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Renormalized volume of minimally bounded regions in asymptotically hyperbolic Einstein spaces

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We define a renormalized volume for a region in an asymptotically hyperbolic Einstein manifold that is bounded by a Graham-Witten minimal surface and the conformal infinity. We prove a Gauss-Bonnet theorem for the renormalized volume, and compute its derivative under variations of the minimal hypersurface.

1. Introduction

The renormalized volume of an even-dimensional asymptotically hyperbolic Einstein (AHE) manifold (X^{n+1}, g_+) is among its most important global invariants. Introduced in [17] (see also [13]), it is defined by taking the order-zero term in the expansion in ε of the quantity $\mathrm{vol}_{g_+}(\{r > \varepsilon\})$, where

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r is a so-called geodesic defining function for the boundary at infinity, M^n. There are many such defining functions, and the essential property of the renormalized volume V_+ is that it does not depend on which one is chosen. (This is generally not true if X is odd-dimensional.)

One of the basic theorems regarding renormalized volume in dimension four is Anderson's Gauss-Bonnet theorem ([2], see also [6]), which states that

$$4\pi^2\chi(X^4) = 3V_+ + \frac{1}{8}\int_X |W_{g_+}|^2_{g_+} dv_{g_+}. \tag{1}$$

Here W_{g_+} is the Weyl tensor of g_+; since $|W_{g_+}|^2_{g_+}$ is a pointwise conformal invariant of weight -4, the integral is guaranteed to converge notwithstanding the infinite volume of (X, g_+). Anderson used (1) to compute the variation of V_+ with respect to changes in g_+.

In this paper, we establish analogous results for half of an AHE manifold that has been partitioned into two by a minimal surface. Specifically, suppose (X^4, g_+) is an AHE manifold with conformal infinity $(M^3, [\bar{h}])$, and suppose further that $Y^3 \subset X$ is a minimal hypersurface that intersects M in a closed manifold $\Sigma^2 = M \cap Y$; we further assume that Y divides X into two parts, X^+ and X^-, whose intersection is precisely Y (the assignment of $+$ is arbitrary). Such a setting has been much studied in the literature on AHE manifolds, beginning with [16], which defined the *renormalized area* of Y in analogy to the renormalized volume of X; it has also been and remains a setting of much interest in the physics literature, particularly in the context of the AdS/CFT correspondence.

We will be concerned, not with the renormalized area of Y, but with the renormalized *volume* V_+^+ of X^+, which we may define as the constant term in the expansion $\mathrm{vol}_{g_+}(\{x \in X^+ : r(x) > \varepsilon\})$, with r a geodesic defining function. It is not immediately obvious that this quantity is independent of the choice of r: the proof in the global case depends strongly on the product decomposition $[0, \delta)_r \times M$ of a collar neighborhood of M in X, but generically there is no such decomposition of a collar neighborhood of $M^+ = M \cap X^+$ in X^+. One could prove using rather more elaborate versions of the arguments of [13] that V_+^+ is invariant in this context, but our interest is in a Gauss-Bonnet formula, and so we approach the result by a somewhat different path, as described below.

We note that renormalized volume of regions in AH spaces divided in two by hypersurfaces was considered in [12] using quite different techniques. The authors showed that a volume could be defined in quite general circumstances – in particular, not assuming the Einstein or minimality conditions

– but did not show that it is well-defined independent of all choices in the four-dimensional Einstein case.

Let $N \subset \mathring{X}$ be any hypersurface, and let $h = g_+|_{TN}$ be the induced metric on N. Define an extrinsic curvature quantity $\mathcal{C}_N$ on N by the formula

$$\mathcal{C}_N = \frac{1}{2}\mathring{L}_N^{\alpha\beta} R^{g_+}_{\alpha\beta} - \mathring{L}_N^{\alpha\beta} R^h_{\alpha\beta} + \frac{1}{3} H_N |\mathring{L}_N|^2_h - \frac{1}{3}\operatorname{tr}_h \mathring{L}_N^3.$$

Here L_N is the second fundamental form of N and $\mathring{L}_N$ its tracefree part, while $H_N = h^{\alpha\beta} L_{\alpha\beta}$ is its mean curvature. The curvature terms appearing are the Ricci tensors of the respective metrics, and α, β are indices on TN. It is easy to show (and will be shown within) that $\mathcal{C}_N$ is a pointwise conformal invariant of weight -3; indeed, in the notation of [5], $\mathcal{C}_N = -\frac{1}{2}\mathcal{L}_4 - \frac{1}{3}\mathcal{L}_5$.

The first main result of this paper is the following.

Theorem 1.1. *Let (X^4, g_+) be an asymptotically hyperbolic space satisfying the Einstein condition $\operatorname{Ric}(g_+) = -3g_+$, with conformal infinity $(M^3, [\bar{h}])$. Let Y^3 be a complete minimal hypersurface dividing X into two pieces X^+ and X^- such that $X^+ \cap X^- = Y$ and such that $Y \cap M = \Sigma^2 \neq \emptyset$. Let r be a fixed geodesic defining function for M, and let V_+^+ be the constant term in the expansion*

$$\operatorname{vol}_{g_+}\left(\left\{x \in X^+ : r(x) > \varepsilon\right\}\right) = c_0\varepsilon^{-3} + c_2\varepsilon^{-1} + V_+^+ + o(1).$$

Let $\tilde{h} = g_+|_{TY}$. Then

$$\pi^2(4\chi(X^+) - \chi(\Sigma^2)) = 3V_+^+ + \frac{1}{8}\int_{\mathring{X}^+} |W_{g_+}|^2_{g_+} dv_{g_+} + \int_{\mathring{Y}} \mathcal{C}_Y dv_{\tilde{h}}. \tag{2}$$

One then immediately obtains

Corollary 1.2. *The renormalized volume V_+^+ is independent of the choice of geodesic defining function r, and it satisfies (2).*

A natural question about the newly defined renormalized volume is how it changes if Y is varied through minimal surfaces in X. The second main result of the paper is as follows.

Theorem 1.3. *Let $X, M, Y, \Sigma, X^+, g_+$, $\bar{h}$, and V_+^+ be as in Theorem 1.1. Suppose that $\mathcal{F} : (-\varepsilon, \varepsilon)_t \times Y \to X$ is a C^3 variation of Y through minimal surfaces in X, so that $\mathcal{F}(t, \Sigma) \subset M$ for all t. Let $\widetilde{\mathcal{F}} = \mathcal{F}|_{(-\varepsilon,\varepsilon)\times\Sigma}$. Define $\tilde{f} \in C^\infty(\Sigma)$ by $\tilde{f} = \left\langle \frac{d}{dt}\big|_{t=0} \widetilde{\mathcal{F}}, \bar{\nu}_M \right\rangle$, where $\bar{\nu}_M$ is the inward-pointing* *normal vector to* Σ *in* M^+ *with respect to* $\bar{h}$*. Define* $f \in C^\infty(\mathring{Y})$ *by* $f = \left\langle \frac{d}{dt}\big|_{t=0} \mathcal{F}, \mu_Y \right\rangle_{g_+}$*, where* μ_Y *is the* (X^+, g_+)*-inward unit normal vector along* Y*. Let* r *be a geodesic defining function near* M*. Then*

$$\frac{d}{dt}\bigg|_{t=0} V_+^+ = \frac{1}{2} \oint_\Sigma \tilde{f} g^{(3)}(\bar{\nu}_M, \bar{\nu}_M) dv_{\bar{k}} + \frac{1}{3} f.p. \int_{\mathring{Y}} f |\mathring{L}_Y|^2_{\tilde{h}} dv_{\tilde{h}},$$

where $\bar{k} = \bar{h}|_{T\Sigma}$*,* $\tilde{h} = g_+|_{TY}$*,* $g^{(3)}$ *is the nonlocal term in the expansion in* r *of* g_+*, and* $f.p. \int_{\mathring{Y}} f|\mathring{L}_Y|^2_{\tilde{h}} dv_{\tilde{h}}$ *denotes the zeroth-order part, in* ε*, of* $\int_{Y \cap \{r > \varepsilon\}} f |\mathring{L}_Y|^2_{\tilde{h}} dv_{\tilde{h}}$*.*

For more about the nonlocal term $g^{(3)}$, see (5) and the surrounding discussion. We show in Lemma 4.3 that the finite part of the integral over Y can be written as the convergent integral of a rather more complicated expression.

The above theorem is stated for variations of Y through minimal surfaces, whose existence in general we do not assert. However, one can broaden the definition of V_+^+ to any dividing hypersurface by using (2). In that case, Theorem 1.3 remains valid for any variation of Y that preserves minimality to first order; see section 4, where we also explain why C^3-regularity of such a variation is in general optimal.

In considering the existence problem for the variation of Y, the required boundary data would be the induced variation of Σ, so another natural question is whether the derivative $\dot{V}_+^+$ only depends on the induced normal variation $\tilde{f}$. For example, suppose there are two variations of Y through minimal surfaces that induce the same variation of Σ; do the derivatives of V_+^+ with respect to these variations agree? The answer is yes, at least if $|\mathring{L}_Y|^2_{\tilde{h}} \leq 3$ everywhere; see Lemma 4.1.

These theorems may be interpreted physically within the AdS/CFT correspondence of high-energy and condensed matter physics. To do so, we assume that $(M^3, [\bar{h}])$ is a spacelike slice within a static four-dimensional conformal field theory Ω; and that (X^4, g_+) is an Einstein spacelike slice within a static asymptotically anti-de-Sitter Einstein five-dimensional spacetime Z with conformal infinity Ω. The surface Σ is then known as an entangling surface between M^+ and M^-, and Y is the so-called Ryu-Takayanagi surface extending Σ. According to the "volume = complexity" conjecture ([3, 4, 9, 18, 22]), then, V_+^+ encodes the algorithmic complexity of the quantum state of M^+. The above theorems can then be interpreted as giving formulae for this complexity and for its derivative as the entangling surface

Σ is varied continuously, so long as Y also varies continuously. (As demonstrated in [4], the latter will not always be the case.)

The assumption that X and its five-dimensional ambient Lorentzian manifold Z are both Einstein, of course, is rather restrictive. In general physical situations, one might expect that the Ricci tensor of X includes some extrinsic terms. But even if so, these would have well-defined asymptotics due to the asymptotically AdS condition on Z, and it would be straightforward, if tedious, to carry out our calculation the same way in that context.

In section 2, we introduce our setting and notation. In section 3, we prove Theorem 1.1; and in section 4, we prove Theorem 1.3.

2. Setting and notation

Recall that an asymptotically hyperbolic (AH) manifold is a compact manifold X^{n+1} with boundary M^n, equipped on the interior $\mathring{X}$ with a metric g_+ such that, for any defining function φ for M, the metric $\bar{g} = \varphi^2 g_+$ extends to a Riemannian metric on $X = \overline{X}$; and such that, in addition, $|d\varphi|_{\bar{g}} = 1$ along M. The optimal regularity of $\bar{g}$ is in general a delicate question, but in the context of this paper (i.e., X is four dimensional) by a result of Chruściel-Delay-Lee-Skinner [8] we may assume that there is a compactification such that $\bar{g}$ is smooth up to the boundary. The canonical example of an AH manifold is hyperbolic space itself, where X is the unit ball $\mathbb{B}^{n+1}$, and the metric is $g_H = \frac{4|dx|^2}{(1-|x|^2)^2}$. Given an AH metric, the metric $\bar{h} = \bar{g}|_{TM}$ is a metric on M, but is not well defined since the choice of φ is arbitrary. However, the conformal class $[\bar{h}]$ is well defined, and is called the *conformal infinity.*

A defining function r for M is called geodesic if $|dr|_{r^2 g_+} = 1$ on a neighborhood of M. Such a function induces a diffeomorphism

$$\psi : [0, \varepsilon)_r \times M \hookrightarrow X \tag{3}$$

onto a neighborhood of M in X such that

$$\psi^* g_+ = \frac{dr^2 + \bar{h}_r}{r^2}, \tag{4}$$

where $\bar{h}_r$ is a one-parameter family of metrics on M. A lemma of Graham-Lee ([15]) states that geodesic defining functions are in one-to-one correspondence with the representatives $\bar{h}$ of $[\bar{h}]$, according to the correspondence $\bar{h}_0 = \bar{h}$. The form (4) is called the geodesic normal form corresponding to $\bar{h} = \bar{h}_0$. We may assume that any geodesic compactification of X is smooth ([8]).

An AH metric is called Einstein (or AHE) if it satisfies as well the condition $\mathrm{Ric}(g) + ng = 0$. We will be concerned exclusively with four-dimensional AHE spaces, i.e. the case $n = 3$. In this case, it is known ([10, 11, 13]) that in geodesic normal form, $\bar{h}_r$ has the expansion

$$\bar{h}_r = \bar{h} - r^2 P^{\bar{h}} + r^3 g^{(3)} + O(r^4), \tag{5}$$

where $\mathrm{tr}_{\bar{h}}\, g^{(3)} = 0$ and where $P^{\bar{h}}$ is the Schouten tensor of $\bar{h}$, given by

$$P^{\bar{h}}_{\mu\nu} = R^{\bar{h}}_{\mu\nu} - \frac{1}{4} R_{\bar{h}} \bar{h}_{\mu\nu}. \tag{6}$$

Apart from the trace condition, the tensor $g^{(3)}$ is not locally determined by the geometry of $(M^3, \bar{h})$.

The renormalized volume of (X, g_+) is defined as follows ([13, 17]). Choose a metric $\bar{h} \in [\bar{h}]$, and let r be the corresponding geodesic defining function. Then the set $\{r > \varepsilon\}$ has volume

$$\mathrm{vol}_{g_+}(\{r > \varepsilon\}) = c_0 \varepsilon^{-3} + c_2 \varepsilon^{-1} + V_+ + o(1). \tag{7}$$

The renormalized volume is V_+, and it is independent of the choice of $\bar{h}$ (that is, of r).

In our setting of interest, there exists as well an orientable minimal surface $Y^3 \subset X$, intersecting M transversely in a closed two-manifold $\Sigma^2 = Y \cap M$, and dividing X into two connected pieces X^+ and X^- such that $Y = X^+ \cap X^-$. We write $M^+ = X^+ \cap M$ and $M^- = X^- \cap M$, so that $\Sigma = M^+ \cap M^-$. The assignment of the signs $+$ and $-$ is arbitrary, and corresponds to a choice of unit normal vector field on Y.

We now introduce the notations we will use. We let (X^4, M^3, g_+) be an AHE space, and $Y^3 \subset X$ a minimal surface as above. We will let $[\bar{h}]$ be the conformal infinity, and corresponding to the metric $\bar{h}$ will be the geodesic defining function r. The compactified metric is $\bar{g} = r^2 g_+$. Furthermore, X^+, M^+, and Σ^2 will be as above. For $\varepsilon > 0$, we let $X_\varepsilon = \{r > \varepsilon\}$, with $X^+_\varepsilon = X^+ \cap X_\varepsilon$. We set $Y_\varepsilon = \overline{Y \cap X_\varepsilon}$ and $M_\varepsilon = \{r = \varepsilon\}$. Similarly we set $M^+_\varepsilon = X^+ \cap M_\varepsilon$. Finally, $\Sigma_\varepsilon = Y \cap M^+_\varepsilon$.

Next, there are a number of metrics to name. We let $h_\varepsilon = g_+|_{TM_\varepsilon}$, while $\bar{h}_\varepsilon = \varepsilon^2 h_\varepsilon = \bar{g}|_{TM_\varepsilon}$. We let $\tilde{h} = g_+|_{TY}$, while $\bar{\tilde{h}} = r^2 \tilde{h} = \bar{g}|_{TY}$. We let $\bar{k} = \bar{g}|_{T\Sigma}$, while $k_\varepsilon = g_+|_{T\Sigma_\varepsilon}$ and $\bar{k}_\varepsilon = r^2 k_\varepsilon = \varepsilon^2 k_\varepsilon$. The decorations of ε will sometimes change position as needed; for example, we will write $h^\varepsilon_{\mu\nu}$, but $h^{\mu\nu}_\varepsilon$.

Now, near $\Sigma \subset M$, we can uniquely solve the eikonal equation and find $w \in C^\infty(M)$ such that $|dw|^2_{\bar{h}} = 1$ near Σ, $w|_\Sigma = 0$, and $w \geq 0$ on M^+. The metric $\bar{h}$ then takes the form $\bar{h} = dw^2 + \bar{k}_w$, with $\bar{k}_w$ a one-parameter family of metrics on Σ. Near any point $p \in \Sigma$, we can choose coordinates x^1, x^2 on a neighborhood of p in Σ; then by the flow of $\mathrm{grad}_{\bar{h}}\, w$ on M^+, the system $(x^1, x^2, x^3 = w)$ extends to a coordinate system on a neighborhood of p in M. Finally, by the flow of $\mathrm{grad}_{\bar{g}}\, r$, the system $(r = x^0, x^1, x^2, x^3 = w)$ extends to a coordinate system on a neighborhood of p in X. Now, we will regard Y as given by a function

$$w = u(r, x^1, x^2), \tag{8}$$

where $u(0, x^1, x^2) \equiv 0$. This is the same convention as in [16]. In fact, we may regard a neighborhood of Σ in this way as a product $[0, \varepsilon)_r \times \Sigma \times (-\varepsilon, \varepsilon)_w$; when using this product identification, we will use ζ to refer to a point of Σ, so that a generic point may be written (r, ζ, w).

When using index notation locally, we will let $0 \leq i, j \leq 3$ be indices on TX; $1 \leq \mu, \nu \leq 3$ be indices on TM; and $1 \leq a, b \leq 2$ be indices on $T\Sigma$. We also let $0 \leq \alpha, \beta \leq 2$, which we will use when discussing TY.

Turning to extrinsic geometry, we let $\bar{\mu}_{M_\varepsilon}, \bar{\mu}_Y$ be the X^+-inward unit $\bar{g}$-normal to the given hypersurface; the unbarred versions will refer to the unit normal with respect to g_+. We let $\bar{\nu}_{M_\varepsilon}$ be the $\bar{g}$-unit normal to Σ_ε that is directed into M^+, and $\bar{\nu}_{Y_\varepsilon}$, similarly, the Y_ε-inward $\bar{g}$-unit normal to Σ_ε. We let $\overline{L}_{M_\varepsilon}, \overline{L}_Y$ be the second fundamental forms of the indicated hypersurfaces with respect to the inward unit normals $\bar{\mu}_{M_\varepsilon}$ and $\bar{\mu}_Y$, and computed with respect to $\bar{g}$. Thus, for example,

$$\overline{L}_Y(A, B) = -\langle \nabla^{\bar{g}}_A \bar{\mu}_Y, B \rangle.$$

The tracefree parts are denoted $\mathring{\overline{L}}_{M_\varepsilon}$, etc. In all of these, we will sometimes write the hypersurface in the upper position, should it be convenient to do so to place covariant indices; similarly, an unbarred L will refer to the second fundamental form with respect to g_+ instead of $\bar{g}$. We let $\overline{H}_{M_\varepsilon} = \bar{h}_\varepsilon^{\mu\nu} \overline{L}^{M_\varepsilon}_{\mu\nu}$ be the mean curvature of M_ε with respect to $\bar{g}$ (or, if we omit the ε, that of M); similarly for $\overline{H}_Y$, while H_{M_ε} and H_Y are the same quantities with respect to g_+ (recall we assume $H_Y \equiv 0$). We let $\overline{II}_{Y_\varepsilon}$ be the second fundamental form of Σ_ε viewed as a hypersurface of Y_ε with respect to $\bar{\bar{h}}$, while $\overline{II}_{M_\varepsilon}$ is the same for Σ_ε viewed as a hypersurface in M_ε with respect to $\bar{h}_\varepsilon$. The traces of these (i.e., the mean curvatures of Σ_ε viewed as a hypersurface of the respective three-manifold) we denote $\bar{\eta}_{Y_\varepsilon}, \bar{\eta}_{M_\varepsilon}$. Again, the unbarred versions are with respect to the unbarred metrics $\tilde{h}$ and h_ε. We also let $\bar{\eta}_M$ be the mean curvature of $(\Sigma, \bar{k}) \subset (M, \bar{h})$.

We define a smooth function $\theta_0^\varepsilon \in C^\infty(\Sigma_\varepsilon)$ to be the angle, at each point, between Y and M_ε; that is, $\cos(\theta_0^\varepsilon) = -\langle \bar\mu_Y, \bar\mu_{M_\varepsilon}\rangle$. If the ε is omitted, then it denotes the angle between M and Y at a point of Σ. Since θ_0^ε is manifestly a conformal invariant, we do not distinguish between barred and unbarred versions.

Our curvature convention is such that the Ricci tensor is given by $R_{ij} = R^k{}_{ikj}$.

If A is a vector or tensor field, we write $A = O_{\bar g}(\varphi)$, for φ a function, whenever $|A|_{\bar g} = O(\varphi)$.

3. The Gauss-Bonnet formula

We now prove Theorem 1.1. We do so by using a form of the Gauss-Bonnet formula that has good conformal invariance properties, which allows us to compute using $\bar g$ instead of g_+.

Proof of Theorem 1.1. Let (X, M, g_+) be an AHE space with conformal infinity $[\bar h]$, and let Y be as in the previous section. Let $\bar h \in [\bar h]$, and let r be the corresponding geodesic defining function. Let $\varepsilon > 0$. Then $\overline{X_\varepsilon^+}$ is a four-manifold with codimension-two corner Σ_ε, and boundary hypersurfaces M_ε^+ and Y_ε (see section 2 for all notation). The Gauss-Bonnet theorem for Riemannian manifolds with corners (in this case X_ε^+), proven first in [1] (and see [7]), can be rewritten in the following conformally useful way ([21], building on [5]).

$$
\begin{aligned}
4\pi^2\chi(X_\varepsilon^+) &= \int_{X_\varepsilon^+} \left(\frac{1}{8}|W_{g_+}|_{g_+}^2 + \frac{1}{2}Q_{g_+}\right) dv_{g_+} + \int_{Y_\varepsilon} (\mathcal{L}_Y + T_Y)\, dv_{\tilde h} \\
&\quad + \int_{M_\varepsilon^+} (\mathcal{L}_{M_\varepsilon} + T_{M_\varepsilon})\, dv_h + \oint_{\Sigma_\varepsilon} (U_{\Sigma_\varepsilon} + G_{\Sigma_\varepsilon})\, dv_{k_\varepsilon}.
\end{aligned}
\tag{9}
$$

Here, W_{g_+} is the Weyl tensor of g_+, and the norm in question is its two-tensor norm $W^{ijkl}W_{ijkl}$. Meanwhile, Q_{g_+} is the Q-curvature of g_+, defined for any metric g by

$$
Q_g = -\frac{1}{6}\Delta_g R_g + \frac{1}{6}R_g^2 - \frac{1}{2}R_g^{ij}R_{ij}^g.
$$

Here, the Laplacian is a negative operator and the curvatures are respectively the scalar and Ricci curvatures of g. For any metric g, the quantity

$|W_g|_g^2 dv_g$ is a pointwise conformal invariant of weight zero. Under a conformal transformation $\tilde{g} = e^{2\omega} g$, the Q curvature transforms as

$$e^{4\omega} Q_{\tilde{g}} = Q_g + P_4^g \omega,$$

where P_4^g is the *Paneitz operator* associated to g; we will not use the Paneitz operator and so omit it here.

We give the definition of $\mathcal{L}_N$ and T_N, due to [5], for an arbitrary boundary hypersurface (N^3, h) embedded in a four-manifold endowed with metric g. The definition is

$$\mathcal{L}_N = \mathring{L}_N^{\mu\nu} R^g_{\mu\nu} - 2\mathring{L}_N^{\mu\nu} R^h_{\mu\nu} + \frac{2}{3} H_N |\mathring{L}_N|_h^2 - \operatorname{tr}_h \mathring{L}_N^3, \tag{10}$$

where L_N and H_N are the second fundamental form and the mean curvature as before, and μ, ν are indices on TN. Similarly, the T-curvature is defined by

$$\begin{aligned} T_N = & -\frac{1}{12}\mu(R_g) - \mathring{L}_N^{\mu\nu} R^g_{\mu\nu} + \mathring{L}_N^{\mu\nu} R^h_{\mu\nu} - \frac{1}{2} H_N |\mathring{L}_N|_h^2 + \frac{2}{3} \operatorname{tr}_h \mathring{L}_N^3 \\ & + \frac{1}{6} R_h H_N - \frac{1}{27} H_N^3 - \frac{1}{3}\Delta_h H_N, \end{aligned} \tag{11}$$

where μ is the inward-pointing unit normal to N. Under the conformal change $\tilde{g} = e^{2\omega} g$, this transforms according to the equation

$$e^{3\omega} \widetilde{T}_N = T_N + P_3^g \omega, \tag{12}$$

where $P_3^g : C^\infty(X) \to C^\infty(N)$ is the conformally covariant boundary operator

$$\begin{aligned} P_3^g f = & \frac{1}{2}\mu \Delta_g f + \Delta_h \mu(f) - \frac{1}{3} H_N \Delta_h f + \mathring{L}_N^{\mu\nu} \nabla^h_\mu \nabla^h_\nu f + \frac{1}{3} H_N^\mu f_\mu \\ & + \left(\frac{1}{6} R_g - \frac{1}{2} R_h - \frac{1}{2} |\mathring{L}_N|_h^2 + \frac{1}{3} H_N^2 \right) \mu(f). \end{aligned} \tag{13}$$

(We note that this formula differs from that in [21]; that paper and others in the literature contain misprints in the formula, which we have corrected by [5].)

Next we turn to the corner quantities. For a corner (Ξ, k) that forms the intersection between two boundary hypersurfaces N and S making angle

$\theta_0 \in C^\infty(\Xi)$, G is defined by

$$G_\Xi = \frac{1}{2}\cot(\theta_0)(|\mathring{II}_N|_k^2 + |\mathring{II}_S|_k^2) - \csc(\theta_0)\mathring{II}^N_{ab}\mathring{II}_S^{ab}, \tag{14}$$

where II, etc., are as in section 2. The G curvature is a pointwise conformal invariant of weight -2 (when the ambient metric on the four-manifold is changed conformally). Next, U_Ξ is defined by

$$\begin{aligned} U_\Xi = (\pi - \theta_0)K_\Xi &- \frac{1}{4}\cot(\theta_0)(\eta_N^2 + \eta_S^2) \\ &+ \frac{1}{2}\csc(\theta_0)\eta_N\eta_S - \frac{1}{3}(\nu_N H_N + \nu_S H_S). \end{aligned} \tag{15}$$

Here, K_Ξ is the Gaussian curvature of Ξ, and the other quantities are defined analogously to those in the previous section. Under a global conformal change $\tilde{g} = e^{2\omega}g$, U transforms according to the equation

$$e^{2\omega}\widetilde{U}_\Xi = U_\Xi + P_2^g\omega, \tag{16}$$

where $P_2^g : C^\infty(X) \to C^\infty(\Xi)$ is the conformally covariant operator

$$\begin{aligned} P_2^g f = (\theta_0 - \pi)\Delta_k f &+ \nu_N\mu_N f + \nu_S\mu_S f \\ &+ \cot(\theta_0)(\eta_N\nu_N f + \eta_S\nu_S f) - \csc(\theta_0)(\eta_S\nu_N f + \eta_N\nu_S f) \\ &+ \frac{1}{3}(H_N\nu_N f + H_S\nu_S f). \end{aligned} \tag{17}$$

We now analyze formula (9) in the context of our space (X_ε^+, g_+). Because $|W_{g_+}|^2_{g_+} dv_{g_+}$ is a pointwise conformal invariant of weight zero, its integral converges as $\varepsilon \to 0$ to $\int_{X^+} |W_{\bar{g}}|^2_{\bar{g}} dv_{\bar{g}}$, which in particular is finite.

In our setting, $R^{g_+}_{ij} = -3g^+_{ij}$ and $R_{g_+} \equiv -12$, so $\Delta_{g_+}R_{g_+} \equiv 0$ and $Q_{g_+} \equiv 6$. The integral of $\frac{1}{2}Q_{g_+}$ therefore is simply the integral of 3, so the second integral over X_+ becomes simply $3\,\mathrm{vol}_{g_+}(\{r > \varepsilon\} \cap X^+)$, which is the same quantity considered in (7), except that the latter is over all of X instead of X^+. To compute the contribution from this integral, we consider four different regions of X. First, let $r_0 > 0$ be small – sufficiently small, in particular, that the geodesic normal form (4) holds for $r < 2r_0$, and that the region $\mathcal{U} = \{r < 2r_0, -2r_0 < w < 2r_0\}$ has the decomposition $[0, 2r_0) \times \Sigma \times (-2r_0, 2r_0)$, with $|u(r,\zeta)| < \frac{1}{2}r_0$ on $\mathcal{U}$. Having chosen r_0, we will leave it fixed for all time.

The first region of interest to us is then $A = \{p \in X^+ : r(p) \geq r_0\}$. (This set does not depend on ε, which we assume is smaller than r_0.) Next, we want to capture the points near the boundary M_ε^+. The obvious set to consider is $B_\varepsilon = (\varepsilon, r_0) \times M^+$. The problem is that this may omit points that are contained in X^+ or include points contained in X^-, because Y is given not by $w = 0$ but by $w = u(r, \zeta)$, where u may be positive or negative away from $\{0\} \times \Sigma$. To address this, we need to add the volume of the omitted points, C_ε, and subtract the volume of the over-included points D_ε, viz.,

$$X_\epsilon^+ = (A \cup B_\epsilon \cup C_\epsilon) \setminus D_\epsilon.$$

To proceed, we analyze the volume form dv_{g_+}. First, at all points, we have $dv_{g_+} = r^{-4} dv_{\bar{g}}$. Near M, we can write

$$dv_{\bar{g}} = dv_{\bar{h}_r} dr$$

using the normal-form identification (3). Now in local coordinates (r, x^1, x^2, x^3) near M, we may write

$$dv_{\bar{h}_r} = \sqrt{\frac{\det(\bar{h}_r)}{\det(\bar{h})}} dv_{\bar{h}}.$$

As shown for example in [13], we have the expansion

$$\sqrt{\frac{\det(\bar{h}_r)}{\det(\bar{h})}} = 1 + v^{(2)} r^2 + v^{(4)} r^4 + O(r^5),$$

where $v^{(2)}, v^{(4)} \in C^\infty(M)$ are the so-called *renormalized volume coefficients.* Either by direct computation using (5) or by using equation (4.5) and the equation at the top of the same page of ([14]) (remembering that M is totally geodesic with respect to $\bar{g}$ and that the singular Yamabe metric for $\bar{g}$ is g_+), we may show that $v^{(2)} = -\frac{1}{8} R_{\bar{h}}$. Thus,

$$\begin{aligned} dv_{g_+} &= r^{-4} \left(1 - \frac{1}{8} r^2 R_{\bar{h}} + O(r^4)\right) dv_{\bar{h}} dr \\ &= \left(r^{-4} - \frac{1}{8} r^{-2} R_{\bar{h}} + O(1)\right) dv_{\bar{h}} dr. \end{aligned}$$

We next derive an expression for $dv_{\bar{g}}$ (and thus dv_{g_+}) near Σ. Since $\bar{h} = dw^2 + \bar{k}_w$ near Σ, we have

$$\begin{aligned} dv_{\bar{h}} &= \sqrt{\frac{\det(\bar{k}_w)}{\det(\bar{k})}} dv_{\bar{k}} dw \\ &= (1 + O(w)) dv_{\bar{k}} dw. \end{aligned}$$

Hence, near Σ, we have

$$dv_{g_+} = \left(r^{-4} - \frac{1}{8} r^{-2} R_{\bar{h}} + O(1) \right) (1 + O(w)) dv_{\bar{k}} dw dr.$$

We then have

$$\begin{aligned} \mathrm{vol}_{g_+}(X_\varepsilon^+) &= \mathrm{vol}_{g_+}(A) + \mathrm{vol}_{g_+}(B_\varepsilon) + \mathrm{vol}_{g_+}(C_\varepsilon) - \mathrm{vol}_{g_+}(D_\varepsilon) \\ &= \mathrm{vol}_{g_+}(A) + \int_{M^+} \int_\varepsilon^{r_0} \left(r^{-4} - \frac{1}{8} r^{-2} R_{\bar{h}} + O(1) \right) dr dv_{\bar{h}} \\ &\quad - \oint_\Sigma \int_\varepsilon^{r_0} \int_0^{u(r,\zeta)} \left(r^{-4} + O(r^{-2}) \right) (1 + O(w)) dw dr dv_{\bar{k}}(\zeta), \end{aligned} \tag{18}$$

where the last integral represents $\mathrm{vol}_{g_+}(C_\varepsilon) - \mathrm{vol}_{g_+}(D_\varepsilon)$. Now, by equations (2.13) and (2.14) in [16],

$$u(r, \zeta) = \frac{1}{4} r^2 \overline{\eta}_M(\zeta) + r^4 \log(r) v(\zeta) + O(r^4), \tag{19}$$

where $\overline{\eta}_M$ is the mean curvature of Σ viewed as a hypersurface of $(M, \bar{h})$ and $v \in C^\infty(\Sigma)$. Thus, we find

$$\begin{aligned} 3\,\mathrm{vol}_{g_+}(X_\varepsilon^+) &= 3\,\mathrm{vol}_{g_+}(A) + 3 \int_M \int_\varepsilon^{r_0} \left(r^{-4} - \frac{1}{8} r^{-2} R_{\bar{h}} + O(1) \right) dr dv_{\bar{h}} \\ &\quad - 3 \oint_\Sigma \int_\varepsilon^{r_0} \left(\frac{1}{4} r^{-2} \overline{\eta}_M + v \log(r) + O(1) \right) dr dv_{\bar{k}} \\ &= \varepsilon^{-3} \mathrm{vol}_{\bar{h}}(M^+) - \varepsilon^{-1} \left(\frac{3}{8} \int_{M^+} R_{\bar{h}} dv_{\bar{h}} + \frac{3}{4} \oint_\Sigma \overline{\eta}_M dv_{\bar{k}} \right) \\ &\quad + 3V_+^+ + o(1). \end{aligned} \tag{20}$$

Here V_+^+ is the collection of all the order-zero terms in ε in the volume expansion, and is defined to be the renormalized volume; of course, we have not shown so far that V_+^+ is independent of the choice of $\bar{h} \in [\bar{h}]$ (or equivalently, of r).

Since (as we saw above) $Q_{g_+} = 6$, the above right-hand side is thus the integral $\int_{X^+} \frac{1}{2} Q_{g_+} dv_{g_+}$. We next turn to the boundary integrals over Y_ε and M_ε, beginning with Y_ε. We will analyze $\mathcal{L}_Y$ and T_Y with respect to the metric g_+; of course, since $\mathcal{L}_Y$ is a pointwise conformal invariant, it is automatic that the integral of $\mathcal{L}_Y$ over Y_ε will converge as $\varepsilon \to 0$. Now, because g_+ is Einstein and Y is minimal in (X, g_+), the first and third terms in (10) vanish in this case. Thus, we get simply $\mathcal{L}_Y = -2\mathring{L}_Y^{\alpha\beta} R_{\alpha\beta}^{\tilde{h}} - \operatorname{tr}_{\tilde{h}} \mathring{L}_Y^3$.

Next turning to T_Y, we again compute with respect to the ambient metric g_+, i.e., with respect to the non-compactified setting. Again, due to the Einstein condition of g_+ and the minimal condition on Y, the first, second, fourth, sixth, seventh, and eighth terms of (11) vanish, so we get

$$\begin{aligned} T_Y &= \mathring{L}^{\alpha\beta} R_{\alpha\beta}^{\tilde{h}} + \frac{2}{3} \operatorname{tr}_{\tilde{h}} \mathring{L}_Y^3 \\ &= -\frac{1}{2}\mathcal{L}_Y + \frac{1}{6} \operatorname{tr}_{\tilde{h}} \mathring{L}_Y^3. \end{aligned}$$

Now, $\mathcal{L}_Y$ and $\operatorname{tr}_{\tilde{h}} \mathring{L}_Y^3$ are both pointwise conformal invariants of weight -3, so we have exhibited T_Y itself as such a pointwise conformal invariant. We define

$$\mathcal{C}_Y = \frac{1}{2}\mathcal{L}_Y + \frac{1}{6} \operatorname{tr}_{\tilde{h}} \mathring{L}_Y^3.$$

This is a pointwise conformal invariant, and the upshot of the above remarks is that

$$\int_{Y_\varepsilon} (\mathcal{L}_Y + T_Y)\, dv_{\tilde{h}} = \int_{Y_\varepsilon} \mathcal{C}_Y dv_{\tilde{h}} = \int_{\mathring{Y}} \mathcal{C}_Y dv_{\tilde{h}} + O(\varepsilon). \tag{21}$$

We now turn to the integral over M_ε^+ in (9). Here, we will compute $\overline{T}_{M_\varepsilon}$ and $\overline{\mathcal{L}}_{M_\varepsilon}$, the extrinsic curvature quantities with respect to the *compactified* metrics $\bar{g}$ and $\bar{h}_\varepsilon$; then we will compute the transformation to g_+, h_ε using equation (12), which in particular implies that

$$\int_{M_\varepsilon^+} (\mathcal{L}_M + T_M) dv_{g_+} = \int_{M_\varepsilon^+} (\overline{\mathcal{L}}_M + \overline{T}_M + P_3^{\bar{g}}(-\log r)) dv_{\bar{g}}.$$

Our goal is thus to compute the right-hand side of this equation. We begin by computing some basic quantities. Recalling that $\bar{g} = dr^2 + \bar{h}_r$ and $M_\varepsilon = \{r = \varepsilon\}$, we find that

$$\overline{L}_{M_\varepsilon} = -\frac{1}{2}\partial_r \bar{h}_r|_{r=\varepsilon} = \varepsilon P^{\bar{h}} + O(\varepsilon^2),$$

where $P^{\bar{h}}$ is the Schouten tensor of $\bar{h}$, and we have used (5). Thus,

$$\overline{H}_{M_\varepsilon} = \varepsilon (P_{\bar{h}})^\mu_\mu + O(\varepsilon^3) = \frac{1}{4}\varepsilon R_{\bar{h}} + O(\varepsilon^3). \tag{22}$$

The reason the error is $O(\varepsilon^3)$ is that the r^3 term in the expansion of $\bar{h}_r$ is trace-free. We also have

$$\mathring{\overline{L}}_{M_\varepsilon} = \varepsilon \mathring{P}^{\bar{h}} + O(\varepsilon^2).$$

We next wish to compute $R_{\bar{g}}$ on M_ε. To do this, we use the fact that $R_{g_+} \equiv -12$ and that $g_+ = r^{-2}\bar{g}$. Thus, we will use the conformal transformation formula for scalar curvature. Let $\omega = -\log(r)$. It will be useful to record that

$$\Delta_{\bar{g}}\omega = r^{-2} + \frac{1}{4}R_{\bar{h}} + O(r^2), \tag{23}$$

which follows easily from (5). Thus, from the conformal change formula, we find

$$\begin{aligned} -12 &= r^2(R_{\bar{g}} - 6\Delta_{\bar{g}}\omega - 6|d\omega|^2_{\bar{g}}) \\ &= r^2\left(R_{\bar{g}} - 6r^{-2} - \frac{3}{2}R_{\bar{h}} - 6r^{-2} + O(r^2)\right), \end{aligned}$$

whence

$$R_{\bar{g}} = \frac{3}{2}R_{\bar{h}} + O(r^2).$$

We next compute the tracefree tangential Ricci tensor $\mathring{R}^{\bar{g}}_{\mu\nu}$. We will use again the same technique of conformal transformation and the fact that $\mathrm{Ric}(g_+) = -3g_+$. We first find using (5) that

$$\nabla^{\bar{g}}_\mu \nabla^{\bar{g}}_\nu \omega = P^{\bar{h}}_{\mu\nu} + O(r).$$

It then follows from the equation

$$R^{g_+}_{\mu\nu} = R^{\bar{g}}_{\mu\nu} - 2\nabla^{\bar{g}}_\mu\nabla^{\bar{g}}_\nu\omega + 2\omega_\mu\omega_\nu - (\Delta_{\bar{g}}\omega - 2|d\omega|^2_{\bar{g}})\bar{g}_{\mu\nu}$$

that

$$\mathring{R}^{\bar{g}}_{\mu\nu} = 2\mathring{P}^{\bar{h}}_{\mu\nu} + O(r).$$

We are ready to analyze the curvature integrands on M_ε. First, we easily find using (10) and the above that

$$\overline{\mathcal{L}}_{M_\varepsilon} = O(\varepsilon),$$

where the first-order contribution is from the first two terms of (10), and the last two terms provide contributions of order $O(\varepsilon^3)$. Next, we compute $\overline{T}_{M_\varepsilon}$, recalling that $\bar{\mu}_{M_\varepsilon} = \frac{\partial}{\partial r}$. Then it again follows from the above computations that

$$\overline{T}_{M_\varepsilon} = O(\varepsilon).$$

The lowest-order contributions come once again from the first three terms of (11), as well as the sixth.

We next turn to computing $P_3^{\bar{g}}(\omega) = -P_3^{\bar{g}}(\log(r))$ for $P_3^{\bar{g}}$ associated to M_ε. First, observe that $\omega|_{M_\varepsilon} \equiv -\log(\varepsilon)$, and $\bar{\mu}_{M_\varepsilon}(\omega) \equiv \frac{1}{\varepsilon}$. Thus, all tangential derivatives of both quanties vanish, which means the second through fifth terms of (13) vanish. Thus, only the first and last remain. It follows from (23) that

$$\frac{1}{2}\bar{\mu}_{M_\varepsilon}\Delta_{\bar{g}}\omega = -\varepsilon^{-3} + O(\varepsilon).$$

Next, using again the facts that $R_{\bar{g}} = \frac{3}{2}R_{\bar{h}} + O(r^2)$ and our above calculations, we find that the last term of (13) simplifies to

$$\left(\frac{1}{6}R_{\bar{g}} - \frac{1}{2}R_{\bar{h}_\varepsilon} - \frac{1}{2}|\mathring{\overline{L}}_{M_\varepsilon}|^2_{\bar{h}_\varepsilon} + \frac{1}{3}\overline{H}^2_{M_\varepsilon}\right)\bar{\mu}(-\log(r)) = \frac{1}{4}\varepsilon^{-1} + O(\varepsilon).$$

Now, we wish to perform the integral over M_ε^+, not M_ε. Just as for the interior integral, the simplest approach will be first to compute the integral over $\{\varepsilon\} \times M^+$, and then subtract or add whatever was missed near the corner due to turning of Y away from Σ. First, we observe that from our above computations, it is clear that

$$\int_{M_\varepsilon^+} (\overline{T}_{M_\varepsilon} + \overline{\mathcal{L}}_{M_\varepsilon} + P_3^{\bar{g}}(-\log(r)))dv_{\bar{h}_\varepsilon} = \int_{M_\varepsilon^+} P_3^{\bar{g}}(-\log(r))dv_{\bar{h}_\varepsilon} + O(\varepsilon).$$

We may focus therefore only on contributions from $P_3^{\bar{g}}(-\log(r))$. We write

$$\int_{M_\varepsilon^+} P_3^{\bar{g}}(\omega)dv_{\bar{h}_\varepsilon} = \int_{\{\varepsilon\}\times M^+} P_3^{\bar{g}}(\omega)dv_{\bar{h}_\varepsilon} - \oint_\Sigma \int_0^{u(\varepsilon,\zeta)} P_3^{\bar{g}}(\omega)(1 + O(w))dw dv_{\bar{k}}(\zeta).$$

(Compare (18).) We compute the first term first. Recall that $dv_{\bar{h}_\varepsilon} = (1 - \frac{1}{8}\varepsilon^2 R_{\bar{h}} + O(\varepsilon^4))dv_{\bar{h}}$. Thus,

$$\int_{\{\varepsilon\}\times M^+} P_3^{\bar{g}}(\omega) = \int_{M^+} \left(-\varepsilon^{-3} + \frac{1}{4}\varepsilon^{-1}R_{\bar{h}} + O(\varepsilon)\right)\left(1 - \frac{1}{8}\varepsilon^2 R_{\bar{h}} + O(\varepsilon^4)\right) dv_{\bar{h}}$$
$$= -\varepsilon^{-3}\operatorname{vol}_{\bar{h}}(M^+) + \frac{3}{8}\varepsilon^{-1}\int_{M^+} R_{\bar{h}} dv_{\bar{h}} + O(\varepsilon).$$

As for the corner integral, we find using (19)

$$\oint_\Sigma \int_0^{u(\varepsilon,\zeta)} P_3^{\bar{g}}(\omega)(1+O(w))dw dv_{\bar{k}}(\zeta) = \oint_\Sigma \left(-\varepsilon^{-3} + O(\varepsilon^{-1})\right)\cdot$$
$$\cdot\left(\frac{1}{4}\varepsilon^2\overline{\eta}_M + O(\varepsilon^4\log(\varepsilon))\right) dv_{\bar{k}}$$
$$= -\frac{1}{4}\varepsilon^{-1}\oint_\Sigma \overline{\eta}_M dv_{\bar{k}} + O(\varepsilon\log\varepsilon).$$

Thus, we have found that

$$\begin{aligned} \int_{M_\varepsilon^+} (T_M + \mathcal{L}_M) dv_{g_+} = &- \varepsilon^{-3}\operatorname{vol}_{\bar{h}}(M^+) \\ &+ \varepsilon^{-1}\left(\frac{3}{8}\int_{M^+} R_{\bar{h}} dv_{\bar{h}} + \frac{1}{4}\oint_\Sigma \overline{\eta}_M dv_{\bar{k}}\right) + o(1). \end{aligned} \tag{24}$$

We are finally ready to evaluate the corner terms U_{Σ_ε} and G_{Σ_ε} in (9). Just as for M_ε^+, our strategy will be to evaluate first with respect to $\bar{g}$, and then use the conformal transformation formula (16) and the pointwise conformal invariance of G. Thus, we will find

$$\oint_{\Sigma_\varepsilon} (G_k + U_k) dv_k = \oint_{\Sigma_\varepsilon} (\overline{G}_{\Sigma_\varepsilon} + \overline{U}_{\Sigma_\varepsilon} + P_2^{\bar{g}}(-\log r)) dv_{\bar{k}_\varepsilon}.$$

To begin, we wish to estimate θ_0^ε, which enters the formulas for U, G, and P_2. To do this, we find normal vectors $\bar{\mu}_{M_\varepsilon}$ and $\bar{\mu}_Y$. The first is easy: $\bar{\mu}_{M_\varepsilon} = \frac{\partial}{\partial r}$. For the second, we observe that, for ε small, we can write Y as the zero level set of $F = w - u(r,\zeta)$ (where, again, $\zeta \in \Sigma$). Now,

$$\operatorname{grad}_{\bar{g}} F = (1+O(r^2))\frac{\partial}{\partial w} - \frac{\partial u}{\partial r}\frac{\partial}{\partial r} - \bar{k}^{ab}\frac{\partial u}{\partial x^a}\frac{\partial}{\partial x^b} + O^i(r^3\log(r))\partial_i$$
$$= (1+O(r^2))\frac{\partial}{\partial w} - \frac{1}{2}r\overline{\eta}_M\frac{\partial}{\partial r} - \frac{1}{4}r^2\bar{k}^{ab}\frac{\partial\overline{\eta}_M}{\partial x^a}\frac{\partial}{\partial x^b} + O_{\bar{g}}(r^3\log(r)).$$

Since $\left|\frac{\partial}{\partial w}\right|_{\bar{g}} = 1 + O(r^2)$, we have

$$|\operatorname{grad}_{\bar{g}} F|_{\bar{g}} = 1 + O(r^2).$$

Consequently,

$$\begin{aligned}\bar{\mu}_Y = \frac{\operatorname{grad}_{\bar{g}} F}{|\operatorname{grad}_{\bar{g}} F|_{\bar{g}}} &= (1 + O(r^2))\frac{\partial}{\partial w} \\ &\quad - \left(\frac{1}{2} r\overline{\eta}_M + O(r^3 \log(r))\right)\frac{\partial}{\partial r} + O_{\bar{g}}(r^2).\end{aligned} \tag{25}$$

Thus,

$$\cos(\theta_0^\varepsilon) = -\langle \bar{\mu}_{M_\varepsilon}, \bar{\mu}_Y\rangle = \frac{1}{2}\varepsilon\overline{\eta}_M + O(\varepsilon^3 \log(\varepsilon)).$$

Next we wish to estimate the second fundamental form $\overline{II}_{Y_\varepsilon}$ of Σ_ε viewed as a submanifold of Y_ε. To do this, we first want to know the inward-pointing unit normal vector $\bar{\nu}_{Y_\varepsilon}$ to Σ_ε in Y_ε. By inspection, we can see that

$$V = \frac{\partial}{\partial r} - \frac{\partial F}{\partial r}\frac{\operatorname{grad}_{\bar{g}} F}{|dF|^2_{\bar{g}}}$$

is normal to Σ_ε and tangent to Y_ε, so

$$\bar{\nu}_{Y_\varepsilon} = \frac{V}{|V|_{\bar{g}}} = (1 + O(\varepsilon^2))\frac{\partial}{\partial r} + \frac{1}{2}\varepsilon\overline{\eta}_M\frac{\partial}{\partial w} + O(\varepsilon^3 \log \varepsilon). \tag{26}$$

Now, a local frame for $T\Sigma_\varepsilon$ is given by $\{X_1, X_2\}$, where

$$X_a = \frac{\partial}{\partial x^a} - \frac{\partial F}{\partial x^a}\frac{\partial}{\partial w}.$$

Since $\nabla^{\bar{g}}_{\partial_a}\partial_r = O^i(\varepsilon)\partial_i$ (which is easy to check), we may conclude that $\langle \nabla^{\bar{g}}_{X_a}\bar{\nu}_{Y_\varepsilon}, X_b\rangle_{\bar{g}} = O(\varepsilon)$. Thus, by Weingarten's equation,

$$|\overline{II}_{Y_\varepsilon}|_{\bar{g}} = O(\varepsilon).$$

It now follows that $\overline{G}_{\Sigma_\varepsilon} = O(\varepsilon)$: the first term in (14) because $\cot(\theta_0^\varepsilon) = O(\varepsilon)$, and the second because of the estimate on $\overline{II}_{Y_\varepsilon}$.

We next turn to $\overline{U}_{\Sigma_\varepsilon}$. The second and third terms in (15) are $O(\varepsilon)$ for the same reason. Turning to the fourth term, $\bar{\nu}_M\overline{H}_{M_\varepsilon} = O(\varepsilon)$ by (22). To compute $\bar{\nu}_{Y_\varepsilon}\bar{H}_Y$, we first compute $\overline{H}_Y$ using the conformal change formula.

Recall that $H_Y \equiv 0$. Then again taking $\omega = -\log r$, we find from the conformal transformation formula $H_Y = e^{-\omega}(\overline{H}_Y - 3\bar{\mu}_Y(\omega))$ that

$$0 = r(\overline{H}_Y - \frac{3}{2}\overline{\eta}_M + O(r^2\log(r))),$$

whence

$$\overline{H}_Y = \frac{3}{2}\overline{\eta}_M + O(r^2\log(r)).$$

Thus, $\bar{\nu}_{Y_\varepsilon}\overline{H}_Y = O(\varepsilon\log(\varepsilon))$; so since $\theta_0^\varepsilon = \frac{\pi}{2} + O(\varepsilon)$, we have

$$\overline{U}_{\Sigma_\varepsilon} = \frac{\pi}{2}K_{\bar{k}} + O(\varepsilon\log\varepsilon).$$

Consequently,

$$\oint_{\Sigma_\varepsilon}(\overline{G}_{\Sigma_\varepsilon} + \overline{U}_{\Sigma_\varepsilon})dv_{\bar{k}_\varepsilon} = \pi^2\chi(\Sigma) + O(\varepsilon\log\varepsilon).$$

We still need to compute the integral of $P_2^{\bar{g}}(-\log r)$. First, still letting $\omega = -\log r$, observe that $\omega|_{M_\varepsilon} \equiv -\log\varepsilon$ and that $\bar{\mu}_M\omega \equiv -\frac{1}{\varepsilon}$. Thus, the first and second terms of (17) in $P_2^{\bar{g}}(\omega)$ vanish identically, as do the terms $\overline{\eta}_{M_\varepsilon}\bar{\nu}_{M_\varepsilon}\omega$, $\overline{\eta}_{Y_\varepsilon}\bar{\nu}_{M_\varepsilon}\omega$, and $\overline{H}_{M_\varepsilon}\bar{\nu}_{M_\varepsilon}\omega$.

Now, the third term takes the form

$$\begin{aligned}\bar{\nu}_{Y_\varepsilon}\bar{\mu}_Y\omega &= \bar{\nu}_{Y_\varepsilon}\left(\frac{1}{2}\overline{\eta}_M + O(r^2\log(r))\right)\\ &= \frac{1}{4}\varepsilon\overline{\eta}_M\partial_w\overline{\eta}_M + O(\varepsilon\log\varepsilon)\\ &= O(\varepsilon\log\varepsilon).\end{aligned}$$

Next, $\bar{\nu}_{Y_\varepsilon}\omega = -\frac{1}{\varepsilon} + O(\varepsilon)$, so $\cot(\theta_0^\varepsilon)\overline{\eta}_{Y_\varepsilon}\bar{\nu}_{Y_\varepsilon}\omega = O(\varepsilon)$. On the other hand, $-\csc(\theta_0^\varepsilon)\overline{\eta}_{M_\varepsilon}\bar{\nu}_{Y_\varepsilon}\omega = \varepsilon^{-1}\overline{\eta}_M + O(\varepsilon)$, since $\overline{\eta}_{M_\varepsilon} = \overline{\eta}_M + O(\varepsilon^2)$ and $\csc(\theta_0^\varepsilon) = 1 + O(\varepsilon^2)$.

Finally,

$$\frac{1}{3}\overline{H}_Y\bar{\nu}_{Y_\varepsilon}\omega = -\frac{1}{2}\varepsilon^{-1}\overline{\eta}_M + O(\varepsilon\log\varepsilon).$$

Adding together all these terms, we therefore find that $P_2^{\bar g}\omega = \frac{1}{2}\varepsilon^{-1}\overline{\eta}_M + O(\varepsilon \log \varepsilon)$. Thus,

$$\begin{aligned}\oint_{\Sigma_\varepsilon} \left(\overline{G}_\varepsilon + \overline{U}_\varepsilon + P_2^{\bar g}(-\log r)\right) dv_{\bar k_\varepsilon} \\ = \frac{1}{2}\varepsilon^{-1} \oint_{\Sigma} \overline{\eta}_M dv_{\bar k} + \pi^2 \chi(\Sigma) + O(\varepsilon \log \varepsilon).\end{aligned} \tag{27}$$

Combining (9), (20), (21), (24), and (27), we find

$$\pi^2(4\chi(X_\varepsilon^+) - \chi(\Sigma)) = 3V_+^+ + \frac{1}{8}\int_{X_\varepsilon^+} |W_{g_+}|^2_{g_+} dv_{g_+} + \int_{Y_\varepsilon} \mathcal{C}_Y dv_{\tilde h} + O(\varepsilon \log \varepsilon).$$

Letting $\varepsilon \to 0$ yields the result. □

4. Variation of renormalized volume

In this section we give a proof of Theorem 1.3. Since this will require extensive calculations we begin by establishing some new notational conventions.

In addition to using the coordinate system (r, x^1, x^2, w), it will be convenient to use the system $(x^{\tilde 0}, x^{\tilde 1}, x^{\tilde 2}, x^{\tilde 3}) = (r, x^1, x^2, w - u)$, where u is as in (8). We will still use $0 \le i, j \le 3$ to refer to coordinate fields on X, but will use $0 \le \tilde\alpha, \tilde\beta \le 2$ to refer to the coordinate fields tangent to Y. It will also be useful on the interior $\mathring{X}$ to let $x^{\hat n}$ be the g_+-distance to $\mathring{Y}$, so that $\frac{\partial}{\partial x^{\hat n}} = \mu_Y$ is the g_+-unit inward normal vector to $\mathring{Y}$. The system $(r, x^1, x^2, x^{\hat n})$ is clearly another coordinate system near $\mathring{Y}$, and the corresponding coordinate vector fields tangent to Y are the same.

As in the introduction, suppose $\mathcal{F} : (-\varepsilon, \varepsilon)_t \times Y \to X$ is a C^3 variation of Y through minimal surfaces in X such that $\mathcal{F}(t, \Sigma) \subset M$ for all t. For each $t \in (-\varepsilon, \varepsilon)$, $\mathcal{F}_t(Y) = Y^t$ splits X into two disjoint sets, X_t^+, X_t^- and we can make our choice of X_t^+ consistent by fixing a point $p \in X_0^+$ and requiring that $p \in X_t^+$ for t in a possibly smaller time interval $t \in (-\delta, \delta)$. Let $V_+^+(t) = V_+^+(X_t^+)$. We will also use the notation $V_+^+(\mathcal{F}_t(Y))$. Our goal is to use the formula (2) to compute a formula for the first variation, $\dot V_+^+$.

Before proceeding we recall that strictly speaking, the formula for V_+^+ given by (2) only holds for minimal Y. However, as we remarked in the introduction, one can use this formula to define V_+^+ for any dividing hypersurface, in particular for $Y^t = \mathcal{F}_t(Y)$, where $\mathcal{F}_t$ is a general variation of Y.

We begin by making two simplifying assumptions about the variation $\mathcal{F}$. First, we show that it suffices to consider normal variations of Y. We then weaken the assumption that $Y^t = \mathcal{F}_t(Y)$ is minimal for each t, and only assume that minimality is preserved infinitesimally. The latter assumption will suffice to establish the theorem.

To see why it suffices to consider normal variations, let $Z = \frac{d}{dt}\mathcal{F}_t\big|_{t=0}$ be the variation field of $\mathcal{F}$. Write $Z = Z^\perp + Z^\top$, with the two uniquely defined fields respectively normal and tangential to Y. Now, because $\mathcal{F}_t(\Sigma) \subset M$ for all t, along Σ we have $Z^\top \in TY \cap TM$, and it follows that $Z^\top$ is tangential to Σ along the boundary. Thus, by Theorem 9.34 of [20] and the fact that $\overline{Y}$ is compact, there exists a unique global flow $\mathcal{G} : \mathbb{R} \times Y \to Y$ such that $\frac{d}{dt}\mathcal{G}\big|_{t=0} = -Z^\top$. Define $\widehat{\mathcal{F}} : (-\varepsilon, \varepsilon) \times Y \to X$ by $\widehat{\mathcal{F}}(t, y) = \mathcal{F}(t, \mathcal{G}(t, y))$. By the chain rule, $\frac{d}{dt}\widehat{\mathcal{F}}_t\Big|_{t=0} = Z^\perp$. On the other hand, $\widehat{\mathcal{F}}_t(Y) = \mathcal{F}_t(Y)$ for all t, so it remains a flow through minimal surfaces, and the renormalized volume at each time t is identical. Thus, it suffices to compute the variation for (initially) normal variation fields, i.e., those satisfying

$$\frac{d}{dt}\mathcal{F}_t\Big|_{t=0} \perp TY.$$

As mentioned, we will also assume

$$\frac{d}{dt}H_{Y^t}\Big|_{t=0} = 0, \tag{28}$$

where H_{Y^t} is the mean curvature of Y^t viewed (via pullback by $\mathcal{F}_t$) as a function on Y.

Let $\mathcal{F} : (-\varepsilon, \varepsilon) \times Y \to X$, be a C^3 normal variation satisfying (28). As in the statement of Theorem 1.3, we let $f = \left\langle \mu_Y, \frac{d}{dt}\big|_{t=0}\mathcal{F}\right\rangle_{g_+}$, where μ_Y is the (X^+, g_+)-inward unit normal vector along Y. Since $\mathcal{F}$ is normal, we can write

$$\frac{d}{dt}\Big|_{t=0}\mathcal{F}_t = f\mu_Y. \tag{29}$$

Also, let $\widetilde{\mathcal{F}} = \mathcal{F}|_{(-\varepsilon,\varepsilon)\times\Sigma}$. Then $\tilde{\mathcal{F}}$ determines $\tilde{f} \in C^\infty(\Sigma)$ given by

$$\tilde{f} = \left\langle \frac{d}{dt}\Big|_{t=0}\widetilde{\mathcal{F}}, \bar{\nu}_M \right\rangle, \tag{30}$$

where $\bar{\nu}_M$ is the inward-pointing normal vector to Σ in M^+ with respect to $\bar{h}$.

From now on, to simplify notation we will let primes denote $\frac{d}{dt}|_{t=0}$. By the formulas (80), (87), and (88) in the appendix, the variations of the induced metric, second fundamental form, and mean curvature of Y are given by

$$\begin{aligned} \tilde{h}'_{\tilde{\alpha}\tilde{\beta}} &= -2fL_{\tilde{\alpha}\tilde{\beta}}, \\ L'_{\tilde{\alpha}\tilde{\beta}} &= \nabla^{\tilde{h}}_{\tilde{\alpha}}\nabla^{\tilde{h}}_{\tilde{\beta}}f - \tilde{h}^{\gamma\tilde{\delta}}L_{\tilde{\alpha}\tilde{\gamma}}L_{\tilde{\beta}\tilde{\delta}}f + R^{g_+}_{\tilde{\alpha}\hat{n}\tilde{\beta}\hat{n}}f, \\ H' &= \Delta_{\tilde{h}}f + (|L_Y|^2_{\tilde{h}} - 3)f. \end{aligned} \tag{31}$$

By (28), $H' = 0$, so the last formula above implies that f must satisfy

$$\Delta_{\tilde{h}}f = (3 - |L_Y|^2_{\tilde{h}})f. \tag{32}$$

Lemma 4.1. *$f \in C^\infty(\mathring{Y})$ has an asymptotic expansion of the form*

$$f = r^{-1}\tilde{f} + o(1), \tag{33}$$

where $\tilde{f} \in C^\infty(\Sigma)$ is given by (30).

Conversely, if $|\mathring{L}_Y|^2_{\tilde{h}} \leq 3$ on $\mathring{Y}$, then given $\tilde{f} \in C^\infty(\Sigma)$, there is a unique solution f to (32) *satisfying the expansion (33).*

Proof. We first observe that near M,

$$|L_Y|^2_{\tilde{h}} = O(r^2). \tag{34}$$

This follows from (67) below, but it can also be seen by using the fact that L_Y is trace-free (since Y is minimal), and the the trace-free second fundamental form is a conformal invariant (of weight 1). Using (34), it is easy to see that the indicial roots of the operator

$$\mathcal{P} = \Delta_{\tilde{h}} - (3 - |L_Y|^2_{\tilde{h}})$$

are -1 and 3. It follows that f has an expansion of the form

$$f = r^{-1}f_{-1} + O(1),$$

for some $f_{-1} \in C^\infty(\Sigma)$. However, using the expansion of the metric $\tilde{h}$ near M in (5), we have $h^{\tilde{0}\tilde{0}} = 1 + O(r^2)$, and using this it is easy to see that

$$f - r^{-1}f_{-1} = o(1).$$

as in (33). Since $\mu_Y = r\bar{\mu}_Y$, (29) implies

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} \mathcal{F}_t &= f\mu_Y \\ &= \left[r^{-1} f_{-1} + o(1)\right] \, r\bar{\mu}_Y \\ &= f_{-1}\bar{\mu}_Y + o(r),\end{aligned}$$

and it follows from (30) and the definition of $\tilde{\mathcal{F}}$ that $f_{-1} = \tilde{f}$.

Conversely, given $\tilde{f}$, if we let

$$f_{-1} = r^{-1}\tilde{f}$$

then $\mathcal{P}f_{-1} = O(1)$. It then follows from standard arguments (see [19]) that there is a unique solution of $\mathcal{P}f = 0$ with $f = r^{-1}f_{-1} + O(1)$. Again using the expansion of the metric it is readily checked that $f = r^{-1}\tilde{f} + o(1)$. □

Remark 4.2. *Although $f \in C^\infty(\mathring{Y})$, since the indicial roots of the equation satisfied by f are -1 and 3, the expansion of f must in general be expected to have a term $r^3 \log r$, so $rf \in C^{3,\alpha}(\bar{Y})$, and optimal regularity of $\mathcal{F}$ is C^3.*

Proof of Theorem 1.3. The statement of Theorem 1.3 consists of two claims: the formula for the derivative of V_+^+, and the assertion that $\tilde{f}$ determines f. Since the latter follows from the uniqueness claim in Lemma 4.1, to complete the proof of the theorem we just need to carry out the calculation of $\dot{V}_+^+$.

By Theorem 1.1,

$$3V_+^+(X_t) = \pi^2(4\chi(X_t^+) - \chi(\partial Y^t)) - \frac{1}{8}\int_{X_t^+} |W_{g_+}|^2_{g_+} dv_{g_+} - \int_{Y^t} \mathcal{C}_{Y^t} dv_{\tilde{h}_t}.$$

We let $\tilde{h}_t = g_+|_{T\mathring{Y}_t}$. For $\varepsilon > 0$ small, recall that $X_\varepsilon = \{x \in X : r(x) > \varepsilon\}$. We let $Y_\varepsilon^t = Y^t \cap X_\varepsilon$, and define

$$\begin{aligned}3V_\varepsilon(t) = \pi^2(4\chi(X_t^+ \cap X_\epsilon) - \chi(\partial Y_\epsilon^t)) \\ - \frac{1}{8}\int_{X_t^+\cap X_\epsilon} |W_{g_+}|^2_{g_+} \, dv_{g_+} - \int_{Y_\varepsilon^t} \mathcal{C}_{Y^t} \, dv_{\tilde{h}_t}.\end{aligned}$$

Then

$$3\frac{d}{dt}V_\varepsilon(t)\Big|_{t=0} = -\frac{1}{8}\frac{d}{dt}\int_{X_t^+\cap X_\varepsilon} |W_{g_+}|^2_{g_+} \, dv_{g_+}\Big|_{t=0} - \frac{d}{dt}\int_{Y_\varepsilon^t} \mathcal{C}_{Y^t} \, dv_{\tilde{h}_t}\Big|_{t=0}.$$

For the first integral,

$$-\frac{1}{8}\frac{d}{dt}\bigg|_{t=0}\int_{X_t^+\cap X_\varepsilon}|W_{g_+}|^2_{g_+}dv_{g_+}=\frac{1}{8}\int_{Y_\varepsilon}|W_{g_+}|^2_{g_+}f dv_{\tilde{h}}. \tag{35}$$

To analyze the second integral, we let $dv^\varepsilon_{\tilde{h}_t}=\psi dv_{\tilde{h}_t}$, where $\psi=\theta(r-\varepsilon)$, with θ the Heaviside function. Then

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0}\int_{Y^t_\varepsilon}\mathcal{C}_{Y^t}dv_{\tilde{h}_t} &= \frac{d}{dt}\bigg|_{t=0}\int_{Y^t}\mathcal{C}_{Y^t}dv^\varepsilon_{\tilde{h}_t}\\
&=\lim_{\tau\to0}\frac{1}{\tau}\bigg[\int_Y(\mathcal{C}_{Y^\tau}\circ\mathcal{F}_\tau)(\psi\circ\mathcal{F}_\tau)(\mathcal{F}^*_\tau dv_{\tilde{h}_\tau}-dv_{\tilde{h}})\\
&\quad+\int_Y(\mathcal{C}_{Y^\tau}\circ\mathcal{F}_\tau-\mathcal{C}_Y)(\psi\circ\mathcal{F}_\tau)dv_{\tilde{h}}\\
&\quad+\int_Y\mathcal{C}_Y(\psi\circ\mathcal{F}_\tau-\psi)dv_{\tilde{h}}\bigg]\\
&=\int_{Y_\varepsilon}\mathcal{C}_Y\left(\frac{d}{dt}dv_{\tilde{h}_t}\bigg|_{t=0}\right)\\
&\quad+\int_{Y_\varepsilon}\frac{d}{dt}\mathcal{C}_{Y^t}\bigg|_{t=0}dv_{\tilde{h}}+\lim_{\tau\to0}\frac{1}{\tau}\int_Y\mathcal{C}_Y(\psi\circ\mathcal{F}_t-\psi)dv_{\tilde{h}}.\end{aligned}$$

Now by the Implicit Function Theorem, the equation $r(\mathcal{F}(t(p),r(p),\zeta(p)))=\varepsilon$ can be written as $r=\xi(t,\zeta)$ for some smooth $\xi:(-\delta,\delta)\times\Sigma\to\mathbb{R}$. Let $\bar{k}_\varepsilon$ be the metric induced on Σ_ε by $\bar{g}$. Writing $dv_{\tilde{h}}=\eta r^{-3}dr dv_{\bar{k}_\varepsilon}$ for some smooth correction factor η that is one on Σ_ε, we may use the fundamental theorem of calculus to write the last term as

$$\begin{aligned}&\lim_{\tau\to0}\frac{1}{\tau}\int_Y\mathcal{C}_Y(\psi\circ\mathcal{F}_\tau-\psi)dv_{\tilde{h}}\\
&\quad=-\lim_{\tau\to0}\frac{1}{\tau}\int_{\Sigma_\varepsilon}\int_\varepsilon^{\xi(\tau,\zeta)}\mathcal{C}_Y(r,\zeta)\eta(r,\zeta)r^{-3}dr dv_{\bar{k}_\varepsilon}(\zeta)\\
&\quad=-\int_{\Sigma_\varepsilon}\frac{d}{dt}\bigg|_{t=0}\int_\varepsilon^{\xi(t,\zeta)}\mathcal{C}_Y(r,\zeta)\eta(r,\zeta)r^{-3}dr dv_{\bar{k}_\varepsilon}(\zeta)\\
&\quad=-\int_{\Sigma_\varepsilon}\mathcal{C}_Y(\varepsilon,\zeta)\varepsilon^{-3}\frac{\partial\xi}{\partial t}\bigg|_{t=0}dv_{\bar{k}_\varepsilon}(\zeta)\\
&\quad=\int_{\Sigma_\varepsilon}\mathcal{C}_Y\varepsilon^{-1}dr(f\mu_Y)dv_{k_\varepsilon}\\
&\quad=\int_{\Sigma_\varepsilon}\mathcal{C}_Y\langle r\partial_r,f\mu_Y\rangle_{g_+}dv_{k_\varepsilon}.\end{aligned}$$

Therefore

$$\frac{d}{dt}\int_{Y^t_\varepsilon}\mathcal{C}_{Y^t}dv_{\tilde{h}_t}\Big|_{t=0}=\int_{Y_\varepsilon}\big(\frac{d}{dt}\mathcal{C}_{Y^t}\big|_{t=0}\big)dv_{\tilde{h}}+\int_{Y_\varepsilon}\mathcal{C}_Y\big(\frac{d}{dt}dv_{\tilde{h}_t}\big|_{t=0}\big)+\int_{\Sigma_\varepsilon}\mathcal{C}_Y\langle r\partial_r,f\mu_Y\rangle_{g_+}dv_{k_\varepsilon}. \tag{36}$$

We dispose of the last term with

Claim 1.

$$\lim_{\varepsilon\to 0}\int_{\Sigma_\varepsilon}\mathcal{C}_Y\langle r\partial_r,f\mu_Y\rangle_{g_+}dv_{k_\varepsilon}=0.$$

Proof. We know that $\mu_Y=r\bar{\mu}_Y$ and that $\mathcal{C}^{g_+}_Y=r^3\mathcal{C}^{\bar{g}}_Y$. We also know from (25) that

$$\langle r\partial_r,\bar{\mu}_Y\rangle_{\bar{g}}=O(\varepsilon^2).$$

So we get

$$\mathcal{C}^{g_+}_Y\langle r\partial_r,\mu_Y\rangle_{g_+}=r^3\mathcal{C}^{\bar{g}}_Y\langle r\partial_r,\mu_Y\rangle_{g_+}=O(\varepsilon^4).$$

Therefore, taking into account the asymptotics of f, we get

$$\int_{\Sigma_\varepsilon}\mathcal{C}_Y\langle r\partial_r,f\mu_Y\rangle_{g_+}dv_{k_\varepsilon}=O(\varepsilon). \tag{37}$$

□

By (28) and the formula for the variation of the volume form (89) in the appendix we have

$$\frac{d}{dt}dv_{\tilde{h}_t}\big|_{t=0}=H_Y dv_{\tilde{h}}=0, \tag{38}$$

since Y is minimal. The minimality of Y to first order also implies $H_{Y^t}=O(t^2)$. Since g_+ is Einstein, the formula for $\mathcal{C}_{Y^t}$ thus simplifies to

$$\mathcal{C}_{Y^t}=-(L_{Y^t})^{\tilde{\alpha}\tilde{\beta}}R^{\tilde{h}_t}_{\tilde{\alpha}\tilde{\beta}}-\frac{1}{3}\operatorname{tr}_{\tilde{h}_t}(L_{Y^t})^3+O(t^2), \tag{39}$$

where L_{Y^t} is the second fundamental form of Y^t with respect to μ_Y and $R^{\tilde{h}_t}$ is the Ricci tensor of $\tilde{h}_t$. Combining (36), (37), (38) and (39) we obtain

$$\frac{d}{dt}\int_{Y^t_\varepsilon}\mathcal{C}_{Y^t}\,dv_{\tilde{h}_t}\Big|_{t=0}=-\int_{Y_\varepsilon}\frac{d}{dt}\big((L^{Y^t})^{\tilde{\alpha}\tilde{\beta}}R^{\tilde{h}_t}_{\tilde{\alpha}\tilde{\beta}}\big)\big|_{t=0}\,dv_{\tilde{h}}-\frac{1}{3}\int_{Y_\varepsilon}\frac{d}{dt}\operatorname{tr}_{\tilde{h}_t}(L^{Y^t})^3\big|_{t=0}\,dv_{\tilde{h}}+O(\varepsilon).$$

We intend to apply integration by parts to the integrand of this expression to write quantities in terms of boundary integrals on Σ. We first write the integrands in terms of geometric quantities on Y.

Define

$$A = (L_{Y^t})^{\tilde{\alpha}\tilde{\beta}} R^{\tilde{h}_t}_{\tilde{\alpha}\tilde{\beta}}$$
$$B = \operatorname{tr}_{\tilde{h}_t}(L_{Y^t})^3.$$

Differentiating A gives

$$A' = \tilde{h}^{\tilde{\alpha}\tilde{\gamma}}\tilde{h}^{\tilde{\beta}\tilde{\delta}} R^{\tilde{h}}_{\tilde{\gamma}\tilde{\delta}} \nabla^{\tilde{h}}_{\tilde{\alpha}}\nabla^{\tilde{h}}_{\tilde{\beta}} f + 3f(L^2)^{\tilde{\alpha}\tilde{\beta}} R^{\tilde{h}}_{\tilde{\alpha}\tilde{\beta}} + f\tilde{h}^{\tilde{\alpha}\tilde{\gamma}}\tilde{h}^{\tilde{\beta}\tilde{\delta}} R^{\tilde{h}}_{\tilde{\gamma}\tilde{\delta}} R^{g_+}_{\tilde{\alpha}\hat{n}\tilde{\beta}\hat{n}} + \tilde{h}^{\tilde{\alpha}\tilde{\gamma}}\tilde{h}^{\tilde{\beta}\tilde{\delta}} L_{\tilde{\gamma}\tilde{\delta}} (R^{\tilde{h}}_{\tilde{\alpha}\tilde{\beta}})'. \tag{40}$$

A standard formula for the variation of the Ricci tensor (see e.g. [23]) gives us

$$(R^{\tilde{h}}_{\tilde{\alpha}\tilde{\beta}})' = -\frac{1}{2}\left[\Delta_{\tilde{h}} \tilde{h}'_{\tilde{\alpha}\tilde{\beta}} - \nabla^{\tilde{h}}_{\tilde{\alpha}}(\tilde{\delta}_{\tilde{\beta}}\tilde{h}') - \nabla^{\tilde{h}}_{\tilde{\beta}}(\tilde{\delta}_{\tilde{\alpha}}\tilde{h}') + \nabla^{\tilde{h}}_{\tilde{\alpha}}\nabla^{\tilde{h}}_{\tilde{\beta}}(\operatorname{tr}_{\tilde{h}}\tilde{h}')\right] - \tilde{h}^{\tilde{\gamma}\tilde{\eta}}\tilde{h}^{\tilde{\delta}\tilde{\zeta}} R^{\tilde{h}}_{\tilde{\alpha}\tilde{\gamma}\tilde{\beta}\tilde{\delta}} \tilde{h}'_{\tilde{\eta}\tilde{\zeta}} + \frac{1}{2}\tilde{h}^{\tilde{\eta}\tilde{\zeta}} R^{\tilde{h}}_{\tilde{\alpha}\tilde{\eta}} \tilde{h}'_{\tilde{\beta}\tilde{\zeta}} + \frac{1}{2}\tilde{h}^{\tilde{\eta}\tilde{\zeta}} R^{\tilde{h}}_{\tilde{\beta}\tilde{\eta}} \tilde{h}'_{\tilde{\alpha}\tilde{\zeta}}. \tag{41}$$

Here $\tilde{\delta}$ is the divergence with respect to $\tilde{h}$. Now, by (31), $\Delta_{\tilde{h}} \tilde{h}'_{\tilde{\alpha}\tilde{\beta}} = -2\Delta^{\tilde{h}}(fL_{\tilde{\alpha}\tilde{\beta}})$. By the same equation,

$$\operatorname{tr}_{\tilde{h}}\tilde{h}' = 0.$$

Taking the divergence of both sides of (31) gives us

$$\begin{aligned} \tilde{\delta}_{\tilde{\beta}}\tilde{h}' &= \tilde{h}^{\tilde{\alpha}\tilde{\gamma}}\nabla^{\tilde{h}}_{\tilde{\gamma}}\tilde{h}'_{\tilde{\alpha}\tilde{\beta}} \\ &= -\nabla^{\tilde{\alpha}}_{\tilde{h}}(2fL_{\tilde{\alpha}\tilde{\beta}}) \\ &= -2f\nabla^{\tilde{\alpha}}_{\tilde{h}}L_{\tilde{\alpha}\tilde{\beta}} - 2L_{\tilde{\alpha}\tilde{\beta}}\nabla^{\tilde{\alpha}}_{\tilde{h}}f. \end{aligned} \tag{42}$$

Now by Codazzi, we have

$$R^{g_+}_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\hat{n}} = \nabla^{\tilde{h}}_{\tilde{\beta}}L_{\tilde{\alpha}\tilde{\gamma}} - \nabla^{\tilde{h}}_{\tilde{\alpha}}L_{\tilde{\beta}\tilde{\gamma}}$$

along Y. Contracting $\tilde{\alpha}$ and $\tilde{\gamma}$ and using the Einstein condition on g_+ along with the fact that Y is minimal gives

$$0 = R^{g_+}_{\tilde{\beta}\hat{n}} = -\tilde{h}^{\tilde{\alpha}\tilde{\gamma}}\nabla^{\tilde{h}}_{\tilde{\alpha}}L_{\tilde{\beta}\tilde{\gamma}} + \tilde{h}^{\tilde{\alpha}\tilde{\gamma}}\nabla^{\tilde{h}}_{\tilde{\beta}}L_{\tilde{\alpha}\tilde{\gamma}} = -\nabla^{\tilde{\gamma}}_{\tilde{h}}L_{\tilde{\beta}\tilde{\gamma}} + \nabla^{\tilde{h}}_{\tilde{\beta}}H = -\nabla^{\tilde{\alpha}}_{\tilde{h}}L_{\tilde{\beta}\tilde{\alpha}}.$$

Hence

$$\nabla^{\tilde{\alpha}}_{\tilde{h}} L_{\tilde{\beta}\tilde{\alpha}} = 0 \tag{43}$$

and

$$\tilde{\delta}_{\tilde{\beta}} \tilde{h}' = -2L_{\tilde{\beta}\tilde{\gamma}} \nabla^{\tilde{\gamma}} f.$$

Turning to the fifth term of (41), we consider the Riemann tensor on Y. As the dimension of Y is three, it follows that the Weyl tensor of $\tilde{h}$ vanishes, giving us

$$\begin{aligned} R^{\tilde{h}}_{\tilde{\alpha}\tilde{\gamma}\tilde{\beta}\tilde{\delta}} = \tilde{h}_{\tilde{\alpha}\tilde{\beta}} R^{\tilde{h}}_{\tilde{\gamma}\tilde{\delta}} - \tilde{h}_{\tilde{\alpha}\tilde{\delta}} R^{\tilde{h}}_{\tilde{\beta}\tilde{\gamma}} - \tilde{h}_{\tilde{\beta}\tilde{\gamma}} R^{\tilde{h}}_{\tilde{\alpha}\tilde{\delta}} + \tilde{h}_{\tilde{\gamma}\tilde{\delta}} R^{\tilde{h}}_{\tilde{\alpha}\tilde{\beta}} - \frac{1}{2} R^{\tilde{h}} \tilde{h}_{\tilde{\alpha}\tilde{\beta}} \tilde{h}_{\tilde{\gamma}\tilde{\delta}} \\ + \frac{1}{2} R^{\tilde{h}} \tilde{h}_{\tilde{\alpha}\tilde{\delta}} \tilde{h}_{\tilde{\beta}\tilde{\gamma}}. \end{aligned}$$

Thus,

$$-\tilde{h}^{\tilde{\gamma}\tilde{\eta}} \tilde{h}^{\tilde{\delta}\tilde{\zeta}} R^{\tilde{h}}_{\tilde{\alpha}\tilde{\gamma}\tilde{\beta}\tilde{\delta}} \tilde{h}'_{\tilde{\eta}\tilde{\zeta}} = -R^{\tilde{\zeta}\tilde{\eta}}_{\tilde{h}} \tilde{h}'_{\tilde{\eta}\tilde{\zeta}} \tilde{h}_{\tilde{\alpha}\tilde{\beta}} + \tilde{h}^{\tilde{\eta}\tilde{\zeta}} R^{\tilde{h}}_{\tilde{\beta}\tilde{\eta}} \tilde{h}'_{\tilde{\alpha}\tilde{\zeta}} + \tilde{h}^{\tilde{\eta}\tilde{\zeta}} R^{\tilde{h}}_{\tilde{\alpha}\tilde{\eta}} \tilde{h}'_{\tilde{\beta}\tilde{\zeta}} - \frac{1}{2} R^{\tilde{h}} \tilde{h}'_{\tilde{\alpha}\tilde{\beta}}.$$

So we can write the last three terms of (41) as

$$\begin{aligned} &- \tilde{h}^{\tilde{\gamma}\tilde{\eta}} \tilde{h}^{\tilde{\delta}\tilde{\zeta}} R^{\tilde{h}}_{\tilde{\alpha}\tilde{\gamma}\tilde{\beta}\tilde{\delta}} \tilde{h}'_{\tilde{\eta}\tilde{\zeta}} + \frac{1}{2} \tilde{h}^{\tilde{\eta}\tilde{\zeta}} R^{\tilde{h}}_{\tilde{\alpha}\tilde{\eta}} \tilde{h}'_{\tilde{\beta}\tilde{\zeta}} + \frac{1}{2} \tilde{h}^{\tilde{\eta}\tilde{\zeta}} R^{\tilde{h}}_{\tilde{\beta}\tilde{\eta}} \tilde{h}'_{\tilde{\alpha}\tilde{\zeta}} = \\ &\quad - R^{\tilde{h}}_{\tilde{\eta}\tilde{\zeta}} (\tilde{h}^{\tilde{\eta}\tilde{\zeta}})' \tilde{h}_{\tilde{\alpha}\tilde{\beta}} + \frac{3}{2} \tilde{h}^{\tilde{\eta}\tilde{\zeta}} R^{\tilde{h}}_{\tilde{\beta}\tilde{\eta}} \tilde{h}'_{\tilde{\alpha}\tilde{\zeta}} + \frac{3}{2} \tilde{h}^{\tilde{\eta}\tilde{\zeta}} R^{\tilde{h}}_{\tilde{\alpha}\tilde{\eta}} \tilde{h}'_{\tilde{\beta}\tilde{\zeta}} - \frac{1}{2} R^{\tilde{h}} \tilde{h}'_{\tilde{\alpha}\tilde{\beta}}. \end{aligned}$$

Therefore we have found

$$\begin{aligned} (R^{\tilde{h}}_{\tilde{\alpha}\tilde{\beta}})' = \Delta^{\tilde{h}} (f L_{\tilde{\alpha}\tilde{\beta}}) - \nabla^{\tilde{h}}_{\tilde{\alpha}} (L_{\tilde{\beta}\tilde{\gamma}} \nabla^{\tilde{\gamma}} f) - \nabla^{\tilde{h}}_{\tilde{\beta}} (L_{\tilde{\alpha}\tilde{\gamma}} \nabla^{\tilde{\gamma}} f) + 2f (R^{\tilde{h}}_{\tilde{\eta}\tilde{\zeta}} L^{\tilde{\eta}\tilde{\zeta}}) \tilde{h}_{\tilde{\alpha}\tilde{\beta}} \\ - 3f L_{\tilde{\alpha}}{}^{\tilde{\gamma}} R^{\tilde{h}}_{\tilde{\beta}\tilde{\gamma}} - 3f L_{\tilde{\beta}}{}^{\tilde{\gamma}} R^{\tilde{h}}_{\tilde{\alpha}\tilde{\gamma}} + f R^{\tilde{h}} L_{\tilde{\alpha}\tilde{\beta}}. \end{aligned}$$

This then lets us write down an expression for $\langle L, (\mathrm{Ric}^{\tilde{h}})' \rangle_{\tilde{h}}$:

$$L^{\tilde{\alpha}\tilde{\beta}} (R^{\tilde{h}}_{\tilde{\alpha}\tilde{\beta}})' = L^{\tilde{\alpha}\tilde{\beta}} \Delta_{\tilde{h}} (f L_{\tilde{\alpha}\tilde{\beta}}) - 2L^{\tilde{\alpha}\tilde{\beta}} \nabla^{\tilde{h}}_{\tilde{\beta}} (L_{\tilde{\alpha}\tilde{\gamma}} \nabla^{\tilde{\gamma}} f) - 6f (L^2)^{\tilde{\alpha}\tilde{\beta}} R^{\tilde{h}}_{\tilde{\alpha}\tilde{\beta}} + f R^{\tilde{h}} |L|^2;$$

hence

$$\begin{aligned} A' = R^{\tilde{\alpha}\tilde{\beta}}_{\tilde{h}} \nabla^{\tilde{h}}_{\tilde{\alpha}} \nabla^{\tilde{h}}_{\tilde{\beta}} f - 3f (L^2)^{\tilde{\alpha}\tilde{\beta}} R^{\tilde{h}}_{\tilde{\alpha}\tilde{\beta}} + f R^{\tilde{\alpha}\tilde{\beta}}_{\tilde{h}} R^{g_+}_{\tilde{\alpha}\hat{n}\tilde{\beta}\hat{n}} \\ + L^{\tilde{\alpha}\tilde{\beta}} \Delta_{\tilde{h}} (f L_{\tilde{\alpha}\tilde{\beta}}) - 2L^{\tilde{\alpha}\tilde{\beta}} \nabla^{\tilde{h}}_{\tilde{\alpha}} (L_{\tilde{\beta}\tilde{\gamma}} \nabla^{\tilde{\gamma}}_{\tilde{h}} f) + f R^{\tilde{h}} |L|^2. \end{aligned}$$

Using formula (87) in the appendix for the the variation of the second fundamental form, it is straightforward to see that B' is given by

$$\begin{aligned} B' = (\operatorname{tr} L^3)' &= 3(\tilde{h}^{\tilde{\alpha}\tilde{\gamma}})'\tilde{h}^{\tilde{\beta}\tilde{\eta}}\tilde{h}^{\tilde{\delta}\tilde{\zeta}}L_{\tilde{\alpha}\tilde{\beta}}L_{\tilde{\gamma}\tilde{\delta}}L_{\tilde{\eta}\tilde{\zeta}} + 3\tilde{h}^{\tilde{\alpha}\tilde{\gamma}}\tilde{h}^{\tilde{\beta}\tilde{\eta}}\tilde{h}^{\tilde{\delta}\tilde{\zeta}}L'_{\tilde{\alpha}\tilde{\beta}}L_{\tilde{\gamma}\tilde{\delta}}L_{\tilde{\eta}\tilde{\zeta}} \\ &= 6f|L^2|^2_{\tilde{h}} + 3\tilde{h}^{\tilde{\alpha}\tilde{\gamma}}\tilde{h}^{\tilde{\beta}\tilde{\eta}}\tilde{h}^{\tilde{\delta}\tilde{\zeta}}\big[\nabla^{\tilde{h}}_{\tilde{\alpha}}\nabla^{\tilde{h}}_{\tilde{\beta}}f - L^2_{\tilde{\alpha}\tilde{\beta}}f \\ &\quad + (R^{g_+}_{\tilde{\alpha}\hat{n}\tilde{\beta}\hat{n}}L^{\tilde{\gamma}}_{\tilde{\alpha}}L_{\tilde{\beta}\tilde{\gamma}})f\big]L_{\tilde{\gamma}\tilde{\delta}}L_{\tilde{\eta}\tilde{\zeta}} \\ &= 3f|L^2|^2_{\tilde{h}} + 3(\nabla^{\tilde{h}}_{\tilde{\alpha}}\nabla^{\tilde{h}}_{\tilde{\beta}}f)(L^2)^{\tilde{\alpha}\tilde{\beta}} + 3fR^{g_+}_{\tilde{\alpha}\hat{n}\tilde{\beta}\hat{n}}(L^2)^{\tilde{\alpha}\tilde{\beta}}. \end{aligned}$$

It will be useful to record two consequences of the Gauss curvature equation. First, using the Einstein condition, the Ricci curvature of $\tilde{h}$ can be expressed as

$$R^{\tilde{h}}_{\tilde{\eta}\tilde{\zeta}} = -3\tilde{h}_{\tilde{\eta}\tilde{\zeta}} - R^{g_+}_{\tilde{\eta}\hat{n}\tilde{\zeta}\hat{n}} - (L^2)_{\tilde{\eta}\tilde{\zeta}}. \tag{44}$$

It follows that the scalar curvature of $\tilde{h}$ is given by

$$R_{\tilde{h}} = -6 - |L|^2. \tag{45}$$

We now focus on rewriting four terms in A' and B' to make them amenable to integration by parts. We thus make the following definitions:

$$\begin{aligned} D_1 &= \int_{Y_\varepsilon} R^{\tilde{\alpha}\tilde{\beta}}_{\tilde{h}}\nabla^{\tilde{h}}_{\tilde{\alpha}}\nabla^{\tilde{h}}_{\tilde{\beta}}f\,dv_{\tilde{h}} \\ D_2 &= \int_{Y_\varepsilon} L^{\tilde{\alpha}\tilde{\beta}}\Delta_{\tilde{h}}(fL_{\tilde{\alpha}\tilde{\beta}})dv_{\tilde{h}} \\ D_3 &= -\int_{Y_\varepsilon} 2L^{\tilde{\alpha}\tilde{\beta}}\nabla^{\tilde{h}}_{\tilde{\alpha}}(L_{\tilde{\beta}\tilde{\gamma}}\nabla^{\tilde{\gamma}}_{\tilde{h}}f)dv_{\tilde{h}} \\ D_4 &= \int_{Y_\varepsilon} 3(L^2)^{\tilde{\alpha}\tilde{\beta}}\nabla^{\tilde{h}}_{\tilde{\alpha}}\nabla^{\tilde{h}}_{\tilde{\beta}}f\,dv_{\tilde{h}}. \end{aligned}$$

We will write each of the above terms as an integral over Y_ε plus an integral over Σ_ε. Recall that ν_{Y_ε} is the inward pointing $\tilde{h}$ unit-normal vector field to Σ_ε in Y_ε. Integrating by parts then applying the second contracted Bianchi identity and (45), we find

$$
\begin{aligned}
D_1 &= \int_{Y_\varepsilon} \tilde{h}^{\tilde{\alpha}\tilde{\gamma}}\tilde{h}^{\tilde{\beta}\tilde{\delta}} R^{\tilde{h}}_{\tilde{\gamma}\tilde{\delta}} \nabla^{\tilde{h}}_{\tilde{\alpha}} \nabla^{\tilde{h}}_{\tilde{\beta}} f \, dv_{\tilde{h}} \\
&= -\int_{Y_\varepsilon} \tilde{h}^{\tilde{\alpha}\tilde{\gamma}}\tilde{h}^{\tilde{\beta}\tilde{\delta}} \nabla^{\tilde{h}}_{\tilde{\alpha}} R^{\tilde{h}}_{\tilde{\gamma}\tilde{\delta}} \nabla^{\tilde{h}}_{\tilde{\beta}} f \, dv_{\tilde{h}} - \oint_{\Sigma_\varepsilon} R^{\tilde{h}}_{\tilde{\alpha}\tilde{\beta}} \nu^{\tilde{\alpha}}_{Y_\varepsilon} \nabla^{\tilde{\beta}}_{\tilde{h}} f \, dv_{k_\varepsilon} \\
&= -\int_{Y_\varepsilon} \frac{1}{2} (\nabla^{\tilde{\alpha}}_{\tilde{h}} R_{\tilde{h}}) \nabla^{\tilde{h}}_{\tilde{\alpha}} f \, dv_{\tilde{h}} - \oint_{\Sigma_\varepsilon} R^{\tilde{h}}_{\tilde{\alpha}\tilde{\beta}} \nu^{\tilde{\alpha}}_{Y_\varepsilon} \nabla^{\tilde{\beta}}_{\tilde{h}} f \, dv_{k_\varepsilon} \\
&= \int_{Y_\varepsilon} \frac{1}{2} R_{\tilde{h}} \Delta_{\tilde{h}} f \, dv_{\tilde{h}} + \oint_{\Sigma_\varepsilon} \left[\frac{1}{2} R_{\tilde{h}} \nu_{Y_\varepsilon}(f) - \mathrm{Ric}_{\tilde{h}}(\nu_{Y_\varepsilon}, \nabla^{\tilde{h}} f) \right] dv_{k_\varepsilon} \\
&= \int_{Y_\varepsilon} \left(\frac{6 + |L|^2}{2} \right) (|L|^2 - 3) f \, dv_{\tilde{h}} \\
&\quad - \oint_{\Sigma_\varepsilon} \left[\mathrm{Ric}_{\tilde{h}}(\nu_{Y_\varepsilon}, \nabla^{\tilde{h}} f) - \frac{1}{2} R_{\tilde{h}} \nu_{Y_\varepsilon}(f) \right] dv_{k_\varepsilon} \\
&= \int_{Y_\varepsilon} \left(\frac{|L|^4}{2} + \frac{3|L|^2}{2} - 9 \right) f \, dv_{\tilde{h}} - \oint_{\Sigma_\varepsilon} \big(\mathrm{Ric}_{\tilde{h}} - \frac{1}{2} R_{\tilde{h}} \tilde{h} \big) \Big(\nabla^{\tilde{h}} f, \nu_{Y_\varepsilon} \Big) dv_{k_\varepsilon}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
D_2 &= \int_{Y_\varepsilon} L^{\tilde{\alpha}\tilde{\beta}} \Delta_{\tilde{h}} (f L_{\tilde{\alpha}\tilde{\beta}}) \, dv_{\tilde{h}} \\
&= \int_{Y_\varepsilon} \left[|L|^2 \Delta_{\tilde{h}} f + f L^{\tilde{\alpha}\tilde{\beta}} \Delta_{\tilde{h}} L_{\tilde{\alpha}\tilde{\beta}} + 2 L^{\tilde{\alpha}\tilde{\beta}} \nabla^{\tilde{\gamma}}_{\tilde{h}} f \nabla^{\tilde{h}}_{\tilde{\gamma}} L_{\tilde{\alpha}\tilde{\beta}} \right] dv_{\tilde{h}} \\
&= \int_{Y_\varepsilon} \left[|L|^2 \Delta_{\tilde{h}} f + f L^{\tilde{\alpha}\tilde{\beta}} \Delta_{\tilde{h}} L_{\tilde{\alpha}\tilde{\beta}} + \langle \nabla^{\tilde{h}} f, \nabla^{\tilde{h}} |L|^2 \rangle \right] dv_{\tilde{h}} \\
&= \int_{Y_\varepsilon} f L^{\tilde{\alpha}\tilde{\beta}} \Delta_{\tilde{h}} L_{\tilde{\alpha}\tilde{\beta}} \, dv_{\tilde{h}} - \oint_{\Sigma_\varepsilon} |L|^2 \nu_{Y_\varepsilon}(f) \, dv_{k_\varepsilon}.
\end{aligned}
$$

We want to use a Simons-type identity to replace the term $\Delta_{\tilde{h}} L_{\tilde{\alpha}\tilde{\beta}}$. By the Codazzi equation,

$$
R^{g_+}_{\tilde{\gamma}\tilde{\alpha}\tilde{\beta}\hat{n}} = \nabla^{\tilde{h}}_{\tilde{\alpha}} L_{\tilde{\gamma}\tilde{\beta}} - \nabla^{\tilde{h}}_{\tilde{\gamma}} L_{\tilde{\alpha}\tilde{\beta}},
$$

so we may write

$$
\begin{aligned}
\tilde{h}^{\tilde{\delta}\tilde{\gamma}} \nabla^{\tilde{h}}_{\tilde{\delta}} R^{g_+}_{\tilde{\gamma}\tilde{\alpha}\tilde{\beta}\hat{n}} &= \tilde{h}^{\tilde{\delta}\tilde{\gamma}} \nabla^{\tilde{h}}_{\tilde{\delta}} \nabla^{\tilde{h}}_{\tilde{\alpha}} L_{\tilde{\gamma}\tilde{\beta}} - \tilde{h}^{\tilde{\delta}\tilde{\gamma}} \nabla^{\tilde{h}}_{\tilde{\delta}} \nabla^{\tilde{h}}_{\tilde{\gamma}} L_{\tilde{\alpha}\tilde{\beta}} \\
&= \nabla^{\tilde{\gamma}}_{\tilde{h}} \nabla^{\tilde{h}}_{\tilde{\alpha}} L_{\tilde{\gamma}\tilde{\beta}} - \Delta_{\tilde{h}} L_{\tilde{\alpha}\tilde{\beta}}.
\end{aligned} \tag{46}
$$

Now we want to commute the covariant derivatives in the first term on the right-hand side of this equation. By the Ricci identity,

$$\nabla^{\tilde h}_{\tilde\delta}\nabla^{\tilde h}_{\tilde\alpha}L_{\tilde\gamma\tilde\beta}-\nabla^{\tilde h}_{\tilde\alpha}\nabla^{\tilde h}_{\tilde\delta}L_{\tilde\gamma\tilde\beta}=R^{\tilde h}_{\tilde\delta\tilde\alpha\tilde\gamma}{}^{\tilde\eta}L_{\tilde\eta\tilde\beta}+R^{\tilde h}_{\tilde\delta\tilde\alpha\tilde\beta}{}^{\tilde\eta}L_{\tilde\eta\tilde\gamma}.$$

Contracting $\tilde\delta$ and $\tilde\gamma$ and using (43) gives

$$\tilde h^{\tilde\delta\tilde\gamma}\nabla^{\tilde h}_{\tilde\delta}\nabla^{\tilde h}_{\tilde\alpha}L_{\tilde\gamma\tilde\beta}=\tilde h^{\tilde\delta\tilde\gamma}R^{\tilde h}_{\tilde\delta\tilde\alpha\tilde\beta}{}^{\tilde\eta}L_{\tilde\eta\tilde\gamma}+R^{\tilde h}_{\tilde\alpha\tilde\gamma}L^{\tilde\gamma}{}_{\tilde\beta}. \tag{47}$$

Combining (47) and (46), we get

$$\Delta_{\tilde h}L_{\tilde\alpha\tilde\beta}=\tilde h^{\tilde\delta\tilde\gamma}R^{\tilde h}_{\tilde\delta\tilde\alpha\tilde\beta}{}^{\tilde\eta}L_{\tilde\eta\tilde\gamma}+R^{\tilde h}_{\tilde\alpha\tilde\gamma}L^{\tilde\gamma}{}_{\tilde\beta}-\tilde h^{\tilde\delta\tilde\gamma}\nabla^{\tilde h}_{\tilde\delta}R^{g_+}_{\tilde\gamma\tilde\alpha\tilde\beta\hat n}.$$

Therefore,

$$\begin{aligned} D_2=&\int_{Y_\varepsilon}\left[L^{\tilde\alpha\tilde\beta}L^{\tilde\gamma\tilde\delta}R^{\tilde h}_{\tilde\delta\tilde\alpha\tilde\beta\tilde\gamma}+(L^2)^{\tilde\alpha\tilde\beta}R^{\tilde h}_{\tilde\alpha\tilde\beta}-L^{\tilde\alpha\tilde\beta}\tilde h^{\tilde\delta\tilde\gamma}\nabla^{\tilde h}_{\tilde\delta}R^{g_+}_{\tilde\gamma\tilde\alpha\tilde\beta\hat n}\right]f\,dv_{\tilde h}\\ &-\oint_{\Sigma_\varepsilon}|L|^2\nu_{Y_\varepsilon}(f)\,dv_{k_\varepsilon}. \end{aligned}\tag{48}$$

Applying integration by parts to D_3 and using (43) yields

$$\begin{aligned} D_3&=-\int_{Y_\varepsilon}2L^{\tilde\alpha\tilde\beta}\nabla^{\tilde h}_{\tilde\alpha}(L_{\tilde\beta\tilde\gamma}\nabla^{\tilde\gamma}_{\tilde h}f)\,dv_{\tilde h}\\ &=\oint_{\Sigma_\varepsilon}2L^2(\nabla^{\tilde h}f,\nu_{Y_\varepsilon})\,dv_{k_\varepsilon}. \end{aligned}\tag{49}$$

Now again using integration by parts and applying (43) we see

$$\begin{aligned} D_4&=\int_{Y_\varepsilon}3(L^2)^{\tilde\alpha\tilde\beta}\nabla^{\tilde h}_{\tilde\alpha}\nabla^{\tilde h}_{\tilde\beta}f\,dv_{\tilde h}\\ &=-\int_{Y_\varepsilon}3L^{\tilde\alpha\tilde\gamma}\nabla^{\tilde h}_{\tilde\alpha}L_{\tilde\gamma}{}^{\tilde\beta}\nabla^{\tilde h}_{\tilde\beta}f\,dv_{\tilde h}-\oint_{\Sigma_\varepsilon}3L_{\tilde\alpha\tilde\gamma}L^{\tilde\gamma\tilde\beta}\nu^{\tilde\alpha}_{Y_\varepsilon}\nabla^{\tilde h}_{\tilde\beta}f\,dv_{k_\varepsilon}. \end{aligned}\tag{50}$$

In order to rewrite the first term on the right, we consider the following:

$$
\begin{aligned}
(51)\qquad -3\tilde{h}^{\tilde{\beta}\tilde{\delta}}L^{\tilde{\alpha}\tilde{\gamma}}\nabla^{\tilde{h}}_{\tilde{\alpha}}L_{\tilde{\delta}\tilde{\gamma}}\nabla^{\tilde{h}}_{\tilde{\beta}}f &= -3\tilde{h}^{\tilde{\beta}\tilde{\delta}}L^{\tilde{\alpha}\tilde{\gamma}}(\nabla^{\tilde{h}}_{\tilde{\alpha}}L_{\tilde{\delta}\tilde{\gamma}} - \nabla^{\tilde{h}}_{\tilde{\delta}}L_{\tilde{\alpha}\tilde{\gamma}})\nabla^{\tilde{h}}_{\tilde{\beta}}f \\
&\quad - 3\tilde{h}^{\tilde{\beta}\tilde{\delta}}L^{\tilde{\alpha}\tilde{\gamma}}\nabla^{\tilde{h}}_{\tilde{\delta}}L_{\tilde{\alpha}\tilde{\gamma}}\nabla^{\tilde{h}}_{\tilde{\beta}}f \\
&= -3L^{\tilde{\alpha}\tilde{\gamma}}R^{g_+}_{\tilde{\beta}\tilde{\alpha}\tilde{\gamma}\hat{n}}\nabla^{\tilde{\beta}}_{\tilde{h}}f - \frac{3}{2}\nabla_{\tilde{\beta}}|L|^2\nabla^{\tilde{\beta}}_{\tilde{h}}f.
\end{aligned}
$$

Using the above formula and then applying integration by parts again we see

$$
\begin{aligned}
(52)\qquad & -\int_{Y_\varepsilon} 3L^{\tilde{\alpha}\tilde{\gamma}}\nabla^{\tilde{h}}_{\tilde{\alpha}}L_{\tilde{\gamma}}{}^{\tilde{\beta}}\nabla^{\tilde{h}}_{\tilde{\beta}}f\,dv_{\tilde{h}} \\
&= \int_{Y_\varepsilon}\left(-3L^{\tilde{\alpha}\tilde{\gamma}}R^{g_+}_{\tilde{\beta}\tilde{\alpha}\tilde{\gamma}\hat{n}}\nabla^{\tilde{\beta}}_{\tilde{h}}f - \frac{3}{2}\nabla_{\tilde{\beta}}|L|^2\nabla^{\tilde{\beta}}_{\tilde{h}}f\right)dv_{\tilde{h}} \\
&= 3\int_{Y_\varepsilon}\nabla^{\tilde{\beta}}_{\tilde{h}}L^{\tilde{\alpha}\tilde{\gamma}}R^{g_+}_{\tilde{\beta}\tilde{\alpha}\tilde{\gamma}\hat{n}}f\,dv_{\tilde{h}} + 3\int_{Y_\varepsilon}L^{\tilde{\alpha}\tilde{\gamma}}\nabla^{\tilde{\beta}}_{\tilde{h}}R^{g_+}_{\tilde{\beta}\tilde{\alpha}\tilde{\gamma}\hat{n}}f\,dv_{\tilde{h}} \\
&\quad + \int_{Y_\varepsilon}\frac{3}{2}|L|^2\nabla^{\tilde{\beta}}\nabla_{\tilde{\beta}}f\,dv_{\tilde{h}} + 3\oint_{\Sigma_\varepsilon}L^{\tilde{\alpha}\tilde{\gamma}}\nu^{\tilde{\delta}}_{Y_\varepsilon}R^{g_+}_{\tilde{\delta}\tilde{\alpha}\tilde{\gamma}\hat{n}}f \\
&\quad + \frac{3}{2}\oint_{\Sigma_\varepsilon}|L|^2\nabla_{\nu_{Y_\varepsilon}}f\,dv_{k_\varepsilon}
\end{aligned}
$$

We also observe that

$$
\begin{aligned}
(53)\qquad 3f\nabla^{\tilde{\beta}}_{\tilde{h}}L^{\tilde{\alpha}\tilde{\gamma}}R^{g_+}_{\tilde{\beta}\tilde{\alpha}\tilde{\gamma}\hat{n}} &= \frac{3}{2}f(\nabla^{\tilde{h}}_{\tilde{\beta}}L_{\tilde{\alpha}\tilde{\gamma}} - \nabla^{\tilde{h}}_{\tilde{\alpha}}L_{\tilde{\beta}\tilde{\gamma}})R_{g_+}{}^{\tilde{\beta}\tilde{\alpha}\tilde{\gamma}}{}_{\hat{n}} \\
&= \frac{3}{2}fR^{g_+}_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\hat{n}}R_{g_+}{}^{\tilde{\beta}\tilde{\alpha}\tilde{\gamma}}{}_{\hat{n}} \\
&= -\frac{3}{2}fW^{g_+}_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\hat{n}}W_{g_+}{}^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}{}_{\hat{n}}.
\end{aligned}
$$

If we use the above formula to re-write the first term on the right-hand side of (52), and use the resulting formula to re-write (50), we get

$$
\begin{aligned}
D_4 = \int_{Y_\varepsilon}&\Big[3fL^{\tilde{\alpha}\tilde{\beta}}\tilde{h}^{\tilde{\delta}\tilde{\gamma}}\nabla^{\tilde{h}}_{\tilde{\delta}}R^{g_+}_{\tilde{\gamma}\tilde{\alpha}\tilde{\beta}\hat{n}} \\
&- \frac{3}{2}fW^{g_+}_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\hat{n}}W_{g_+}{}^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}{}_{\hat{n}} + \frac{3}{2}|L|^2\Delta_{\tilde{h}}f\Big]dv_{\tilde{h}} \\
&+ \oint_{\Sigma_\varepsilon}\Big[3fL^{\tilde{\alpha}\tilde{\gamma}}R^{g_+}_{\tilde{\beta}\tilde{\alpha}\tilde{\gamma}\hat{n}}\nu^{\tilde{\beta}}_{Y_\varepsilon} + \frac{3}{2}|L|^2\nu_{Y_\varepsilon}(f) - 3L^2(\nabla^{\tilde{h}}f,\nu_{Y_\varepsilon})\Big]dv_{k_\varepsilon}.
\end{aligned}
$$

It is interesting to note that the cancellation of the first term in D_4 with the last interior term of D_2 accounts for the absence of any derivatives of Weyl terms in our final formula.

Now we want to compute $\int_{Y_\varepsilon} \mathcal{C}' dv_{\tilde{h}} = -\int_{Y_\varepsilon} A' dv_{\tilde{h}} - \frac{1}{3}\int_{Y_\varepsilon} B' dv_{\tilde{h}}$. Using our expressions for D_1, D_2, D_3 and D_4 and gathering together all of the terms that appear as integrals over Y_ε we get:

$$
\begin{aligned}
I_Y := -\int_{Y_\varepsilon} \Bigg[& \left(\frac{3|L|^2}{2} + \frac{|L|^4}{2} - 9\right) f \\
& + L^{\tilde{\alpha}\tilde{\beta}} L^{\tilde{\gamma}\tilde{\delta}} R^{\tilde{h}}_{\tilde{\gamma}\tilde{\alpha}\tilde{\beta}\tilde{\delta}} f + R^{\tilde{h}}_{\tilde{\alpha}\tilde{\beta}} (L^2)^{\tilde{\alpha}\tilde{\beta}} f - L^{\tilde{\alpha}\tilde{\beta}} \tilde{h}^{\tilde{\delta}\tilde{\gamma}} \nabla^{\tilde{h}}_{\tilde{\delta}} R^{g_+}_{\tilde{\gamma}\tilde{\alpha}\tilde{\beta}\hat{n}} f \\
& + f L^{\tilde{\alpha}\tilde{\beta}} \tilde{h}^{\tilde{\delta}\tilde{\gamma}} \nabla^{\tilde{h}}_{\tilde{\delta}} R^{g_+}_{\tilde{\gamma}\tilde{\alpha}\tilde{\beta}\hat{n}} \\
& - \frac{1}{2} f W^{g_+}_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\hat{n}} {W_{g_+}}^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}{}_{\hat{n}} + \frac{1}{2}|L|^2 \Delta_{\tilde{h}} f \\
& - 3f (L^2)^{\tilde{\alpha}\tilde{\beta}} R^{\tilde{h}}_{\tilde{\alpha}\tilde{\beta}} + f R^{\tilde{\alpha}\tilde{\beta}}_{\tilde{h}} R^{g_+}_{\tilde{\alpha}\hat{n}\tilde{\beta}\hat{n}} + f R^{\tilde{h}} |L|^2 \\
& + f|L^2|^2_{\tilde{h}} + f R^{g_+}_{\tilde{\alpha}\hat{n}\tilde{\beta}\hat{n}} (L^2)^{\tilde{\alpha}\tilde{\beta}} \Bigg] dv_{\tilde{h}}
\end{aligned} \tag{54}
$$

$$
\begin{aligned}
= -\int_{Y_\varepsilon} \Bigg[& \left(\frac{3|L|^2}{2} + \frac{|L|^4}{2} - 9\right) f \\
& + L^{\tilde{\alpha}\tilde{\beta}} L^{\tilde{\gamma}\tilde{\delta}} R^{\tilde{h}}_{\tilde{\gamma}\tilde{\alpha}\tilde{\beta}\tilde{\delta}} f \\
& - \frac{1}{2} f W^{g_+}_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\hat{n}} {W_{g_+}}^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}{}_{\hat{n}} + \frac{1}{2}|L|^2 \Delta_{\tilde{h}} f \\
- & 2f (L^2)^{\tilde{\alpha}\tilde{\beta}} R^{\tilde{h}}_{\tilde{\alpha}\tilde{\beta}} + f R^{\tilde{\alpha}\tilde{\beta}}_{\tilde{h}} R^{g_+}_{\tilde{\alpha}\hat{n}\tilde{\beta}\hat{n}} + f R^{\tilde{h}} |L|^2 \\
& + f|L^2|^2_{\tilde{h}} + f R^{g_+}_{\tilde{\alpha}\hat{n}\tilde{\beta}\hat{n}} (L^2)^{\tilde{\alpha}\tilde{\beta}} \Bigg] dv_{\tilde{h}}.
\end{aligned} \tag{55}
$$

Next, decomposing the Riemann tensor of g_+ gives

$$
R^{g_+}_{\tilde{\alpha}\hat{n}\tilde{\beta}\hat{n}} = W^{g_+}_{\tilde{\alpha}\hat{n}\tilde{\beta}\hat{n}} - \tilde{h}_{\tilde{\alpha}\tilde{\beta}}. \tag{56}
$$

Applying (56) to (44) gives

$$
R^{\tilde{h}}_{\tilde{\alpha}\tilde{\beta}} = -L^2_{\tilde{\alpha}\tilde{\beta}} - 2\tilde{h}_{\tilde{\alpha}\tilde{\beta}} - W^{g_+}_{\tilde{\alpha}\hat{n}\tilde{\beta}\hat{n}}. \tag{57}
$$

Decomposing the Riemann tensor of $\tilde{h}$ allows us to write

$$R^{\tilde{h}}_{\tilde{\delta}\tilde{\alpha}\tilde{\beta}\tilde{\zeta}} L^{\tilde{\zeta}\tilde{\delta}} L^{\tilde{\alpha}\tilde{\beta}} = L^{\tilde{\zeta}\tilde{\delta}} L^{\tilde{\alpha}\tilde{\beta}} [\tilde{h}_{\tilde{\delta}\tilde{\beta}} R^{\tilde{h}}_{\tilde{\alpha}\tilde{\zeta}} - \tilde{h}_{\tilde{\delta}\tilde{\zeta}} R^{\tilde{h}}_{\tilde{\alpha}\tilde{\beta}} - \tilde{h}_{\tilde{\alpha}\tilde{\beta}} R^{\tilde{h}}_{\tilde{\delta}\tilde{\zeta}} + \tilde{h}_{\tilde{\alpha}\tilde{\zeta}} R^{\tilde{h}}_{\tilde{\delta}\tilde{\beta}} - \frac{1}{2} R_{\tilde{h}} \tilde{h}_{\tilde{\delta}\tilde{\beta}} \tilde{h}_{\tilde{\alpha}\tilde{\zeta}} + \frac{1}{2} R_{\tilde{h}} \tilde{h}_{\tilde{\delta}\tilde{\zeta}} \tilde{h}_{\tilde{\alpha}\tilde{\beta}}]. \tag{58}$$

Next we apply (57) and then (45) to get

$$\begin{aligned} R^{\tilde{h}}_{\tilde{\delta}\tilde{\alpha}\tilde{\beta}\tilde{\zeta}} L^{\tilde{\zeta}\tilde{\delta}} L^{\tilde{\alpha}\tilde{\beta}} &= 2(L^2)^{\tilde{\alpha}\tilde{\beta}} R^{\tilde{h}}_{\tilde{\alpha}\tilde{\beta}} - \frac{1}{2}|L|^2 R_{\tilde{h}} \\ &= -|L|^4 - 4|L|^2 - 2(L^2)^{\tilde{\alpha}\tilde{\beta}} W^{g_+}_{\tilde{\alpha}\hat{n}\tilde{\beta}\hat{n}} + 3|L|^2 + \frac{1}{2}|L|^4. \end{aligned}$$

Note that we also use here the fact that $|L^2|^2 = \frac{1}{2}|L|^4$, which holds because $H_Y = 0$.

Simplifying gives

$$L^{\tilde{\alpha}\tilde{\beta}} L^{\tilde{\gamma}\tilde{\delta}} R^{\tilde{h}}_{\tilde{\gamma}\tilde{\alpha}\tilde{\beta}\tilde{\delta}} = -|L|^2 - \frac{1}{2}|L|^4 - 2(L^2)^{\tilde{\alpha}\tilde{\beta}} W^{g_+}_{\tilde{\alpha}\hat{n}\tilde{\beta}\hat{n}}.$$

Applying this to re-write $L^{\tilde{\alpha}\tilde{\beta}} L^{\tilde{\gamma}\tilde{\delta}} R^{\tilde{h}}_{\tilde{\gamma}\tilde{\alpha}\tilde{\beta}\tilde{\delta}}$ and using (32) to re-write $\Delta^{\tilde{h}} f$ gives

$$\begin{aligned} I_Y = -\int_{Y_\varepsilon} \Bigg[&\left(\frac{3|L|^2}{2} + \frac{|L|^4}{2} - 9 \right) f \\ &- |L|^2 f - 2(L^2)^{\tilde{\alpha}\tilde{\beta}} W^{g_+}_{\tilde{\alpha}\hat{n}\tilde{\beta}\hat{n}} f - \frac{1}{2}|L|^4 f \\ &- \frac{1}{2} f W^{g_+}_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\hat{n}} W_{g_+}{}^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}{}_{\hat{n}} + \frac{3}{2}|L|^2 f - \frac{1}{2}|L|^4 f \\ &+ 2f|L^2|^2 + 4|L|^2 f + 2(L^2)^{\tilde{\alpha}\tilde{\beta}} W^{g_+}_{\tilde{\alpha}\hat{n}\tilde{\beta}\hat{n}} - 6f|L|^2 - f|L|^4 \\ &- (L^2)^{\tilde{\alpha}\tilde{\beta}} W_{\tilde{\alpha}\hat{n}\tilde{\beta}\hat{n}} f - W_{\tilde{\alpha}\hat{n}\tilde{\beta}\hat{n}} W^{\tilde{\alpha}}{}_{\hat{n}}{}^{\tilde{\beta}}{}_{\hat{n}} f + |L|^2 f + 6f \\ &+ f|L^2|^2_{\tilde{h}} + f W^{g_+}_{\tilde{\alpha}\hat{n}\tilde{\beta}\hat{n}} (L^2)^{\tilde{\alpha}\tilde{\beta}} - f|L|^2 \Bigg] dv_{\tilde{h}} \\ = \int_{Y_\varepsilon} &\left(3f + \frac{1}{2} f W_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\hat{n}} W^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}{}_{\hat{n}} + f W_{\tilde{\alpha}\hat{n}\tilde{\gamma}\hat{n}} W^{\tilde{\alpha}}{}_{\hat{n}}{}^{\tilde{\gamma}}{}_{\hat{n}} \right) dv_{\tilde{h}}. \end{aligned} \tag{59}$$

We may simplify this helpfully:

Claim 2.

$$\frac{1}{2} W^{g_+}_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\hat{n}} W_{g_+}{}^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}{}_{\hat{n}} + W^{g_+}_{\tilde{\alpha}\hat{n}\tilde{\beta}\hat{n}} W^{\tilde{\alpha}\hat{n}\tilde{\beta}\hat{n}}_{g_+} = \frac{1}{8}|W_{g_+}|^2_{g_+}. \tag{60}$$

Proof. Observe that

$$\begin{aligned}|W_{g_+}|^2_{g_+} &= W^{g_+}_{ijkl}W^{ijkl}_{g_+}\\ &= 4W^{g_+}_{\hat{n}\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}W^{\hat{n}\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}_{g_+} + 4W^{g_+}_{\hat{n}\tilde{\alpha}\hat{n}\tilde{\beta}}W^{\hat{n}\tilde{\alpha}\hat{n}\tilde{\beta}}_{g_+} + W^{g_+}_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}}W^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}}_{g_+}.\end{aligned}$$

Now, $W^{g_+}_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}}$ is an algebraic curvature tensor on Y, a three-manifold, and (omitting g_+ for clarity) its trace is given by

$$W_{\tilde{\alpha}\tilde{\beta}}{}^{\tilde{\alpha}}{}_{\tilde{\delta}} = -W_{\hat{n}\tilde{\beta}}{}^{\hat{n}}{}_{\tilde{\delta}}.$$

But (by, e.g., Prop. 7.23 and Corollary 7.25 of [20]), an algebraic curvature tensor on a three-manifold is determined by its trace; in this case, the formula reads

$$W_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}} = W_{\hat{n}\tilde{\alpha}}{}^{\hat{n}}{}_{\tilde{\delta}}\tilde{h}_{\tilde{\beta}\tilde{\gamma}} + W_{\hat{n}\tilde{\beta}}{}^{\hat{n}}{}_{\tilde{\gamma}}\tilde{h}_{\tilde{\alpha}\tilde{\delta}} - W_{\hat{n}\tilde{\alpha}}{}^{\hat{n}}{}_{\tilde{\gamma}}\tilde{h}_{\tilde{\beta}\tilde{\delta}} - W_{\hat{n}\tilde{\beta}}{}^{\hat{n}}{}_{\tilde{\delta}}\tilde{h}_{\tilde{\alpha}\tilde{\gamma}}.$$

It follows that

$$W^{g_+}_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}}W^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}}_{g_+} = 4W^{\hat{n}\tilde{\alpha}\hat{n}\tilde{\beta}}_{g_+}W^{g_+}_{\hat{n}\tilde{\alpha}\hat{n}\tilde{\beta}}.$$

So

$$|W_{g_+}|^2_{g_+} = 4W^{g_+}_{\hat{n}\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}W^{\hat{n}\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}_{g_+} + 8W^{g_+}_{\hat{n}\tilde{\alpha}\hat{n}\tilde{\beta}}W^{\hat{n}\tilde{\alpha}\hat{n}\tilde{\beta}}_{g_+}.$$

□

It follows from the previous claim and (32) that (59) is equal to

$$I_Y = \int_{Y_\varepsilon}\left[|L|^2 f + \frac{1}{8}|W_{g_+}|^2_{g_+} f\right]dv_{\tilde{h}} - \oint_{\Sigma_\varepsilon}\nu_{Y_\varepsilon}(f)dv_{k_\varepsilon}. \tag{61}$$

Gathering the boundary terms from D_1, D_2, D_3 and D_4 and the normal derivative term on the above line we get

$$\begin{aligned}\oint_{\Sigma_\varepsilon}\Bigg[&\left(\operatorname{Ric}_{\tilde{h}} - \frac{1}{2}R_{\tilde{h}}\tilde{h}\right)(\nabla^{\tilde{h}}f,\nu_{Y_\varepsilon}) + |L|^2\nu_{Y_\varepsilon}(f) - 2L^2(\nabla^{\tilde{h}}f,\nu_{Y_\varepsilon})\\ &- fL^{\tilde{\alpha}\tilde{\gamma}}R^{g_+}_{\tilde{\beta}\tilde{\alpha}\tilde{\gamma}\hat{n}}\nu^{\tilde{\beta}}_{Y_\varepsilon} - \frac{1}{2}|L|^2\nu_\varepsilon(f) + L^2(\nabla f,\nu_\varepsilon) - \nu_{Y_\varepsilon}(f)\Bigg]dv_{k_\varepsilon}.\end{aligned} \tag{62}$$

Now we apply (57) to the first term and use $R^{g_+}_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\hat{n}} = W^{g_+}_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\hat{n}}$ to re-write $fL^{\tilde{\alpha}\tilde{\gamma}}R^{g_+}_{\tilde{\beta}\tilde{\alpha}\tilde{\gamma}\hat{n}}\nu^{\tilde{\beta}}_{\varepsilon}$, giving us

$$\oint_{\Sigma_\varepsilon}\Big[-2L^2(\nabla f,\nu_{Y_\varepsilon}) + fL^{\tilde{\alpha}\tilde{\gamma}}W^{g_+}_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\hat{n}}\nu^{\tilde{\beta}}_{Y_\varepsilon} - W^{g_+}_{\tilde{\alpha}\hat{n}\tilde{\beta}\hat{n}}f^{\tilde{\alpha}}\nu^{\tilde{\beta}}_{\varepsilon} + |L|^2\nu_{Y_\varepsilon}(f)\Big]dv_{k_\varepsilon}.$$

Combining this with (35), (36) and (61) gives us

$$\begin{aligned} 3\frac{d}{dt}V_\epsilon(t)\big|_{t=0} = \int_{Y_\varepsilon}|L|^2 f dv_{\tilde{h}} + \oint_{\Sigma_\varepsilon}\Big[&- W^{g_+}_{\tilde{\alpha}\hat{n}\tilde{\beta}\hat{n}}\nu^{\tilde{\alpha}}_{Y_\varepsilon}\nabla^{\tilde{\beta}}_{\tilde{h}}f + |L|^2\nu_{Y_\varepsilon}(f) \\ &- 2L_{\tilde{\alpha}\tilde{\gamma}}L^{\tilde{\gamma}}{}_{\tilde{\beta}}\nu^{\tilde{\alpha}}_{Y_\varepsilon}\nabla^{\tilde{\beta}}_{\tilde{h}}f + L^{\tilde{\gamma}\tilde{\delta}}W^{g_+}_{\tilde{\gamma}\tilde{\alpha}\tilde{\delta}\hat{n}}f\nu^{\tilde{\alpha}}_{Y_\varepsilon}\Big]dv_{k_\varepsilon} + O(\varepsilon). \end{aligned} \tag{63}$$

Next we will examine the asymptotics of the term $-W^{g_+}_{\tilde{\alpha}\hat{n}\tilde{\beta}\hat{n}}\nu^{\tilde{\alpha}}_{Y_\varepsilon}f^{\tilde{\beta}}$. Now, it follows from (5), (6), and the second-last equation on the bottom of page 52 of [11] that

$$W^{\bar{g}}_{0\mu 0\nu} = O(r).$$

Moreover, from the first equation on p. 53 of the same book, we may conclude that

$$W^{\bar{g}}_{0\mu 0\nu} = -\frac{3}{2}rg^{(3)}_{\mu\nu} + O(r^2)$$

with $g^{(3)}$ as in (5). By the conformal change formula for the Weyl tensor, therefore, we find

$$W^{g_+}_{\mu 0\nu 0} = -\frac{3}{2}r^{-1}g^{(3)}_{\mu\nu} + O_{\bar{g}}(1). \tag{64}$$

Now by (33),

$$\begin{aligned} -W_{g_+}(\nu_{Y_\varepsilon},\mu_Y,\nabla^{\tilde{h}}f,\mu_Y) &= -r^3W_{g_+}(\bar{\nu}_{Y_\epsilon},\bar{\mu}_Y,\nabla^Y f,\bar{\mu}_Y) \\ &= -r^3W^{g_+}_{\tilde{\alpha}\bar{n}\tilde{\beta}\bar{n}}\bar{\nu}^{\tilde{\alpha}}_{Y_\epsilon}f^{\tilde{\beta}} \\ &= -r^5W^{g_+}_{\tilde{\alpha}\bar{n}\tilde{\beta}\bar{n}}\bar{\nu}^{\tilde{\alpha}}_{Y_\epsilon}\bar{g}^{\tilde{\beta}\tilde{\gamma}}\partial_{\tilde{\gamma}}f \\ &= r^3W_{\tilde{0}\bar{n}\tilde{0}\bar{n}}\bar{\nu}^{\tilde{0}}_{Y_\epsilon}\tilde{f} + O(r^4), \end{aligned}$$

where $\bar{n}$ corresponds to $\bar{\mu}_Y$. Taking (25), (26), (64), and (33), we see that the first corner term of (63) may be written

$$\oint_{\Sigma_\varepsilon} -W_{g_+}(\nu_{Y_\epsilon},\mu_Y,\nabla^{\tilde{h}}f,\mu_Y)dv_{k_\varepsilon} = \oint_{\Sigma}\frac{3}{2}g^{(3)}(\bar{\nu}_M,\bar{\nu}_M)\tilde{f}dv_{\bar{k}} + O(\varepsilon). \tag{65}$$

We now simplify the remaining terms of (63):

Claim 3.

$$\int_{Y_\varepsilon} |L|^2 f dv_{\tilde{h}} + \oint_{\Sigma_\varepsilon} \Big[|L|^2 \nu_{Y_\varepsilon}(f) - 2L_{\tilde{\alpha}\tilde{\gamma}} L^{\tilde{\gamma}}{}_{\tilde{\beta}} \nu^{\tilde{\alpha}}_{Y_\varepsilon} \nabla^{\tilde{\beta}}_{\tilde{h}} f + L^{\tilde{\gamma}\tilde{\delta}} W^{g_+}_{\tilde{\gamma}\tilde{\alpha}\tilde{\delta}\hat{n}} f \nu^{\tilde{\alpha}}_{Y_\varepsilon} \Big] dv_{k_\varepsilon} = f.p. \int_{\mathring{Y}} |L|^2 f dv_{\tilde{h}} + O(\varepsilon \log \varepsilon). \tag{66}$$

Proof. Observe that

$$L_{\tilde{\alpha}\tilde{\beta}} = \frac{\overline{L}_{\tilde{\alpha}\tilde{\beta}}}{r} + \frac{\bar{\mu}_Y(r)}{r^2} \bar{g}_{\tilde{\alpha}\tilde{\beta}}. \tag{67}$$

Now

$$\bar{\mu}_Y = (1 + O(r^2))\partial_w - \left(\frac{\overline{\eta}_M r}{2} + O(r^3 \log r) \right) \partial_r + O^a(r^2)\partial_a. \tag{68}$$

Therefore

$$\bar{\mu}_Y(r) = -\frac{1}{2}\big[\overline{\eta}_M r + O(r^3 \log r)\big]. \tag{69}$$

Now using the fact that $\bar{g}_{\tilde{\alpha}\tilde{\beta}} = \bar{g}_{\alpha\beta} + O(r^2)$ we may write

$$\overline{L}_{\tilde{\alpha}\tilde{\beta}} = -\frac{1}{2}\bar{\mu}_Y \bar{g}_{\alpha\beta} + O(r^2) = -\frac{1}{2}\partial_w \bar{g}_{\alpha\beta} + O(r^2). \tag{70}$$

Therefore we may write

$$L_{\tilde{\alpha}\tilde{\beta}} = -\frac{\partial_w \bar{g}_{\alpha\beta}}{2r} - \frac{\overline{\eta}_M \bar{g}_{\alpha\beta}}{2r} + O(r \log r). \tag{71}$$

Hence

$$\begin{aligned} |L|^2_{\tilde{h}} \nu_{Y_\varepsilon}(f) dv_{k_\varepsilon} &= \varepsilon^4 \bar{g}^{\alpha\gamma} \bar{g}^{\beta\delta} \left[\frac{\partial_w \bar{g}_{\alpha\beta}}{2\varepsilon} + \frac{\overline{\eta}_M \bar{g}_{\alpha\beta}}{2\varepsilon} + O(\varepsilon \log \varepsilon) \right] \cdot \\ &\quad \cdot \left[\frac{\partial_w \bar{g}_{\gamma\delta}}{2\varepsilon} + \frac{\overline{\eta}_M \bar{g}_{\gamma\delta}}{2\varepsilon} + O(\varepsilon \log \varepsilon) \right] \left[-\tilde{f}\varepsilon^{-1} + O(\varepsilon) \right] dv_{k_\varepsilon} \\ &= \big[-\varepsilon^{-1} |\mathring{\overline{II}}_M|^2_k \tilde{f} + O(\varepsilon \log \varepsilon) \big] dv_{\bar{k}_\varepsilon}, \end{aligned} \tag{72}$$

so we may write

$$\oint_{\Sigma_\varepsilon} |L|^2_{\tilde{h}} \nu_{Y_\varepsilon}(f) dv_{k_\varepsilon} = -\oint_\Sigma |\overset{\circ}{\overline{II}}_M|^2_k \tilde{f} dv_k \varepsilon^{-1} + O(\varepsilon \log \varepsilon).$$

Now,

$$(73)\qquad \begin{aligned} |L|^2_{\tilde{h}} f dv_{\tilde{h}} &= r^4 \bar{g}^{\alpha\gamma} \bar{g}^{\beta\delta} \left[\frac{\partial_w \bar{g}_{\alpha\beta}}{2r} + \frac{\overline{\eta}_M \bar{g}_{\alpha\beta}}{2r} + O(r \log r) \right] \cdot \\ &\quad \cdot \left[\frac{\partial_w \bar{g}_{\gamma\delta}}{2r} + \frac{\overline{\eta}_M \bar{g}_{\gamma\delta}}{2r} + O(r \log r) \right] \left[\tilde{f} r^{-1} + O(r) \right] dv_{\tilde{h}} \\ &= \left[r^{-2} |\overset{\circ}{\overline{II}}_M|^2_k \tilde{f} + O(\log r) \right] dv_{\bar{h}}, \end{aligned}$$

so

$$(74)\qquad \begin{aligned} \int_{Y_\varepsilon} |L|^2_{\tilde{h}} f dv_{\tilde{h}} &= C + \int_\varepsilon^{r_0} \oint_\Sigma |L|^2_{\tilde{h}} f dv_k dr \\ &= C + \int_\varepsilon^{r_0} \oint_\Sigma r^{-2} |\overset{\circ}{\overline{II}}_M|^2_k \tilde{f} + O(\log r) dv_k dr \\ &= C' + \varepsilon^{-1} \oint_\Sigma |\overset{\circ}{\overline{II}}_M|^2_k \tilde{f} dv_k + O(\varepsilon \log \varepsilon) \end{aligned}$$

for some constants C and C' and $r_0 > 0$ chosen small enough. Observe that

$$C' = f.p. \int_{\overset{\circ}{Y}} |L|^2 f dv_{\tilde{h}}.$$

By (71) we can write

$$(L^2)_{\tilde{\alpha}\tilde{\beta}} = O(1),$$

$$(L^2)_{\tilde{\alpha}\tilde{0}} = O(r).$$

Also observe by (26) that $\nu_\varepsilon^{\tilde{\alpha}} = O(r^2)$ unless $\tilde{\alpha} = \tilde{0}$, in which case $\nu_\varepsilon^{\tilde{0}} = O(r)$. Now if we let a run over the indices $1, 2$ we can write

$$\begin{aligned} L_{\tilde{\alpha}\tilde{\gamma}} L^{\tilde{\gamma}}_{\tilde{\beta}} \nu_\varepsilon^{\tilde{\alpha}} \nabla^{\tilde{\beta}} f &= h^{\tilde{\beta}\tilde{\delta}} (L^2)_{\tilde{a}\tilde{\beta}} \nu_\varepsilon^{\tilde{a}} f_{\tilde{\delta}} + h^{\tilde{\beta}\tilde{\delta}} (L^2)_{\tilde{0}\tilde{\beta}} \nu_\varepsilon^{\tilde{0}} f_{\tilde{\delta}} \\ &= O(r^3) \end{aligned}$$

It follows that

$$L_{\tilde{\alpha}\tilde{\gamma}} L^{\tilde{\gamma}}_{\tilde{\beta}} \nu_\varepsilon^{\tilde{\alpha}} \nabla^{\tilde{\beta}} f dv_k = O(\varepsilon) dv_{\bar{k}},$$

so

$$\oint_{\Sigma_\varepsilon} L_{\tilde{\alpha}\tilde{\gamma}} L^{\tilde{\gamma}}_{\tilde{\beta}} \nu^{\tilde{\alpha}}_\varepsilon \nabla^{\tilde{\beta}} f dv_k = O(\varepsilon).$$

Now we turn our attention to the term $L^{\tilde{\gamma}\tilde{\delta}} W^{g_+}_{\gamma\alpha\delta n} f \nu^\alpha_\varepsilon$. First observe that

$$W^{g_+}_{\tilde{\gamma}\tilde{\alpha}\tilde{\delta}\hat{n}} = r W^{g_+}_{\tilde{\gamma}\tilde{\alpha}\tilde{\delta}\hat{\hat{n}}}$$

and

$$W^{g_+}_{\tilde{\gamma}\tilde{\alpha}\tilde{\delta}\hat{\hat{n}}} = \frac{W^{\bar{g}}_{\tilde{\gamma}\tilde{\alpha}\tilde{\delta}\hat{\hat{n}}}}{r^2},$$

where $\hat{\hat{n}}$ corresponds to $\bar{\mu}_Y$. Now

$$\begin{aligned}
W^{\bar{g}}_{\tilde{\gamma}\tilde{0}\tilde{\delta}\bar{\bar{n}}} &= R^{\bar{g}}_{\tilde{\gamma}\tilde{0}\tilde{\delta}\hat{\hat{n}}} \\
&= \nabla^{\bar{g}}_{\tilde{0}} L^{\bar{g}}_{\tilde{\gamma}\tilde{\delta}} - \nabla^{\bar{g}}_{\tilde{\gamma}} L^{\bar{g}}_{\tilde{0}\tilde{\delta}} \\
&= \partial_{\hat{0}} \overline{L}_{\tilde{\gamma}\tilde{\delta}} - \Gamma^{\tilde{\beta}}_{\tilde{0}\tilde{\gamma}} \overline{L}_{\tilde{\beta}\tilde{\delta}} - \Gamma^{\tilde{\beta}}_{\tilde{0}\tilde{\delta}} \overline{L}_{\tilde{\beta}\tilde{\gamma}} - (\partial_{\tilde{\gamma}} \overline{L}_{\tilde{0}\tilde{\delta}} - \Gamma^{\tilde{\beta}}_{\tilde{0}\tilde{\gamma}} \overline{L}_{\tilde{\beta}\tilde{\delta}} - \Gamma^{\tilde{\beta}}_{\tilde{\gamma}\tilde{\delta}} \overline{L}_{\tilde{\beta}\tilde{0}}) \\
&= \partial_{\tilde{0}} \overline{L}_{\tilde{\gamma}\tilde{\delta}} - \Gamma^{\tilde{\beta}}_{\tilde{0}\tilde{\delta}} \overline{L}_{\tilde{\beta}\tilde{\gamma}} - (\partial_{\tilde{\gamma}} \overline{L}_{\tilde{0}\tilde{\delta}} - \Gamma^{\tilde{\beta}}_{\tilde{\gamma}\tilde{\delta}} \overline{L}_{\tilde{\beta}\tilde{0}}) \\
&= O(r).
\end{aligned}$$

This gives us

$$L^{\tilde{\gamma}\tilde{\delta}} W^{g_+}_{\tilde{\gamma}\tilde{0}\tilde{\delta}\hat{\hat{n}}} f \nu^{\tilde{0}}_\varepsilon = O(r^3)$$

and

$$\begin{aligned}
L^{\tilde{\gamma}\tilde{\delta}} W^{g_+}_{\tilde{\gamma}\tilde{\alpha}\tilde{\delta}\hat{\hat{n}}} f \nu^{\tilde{\alpha}}_\varepsilon &= L^{\tilde{\gamma}\tilde{\delta}} W_{\tilde{\gamma}\tilde{0}\tilde{\delta}\hat{\hat{n}}} f \nu^{\tilde{0}}_\varepsilon + L^{\tilde{\gamma}\tilde{\delta}} W^{g_+}_{\tilde{\gamma}\tilde{b}\tilde{\delta}\hat{\hat{n}}} f \nu^{\tilde{b}}_\varepsilon \\
&= L^{\tilde{\gamma}\tilde{\delta}} W^{g_+}_{\tilde{\gamma}\tilde{0}\tilde{\delta}\hat{\hat{n}}} f \nu^{\tilde{0}}_\varepsilon + O(r^3) \\
&= O(r^3).
\end{aligned}$$

Therefore we may write

$$L^{\tilde{\gamma}\tilde{\delta}} W^{g_+}_{\tilde{\gamma}\tilde{\alpha}\tilde{\delta}\hat{n}} f \nu^{\tilde{\alpha}}_\varepsilon dv_k = O(\varepsilon) dv_{\bar{k}}.$$

We then get that

$$\oint_{\Sigma_\varepsilon} L^{\tilde{\gamma}\tilde{\delta}} W^{g_+}_{\tilde{\gamma}\tilde{\alpha}\tilde{\delta}\hat{n}} \nu^{\tilde{\alpha}}_\varepsilon f dv_k = O(\varepsilon).$$

This proves the claim. □

Combining Claim 3 with (63) and (65) and letting $\varepsilon \to 0$ yields the theorem. $\square$

As promised in the introduction, we show that the finite part can be written as a convergent integral.

Lemma 4.3. *With notation as above, we obtain*

$$f.p. \int_{Y_\varepsilon} |L|^2 f dv_{\tilde{h}} = \int_Y \left(\Delta_{\tilde{h}}(|L|^2 f) + |L|^2 f\right) dv_{\tilde{h}},$$

where the right side is a convergent integral.

Proof. By (73) we know

$$|L|^2_{\tilde{h}} f = |\mathring{\overline{II}}_M|^2_{\overline{k}} \tilde{f} r + O(r^3 \log(r)) \tag{75}$$

which implies

$$\begin{aligned}\nabla_{\nu_\varepsilon} |L|^2_{\tilde{h}} f &= \nabla_{\nu_\varepsilon}\left[|\mathring{\overline{II}}_M|^2_{\overline{k}} \tilde{f} r + O(r^3 \log(r))\right] \\ &= |\mathring{\overline{II}}_M|^2_{\overline{k}} \tilde{f} \varepsilon + O(\varepsilon^3 \log(\varepsilon)).\end{aligned}$$

It follows that

$$\oint_{\Sigma_\epsilon} \nabla_{\nu_\varepsilon}(|L|^2_{\tilde{h}} f) dv_{k_\varepsilon} = \varepsilon^{-1} \oint_\Sigma |\mathring{\overline{II}}_M|^2_{\overline{k}} \tilde{f} dv_{\overline{k}} + O(\varepsilon \log(\varepsilon)), \tag{76}$$

where we have used that fact that $\sqrt{\det \overline{k}_\varepsilon}$ has vanishing first derivative at $r = 0$. By Stokes's theorem,

$$\int_{Y_\varepsilon} \Delta^Y (|L|^2_{\tilde{h}} f) dv_{\tilde{h}} = -\oint_{\Sigma_\epsilon} \nabla_{\nu_\varepsilon}(|L|^2_{\tilde{h}} f) dv_{\overline{k}_\varepsilon}. \tag{77}$$

Also recall (74):

$$\begin{aligned}\int_{Y_\varepsilon} |L|^2_{\tilde{h}} f dv_{\tilde{h}} &= C + \oint_\Sigma \int_\varepsilon^{r_0} |\mathring{\overline{II}}_M|^2_{\overline{k}} \tilde{f} r^{-2} + O(1) dr dv_{\overline{k}} \\ &= f.p. \int_{Y_\varepsilon} |L|^2 f dv_{\tilde{h}} + \varepsilon^{-1} \oint_\Sigma |\mathring{\overline{II}}_M|^2_{\overline{k}} \tilde{f} dv_{\overline{k}} + O(\varepsilon).\end{aligned}$$

The result now follows. $\square$

5. Appendix

In this appendix we give a brief summary of the formulas needed in the proof of Theorem 1.3, based on notes provided by Nicholas Edelen. Although they are all standard, due to differences in notation and convention we have decided to present a summary of the calculations.

Let (X, g) be a Riemannian manifold of dimension $n+1$, and ∇ denote the Riemannian connection. Let Y be a smooth manifold of dimension n, and consider a one-parameter family of smooth immersions $\mathcal{F} : (-\epsilon, \epsilon) \times Y \to X$. Let $h = (\mathcal{F}_t)^* g$ be the induced metric on Y, and ∇^Y the corresponding connection.

Let V denote the variation field of $\mathcal{F}_t$:

$$V = \frac{d}{dt}\mathcal{F}_t\Big|_{t=0}.$$

Eventually we will assume that $\mathcal{F}_t$ is a normal variation; i.e., if ν is a choice of unit to Y then there is a function $f \in C^\infty(Y)$ such that $V = f\nu$.

Let $\{x^1, \dots, x^n\}$ be local coordinates near a point $0 \in Y$. They induce coordinates on $\mathcal{F}_t(Y)$ defined via $(t, x^1, \dots, x^n) \mapsto \mathcal{F}_t(x^1, \dots, x^n)$, and we have the corresponding coordinate vector fields $\{\partial_1, \dots, \partial_n\}$, along with $\partial_t = V$. Let

$$h_{\alpha\beta}(t, x) = g_{\mathcal{F}_t(Y)}(\partial_\alpha, \partial_\beta).$$

Then

$$\begin{aligned} h'_{\alpha\beta} &= \frac{\partial}{\partial t} h_{\alpha\beta}\Big|_{t=0} \\ &= g(\nabla_{\partial_t}\partial_\alpha, \partial_\beta) + g(\partial_\alpha, \nabla_{\partial_t}\partial_\beta) \\ &= g(\nabla_{\partial_\alpha} V, \partial_\beta) + g(\partial_\alpha, \nabla_{\partial_\beta} V). \end{aligned}$$

If $V = f\nu$, then this becomes

$$h'_{\alpha\beta} = fg(\nabla_{\partial_\alpha}\nu, \partial_\beta) + g(\partial_\alpha, \nabla_{\partial_\beta}\nu). \tag{78}$$

Given a choice of normal ν our definition of the second fundamental form of Y is

$$L(\partial_\alpha, \partial_\beta) = g(\nu, \nabla_{\partial_\alpha}\partial_\beta) = -g(\nabla_{\partial_\alpha}\nu, \partial_\beta). \tag{79}$$

Therefore, by (78) we conclude

$$h'_{\alpha\beta} = -2fL_{\alpha\beta}. \tag{80}$$

By the standard formula for the inverse, this implies

$$(h^{\alpha\beta})' = 2fL^{\alpha}{}_{\gamma}L^{\beta\gamma}. \tag{81}$$

By our definition of second fundamental form,

$$\begin{aligned} L'_{\alpha\beta} &= \frac{\partial}{\partial t} L_{\alpha\beta}\Big|_{t=0} \\ &= g(\nabla_{\partial_t}\nu, \nabla_{\partial_\alpha}\partial_\beta) + g(\nu, \nabla_{\partial_t}\nabla_{\partial_\alpha}\partial_\beta). \end{aligned} \tag{82}$$

The first term on the right is easily seen to vanish, since $0 = \partial_t g(\nu,\nu) = 2g(\nabla_{\partial_t}\nu, \nu)$ implies that

$$g(\nabla_{\partial_t}\nu, \nabla_{\partial_\alpha}\partial_\beta) = -L_{\alpha\beta}\, g(\nabla_{\partial_t}\nu, \nu) = 0. \tag{83}$$

For the second term, we commute derivatives to get

$$\begin{aligned} g(\nu, \nabla_{\partial_t}\nabla_{\partial_\alpha}\partial_\beta) &= g(\nu, \nabla_{\partial_\alpha}\nabla_{\partial_t}\partial_\beta) + R(V, \partial_\alpha, \partial_\beta, \nu) \\ &= g(\nu, \nabla_{\partial_\alpha}\nabla_{\partial_\beta}V) + R(V, \partial_\alpha, \partial_\beta, \nu), \end{aligned} \tag{84}$$

where R is the curvature tensor of g. If $V = f\nu$ then by (83) and (84), (82) simplifies to

$$\begin{aligned} L'_{\alpha\beta} &= g(\nu, \nabla_{\partial_t}\nabla_{\partial_\alpha}\partial_\beta) \\ &= g(\nu, \nabla_{\partial_\alpha}\nabla_{\partial_\beta}(f\nu)) + fR(\nu, \partial_\alpha, \partial_\beta, \nu) \\ &= \nabla^Y_\alpha\nabla^Y_\beta f + g(\nu, \partial_\alpha f\nabla_{\partial_\beta}\nu + \partial_\beta f\nabla_{\partial_\alpha}\nu + f\nabla_{\partial_\alpha}\nabla_{\partial_\beta}\nu) \\ &\quad + fR(\nu, \partial_\alpha, \partial_\beta, \nu) \\ &= \nabla^Y_\alpha\nabla^Y_\beta f + fg(\nu, \nabla_{\partial_\alpha}\nabla_{\partial_\beta}\nu) + fR(\nu, \partial_\alpha, \partial_\beta, \nu), \end{aligned} \tag{85}$$

where in the last line we used the fact that $\partial_\alpha g(\nu,\nu) = 0$. Using this fact again we also find

$$g(\nu, \nabla_{\partial_\alpha}\nabla_{\partial_\beta}\nu) = -g(\nabla_{\partial_\alpha}\nu, \nabla_{\partial_\beta}\nu). \tag{86}$$

It follows from the definition of the second fundamental form that

$$\nabla_{\partial_\alpha}\nu = -L^\gamma_\alpha\,\partial_\gamma,$$

hence

$$-g(\nabla_{\partial_\alpha}\nu, \nabla_{\partial_\beta}\nu) = -L_\alpha^\gamma L_{\beta\gamma}.$$

Substituting this into (86) and combining with (85), we arrive at

$$L'_{\alpha\beta} = \nabla_\alpha^Y \nabla_\beta^Y f - f\, L_\alpha^\gamma L_{\beta\gamma} + f\, R(\nu, \partial_\alpha, \partial_\beta, \nu). \tag{87}$$

For the variation of the mean curvature $H = h^{\alpha\beta} L_{\alpha\beta}$ we use (81) and (87) to obtain

$$H' = \Delta_Y f + \big(|L|^2 + \mathrm{Ric}(\nu,\nu)\big) f. \tag{88}$$

Finally, using the standard formula for the derivative of the volume form, we have

$$(dv_h)' = -fH\, dv_h. \tag{89}$$

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