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Einstein, σ-model and Ernst-type equations and non-isospectral GBDT version of Darboux transformation

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We present a non-isospectral GBDT version of Bäcklund-Darboux transformation for the gravitational and σ-model equations. New families of explicit solutions correspond to the case of GBDT with non-diagonal generalized matrix eigenvalues. An interesting integrable Ernst-type system, the auxiliary linear systems of which are non-isospectral canonical systems, is studied as well.

1. Introduction

The study of the integrable reductions of Einstein field equations goes back to the seminal paper [6] (see also [23]). The survey [3] includes several references to the interesting articles which precede [6] and a bibliography of the related works during thirty years after its publication. For the recent references one can turn, for instance, to [21]. Following the publication of [6], a closely related σ-model equation was studied in [26]. Gravitational (Einstein) equation and σ-model equation both belong to the so called non-isospectral case where the spectral parameter depends on other variables (see, e.g., [7, 8] on this topic). We apply to the gravitational (Einstein) equation and σ-model equation the non-isospectral GBDT version of Bäcklund-Darboux transformation. This version of Bäcklund-Darboux transformation is especially suitable for the explicit construction of the wave functions and solutions of those equations. Generalized matrix eigenvalues $\mathcal{A}$ are used in GBDT instead of the usual eigenvalues, and *new classes of explicit solutions* appear when we deal with the non-diagonal $\mathcal{A}$ (e.g., $\mathcal{A}$ in the normal Jordan form).

Hamiltonian evolution equations are related to Einstein and σ-model equations (and play an essential role in their study), see, for instance, [4, 5, 18]. In this paper, we investigate an interesting Ernst-type integrable nonlinear system:

$$H(\xi,\eta) - \mathcal{H}(\xi,\eta) = \mathrm{i}\big(\mathcal{H}(\xi,\eta)JH(\xi,\eta) - H(\xi,\eta)J\mathcal{H}(\xi,\eta)\big),$$
$$H_\eta(\xi,\eta) = \mathcal{H}_\xi(\xi,\eta) \quad (H \geq 0, \quad \mathcal{H} \geq 0), \quad J = \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix},$$

where the Hamiltonians H and $\mathcal{H}$ are $2p \times 2p$ matrix functions and the auxiliary (to our Ernst-type system) linear systems are non-isospectral canonical systems with these Hamiltonians. As usual, i above stands for the imaginary unit ($\mathrm{i}^2 = -1$) and I_p is the $p \times p$ identity matrix.

Gravitational (Einstein) equation in light-cone coordinates has the form

$$\big(\alpha(\xi,\eta)u_\xi(\xi,\eta)u(\xi,\eta)^{-1}\big)_\eta + \big(\alpha(\xi,\eta)u_\eta(\xi,\eta)u(\xi,\eta)^{-1}\big)_\xi = 0, \ \alpha_{\xi\eta} = 0, \tag{1.1}$$

where α is a scalar function, u is a 2×2 matrix function, and $u_\xi = \frac{\partial}{\partial \xi} u$. Physically meaningful solutions u of (1.1) have the properties [6, 7]:

$$\alpha \in \mathbb{R}, \quad u(\xi,\eta) \in \mathrm{GL}(2,\mathbb{R}), \tag{1.2}$$

where $\mathbb{R}$ is the real axis and $\mathrm{GL}(2,\mathbb{R})$ stands for the set of 2×2 invertible matrix functions with real-valued entries. The solutions satisfying an additional property

$$\det(u) = \alpha^2. \tag{1.3}$$

are constructed via the multiplication of the solutions u of (1.1) satisfying (1.2) by certain real-valued scalar functions (see (2.31) or [6, (2.17)]).

In the important paper [26], the authors wrote down σ-model equation in the form (1.1), where u are $m \times m$ invertible matrix functions with complex-valued entries ($u \in \mathrm{GL}(m,\mathbb{C})$). More precisely, it is supposed that the relations

$$\alpha(\xi,\eta) \in \mathbb{R}, \quad u(\xi,\eta) \in \mathrm{GL}(m,\mathbb{C}), \quad u(\xi,\eta)^* J u(\xi,\eta) \equiv J \tag{1.4}$$

hold. Here, $\mathbb{C}$ stands for the complex plane, u^* means complex conjugate transpose of u, and we assume further that the $m \times m$ matrix J satisfies relations

$$J = J^* = J^{-1}. \tag{1.5}$$

The paper consists of five sections. Some basic GBDT relations for the equation (1.1) are given in Section 2. Using these relations, we express (in

Section 3) wide families of solutions of the σ-model and gravitational equations (so called *transformed* solutions) via some initial solutions. The discussed above Ernst-type equation is studied in Section 4. Finally, explicit formulas and new explicit solutions are presented in Section 5. Several auxiliary results are proved in the Appendices A and B.

Some notations have been introduced above and further notations are explained here. The notation $\mathbb{N}$ means the set of positive integer numbers, $\overline{\alpha}$ stands for the complex conjugate of α, and the inequality $H \geq 0$ for some matrix H means that $H = H^*$ and has nonnegative eigenvalues. The set of $i \times k$ matrices with real-valued entries in denoted by $\mathbb{R}^{i\times k}$. The spectrum of matrix $\mathcal{A}$ is denoted by $\sigma(\mathcal{A})$. We say that the function is continuously differentiable if its first derivatives exist and are continuous (in the topology $\mathbb{R}^k$ if it is a function of k variables).

2. Preliminaries

1. Bäcklund-Darboux transformations and related commutation methods present an important tool in spectral, gauge and soliton theories (see, e.g., [9, 11, 14–16, 20, 24, 25, 27, 44]). Our GBDT version of Bäcklund–Darboux transformation was first introduced in [29] (see further results and references in the papers [19, 20, 30, 32, 35] and in the book [37]).

In this section, we derive some important relations for the equation (1.1) from our more general GBDT results in [30, Sections 2,3]. We study the case (1.1), (1.4). We discuss also the modification of the solution of (1.1), (1.2) such that (1.3) holds.

Integrable linear equations are often considered in the so called zero curvature form

$$\frac{d}{d\eta}G - \frac{d}{d\xi}F + [G, F] = 0 \quad ([G, F] := GF - FG), \tag{2.1}$$

which is the compatibility condition of the auxiliary linear systems

$$\begin{aligned} \frac{d}{d\xi}w(\xi, \eta, \lambda) &= G(\xi, \eta, \lambda)w(\xi, \eta, \lambda), \\ \frac{d}{d\eta}w(\xi, \eta, \lambda) &= F(\xi, \eta, \lambda)w(\xi, \eta, \lambda), \end{aligned} \tag{2.2}$$

where ξ and η are independent variables, λ is the spectral parameter and w is an $m \times m$ non-degenerate matrix function (fundamental solution). In the so called isospectral case, where λ does not depend on ξ and η, one can write, for instance, $G_\eta = \frac{\partial}{\partial\eta}G$ instead of $\frac{d}{d\eta}G$. In the non-isospectral case, where λ depends on ξ and (or) η, we need the total derivatives $\frac{d}{d\xi}$ and (or) $\frac{d}{d\eta}$ with respect to these variables.

Here, relation (2.1) easily follows from (2.2) but the fact that (2.1) yields the existence of w satisfying (2.2) is somewhat more complicated, see [34] and references therein. Clearly, zero curvature representation [1, 13, 43] is closely related to Lax pairs.

According to [7], equation (1.1) is equivalent to (2.1) in the case

$$G(\xi,\eta,\lambda) = -\frac{1}{\lambda-1}q(\xi,\eta), \quad F(\xi,\eta,\lambda) = -\frac{1}{\lambda+1}Q(\xi,\eta). \tag{2.3}$$

Moreover, the case is non-isospectral, that is, λ is a scalar function depending on the variables ξ and η and on the "hidden spectral parameter" z. The dependence of λ on ξ and η is given by the equations [7, 30]:

$$\lambda_\xi = -\frac{\alpha_\xi}{\alpha}\lambda\frac{\lambda+1}{\lambda-1} = -\frac{\alpha_\xi}{\alpha}\lambda - \frac{2\alpha_\xi}{\alpha} - \frac{2\alpha_\xi}{\alpha(\lambda-1)}, \tag{2.4}$$

$$\lambda_\eta = -\frac{\alpha_\eta}{\alpha}\lambda\frac{\lambda-1}{\lambda+1} = -\frac{\alpha_\eta}{\alpha}\lambda + \frac{2\alpha_\eta}{\alpha} - \frac{2\alpha_\eta}{\alpha(\lambda+1)}. \tag{2.5}$$

The equality $\alpha_{\xi\eta} = 0$ in (1.1) means that α admits representation:

$$\alpha(\xi,\eta) = f(\xi) + h(\eta). \tag{2.6}$$

Since $\alpha = \overline{\alpha}$ (see (1.2) and (1.4)), we assume further that

$$f(\xi) = \overline{f(\xi)}, \quad h(\eta) = \overline{h(\eta)} \tag{2.7}$$

in (2.6). It follows from (2.4)–(2.6) (see [7]) that one can choose

$$\lambda(\xi,\eta,z) = \frac{h(\eta) - f(\xi) - z + \sqrt{(z-2h(\eta))(z+2f(\xi))}}{f(\xi)+h(\eta)}. \tag{2.8}$$

Remark 2.1. *The functions f, h and the branch of the square root in* (2.8) (*or, equivalently, in* (3.6)) *should be chosen so that $\lambda(\xi,\eta)$ is well-defined and continuously differentiable. For this purpose, we may also either restrict the domains of ξ and η or turn to the Riemann surfaces (see, e.g., [26, p. 510]).*

We note that the matrix functions q and Q in (2.3) are connected with the solution u of the corresponding equation (1.1) by the equalities

$$q(\xi,\eta) = u_\xi(\xi,\eta)u(\xi,\eta)^{-1}, \quad Q(\xi,\eta) = -u_\eta(\xi,\eta)u(\xi,\eta)^{-1} \tag{2.9}$$

(see [7] or [30, (44)]).

Remark 2.2. *In fact, taking into account the property* (3.7) *of* λ, *we rewrite* (2.1) *in the form*

$$q_\eta + Q_\xi = [q, Q], \quad (\alpha q)_\eta = (\alpha Q)_\xi. \tag{2.10}$$

The existence of u *satisfying* (2.9) *and the fact that* (1.1) *holds for this* u *easily follow from* (2.10).

2. In this paper, each generalized Bäcklund-Darboux transformation (GBDT) is determined by some initial system (2.2), (2.3) (to which GBDT is applied) and by a triple of matrices $\{\mathcal{A}, S(0,0), \Pi(0,0)\}$, where $\mathcal{A}$ and $S(0,0)$ are $n\times n$ matrices ($n \in \mathbb{N}$), $\Pi(0,0)$ is an $n \times m$ matrix and the matrix identity

$$\mathcal{A}S(0,0) - S(0,0)\mathcal{A}^* = \mathrm{i}\Pi(0,0)J\Pi(0,0)^*, \quad J = J^* = J^{-1}. \tag{2.11}$$

holds. Clearly, instead of the initial system (2.2), (2.3), we may fix the functions $\alpha(\xi,\eta)$, $q(\xi,\eta)$ and $Q(\xi,\eta)$ generating (2.2), (2.3) or the functions $\alpha(\xi,\eta)$ and $u(\xi,\eta)$ satisfying (1.1) (in which case q and Q are given by (2.9)).

We assume that

$$\alpha = \overline{\alpha}, \quad S(0,0) = S(0,0)^*, \quad qJ = -Jq^*, \quad QJ = -JQ^*. \tag{2.12}$$

Below, we show that similar to the isospectral case, the so called Darboux matrix function w_A has at each ξ and η the form of the transfer matrix function:

$$w_A(\xi,\eta,\lambda) = I_m - \mathrm{i}J\Pi(\xi,\eta)^*S(\xi,\eta)^{-1}\big(A(\xi,\eta) - \lambda I_n\big)^{-1}\Pi(\xi,\eta). \tag{2.13}$$

The transfer matrix function was introduced in this form by Lev Sakhnovich in [38] (see also [37, 40]). The corresponding matrix functions $A(\xi,\eta)$, $\Pi(\xi,\eta)$ and $S(\xi,\eta)$ in (2.13) are defined by the values $A(0,0) = \mathcal{A}$, $\Pi(0,0)$ and

$S(0,0)$, respectively, and by the linear equations:

$$A_\xi = -\frac{\alpha_\xi}{\alpha}A - \frac{2\alpha_\xi}{\alpha}I_n - \frac{2\alpha_\xi}{\alpha}(A-I_n)^{-1}, \tag{2.14}$$

$$A_\eta = -\frac{\alpha_\eta}{\alpha}A + \frac{2\alpha_\eta}{\alpha}I_n - \frac{2\alpha_\eta}{\alpha}(A+I_n)^{-1}, \tag{2.15}$$

$$\Pi_\xi = (A-I_n)^{-1}\Pi q, \quad \Pi_\eta = (A+I_n)^{-1}\Pi Q, \tag{2.16}$$

$$\begin{aligned} S_\xi = {} & \frac{\alpha_\xi}{\alpha}\big(S - 2(A-I_n)^{-1}S(A^*-I_n)^{-1}\big) \\ & - \mathrm{i}(A-I_n)^{-1}\Pi q J \Pi^*(A^*-I_n)^{-1}, \end{aligned} \tag{2.17}$$

$$\begin{aligned} S_\eta = {} & \frac{\alpha_\eta}{\alpha}\big(S - 2(A+I_n)^{-1}S(A^*+I_n)^{-1}\big) \\ & - \mathrm{i}(A+I_n)^{-1}\Pi Q J \Pi^*(A^*+I_n)^{-1}. \end{aligned} \tag{2.18}$$

Recall that (for our non-isospectral case) $\lambda = \lambda(\xi,\eta,z)$ in (2.13). The Darboux matrix function w_A transforms the fundamental solution w of the initial system (2.2) into the fundamental solution $w_A w$ of the transformed system.

In view of (2.14) and (2.15) we have

$$A_{\xi\eta} = A_{\eta\xi} = 2\frac{\alpha_\xi\alpha_\eta}{\alpha^2}A^3(A-I_n)^{-1}(A+I_n)^{-1},$$

and so the compatibility condition for systems (2.14), (2.15) is fulfilled. In order to see that equations (2.16) are compatible, we take into account (2.14), (2.15) and differentiate Π_ξ with respect to η and Π_η with respect to ξ. It follows that

$$\begin{aligned} \alpha\Pi_{\xi\eta} &= (A-I_n)^{-1}(A+I_n)^{-1}\big(\alpha_\eta A\Pi q + \alpha\Pi Q q + \alpha(A+I_n)\Pi q_\eta\big), \\ \alpha\Pi_{\eta\xi} &= (A-I_n)^{-1}(A+I_n)^{-1}\big(\alpha_\xi A\Pi Q + \alpha\Pi q Q + \alpha(A-I_n)\Pi Q_\xi\big). \end{aligned}$$

Now, the compatibility condition $\Pi_{\xi\eta} = \Pi_{\eta\xi}$ is immediate from (2.10). The equality $S_{\xi\eta} = S_{\eta\xi}$ is proved in a similar way although more complicated calculations are required for that purpose. See some further details in Appendix A.

Equations (2.14)–(2.16) are derived from the more general formulas considered in [30, pp. 1252-1254]. The equality $\alpha = \overline{\alpha}$ enabled us to set (for our special case) $A_1 = A$ and $A_2 = A^*$ in the formula [30, (19)]. After substitution $\Pi_2(0,0)^* = \mathrm{i}J\Pi_(0,0)^*$, formula [30, (6)] at the point $(0,0)$ took the form (2.11). Next, we used the last two equalities in (2.12) in order to set

$\Pi_1 \equiv \Pi$ and $\Pi_2^* \equiv \mathrm{i}J\Pi^*$ in [30, (5), (19), (20))]. Equations (2.14)–(2.16) followed. Now, from (2.11)–(2.18) one obtains (see [30, (6)]):

$$A(\xi,\eta)S(\xi,\eta) - S(\xi,\eta)A(\xi,\eta)^* = \mathrm{i}\Pi(\xi,\eta)J\Pi(\xi,\eta)^*. \tag{2.19}$$

Finally, formulas (11)–(13) in [30] imply that

$$\frac{d}{d\xi}w_A(\xi,\eta,\lambda) = \widehat{G}(\xi,\eta,\lambda)w_A(\xi,\eta,\lambda) - w_A(\xi,\eta,\lambda)G(\xi,\eta,\lambda), \tag{2.20}$$

$$\frac{d}{d\eta}w_A(\xi,\eta,\lambda) = \widehat{F}(\xi,\eta,\lambda)w_A(\xi,\eta,\lambda) - w_A(\xi,\eta,\lambda)F(\xi,\eta,\lambda), \tag{2.21}$$

$$\widehat{G}(\xi,\eta,\lambda) = -\frac{1}{\lambda-1}\widehat{q}(\xi,\eta), \quad F(\xi,\eta,\lambda) = -\frac{1}{\lambda+1}\widehat{Q}(\xi,\eta). \tag{2.22}$$

Here, according to [30, (10), (13)] we have the following expressions for the transformed coefficients $\widehat{q}$ and $\widehat{Q}$ (denoted by $\widehat{q}_{11}$ and $\widehat{Q}_{11}$ in [30]):

$$\begin{aligned}\widehat{q} =& \big(I_m - \mathrm{i}J\Pi^*S^{-1}(A-I_n)^{-1}\Pi\big)q\big(I_m + \mathrm{i}J\Pi^*(A^*-I_n)^{-1}S^{-1}\Pi\big)\\ &- 2\mathrm{i}(\alpha_\xi/\alpha)J\Pi^*S^{-1}(A-I_n)^{-1}S(A^*-I_n)^{-1}S^{-1}\Pi,\end{aligned} \tag{2.23}$$

$$\begin{aligned}\widehat{Q} =& \big(I_m - \mathrm{i}J\Pi^*S^{-1}(A+I_n)^{-1}\Pi\big)Q\big(I_m + \mathrm{i}J\Pi^*(A^*+I_n)^{-1}S^{-1}\Pi\big)\\ &- 2\mathrm{i}(\alpha_\eta/\alpha)J\Pi^*S^{-1}(A+I_n)^{-1}S(A^*+I_n)^{-1}S^{-1}\Pi.\end{aligned} \tag{2.24}$$

When we invert S above, we consider the corresponding formulas in the points of invertibility of S. Note that equalities (2.17)–(2.12) yield $S(\xi,\eta) = S(\xi,\eta)^*$, and so $\widehat{q}J$ and $\widehat{Q}J$ given by (2.23) and (2.24) satisfy skew-self-adjointness conditions similar to the last two equalities in (2.12) for qJ and QJ:

$$\widehat{q}J = -J(\widehat{q})^*, \quad \widehat{Q}J = -J(\widehat{Q})^*. \tag{2.25}$$

Recall that α and u satisfy (1.1), and so (2.1) holds. Hence, according to [37, Theorem 6.1] the initial system (2.2) is compatible.

Remark 2.3. *For simplicity, we assume in the text that G and F are continuously differentiable and, for this purpose, $\alpha(\xi,\eta) = f(\xi) + h(\eta)$, $q(\xi,\eta)$ and $Q(\xi,\eta)$ are continuously differentiable* (*or, instead of the requirements on q and Q, that u is two times continuously differentiable*)*. In fact, the conditions which we need in order that* (2.1) *yields the compatibility of systems* (2.2) *are weaker* (*see, e.g., [37, Theorem 6.1]*)*.*

In view of (2.2) and taking into account (2.20), (2.21) we see that the matrix function $\widehat{w}(\xi,\eta,\lambda) = w_A(\xi,\eta,\lambda)w(\xi,\eta,\lambda)$ satisfies the system

$$\begin{aligned} \frac{d}{d\xi}\widehat{w}(\xi,\eta,\lambda) &= \widehat{G}(\xi,\eta,\lambda)\widehat{w}(\xi,\eta,\lambda), \\ \frac{d}{d\eta}\widehat{w}(\xi,\eta,\lambda) &= \widehat{F}(\xi,\eta,\lambda)\widehat{w}(\xi,\eta,\lambda). \end{aligned} \tag{2.26}$$

Thus, the transformed system (2.26) is compatible and the compatibility condition holds:

$$\frac{d}{d\eta}\widehat{G} - \frac{d}{d\xi}\widehat{F} + [\widehat{G}, \widehat{F}] = 0. \tag{2.27}$$

Moreover, relations (2.26) (or, equivalently, relations (2.20) and (2.21)) show that w_A of the form (2.13) is, indeed, a Darboux matrix function.

3. Finally, we describe a way to modify a solution of (1.1) (when $m = 2$) so that the modified solution u satisfies the equality (1.3) It is easy to see that a 2×2 matrix function $u = \{u_{ik}\}_{i,k=1}^2$ has a property:

$$u_\xi u^{-1} = \frac{1}{\det(u)}\begin{bmatrix} (u_{11})_\xi u_{22} - (u_{12})_\xi u_{21} & * \\ * & (u_{22})_\xi u_{11} - (u_{21})_\xi u_{12} \end{bmatrix}. \tag{2.28}$$

Thus, considering traces "tr" of both sides of (2.28) we have

$$\mathrm{tr}\big(u_\xi u^{-1}\big) = \frac{(\det u)_\xi}{\det(u)}. \tag{2.29}$$

Clearly, a similar to (2.29) formula is valid for $\mathrm{tr}\big(u_\eta u^{-1}\big)$. Hence, taking traces in (1.1) one obtains

$$\left(\alpha\frac{(\det u)_\xi}{\det(u)}\right)_\eta + \left(\alpha\frac{(\det u)_\eta}{\det(u)}\right)_\xi = 0. \tag{2.30}$$

Now, assuming that α and some 2×2 matrix function $\breve{u}$ satisfy (1.1), (1.2), and that $\det(\breve{u}) > 0$, one (using standard calculations) derives that α and the matrix function

$$u := \alpha(\det \breve{u})^{-1/2}\breve{u} \tag{2.31}$$

satisfy (1.1), (1.2) and equality (1.3).

3. σ-model and gravitational equations

1. Clearly, (1.1) remains valid if we multiply u (from the right) by some constant $m \times m$ matrix. Hence, without loss of generality one may assume that

$$u(0,0) = I_m. \tag{3.1}$$

First, we prove the following theorem on the construction of solutions of the σ-model equation.

Theorem 3.1. *Let α and u satisfy equation* (1.1), *let a triple of matrices $\{\mathcal{A}, S(0,0), \Pi(0,0)\}$ satisfying* (2.11) *be given and assume that relations* (2.12) *and* (3.1) *hold, where q and Q in* (2.12) *are given by* (2.9). *Set*

$$\mathcal{U}(\xi,\eta) := I_m - \mathrm{i}J\Pi(\xi,\eta)^* S(\xi,\eta)^{-1} A(\xi,\eta)^{-1}\Pi(\xi,\eta), \tag{3.2}$$

where the matrix functions $A(\xi,\eta)$, $\Pi(\xi,\eta)$ and $S(\xi,\eta)$ are introduced by the linear equations (2.14)–(2.18).

Then, the scalar function α and the matrix function

$$\widehat{u}(\xi,\eta) = \mathcal{U}(\xi,\eta) u(\xi,\eta) \tag{3.3}$$

satisfy equation (1.1), *that is,*

$$\big(\alpha(\xi,\eta)\widehat{u}_\xi(\xi,\eta)\widehat{u}(\xi,\eta)^{-1}\big)_\eta + \big(\alpha(\xi,\eta)\widehat{u}_\eta(\xi,\eta)\widehat{u}(\xi,\eta)^{-1}\big)_\xi = 0. \tag{3.4}$$

Moreover, $\widehat{u}$ is J-unitary:

$$\widehat{u}(\xi,\eta)^* J \widehat{u}(\xi,\eta) \equiv J. \tag{3.5}$$

Proof. Fixing the branch of the square root we rewrite (2.8) in the form

$$\lambda(\xi,\eta,z) = \frac{\big(\sqrt{z-2h(\eta)} - \sqrt{z+2f(\xi)}\big)^2}{z-2h(\eta) - (z+2f(\xi))}$$

$$= \frac{\sqrt{z-2h(\eta)} - \sqrt{z+2f(\xi)}}{\sqrt{z-2h(\eta)} + \sqrt{z+2f(\xi)}}. \tag{3.6}$$

Thus, we have

$$\lambda \to 0 \quad \text{for} \quad z \to \infty. \tag{3.7}$$

Relations (2.4), (2.5) and (3.7) yield

$$\lambda_\xi \to 0, \quad \lambda_\eta \to 0 \quad \text{for} \quad z \to \infty. \tag{3.8}$$

For a fixed constant $\mu = \lambda(\xi, \eta, z_\mu)$, from the definition (2.13) we obtain

$$\begin{aligned} &\big(I_m - \mathrm{i}J\Pi(\xi,\eta)^* S(\xi,\eta)^{-1}(A(\xi,\eta) - \mu I_n)^{-1}\Pi(\xi,\eta)\big)_\xi \\ &\quad = \frac{d}{d\xi} w_A(\xi,\eta,\lambda) \\ &\qquad + \mathrm{i}\lambda_\xi(\xi,\eta,z_\mu) J\Pi(\xi,\eta)^* S(\xi,\eta)^{-1}(A(\xi,\eta) - \mu I_n)^{-2}\Pi(\xi,\eta); \end{aligned} \tag{3.9}$$

$$\begin{aligned} &\big(I_m - \mathrm{i}J\Pi(\xi,\eta)^* S(\xi,\eta)^{-1}(A(\xi,\eta) - \mu I_n)^{-1}\Pi(\xi,\eta)\big)_\eta \\ &\quad = \frac{d}{d\eta} w_A(\xi,\eta,\lambda) \\ &\qquad + \mathrm{i}\lambda_\eta(\xi,\eta,z_\mu) J\Pi(\xi,\eta)^* S(\xi,\eta)^{-1}(A(\xi,\eta) - \mu I_n)^{-2}\Pi(\xi,\eta). \end{aligned} \tag{3.10}$$

When z_μ tends to infinity, formulas (2.20)–(2.22) and (3.7)–(3.10) yield

$$\mathcal{U}_\xi(\xi,\eta) = \widehat{q}(\xi,\eta)\mathcal{U}(\xi,\eta) - \mathcal{U}(\xi,\eta)q(\xi,\eta), \tag{3.11}$$

$$\mathcal{U}_\eta(\xi,\eta) = -\widehat{Q}(\xi,\eta)\mathcal{U}(\xi,\eta) + \mathcal{U}(\xi,\eta)Q(\xi,\eta). \tag{3.12}$$

Using equalities (2.9), (3.11) and (3.12) we derive

$$\widehat{u}_\xi(\xi,\eta) = \widehat{q}(\xi,\eta)\widehat{u}(\xi,\eta), \quad \widehat{u}_\eta(\xi,\eta) = -\widehat{Q}(\xi,\eta)\widehat{u}(\xi,\eta), \tag{3.13}$$

where $\widehat{q}$, $\widehat{Q}$ and $\widehat{u}$ are given by (2.23), (2.24) and (3.3), respectively. Applying Remark 2.2 to the zero curvature equation (2.27) and taking into account (3.13), we see that $\widehat{u}$ satisfies equation (3.4). Moreover, relations (1.5), (2.25) and (3.13) imply that

$$\big(\widehat{u}(\xi,\eta)^* J\widehat{u}(\xi,\eta)\big)_\xi = 0, \quad \big(\widehat{u}(\xi,\eta)^* J\widehat{u}(\xi,\eta)\big)_\eta = 0. \tag{3.14}$$

From (2.13) and (2.19) one obtains

$$w_A(\xi,\eta,\overline{\lambda})^* J\, w_A(\xi,\eta,\lambda) \equiv J \tag{3.15}$$

(see, e.g., [38] or [37, (1.84)]). In particular, we have

$$\mathcal{U}(\xi,\eta)^* J\mathcal{U}(\xi,\eta) = J. \tag{3.16}$$

By virtue of (3.1), (3.3) and (3.16), the equality

$$\widehat{u}(0,0)^*J\widehat{u}(0,0)=J \tag{3.17}$$

holds. Finally, formulas (3.14) and (3.17) yield (3.5). □

2. Setting in Theorem 3.1 $m=2$, we easily obtain the following corollary for the gravitational equation.

Corollary 3.2. *Let α and u satisfy* (1.1) *and* (1.2)*, let a triple of matrices* $\{\mathcal{A}, S(0,0), \Pi(0,0)\}$*, which satisfies the matrix identity* (2.11)*, be given and assume that the relations*

$$\mathcal{A},\ S(0,0)\in \mathrm{GL}(n,\mathbb{R}), \quad \Pi(0,0)\in\mathbb{R}^{n\times 2}, \quad S(0,0)=S(0,0)^*, \tag{3.18}$$

$$\mathrm{i}J\in \mathrm{GL}(2,\mathbb{R}), \quad J=J^*=J^{-1}, \quad qJ=-Jq^*, \quad QJ=-JQ^* \tag{3.19}$$

are valid, where q and Q in (3.19) *are given by* (2.9)*. Assume additionally that*

$$d:=\det\Big(\big(I_2-\mathrm{i}J\Pi(0,0)^*S(0,0)^{-1}\mathcal{A}^{-1}\Pi(0,0)\big)u(0,0)\Big)>0. \tag{3.20}$$

Then, the scalar function α and the matrix function $\widetilde{u}$ of the form

$$\widetilde{u}(\xi,\eta)=\alpha(\xi,\eta)d^{-1/2}\mathcal{U}(\xi,\eta)u(\xi,\eta), \tag{3.21}$$

where $\mathcal{U}(\xi,\eta)$ is given by (3.2)*, satisfy* (1.1)–(1.3).

Proof. According to (1.2) and (2.9), we have

$$q(\xi,\eta),\ Q(\xi,\eta)\in\mathbb{R}^{2\times 2}. \tag{3.22}$$

Relations (2.14)–(2.18), (3.18), (3.19) and (3.22) show that

$$A(\xi,\eta),\ S(\xi,\eta)\in\mathbb{R}^{n\times n}, \quad \Pi(\xi,\eta)\in\mathbb{R}^{n\times 2}. \tag{3.23}$$

It follows from (1.2), (3.2), the first relation in (3.19), and (3.23) that

$$\mathcal{U}(\xi,\eta)\in\mathbb{R}^{2\times 2}, \quad \widehat{u}(\xi,\eta)\in\mathbb{R}^{2\times 2}. \tag{3.24}$$

where $\widehat{u}(\xi,\eta)=\mathcal{U}(\xi,\eta)u(\xi,\eta)$. Moreover, equalities (2.9) and the last two equalities in (3.19) imply that

$$\big(u(\xi,\eta)^*Ju(\xi,\eta)\big)_\xi=0, \quad \big(u(\xi,\eta)^*Ju(\xi,\eta)\big)_\eta=0,$$

that is,

$$u(\xi,\eta)^*Ju(\xi,\eta) = u(0,0)^*Ju(0,0). \tag{3.25}$$

By virtue of (3.16) and (3.25) we have $\widehat{u}(\xi,\eta)^*J\widehat{u}(\xi,\eta) = u(0,0)^*Ju(0,0)$. Hence, taking taking into account that $\widehat{u}(\xi,\eta)$ is continuous, $\widehat{u}(\xi,\eta) \in \mathbb{R}^{2\times 2}$ and (3.20) holds we obtain

$$\det \widehat{u}(\xi,\eta) \equiv d > 0. \tag{3.26}$$

According to Theorem 3.1, α and $\widehat{u}$ satisfy (1.1). In view of (3.24) and (3.26), $\widehat{u}$ satisfies (1.2) and $\det \widehat{u}(\xi,\eta) > 0$. Now, compare (2.31) and (3.21) in order to see that α and $\widetilde{u}$ satisfy (1.1)–(1.3). □

4. Ernst-type equations

1. Non-isospectral (or modified) canonical system has the form

$$w_\xi(\xi,z) = \mathrm{i}\lambda JH(\xi)w(\xi,z) \quad (\lambda = (z-\xi)^{-1}), \tag{4.1}$$

$$H(\xi) = H(\xi)^* \in \mathbb{C}^{m\times m}, \quad J = J^* = J^{-1} \in \mathbb{C}^{m\times m}, \tag{4.2}$$

where $H(\xi) \geq 0$. This system (or the corresponding multiplicative integrals) appeared, e.g., in the works by M.S. Livšic [22], by V.P. Potapov [28], by Yu.P. Ginzburg and by L.A. Sakhnovich (see the review [39, pp. 37, 38]). It is closely connected (see [39, pp. 34–39]) with the Riemann-Hilbert problem for random matrices presented in [12] and with Wiener-Masani problem in prediction theory (as discussed in [41]).

GBDT for the canonical system, that is, for system (4.1), (4.2) ($H \geq 0$), where the spectral parameter λ does not depend on z and ξ was treated in [31]. GBDT for the non-isospectral system (4.1), (4.2) was studied in [33].

We note that Bäcklund transformation for Ernst equation was first introduced in [17], and for the matrix form of Ernst equation see, for instance, [2, 42]. In particular, G.A. Alekseev [2] considered Ernst equation as the compatibility condition for the systems

$$w_\xi = (z-\xi)^{-1}U(\xi,\eta)w, \quad w_\eta = (z-\eta)^{-1}V(\xi,\eta))w, \tag{4.3}$$

where U and V have real-valued entries (for the hyperbolic case) and some special structure (see [2, (8),(11)]).

Somewhat modifying systems (4.3), we consider the compatibility condition of the auxiliary linear systems

$$w_\xi = (z-\xi-\eta)^{-1}U(\xi,\eta)w, \quad w_\eta = (z-\xi-\eta)^{-1}V(\xi,\eta)w, \tag{4.4}$$

and obtain an Ernst-type integrable nonlinear system (non-isospectral case):

$$U(\xi,\eta) - V(\xi,\eta) + [U(\xi,\eta), V(\xi,\eta)] = 0, \quad U_\eta(\xi,\eta) = V_\xi(\xi,\eta). \tag{4.5}$$

Indeed, the compatibility condition (2.1) for systems (4.4) takes the form

$$\begin{aligned} &(z-\xi-\eta)^{-1}\big(U_\eta(\xi,\eta) - V_\xi(\xi,\eta)\big) \\ &\quad + (z-\xi-\eta)^{-2}\big(U(\xi,\eta) - V(\xi,\eta) + [U(\xi,\eta), V(\xi,\eta)]\big) = 0, \end{aligned} \tag{4.6}$$

which is equivalent to (4.5). *We note that in the case of the systems* (4.4) *it is convenient to get rid of the spectral parameter λ and use the expression $(z-\xi-\eta)^{-1}$ instead of it.* Thus, we deal with $w(\xi,\eta,z)$ where z is the independent "hidden" spectral parameter.

Further we set

$$U(\xi,\eta) = \mathrm{i}JH(\xi,\eta), \quad V(\xi,\eta) = \mathrm{i}J\mathcal{H}(\xi,\eta) \quad (\mathcal{H} = \mathcal{H}^*), \tag{4.7}$$

and assume that (4.2) holds. System (4.5) takes the form

$$\begin{aligned} &JH(\xi,\eta) - J\mathcal{H}(\xi,\eta) + \mathrm{i}[JH(\xi,\eta), J\mathcal{H}(\xi,\eta)] = 0, \\ &H_\eta(\xi,\eta) = \mathcal{H}_\xi(\xi,\eta). \end{aligned} \tag{4.8}$$

2. In order to construct Darboux matrix corresponding to the system (4.8), we fix a triple $\{\mathcal{A}, S(0,0), \Pi(0,0)\}$ satisfying (2.11) and set

$$A(\xi,\eta) = \big(\mathcal{A} - (\xi+\eta)I_n\big)^{-1} \quad \text{i.e.,} \quad A_\xi = A_\eta = A^2. \tag{4.9}$$

We introduce $\Pi(\xi,\eta)$ and $S(\xi,\eta)$ by the linear equations

$$\Pi_\xi = -\mathrm{i}A\Pi JH, \quad \Pi_\eta = -\mathrm{i}A\Pi J\mathcal{H}; \tag{4.10}$$

$$S_\xi = \Pi JHJ^*\Pi^* - (AS + SA^*), \quad S_\eta = \Pi J\mathcal{H}J^*\Pi^* - (AS + SA^*). \tag{4.11}$$

It is easily checked that by virtue of (4.8), (4.10) and the last two equalities in (4.9) we have $\Pi_{\xi\eta} = \Pi_{\eta\xi}$, that is, the compatibility condition (for systems (4.10)) is fulfilled. Moreover, the identity (2.19) is valid (see [33, (2.4)]).

Now, we introduce a matrix function

$$v(\xi,\eta,z) := w_0(\xi,\eta)^{-1}w_A\big(\xi,\eta,(z-\xi-\eta)^{-1}\big); \tag{4.12}$$

where w_A is given by (2.13) and

$$\frac{\partial}{\partial\xi}w_0(\xi,\eta) = \widetilde{G}_0(\xi,\eta)w_0(\xi,\eta), \quad \frac{\partial}{\partial\eta}w_0(\xi,\eta) = \widetilde{F}_0(\xi,\eta)w_0(\xi,\eta), \tag{4.13}$$

$$\widetilde{G}_0 = -\mathrm{i}J\Pi^*S^{-1}\Pi - [J\Pi^*S^{-1}\Pi, JH], \tag{4.14}$$

$$\widetilde{F}_0 = -\mathrm{i}J\Pi^*S^{-1}\Pi - [J\Pi^*S^{-1}\Pi, J\mathcal{H}], \quad w_0(0,0)^*Jw_0(0,0) = J. \tag{4.15}$$

Using matrix functions A, Π, S and w_0 from above and taking into account our results for GBDT of the non-isospectral canonical system [33], we prove the following theorem.

Theorem 4.1. *Let H and $\mathcal{H}$ satisfy* (4.8)*, let the equalities* (4.2) *hold for H and J and assume that $\mathcal{H} = \mathcal{H}^*$. Then, the matrix function $v(\xi,\eta,z)$ of the form* (4.12) *is the corresponding Darboux matrix, that is, it satisfies the systems:*

$$v_\xi(\xi,\eta,z) = \mathrm{i}(z-\xi-\eta)^{-1}\big(J\widetilde{H}(\xi,\eta)v(\xi,\eta,z) - v(\xi,\eta,z)JH(\xi,\eta)\big), \tag{4.16}$$

$$v_\eta(\xi,\eta,z) = \mathrm{i}(z-\xi-\eta)^{-1}\big(J\widetilde{\mathcal{H}}(\xi,\eta)v(\xi,\eta,z) - v(\xi,\eta,z)J\mathcal{H}(\xi,\eta)\big), \tag{4.17}$$

where

$$\begin{aligned} \widetilde{H}(\xi,\eta) &= w_0(\xi,\eta)^*H(\xi,\eta)w_0(\xi,\eta), \\ \widetilde{\mathcal{H}}(\xi,\eta) &= w_0(\xi,\eta)^*\mathcal{H}(\xi,\eta)w_0(\xi,\eta). \end{aligned} \tag{4.18}$$

Proof. For each fixed ξ or η we substitute into [33] $z-\xi$ instead z and $(\mathcal{A}-(\xi+\eta)I_n)^{-1}$ instead of $A(\eta)$ or $z-\eta$ instead z and $(\mathcal{A}-(\xi+\eta)I_n)^{-1}$ instead of $A(\xi)$, respectively, and use [33, (2.14), (2.15)] in order to derive

$$\begin{aligned} &\frac{d}{d\xi}w_A\big(\xi,\eta,(z-\xi-\eta)^{-1}\big) \\ &\quad = \mathrm{i}(z-\xi-\eta)^{-1}\Big(JH(\xi,\eta)w_A\big(\xi,\eta,(z-\xi-\eta)^{-1}\big) \\ &\qquad - w_A\big(\xi,\eta,(z-\xi-\eta)^{-1}\big)JH(\xi,\eta)\Big) \\ &\quad + \widetilde{G}_0(\xi,\eta)w_A\big(\xi,\eta,(z-\xi-\eta)^{-1}\big) \end{aligned} \tag{4.19}$$

and

$$\begin{aligned}(4.20)\qquad \frac{d}{d\eta}w_A\big(\xi,\eta,(z-\xi-\eta)^{-1}\big)& \\ = \mathrm{i}(z-\xi-\eta)^{-1}\Big(&J\mathcal{H}(\xi,\eta)w_A\big(\xi,\eta,(z-\xi-\eta)^{-1}\big) \\ &- w_A\big(\xi,\eta,(z-\xi-\eta)^{-1}\big)J\mathcal{H}(\xi,\eta)\Big) \\ &+ \widetilde{F}_0(\xi,\eta)w_A\big(\xi,\eta,(z-\xi-\eta)^{-1}\big),\end{aligned}$$

where $\widetilde{G}_0$ and $\widetilde{F}_0$ are given by (4.14) and (4.15). Here, we took into account that the definition [33, (2.5)] of w_A slightly differs from the definition in this paper. Using (4.9)–(4.11) and (2.19), one could also derive (4.19) and (4.20) directly.

According to (4.12), (4.13) and (4.19), (4.20), we have (4.16) and (4.17), where

$$(4.21)\qquad \widetilde{H}=Jw_0^{-1}JHw_0, \quad \widetilde{\mathcal{H}}=Jw_0^{-1}\mathcal{H}w_0 \quad (J=J^*=J^{-1}).$$

In view of (4.13)–(4.15), differentiating $w_0^*Jw_0$ we obtain

$$(4.22)\qquad w_0(\xi,\eta)^*Jw_0(\xi,\eta)\equiv J.$$

Finally, formulas (4.21) and (4.22) imply (4.18). □

Remark 4.2. *If A is invertible, we may set*

$$(4.23)\quad w_0(\xi,\eta)=w_A(\xi,\eta,0)=I_m-\mathrm{i}J\Pi(\xi,\eta)^*S(\xi,\eta)^{-1}A(\xi,\eta)^{-1}\Pi(\xi,\eta).$$

Indeed, in view of [33, Remark 1] $w_A(\xi,\eta,0)$ satisfies (4.13)*. Recall also that relation* (2.19) *yields the identity* (3.15)*, and so the matrix $w_A(0,0,0)$ satisfies the last equality in* (4.15)*.*

Remark 4.3. *According to* (4.21)*, $J\widetilde{H}(\xi,\eta)$ is linear similar to $JH(\xi,\eta)$ and $J\widetilde{\mathcal{H}}(\xi,\eta)$ is linear similar to $J\mathcal{H}(\xi,\eta)$. Moreover, in view of* (4.18) *the inequality $H(\xi,\eta)\geq 0$ implies $\widetilde{H}(\xi,\eta)\geq 0$ and the inequality $\mathcal{H}(\xi,\eta)\geq 0$ implies $\widetilde{\mathcal{H}}(\xi,\eta)\geq 0$.*

5. Some examples

1. Explicit constructions are of special interest in our theory. Recall that λ of the form (3.6) satisfies (2.4) and (2.5). Compare (2.4) and (2.5) with (2.14) and (2.15), respectively, in order to see that the matrix function $A(\xi,\eta)$ in Theorem 3.1 may be given explicitly. Namely, in view of Proposition B.1 the following proposition is also valid.

Proposition 5.1. *Let the matrix $\mathcal{A}\in\mathbb{C}^{n\times n}$ be given. Assume that*

$$\lambda_k - 2h(\eta) \neq 0, \quad \lambda_k + 2f(\xi) \neq 0$$

for the values $\lambda_k\in\sigma(\mathcal{A})$. Then, the matrix function

$$A(\xi,\eta) = \Big(\mathcal{R}\big(2h(\eta)\big) - \mathcal{R}\big(-2f(\xi)\big)\Big)\Big(\mathcal{R}\big(2h(\eta)\big) + \mathcal{R}\big(-2f(\xi)\big)\Big)^{-1}, \tag{5.1}$$

where $\mathcal{R}$ is constructed in the proof of Proposition B.1, satisfies (2.14) *and* (2.15).

We note that f,h and the square roots $\sqrt{\lambda_k - 2h(\eta)}$ and $\sqrt{\lambda_k + 2f(\xi)}$ in the construction of $\mathcal{R}$ should be chosen so that $\mathcal{R}\big(2h(\eta)\big) + \mathcal{R}\big(-2f(\xi)\big)$ is invertible and $\sqrt{\lambda_k - 2h(\eta)}$ and $\sqrt{\lambda_k + 2f(\xi)}$ are continuously differentiable.

Remark 5.2. *Given $m=2p$ and α of the form* (2.6), (2.7), *we may choose*

$$u(\xi,\eta) = \mathrm{e}^{\big(f(\xi)-h(\eta)\big)j}, \quad j = \begin{bmatrix} I_p & 0 \\ 0 & -I_p \end{bmatrix}, \tag{5.2}$$

and

$$J = \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix} \quad \text{or} \quad J = \begin{bmatrix} 0 & -\mathrm{i}I_p \\ \mathrm{i}I_p & 0 \end{bmatrix} \tag{5.3}$$

in Theorem 3.1. Indeed, in view of (2.9) *and* (5.2) *we have*

$$q(\xi,\eta) \equiv f'(\xi)j, \quad Q(\xi,\eta) \equiv h'(\eta)j \quad \Big(f' = \frac{d}{d\xi}f\Big), \tag{5.4}$$

and the corresponding equalities in (2.12) *hold. Clearly,* (3.1) *holds as well. Finally, substituting* (2.6) *and* (5.2) *into the left-hand side of* (1.1) *we rewrite* (1.1) *in the form $h'(\eta)f'(\xi)j - f'(\xi)h'(\eta)j = 0$, and so* (1.1) *is valid. In other words, α and u given by* (2.6) *and* (5.2), *respectively, satisfy* (1.1).

Our next remark suggests the choice of J and u in Corollary 3.2.

Remark 5.3. *Pauli matrix* $\sigma_2 = J = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}$ *gives a simple example of the matrix* J *such that the corresponding relations in* (3.19) *hold.*

Setting (*in Remark 5.2*) $p = 1$ *and* $J = \sigma_2$, *we see that relations* (1.2) *and* (3.19) *are valid. According to Remark 5.2, the pair* α, u *satisfies* (1.1).

2. Consider the case

$$f(\xi) = -\xi, \quad h(\eta) = \eta, \tag{5.5}$$

which was studied in [26]. According to (5.4) and (5.5) we have

$$q \equiv -j, \quad Q \equiv j. \tag{5.6}$$

New explicit solutions appear when the parameter matrix $\mathcal{A}$ is non-diagonal. In the next example, we deal with the simplest of such cases ($m = 2p$, $n = 2$).

Example 5.4. *Let* $\mathcal{A}$ *be a* 2×2 *Jordan block:*

$$\mathcal{A} = \begin{bmatrix} c & 1 \\ 0 & c \end{bmatrix}. \tag{5.7}$$

Then, in view of (B.5) *and* (B.7) *we have*

$$\mathcal{R}(2\eta) = \begin{bmatrix} \omega(\eta) & \frac{1}{2\omega(\eta)} \\ 0 & \omega(\eta) \end{bmatrix}, \quad \mathcal{R}(2\xi) = \begin{bmatrix} \nu(\xi) & \frac{1}{2\nu(\xi)} \\ 0 & \nu(\xi) \end{bmatrix}; \tag{5.8}$$

$$\omega(\eta) := \sqrt{c - 2\eta}, \quad \nu(\xi) := \sqrt{c - 2\xi}. \tag{5.9}$$

After some simple calculations, using (5.1), (5.5) *and* (5.8) *we derive*

$$A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}, \quad a(\xi,\eta) = \frac{\omega(\eta) - \nu(\xi)}{\omega(\eta) + \nu(\xi)}, \quad b(\xi,\eta) = -\frac{a(\xi,\eta)}{\nu(\xi)\omega(\eta)}. \tag{5.10}$$

$$\big(A(\xi,\eta) - I_2\big)^{-1} = -\frac{\omega(\eta) + \nu(\xi)}{2\nu(\xi)} \begin{bmatrix} 1 & \frac{\nu(\xi)-\omega(\eta)}{2\omega(\eta)\nu(\xi)^2} \\ 0 & 1 \end{bmatrix}, \tag{5.11}$$

$$\big(A(\xi,\eta) + I_2\big)^{-1} = \frac{\omega(\eta) + \nu(\xi)}{2\omega(\eta)} \begin{bmatrix} 1 & \frac{\omega(\eta)-\nu(\xi)}{2\nu(\xi)\omega(\eta)^2} \\ 0 & 1 \end{bmatrix}. \tag{5.12}$$

Partition $\Pi(\xi,\eta)$ *into two* $2\times p$ *blocks:* $\Pi(\xi,\eta)=\begin{bmatrix}\Lambda_1(\xi,\eta) & \Lambda_2(\xi,\eta)\end{bmatrix}$, *recall that* $\Pi(0,0)$ *is assumed to be given* (*it belongs to the triple, which determines GBDT*) *and set*

$$\Lambda_1(\xi,\eta)=\exp\left\{-\frac{1}{4}\begin{bmatrix}\big(\nu(\xi)+\omega(\eta)\big)^2 & \frac{\nu(\xi)}{\omega(\eta)}+\frac{\omega(\eta)}{\nu(\xi)}\\ 0 & \big(\nu(\xi)+\omega(\eta)\big)^2\end{bmatrix}\right\}\Lambda_1(0,0), \tag{5.13}$$

$$\Lambda_2(\xi,\eta)=\exp\left\{\frac{1}{4}\begin{bmatrix}\big(\nu(\xi)+\omega(\eta)\big)^2 & \frac{\nu(\xi)}{\omega(\eta)}+\frac{\omega(\eta)}{\nu(\xi)}\\ 0 & \big(\nu(\xi)+\omega(\eta)\big)^2\end{bmatrix}\right\}\Lambda_2(0,0), \tag{5.14}$$

where ω *and* ν *are introduced in* (5.9). *Direct differentiation in* (5.13), (5.14) *and formulas* (5.6), (5.11) *and* (5.12) *show that we constructed* $\Pi(\xi,\eta)$ *correctly and it satisfies* (2.16). *Formulas* (5.13) *and* (5.14) *may be simplified*

$$\Lambda_1(\xi,\eta)=\exp\left\{-\big(\nu(\xi)+\omega(\eta)\big)^2/4\right\}\left(I_2-\frac{1}{4}\begin{bmatrix}0 & \frac{\nu(\xi)}{\omega(\eta)}+\frac{\omega(\eta)}{\nu(\xi)}\\ 0 & 0\end{bmatrix}\right)\Lambda_1(0,0), \tag{5.15}$$

$$\Lambda_2(\xi,\eta)=\exp\left\{\big(\nu(\xi)+\omega(\eta)\big)^2/4\right\}\left(I_2+\frac{1}{4}\begin{bmatrix}0 & \frac{\nu(\xi)}{\omega(\eta)}+\frac{\omega(\eta)}{\nu(\xi)}\\ 0 & 0\end{bmatrix}\right)\Lambda_2(0,0), \tag{5.16}$$

Finally, we note that under the condition $\omega(\eta)\overline{\nu(\xi)}\neq\nu(\xi)\overline{\omega(\eta)}$ *the entries of the* 2×2 *matrix function* $S(\xi,\eta)=\{S_{ik}(\xi,\eta)\}_{i,k=1}^2$ *are uniquely successively recovered from the identity* (2.19) (*and from formula* (5.10)) :

$$S_{22}=(a-\overline{a})^{-1}K_{22},\quad S_{21}=(a-\overline{a})^{-1}(K_{21}+\overline{b}S_{22}),$$
$$S_{12}=(a-\overline{a})^{-1}(K_{12}-bS_{22}),\quad S_{11}=(a-\overline{a})^{-1}(K_{11}+\overline{b}S_{12}-bS_{21}),$$

where K_{ik} *are the entries of* $K:=\mathrm{i}\Pi J\Pi^*$ *and* $\Pi(\xi,\eta)$ *is explicitly constructed above. Now, our main formulas* (3.2) *and* (3.3) *provide a corresponding family of explicit solutions* $\widehat{u}$ *of the* σ*-model.*

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Appendix A. Compatibility condition for systems on S

Heuristically, the compatibility condition

$$S_{\xi\eta}=S_{\eta\xi} \tag{A.1}$$

for the systems (2.17) and (2.18) may be deduced from the unique solvability (under natural assumptions) of the identity (2.19) on S.

In order to prove (A.1) rigorously, we rewrite (2.14) and (2.15) in the forms

$$A_\xi = -\frac{\alpha_\xi}{\alpha}A(A+I_n)(A-I_n)^{-1}, \quad A_\eta = -\frac{\alpha_\eta}{\alpha}A(A-I_n)(A+I_n)^{-1}. \tag{A.2}$$

It is easy to see that the identities

$$2A(A-I_n)^{-1}(A+I_n)^{-1} = (A-I_n)^{-1} + (A+I_n)^{-1}, \tag{A.3}$$

$$2(A-I_n)^{-1}(A+I_n)^{-1} = (A-I_n)^{-1} - (A+I_n)^{-1}. \tag{A.4}$$

are valid. Now, we differentiate both sides of (2.17) with respect to η and both sides of (2.18) with respect to ξ using (2.18) and (2.17), respectively, as well as the equalities (2.16) and (A.2). We simplify the right-hand sides of the obtained relations using (2.12), (A.3) and (A.4). Then, reducing similar terms we derive

$$\begin{aligned} S_{\xi\eta} - S_{\eta\xi} = -\frac{\mathrm{i}}{2\alpha}\Big(&\alpha_\xi\big((A+I_n)^{-1}\Pi QJ\Pi^*(A^*+I_n)^{-1} \\ &- (A-I_n)^{-1}\Pi QJ\Pi^*(A^*-I_n)^{-1}\big) \\ &+ \alpha_\eta\big((A+I_n)^{-1}\Pi qJ\Pi^*(A^*+I_n)^{-1} \\ &- (A-I_n)^{-1}\Pi qJ\Pi^*(A^*-I_n)^{-1}\big) \\ &+ (A-I_n)^{-1}\Pi\big(2(\alpha q)_\eta + \alpha[Q,q]\big)J\Pi^*(A^*-I_n)^{-1} \\ &- (A+I_n)^{-1}\Pi\big(2(\alpha Q)_\xi + \alpha[Q,q]\big)J\Pi^*(A^*+I_n)^{-1}\Big). \end{aligned} \tag{A.5}$$

Next, we multiply (A.5) by $2\mathrm{i}\alpha(A-I_n)(A+I_n)$ from the left and by $(A^*+I_n)(A^*-I_n)$ from the right. Taking into account that $(\alpha q)_\eta = (\alpha Q)_\xi$, we obtain

$$\begin{aligned} &2\mathrm{i}\alpha(A-I_n)(A+I_n)\big(S_{\xi\eta} - S_{\eta\xi}\big)(A^*+I_n)(A^*-I_n) \\ &\quad = -2\alpha_\xi\big(A\Pi QJ\Pi^* + \Pi QJ\Pi^*A^*\big) \\ &\qquad - 2\alpha_\eta\big(A\Pi qJ\Pi^* + \Pi qJ\Pi^*A^*\big) \\ &\qquad + 2A\Pi\big((\alpha q)_\eta + (\alpha Q)_\xi + \alpha[Q,q]\big)J\Pi^* \\ &\qquad + 2\Pi\big((\alpha q)_\eta + (\alpha Q)_\xi + \alpha[Q,q]\big)J\Pi^*A^*. \end{aligned} \tag{A.6}$$

Finally, the first equation in (2.10) implies that the right-hand side of (A.6) equals zero. Thus, (A.1) follows.

Appendix B. Matrix square roots

Modifying the proof of [35, Proposition 3.3], we obtain the following proposition.

Proposition B.1. *Let $\mathcal{A}\in \mathbb{C}^{n\times n}$ admit representation $E\mathcal{J}E^{-1}$, where $\mathcal{J}$ is the Jordan normal form of $\mathcal{A}$. Then, there is a matrix function $\mathcal{R}(\mu)$ $(\mu\in\mathbb{R},\ \mu\notin\sigma(\mathcal{A}))$ such that*

$$\mathcal{R}(\mu)^2=\mathcal{A}-\mu I_n, \quad \mathcal{R}(\mu)=E\mathcal{D}(\mu)E^{-1}, \tag{B.1}$$

$$\mathcal{R}(\mu_1)\mathcal{R}(\mu_2)=\mathcal{R}(\mu_2)\mathcal{R}(\mu_1) \quad (\mu_1,\mu_2\in\mathbb{R}), \tag{B.2}$$

where $\mathcal{D}$ is a block diagonal matrix with the blocks of the same orders as the corresponding Jordan blocks of $\mathcal{J}$. Moreover, the blocks of $\mathcal{D}$ are upper triangular Toeplitz matrices (or scalars if the corresponding blocks of $\mathcal{J}$ are scalars).

Proof. Clearly, the statement of proposition holds for $n=1$. Consider the case, where $\mathcal{A}$ is an $n\times n$ Jordan block $(n\geq 2)$:

$$\mathcal{A}=\begin{bmatrix}\lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1\\ & & & \lambda\end{bmatrix}. \tag{B.3}$$

For this $\mathcal{A}$ and $\mu\in\mathbb{R}$, we construct upper triangular Toeplitz matrices $\mathcal{R}(\mu)$ satisfying the first equality in (B.1). First, we introduce the shift matrices

$$\mathcal{S}_i:=\{\delta_{k-l+i}\}_{k,l=1}^n, \quad \mathcal{S}_i\mathcal{S}_j=\mathcal{S}_{i+j}. \tag{B.4}$$

where δ_s is Kronecker delta, and $\mathcal{S}_i=0$ for $i\geq n$. Let us write down the representations

$$\mathcal{A}-\mu I_n=(\lambda-\mu)I_n+\mathcal{S}_1, \quad \mathcal{R}(\mu)=c_0I_n+c_1\mathcal{S}_1+\ldots+c_{n-1}\mathcal{S}_{n-1}. \tag{B.5}$$

According to (B.4) and (B.5), we have

$$\begin{aligned}\mathcal{R}(\mu)^2&=c_0^2I_n+2c_0c_1\mathcal{S}_1+(2c_0c_2+c_1^2)\mathcal{S}_2\\ &\quad+\sum_{i=3}^{n-1}\big(2c_0c_i+c_1c_{i-1}+\ldots+c_{i-1}c_1\big)\mathcal{S}_i.\end{aligned} \tag{B.6}$$

Now, we set

$$c_0 = \sqrt{\lambda - \mu} \neq 0, \quad c_1 = 1/(2c_0), \tag{B.7}$$

and choose successively the values $c_2, \dots$ so that the coefficients before the shift matrices $\mathcal{S}_i$ $(i \geq 2)$ on the right-hand side of (B.6) turn to zero. (In this way, the values c_i $(i \geq 2)$ are uniquely defined, and the upper triangular Toeplitz matrix $\mathcal{R}(\mu)$ given by (B.5) satisfies the first equality in (B.1).)

When $\mathcal{A}$ is a Jordan matrix $\mathcal{J}$, we construct block diagonal matrix $\mathcal{D}(\mu)$, each block of which is generated by the corresponding Jordan block of $\mathcal{J}$ in a way described above. It is easy to see that (B.1) holds for $\mathcal{A} = \mathcal{J}$ and $\mathcal{R}(\mu) = \mathcal{D}(\mu)$. Finally, if $\mathcal{A} = E\mathcal{J}E^{-1}$, we set $\mathcal{R}(\mu) = E\mathcal{D}(\mu)E^{-1}$ (as in the second equality in (B.1)), and the first equality in (B.1) for $\mathcal{A}$ and $\mathcal{R}$ follows from the first equality in (B.1) for $\mathcal{J}$ and $\mathcal{D}$.

Since the blocks of $\mathcal{J}$ and $\mathcal{D}(\mu)$ are upper triangular Toeplitz matrices, $\mathcal{J}$ and $\mathcal{D}(\mu)$ commute (see, e.g., [10] on the properties of triangular Toeplitz matrices). Hence, (B.2) is valid. $\square$

Although $\mu \in \mathbb{R}$ is required is Section 5, it is easy to see that the construction above works also for $\mu \in \mathbb{C}$. A closely related construction of the matrix root $f(A)^{1/\ell}$ $(\ell \in \mathbb{N})$, which commutes with the matrix A, is presented in [36] together with some references.

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