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Asymptotics of the eigenvalues and Abel basis property of the root functions of new type Sturm-Liouville problems

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The main goal of this study is to investigate the main spectral properties of a new type Sturm-Liouville problems(SLP's). The problems studied here differs from the classical SLP's in that, the equation contain an abstract linear operator which can be non-selfadjoint and unbounded in the Hilbert space of square-integrable functions, and the boundary conditions contain an additional transmission conditions at an internal singular point. So, SLP's under consideration are not purely differential.

We emphasize that this type of non-classical SLP's which includes an abstract linear operator in differential equation, was studied by the authors of this work for the first time in the literature. Naturally, the study of such type non-classical SLP's are much more complicated than the classical purely differential SLP's, because it is not clear how to apply the known methods of the Sturm-Liouville theory to problems of this type. The main difficulties lie in the derivation of such important spectral properties as the discreteness of the spectrum and the completeness of the corresponding eigenfunctions.

First, we establish isomorphism and coerciveness with respect to the spectral parameter for the corresponding nonhomogeneous problem. Based on these results and using our own approaches we prove that the spectrum of the main problem is discrete. Then we derive some asymptotic formulas for the eigenvalues. Finally it is shown that the system of root functions (i.e. eigen and associated functions) form an Abel basis of order α, for all $\alpha > 1$. The obtained results are new even in the case when the problem under consideration does not contain an additional transmission conditions.

1. Introduction

Self-adjoint boundary value problems(BVPs, for short) are of significant importance in many models of applied mathematics and quantum mechanics in spherical and cylindrical geometries. Among these BVPs, the Sturm-Liouville problems is a typical one. Many physical processes, such as the vibration of strings, the interaction of atomic particles, electrodynamics of complex medium, aerodynamics, polymer rheology or the earth's free oscillations yields Sturm-Liouville eigenvalue problems(see, for example, [14, 23, 33, 35–38] and references cited therein). Generally, the separation of variables method was applied on the two-order partial differential equation to obtain a Sturm-Liouville problem for each independent variable. This method is a cornerstone in the study of partial differential equations, and is a major element in physical problems. For example, consider a boundary value problem for the one-dimensional wave equation

$$\rho_0 u_{tt} = (k u_x)_x, \quad 0 \leq x \leq L,$$
$$u(0,t) = u(L,t) = 0,$$

for the longitudinal displacement $u(x;t)$ of a string of length L with mass-density $\rho_0(x)$ and stiffness $k(x)$, both of which we assume are smooth, strictly positive functions on $0 \leq x \leq L$. Looking for separable time-periodic solutions we get Sturm-Liouville problem

$$-(k\varphi')' = \lambda \rho_0 \varphi, \quad \varphi(0) = \varphi(L) = 0.$$

Sturm Liouville theory was developed collaboratively by Charles-Franciois Sturm (1803- 1855) and Joseph Liouville (1809-1882) in order to generalise a relatively disorganised array of second order linear differential equations used to model physical problems. These included Bernoulli's work on vibrating strings and Liouville's own work on heat conduction [25]. In 1910 Hermann Weyl [43] gave the first rigorous treatment, in the case of an equation of Sturm-Liouville type, of cases where continuous spectra can occur. The theory was particularly significant because it provided the first qualitative theory of differential equations, and was thus very useful for solutions that could not be solved explicitly. These problems involve self-adjoint (differential) operators which play an important role in the spectral theory of linear operators and the existence of the eigenfunctions. The development of classical, rather than the operatoric, Sturm-Liouville theory in the years after 1950 can be found in various sources; in particular in the texts of Atkinson [4], Coddington and Levinson [12], Levitan and Sargsjan [26] and Naimark [32]. Spectral problems associated with differential operators having only a discrete spectrum and depending polynomially on the spectral parameter have been considered by Gohberg and Krein [15], and by Keldysh [22]. They studied the spectrum and principal functions of such problems and showed the completeness of the principal functions in the corresponding Hilbert function space. There are a lot of studies about the spectrum of such operators [1, 2]. For the background and applications of the boundary value problems to different areas, we refer the reader to the monographs and some recent contributions as [1, 3, 5–9, 11, 17, 19, 20, 27, 30, 31, 39]. Note that in recent years, there has been growing interest in boundary- value problems with interior singularities(see, for example, [6, 7, 9, 11, 16, 30, 41] and references cited therein).

In this study we shall investigate some spectral aspects of a new type Sturm-Liouville equation involving an abstract linear operator A, namely the "differential" equation

$$Lu \equiv p(x)u'' + Au = \lambda u \quad x \in [-1,0) \cup (0,1] \tag{1.1}$$

together with boundary conditions at the end-points $x = -1, 1$ given by

$$L_1 u \equiv \alpha_0 u(-1) + \alpha_1 u'(-1) = 0 \tag{1.2}$$

$$L_2 u \equiv \beta_0 u(1) + \beta_1 u'(1) = 0 \tag{1.3}$$

and the transmission conditions at the point of singularity $x = 0$ given by

$$u'(0^-) = \gamma_0 u(0^-) + \delta_0 u(0^+) = 0 \tag{1.4}$$

$$u'(0^+) = \gamma_1 u(0^-) + \delta_1 u(0^+) = 0 \tag{1.5}$$

where $p(x) = p_1$ for $x \in [-1,0)$, $p(x) = p_2$ for $x \in (0,1]$; $p_i, \alpha_i, \beta_{ij}, \delta_i, \gamma_i$ $(i = 0,1)$ are real numbers; $p_1 \neq 0, p_2 \neq 0$, $|\alpha_0| + |\alpha_1| \neq 0$, $|\beta_0| + |\beta_1| \neq 0$; λ is a complex spectral parameter. Transmission problems appear frequently in various fields of physics and technics [38, 41, 42]. For instance, in electrostatics and magnetostatics the model problem which describes the heat transfer through an infinitely conductive layer is a transmission problem (see, [34] and the references listed therein). Another completely different field is that of “hydraulic fracturing” (see, [10]) used in order to increase the flow of oil from a reservoir into a producing oil well. Further examples can be found in Dautray and Lions [13, 24].

1.1. Examples

We emphasize that the non-classical Sturm-Liouville problems of the form (1.1)–(1.5) containing an abstract linear operator A in the equation was investigated by the authors of this study for the first time in the literature. The results obtained in this paper are applicable to a wide class of boundary value problems, the spectral properties of which have not been studied previously. To verify this let us give some interesting examples.

1. The "Sturm-Liouville equations" s of the forms

$$p(x)u''(x) + q(x)u'(\frac{x}{2}) + r(x)u(\frac{x}{3}) = \lambda u(x), \ x \in [-1,0) \cup (0,1]$$

and

$$p(x)u''(x) + q(x)u'(\xi_1) + r(x)u(\xi_0) = \lambda u(x), \ x \in [-1,0) \cup (0,1]$$

or more general equations of the form

$$p(x)u''(x) + q(x)u'(\varphi_1(x)) + r(x)u(\varphi_0(x)) = \lambda u(x), \ x \in [-1,0) \cup (0,1]$$

together with the same boundary and transmission conditions (1.2)–(1.5) are the special cases of the considered problem (1.1)–(1.5), because the linear operator

$$\mathbb{A}u := q(x)u'(\varphi_1(x)) + r(x)u(\varphi_0(x))$$

acted compactly from the Hilbert space $W_2^2(-1,0) \oplus W_2^2(0,1)$ into the Hilbert space $L_2(-1,0) \oplus L_2(0,1)$ and acted boundedly from the Hilbert space $W_2^1(-1,0) \oplus W_2^1(0,1)$ into the Hilbert space $L_2(-1,0) \oplus L_2(0,1)$(i.e. this operator satisfies the conditions of the main Theorems 4.3, 4.4 and 5.2), where $p(x) = p_1$ for $x \in [-1,0)$, $p(x) = p_2$ for $x \in (0,1]$, $p_1 \neq 0, p_2 \neq 0$; p_1, p_2 are real numbers; $\xi_0, \xi_1 \in (-1,0) \cup (0,1)$ are arbitrary points; the real-valued functions $q(x), r(x), \varphi_0(x)$ and $\varphi_1(x)$ are continuous on $[-1,0) \cup (0,1]$ with the finite limits $q(0\pm), r(0\pm), \varphi_0(0\pm)$ and $\varphi_1(0\pm)$; $\varphi_0(x)$ and $\varphi_1(x)$ are mappings of $(-1,0) \cup (0,1)$ into itself.

2. The "Sturm-Liouville equations" of the forms

$$p(x)u''(x) + q(x)u'(\varphi_1(x)) + r(x)u(\varphi_0(x)) + \sum_{i=0}^{1}\left(\int_{-1}^{0-} K_{1i}(x,t)u^{(i)}(t)dt + \int_{0+}^{1} K_{2i}(x,t)u^{(i)}(t)dt\right) = \lambda u(x),\ x \in [-1,0) \cup (0,1]$$

together with the same boundary and transmission conditions (1.1)–(1.5) are special cases of the problem (1.1)–(1.5) where the Kernels $K_{ji}(x,t)(i = 0,1, j = 1,2)$ $x,t \in [-1,0) \cup (0,1]$ are continuous functions.

1.2. Some remarks

It is well known that the classical Sturm-Liouville problems has infinitely many real eigenvalues and the corresponding eigenfunctions forms an orthonormal basis in the Hilbert space of square-integrable functions. But the eigenvalues of the problem (1.1)–(1.5) may be also nonreal complex numbers. However, the leading term in the asymptotic formulas of the eigenvalues of the problem (1.1)–(1.5) is the same as for the classical Sturm-Liouville problems and the corresponding eigen -and associated functions forms an Abel basis in the Hilbert space of square-integrable functions (see, Theorems 4.3, 4.4 and 5.2 below). It is also known that the Abel basis property is stronger than the completeness, but weaker than the basis with parantheses(see, [44]). Moreover, in the case when the linear operator A acts boundedly from the Hilbert space $W_2^1(-1,0) \oplus W_2^1(0,1)$ into the Hilbert space $L_2(-1,0) \oplus L_2(0,1)$ the asymptotic term for our problem (1.1)–(1.5) is the same as for the classical SLP's(i.e. this asymptotic term has the form $O(n)$), but when this operator acts compactly from the Hilbert space $W_2^2(-1,0) \oplus W_2^2(0,1)$ into the Hilbert space $L_2(-1,0) \oplus L_2(0,1)$, then the asymptotic term for our problem (1.1)–(1.5) takes on a weaker form $o(n^2)$(see, Theorems 4.3 and 4.4 below).

2. Some auxiliary facts and results

Let E and F be two Banach spaces, for which a set-theoretical inclusion $E \subset F$ holds, and the linear space F induces on E the linear structure coinciding with the structure of the linear space E and let J be the embedding operator from E to F, i.e. $Jx = x$ for all $x \in E$. If this operator is continuous, we say that the embedding $E \subset F$ is continuous. Similarly if the operator J is compact then the embedding $E \subset F$ is said to be a compact embedding. If $\overline{E} = F$, i.e. the subset $E \subset F$ is dense in the Banach space F, then we say that the embedding $E \subset F$ is dense.

Throughout in this study, the notation of inclusion "$\subset$" must be understood in the set-theoretical and in the topological meaning. Let E_1 and E_2 be two complex Banach spaces, both linearly and continuously embedded in a Banach space E. Then the pair $\{E_1, E_2\}$ are said to be an interpolation couple. Let us define the linear space $E_1 + E_2$ by

$$E_1 + E_2 = \{u \in E | \text{there are } u_1 \in E_1 \text{ and } u_2 \in E_2 \text{ such that } u = u_1 + u_2\}$$

It is known that this linear space forms the Banach space (see,[40]) with respect to the norm given by

$$\|u\|_{E_1+E_2} = \inf\{\|u_1\|_{E_1} + \|u_2\|_{E_2}| \ u_1 \in E_1, u_2 \in E_2 \ \ u_1 + u_2 = u\}$$

where the infimum is taken over all representations $u = u_1 + u_2$ in the described way. It is easy to see that for any $t > 0$ the functional $K(t, u)$ defined on $E_1 + E_2$ by

$$K(t, u) = \inf\{\|u_1\|_{E_1} + t\|u_2\|_{E_2}| \ u_1 \in E_1, u_2 \in E_2 \ \ u_1 + u_2 = u\}$$

is an equivalent norm in the Banach space $E_1 + E_2$. An interpolation space for interpolation couple $\{E_1, E_2\}$ by K-method is defined as follows(see [40])

$$\{E_1, E_2\}_{\theta,p} := \{u : u \in E_1 + E, \ \|u\|_{\{E_1,E_2\}} := (\int\limits_0^\infty \frac{K^p(t, u)}{t^{1+\theta p}} dt)^{\frac{1}{p}} < \infty\}$$

Due to [[40], Triebel 1.3.3] there exists a positive number $C_{\theta,p}$, such that for all $u \in E_1 \cap E_2$

$$\|u\|_{(E_1,E_2)_{\theta,p}} \leq C_{\theta,p} \|u\|^{1-\theta}_{E_1} \|u\|^{\theta}_{E_2}. \tag{2.1}$$

Applying the well-known Young inequality to the right hand of the last inequality we have that for each $\epsilon > 0$ there exists $C(\epsilon) > 0$ such that for all $u \in E_0 \cap E_1$

$$\|u\|_{(E_1,E_2)_{\theta,p}} \leq \epsilon \|u\|_{E_1} + C(\epsilon)\|u\|_{E_2}. \tag{2.2}$$

Definition 2.1. *The Sobolev space $W_2^n(a,b)(n = 0, 1, 2, ...)$ is the Hilbert space consisting of all functions $f \in L_2(a,b)$ that have square-integrable generalized derivatives $f', f'', ..., f^{(n)}$ on (a,b) with the inner-product*

$$\langle f, g\rangle_{W_2^n(a,b)} = \sum_{k=0}^{n} \langle f^{(k)}, g^{(k)}\rangle_{L_2(a,b)}.$$

where that under $W_2^0(a,b)$ we mean $L_2[a,b]$.

Definition 2.2. *Let [a,b] be any finite interval, $0 < s \neq$ integer be any real number and let n be any integer such that $n > s$. For such s the interpolation space $W_2^s(a,b)$ is defined as*

$$W_2^s(a,b) := (W_2^n(a,b), L_2(a,b))_{1-\frac{s}{n},2} \tag{2.3}$$

Remark 2.3. *It is known that(see,for example [40]) the equality (2.3) is hold even in the case when s is also integer.*

Below we shall use the direct sum of Sobolev spaces $W_2^s(-1,0) \oplus W_2^s(0,1)$ of functions on $(-1,0) \cup (0,1)$ belonging to $W_2^s(-1,0)$ and $W_2^s(0,1)$ in $(-1,0)$ and $(0,1)$ respectively, with the norm

$$\|f\|_{W_2^s} := (\|f\|^2_{W_2^s(-1,0)} + \|f\|^2_{W_2^s(0,1)})^{1/2}.$$

From inequalities (2.1) and (2.3) we have the following Lemma.

Lemma 2.1. *Let $0 \leq s \leq 2$. Then there is a constant $C > 0$ such that for all $u \in W_2^2(-1,0) \oplus W_2^2(0,1)$ and $\lambda \in \mathbb{C}$ the following inequality holds:*

$$|\lambda|^{2-s} \|u\|_{W_2^s} \leq C(\|u\|_{W_2^2} + |\lambda|^2 \|u\|_{L_2}) \tag{2.4}$$

By using (2.2) and (2.3) we have

Lemma 2.2. *Let $k \geq 0$ any real number. Then for each $\epsilon > 0$ there is a constant $C(\epsilon) > 0$ such that for all $u \in W_2^{k+\frac{1}{2}}(-1,0) \oplus W_2^{k+\frac{1}{2}}(0,1)$ the following inequality holds*

$$\|u\|_{W_2^k} \leq \epsilon \|u\|_{W_2^{k+\frac{1}{2}}} + C(\epsilon)\|u\|_{L_2} \tag{2.5}$$

Lemma 2.3. *Let the following conditions be satisfied:*

i)H_1 and H_2 are separable Hilbert spaces and $H_1 \subset H_2$

ii)The embedding operator $J : H_1 \to H_2$ is densely defined and continuous.

iii)The operator B acts compactly from H_1 into H_2. Then for any $\epsilon > 0$ there exist a constant $C(\epsilon) > 0$ such that

$$\|Bu\|_{H_2} \leq \epsilon\|u\|_{H_1} + C(\epsilon)\|u\|_{H_2} \tag{2.6}$$

for all $u \in H$.

Proof. The proof follows immediately from [[31], Lemma 1.2.8/3]. □

Denoting

$$L_3u = u'(0^-) - \gamma_0 u(0^-) - \delta_0 u(0^+) = 0$$

and

$$L_4u = u'(0^+) - \gamma_1 u(0^-) - \delta_1 u(0^+) = 0$$

we shall define the operator $\pounds_0$ in the Hilbert space $L_2(-1,0) \oplus L_2(0,1)$ by domain of definition

$$D(\pounds_0) = \left\{ u | u \in W_2^2(-1,0) \oplus W_2^2(0,1), L_\nu u = 0, \ \nu = 1 \div 4 \right\} \tag{2.7}$$

and action low $\pounds_0 u = p(x)u''$. Throughout in below we shall assume that $D(A) \supset D(\pounds_0)$ and define the operator $\pounds$ in the Hilbert space $L_2(-1,0) \oplus L_2(0,1)$ by domain of definition $D(\pounds) = D(\pounds_0)$ and action low

$$\pounds(u) = p(x)u'' + Au.$$

Remark 2.4. *Note that under spectrum and root functions of the problem (1.1)–(1.5) we mean the spectrum and root functions of the operator $\pounds$, respectively.*

Theorem 2.5. *If $p_1\delta_0 = a_2\gamma_1$, then the operator $\pounds_0$ is densely defined and symmetric.*

Proof. Denote by $C_0^\infty[-1,0) \oplus C_0^\infty(0,1]$ the set of infinitely differentiable functions in $[-1,0) \cup (0,1]$, each of which vanishes on some neighborhoods of the points $x = -1, x = 0$ and $x = 1$. It is well-known that this set is dense in the Hilbert space $L_2(-1,0) \oplus L_2(0,1)$. Since $C_0^\infty[-1,0) \oplus C_0^\infty(0,1] \subset D(\pounds_0)$ we have that $D(\pounds_0)$ is also dense in the Hilbert space $L_2(-1,0) \oplus L_2(0,1)$.

Further, by using Lagrange's formula we can derive easily that

(2.8) $(\pounds_0 u, v)_{L_2} = (u, \pounds_0 v)_{L_2} + p_1 W(u, \overline{v}; x)|_{-1}^{0^-} + p_2 W(u, \overline{v}; x)|_{1}^{0^+}$.

Since both u and $\overline{v}$ satisfy the boundary conditions (1.2)–(1.3) it follows that

$$W(u, \overline{v}; -1) = W(u, \overline{v}; 1) = 0. \tag{2.9}$$

Furthermore, from the transmission conditions (1.4)–(1.5) we have

$$\begin{aligned} & p_1 W(u, \overline{v}; 0^-) + p_2 W(u, \overline{v}; 0^+) = -p_1(\gamma_0) u(0^-) + \delta_0 u(0^+) \overline{v}(0^-) \\ & + p_1(\gamma_0) \overline{v}(0^-) + \delta_0 \overline{v}(0^+) u(0^-) - p_2(\gamma_1) u(0^-) + \delta_1 u(0^+) \overline{v}(0^+) \\ & + p_2(\gamma_1) \overline{v}(0^-) + \delta_1 \overline{v}(0^+) u(0^+) \\ & = -(p_1 \delta_0 - p_2 \gamma_1)(u(0^+) \overline{v}(0^-) - u(0^-) \overline{v}(0^+)) = 0. \end{aligned} \tag{2.10}$$

Putting (2.9) and (2.10) in (2.8) yields

$$(\pounds_0 u, v)_{L_2} = (u, \pounds_0 v)_{L_2}, \tag{2.11}$$

for all $u, v \in D(\pounds_0)$. The proof is complete. □

3. Separation results for the corresponding nonhomogeneous problem

Consider the following nonhomogeneous problem

$$Lu - \lambda u(x) = f(x), \quad L_v u = f_v, \quad v = 1 \div 4. \tag{3.1}$$

for arbitrary $f \in L_2(-1, 0) \oplus L_2(0, 1)$, $f_v \in \mathbb{C}$, $v = 1 \div 4$. The next theorem is crucial for further consideration.

Theorem 3.1. *Let the following conditions be satisfied:*

1. $p_1 \delta_0 = p_2 \gamma_1$

2. The operator A acts compactly from the Hilbert space $W_2^2(-1, 0) \oplus W_2^2(0, 1)$ to the Hilbert space $L_2(-1, 0) \oplus L_2(0, 1)$.

Then for any $\epsilon > 0$ (small enough) there exists $R_\epsilon > 0$ such that for all $\lambda \in \mathbb{C}$ satisfying $|\arg \lambda \pm \frac{\pi}{2}| > \epsilon$, $|\lambda| > R_\epsilon$, the operator $\Im(\lambda) : u \to ((\pounds - \lambda I)u, L_1 u, L_2 u, L_3 u, L_4 u)$ is an isomorphism from $W_2^2(-1, 0) \oplus W_2^2(0, 1)$

onto $L_2(-1,0) \oplus L_2(0,1) \oplus \mathbb{C}^4$ *and the following coercive estimate holds*

$$|\lambda| \|u\|_{L_2} + |\lambda|^{\frac{1}{2}} \|u\|_{W_2^1} + \|u\|_{W_2^2} \leq C(\epsilon)(\|f\|_{L_2} + |\lambda|^{\frac{1}{4}}(|f_1| + |f_2| + |f_3| + |f_4|)) \tag{3.2}$$

where $C(\epsilon)$ *is the constant which is dependent only on* $\epsilon > 0$

Proof. Let us define the linear functionals $\ell_i(u)(i = 1 \div 4)$ by the equalities $\ell_1 u = \alpha_0 u(-1)$, $\ell_2 u = \beta_0 u(1)$, $\ell_3 u = -\gamma_0 u(0^-) - \delta_0 u(0^-)$ and $\ell_4 u = -\gamma_1 u(0^-) - \delta_1 u(0^+)$. Let $u \in W_2^2(-1,0) \oplus W_2^2(0,1)$ be any solution of the problem (3.1). Denoting $g(x) = f(x) - (Au)(x)$, $g_i = f_i - \ell_i(u)(i = 1 \div 4)$ consider the differential equation

$$p(x)u'' - \lambda u = g(x), \quad x \in [-1,0) \cup (0,1] \tag{3.3}$$

together with boundary conditions

$$\alpha_1 u'(-1) = g_1, \quad \beta_1 u'(1) = g_2 \tag{3.4}$$

and with transmission conditions

$$u'(0^-) = g_3, \quad u'(0^+) = g_4 \tag{3.5}$$

By applying the Theorem 3 in [31] to the problem (3.3)–(3.5) we obtain immediately the next a priori estimate

$$\begin{aligned} &|\lambda| \|u\|_{L_2} + |\lambda|^{\frac{1}{2}} \|u\|_{W_2^1} + \|u\|_{W_2^2} \leq C(\epsilon)(\|g\|_{L_2} + |\lambda|^{\frac{1}{4}} \sum_{i=1}^{4} |g_i|) \\ &\leq C(\epsilon)[(\|f\|_{L_2} + |\lambda|^{\frac{1}{4}} \sum_{i=1}^{4} |f_i|) + (\|Au\|_{L_2} + |\lambda|^{\frac{1}{4}} \sum_{i=1}^{4} |\ell_i u|)] \end{aligned} \tag{3.6}$$

for the solution of problem (3.1). Let us estimate the right hand of this inequality. Since the operator A is compact from $W_2^2(-1,0) \oplus W_2^2(0,1)$ into $L_2(-1,0) \oplus L_2(0,1)$, by virtue of Lemma 2.3 for any $\delta > 0$ there exist a constant $C(\delta)$ such that

$$\|Au\|_{L_2} \leq \delta \|u\|_{W_2^2} + C(\delta) \|u\|_{L_2} \tag{3.7}$$

By virtue of (3.6) and Lemma 2.3 it follows that, for any $\delta > 0$ there is a constant $C(\delta) > 0$ such that

$$\|Au\|_{L_2} \leq (\delta + C(\delta)|\lambda|^{-1})(\|u\|_{W_2^2} + |\lambda|^{\frac{1}{2}} \|u\|_{L_2}) \tag{3.8}$$

Since the embedding $C[a,b] \subset W_2^1[a,b]$ is continuous for arbitrary finite interval $[a,b]$, then by virtue of Lemma 2.2 we find that for each $\delta > 0$ there exists a constant $C = C(\delta) > 0$ such that

$$|\ell_v u| \leq C\|u\|_{W_2^1} \leq \delta \|u\|_{W_2^{\frac{3}{2}}} + C(\delta)\|u\|_{L_2} \tag{3.9}$$

for all $u \in W_2^{\frac{3}{2}}(-1,0) \oplus W_2^{\frac{3}{2}}(0,1)$. Now applying Lemma 2.1 we find

$$\|u\|_{W_2^{\frac{3}{2}}} \leq |\lambda|^{-\frac{1}{4}}(\|u\|_{W_2^2} + |\lambda|\|u\|_{L_2}). \tag{3.10}$$

Putting this in the previous inequality gives us the inequality

$$|\lambda|^{\frac{1}{4}}|\ell_v u| \leq (\delta + C(\delta)|\lambda|^{-\frac{3}{4}})(\|u\|_{W_2^2} + |\lambda|\|u\|_{L_2}) \tag{3.11}$$

Making use the inequalities (3.6)–(3.11) we get

$$\sum_{k=0}^{2} |\lambda|^{1-\frac{k}{2}}\|u\|_{W_2^k} \leq C(\epsilon)(\|f\|_{L_2} + |\lambda|^{\frac{1}{4}} \sum_{v=1}^{4} |f_v|)$$
$$+ C(\epsilon)(\delta + C(\delta)|\lambda|^{\frac{-1}{4}})(\|u\|_{W_2^2} + |\lambda|\|u\|_{L_2})$$

in the angle $G_\epsilon^{\pm} := \{\lambda \in \mathbb{C} \mid |arg\lambda \pm \frac{\pi}{2}| > \epsilon\}$ for sufficiently large $|\lambda|$. It is obvious that for any fixed $\epsilon > 0$ we can choose $\delta > 0$ so small and $|\lambda|$ so large that $C(\epsilon)(\delta + C(\delta)|\lambda|^{\frac{-1}{4}}) < 1$. Consequently, for $\lambda \in G_\epsilon^{\pm}$ sufficiently large in modulus we obtain a priori estimate (3.2). From this estimation it follows that for $\lambda \in G_\epsilon^{\pm}$, sufficiently large in modulus a solution of the problem (3.1) in $W_2^2(-1,0) \oplus W_2^2(0,1)$ is unique. Now by applying the Theorem 2 from [31] we have that for such λ the operator $\Im(\lambda)$ from $W_2^2(-1,0) \oplus W_2^2(0,1)$ into $(L_2(-1,0) \oplus L_2(0,1)) \oplus \mathbb{C}^4$ is Fredholm i.e. the range of $\Im(\lambda)$ is closed subset of the space $(L_2(-1,0) \oplus L_2(0,1)) \oplus \mathbb{C}^4$ and $\dim\ker \pounds(\lambda) = \dim co\ker \pounds(\lambda) < \infty$. Consequently, the range of $\Im(\lambda)$ coincide with the whole space $(L_2(-1,0) \oplus L_2(0,1)) \oplus \mathbb{C}^4$. From this and the fact that the operator $\pounds(\lambda)$ is injective, the statement of the theorem follows. □

Corollary 3.1. *Let the conditions of the previous theorem be satisfied. Then for any $\epsilon > 0$ there exists $R_\epsilon > 0$ such that all complex numbers λ satisfying $|\arg\lambda \pm \frac{\pi}{2}| > \epsilon$, $|\lambda| > R_\epsilon$ are regular values of the operator $\pounds$ and for the resolvent operator $R(\lambda, \pounds) : L_2(-1,0) \oplus L_2(0,1) \to L_2(-1,0) \oplus L_2(0,1)$*

the following inequality holds:

$$\|R(\lambda, \pounds)\| \leq C(\epsilon)|\lambda|^{-1}.$$

Proof. Putting $f_1 = f_2 = f_3 = f_4$ in (3.2) we have, in particular, that

$$|\lambda|\|u\|_{L_2} \leq C(\epsilon)\|f\|_{L_2},$$

that is,

$$|\lambda|\|R(\lambda, \pounds)f\|_{L_2} \leq C(\epsilon)\|f\|_{L_2}.$$

The proof is complete. □

Corollary 3.2. *Under conditions of the Theorem 3.1 the resolvent operator $R(\lambda, \pounds)$ acts boundedly from $L_2(-1,0) \oplus L_2(0,1)$ into $W_2^2(-1,0) \oplus W_2^2(0,1)$.*

Proof. Again, $f_1 = f_2 = f_3 = f_4$ in (3.2) we have, in particular, that

$$\|u\|_{W_2^2} \leq C(\epsilon)\|f\|_{L_2}$$

so

$$|\lambda|\|R(\lambda, \pounds)f\|_{W_2^2} \leq C(\epsilon)\|f\|_{L_2}$$

which completes the proof. □

By using the above results we can prove the next result.

Theorem 3.2. *Let $p_1\delta_0 = p_2\delta_1$. Then the operator $\pounds_0$ is self-adjoint.*

Proof. By Theorem 2.5 that the operator $\pounds_0$ is densely defined and symmetric. Taking into account the corollary 3.1 and applying the familiar theorem of Functional Analysis about the extensions of symmetric operators. (see [40]) it follows that the operator $\pounds_0$ is closed symmetric operator and both index defect of this operator is equal to zero, i.e. the operator $\pounds_0$ is self-adjoint. □

4. Discreteness of the spectrum and asymptotic behaviour of eigenvalues

At first let us give some needed definitions. Let S be unbounded closed linear operator in separable complex Hilbert space H and let λ_0 be any eigenvalue of this operator. Then the linear manifold $M_{\lambda_0} := \cup_{n=1}^{\infty} Ker(S-\lambda_0 I)^n$ is called a root lineal of S according to eigenvalue λ_0. Elements of this lineal are called a root vectors of S. The dimension of the lineal M_{λ_0} is called an algebraic multiplicity of the eigenvalue λ_0. The spectrum of operator S is called discrete if whole spectrum $\sigma(S)$ consist only of eigenvalues with finite multiplicity and the set of eigenvalues has not finite limit point. For such operators by $N(r,S)$ we denote the number of eigenvalues belonging to the ball $|\lambda| \leq r$ provided that each of eigenvalues counted according to their algebraic multiplicity. Let Φ be any subset of the complex plane $\mathbb{C}$. Then by $N(r,\Phi,S)$ we denote the number of eigenvalues of operator S belonging to $\{\lambda \in \mathbb{C} \mid |\lambda| \leq r\} \cap \Phi$ provided thateach of eigenvalues counted according to their algebraic multiplicity;

$$(N(r,\Phi,S) = \sum_{\lambda_n \in \{\lambda : |\lambda| \leq r\} \cap \Phi} 1 \quad)$$

Denote $\psi_\alpha^{\pm} = \{\lambda : |\arg(\pm\lambda)| < \alpha\}$, $R_+ = \{x \in \mathbb{R} : x > 0\}$ and $R_- = \{x \in \mathbb{R} : x < 0\}$. If there is no danger of confusion, we shall write $N_{\pm}(r,\alpha,S)$ instead of $N(r,\psi_\alpha^{\pm},S)$ and $N_{\pm}(r,\alpha)$ instead of $N(r,R_{\pm},S)$. The operator B is called p-subordinate(where $0 \leq p \leq 1$) to S if its domain $D(B) \supset D(S)$ and if there exist $b > 0$ such that

$$\|Bu\| \leq b\|Su\|^p \|u\|^{1-p} \; forall \; u \in D(S). \tag{4.1}$$

It is known that if S is self-adjoint with discrete spectrum and the operator B is p-subordinate to $S (0 \leq p < 1)$, then the spectrum of $S+B$ is also discrete (see [15], Lemma V.10.1).

Lemma 4.1. *Let S be self-adjoint with discrete spectrum and let B is p-subordinate$(0 \leq p < 1)$ to S and $T = S+B$. Then the spectrum T lies in the set $|Im\lambda| \leq b|\lambda|^p$ (b is the same constant as in (4.1)) and for all $\delta > 0$ and α with $0 < \alpha < \frac{\pi}{2}$, there are $c_1 > 0$ and $c_2 > 0$ such that*

$$\begin{aligned} \|N_{\pm}(\tau,\alpha,T) - N_{\pm}(r,\alpha,S)\| &\leq c_1(N_{\pm}(\tau + b(1+\delta)\tau^p, S) \\ &- N_{\pm}(\tau - b(1+\delta)\tau^p, S) + c_2) \end{aligned} \tag{4.2}$$

Proof. The proof follows immediately from the propositions [[28] Lemma 2.1], Theorem 2.3 and Remark 2.4 □

Suppose that the operators has at least one regular point λ_0. Then the operator B is called compact with respect to the operator S , if $D(B) \supset D(S)$

and if $BR(\lambda_0, S)$ is compact. If S is self-adjoint with discrete spectrum and B is p-subordinate$(0 \leq p < 1)$ to S , then B is compact with respect to the operator S. It is known that, if S is self-adjoint with discrete spectrum and B is compact with respect to the operator S, then the operator $S + B$ has also discrete spectrum. (see, [15], Lemma V.10.1)

Lemma 4.2. *[28] Let S be self-adjoint with discrete spectrum and let B be compact with respect to S and $T = S + B$ Then, if the number of positive eigenvalues of the operator S is infinite and*

$$\lim_{\substack{r \longrightarrow \infty \\ \varepsilon \longrightarrow 0}} \frac{N_+\left(r(1+\varepsilon), S\right)}{N_+(r, S)} = 1.$$

then for each $\alpha(0 < \alpha < \frac{\pi}{2})$ the relation

$$\lim_{r \longrightarrow \infty} \frac{N_+\left(r, \alpha, T\right)}{N_+(r, S)} = 1. \tag{4.3}$$

is hold

Now we shall derive asymptotic formulas for eigenvalues of the problem (1.1)–(1.5) for various type abstract operators A appearing in the equation. In particular case, we shall prove that there are infinitely many eigenvalues.

Theorem 4.3. *Let us satisfy the following conditions*

1. $p_1\delta_0 = p_2\gamma_1$.

2. The operator A acted boundedly from $W_2^1(-1, 0) \oplus W_2^1(0, 1)$ to $L_2(-1, 0) \oplus L_2(0, 1)$, i.e. there is $C > 0$ such that $\|Au\|_{L_2} \leq C\|u\|_{W_2^1}$ for all $u \in W_2^1$. Case1. If $p_1p_2 < 0$ then the eigenvalues of (1.1)–(1.5) can be arranged as one two sequences $\{\lambda_{n,1}\}_1^\infty$ and $\{\lambda_{n,2}\}_1^\infty$ with asymptotic behaviour

$$\lambda_{n,1} = -p_1\pi^2n^2 + O(n), \quad \lambda_{n,2} = -p_2\pi^2n^2 + O(n) \tag{4.4}$$

Case 2. If $p_1p_2 > 0$ then the eigenvalues of (1.1)–(1.5) can be arranged as sequence $\{\lambda_n\}_1^\infty$ with asymptotic behaviours

$$\lambda_n = -\frac{p_1p_2}{p_1 + 2\sqrt{p_1p_2} + p_2}\pi^2n^2 + O(n). \tag{4.5}$$

Proof. Since the embedding $W_2^2(-1, 0) \oplus W_2^2(0, 1) \subset L_2(-1, 0) \oplus L_2(0, 1)$ is compact (see, [18]) by virtue of corollary 3.2 the resolvent operator $R(\lambda, \pounds)$

is compact in the space $L_2(-1,0) \oplus L_2(0,1)$. Consequently spectrum of the operators $\pounds_0$ and $\pounds$ are discrete. In view of Theorem 3.2 the operator $\pounds_0$ is self-adjoint. At the other hand, by applying the multiplicative inequality (2.1) we have

$$\|u\|_{W_2^1} \leq C\|u\|_{W_2^2}^{1/2}\|u\|_{L_2}^{1/2}, \quad u \in W_2^1(-1,0) \oplus W_2^1(0,1) \tag{4.6}$$

Without loss of generality we shall assume that $\lambda = 0$ is not eigenvalue of $\pounds_0$. Otherwise we can find a real value $\mu_0 \notin \sigma(\pounds_0)$ and replace the spectral parameter λ by $\lambda - \mu_0$. Applying corollary 3.2 we find that

$$\|Au\|_{L_2} \leq C_1\|u\|_{W_2^1} \leq C_2\|u\|_{W_2^1}^{1/2}\|u\|_{L_2}^{1/2} \leq C_3\|\pounds_0 u\|_{L_2}^{1/2}\|u\|_{L_2}^{1/2} \tag{4.7}$$

for some $C_i = const(i = 1,2,3)$ i.e. the operator A is $\frac{1}{2}$-subordinate to $\pounds_0$. Let us find asymptotic behaviour of $N_\pm(r,\pounds_0)$ for $r \to \pm\infty$. Consider the case $p_1p_2 < 0$. Let $p_1 < 0, p_2 > 0$(the other case $p_1 > 0, p_2 < 0$ is totaly similar). The eigenvalues of the operator $\pounds_0$ can be arranged as two infinite series $\{\widetilde{\lambda}_{n,1}\}$ and $\{\widetilde{\lambda}_{n,2}\}$ with asymptotics

$$\widetilde{\lambda}_{n,1} = -p_1\pi^2 n^2 + O(n), \quad \widetilde{\lambda}_{n,2} = -p_2\pi^2 n^2 + O(n) \tag{4.8}$$

(see, [?]). Then from (4.8) we can derive easily that

$$N_+(r,\pounds_0) = \sum_{\widetilde{\lambda}_{n,1} \leq r} 1 = \frac{1}{\sqrt{-p_1\pi}}\sqrt{r} + o(1), \quad r \to \infty \tag{4.9}$$

and

$$N_-(r,\pounds_0) = \sum_{-\widetilde{\lambda}_{n,2} \leq r} 1 = \frac{1}{\sqrt{p_2\pi}}\sqrt{r} + o(1), \quad r \to \infty. \tag{4.10}$$

Further, applying Lemma 4.2 to the operators $\pounds_0$ and A we get that there is $b > 0$ such that

$$\begin{aligned} |N_\pm(r,\alpha,\pounds) - N_\pm(r,\alpha,\pounds_0)| &\leq C_1(N_\pm(r + b\sqrt{r},\pounds_0) - N_\pm(r - b\sqrt{r},\pounds_0)) \\ &+ C_2 \end{aligned} \tag{4.11}$$

for arbitrary α with $0 < \alpha < \frac{\pi}{2}$, where C_1 and C_2 are some constants depending only on α. Since

$$\sqrt{r + b\sqrt{r}} - \sqrt{r - b\sqrt{r}} = O(1), \ as\ r \to \infty \tag{4.12}$$

from (4.10) and (4.11) it follows that

$$(4.13) \qquad (N_{\pm}(r + b\sqrt{r}, \pounds_0) - N_{\pm}(r - b\sqrt{r}, \pounds_0)) \leq C, \quad r \to \infty$$

for some $C > 0$. Hence, by virtue of (4.12) for arbitrary α with $0 < \alpha < \frac{\pi}{2}$, there is a constant C_α such that

$$(4.14) \qquad |N_{\pm}(r, \alpha, \pounds) - N_{\pm}(r, \alpha, \pounds_0)| \leq C_\alpha$$

Taking in view the fact that the spectrum of $\pounds_0$ is discrete and using Corollaries 3.1 and 3.2 we have that for all α, $0 < \alpha < \frac{\pi}{2}$, the number of eigenvalues of $\pounds$ which are lying outside the angle $\psi_\alpha^{\pm} = \{\lambda : |\arg(\pm\lambda)| < \alpha\}$ is finite. Therefore from (4.10), (4.11) and (4.12) it follows that

$$(4.15) \qquad N_{+}(r, \frac{\pi}{2}, \pounds) = \frac{1}{\sqrt{-p_1\pi}}\sqrt{r} + O(1), \quad r \to \infty$$

and

$$(4.16) \qquad N_{-}(r, \frac{\pi}{2}, \pounds) = \frac{1}{\sqrt{p_2\pi}}\sqrt{r} + O(1), \quad r \to \infty$$

Consequently in both left- and right half-plane the operator $\pounds$ has infinitely many eigenvalues. Denote by $\{\lambda_{n,1}\}_1^\infty$ and $\{\lambda_{n,2}\}_1^\infty$ all eigenvalues of operator $\pounds$, which lies in the right and left half-plane respectively and arranged as $|\lambda_{1,i}| \leq |\lambda_{2,i}| \leq ...$ (i=1,2) according counted with their algebraic multiplicity. Then from (4.15) and (4.16) we have

$$(4.17) \qquad |\lambda_{n,i}| = |p_i|\pi^2 n^2 + O(n), \quad n \to \infty (i = 1, 2)$$

Further, by virtue of Lemma 4.1, there is $C > 0$ such that

$$(4.18) \qquad |Im\lambda_{n,i}|^2 \leq C|\lambda_{n,i}|, \ (i = 1, 2)$$

and therefore for sufficiently large n(in fact, when $|\lambda_{n,i}| \geq C$) we have

$$(4.19) \qquad \begin{aligned} |Re\lambda_{n,i}|^2 &= |\lambda_{n,i}|^2 - |Im\lambda_{n,i}|^2 \\ &\geq |\lambda_{n,i}|^2 - C|\lambda_{n,i}| \geq (|\lambda_{n,i}| - C)^2, \quad C = const \end{aligned}$$

Consequently,

$$(4.20) \qquad |Re\lambda_{n,i}| = |p_i|\pi^2 n^2 + O(n), \ \text{and} \ |Im\lambda_{n,i}| = O(n)$$

i.e.

$$\lambda_{n,i} = -p_i\pi^2 n^2 + O(n), \quad (i = 1, 2) \tag{4.21}$$

The proof for the case $p_1p_2 > 0$ is totally similar. The proof is complete. □

Theorem 4.4. *Let the condition 1. of Theorem 4.3 be satisfied and let the operator A from $W_2^2(-1,0) \oplus W_2^2(0,1)$ into $L_2(-1,0) \oplus L_2(0,1)$ acts compactly.*

Case1. If $p_1p_2 < 0$ then the eigenvalues of the problem (1.1)–(1.5) can be arranged as two sequence $\{\lambda_{n,1}\}_1^\infty$ and $\{\lambda_{n,2}\}_1^\infty$ with asymptotics

$$\lambda_{n,1} = -p_1\pi^2 n^2 + o(n^2), \quad \lambda_{n,2} = -p_2\pi^2 n^2 + o(n^2) \tag{4.22}$$

Case2. If $p_1p_2 > 0$ then the eigenvalues of the problem (1.1)–(1.5) can be arranged as one sequence $\{\lambda_n\}_1^\infty$ with asymptotics

$$\lambda_n = -\frac{p_1p_2}{p_1 + 2\sqrt{p_1p_2} + p_2}\pi^2 n^2 + o(n^2). \tag{4.23}$$

Proof. We are already shown in the proof the Theorem 4.3, that the operator $\pounds_0$ is self-adjoint with discrete spectrum and for the eigenvalues of this operator the asymptotic formulas (4.4)(for $p_1p_2 < 0$) and (4.5) (for $p_1p_2 > 0$) are hold. At the other and, by virtue of the Corollary 3.2 the operator $R(\lambda, \pounds_0)$ is compact. Let us consider the case $p_1 < 0$, $p_2 > 0$. From (4.9) and (4.10) it follows that

$$\lim_{\substack{r\to\infty \\ \varepsilon\to 0}} \frac{N_\pm\left(r(1+\varepsilon), \pounds_0\right)}{N_\pm(r, \pounds_0)} \quad = \quad 1. \tag{4.24}$$

By virtue of the lemma 4.2, from (4.9) and (4.10) it follows that

$$N_\pm(r, \alpha, \pounds) = N_\pm(r, \alpha, \pounds_0) + o(\sqrt{r}), \;\; r \to \infty \tag{4.25}$$

for all $\alpha(0 < \alpha < \frac{\pi}{2})$.Taking in view the Corollary 3.1 we see that the relation (4.25) is equivalent to the following asymptotic relation

$$|\lambda_{n,i}| = |p_i|\pi^2 n^2 + O(n^2), \quad n \to \infty (i = 1, 2) \tag{4.26}$$

Further from Corollary 3.1 it follows that for all $\alpha(0 < \alpha < \frac{\pi}{2})$ there is natural number n_α such that for all $n \geq n_\alpha$

$$\frac{|Re\lambda_{n,i}|}{|\lambda_{n,i}|} > \cos\alpha, \quad \frac{|Im\lambda_{n,i}|}{|\lambda_{n,i}|} < \sin\alpha \quad (i = 1, 2) \tag{4.27}$$

Consequently

$$\lim_{n\to\infty}\frac{|Re\lambda_{n,i}|}{|\lambda_{n,i}|}=1,\quad \lim_{n\to\infty}\frac{|Im\lambda_{n,i}|}{|\lambda_{n,i}|}=0\quad (i=1,2) \tag{4.28}$$

This means that

$$|Re\lambda_{n,i}|=|p_i|\pi^2n^2+O(n),\ |Im\lambda_{n,i}|=O(n) \tag{4.29}$$

i.e.

$$\lambda_{n,i}=-p_i\pi^2n^2+O(n),\quad (i=1,2) \tag{4.30}$$

The proof is complete. □

5. The Abel basis of root functions of the problem

(1.1)–(1.5) Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{S}$ a unbounded closed linear operator acting in this space with a dence domain $D(\mathcal{S})$. Assume that the spectrum of $\mathcal{S}$ is discrete and $\{\lambda_j\}(j=1\div\infty)$ its eigenvalues which arranged as $|\lambda_1|\le|\lambda_2|\le ....$ Denote by m_j the dimension of root lineal M_{λ_j} and let $f_1^j, f_2^{(j)},..., f_{m_{(j)}}^{(j)}$ be any orthonormal basis of this root lineal. Let $\epsilon_j>0$ any real numbers, so that $\epsilon_j<\min_{i\neq j}|\lambda_i-\lambda_j|$. Obviously the contour $|\lambda_i-\lambda_j|=\epsilon_j$ surrounds only one eigenvalue(namely the eigenvalue λ_j) It is known that (see, [18]) the range of the projection operator $P_{\lambda_k}(S)$ defined as

$$P_{\lambda_k}(S):=-\frac{1}{2\pi i}\oint\limits_{|\lambda-\lambda_k|=\epsilon_k}(\lambda I-\mathcal{S})^{-1}d\lambda$$

is contained in the root lineal $M_{\lambda_j}(S)$ and can be represented as

$$P_{\lambda_k}(S)f=\sum_{i=1}^{m_k}c_i^{(k)}f_i^{(k)}$$

for each $f\in\mathcal{H}$. Under above assumptions the series(not necessarily convergent)

$$f\sim\sum_{j=1}^{\infty}(-\frac{1}{2\pi i}\oint\limits_{|\lambda-\lambda_j|=\epsilon_j}(\lambda I-\mathcal{S})^{-1}fd\lambda)=\sum_{j=1}^{\infty}(\sum_{i=1}^{m_j}c_i^{(j)}f_i^{(j)})$$

is said to be a formal expansion of the vector $f\in\mathcal{H}$ in the series of root vectors of $\mathcal{S}$. Let θ and α any real positive numbers such that $\theta<\frac{\pi}{2}$ and

$\alpha < \frac{\pi}{2\theta}$. Assume that the eigenvalues λ_j(without, at least, finite number) of the operator $\mathcal{S}$ are contained in the angle $\chi_\theta = \{\lambda \in \mathbb{C} | |arg\lambda| < \theta\}$. Then for λ^α in this angle we mean $\lambda^\alpha := |\lambda|^\alpha e^{i\alpha arg\lambda}$. Consequently for each constant $t > 0$ the function $|e^{-\lambda^\alpha t}|$ exponentially tends to zero in the angle χ_θ for $|\lambda| \to \infty$. If

$$\lim_{t\to+0} \|f - \sum_{j=1}^{\infty}(-\frac{1}{2\pi i} \oint_{|\lambda-\lambda_j|=\epsilon_j} e^{-\lambda_j^\alpha t}(\lambda I - \mathcal{S})^{-1} f d\lambda)\| = 0$$

then the system of root vectors of $\mathcal{S}$ is said to be an Abel basis of order α where for $\lambda_k \notin \chi_\theta$ the expression $e^{-\lambda_k^\alpha t}$ is replace by 1.

Theorem 5.1. *[28] If $\mathcal{S}$ is self-adjoint operator with discrete spectrum in the Hilbert space and*

$$\liminf(N(r, R, \mathcal{S})/r^s) < \infty \tag{5.1}$$

for some $s > 0$ and if B is p-subordinate to $\mathcal{S}(0 \le p < 1)$, then for each $\alpha > \max\{s - p + 1, 0\}$ the system of root vectors of the operator $\mathcal{S} + B$ forms an Abel basis of order α in the Hilbert space $\mathcal{H}$.

By using this theorem and the Theorem 4.4 we shall prove the next result.

Theorem 5.2. *Let the following conditions be satisfied:*

1. $p_1\delta_0 = p_2\gamma_1$.

2. The operator A acts boundedly from the Hilbert space $W_2^1(-1,0) \oplus W_2^1(0,1)$ into the Hilbert space $L_2(-1,0) \oplus L_2(0,1)$.

Then the system of root functions(i.e. eigen and associated functions) of the main problem (1.1)–(1.5) forms an Abel basis of order α in the Hilbert space $L_2(-1,0) \oplus L_2(0,1)$ for arbitrary $\alpha > 1$.

Proof. Consider the case $p_1 < 0, p_2 > 0$(the other cases are similar). Then from (4.9) and (4.10) it follows that the condition (5.1) is satisfied for $s = \frac{1}{2}$. Moreover, similarly to the proof of the Theorem 4.3 we can prove that the operator A is $\frac{1}{2}$ -subordinate to $\pounds_0$. Consequently, it is enough to apply the Theorem 5.2 to the operators $\pounds_0$ and A to complete the proof. □

Remark 5.3. *It is known that the property of a system of root vectors to form an Abel basis of some order $\alpha > 0$ is the internal property between the completeness of root vectors and a basis with parentheses. Note that the concept of an Abel basis was first introduced in [44].*

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