$ is even, one can assume that this on-site state is even.

different from each other and $\lambda \in \mathbb{C}$ is such that $\|A\|_{\mathcal{U}} = 1$, then $\Phi^A \in \mathcal{W}_1$ is the interaction defined by

$$\Phi^A_{\{x\}} = \lambda a_{x,\mathrm{s}_1} a_{x,\mathrm{s}_2} a_{x,\mathrm{s}_3} a_{x,\mathrm{s}_4}, \quad x \in \mathbb{Z}^d, \qquad \text{and} \qquad \Phi^A_\Lambda = 0 \quad \text{otherwise.}$$

Observe that the constant $\lambda \in \mathbb{C}$ is fixed in such a way $\Phi^A \in \mathbb{S} \subseteq \mathcal{W}_1$, i.e., $\|\Phi^A\|_{\mathcal{W}_1} = \|A\|_{\mathcal{U}} = 1$. Let $\mathfrak{a}_+$ be defined, for all Borel subset $\mathfrak{B} \subseteq \mathbb{S}$, by

$$\mathfrak{a}_+ (\mathfrak{B}) = \mathbf{1} \left[\Phi^A \in \mathfrak{B} \right] \ .$$

Since $\hat{\rho}_0, \hat{\rho}_1, \hat{\rho}_2 \in \mathcal{E}(E_1)$ are all extreme states, thanks to [11, Lemma 9.7], there is $\Phi \in \mathcal{W}_1^{\mathbb{R}}$ such that $\hat{\rho}_0, \hat{\rho}_1, \hat{\rho}_2 \in M_\Phi$, see (39). Let $\mathfrak{m}_A \doteq (\Phi, \mathfrak{a}_+) \in \mathcal{M}_1$, which is a purely repulsive translation-invariant long-range model. By convexity of M_Φ,

$$\text{(79)} \qquad \omega_1 \doteq \frac{1}{2}\hat{\rho}_0 + \frac{1}{2}\hat{\rho}_1 \in M_\Phi \qquad \text{and} \qquad \omega_2 \doteq \frac{1}{2}\hat{\rho}_0 + \frac{1}{2}\hat{\rho}_2 \in M_\Phi \ .$$

By assumption, $\hat{\rho}_1 \neq \hat{\rho}_2$ and thus, $\omega_1 \neq \omega_2$.

Consider the convex and weak*-lower semi-continuous functional $g_{\mathfrak{m}_A} : E_1 \to \mathbb{R}$ defined by

$$g_{\mathfrak{m}_A}(\rho) = |\rho(A)|^2 + f_\Phi(\rho) \ , \qquad \rho \in E_1.$$

It turns out that the generalized equilibrium states of $\mathfrak{m}_A$ are exactly the minimizers of this functional, by [11, Theorem 2.25 (+)]. From Equations (17)–(19), (60)–(62), (64) and (77)–(79), it follows that $\omega_1, \omega_2 \in \Omega_{\mathfrak{m}_A}$ and any minimizer $\omega \in E_1$ of $g_{\mathfrak{m}_A}$ satisfies $e_{\Phi^A}(\omega) = 0$. Thus, generalized equilibrium states $\omega \in \Omega_{\mathfrak{m}_A}$ have to satisfy the equality $e_{\Phi^A}(\omega) = 0$. Observe that this refers to the (static) self-consistency of generalized equilibrium states of long-range models. See, for instance, Theorem 4.2 (ii) with $\mathcal{C}_{\mathfrak{m}} = \{0\}$ and $\mathrm{r}_+(0) = 0$.

Now, using (79) and Theorem 2.1 (in particular that E_1 is a Choquet simplex), we deduce that

$$\omega_1, \omega_2 \in \mathcal{E}(\Omega_{\mathfrak{m}_A}) \subseteq E_1 \ ,$$

by construction of the long-range model $\mathfrak{m}_A$. In fact, if ω_1, ω_2 were not both extreme in $\Omega_{\mathfrak{m}_A}$ then we would have states $\omega_1', \omega_1'', \omega_2', \omega_2'' \in \Omega_{\mathfrak{m}_A}$, $\omega_1' \neq \omega_1''$,

$\omega_2' \neq \omega_2''$, such that

$$\omega_1 = \frac{1}{2}\omega_1' + \frac{1}{2}\omega_1'' \qquad \text{and} \qquad \omega_2 = \frac{1}{2}\omega_2' + \frac{1}{2}\omega_2'' \ .$$

In this case, as $\hat{\rho}_0, \hat{\rho}_1 \in \mathcal{E}(E_1)$, the supports of the (Choquet) measures $\mu_{\omega_1'}$ and $\mu_{\omega_1''}$ decomposing ω_1' and ω_1'' in E_1 are contained in $\{\hat{\rho}_0, \hat{\rho}_1\}$. This means that ω_1' is a convex combination of $\hat{\rho}_0$ and $\hat{\rho}_1$. But, because of the above (static) self-consistency condition for generalized equilibrium states, the unique convex combination of $\hat{\rho}_0$ and $\hat{\rho}_1$ which is an element of $\Omega_{\mathfrak{m}_A}$, is ω_1 itself. From this we would conclude that $\omega_1' = \omega_1$ and, hence, $\omega_1' = \omega_1''$. Using exactly the same argument for the state ω_2, we would arrive at $\omega_2' = \omega_2''$.

Finally, by [15, Lemma 4.1.19 and Definition 4.1.20], ω_1 and ω_2, which are two *different* elements of $\mathcal{E}(\Omega_{\mathfrak{m}_A})$, are *not* orthogonal because $\hat{\rho}_0/2 \leq \omega_1$, $\hat{\rho}_0/2 \leq \omega_2$ and $\hat{\rho}_0/2$ is a non-zero positive functional (as $\hat{\rho}_0$ is a state). In particular, the state

$$\omega_0 \doteq \frac{1}{2}\omega_1 + \frac{1}{2}\omega_2 = \frac{1}{2}\hat{\rho}_0 + \frac{1}{4}\hat{\rho}_1 + \frac{1}{4}\hat{\rho}_2 \in \Omega_{\mathfrak{m}_A} \backslash \mathcal{E}(\Omega_{\mathfrak{m}_A})$$

has a non-orthogonal (Choquet) decomposition on the set $\mathcal{E}(\Omega_{\mathfrak{m}})$ of extreme points of $\Omega_{\mathfrak{m}}$. □

By Theorem 4.2 (i), note meanwhile that, for all translation-invariant long-range models $\mathfrak{m} \in \mathcal{M}_1$,

$$(80) \quad \Omega_{\mathfrak{m}} \subseteq \overline{\mathrm{co}}\,(\mathbf{M}_{\mathfrak{m}}) \subseteq E_1 \qquad \text{with} \qquad \mathbf{M}_{\mathfrak{m}} \doteq \bigcup_{d_- \in \mathcal{C}_{\mathfrak{m}}} M_{\Phi_{\mathfrak{m}}(d_- + \mathrm{r}_+(d_-))} \subseteq E_1 \ .$$

Therefore, we can alternatively decompose generalized equilibrium states within the weak*-compact convex set $\overline{\mathrm{co}}\,(\mathbf{M}_{\mathfrak{m}})$. In contrast to the (Choquet) decomposition within $\Omega_{\mathfrak{m}}$, in this situation the decomposition of an arbitrary generalized equilibrium state is always orthogonal, for it coincides with its ergodic decomposition:

Theorem 4.7 (Ergodic decomposition of equilibrium states). *For any $\mathfrak{m} \in \mathcal{M}_1$, $\overline{\mathrm{co}}\,(\mathbf{M}_{\mathfrak{m}})$ is a weak*-closed face of E_1 and, for any $\omega \in \Omega_{\mathfrak{m}} \subseteq \overline{\mathrm{co}}\,(\mathbf{M}_{\mathfrak{m}}) \subseteq E_1$, the unique Choquet probability measure μ_ω on $\overline{\mathrm{co}}\,(\mathbf{M}_{\mathfrak{m}})$ (or E_1) representing ω (see Theorem 2.1) satisfies*

$$\mu_\omega\left(\mathcal{E}(E_1) \cap \mathbf{M}_{\mathfrak{m}}\right) = 1 \ .$$

Proof. Fix without loss of generality any long-range model $\mathfrak{m} = (\Phi, \mathfrak{a}) \in \mathcal{M}_1$ such that $\mathfrak{a}_\pm \neq 0$. (The cases $\mathfrak{a}_- = 0$ or $\mathfrak{a}_+ = 0$ are clearly simpler.) Assume that the set $\mathbf{M}_{\mathfrak{m}}$ is weak*-closed. By the Milman theorem [11, Theorem 10.13 (ii)], $\mathcal{E}(\overline{\mathrm{co}}\,(\mathbf{M}_{\mathfrak{m}})) \subseteq \mathbf{M}_{\mathfrak{m}}$. Thus, since, for any $\Psi \in \mathcal{W}_1^{\mathbb{R}}$, the non-empty convex set M_Ψ is a (weak*-closed) face of E_1, one has

$$\mathcal{E}(\overline{\mathrm{co}}\,(\mathbf{M}_{\mathfrak{m}})) \subseteq \bigcup_{d_- \in \mathcal{C}_{\mathfrak{m}}} \mathcal{E}\left(M_{\Phi_{\mathfrak{m}}(d_- + \mathrm{r}_+(d_-))}\right) \subseteq \mathcal{E}\left(E_1\right) \ . \tag{81}$$

Hence, $\overline{\mathrm{co}}\,(\mathbf{M}_{\mathfrak{m}})$ is a weak*-closed face of the simplex E_1. Therefore, by Theorem 2.1, for any $\omega \in \Omega_{\mathfrak{m}} \subseteq \overline{\mathrm{co}}\,(\mathbf{M}_{\mathfrak{m}})$, there is a unique probability measure μ_ω on $\mathbf{M}_{\mathfrak{m}}$ such that

$$\mu_\omega(\mathcal{E}(\overline{\mathrm{co}}\,(\mathbf{M}_{\mathfrak{m}}))) = 1 \quad \text{and} \quad \omega\left(A\right) = \int_{\mathcal{E}(\overline{\mathrm{co}}(\mathbf{M}_{\mathfrak{m}}))} \hat{\omega}\left(A\right)\ \mu_\omega(\mathrm{d}\hat{\omega}) \ , \quad A \in \mathcal{U} \ . \tag{82}$$

By Equation (81), μ_ω is supported on $\mathcal{E}(E_1)$. Therefore, the theorem follows from (82), provided one proves the weak*-closedness of the set $\mathbf{M}_{\mathfrak{m}}$.

In order to prove that $\mathbf{M}_{\mathfrak{m}}$ is indeed weak*-closed, take any sequence $(\omega_n)_{n=1}^\infty \subseteq \mathbf{M}_{\mathfrak{m}}$ converging in the weak*-topology to $\omega_\infty \in E_1$. (Note that E_1 is weak*-closed, being even weak*-compact, and the use of nets is here not necessary, as the weak* topology of E_1 is metrizable.) Then, by Equation (80), for every $n \in \mathbb{N}$, there is an element $d_-^{(n)} \in \mathcal{C}_{\mathfrak{m}}$ such that

$$\omega_n \in M_{\Phi_{\mathfrak{m}}(d_-^{(n)} + \mathrm{r}_+(d_-^{(n)}))} \ .$$

Since $\mathcal{C}_{\mathfrak{m}} \subseteq L_-^2(\mathbb{S};\mathbb{C};|\mathfrak{a}|)$ is a (non-empty) norm-bounded and weakly compact set [11, Lemma 8.4 (♯)]], the sequence $(d_-^{(n)})_{n=1}^\infty$ converges (along a subsequence again denoted by $(d_-^{(n)})_{n=1}^\infty$) in the weak topology to an element $d_-^{(\infty)} \in \mathcal{C}_{\mathfrak{m}}$ within a ball $\mathcal{B}_R\left(0\right) \subseteq L^2(\mathbb{S};\mathbb{C};|\mathfrak{a}|)$ of sufficiently large radius $R > 0$. Since, by [11, Lemma 8.8], the thermodynamic decision rule (69) defines a continuous mapping from $L_-^2(\mathbb{S};\mathbb{C})$ to $L_+^2(\mathbb{S};\mathbb{C})$ with $L_-^2(\mathbb{S};\mathbb{C})$ and $L_+^2(\mathbb{S};\mathbb{C})$ endowed with the weak and norm topologies, respectively, we then infer from [11, Proposition 7.1 (ii)] that

$$\lim_{n\to\infty} \mathrm{P}_{\Phi_{\mathfrak{m}}(d_-^{(n)} + \mathrm{r}_+(d_-^{(n)})) + \Psi} = \mathrm{P}_{\Phi_{\mathfrak{m}}(d_-^{(\infty)} + \mathrm{r}_+(d_-^{(\infty)})) + \Psi} \ , \qquad \Psi \in \mathcal{W}_1^{\mathbb{R}} \ . \tag{83}$$

(Recall the notation given by (65).) Now, by [11, Theorem 2.28] applied to models with vanishing long-range components, for any $n \in \mathbb{N}$,

$$\mathrm{P}_{\Phi_{\mathfrak{m}}(d_-^{(n)} + \mathrm{r}_+(d_-^{(n)})) + \Psi} - \mathrm{P}_{\Phi_{\mathfrak{m}}(d_-^{(n)} + \mathrm{r}_+(d_-^{(n)}))} \geq -e_\Psi(\omega_n) \ , \qquad \Psi \in \mathcal{W}_1^{\mathbb{R}} \ ,$$

which, combined with (35)-(36) and (83), implies that

$$\mathrm{P}_{\Phi_{\mathfrak{m}}(d_-^{(\infty)}+\mathrm{r}_+(d_-^{(\infty)}))+\Psi}-\mathrm{P}_{\Phi_{\mathfrak{m}}(d_-^{(\infty)}+\mathrm{r}_+(d_-^{(\infty)}))}\geq -e_{\Psi}(\omega_\infty)\ ,\qquad \Psi\in\mathcal{W}_1^{\mathbb{R}}\ .$$

Finally, again by [11, Theorem 2.28], we deduce that

$$\omega_\infty \in M_{\Phi(d_-^{(\infty)}+\mathrm{r}_+(d_-^{(\infty)}))}\qquad \text{with}\qquad d_-^{(\infty)}\in\mathcal{C}_{\mathfrak{m}}\ .$$

It follows from Equation (80) that $\omega_\infty \in \mathbf{M}_{\mathfrak{m}}$. In other words, $\mathbf{M}_{\mathfrak{m}}$ is weak*-closed. □

Corollary 4.8 (Equilibrium states as faithful modular states). *For all $\mathfrak{m}\in\mathcal{M}_1$, generalized equilibrium states $\omega\in\varOmega_{\mathfrak{m}}$ are faithful and modular.*

Proof. Let $\mathfrak{m}\in\mathcal{M}_1$. Since μ_ω is an orthogonal measure (Theorem 2.1) for any $\omega\in\varOmega_{\mathfrak{m}}$, one can use the theory of direct integrals of GNS representations of families of states described in [12, Section 5.6]. Since, in each fiber, the corresponding state $\hat{\omega}\in\mathcal{E}(\overline{\mathrm{co}}\,(\mathbf{M}_{\mathfrak{m}}))$ is faithful and modular, by Corollary 3.2 and Equation (81), the corollary easily follows. □

The property of generalized equilibrium states of translation-invariant long-range models stated in Corollary 4.8 is important, because it allows to use the Tomita-Takesaki modular theory [15, Section 2.5].

Note that the situation is much simpler for purely attractive or simple long-range models (see Section 4.2 and Definition 4.4), because Theorem 4.2 combined with Propositions 4.3 and 4.5 directly yields the following assertion, which is merely a reformulation of Equation (74).

Lemma 4.9 (Choquet decompositions of equilibrium states). *If $\mathfrak{m}\in\mathcal{M}_1$ is purely attractive or simple then $\varOmega_{\mathfrak{m}}=\overline{\mathrm{co}}\,(\mathbf{M}_{\mathfrak{m}})$, the set $\mathbf{M}_{\mathfrak{m}}$ being defined by (80). In particular, the Choquet decomposition in $\varOmega_{\mathfrak{m}}$ of any generalized equilibrium state of $\mathfrak{m}$ coincides with its ergodic decomposition.*

Proof. Let $\mathfrak{m}\in\mathcal{M}_1$ be a purely attractive or simple translation-invariant long-range model. From Proposition 4.3 or 4.5, in both cases,

$$\varOmega_{\mathfrak{m}}(d_-+\mathrm{r}_+(d_-))=M_{\Phi_{\mathfrak{m}}(d_-+\mathrm{r}_+(d_-))}\ ,\qquad d_-\in\mathcal{C}_{\mathfrak{m}}\ .$$

The assertion then follows from Equation (80) and Theorem 4.2 (i). □

To sum up, there are two natural ways to perform Choquet decompositions of generalized equilibrium states $\omega \in \Omega_{\mathfrak{m}} \subseteq E_1$, for any fixed translation-invariant long-range model $\mathfrak{m} \in \mathcal{M}_1$:

- We can decompose ω in the weak*-compact convex set $\Omega_{\mathfrak{m}}$ of all generalized equilibrium states of the long-range model $\mathfrak{m}$. The advantage of doing so is that the extreme states of $\Omega_{\mathfrak{m}}$ always satisfy the (static) self-consistency condition (73). A drawback, however, is that the decomposition may not be orthogonal (Theorem 4.6) and the Effros Theorem [15, Theorem 4.4.9] cannot be used. See [12, Section 5.6] for more details.
- We can decompose ω in the weak*-compact convex set $\overline{\mathrm{co}}\,(\mathbf{M}_{\mathfrak{m}})$, which turns out to be equivalent to decompose it in E_1 (Theorem 4.7). In this case, in contrast to the previous strategy, the decomposition is always orthogonal while the extreme (ergodic) states of $\overline{\mathrm{co}}\,(\mathbf{M}_{\mathfrak{m}})$ are still equilibrium states of approximating (short-range) interactions. The drawback is now that elements of $\mathcal{E}(\overline{\mathrm{co}}\,(\mathbf{M}_{\mathfrak{m}}))$ may not anymore satisfy the (static) self-consistency condition (73), which turns out to be essential in the present study, as it is apparent in the next section.

If $\mathfrak{m} \in \mathcal{M}_1$ is purely attractive or simple then the situation becomes simpler because the Choquet decompositions of generalized equilibrium states in $\Omega_{\mathfrak{m}}$ coincide with those in $\overline{\mathrm{co}}\,(\mathbf{M}_{\mathfrak{m}})$. Thus, we have, in this case, all the good properties of both types of extreme decomposition. In this situation, we obtain in Section 5 two extensions of Theorem 3.1 to long-range models: Corollary 5.5 and Theorem 5.8.

5. Main results: generalized equilibrium states and KMS conditions

Recall that in Section 3.3.4, we assert that equilibrium states of lattice fermion systems with short-range interactions, as translation-invariant minimizers of the free energy density functional, are KMS states associated with the (well-defined) infinite volume dynamics on the CAR C^*-algebra $\mathcal{U}$. See Theorem 3.1, which is a direct consequence of results of [4]. Here, we aim at contributing an extension of this result to generalized equilibrium states of long-range models.

Note first that, for a general long-range model $\mathfrak{m} \in \mathcal{M}$, at fixed $A \in \mathcal{U}$ and $t \in \mathbb{R}$, the sequence $\tau_t^{(L,\mathfrak{m})}(A)$, $L \in \mathbb{N}$, where (the finite volume dynamics) $\tau_t^{(L,\mathfrak{m})}$ is defined by (53), does not necessarily converge in the C^*-algebra

$\mathcal{U}$, in contrast to the short-range case. See Sections 3.2.2 and 4.3.3. In other words, for a general long-range model, we do not have at our disposal a well-defined infinite volume dynamics on the CAR C^*-algebra $\mathcal{U}$. There are two ways to get around this problem:

- The first one is to relax the (global) KMS property by considering it only "fiberwise", using the ergodic decomposition of Theorem 2.1. This is performed in Section 5.1.
- The second approach is to use a faithful representation of the C^*-algebra $\mathcal{U}$ in order to make sense of the infinite volume long-range dynamics, by considering a different topology from the one associated with the norm, typically the strong[11] or the σ-weak operator topology. This is performed in Section 5.2.

Note in this context that Theorem 4.7 is absolutely crucial to obtain the main outcomes of the present section, for general long-range models, while Theorem 4.6 (and its proof) is pivotal to understand the restriction of our study to long-range models that are either purely attractive (Section 4.2) or simple (Definition 4.4). Moreover, we only consider long-range models within a dense subspace of translation-invariant models, that is, $\mathcal{M}_0 \cap \mathcal{M}_1$ (see Equations (45) and (48)), but this is a very mild technical restriction, which is related to the well-posedness of (dynamical) self-consistency equations (56).

5.1. Invariant self-consistently KMS states

As explained above, in this section we relax the (global) KMS property by considering it with respect to the ergodic decomposition of translation-invariant states. This leads to the concept of *self-consistently* KMS states of translation-invariant long-range models.

Definition 5.1 (Self-consistently KMS states).
Take any $\rho \in E_1$ and let μ_ρ be the unique (Choquet) measure (Theorem 2.1) on ergodic states, whose barycenter is the translation-invariant state ρ. We say that ρ is a (translation-invariant) self-consistently KMS state of $\mathfrak{m} \in \mathcal{M}_1$

[11][12, Theorem 4.3 and Corollary 4.5] show that, within the cyclic representation associated with the initial state, the infinite volume limit of dynamics is well-defined in the σ-weak operator topology. Using stronger norms on interactions together with additional estimates, one could improve [12, Theorem 4.3 and Corollary 4.5] to get the convergence in the strong operator topology.

iff $\hat{\rho} \in E_1$ is μ_ρ-almost surely a KMS state for the strongly continuous group of automorphisms generated by the approximating interaction $\Phi_{\mathfrak{m}}(e_{(\cdot)}(\hat{\rho}))$ (see (54) and (71)), at inverse temperature $\beta \in \mathbb{R}^+$. We denote by $\mathrm{K}_{\mathfrak{m}} \subseteq E_1$ the set of self-consistently KMS states of $\mathfrak{m} \in \mathcal{M}_1$.

The above notion is reminiscent of van Hemmen's approach [10] to infinite volume equilibrium states for mean-field models. If the long-range component of $\mathfrak{m}$ is trivial, i.e., $\mathfrak{m} = (\Phi, 0)$ for some $\Phi \in \mathcal{W}_1^{\mathbb{R}}$, then $\Phi_{\mathfrak{m}}(e_{(\cdot)}(\rho)) = \Phi$ for any $\rho \in E_1$ and one thus has $\mathrm{K}_{(\Phi,0)} = K_\Phi$ in this case, see Equation (41). Therefore, the above definition generalizes the notion of KMS states to long-range models.

Note that, even for purely attractive long-range models, a self-consistently KMS state is not necessarily a generalized equilibrium state. To see this explicitly, one may use, for instance, the strong-coupling BCS model explained in [13] and consider the equilibrium states of the same model without its mean-field component. These states are always ergodic, gauge-invariant and self-consistently KMS (for the original model with mean-field interactions), but they violate the gap equations (73) at sufficiently large (inverse temperature) β. This fact leads us to introduce the notion of *Bogoliubov* states, which is crucial in order to establish a relation between the self-consistent KMS condition and generalized equilibrium states of translation-invariant long-range models.

Definition 5.2 (Bogoliubov states).
Take any $\rho \in E_1$ and let μ_ρ be the unique (Choquet) measure (Theorem 2.1) on ergodic states, whose barycenter is the translation-invariant state ρ. We say that ρ is a Bogoliubov state of $\mathfrak{m} \in \mathcal{M}_1$ iff the gap equation $e_{(\cdot)}(\hat{\rho}) = d_- + \mathrm{r}_+(d_-)$ holds true for some $d_- \in \mathcal{C}_{\mathfrak{m}}$, μ_ρ-almost surely for $\hat{\rho} \in E_1$. We denote by $\mathrm{B}_{\mathfrak{m}} \subseteq E_1$ the set of all Bogoliubov states of $\mathfrak{m} \in \mathcal{M}_1$.

Compare with Equation (73), recalling that r_+ is the thermodynamic decision rule (69) of $\mathfrak{m} \in \mathcal{M}_1$, while $e_{(\cdot)}(\rho) : \mathbb{S} \to \mathbb{C}$ is the continuous and bounded mapping defined from (35)–(36) by Equation (71) for any state $\rho \in E_1$.

Note that if the long-range component of $\mathfrak{m}$ is trivial, i.e., $\mathfrak{m} = (\Phi, 0)$ for some $\Phi \in \mathcal{W}_1$, then the equality $e_{(\cdot)}(\rho) = d_- + \mathrm{r}_+(d_-)$ holds trivially true for any $\rho \in E_1$, since $L^2_-(\mathbb{S};\mathbb{C}) = L^2_+(\mathbb{S};\mathbb{C}) = L^2(\mathbb{S};\mathbb{C}) = \{0\}$, the unique equivalent class of all complex-valued functions on $\mathbb{S}$. Thus, $\mathrm{B}_{(\Phi,0)} = E_1$ in this case. Recall that, by Lemma 4.9, if $\mathfrak{m} \in \mathcal{M}_1$ is purely attractive or simple then the Choquet decompositions of generalized equilibrium states in $\Omega_{\mathfrak{m}}$ coincide with those in $\overline{\mathrm{co}}\,(\mathbf{M}_{\mathfrak{m}})$. In this case, all generalized equilibrium states are Bogoliubov states:

Lemma 5.3 (Equilibrium states as Bogoliubov states).
If $\mathfrak{m} \in \mathcal{M}_1$ *is purely attractive or simple then* $\Omega_{\mathfrak{m}} \subseteq \mathrm{B}_{\mathfrak{m}}$*.*

Proof. Let $\mathfrak{m} \in \mathcal{M}_1$ be a purely attractive or simple translation-invariant long-range model. Suppose that $\omega \in \Omega_{\mathfrak{m}}$. From Theorems 4.2, 4.7 and Lemma 4.9, for μ_ω-almost all $\rho \in E_1$, $\rho \in \Omega_{\mathfrak{m}}(d_- + \mathrm{r}_+(d_-))$ for some $d_- \in \mathcal{C}_{\mathfrak{m}}$, that is, $\rho \in M_{\Phi_{\mathfrak{m}}(d_- + \mathrm{r}_+(d_-))}$ and $e_{(\cdot)}(\rho) = d_- + \mathrm{r}_+(d_-)$. See Equation (70) and Definition 5.2. □

From the proof of Theorem 4.6, note that, in general, generalized equilibrium states of long-range models are not Bogoliubov states of these models. This results from the fact that one can construct (uncountably many) models $\mathfrak{m} \in \mathcal{M}_1$ whose set $\Omega_{\mathfrak{m}}$ of generalized equilibrium states is not a face of E_1. See [11, Lemma 9.8]. This singular case is however still important, since it is also related to long-range order of equilibrium states, as shown in [11, Section 2.9]. In fact, in the general case, we have the following result on the relation between self-consistently KMS, Bogoliubov and generalized equilibrium states:

Theorem 5.4 (Self-cons. KMS, Bog. and equilibrium states).
For any translation-invariant long-range model $\mathfrak{m} \in \mathcal{M}_1$*,* $\Omega_{\mathfrak{m}} \cap \mathrm{B}_{\mathfrak{m}} = \mathrm{K}_{\mathfrak{m}} \cap \mathrm{B}_{\mathfrak{m}}$*.*

Proof. Let $\mathfrak{m} \in \mathcal{M}_1$. Assume that $\Omega_{\mathfrak{m}} \cap \mathrm{B}_{\mathfrak{m}} \neq \emptyset$ and take $\omega \in \Omega_{\mathfrak{m}} \cap \mathrm{B}_{\mathfrak{m}}$. From Theorem 4.7, for μ_ω-almost all $\rho \in E_1$, $\rho \in M_{\Phi_{\mathfrak{m}}(d_- + \mathrm{r}_+(d_-))} = M_{\Phi_{\mathfrak{m}}(e_{(\cdot)}(\rho))}$ for some $d_- \in \mathcal{C}_{\mathfrak{m}}$. Then, thanks to Theorem 3.1, $\rho \in K_{\Phi_{\mathfrak{m}}(e_{(\cdot)}(\rho))}$ μ_ω-almost surely. Therefore, $\omega \in \mathrm{K}_{\mathfrak{m}}$.

Assume that $\mathrm{K}_{\mathfrak{m}} \cap \mathrm{B}_{\mathfrak{m}} \neq \emptyset$ and take $\omega \in \mathrm{K}_{\mathfrak{m}} \cap \mathrm{B}_{\mathfrak{m}}$. By Definition 5.1, it means in particular that, for μ_ω-almost all $\rho \in E_1$, $\rho \in K_{\Phi_{\mathfrak{m}}(e_{(\cdot)}(\rho))}$ and $e_{(\cdot)}(\rho) = d_- + \mathrm{r}_+(d_-)$ for some $d_- \in \mathcal{C}_{\mathfrak{m}}$. Thanks again to Theorem 3.1, it follows that, for μ_ω-almost all $\rho \in E_1$, $\rho \in M_{\Phi_{\mathfrak{m}}(d_- + \mathrm{r}_+(d_-))}$ and $\rho \in \Omega_{\mathfrak{m}}(d_- + \mathrm{r}_+(d_-))$ for some $d_- \in \mathcal{C}_{\mathfrak{m}}$, by Equation (70). We then conclude from Theorem 4.2 (i) that $\omega \in \Omega_{\mathfrak{m}}$. □

In the case of purely attractive or simple long-range models we have the following improvement of Theorem 5.4:

Corollary 5.5 (Equil. states as self-cons. KMS and Bog. states).
If $\mathfrak{m} \in \mathcal{M}_1$ *is purely attractive or simple then* $\Omega_{\mathfrak{m}} = \mathrm{K}_{\mathfrak{m}} \cap \mathrm{B}_{\mathfrak{m}}$*.*

Proof. The assertion directly follows from Lemma 5.3 and Theorem 5.4. □

For any self-adjoint translation-invariant and short-range interaction $\Phi \in \mathcal{W}_1^{\mathbb{R}}$, recall that $\mathrm{B}_{(\Phi,0)} = E_1$, $\mathrm{K}_{(\Phi,0)} = K_\Phi$ and $\Omega_{(\Phi,0)} = M_\Phi$. It means that Corollary 5.5 is an extension of Theorem 3.1 to translation-invariant long-range models. This result is complemented by Theorem 5.8 below, which asserts that (under the mild technical condition that $\mathfrak{m} \in \mathcal{M}_0$) the elements of $\Omega_{\mathfrak{m}} = \mathrm{K}_{\mathfrak{m}} \cap \mathrm{B}_{\mathfrak{m}}$ are KMS states in the usual (or global) sense, with respect to the infinite volume dynamics generated by the long-range model $\mathfrak{m}$ in a given representation of the CAR C^*-algebra $\mathcal{U}$.

5.2. The modular group of generalized equilibrium states

Recall that $\mathcal{M}_0$ is the dense subspace of the Banach space $\mathcal{M}$ of long-range models defined by Equation (45), while $\mathcal{M}_1 \subseteq \mathcal{M}$ is the Banach space of all translation-invariant long-range models defined by Equation (48). A long-range model $\mathfrak{m} = (\Phi, \mathfrak{a}) \in \mathcal{M}$ is purely attractive whenever $\mathfrak{a}_+ = 0$, see Section 4.2. Simple models are defined by Definition 4.4. Both situations are important here because in these cases, when $\mathfrak{m} \in \mathcal{M}_1$, the Choquet decomposition in $\Omega_{\mathfrak{m}}$ of any generalized equilibrium state of $\mathfrak{m}$ is the same as its ergodic decomposition (in E_1), by Lemma 4.9. In particular, $\mathcal{E}(E_1) \cap \Omega_{\mathfrak{m}} \neq \emptyset$. For such long-range models, we first prove the stationarity of generalized equilibrium states, in the Schrödinger picture of quantum mechanics (see Equation (58)).

Proposition 5.6 (Stationarity of generalized equilibrium states). *If* $\mathfrak{m} \in \mathcal{M}_0 \cap \mathcal{M}_1$ *is purely attractive or simple then*

$$\lim_{L\to\infty} \omega_t^{(L)}(A) \doteq \lim_{L\to\infty} \omega \circ \tau_t^{(L,\mathfrak{m})}(A) = \omega(A) \ , \qquad A \in \mathcal{U},\ \omega \in \Omega_{\mathfrak{m}} \ .$$

Proof. Let $\mathfrak{m} \in \mathcal{M}_0 \cap \mathcal{M}_1$ be a purely attractive or simple translation-invariant long-range model. If $\hat{\omega} \in \mathcal{E}(E_1) \cap \Omega_{\mathfrak{m}}$ is an ergodic generalized equilibrium state, then we infer from Lemma 4.9 the existence of a unique $d_- \in \mathcal{C}_{\mathfrak{m}}$ such that

$$d \doteq d_- + \mathrm{r}_+(d_-) = e_{(\cdot)}(\hat{\omega}) \doteq \hat{\omega}(\mathfrak{e}_{(\cdot)}) \qquad \text{and} \qquad \hat{\omega} \in M_{\Phi_{\mathfrak{m}}(d)} \ , \tag{84}$$

recalling that $e_{(\cdot)}(\rho) : \mathbb{S} \to \mathbb{C}$ is the continuous and bounded mapping defined from (35)–(36) by Equation (71) for any state $\rho \in E_1$. Thanks to Theorem 3.1, $\hat{\omega} \in K_{\Phi_{\mathfrak{m}}(d)}$, i.e., $\hat{\omega}$ is a $(\tau^{\Phi_{\mathfrak{m}}(d)}, \beta)$-KMS state, keeping in mind that $\beta \in \mathbb{R}^+$ is fixed in all the paper. It is well-known that KMS states are stationary, see, e.g., [1, Proposition 5.3.3]. In particular, the KMS state $\hat{\omega}$ is $\tau^{\Phi_{\mathfrak{m}}(d)}$-invariant, i.e.,

$$\hat{\omega} = \hat{\omega} \circ \tau_t^{\Phi_{\mathfrak{m}}(d)} \ , \qquad t \in \mathbb{R} \ . \tag{85}$$

It follows from Equation (84) and [9, Lemma 7.2] that, for all $t \in \mathbb{R}$,

$$\varpi^{\mathfrak{m}}(t;\hat{\omega}) = \hat{\omega} \qquad \text{and} \qquad \Phi^{\mathfrak{m},\hat{\omega}}(t) = \Phi_{\mathfrak{m}}(\hat{\omega}(\mathfrak{e}_{(\cdot)})) = \Phi_{\mathfrak{m}}(d) \ , \tag{86}$$

where $\varpi^{\mathfrak{m}}$ is the unique continuous mapping from $\mathbb{R}$ to the space of automorphisms of E satisfying (56) and $\Phi^{(\mathfrak{m},\hat{\omega})}(t)$ is defined by Equation (57) with $\rho = \hat{\omega}$. The proposition then follows from Equations (59) and (86). □

The stationarity of the state of a given physical system is the minimal requirement characterizing the thermodynamic equilibrium of that system. In this section, we contribute a much stronger result, which complements Corollary 5.5. In fact, we show below that generalized equilibrium states of purely attractive or simple long-range models can also be seen as KMS states, in the usual sense. We first need to provide a well-defined infinite volume dynamics for long-range models, in order to be able to study the KMS property of their (generalized) equilibrium states.

Proposition 5.7 (Long-range dynamics for equilibrium states). *If $\mathfrak{m} \in \mathcal{M}_0 \cap \mathcal{M}_1$ is purely attractive or simple then, for any $\omega \in \Omega_{\mathfrak{m}}$ with cyclic representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$, there exists a unique σ-weakly continuous*[12] *group $(\mathbf{\Lambda}_t^\omega)_{t\in\mathbb{R}}$ of $*$-automorphisms of the von Neumann algebra $\pi_\omega(\mathcal{U})''$ such that, with respect to the σ-weak topology,*

$$\lim_{L\to\infty} \pi_\omega\left(\tau_t^{(L,\mathfrak{m})}(A)\right) = \mathbf{\Lambda}_t^\omega\left(\pi_\omega(A)\right) \ , \qquad A \in \mathcal{U},\ t \in \mathbb{R} \ ,$$

where, in the special case of ergodic generalized equilibrium states,

$$\mathbf{\Lambda}_t^{\hat{\omega}}\left(\pi_{\hat{\omega}}(A)\right) = \pi_{\hat{\omega}} \circ \tau_t^{\Phi_{\mathfrak{m}}(\hat{\omega}(\mathfrak{e}_{(\cdot)}))}(A) \ , \qquad A \in \mathcal{U},\ t \in \mathbb{R},\ \hat{\omega} \in \mathcal{E}(E_1) \cap \Omega_{\mathfrak{m}} \ .$$

Proof. Let $\mathfrak{m} \in \mathcal{M}_0 \cap \mathcal{M}_1$ be a purely attractive or simple translation-invariant long-range model. In the special case $\hat{\omega} \in \mathcal{E}(E_1) \cap \Omega_{\mathfrak{m}}$, with associated cyclic representation $(\mathcal{H}_{\hat{\omega}}, \pi_{\hat{\omega}}, \Omega_{\hat{\omega}})$, we infer from [12, Theorem 5.8]

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[12] This means here that, for any fixed $t \in \mathbb{R}$ and $A \in \pi_\omega(\mathcal{U})''$, the mappings $\mathbf{\Lambda}_t^\omega(\cdot) : \pi_\omega(\mathcal{U})'' \to \pi_\omega(\mathcal{U})''$ and $\mathbf{\Lambda}_{(\cdot)}^\omega(A) : \mathbb{R} \to \pi_\omega(\mathcal{U})''$ are σ-weak continuous.

and Equation (86) that

$$\lim_{L\to\infty} \pi_{\hat{\omega}} \left(\tau_t^{(L,\mathfrak{m})}(A)\right) = \pi_{\hat{\omega}} \left(\tau_t^{\Phi_{\mathfrak{m}}(\hat{\omega}(\mathfrak{e}_{(\cdot)}))}(A)\right) , \qquad A \in \mathcal{U} , \ t \in \mathbb{R} ,$$

with respect to the σ-weak topology. To study general (possibly non-ergodic) generalized equilibrium states, one applies the theory of direct integrals of measurable families of Hilbert spaces, operators, von Neumann algebras and C^*-algebra representations, as already done in [12]. In fact, we consider the C^*-algebra $\mathfrak{U} \doteq C(E;\mathcal{U})$, whose norm is

$$\|f\|_{\mathfrak{U}} \doteq \max_{\rho\in E} \|f(\rho)\|_{\mathcal{U}} , \qquad f \in \mathfrak{U} .$$

The CAR C^*-algebra $\mathcal{U}$ is canonically identified with the subalgebra of constant functions of $\mathfrak{U}$, i.e., $\mathcal{U} \subseteq \mathfrak{U}$. Fix now, once and for all in the proof, $\omega \in \Omega_{\mathfrak{m}} \subseteq E_1$. From [12, Proposition 4.2], there exists a unique representation Π_ω of $\mathfrak{U}$ on $\mathcal{H}_\omega$ such that $\Pi_\omega|_{\mathcal{U}} = \pi_\omega$ and $(\mathcal{H}_\omega, \Pi_\omega, \Omega_\omega)$ is a cyclic representation associated with the state ω, seen as a state of $\mathfrak{U}$ via the definition

$$\omega(f) \doteq \int_{\mathcal{E}(E_1)\cap\Omega_{\mathfrak{m}}} \hat{\omega}(f(\hat{\omega}))\, \mu_\omega(\mathrm{d}\hat{\omega}) , \qquad f \in \mathfrak{U} . \tag{87}$$

See Theorem 2.1 and Lemma 4.9. Moreover, one has

$$\Pi_\omega(\mathfrak{U})'' = \pi_\omega(\mathcal{U})'' . \tag{88}$$

From [12, Theorem 4.3] and Equation (86), for any time $t \in \mathbb{R}$ and all elements $A \in \mathcal{U} \subseteq \mathfrak{U}$,

$$\lim_{L\to\infty} \pi_\omega \left(\tau_t^{(L,\mathfrak{m})}(A)\right) = \Pi_\omega\left(\mathfrak{T}_t^{\mathfrak{m}}(A)\right) \in \mathcal{B}(\mathcal{H}_\omega) \tag{89}$$

with respect to the σ-weak topology, where $\mathfrak{T}^{\mathfrak{m}} = (\mathfrak{T}_t^{\mathfrak{m}})_{t\in\mathbb{R}}$ is a strongly continuous group of $*$-automorphisms of $\mathfrak{U}$ defined by

$$\mathfrak{T}_t^{\mathfrak{m}}(f)(\rho) \doteq \tau_{t,0}^{\Phi^{(\mathfrak{m},\rho)}}(f(\rho)) , \qquad \rho \in E_1, \ f \in \mathfrak{U} , \ t \in \mathbb{R},$$

the interaction $\Phi^{(\mathfrak{m},\rho)}$ being defined by (57). We deduce from (87) combined with (85)–(86) that, for any $t \in \mathbb{R}$ and $f \in \mathfrak{U}$,

$$\begin{aligned}
\omega\left(\mathfrak{T}_t^{\mathfrak{m}}(f)\right) &= \int_{\mathcal{E}(E_1)\cap\Omega_{\mathfrak{m}}} \hat{\omega}\left(\left(\mathfrak{T}_t^{\mathfrak{m}} f\right)(\hat{\omega})\right) \mu_\omega(\mathrm{d}\hat{\omega}) \\
&= \int_{\mathcal{E}(E_1)\cap\Omega_{\mathfrak{m}}} \hat{\omega}\left(\tau_{t,0}^{\Phi^{(\mathfrak{m},\hat{\omega})}}\left(f(\hat{\omega})\right)\right) \mu_\omega(\mathrm{d}\hat{\omega}) \\
&= \int_{\mathcal{E}(E_1)\cap\Omega_{\mathfrak{m}}} \hat{\omega}\left(f(\hat{\omega})\right) \mu_\omega(\mathrm{d}\hat{\omega}) = \omega(f).
\end{aligned}$$

In other words, $\omega \in \mathfrak{U}^*$ is $\mathfrak{T}^{\mathfrak{m}}$-invariant. By [15, Corollary 2.3.17], there exists a unique strongly continuous family $(U_t)_{t\in\mathbb{R}}$ of unitary operators on $\mathcal{B}(\mathcal{H}_\omega)$ such that

$$\Pi_\omega\left(\mathfrak{T}_t^{\mathfrak{m}}(A)\right) = U_t \pi_\omega(A) U_t^{-1}, \qquad A \in \mathcal{U},\ t \in \mathbb{R}. \tag{90}$$

For any time $t \in \mathbb{R}$, define

$$\mathbf{\Lambda}_t^\omega(A) \doteq U_t A U_t^{-1}, \qquad A \in \pi_\omega(\mathcal{U})''. \tag{91}$$

Observe in particular that, for any fixed $t \in \mathbb{R}$,

$$\mathbf{\Lambda}_t^\omega(\cdot) : \pi_\omega(\mathcal{U})'' \to \mathcal{B}(\mathcal{H}_\omega)$$

is σ-weakly continuous. Note from Equation (88) that

$$\mathbf{\Lambda}_t^\omega(\pi_\omega(\mathcal{U})) \subseteq \Pi_\omega(\mathfrak{U}) \subseteq \pi_\omega(\mathcal{U})'', \qquad t \in \mathbb{R}.$$

Since the bicommutant $\pi_\omega(\mathcal{U})''$ is the closure of $\pi_\omega(\mathcal{U})$ in the σ-weak topology and $\mathbf{\Lambda}_t^\omega(\cdot)$ is continuous in this topology at any fixed $t \in \mathbb{R}$, we arrive at $\mathbf{\Lambda}_t^\omega(\pi_\omega(\mathcal{U})'') \subseteq \pi_\omega(\mathcal{U})''$. Then, from the strong continuity of $(U_t)_{t\in\mathbb{R}}$, $(\mathbf{\Lambda}_t^\omega)_{t\in\mathbb{R}}$ is a σ-weakly continuous group of $*$-automorphisms of the von Neumann algebra $\pi_\omega(\mathcal{U})''$. Finally, note from Equations (89) and (90) that any σ-weakly continuous group of $*$-automorphisms of $\pi_\omega(\mathcal{U})''$ implementing the infinite volume dynamics should be equal to $(\mathbf{\Lambda}_t^\omega)_{t\in\mathbb{R}}$ on $\pi_\omega(\mathcal{U})$, and thus on $\pi_\omega(\mathcal{U})''$, by the σ-weak density of $\pi_\omega(\mathcal{U})$ in $\pi_\omega(\mathcal{U})''$. □

Having an appropriate and natural notion of infinite volume long-range dynamics, as given by Proposition 5.7, we can now study the (global) KMS property of generalized equilibrium states for a fixed translation-invariant long-range model $\mathfrak{m} = (\Phi, \mathfrak{a}) \in \mathcal{M}_1 \cap \mathcal{M}_0$ that is either purely attractive or simple. The KMS property is in this case defined as follows: Given a generalized equilibrium state $\omega \in \Omega_{\mathfrak{m}}$ with associated cyclic representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$, we define $\tilde{\omega} \doteq \langle \Omega_\omega, (\cdot)\Omega_\omega \rangle_{\mathcal{H}_\omega}$, which is the unique normal extension[13] of ω to the von Neumann algebra $\pi_\omega(\mathcal{U})''$. Considering the σ-weakly continuous group $\mathbf{\Lambda}^\omega \equiv (\mathbf{\Lambda}_t^\omega)_{t\in\mathbb{R}}$ of $*$-automorphisms of the von Neumann algebra $\pi_\omega(\mathcal{U})''$ of Proposition 5.7, for a fixed inverse temperature $\beta \in \mathbb{R}^+$, we say that $\tilde{\omega}$ is a $(\mathbf{\Lambda}^\omega, \beta)$-KMS state if

$$\int_{\mathbb{R}} f(t - i\beta)\, \tilde{\omega}\left(A \mathbf{\Lambda}_t^\omega (B)\right) \mathrm{d}t = \int_{\mathbb{R}} f(t)\, \tilde{\omega}\left(\mathbf{\Lambda}_t^\omega (B)\, A\right) \mathrm{d}t \tag{92}$$

for all $A, B \in \pi_\omega(\mathcal{U})''$ and any function f being the (holomorphic) Fourier transform of a smooth function with compact support. Compare this definition of KMS states with the one related to Equality (40) on the CAR C^*-algebra $\mathcal{U}$. Note that the integrals of Equation (92) are well-defined since, for any $A, B \in \pi_\omega(\mathcal{U})''$, the functions

$$\tilde{\omega}(A\mathbf{\Lambda}_{(\cdot)}^\omega (B)), \tilde{\omega}(\mathbf{\Lambda}_{(\cdot)}^\omega (B)\, A) : \mathbb{R} \to \mathbb{C}$$

are clearly continuous and bounded. We can now see generalized equilibrium states of long-range models as KMS states in the following sense:

Theorem 5.8 (Generalized equilibrium states as KMS states). *Fix $\beta \in \mathbb{R}^+$ and a translation-invariant long-range model $\mathfrak{m} \in \mathcal{M}_1 \cap \mathcal{M}_0$ that is purely attractive or simple. Given $\omega \in \Omega_{\mathfrak{m}}$ with associated cyclic representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$, let $\tilde{\omega}$ be the normal extension of ω to $\pi_\omega(\mathcal{U})''$ and $\mathbf{\Lambda}^\omega \equiv (\mathbf{\Lambda}_t^\omega)_{t\in\mathbb{R}}$ be the σ-weakly continuous group of $*$-automorphisms of $\pi_\omega(\mathcal{U})''$, whose existence and uniqueness are stated in Proposition 5.7. Then, $\tilde{\omega}$ is a $(\mathbf{\Lambda}^\omega, \beta)$-KMS state.*

Proof. Fix all parameters of the theorem. Like in the proof of Proposition 5.7, consider first an arbitrary ergodic generalized equilibrium state $\hat{\omega} \in \mathcal{E}(E_1) \cap \Omega_{\mathfrak{m}}$. By Equation (84) combined with Theorems 3.1 and 4.7, $\hat{\omega} \in K_{\Phi_{\mathfrak{m}}(\hat{\omega}(\mathfrak{e}_{(\cdot)}))}$. Recall that the strong convergence of a net of bounded operators implies the boundedness of this net (by the Banach-Steinhaus uniform boundedness principle), as well as its σ-weak convergence. Thus, from the strong density of $\pi_{\hat{\omega}}(\mathcal{U})$ in $\pi_{\hat{\omega}}(\mathcal{U})''$, by Lebesgue's dominated convergence

[13] In fact, we say here that the (generalized equilibrium) state ω, which is a state on the C^*-algebra $\mathcal{U}$, has an extension to $\pi_\omega(\mathcal{U})''$, because, by Corollary 4.8, the representation π_ω is faithful and the C^*-algebras $\mathcal{U}$ and $\pi_\omega(\mathcal{U})$ can thus be canonically identified with each other.

theorem and the σ-weak continuity of $\mathbf{\Lambda}^{\hat{\omega}}$ (Proposition 5.7), it follows that the normal extension of $\hat{\omega} \in \mathcal{E}(E_1) \cap \mathit{\Omega}_{\mathfrak{m}}$ to $\pi_{\hat{\omega}}(\mathcal{U})''$ is a $(\mathbf{\Lambda}^{\hat{\omega}}, \beta)$-KMS state. To study general (possibly non-ergodic) generalized equilibrium states, one applies again the theory of direct integrals of measurable families of Hilbert spaces, operators, von Neumann algebras and C^*-algebra representations, as in [12, Sections 5-6] and in the proof of Proposition 5.7: The ergodic decomposition μ_ρ of any translation-invariant state $\rho \in E_1$ is an orthogonal measure (Theorem 2.1) and, thanks to the Effros theorem (see, e.g., [12, Corollary 5.14]), for any $\rho \in E_1$, the direct integral

(93)

$$\left(\mathcal{H}_\rho^\oplus \equiv \int_{\mathcal{E}(E_1)} \mathcal{H}_{\hat{\rho}} \mu_\rho(\mathrm{d}\hat{\rho}),\ \pi_\rho^\oplus \equiv \int_{\mathcal{E}(E_1)} \pi_{\hat{\rho}} \mu_\rho(\mathrm{d}\hat{\rho}),\ \Omega_\rho^\oplus \equiv \int_{\mathcal{E}(E_1)} \Omega_{\hat{\rho}} \mu_\rho(\mathrm{d}\hat{\rho})\right)$$

of the GNS representations $(\mathcal{H}_{\hat{\rho}}, \pi_{\hat{\rho}}, \Omega_{\hat{\rho}})$ of $\mathcal{U}$ associated with the extreme states $\hat{\rho} \in \mathcal{E}(E_1)$ is a cyclic representation of the C^*-algebra $\mathcal{U}$, associated with the state $\rho \in E_1$. Moreover, from [12, Equation (158)], one has the inclusion

$$\pi_\rho^\oplus (\mathcal{U})'' \subseteq \int_{E_1} \pi_{\hat{\rho}} (\mathcal{U})'' \mu_\rho(\mathrm{d}\hat{\rho}) , \qquad \rho \in E_1 .$$

In particular, any element $A \in \pi_\rho(\mathcal{U})''$ can be identified with an element $(A_{\hat{\rho}})_{\hat{\rho} \in E_1}$ of

$$\int_{E_1} \pi_{\hat{\rho}} (\mathcal{U})'' \mu_\rho(\mathrm{d}\hat{\rho}) .$$

Fix now $\omega \in \mathit{\Omega}_{\mathfrak{m}} \subseteq E_1$. From Lemma 4.9 and Proposition 5.7,

$$\mathbf{\Lambda}_t^\omega(A) = \int_{\mathcal{E}(E_1)\cap \mathit{\Omega}_{\mathfrak{m}}} \mathbf{\Lambda}_t^{\hat{\omega}}(A_{\hat{\omega}}) \mu_\omega(\mathrm{d}\hat{\omega}) , \qquad A \in \pi_\omega(\mathcal{U})'',\ t \in \mathbb{R} .$$

Note that this identity follows from the uniqueness of the σ-weakly continuous group $(\mathbf{\Lambda}_t^\omega)_{t\in\mathbb{R}}$ stated in Proposition 5.7 together with general properties of direct integrals that can be found, for instance, in [12, Section 6]. Since in each fiber the normal extension of $\hat{\omega} \in \mathcal{E}(E_1) \cap \mathit{\Omega}_{\mathfrak{m}}$ to $\pi_{\hat{\omega}}(\mathcal{U})''$ is a $(\mathbf{\Lambda}^{\hat{\omega}}, \beta)$-KMS state, again denoted by $\hat{\omega}$ to simplify the notation, it follows from Lemma 4.9 and Fubini's theorem that, for all $A, B \in \pi_\omega(\mathcal{U})''$ and any function f being the (holomorphic) Fourier transform of a smooth function with compact support,

$$\begin{aligned}
&\int_{\mathbb{R}} f\left(t-i\beta\right)\tilde{\omega}(A\mathbf{\Lambda}_t^{\omega}(B))\mathrm{d}t \\
&\quad = \int_{\mathbb{R}} f\left(t-i\beta\right)\left(\int_{\mathcal{E}(E_1)\cap\Omega_{\mathfrak{m}}}\hat{\omega}(A_{\hat{\omega}}\mathbf{\Lambda}_t^{\hat{\omega}}(B_{\hat{\omega}}))\mu_{\omega}(\mathrm{d}\hat{\omega})\right)\mathrm{d}t \\
&\quad = \int_{\mathcal{E}(E_1)\cap\Omega_{\mathfrak{m}}}\left(\int_{\mathbb{R}} f\left(t-i\beta\right)\hat{\omega}(A_{\hat{\omega}}\mathbf{\Lambda}_t^{\hat{\omega}}(B_{\hat{\omega}}))\mathrm{d}t\right)\mu_{\omega}(\mathrm{d}\hat{\omega}) \\
&\quad = \int_{\mathcal{E}(E_1)\cap\Omega_{\mathfrak{m}}}\left(\int_{\mathbb{R}} f\left(t\right)\hat{\omega}(\mathbf{\Lambda}_t^{\hat{\omega}}(B_{\hat{\omega}})A_{\hat{\omega}})\mathrm{d}t\right)\mu_{\omega}(\mathrm{d}\hat{\omega}) \\
&\quad = \int_{\mathbb{R}} f\left(t\right)\left(\int_{\mathcal{E}(E_1)\cap\Omega_{\mathfrak{m}}}\hat{\omega}(\mathbf{\Lambda}_t^{\hat{\omega}}(B_{\hat{\omega}})A_{\hat{\omega}})\mu_{\omega}(\mathrm{d}\hat{\omega})\right)\mathrm{d}t \\
&\quad = \int_{\mathbb{R}} f\left(t\right)\tilde{\omega}(\mathbf{\Lambda}_t^{\omega}(B)A)\mathrm{d}t\ .
\end{aligned}$$

In other words, the normal extension $\tilde{\omega}$ of ω to $\pi_\omega(\mathcal{U})''$ is a $(\mathbf{\Lambda}^\omega, \beta)$-KMS state. □

Theorem 5.8 is an(other) extension of Theorem 3.1 to translation-invariant long-range models, which complements Corollary 5.5. These results pave the way to the use of the Tomita-Takesaki modular theory [15, Section 2.5] and the KMS theory [1, Sections 5.3-5.4] in the study of mean-field models, like the BCS model of (conventional) superconductivity. For instance, by Corollary 4.8, recall that, for $\mathfrak{m} \in \mathcal{M}_1$, any generalized equilibrium state $\omega \in \Omega_{\mathfrak{m}}$, with associated cyclic representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$, is a modular state, i.e., the vector $\Omega_\omega \in \mathcal{H}_\omega$ is cyclic and separating for the von Neumann algebra $\pi_\omega(\mathcal{U})'' \subseteq \mathcal{B}(\mathcal{H}_\omega)$. Therefore, the modular $*$-automorphism group associated with Ω_ω and the von Neumann algebra $\pi_\omega(\mathcal{U})''$ is well-defined. See, e.g.,[15, Section 2.5.2]. By Proposition 5.7 and Theorem 5.8, it is directly related to the limit long-range dynamics within the cyclic representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ of the corresponding generalized equilibrium state:

Corollary 5.9 (Long-range dynamics as a modular group).
Fix $\beta \in \mathbb{R}^+$ and a translation-invariant long-range model $\mathfrak{m} \in \mathcal{M}_1 \cap \mathcal{M}_0$ that is purely attractive or simple. Given $\omega \in \Omega_{\mathfrak{m}}$ with associated cyclic representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$, let $(\sigma_t^\omega)_{t\in\mathbb{R}}$ be the modular $*$-automorphism group associated with Ω_ω and the von Neumann algebra $\pi_\omega(\mathcal{U})''$. Then, with respect to

the σ-weak topology,

$$\lim_{L\to\infty} \pi_\omega \left(\tau_t^{(L,\mathfrak{m})}(A)\right) = \sigma^\omega_{-\beta^{-1}t}\left(\pi_\omega(A)\right) , \qquad A \in \mathcal{U},\ t \in \mathbb{R} .$$

Proof. Fix all parameters of the corollary. By Theorem 5.8, the unique normal extension $\tilde{\omega}$ of the generalized equilibrium state $\omega \in \Omega_{\mathfrak{m}}$ is a $(\mathbf{\Lambda}^\omega, \beta)$-KMS state. As a consequence, $\tilde{\omega}$ is also a $(\tilde{\mathbf{\Lambda}}^\omega, -1)$-KMS state with respect to the rescaled (σ-weakly continuous) group $\tilde{\mathbf{\Lambda}}^\omega \equiv (\mathbf{\Lambda}^\omega_{-\beta t})_{t\in\mathbb{R}}$ of $*$-automorphisms of $\pi_\omega(\mathcal{U})''$. From [1, Theorem 5.3.10], it follows that

$$\mathbf{\Lambda}^\omega_{-\beta t} = \sigma^\omega_t , \qquad t \in \mathbb{R} .$$

By combining this equality with Proposition 5.7 we arrive at the assertion. □

Acknowledgments: W. de Siqueira Pedra has been supported by CNPq (309723/2020-5) and FAPESP (2017/22340-9), and R. S. Yamaguti by CNPq (140782/2020-6). J.-B. Bru is supported by the Basque Government through the grant IT1615-22 and the BERC 2022-2025 program, by the COST Action CA18232 financed by the European Cooperation in Science and Technology (COST), and by the grant PID2020-112948GB-I00 funded by MCIN/AEI/10.13039/501100011033 and by "ERDF A way of making Europe".

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