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The Borel transform and linear nonlocal equations: applications to zeta-nonlocal field models

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We define rigorously operators of the form $f(\partial_t)$, in which f is an analytic function on a simply connected domain. Our formalism is based on the Borel transform on entire functions of exponential type. We study existence and regularity of real-valued solutions for the nonlocal in time equation

$$f(\partial_t)\phi = J(t) \ , \quad t \in \mathbb{R} \ ,$$

and we find its more general solution as a restriction to $\mathbb{R}$ of an entire function of exponential type. As an important special case, we solve explicitly the linear nonlocal zeta field equation

$$\zeta(\partial_t^2 + h)\phi = J(t) \ ,$$

in which h is a real parameter, ζ is the Riemann zeta function, and J is an entire function of exponential type. We also analyse the case in which J is a more general analytic function (subject to some weak technical assumptions). This latter case turns out to be rather delicate: we need to re-interpret the symbol $\zeta(\partial_t^2 + h)$. We prove that in this case the zeta-nonlocal equation above admits an analytic solution on a simply connected domain determined by J.

The linear zeta field equation is a linear version of a field model depending on the Riemann zeta function arising in p-adic string theory [B. Dragovich, Zeta-nonlocal scalar fields, *Theoret. Math. Phys.*, 157 (2008), 1671–1677].

1. Introduction

In this paper a *nonlocal operator* is an expression of the form $f(\partial_t)$, in which f is an analytic function, and a *nonlocal equation* is an equation in which a nonlocal operator appears. It is of course well-known how to define the action of $f(\partial_t)$ on a given class of functions if the "symbol" f is a polynomial. However, it is not obvious how to extend this definition to more general symbols f: for instance, f may be beyond the reach of classical tools used in the study of pseudo-differential operators (*e.g.*, the derivatives of f may not satisfy appropriated bounds, see [25, 31, 45]). We provide a rigorous definition of $f(\partial_t)$ for essentially *any* analytic function f —including the Riemann zeta function ζ— in the main body of this work, using integral transforms.

In our previous work [16] we use Laplace transform as an operator from an appropriate Lebesgue space into a Hardy space in order to define properly $f(\partial_t)$ and to solve the equation $f(\partial_t)\phi = J$ for an appropriate function J. We can think of two reasons why it is worthwhile to extend this approach, in spite of its power:

1. Let us consider the function $p : s \mapsto s + h$, $h \in \mathbb{R}$, and the symbol $(\zeta \circ p)(\partial_t)$. In [16] we show how to solve the equation $(\zeta \circ p)(\partial_t)\phi = \zeta(\partial_t + h)\phi(t) = J(t)$ for appropriate functions J. But, if we take $q : s \mapsto s^2 + h$, $h \in \mathbb{R}$, *we cannot use* the approach presented in [16] to solve the equation $(\zeta \circ q)(\partial_t)\phi = \zeta(\partial_t^2 + h)\phi = J$, as we explain in great detail in Section 2 below. The importance of developing a theory able to deal with the latter equation is that the operator $\zeta(\partial_t^2 + h)$ does appear in string theory research, see [21–24].

2. The extended approach considered herein uses the full power of the Borel transform (introduced in Subsection 3.1 below) instead of the Laplace transform. Roughly speaking, if we use Borel transform we integrate over appropriate closed curves in the plane, while if we use Laplace transform we are restricted to integrating over the x-axis. This flexibility allows us to deal with operators such as $\zeta(\partial_t^2 + h)$ *and* to obtain results on existence of *analytic solutions* to nonlocal equations, instead of L^p-solutions as in [16].

Further motivation for the work carried out herein is given later on in this section. Before going into that, we make some short comments on our general framework: we frame our discussion within classical analytic function theory, see for instance [8]. We believe it is interesting to note that nonlocal operators appear naturally in this abstract setting, for example, in the study of the distribution of zeroes of entire functions, see [13, 14, 35] and references therein. We present the following results (Lemma 1.1 and Theorem 1.2) as an illustration:

The Laguerre-Pólya class, denoted by $\mathcal{LP}$, is defined as the collection of entire functions f having only real zeros, and such that f has the following factorization [8, sections 2.6 and 2.7]:

$$f(z) = cz^m e^{\alpha z - \beta z^2} \prod_k \left(1 - \frac{z}{\alpha_k}\right) e^{z/\alpha_k} \ ,$$

where $c, \alpha, \beta, \alpha_k$ are real numbers, $\beta \geq 0$, $\alpha_k \neq 0$, m is a non-negative integer, and $\sum_{k=1}^{\infty} \alpha_k^{-2} < \infty$.

Let D be the differentiation operator and $\phi \in \mathcal{LP}$; the following lemma presents one important instance in which the nonlocal operator $\phi(D)$ understood via power series, is in fact well defined, see [35, Theorem 8, p. 360].

Lemma 1.1. *Let $\phi, f \in \mathcal{LP}$ such that*

$$\phi(z) = e^{-\alpha z^2} \phi_1(z) \text{ and } f(z) = e^{-\beta z^2} f_1(z) \ ,$$

where ϕ_1, f_1 have genus 0 or 1 and $\alpha, \beta \geq 0$. If $\alpha\beta < 1/4$, then $\phi(D)f \in \mathcal{LP}$.

The notion of the genus of a function is explained in [8, p. 22]. We have, see [14, Theorem 1],

Theorem 1.2. *Let $\phi, f \in \mathcal{LP}$ such that*

$$\phi(z) = e^{-\alpha z^2} \phi_1(z) \text{ and } f(z) = e^{-\beta z^2} f_1(z) \ ,$$

where ϕ_1, f_1 have genus 0 or 1 and $\alpha, \beta \geq 0$. If $\alpha\beta < 1/4$ and ϕ has infinitely many zeros, then $\phi(D)f$ has only simple and real zeros.

We do not use this result explicitly in this paper, but zeroes of entire functions *are* crucial in our theory, see for instance Theorem 3.17 and Example 4.1 below, and so we expect that results such as Theorem 1.2 will play a role in future developments.

Now let us go back to our motivation and goals. As is the case with our previous paper [16], our main motivation for the study of nonlocal equations comes from Physics: nonlocal operators and equations can be found in the Physics literature as far back as the 1950's, see [42]. In this classical paper, Pais and Uhlenbeck present the equation

$$F(\Box)\psi = \rho \, , \tag{1.1}$$

where (their notation)

$$F(\Box) = \Pi_{i=1}^{N}(\Box^i - \kappa_i^2) \, . \tag{1.2}$$

The relevance of this equation for field theory is discussed in depth in Section 1 of [42]. In the same section, the authors state that "we will in fact admit that N in (1.2) may tend to infinity provided the infinite product thus arising is mathematically well defined". If we consider the function $g(s) = \Pi_{i=1}^{N}(s^i - \kappa_i^2)$, then (1.2) is an operator with "symbol" g, and in light of known theorems in complex analysis (see for instance [43]) we may understand Pais and Uhlenbeck's statement as an invitation for the consideration of general analytic functions as symbols. We remark that they do use very general symbols later on in their paper. For example, the symbol $\exp(f(s))\,(s - \kappa^2)$ (a special case of which is of importance for string theory, see Equation (4.8) in [5]) appears in [42, Equation (11)].

Nowadays nonlocal equations are of interest for string theory, cosmology and non-local theories of gravity. We refer the reader to [5–7, 12, 28] for information on the last two topics. With respect to string theory, let us mention two important equations:

$$p^{a\,\partial_t^2}\phi = \phi^p \, , \qquad a > 0 \, , \tag{1.3}$$

and

$$\left((\partial_x^2 + 1)e^{-c\partial_x^2} - 2\right)\phi = \overline{g}\phi^2 \, . \tag{1.4}$$

Equation (1.3) is the equation of motion for the tachyon field in p-adic string theory, see [5, Equation (1.1)], [40, Equation (1.5)], and [21, 48, 49, 51], while equation (1.4) is of interest for open string theory, see [5, Equation (1.2)] and [18, Equation (14)]. These equations have been studied numerically and by means of essentially formal arguments in *e.g.* [3, 4, 18, 40, 41, 48, 50, 51]. A linear version of (1.3) is considered in Section 4 of the inspirational paper [5] by Barnaby and Kamran. Are there other analytic functions that appear as symbols of meaningful nonlocal equations? Yes. In the paper [36] there appear the symbols $\cos(s/k)$ and $2k\sin^2(s/2k)$, $k > 0$, in [34] the authors study nonlocal actions for gravitation that depend on very general analytic symbols $f(s)$, see for instance [34, Equations (2) and (3)] and, more importantly for us, in the intriguing paper [21], see also [22–24], B. Dragovich constructs a field theory starting from (1.3) whose equation of motion (in $1+0$ dimensions) is

$$\zeta\left(-\frac{1}{m^2}\,\partial_t^2 + h\right)\psi = U(\psi)\ . \tag{1.5}$$

Here ζ is the Riemann zeta function as we noted before; m, h are real parameters and U is some nonlinear function. Here and hereafter, we understand the Riemann zeta function as the analytic extension of the infinite series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad Re(s) > 1\ ,$$

to the whole complex plane, except $s = 1$, where it has a pole of order 1. Dragovich explains his construction further, and presents some alternative equations arising from (1.3) that also depend of the function ζ, in the recent preprint [24].

Now we are ready to state the main aim of this work. *Our goal is to develop a general theory for interpreting and solving linear nonlocal equations of the form*

$$f(\partial_t)\phi(t) = g(t)\ , \qquad t \in \mathbb{R}\ , \tag{1.6}$$

in which the symbol f *is an arbitrary analytic function on a simply connected domain* and g is an appropriate (but fairly general) function. We have presented a rigorous setting for (a restricted class of) equations (1.6) in [16], using Laplace transform. However, as we explain in Section 2 below, the important equation

$$\zeta\left(-\frac{1}{m^2}\,\partial_t^2 + h\right)\phi = J(t)\ , \tag{1.7}$$

naturally motivated by (1.5), is beyond the reach of our Laplace transform method. Given the importance of (1.5) and (1.7) for string theory, see [21–24], and the appearance of interesting examples of equations of the general form (1.6) in diverse physical theories, see *e.g.* [5, 6, 26, 34, 42], we believe that our endeavour is indeed fully justified.

We extend our previous methods —so as to encompass as allowable symbols *any* function that is holomorphic in an appropriate domain— essentially by moving from Laplace transform to Borel transform, as we explain at the beginning of this section. We point out that we have previously used the latter in [15], and we have found that "the Borel transform method" is in fact of easy applicability. We remark immediately that the analysis of [15] is restricted to symbols f (see Equation (1.6)) that are entire functions, and to right hand side terms J that are functions of exponential type. Equations such as (1.6) and (1.7) motivate us to remove these two restrictions. Thus, this paper is an extension and generalization of *both* [15] and [16].

We finish this section with a brief summary of our analytic set-up and a description of the contents of this work.

Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain. We denote by $Exp(\Omega)$ the space of entire functions of finite exponential type such that its elements have Borel transform with singularities in Ω (definitions are in Section 3). Now let f be an holomorphic function in Ω; we use the Borel transform to define $f(\partial_t)$ as a linear operator on the space $Exp(\Omega)$ in a way that evokes the definition of classical pseudo-differential operators via Fourier transform, and then we find the most general solution to Equation (1.6) as a restriction to $\mathbb{R}$ of a function in $Exp(\Omega)$. As essentially already announced, our main example is the application of our theory to the following equation, a normalized version of (1.7) with our signature conventions:

$$\zeta(\partial_t^2 + h)\phi = J \ . \tag{1.8}$$

We organize this work as follows. In Section 2 we recall our work [16] and we explain why Equation (1.8) *cannot* be studied with the tools developed therein. In this section we also comment briefly on a possible physical motivation for the study of (1.8). In Section 3 we consider a general analytic symbol f and we define the action of the operator $f(\partial_t)$ on the space of entire functions of exponential type using Borel transform. We also solve Equation (1.6) and we prove that it admits *analytic* solutions. In Section 4 we apply the theory developed in Section 3 to the linear zeta-nonlocal scalar field equation (1.8): we find that the zeroes of the Riemann zeta function play an important role in representing its solution. Also in this section, we introduce the space $\mathcal{L}_>(\mathbb{R}_+)$ of all real functions g with domain $[0, +\infty)$ such that: *(a)* there exist their Laplace transform $\mathcal{L}(g)$, and *(b)* $\mathcal{L}(g)$ has an analytic extension to an angular contour. With its help, we study and solve equation (1.8) for right hand side J in $\mathcal{L}_>(\mathbb{R}_+)$. This study involves some delicate limit procedures that take us outside the class of functions of exponential type that we used as domain for our operators. Finally, in an appendix we formally derive some equations of motion of interest from a mathematical point of view, including the zeta-nonlocal scalar field proposed by B. Dragovich.

2. A preliminary discussion

As is discussed in Dragovich's papers [21–24], see also the appendix of this work, the following nonlocal equation

$$\zeta\left(\frac{\Box}{2m^2}+h\right)\psi = U(\psi) \tag{2.1}$$

in which $U(z)$ is an analytic non-linear function of z, appears naturally in an interesting mathematical modification of p-adic string theory.

Motivated by Equation (2.1), in this section we make a preliminary investigation of linear equations in $(1+0)$ dimensions of the form

$$\zeta(\partial_t^2 + h)\phi = J\ . \tag{2.2}$$

in which we are using a signature so that $\Box = \partial_t^2$, simply for comparison purposes with our previous articles.

Before going into technical details, we comment on the possible physical interest of (2.2). First, we point out that the Riemann zeta function and its generalizations are ubiquitous in contemporary Mathematical Physics, see for instance [27] and [2], and therefore the study of (2.2) seems to us to be of interest on its own. Also, motivated by the analysis in [5, Section 4], let us take a solution ψ_0 to Dragovich's equation (2.1) (particular solutions for a special choice of $U(\phi)$ appear in [21]). Then, linearizing around ψ_0 we obtain the equation

$$\zeta(\partial_t^2 + h)\phi = U'(\psi_0)\phi\ .$$

Equation (2.2) with $J = 0$ coincides with this equation if $U'(\psi_0) = 0$. Thus, Equation (2.2) is a "driven" version of (a special case of) the linearized Dragovich equation. Our theory allows us, in particular, to present rigorous theorems on the explicit representation of solutions to (2.2) for a source function $g(z)$ satisfying diverse technical hypotheses, see Theorems 4.1, 4.2, 4.4 and Proposition 4.8. We finish this paragraph by remarking that of course our theory *is not* tailored to Equation (2.2): it is applicable to *any* equation of the form (1.6), as we show in Section 3.

Now we begin our preliminary analysis of (2.2). We recall that in our previous work [16], we study linear nonlocal equations (and its associated Cauchy problem) using an approach based on Laplace transforms and the Doetsch representation theorem, see [19]: If $L^p([0,+\infty))$ is the standard L^p-Lebesgue space and $H^q(\mathbb{C}_+)$ is the Hardy space, there exists a correspondence between these spaces determined by the Laplace transform $\mathcal{L} : L^p([0,+\infty)) \to H^q(\mathbb{C}_+)$ for appropriated Lebesgue exponents p, q. In this situation, we obtained exact formulas for the representation of the solution for equations such as (1.6). (The approach of [16] supersedes previous work [29, 30]). One of our results is the following theorem (The function r appearing in the statement of the theorem is a "generalized initial condition", see [16, Section 3]):

Theorem 2.1. *Let us fix a function f which is analytic in a region D which contains the half-plane $\{s \in \mathbb{C} : Re(s) > 0\}$. We also fix p and p' such that $1 < p \leq 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$, and we consider a function $J \in L^{p'}(\mathbb{R}_+)$ such that $\mathcal{L}(J) \in H^p(\mathbb{C}_+)$. We assume that the function $(\mathcal{L}(J) + r)/f$ is in the space $H^p(\mathbb{C}_+)$. Then, the linear equation*

$$f(\partial_t)\phi = J \tag{2.3}$$

can be uniquely solved on $L^{p'}(0,\infty)$. Moreover, the solution is given by the explicit formula

$$\phi = \mathcal{L}^{-1}\left(\frac{\mathcal{L}(J) + r}{f}\right) . \tag{2.4}$$

Moreover, using some technical assumptions (see [16, corollary 2.10]), the representation formula (2.4) for the solution can be reduced to

$$\phi = \mathcal{L}^{-1}\left(\frac{\mathcal{L}(J)}{f}\right) + \mathcal{L}^{-1}\left(\frac{r}{f}\right) . \tag{2.5}$$

The theory can be applied to various field models; in particular, it can be applied to zeta-nonlocal field models of the form

$$\zeta(\partial_t + h)\phi = J .$$

for appropriate functions J, see [16, Section 4].

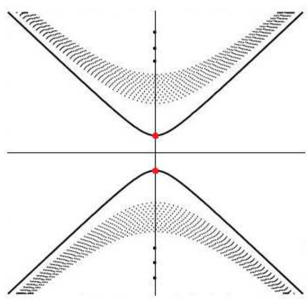
Now, let us denote by $\mathcal{A}(\mathbb{C}_+)$ the class of functions which are analytic in a region D which contains the half-plane $\{s \in \mathbb{C} : Re(s) > 0\}$. We can see that for $h > 1$ the symbol $\zeta(s + h)$ is in the class $\mathcal{A}(\mathbb{C}_+)$ while, if $p(s) := s^2$, the symbol $\zeta_h \circ p(s) := \zeta(s^2 + h)$ is not, as we explain presently. It follows from this observation that for some non-analytic forces J (e.g. piecewise smooth functions with exponential decay) we have $\frac{\mathcal{L}(J)}{\zeta_h \circ p} \notin H^p(\mathbb{C}_+)$, and therefore the representation formula (2.5) of the solution breaks down. We conclude that the study of Equation (2.2) requires a generalization of the theory developed in [16].

First of all, we observe that the properties of the Riemann zeta function (see for instance [16, Section 4]) imply that the symbol

$$\zeta(s^2 + h) = \sum_{n=0}^{\infty} \frac{1}{n^{s^2+h}} \tag{2.6}$$

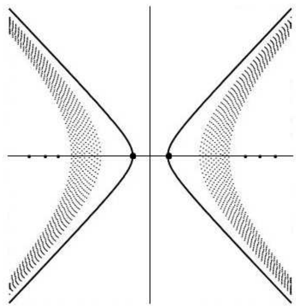
is analytic in the region $\Gamma := \{s \in \mathbb{C} : Re(s)^2 - Im(s)^2 > 1 - h\}$, which is not a half-plane; its analytic extension $\zeta \circ p$ has poles at the vertices of the hyperbolas $Re(s)^2 - Im(s)^2 = 1 - h$, and its critical region is the set $\{s \in \mathbb{C} : -h < Re(s)^2 - Im(s)^2 < 1 - h\}$. In fact, we recall from [16, Section 6] that according to the value of h we have:

i) For $h > 1$, Γ is the region limited by the interior of the dark hyperbola $Re(s)^2 - Im(s)^2 = 1 - h$ containing the real axis:



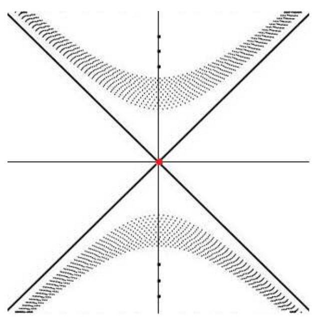
The poles of $\zeta(s^2 + h)$ are the vertices of dark hyperbola, indicated by two thick dots. The trivial zeroes of $\zeta(s^2 + h)$ are indicated by thin dots on the imaginary axis; and the non-trivial zeroes are located on the darker painted region (critical region).

ii) For $h < 1$, Γ is the interior of the dark hyperbola $Re(s)^2 - Im(s)^2 = 1 - h$ containing the imaginary axis:



The poles of $\zeta(s^2 + h)$ are the vertices of dark hyperbola, indicated by two thick dots. The trivial zeroes of $\zeta(s^2 + h)$ are indicated by thin dots on the real axis; the non-trivial zeroes are located on the darker painted region (critical region).

iii) For $h = 1$, Γ is the interior of the cones limited by the curves $y = |x|, y = -|x|$.



The pole of $\zeta(s^2 + 1)$ is the origin (vertex of dark curves $y = |x|, y = -|x|$). The trivial zeroes of $\zeta(s^2 + h)$ are indicated by thin dots on the imaginary axis; the non-trivial zeroes are located on the darker painted region (critical region).

On the other hand, since the Riemann zeta function has an infinite number of nontrivial zeroes in the critical strip (as famously proven by Hadamard and Hardy, see [32] for original references), the function $\zeta_h \circ p(\cdot)$ has also an infinite number of nontrivial zeroes on its critical region. We denote by $\mathcal{Z}$ the set of all such zeroes. By i), ii) and iii) we have that $\mathcal{Z}$ is contained in the corresponding dark dotted region. Moreover

$$\sup_{z\in\mathcal{Z}} |Re(z)| = +\infty.$$

This analysis implies that the function $\mathcal{L}(J)/(\zeta_h \circ p)$ does not necessarily belongs to $H^p(\mathbb{C}_+)$, and therefore the expression $\mathcal{L}^{-1}\left(\frac{\mathcal{L}(J)}{\zeta_h \circ p}\right)$ in the representation of the solution (2.5) does not always make sense.

Thus, a new approach for the study of Equation (2.1) is necessary. As stated in Section 1, the method that we use is based on the Borel transform, see [8, 15, 25, 45] and references therein.

3. The general theory for nonlocal equations

3.1. Entire functions of exponential type

Definition 3.1. *An entire function $\phi : \mathbb{C} \to \mathbb{C}$ is said to be of finite exponential type τ_ϕ and finite order ρ_ϕ if τ_ϕ and ρ_ϕ are the infimum of the positive numbers τ, ρ such that the following inequiality holds:*

$$|\phi(z)| \leq Ce^{\tau |z|^\rho}, \quad \forall z \in \mathbb{C}\ , \ \textit{and some } C > 0.$$

When $\rho_\phi = 1$, the function ϕ is said to be of **exponential type**, or of **exponential type** τ_ϕ, if we need to specify its type. If we know the representation of a entire function ϕ as a power series, then a standard way to calculate its order, see [8, Theorem 2.2.2], is by using the formula

$$\rho = \left(1 - \limsup_{n\to\infty} \frac{\ln |\phi^{(n)}(0)|}{n \ln(n)}\right)^{-1}, \tag{3.1}$$

while its type is calculated as follows (see formula 2.2.12, page 11 in [8]):

$$\sigma = \limsup_{n\to\infty} |\phi^{(n)}(0)|^{1/n}\ . \tag{3.2}$$

The space of functions of exponential type will be denoted by $Exp(\mathbb{C})$.

Definition 3.2. *Let ϕ be an entire function of exponential type τ_ϕ. If $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$; then, the Borel transform of ϕ is defined by*

$$B(\phi)(z) := \sum_{n=0}^{\infty} \frac{a_n n!}{z^{n+1}}\ .$$

It can be checked that $B(\phi)(z)$ converges uniformly for $|z| > \tau_\phi$, see [45, p. 106], and therefore it defines an analytic function on $\{z \in \mathbb{C} : |z| > \tau_\phi\}$.

An alternative way to calculate the Borel transform of an entire function ϕ of exponential type τ_ϕ is the use of the complex Laplace transform, see [8]: if $z = |z| \exp(i\theta)$ is such that $|z| = r > \tau_\phi$, then

$$B(\phi)(re^{i\theta}) = e^{i\theta} \int_0^\infty \phi(te^{i\theta}) e^{-rt} dt \,. \tag{3.3}$$

In particular, if $z \in \mathbb{R}$ is such that $z > \tau_\phi$, then $B(\phi)$ can be obtained as the analytic continuation of its real Laplace transform:

$$\mathcal{L}(\phi)(z) = \int_0^\infty \phi(t) e^{-zt} dt \,. \tag{3.4}$$

For $\phi \in Exp(\mathbb{C})$, we let $s(B(\phi))$ denote the set of singularities of the Borel transform of ϕ, and we also denote by $S(\phi)$ the *conjugate diagram* of $B(\phi)$, this is, the closed convex hull of the set of singularities $s(B(\phi))$. The set $S(\phi)$ is a convex compact subset of $\mathbb{C}$, and we can check that $B(\phi)$ is an analytic function in $\mathbb{S} \setminus S(\phi)$, where $\mathbb{S}$ is the extended complex plane $\mathbb{C} \cup \{\infty\}$ and we have set $B(\phi)(\infty) = 0$.

Remark. *Hereafter we will use the following notation: if $\Omega \subset \mathbb{C}$ is a domain, then Ω^c denotes the complement of Ω in the extended complex plane $\mathbb{S}$.*

Definition 3.3. *Let Ω be a simple connected domain; we define the space $Exp(\Omega)$ as the set of all entire functions ϕ of exponential type such that its Borel transform $B(\phi)$ has all its singularities in Ω and such that $B(\phi)$ admits an analytic continuation to Ω^c. This continuation will continue being denoted by $\mathcal{B}(\phi)$.*

Remark. *Since Ω^c is closed, the fact that $\mathcal{B}(\phi)$ is analytic in Ω^c means that there exists an open set $U \subset \mathbb{S}$ such that $\mathcal{B}(\phi)$ is analytic in U and $\Omega^c \subset U$. Therefore, using the alternative definition of Borel transform* (3.3)*, we understand $\mathcal{B}(\phi)$ as the analytic continuation of its real Laplace transform* (3.4)*.*

Remark. *In what follows,* **the Borel transform of** $\phi \in Exp(\Omega)$ *always refers to the complex function $\mathcal{B}(\phi)$ together with an open set U in the extended plane $\mathbb{S}$ such that $\mathcal{B}(\phi)$ is analytic in U and $\Omega^c \subset U$.*

Definition 3.4. *For a function $\phi \in Exp(\Omega)$, we define the set $H_1(\phi)$ to be the class of closed rectifiable and simple curves in $\mathbb{C}$ which are pairwise homologous and contain the set $s(B(\phi))$ in their bounded regions.*

The following theorem is a classical result about the representation of entire functions of exponential type.

Theorem 3.5. *(Polya's Representation Theorem). Let ϕ be a function of exponential type and let $\gamma \in H_1(\phi)$. Then,*

$$\phi(z) = \frac{1}{2\pi i} \int_{\gamma} e^{sz} \mathcal{B}(\phi)(s) ds.$$

In particular, if ϕ is of type τ and $R > \tau$, then

$$\phi(z) = \frac{1}{2\pi i} \int_{|s|=R} e^{sz} B(\phi)(s) ds.$$

This theorem is discussed for instance in [45, p. 107] and [15]. A proof appears in [8, Theorem 5.3.5].

Definition 3.6. *If d is a distribution with compact support in $\mathbb{C}$, we define the $\mathcal{P}$-transform of d by:*

$$\mathcal{P}(d)(z) := < e^{sz}, d > , \quad z \in \mathbb{C}.$$

The $\mathcal{P}$-transform is called the Fourier-Laplace transform in [45] and the Fourier-Borel transform in Martineau's classical paper [37]. For the particular case in which $d = \mu$ is a complex measure with compact support, the $\mathcal{P}$-transform is

$$\mathcal{P}(\mu)(z) = \int_{\mathbb{C}} e^{sz} d\mu(s) \, , z \in \mathbb{C}.$$

Proposition 3.7. *Let $\mathcal{O} \subset \mathbb{C}$ be a simply connected domain; if μ is a complex measure with compact support contained in $\mathcal{O}$, then $\mathcal{P}(\mu) \in Exp(\mathcal{O})$. Conversely, given any function $\phi \in Exp(\mathcal{O})$, there exists a complex measure μ_ϕ with compact support in $\mathcal{O}$ and such that $\mathcal{P}(\mu_\phi)(z) = \phi(z)$. The measure μ_ϕ is not unique: it can be chosen to have support on any given curve $\gamma \in H_1(\phi)$.*

Proof. Let K be the support of the complex measure μ (which is of finite variation). The $\mathcal{P}$-transform of μ is

$$\mathcal{P}(\mu)(z) = \int_{\mathbb{C}} e^{sz} d\mu(s) ,$$

which is an entire function. Now, if $R = sup_{s\in K}|s|$, we have

$$|\mathcal{P}(\mu)(z)| \leq \int_{\mathbb{C}} e^{R|z|} |d\mu(s)| \leq e^{R|z|} ||\mu|| ,$$

that is, $\mathcal{P}(\mu)$ is an entire function of exponential type.

It remains to show that $s(B(\mathcal{P}(\mu))) \subset \mathcal{O}$. To do that, we compute the Borel transform of $\mathcal{P}(\mu)$ as the analytic continuation of its real Laplace transform. Let z be a real number such that $z > R$. Then, we have

$$\begin{aligned} B(\mathcal{P}(\mu))(z) &= \int_0^{+\infty} e^{-zt} \mathcal{P}(\mu)(t) dt \\ &= \int_0^{+\infty} e^{-zt} \int_K e^{st} d\mu(s) dt \\ &= \int_K \int_0^{+\infty} e^{(s-z)t} dt d\mu(s) \\ &= \int_K \frac{1}{z-s} d\mu(s) , \end{aligned}$$

in which we have used Fubini's theorem. From these computations we have that $\mathcal{P}(\mu) \in Exp(\mathcal{O})$. In fact, the last integral is the analytic continuation $\mathcal{B}(\mathcal{P}(\mu))$.

To prove the converse implication, let $\gamma \in H_1(\phi)$. Then, Polya's representation theorem (Theorem 3.5) means that ϕ can be represented as $\phi(z) = \mathcal{P}(\mu_\phi)(z)$ for the complex measure μ_ϕ defined by

$$d\mu_\phi(s) := \mathcal{B}(\phi)(s) \frac{ds}{2\pi i} , \quad s \in \gamma . \tag{3.5}$$

□

Remark. *The analogous of this proposition for general distributions can be found in* [15]. *We prefer our version with complex measures because in this work we do not use the machinary of distributions.*

3.2. Functions of ∂_t via Borel transform

Let Ω be a simply connected domain (equivalently, let Ω be a Runge domain, see [44, Prop. 17.2]). In what follows we denote by $Hol(\Omega)$ the set of holomorphic functions on Ω.

Definition 3.8. *Let $f \in Hol(\Omega)$, $\phi \in Exp(\Omega)$, and assume that μ_ϕ is the complex measure defined in* 3.5 *with compact support on a curve $\gamma \in H_1(\phi)$ so that $\mathcal{P}(\mu_\phi) = \phi$. We define the operator $f(\partial_t)\phi$ as*

$$f(\partial_t)\phi := \mathcal{P}(f\mu_\phi) .$$

In Definition 3.8 we assume that the curve $\gamma \in H_1(\phi)$, which defines the measure μ_ϕ, is contained in the region Ω. By Cauchy's Theorem, the operator $f(\partial_t)$ is independent of such a γ and therefore is well defined.

In this way, using the new measure $f\mu_\phi$ and the definition of the $\mathcal{P}$-transform, we see that Equation (1.6) is understood as the following integral equation

$$\int_\gamma e^{st} f(s)\mathcal{B}(\phi)(s)\frac{ds}{2\pi i} = g(t), \quad \gamma \in H_1(\phi) , \quad \phi \in Exp(\Omega) . \tag{3.6}$$

We may wonder whether it is necessary to restrict ourselves to Runge domains. Indeed, the counterexample below is proposed in [25, pag. 27] in order to show that $f(\partial_t)\phi$ can be multivalued if Ω is not a Runge domain. We reproduce it here for the sake of completeness.

Example 3.1. Set $\Omega = \mathbb{C} \setminus \{0\}$. We consider the symbol $f(s) = \dfrac{1}{s}$ which is holomorphic in Ω and the function $\phi_\lambda(z) = e^{\lambda z}, \lambda \in \Omega$. Then the set $s(\mathcal{B}(\phi_\lambda)) = \{\lambda\}$ and we have, for a closed curve γ in Ω containing λ, two possible values for $f(\partial_t)\phi_\lambda(z)$: either

$$f(\partial_t)\phi_\lambda(z) = \frac{1}{\lambda}(e^{\lambda z} - 1) ,$$

or

$$f(\partial_t)\phi_\lambda(z) = \frac{1}{\lambda}e^{\lambda z} ,$$

depending on whether γ encloses the point $\{0\}$ or not.

The following proposition justifies some of the formal computations appearing in physical papers (see [5, 6] and references therein). It says that the integral operator $f(\partial_t)$ is *locally* (*i.e.*, whenever f can be expanded as a power series in an adequate ball contained in Ω) a differential operator of infinite order.

Proposition 3.9. *Let $R > 0$ and assume that $B_R(0) \subset \Omega$. Suppose that $\phi \in Exp(\Omega)$ is such that $s(\mathcal{B}(\phi)) \subset B_R(0)$ and take $f \in Hol(\Omega)$ with $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $|z| < R$. Then, there exist a measure μ_ϕ supported on a curve $\gamma \in H_1(\phi)$ contained in $B_R(0)$ such that $\phi = \mathcal{P}(\mu_\phi)$, and moreover*

$$f(\partial_t)\phi(t) = \mathcal{P}(f\mu_\phi)(t) = \lim_{l\to\infty} \sum_{k=0}^{l} a_k(\partial_t^k \phi)(t),$$

uniformly on compact sets.

Proof. We note that, since $s(\mathcal{B}(\phi))$ is a discrete set, there exists a real number $\delta > 0$ such that $dist(s(\mathcal{B}(\phi)), \{s : |s| = R\}) < \delta$. From [8, Theorem 5.3.12] we have that $\tau_\phi = \sup_{\omega \in s(\mathcal{B}(\phi))} |\omega|$ is the type of ϕ, then there exist a curve $\gamma \subset (B_{\tau_\phi}(0))^c \cap B_R(0)$ such that $\gamma \in H_1(f)$. Let μ_ϕ be the measure described by Theorem 3.7 supported on γ, then $\phi = \mathcal{P}(\mu_\phi)$.

Moreover, using the measure μ_ϕ, we compute:

$$\frac{d}{dz}\phi(z) = \frac{d}{dz}\mathcal{P}(\mu_\phi)(z) = \frac{d}{dz}\int_\gamma e^{sz} d\mu_\phi(s) = \int_\gamma s e^{sz} d\mu_\phi(s) = \mathcal{P}(s\mu_\phi).$$

From this, we obtain

$$\sum_{k=0}^{l} a_k \frac{d^k}{dz^k}\phi(z) - \mathcal{P}(f\mu_\phi)(z) = \mathcal{P}\left(\left\{\sum_{k=0}^{l} a_k s^k - f(s)\right\}\mu_\phi\right)(z)\ ,$$

and therefore

$$\left|\sum_{k=0}^{l} a_k \frac{d^k}{dz^k}\phi(z) - \mathcal{P}(f\mu_{\phi,R})(z)\right| = \left|\int_\gamma e^{sz}\{\sum_{k=0}^{l} a_k s^k - f(s)\} d\mu_\phi(s)\right|$$

$$\leq \int_\gamma e^{|z||s|}\left|\sum_{k=0}^{l} a_k s^k - f(s)\right| |d\mu_\phi|(s)\ .$$

Now we take limits as $l \to \infty$. The result follows from the Lebesgue dominated convergence theorem. We note that the convergence is uniform over compact subsets of $\mathbb{C}$. $\square$

Thus, under the hypothesis of this proposition, we have seen that Equation (1.6) becomes *locally* the following infinite order differential equation:

$$\sum_{k=0}^{\infty} a_k \frac{d^k}{dt^k}\phi(t) = g(t). \tag{3.7}$$

Interestingly, Proposition 3.9 also shows that $f(\partial_t)$ is linear on the space of functions ϕ satisfying the hypothesis appearing therein. We now show that linearity is true in general:

Lemma 3.10. *The operator $f(\partial_t) : Exp(\Omega) \to Exp(\Omega)$ is linear.*

Proof. For $\phi, \psi \in Exp(\Omega)$ we have that $s(\mathcal{B}((\phi + \psi))) \subseteq s(\mathcal{B}(\phi)) \cup s(\mathcal{B}(\psi)) \subset \Omega$. Let $\gamma \in H_1(\phi + \psi)$ such that $s(\mathcal{B}(\phi)) \cup s(\mathcal{B}(\psi))$ is enclosed by γ. This implies that $\gamma \in H_1(\phi)$, $\gamma \in H_1(\psi)$; then

$$\begin{aligned} f(\partial_t)(\phi+\psi)(t) = \mathcal{P}(f\mu_{\phi+\psi})(t) &= \int_\gamma e^{st} f(s) B(\phi+\psi)(s) \frac{ds}{2\pi i} \\ &= \int_\gamma e^{st} f(s) B(\phi)(s) \frac{ds}{2\pi i} + \int_\gamma e^{st} f(s) B(\psi)(s) \frac{ds}{2\pi i} \\ &= f(\partial_t)(\phi)(t) + f(\partial_t)(\psi)(t). \end{aligned}$$

$\square$

Remark. *We remark that, if $\phi, \psi \in Exp(\Omega)$ then the inclusions $H_1(\phi + \psi) \subseteq H_1(\phi)$ and $H_1(\phi + \psi) \subseteq H_1(\psi)$ do not necessarily hold. This is why we had to choose an appropriated curve γ to carry out the above proof. For example, let $R > 2$ and set $\lambda \in \mathbb{C}$ with $|\lambda| = 3/2$; also consider two functions $g_1, g_2 \in Exp_{1/2}(\mathbb{C})$; then, the functions $f_1(z) = e^{\lambda z} + g_1(z)$ and $f_2(z) = -e^{\lambda z} + g_2(z)$ are elements in $Exp(B_R(0))$. Furthermore, if $\gamma \in H_1(f_1 + f_2)$ with $\gamma \subset B_{1/2}(0)^c \cap B_1(0)$, then $\gamma \notin H_1(f_1)$ and also $\gamma \notin H_1(f_2)$.*

Remark. *Let us assume that $Exp(\Omega)$ is endowed with the topology of uniform convergence on compact sets. With this topology, the operator $f(\partial_t)$ is not bounded: It is enough to take $\Omega = \mathbb{C}$ and as symbol f the identity map. The following specific example shows less trivially that the linear operator $f(\partial_t)$ is not necessarily continuous:*

Let $\Omega = \mathbb{C} \setminus \mathbb{R}_0^+$ (a Runge Domain) and consider the symbol $f(s) = \dfrac{1}{s}$. Then $f \in Hol(\Omega)$; we consider the sequence $\phi_n(z) = e^{\frac{i}{n}z} - e^{\frac{-i}{n}z}$. We have

$$s(\mathcal{B}(\phi_n)) = \left\{ -\frac{i}{n}, \frac{i}{n} \right\} \subset \Omega \, .$$

We can see that $\phi_n \to 0$ in the topology of uniform convergence on compact sets. On the other hand we have (See Example 3.1)

$$f(\partial_t)(\phi_n)(z) = \frac{n}{i} \left(e^{\frac{i}{n}z} + e^{\frac{-i}{n}z} \right) \, ,$$

and considering the compact ball $\overline{B_k(0)}$ with centre the origin and radius k, we have

$$\sup_{z \in \overline{B_k(0)}} |f(\partial_t)(\phi_n)(z)| \geq |f(\partial_t)(\phi_n)(0)| = 2n$$

which goes to infinity when $n \to \infty$.

The following lemma says that nonlocal equations involving $f(\partial_t)$ can be solved in $Exp(\Omega)$. More precisely, the solution to the equation $f(\partial_t)\phi = g$, $g \in Exp(\Omega)$, is *analytic of exponential type in* Ω:

Lemma 3.11. *The operator $f(\partial_t) : Exp(\Omega) \to Exp(\Omega)$ is surjective.*

Proof. The surjectivity of the operator comes from the solvability of the following equation

$$f(\partial_t)\phi = g, \qquad g \in Exp(\Omega) \, . \tag{3.8}$$

Since the zet of zeros of f, $\mathcal{Z}(f)$ say, is a set of isolated points and $g \in Exp(\Omega)$, there is a curve $\gamma \in H_1(g)$ such that $\mathcal{Z}(f) \cap \gamma = \emptyset$. Also, there is a measure μ_g suported on γ such that $g = \mathcal{P}(\mu_g)$. Set $\phi = \mathcal{P}(\dfrac{\mu_g}{f})$; i.e.

$$\phi(z) = \frac{1}{2\pi i} \int_\gamma \frac{e^{z\eta} \mathcal{B}(g)(\eta)}{f(\eta)} d\eta \, .$$

It is evident that $\phi \in Exp(\mathbb{C})$, now we want to see that $s(\mathcal{B}(\phi)) \subset \Omega$. For that, let us calculate the Borel Transform of ϕ as the analytic continuation of its real Laplace transform. Let $z \in \mathbb{R}$ be sufficiently large so that $|Re(\eta)| < z$

for all $\eta \in \gamma$; we have

$$\begin{aligned}\int_0^{+\infty} e^{-zt}\phi(t)dt &= \int_0^{+\infty} e^{-zt}\frac{1}{2\pi i}\int_\gamma \frac{e^{z\eta}\mathcal{B}(g)(\eta)}{f(\eta)}d\eta dt \\ &= \frac{1}{2\pi i}\int_\gamma \int_0^{+\infty} \frac{e^{t(\eta-z)}\mathcal{B}(g)(\eta)}{f(\eta)}d\eta \\ &= \frac{1}{2\pi i}\int_\gamma \frac{\mathcal{B}(g)(\eta)}{f(\eta)(z-\eta)}d\eta \ ,\end{aligned}$$

in which we have used Fubini's Theorem. Therefore,

$$\mathcal{B}(\phi)(z) = \frac{1}{2\pi i}\int_\gamma \frac{\mathcal{B}(g)(\eta)}{f(\eta)(z-\eta)}d\eta \ ,$$

and using Morera's theorem, we can see that $\mathcal{B}(\phi)$ is analytic in Ω^c; thus $s(\mathcal{B}(\phi)) \subset \Omega$. On the other hand, it is not difficult to see that it satisfies $f(\partial_t)\phi = g$. □

3.3. A representation formula for solutions to $f(\partial_t)\phi = g$

Proposition 3.12. *Let $f \in Hol(\Omega)$ and denote $\mathcal{Z}(f)$ for the set of its zeros. A function $\phi \in Exp(\Omega)$ of exponential type τ_ϕ is solution to the homogeneous equation $f(\partial_t)\phi = 0$ if and only if there exist polynomials p_k of degree less than the multiplicity of the root $s_k \in \mathcal{Z}(f) \cap B_{\tau_\phi}(0)$, such that*

$$\phi(t) = \sum_{\substack{s_k \in \mathcal{Z}(f) \\ |s_k| < \tau_\phi}} p_k(t)e^{ts_k} \ .$$

Proof. (*Sufficiency*) in order to prove that the function

$$\phi(t) = \sum_{\substack{s_k \in \mathcal{Z}(f) \\ |s_k| < \tau_\phi}} p_k(t)e^{ts_k} \ ,$$

is solution to the homogeneous equation $f(\partial_t)\phi = 0$, it is enough to see that for a given k and $s_k \in \mathcal{Z}(f)$ with $|s_k| < \tau_\phi$, the following holds:

$$f(\partial_t)(p_k(t)e^{ts_k}) = 0.$$

Indeed, we first note that for a natural number d and a complex number a_d, the Borel transform of $a_d s^d e^{\lambda s}$ is

$$\mathcal{B}(a_d s^d e^{\lambda s}) = a_d \frac{d!}{(s-\lambda)^{d+1}} \ . \tag{3.9}$$

Now, let s_k be a zero of f of order $d_k + 1$, p_k a polynomial of degree $deg(p_k) \leq d_k$ and suppose that $\gamma_k \in H_1(p_k(z)e^{zs_k})$; then, using linearity of the Borel transform and the Cauchy theorem we have

$$f(\partial_t)\left(p_k(t)e^{ts_k}\right) = \frac{1}{2\pi i}\int_{\gamma_k} e^{t\eta} f(\eta)\mathcal{B}(p_k(z)e^{zs_k})(\eta)d\eta = 0 \ .$$

From these computations, we deduce that

$$f(\partial_t)\left(\sum_{s_k\in\mathcal{Z}(f):|s_k|<\tau_\phi} p_k(t)e^{ts_k}\right) = 0.$$

On the other hand, it is evident that

$$\phi(t) = \sum_{\substack{s_k\in\mathcal{Z}(f)\\|s_k|<\tau_\phi}} p_k(t)e^{ts_k} \ ,$$

has exponential type τ_ϕ and from (3.9) we conclude that $\phi \in Exp(\Omega)$.

Before proving *necessity*, we must ensure finite dimensionality of a vector space to be defined below. We write this fact as a separate result, because it is interesting in its own right; after that we will finish the proof of this proposition. □

We use the following notations. Let $\mathcal{R}$ be the closure of a bounded and simply connected region which does not contain any singularity of f and let γ denotes its boundary. We also denote by $A(\mathcal{R})$ the set of continuous functions that are analytic in the interior of $\mathcal{R}$ endowed with the supremum norm and, for $z \in \mathbb{C}$, we let $E_z : \mathcal{R} \to \mathbb{C}$ be the complex exponential function $E_z(\xi) = e^{z\xi}$. Finally, we let

$$\mathcal{M}_{f,\gamma} := cl(span\{E_z \cdot f : z \in \mathbb{C}\}) \ ,$$

where cl denotes closure in $A(\mathcal{R})$.

Lemma 3.13. *Let* $\{s_k\}_{k=1}^K$ *be an enumeration of all the zeros of* f *in* $\mathcal{R}$ *and let* m_k *denote their corresponding multiplicities. Then*

$$\mathcal{M}_{f,\gamma} = \{\psi \in A(\mathcal{R}) : \psi \text{ is zero at } s_k \text{ with multiplicity } \geq m_k,\ 1 \leq k \leq K\} . \tag{3.10}$$

Proof. It is not difficult to see that $\mathcal{M}_{f,\gamma}$ is a subset of the set appearing in the right hand side of (3.10). Let us prove the other inclusion. If ψ belongs to the right hand side of (3.10), then $\frac{\psi}{f} \in A(\mathcal{R})$. Since $\mathcal{R}$ is compact with connected complement, by Mergelyan's Theorem (see [43, Theorem 20.5] and [38, 39]) we know that the set of polynomials Pol is dense in $A(\mathcal{R})$. Therefore, given $\epsilon > 0$ there is a polynomial $p \in Pol$ such that $||\frac{\psi}{f} - p||_{A(\mathcal{R})} < \epsilon$. It follows that $\psi \in cl(f \cdot Pol)$. Now we note that

$$Pol \subset cl(span\{E_z : z \in \mathbb{C}\}).$$

Indeed, it is sufficient to note that the right hand side is an algebra which contains 1 and ξ for any $\xi \in \mathcal{R}$, and certainly, we have $\xi = \lim_{n\to\infty} \frac{e^{\xi 1/n} - 1}{1/n}$. Therefore

$$\psi \in cl(f \cdot Pol) \subset cl(span\{f \cdot E_z : z \in \mathbb{C}\}) .$$

□

A special case of this lemma is in [15, Lemma 5.4].

Lemma 3.14. *Under the conditions of previous lemma, the space* $A(\mathcal{R})/M_{f,\gamma}$ *has dimension* $M = m_1 + m_2 + \cdots + m_K$.

Proof. We can note that $M_{f,\gamma} = (\prod_{k=1}^K (z - s_k)^{m_k})$ is a closed ideal of $A(\mathcal{R})$. First of all, given a complex number $\omega \in \mathcal{R}$ and an integer number $m > 0$ we claim that the quotient space $A(\mathcal{R})/((z - \omega)^m)$ has dimension m and that a basis is given by the set $\beta_1 := \{\overline{1}, \overline{z - \omega}, \overline{(z - \omega)^2}, \cdots , \overline{(z - \omega)^{m-1}}\}$ (here an overline indicates equivalence class). In fact, let $\alpha_0, \alpha_1, \cdots , \alpha_{m-1}$ be complex numbers, then

$$\sum_{l=0}^{m-1} \alpha_l (z - \omega)^l = 0 \qquad \text{belongs to} \qquad A(\mathcal{R})/((z - \omega)^m)$$

if and only if $\sum_{l=0}^{m-1} \alpha_l (z-\omega)^l \in ((z-\omega)^m)$. Thus, there exists a function $\psi \in ((z-\omega)^m)$ such that $\sum_{l=0}^{m-1} \alpha_l (z-\omega)^l = \psi(z)$, and therefore

$$\alpha_l = \frac{1}{l!}\frac{d^l}{dz^l}\psi(z)|_{z=\omega} = 0 \ , \qquad \text{for} \qquad 0 \leq l \leq m-1 \ .$$

It follows that β_1 is a linearly independent set. Now let $\overline{h} \in A(\mathcal{R})/((z-\omega)^m)$ and consider the complex numbers $\alpha_l = \frac{1}{l!}\frac{d^l}{dz^l}h(z)|_{z=\omega}$, $l = 0, 1, \cdots, m-1$; then

$$h(z) = \sum_{l=0}^{m-1} \alpha_l (z-s_1)^l \qquad \text{belongs to} \qquad A(\mathcal{R})/((z-\omega)^m) \ .$$

As a second step, we show that for any $k \in \{1, 2, \cdots, K\}$ the following equality holds

$$A(\mathcal{R})/M_{f,\gamma} = A(\mathcal{R})/((z-s_k)^{\sum_{j=1}^{K} m_j}) \ .$$

In fact, fix any $k \in \{1, 2, \cdots, K\}$ and set $h \in A(\mathcal{R})/M_{f,\gamma}$, then

$$h(z) = r(z) + \prod_{i=1}^{K} (z-s_i)^{m_i} \psi_0(z) \ .$$

Also, we have

$$\begin{aligned}(z-s_i)^{m_i} &= ((z-s_k) + (s_k - s_i))^{m_i} \\ &= \sum_{n=0}^{m_i} a_n (z-s_k)^{m_i - n}(s_k - s_i)^n = (z-s_k)^{m_i} + p_i(z) \ ,\end{aligned}$$

where the polynomial p_i has degree $m_i - 1$; therefore we obtain

$$h(z) = r(z) + \prod_{i=1}^{K} (z-s_i)^{m_i}\psi_0(z) = r(z) + r_1(z) + (z-s_k)^{\sum_{j=1}^{K} m_j}\psi_0(z),$$

which implies that $h \in A(\mathcal{R})/((z-s_k)^{\sum_{j=1}^{K} m_j})$.

Conversely, let $h \in A(\mathcal{R})/((z-s_k)^{\sum_{j=1}^K m_j})$. Then

$$h(z) = r(z) + (z-s_k)^{\sum_{j=1}^K m_j}\psi_0(z) = r(z) + \prod_{i=1}^{K}(z-s_k)^{m_i}\psi_0(z)\ .$$

Now, as in the previous step, we have

$$\begin{aligned}(z-s_k)^{m_i} &= ((z-s_i)+(s_i-s_k))^{m_i} \\ &= \sum_{n=0}^{m_i} a_n (z-s_i)^{m_i-n}(s_i-s_k)^n = (z-s_i)^{m_i} + p_i(z)\ ,\end{aligned}$$

where the polynomial p_i has degree $m_i - 1$. It follows that

$$h(z) = r(z) + r_1(z) + \prod_{i=1}^{K}(z-s_i)^{m_i}\psi_0(z)\ ,$$

which implies that $h \in A(\mathcal{R})/M_{f,\gamma}$. Therefore, using the first step, we conclude that $m_1 + m_2 + \cdots + m_K$ is the dimension of the quotient space $A(\mathcal{R})/M_{f,\gamma}$, as claimed. □

As an immediate consequence of this lemma we have:

Corollary 3.15. *Let $\Omega \subset \mathbb{C}$ be an unbounded domain and assume that $f \in Hol(\Omega)$ has an infinite number of zeros $\{s_k\}$ with multiplicity m_k for $k \in \{1,2,3,\cdots\}$. Then the space $A(\Omega)/M_f$ has infinite dimension, where*

$$\begin{aligned}\mathcal{M}_f = \{\psi \in Hol(\Omega) : \psi\ &\textit{is zero at}\ s_k \\ &\textit{with multiplicity}\ \geq m_k,\ 1 \leq k < \infty\}\ .\end{aligned}$$

Now we proceed to finish the proof of Proposition 3.12.

Proof. Let $\phi \in Exp(\Omega)$ be given. From [8, Theorem 5.3.12] we have that its type is $\tau_\phi = \max_{\omega \in s(\mathcal{B}(\phi))}|\omega|$. Since $s(\mathcal{B}(\phi)) \subset \Omega$ and is a discrete set, we can find a curve γ in Ω whose enclosed region $\mathcal{R}$ contains the set $s(\mathcal{B}(\phi))$ (i.e $\gamma \in H_1(\phi)$) and such that it also contains all zeros $s_i \in \Omega$ of the symbol f with $|s_i| < \tau_\phi$. Let $\{s_i\}_{i=1}^k$ be an enumeration of the zeros of f in $\mathcal{R} \cap B_{\tau_\phi}(0)$ and let m_i denote their corresponding multiplicities. We note also that (using Proposition 3.7) we know that there exist a measure μ supported on $\gamma \in H_1(\phi)$ such that $\phi = \mathcal{P}(\mu)$.

Now, we note that an element $h \in A(\mathcal{R})/M_{f,\gamma}$ is completely determined by the following set

$$\left\{ \frac{d^j}{dz^j} h(z)|_{z=s_i} : 0 \leq j \leq m_i - 1;\ 1 \leq i \leq k \right\} . \tag{3.11}$$

From Lemma 3.14, we have that $A(\mathcal{R})/M_{f,\gamma}$ has dimension $m_1 + m_2 + \cdots + m_k$; therefore its dual space has the same dimension. Moreover, it is not difficult to see that the following collection of linear functionals

$$\left\{ d_{i,j} = \frac{d^j}{dz^j}|_{z=s_i} : 0 \leq j \leq m_k - 1;\ 1 \leq i \leq k \right\} ,$$

are $m_1 + m_2 + \cdots m_k$-elements in the space $(A(\mathcal{R}))^*$ which annihilate $M_{f,\gamma}$; therefore they induces the following $m_1 + m_2 + \cdots m_k$-linearly independent elements in the dual space of $A(\mathcal{R})/M_{f,\gamma}$

$$\{\widetilde{d_{i,j}} : 0 \leq j \leq m_k - 1;\ 1 \leq i \leq k\} ;$$

where $\widetilde{d_{i,j}}(\overline{\phi}) = d_{i,j}(\phi)$ for $\overline{\phi} \in A(\mathcal{R})/M_{f,\gamma}$. Consequently, every element $\varrho \in (A(\mathcal{R})/M_{f,\gamma})^*$ can be written in the form

$$\varrho = \sum_{i=1}^{k} \sum_{j=0}^{m_i - 1} a_{i,j} \widetilde{d_{i,j}}$$

for some $a_{i,j} \in \mathbb{C}$. Now, given and element $\phi \in A(\mathcal{R})$, there exist a unique $r \in A(\mathcal{R})$ such that $\overline{\phi} = \overline{r}$ in $A(\mathcal{R})/M_{f,\gamma}$, and using the characterization given in (3.11) we have

$$\varrho(\overline{\phi}) = \varrho(\overline{r}) = \sum_{i=1}^{k} \sum_{j=0}^{m_i - 1} a_{i,j} d_{i,j}(r) = \sum_{i=1}^{k} \sum_{j=0}^{m_i - 1} a_{i,j} \frac{d^j}{dz^j}(\phi)|_{z=s_i} .$$

On the other hand, from the equation $\mathcal{P}(f \cdot \mu) = 0$ we have that the measure μ defines a functional on $A(\mathcal{R})$ which annihilates $M_{f,\gamma}$ and it induces a functional $\widetilde{\mu}$ on $A(\mathcal{R})/M_{f,\gamma}$. Therefore, there exist complex numbers $b_{i,j}$ such that

$$\widetilde{\mu} = \sum_{i=1}^{k} \sum_{j=0}^{m_i - 1} b_{i,j} \widetilde{d_{i,j}} .$$

Then, we have

$$\begin{aligned}\phi(t) = \mathcal{P}(\mu)(t) &= \int_\gamma e^{tz} d\mu(z) = \widetilde{\mu}(\overline{e^{tz}}) = \sum_{i=1}^{k} \sum_{j=0}^{m_i-1} a_{i,j} \frac{d^j}{dz^j}(e^{tz})|_{z=s_i} \\ &= \sum_{i=1}^{k} \left(\sum_{j=0}^{m_i-1} a_{i,j} t^j \right) e^{ts_i} \\ &= \sum_{i=1}^{k} p_i(t) e^{ts_i} \ .\end{aligned}$$

The proof of Proposition 3.12 is finished. □

Corollary 3.16. *Let $R > 0$ and $f \in Hol(B_R(0))$. Then, a function $\phi \in Exp(B_R(0))$ of exponential type τ_ϕ is a solution of the homogeneous equation $f(\partial_t)\phi = 0$, if and only if there exist polynomials p_k of degree less than the multiplicity of the root $s_k \in \mathcal{Z}(f) \cap B_{\tau_\phi}(0)$, such that*

$$\phi(t) = \sum_{\substack{s_k \in \mathcal{Z}(f) \\ |s_k| < \tau_\phi}} p_k(t) e^{ts_k} \ .$$

In particular, if the symbol f is an entire function, we deduce Theorem 5.1 in [15] from Corollary 3.16. The following theorem is an easy application of the previous results; it generalizes Proposition 3.12.

Theorem 3.17. *Let $f \in Hol(\Omega)$ and $g \in Exp(\Omega)$. Then a function $\phi \in Exp(\Omega)$ of exponential type τ_ϕ is solution for the non-homogeneous equation $f(\partial_t)\phi = g$ if and only if there exist polynomials p_k of degree less than the multiplicity of the root $s_k \in \mathcal{Z}(f) \cap B_{\tau_\phi}(0)$, such that*

$$\phi(t) = \mathcal{P}\left(\frac{\mu_g}{f}\right)(t) + \sum_{\substack{s_k \in \mathcal{Z}(f) \\ |s_k| < \tau_\phi}} p_k(t) e^{ts_k} \ .$$

4. Linear zeta-nonlocal field equations

We apply the theory developed in the previous section to the following linear zeta-nonlocal field equation:

$$\zeta(\partial_t^2 + h)\phi = g \ , \tag{4.1}$$

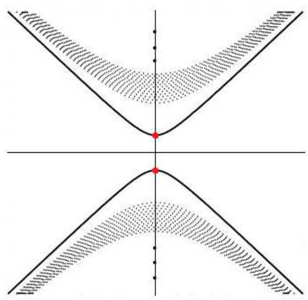
in which h is a real parameter. Its solution depends crucially on the properties of g: we show that if g is of exponential type then so is ϕ, and solving (4.1) explicitly is rather straightforward. However, if the data g is not of exponential type, analysis become very delicate. We consider this problem in 4.2, in which we assume that the Laplace transform $\mathcal{L}(g)$ exists and it has an analytic extension to an appropriated angular contour in the plane.

Hereafter we use freely notation introduced in Section 2.

4.1. Zeta-nonlocal field equation with source function in $Exp(\Omega)$

Equation (4.1) can be solved completely in the space of entire functions of exponential type. Since Equation (4.1) depends on the values of h, we study it in three different cases:

4.1.1. Case $h > 1$. In this case the symbol $\zeta_h \circ p(s) = \zeta(s^2 + h)$ has poles $i\sqrt{h-1}$ and $-i\sqrt{h-1}$. As we have already pointed out, the behavior of $\zeta_h \circ p(s)$ can be represented in the following picture:



The poles of $\zeta(s^2 + h)$ are the vertices of dark hyperbola, indicated by two thick dots. The trivial zeros of $\zeta(s^2 + h)$ are indicated by thin dots on the imaginary axis; and the non-trivial zeros are located on the darker painted region (critical region).

Now, let us consider the simply connected domain

$$\Omega := \mathbb{C} \setminus \left\{ s \in \mathbb{C} : Re(s) \geq 0, \ \ |Im(s)| = \sqrt{h-1} \right\} .$$

We see that the symbol $\zeta_h(s)$ is holomorphic in Ω, and therefore for a source function $g \in Exp(\Omega)$, equation (4.1) is the following integral equation for the measure μ_ϕ:

$$\mathcal{P}((\zeta_h \circ p) \cdot \mu_\phi)(t) = g(t) . \tag{4.2}$$

Theorem 4.1. *Let $g \in Exp(\Omega)$. Then a function $\phi \in Exp(\Omega)$ of exponential type τ_ϕ is solution for the integral equation* (4.2) *if and only if there exist polynomials p_k of degree less than the multiplicity of the root $s_k \in \mathcal{Z}(\zeta_h \circ p) \cap B_{\tau_\phi}(0)$, such that*

$$\phi(t) = \int_\gamma \frac{e^{ts}}{\zeta(s^2+h)} d\mu_g(s) + \sum_{\substack{s_k \in \mathcal{Z}(\zeta_h \circ p) \\ |s_k| < \tau_\phi}} p_k(t) e^{ts_k}.$$

Where $\gamma \in H_1(g)$ and enclose the root $s_k \in \mathcal{Z}(\zeta_h \circ p) \cap B_{\tau_\phi}(0)$.

Remark. *In this theorem (and also in the results that follow) we find that the solution $\varphi(t)$ depends on polynomials $p_k(t)$. These polynomials are calculated using the zeroes (and their orders) of the function $\zeta_h \circ p$, see Proposition 3.6 and Theorem 3.11. We comment further on this in Subsection 4.2.*

On the other hand, we can note that for given $R < \sqrt{h-1}$, the domain Ω contains the ball $B_R(0)$, and since the symbol $\zeta_h \circ p(s)$ is analytic in this ball, it can be expressed there in its Taylor series representation, say

$$\zeta(s^2+h) = \sum_{k=0}^{\infty} a_k(h) s^k \ , |s| < R \ .$$

Therefore, using proposition 3.9, in the space $Exp(B_R(0))$ we have that equation (4.1) can be viewed as the following infinite order differential equation

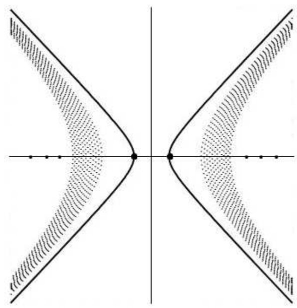
$$\sum_{k=0}^{\infty} a_k(h) \frac{d^k}{dt^k} \phi(t) = g(t) \ . \tag{4.3}$$

In this situation, we have the following result:

Theorem 4.2. *Let $R < \sqrt{h-1}$ and $g \in Exp(B_R(0))$. Then, a function $\phi \in Exp(B_R(0))$ of exponential type τ_ϕ is solution of the infinite order zeta-nonlocal field equation* (4.3) *if and only if there exist polynomials p_k of degree less than the multiplicity of the root $s_k \in \mathcal{Z}(\zeta_h \circ p) \cap B_R(0)$, such that*

$$\phi(t) = \int_{|s|=R} \frac{e^{ts}}{\zeta(s^2+h)} d\mu_g(s) + \sum_{\substack{s_k \in \mathcal{Z}(\zeta_h \circ p) \\ |s_k| < \tau_\phi}} p_k(t) e^{ts_k} \ .$$

4.1.2. Case $h < 1$. In this case we have that $\sqrt{1-h}$ and $-\sqrt{1-h}$ are the poles of the symbol $\zeta_h \circ p$. The behavior of $\zeta_h \circ p$ is represented in the following picture



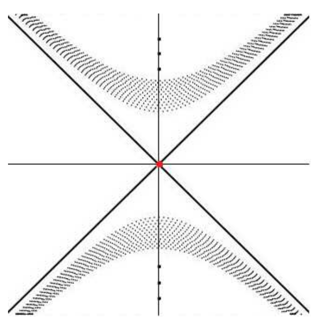
The poles of $\zeta(s^2 + h)$ are the vertices of dark hyperbola, indicated by two thick dots. The trivial zeros of $\zeta(s^2 + h)$ are indicated by thin dots on the real axis; the non-trivial zeros are located on the darker painted region (critical region).

Therefore choosing as our region Ω the following domain:

$$\Omega := \mathbb{C} \setminus \left\{ s \in \mathbb{C} : Im(s) \geq 0, \ \ |Re(s)| = \sqrt{1-h} \right\} ,$$

we can obtain theorems for the equation $\zeta(\partial_t^2 + h)\phi = g$ which are similar to Theorem 4.1 and Theorem 4.2.

4.1.3. Case $h = 1$. In this case there is a pole at $s = 0$. We saw that the behavior of $\zeta_1 \circ p$ is represented in the following picture



The pole of $\zeta(s^2 + 1)$ is the origin (vertex of dark curves $y = |x|, y = -|x|$). The trivial zeros of $\zeta(s^2 + h)$ are indicated by thin dots on the imaginary axis; the non-trivial zeros are located on the darker painted region (critical region).

Let Ω, be the region

$$\Omega := \mathbb{C} \setminus \{s \in \mathbb{C} : Re(s) \geq 0, \ \ |Im(s)| = 0\} \ .$$

Since we cannot construct a ball around the origen in which $\zeta_1 \circ p$ is analytic, we cannot obtain a result analogous to Theorem 4.2 for the equation

$$\zeta(\partial_t^2)\phi = g \ ,$$

this is, we *do not have* an "infinite order equation" but a genuine nonlocal equation. On the other hand, it is possible to state a result analogous to Theorem 4.1. We omit details.

4.2. Zeta-nonlocal field equation with source function in $\mathcal{L}_{>}(\mathbb{R}_{+})$

Now we consider the case in which the source function $g(t), t \geq 0$ is an analytic function not necessarily of exponential type. We assume that it possesses Laplace transform, and therefore there exists a real number a such that the following integral

$$\mathcal{L}(g)(z) = \int_0^\infty e^{-tz} g(t) dt \ ,$$

converges absolutely and uniformly on the half-plane $\{z \in \mathbb{C} : Re(z) > a\}$, and for which the function $z \to \mathcal{L}(g)(z)$ is analytic. We also assume that $\mathcal{L}(g)$ has an analytic extension to the left of $Re(s) = a$ until a singularity a_0, and that this new region of analyticity has an angular contour κ_∞ as its boundary.

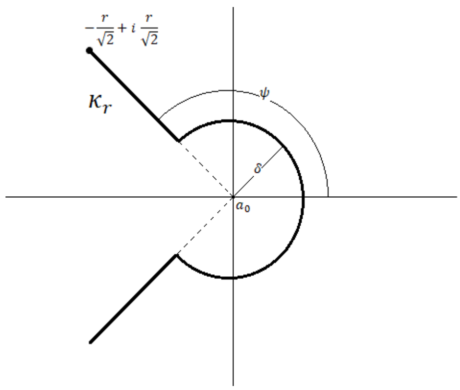
Hereafter we denote by $\mathcal{L}_{>}(\mathbb{R}_{+})$ the space of analytic functions that possess the properties described above.

The problem of interest in this situation is to solve the following equation

$$\zeta(\partial_t^2 + h) f = g, \tag{4.4}$$

for a given $g \in \mathcal{L}_{>}(\mathbb{R}^{+})$, where the operator $\zeta(\partial_t^2 + h)$ needs to be properly defined in order to have a correctly posed problem. The solution of Equation (4.4) if it exists, will not necessarily be an entire function of exponential type.

Let $g \in \mathcal{L}_{>}(\mathbb{R}_{+})$ and let the first singularity of the analytic extension of $\mathcal{L}(g)$ up to an angular contour κ_∞ be $a_0 = 0$. Now consider an angle $\frac{\pi}{2} < \psi \leq \pi$, a positive real number $r > 0$ and let κ_r be a finite angular contour contained in κ_∞. Concretely, κ_r is composed by a circular sector of radius δ centered at the origen and the respective rays of opening $\pm\psi$ as in the following picture:



Now, let us pick the complex measure

$$d\mu_r(s) := \mathcal{X}_{\kappa_r}(s)\mathcal{L}(g)(s)\frac{ds}{2\pi i} ,$$

where $\mathcal{X}_{\kappa_r}$ denotes the characteristic function of the contour κ_r. This measure allows us to define the following function $g_r : \mathbb{C} \to \mathbb{C}$ using $\mathcal{P}$-transform:

$$g_r(z) := \mathcal{P}(\mu_r)(z) = \int_{\kappa_r} e^{zs}\mathcal{L}(g)(s)\frac{ds}{2\pi i} .$$

Lemma 4.3. *We have:*

1) *The function g_r is an entire function of order 1 and exponential type r.*

2) *For each $r > 0$, the analytic continuation of the Borel Transform of g_r is $\mathcal{B}(g_r)(z) = K * \mu_r(z)$, where $K(z) = 1/z$, and its conjugate diagram is the convex hull of the contour κ_r. In particular, if we consider the measure*

$$d\mu_{g_r}(s) = K * \mu_r(s)\frac{ds}{2\pi i} ,$$

then $g_r = \mathcal{P}(\mu_r) = \mathcal{P}(\mu_{g_r})$.

Proof. We prove item *1*. Let $r > 0$ fixed; first, we note that for $n \geq 0$

$$g_r^{(n)}(0) = \int_{\kappa_r} s^n\mathcal{L}(g)(s)\frac{ds}{2\pi i} .$$

Now, defining

$$M_r := \frac{1}{2\pi}\int_{\kappa_r} |\mathcal{L}(g)(s)ds| \ ,$$

we obtain

$$\frac{\ln|g_r^{(n)}(0)|}{n\ln n} \leq \frac{\ln r^n M_r}{n\ln n} = \frac{n\ln r + \ln M_r}{n\ln n} \ ,$$

which approaches zero as $n \to \infty$. Therefore using formula (3.1) we obtain the order of g_r as

$$\rho = \left(1 - \lim_{n\to\infty}\sup \frac{\ln|g_r^{(n)}(0)|}{n\ln(n)}\right)^{-1} = 1$$

With this information, we compute the type of g_r using formula (3.2),

$$\sigma = \lim_{n\to\infty}\sup |g_r^{(n)}(0)|^{1/n}.$$

It is not difficult to see that $\sigma \leq r$; we will conclude that $\sigma = r$ by considering the region of analyticity of the Borel transform of g_r using item *2*.

2. Since κ_r is compact we have

$$g_r(z) = \int_{\kappa_r}\sum_{n=0}^{\infty}\frac{(sz)^n}{n!}\mathcal{L}(g)(s)\frac{ds}{2\pi i} = \sum_{n=0}^{\infty}\frac{z^n}{n!}\int_{\kappa_r} s^n\mathcal{L}(g)(s)\frac{ds}{2\pi i} = \sum_{n=0}^{\infty}\frac{a_n}{n!}z^n \ ,$$

where

$$a_n := \int_{\kappa_r} s^n \mathcal{L}(g)(s)\frac{ds}{2\pi i}$$

and we have used uniform convergence. Now, for $|z| > r$ we have,

$$B(g_r)(z) = \sum_{n=0}^{\infty}\frac{a_n}{z^{n+1}} = \sum_{n=0}^{\infty}\int_{\kappa_r}\frac{1}{z}\left(\frac{s}{z}\right)^n\mathcal{L}(g)(s)\frac{ds}{2\pi i} = \int_{\kappa_r}\frac{1}{z-s}\mathcal{L}(g)(s)\frac{ds}{2\pi i} \ .$$

This calculation means that the analytic continuation of the Borel transform for g_r is

$$\mathcal{B}(g_r)(z) = \int_{\kappa_r}\frac{1}{z-s}\mathcal{L}(g)(s)\frac{ds}{2\pi i} = \int_{\mathbb{C}} K(z-s)d\mu_r(s) = K * \mu_r(z) \ ,$$

which is an analytic function for every $z \in \mathbb{C} - \kappa_r$. As a by product we have that the conjugate diagram of $\mathcal{B}(g_r)$ is the convex hull of the contour κ_r.

Moreover, this means that the type of the function g_r must be $\tau_{g_r} \geq r$, so that by using the calculus in Item *1* we conclude that $\tau_{g_r} = r$. This completes the proof of Item *1*. Finally we note that the description of the Borel transform of g_r implies that g_r is recovered via $\mathcal{P}$-Transform from the measure

$$d\mu_{g_r}(s) = K * \mu_r(s) \frac{ds}{2\pi i} \ .$$

□

4.2.1. The truncated equation. In this subsection we consider the following "truncated" equation

$$\zeta(\partial_t^2 + h) f_r = g_r \ , \qquad h > 1 \ , \tag{4.5}$$

for each $r > 0$. We note that in the case $h > 1$, the poles of the function are $i\sqrt{h-1}$ and $-i\sqrt{h-1}$, and therefore we analyse Equation (4.5), in the domain

$$\Omega := \mathbb{C} \setminus \{s \in \mathbb{C} : Re(s) \geq 0, \ \ |Im(s)| = \sqrt{h-1}\} \ ,$$

which was used in subsection 4.1.

The following theorem shows that Equation (4.5) is well posed in the space $Exp(\Omega)$.

Theorem 4.4. *A general solution to Equation* (4.5) *in the space* $Exp(\Omega)$*, is provided by the function*

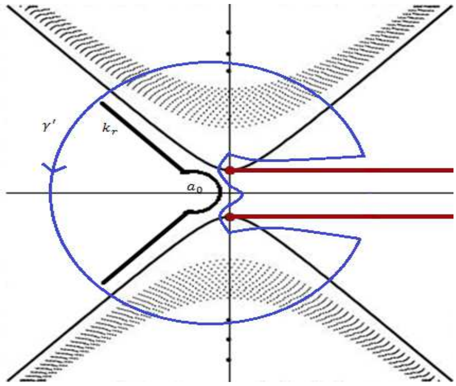
$$\phi_r(z) := \int_{\gamma'} e^{sz} \frac{K * \mu_r(s)}{\zeta(s^2+h)} \frac{ds}{2\pi i} = \int_{\kappa_r} e^{sz} \frac{\mathcal{L}(g)(s)}{\zeta(s^2+h)} \frac{ds}{2\pi i} + \sum_{j=1}^{N_r} p_j(z) e^{\tau_j z} \ , \tag{4.6}$$

where $\gamma' \in H_1(g_r)$ *is such that it encloses the zeros* $\{\tau_j, j = 1, 2, \cdots, N_r\}$ *of the function* $\zeta(s^2+h)$ *which lie in the closed ball* $\overline{B}_r(0)$*, and* $p_j(z)$ *are polynomials of degree* $ord(\tau_j) - 1$.

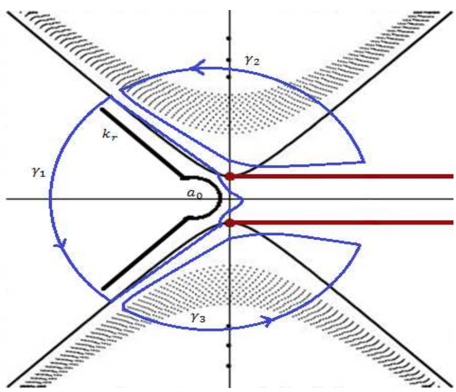
Proof. By Theorem 3.17 we know that a solution for the Equation (4.5) is

$$\int_{\gamma'} e^{sz} \frac{\mathcal{B}(g_r)(s)}{\zeta(s^2+h)} \frac{ds}{2\pi i} = \int_{\gamma'} e^{sz} \frac{K * \mu_r(s)}{\zeta(s^2+h)} \frac{ds}{2\pi i} \ ,$$

where γ' is the curve in the following picture



Since the conjugate diagram S for $\mathcal{B}(g_r)(z)$ is the convex hull of the contour κ_r, we can decompose the circle $\{z : |z|' = r\}$ into three pieces $\gamma_1, \gamma_2, \gamma_3$ in which γ_1 is in the region of analyticity of $\zeta(s^2+h)$ and contains the set S in its interior, while the other two closed paths contain the zeros of $\zeta(s^2+h)$ in its interior, as in the following picture



Therefore,

$$\begin{aligned}
(4.7) \qquad \phi_r(z) &= \int_{|s|'=r} e^{sz} \frac{K * \mu_r(s)}{\zeta(s^2+h)} \frac{ds}{2\pi i} \\
&= \int_{\gamma_1} e^{sz} \frac{K * \mu_r(s)}{\zeta(s^2+h)} \frac{ds}{2\pi i} + \int_{\gamma_2} e^{sz} \frac{K * \mu_r(s)}{\zeta(s^2+h)} \frac{ds}{2\pi i} \\
&\quad + \int_{\gamma_3} e^{sz} \frac{K * \mu_r(s)}{\zeta(s^2+h)} \frac{ds}{2\pi i} \, .
\end{aligned}$$

We compute the first integral. Using Fubini's Theorem and the Cauchy integral formula, we obtain

$$\begin{aligned}\int_{\gamma_1} e^{sz}\frac{K*\mu_r(s)}{\zeta(s^2+h)}\frac{ds}{2\pi i} &= \int_{\gamma_1}\frac{e^{sz}}{\zeta(s^2+h)}\int_{\kappa_r}\frac{1}{s-\omega}\mathcal{L}(g)(\omega)\frac{d\omega}{2\pi i}\frac{ds}{2\pi i}\\ &= \int_{\kappa_r}\mathcal{L}(g)(\omega)\int_{\gamma_1}\frac{e^{sz}}{\zeta(s^2+h)}\frac{1}{s-\omega}\frac{ds}{2\pi i}\frac{d\omega}{2\pi i}\\ &= \int_{\kappa_r}\frac{e^{\omega z}}{\zeta(\omega^2+h)}\mathcal{L}(g)(\omega)\frac{d\omega}{2\pi i}\ .\end{aligned}$$

Now the second integral. Using Fubini's theorem again we have

$$\int_{\gamma_2} e^{sz}\frac{K*\mu_r(s)}{\zeta(s^2+h)}\frac{ds}{2\pi i} = \int_{\kappa_r}\mathcal{L}(g)(\omega)\int_{\gamma_2}\frac{e^{sz}}{\zeta(s^2+h)}\frac{1}{(s-\omega)}\frac{ds}{2\pi i}\frac{d\omega}{2\pi i}\ ,$$

but now we cannot apply Cauchy's integral formula as before, since the zeros of $\zeta(s^2+h)$ are now poles of the function

$$F(s) = \frac{e^{sz}}{\zeta(s^2+h)}\ \frac{1}{(s-\omega)}\ ; \tag{4.8}$$

but, we can use the Residue Theorem. Let τ_j be a zero of the function $\zeta(s^2+h)$ lying inside the region enclosed by the curve γ_2. We have,

$$Res(F,\tau_j) = \sum_{l=0}^{ord(\tau_j)-1} h_l(\omega,\tau_j)z^l e^{\tau_j z}\ ,$$

for some functions h_l. Now we let $N_{2,r}$ be the number of zeros of the function $\zeta(s^2+h)$ inside the region enclosed by the curve γ_2. We conclude that the second integral becomes

$$\int_{\gamma_2} e^{sz}\frac{K*\mu_r(s)}{\zeta(s^2+h)}\frac{ds}{2\pi i} = \int_{\kappa_r}\mathcal{L}(g)(\omega)\sum_{j=1}^{N_{2,r}}\left(\sum_{l=0}^{ord(\tau_j)-1} h_l(\omega,\tau_j)z^l e^{\tau_j z}\right)\frac{d\omega}{2\pi i}$$
$$= \sum_{j=1}^{N_{2,r}}\sum_{l=0}^{ord(\tau_j)-1} z^l e^{\tau_j z}\int_{\kappa_r}\mathcal{L}(g)(\omega)h_l(\omega,\tau_j)\frac{d\omega}{2\pi i}$$
$$= \sum_{j=1}^{N_{2,r}}\sum_{l=0}^{ord(\tau_j)-1} A_l(\tau_j)z^l e^{\tau_j z}$$
$$= \sum_{j=1}^{N_{2,r}} a_j(z)e^{\tau_j z} \ ,$$

where we have defined the polynomials

$$a_j(z) := \sum_{l=0}^{ord(\tau_j)-1} A_l(\tau_j)z^l \ .$$

Finally, let $N_{3,r}$ be the number of zeros of the function $\zeta(s^2+h)$ inside the region enclosed by the curve γ_3. We use the same strategy as above for the third integral in (4.7) and we obtain

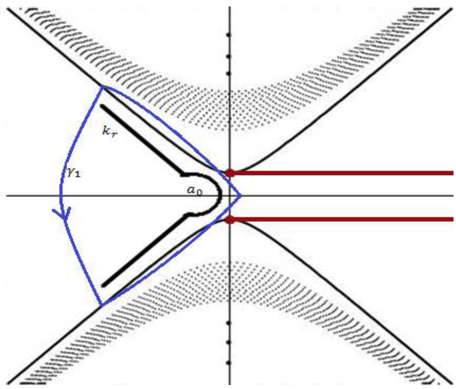
$$\int_{\gamma_3} e^{sz}\frac{K*\mu_r(s)}{\zeta(s^2+h)}\frac{ds}{2\pi i} = \sum_{j=1}^{N_{3,r}} b_j(z)e^{\tau_j z},$$

Putting $N_r = N_{2,r} + N_{3,r}$ as the number of zeros inside of the closed ball $\overline{B}_r(0)$, and setting $p_j = a_j$ for $j = 1, 2, \cdots, N_{2,r}$ and $p_j = b_j$ for $j = 1, 2, \cdots, N_{3,r}$ we obtain equality (4.6) and the theorem is proved. □

In what follows we consider only the particular solution

$$\phi_r(z) = \int_{\kappa_r} e^{sz}\frac{\mathcal{L}(g)(s)}{\zeta(s^2+h)}\frac{ds}{2\pi i} \tag{4.9}$$

to Equation (4.5). This solution is obtained from Theorem 4.4 by using a curve γ_1 as in the following picture



One reason for considering only this expression is that the contribution of the second summand in (4.6) "can be ignored", since it corresponds to a solution of the homogeneous equation $\zeta(\partial_t^2 + h)f_r = 0$. Also, we note that it is still an open problem whether the zeros of the Riemann Zeta function are simple or not (see for example [1, 11, 17]); consequently, we do not even know a precise upper bound for the degree of the polynomials p_j appearing in Theorem (4.4)! Such an information could be used, for example, for the study of the uniform convergence of the sequence of partial sums determined for the second summand in (4.6) for each r.

Remark. *On the other hand, from* [9, 10, 46, 47], *we know that the first zeros of the Riemann zeta function are simple; therefore the first zeros of* $\zeta(s^2 + h)$ *are also simple. Let* $r > 0$ *and suppose that the curve* $\gamma' \in H_1(g_r)$ *encloses the first known simple zeros of* $\zeta(s^2 + h)$*; then, in this situation the full representation formula for the solution given in Theorem 4.4 is more concrete. This situation is treated in the example that follows.*

Example 4.1. From the work [47] (and references therein) we know that at least the first 1.500.000.001 zeros of the Riemann Zeta function are simple and are located at the critical line; therefore the first zeros of $\zeta(s^2 + h)$ are also simple. This implies that the first terms of the sequence of sums in the representation formula (4.6) are easy to calculate.

In fact, let $r > 0$ be such that the curve $\gamma' \in H_1(g_r)$ encloses the first 3.000.000.002 simple zeros of the shifted Riemann Zeta function $\zeta(s^2 + h)$. Let $j \in \{1, 2, 3, \cdots, 3.000.000.002\}$ and let τ_j be the corresponding simple zero. If we define

$$\zeta_j = \lim_{s \to \tau_j} \frac{\zeta(s^2 + h)}{s - \tau_j} \, ,$$

then, applying the residue theorem to the function F defined in Equation (4.8) we obtain

$$Res(F, \tau_j) = \frac{e^{\tau_j z}}{\zeta_j(\tau_j - \omega)} .$$

Therefore, from the proof of Theorem 4.4 we have that the representation formula of the solution is reduced to

$$\phi_r(z) = \int_{\kappa_r} e^{sz} \frac{\mathcal{L}(g)(s)}{\zeta(s^2+h)} \frac{ds}{2\pi i} + \sum_{j=1}^{N_r} c_j e^{\tau_j z} ,$$

where c_j are the following complex numbers:

$$c_j := \frac{1}{\zeta_j} \int_{\kappa_r} \frac{\mathcal{L}(g)(\omega)}{\zeta_j(\tau_j - \omega)} \frac{d\omega}{2\pi i} .$$

4.2.2. A particular solution. The proof of the following lemma can be found in [20, Theorem 36.1]

Lemma 4.5. *Let $g \in \mathcal{L}_>(\mathbb{R}_+)$ and let κ be the angular contour of the domain of the analytic extension of $\mathcal{L}(g)$ with centre $a_0 = 0$ and half-angle of opening ψ, where $\frac{\pi}{2} < \psi \leq \pi$. Then, the function*

$$g_\infty(z) := \int_{\kappa_\infty} e^{zs} \mathcal{L}(g)(s) \frac{ds}{2\pi i},$$

is analytic in an angular region with horizontal bisector and half-angle of opening $\psi - \frac{\pi}{2}$.

Let us denote by D_ψ the angular region with horizontal bisector and half-angle of opening $\psi - \frac{\pi}{2}$ arising in the previous lemma, see [20, figure 32, p. 243]. We note that g_∞ is analytic in D_ψ. We can estimate ψ.

We can see that the real functions $y = |x|$ and $y = -|x|$ are asymptotes to the region which contain the zeros of $\zeta(s^2 + h)$. Therefore, the angle ψ satisfies $\frac{3\pi}{4} < \psi \leq \pi$. This gives us a natural fixed angular region $D_{\frac{3\pi}{4}}$ on which the function g_∞ is analytic, since $D_{\frac{3\pi}{4}} \subset D_\psi$ for all $\frac{3\pi}{4} < \psi \leq \pi$.

Proposition 4.6. *Let $a_0 = 0$ be the first singularity of the analytic extension of $\mathcal{L}(g)$ and also let $\frac{3\pi}{4} < \psi \leq \pi$ be the angle described in lemma 4.5. Then, on compact subsets of $D_\psi \subset \mathbb{C}$ we have:*

1) *The sequence* $\{g_r\}_{r>0}$ *converge uniformly to*

$$g_\infty(z) = \int_{\kappa_\infty} e^{zs}\mathcal{L}(g)(s)\frac{ds}{2\pi i} .$$

2) *The sequence* $\{f_r\}_{r>0}$ *converge uniformly to*

$$f_\infty(z) := \int_{\kappa_\infty} e^{sz}\frac{\mathcal{L}(g)(s)}{\zeta(s^2+h)}\frac{ds}{2\pi i} .$$

In particular both conclusions hold on $D_{\frac{3\pi}{4}}$.

Proof. We prove Item *1*. Let K be a compact subset of D_ψ; since it is closed and bounded, there is a positive number δ such that the distance between D_ψ and ∂K (the topological boundary of K) is at least δ. Also, there exist a positive number A and angles $\theta_1,\ \theta_2$ satisfying $\frac{\pi}{2}-\psi<\theta_1<\theta_2<\psi-\frac{\pi}{2}$, such that for all $z\in K$

a. $|z|\geq A$, and

b. $\theta_1<\theta_z<\theta_2$, where θ_z denotes the angle of z with the real line, $z=|z|\exp(i\theta_z)$.

Therefore, for $z\in K$ we have

$$\begin{aligned}|g_r(z)-g_\infty(z)| &= \left|\int_{\kappa_\infty-\kappa_r} e^{sz}\mathcal{L}(g)(s)\frac{ds}{2\pi i}\right|\\ &\leq \left|\int_r^\infty e^{te^{i\psi}z}\mathcal{L}(g)(te^{i\psi})e^{i\psi}\frac{dt}{2\pi}\right|\\ &\quad+\left|\int_r^\infty e^{te^{-i\psi}z}\mathcal{L}(g)(te^{-i\psi})e^{-i\psi}\frac{dt}{2\pi}\right|.\end{aligned}$$

Let $l_z=|z|$; for the first integral, we have

$$\begin{aligned}\left|\int_r^\infty e^{te^{i\psi}z}\mathcal{L}(g)(te^{i\psi})e^{i\psi}\frac{dt}{2\pi}\right| &= \left|\int_r^\infty e^{tl_ze^{i(\psi+\theta_z)}}\mathcal{L}(g)(te^{i\psi})e^{i\psi}\frac{dt}{2\pi}\right|\\ &\leq \int_r^\infty e^{tl_z\cos(\psi+\theta_z)}\left|\mathcal{L}(g)(te^{i\psi})\right|\frac{dt}{2\pi} .\end{aligned}$$

By the Riemann-Lebesgue Lemma we have that $\mathcal{L}(g)$ is bounded, and therefore $|\mathcal{L}(g)(te^{i\psi})|\leq M_{\mathcal{L}(g)}$ for $t\geq r$. Also, $\frac{\pi}{2}<\theta_1+\psi<\psi+\theta_z<\theta_2+\psi<\frac{3\pi}{2}$, which implies that $\cos(\psi+\theta_z)<-B<0$, for some $B>0$. Therefore, we have

$$\begin{aligned}\int_r^\infty e^{tl_z \cos(\psi+\theta_z)} \left|\mathcal{L}(g)(te^{i\psi})\right| \frac{dt}{2\pi} &\leq M_{\mathcal{L}(g)} \int_r^\infty e^{tl_z \cos(\psi+\theta_z)} \frac{dt}{2\pi} \\ &\leq \int_r^\infty e^{-tl_z B} \frac{dt}{2\pi} = \frac{1}{l_z B} e^{-rl_z B} \\ &\leq \frac{1}{AB} e^{-rAB} \ .\end{aligned}$$

For the second integral, we have $-\frac{3\pi}{2} < \theta_1 - \psi < \theta_z - \psi < \theta_2 - \psi < -\frac{\pi}{2}$, and therefore there is a constant $C > 0$ such that $\cos(\theta_z - \psi) < -C < 0$. Then

$$\begin{aligned}\left|\int_r^\infty e^{te^{-i\psi}z} \mathcal{L}(g)(te^{-i\psi}) e^{-i\psi} \frac{dt}{2\pi}\right| &= \int_r^\infty e^{tl_z \cos(\theta_z-\psi)} \left|\mathcal{L}(g)(te^{-i\psi})\right| \frac{dt}{2\pi} \\ &\leq M_{\mathcal{L}(g)} \int_r^\infty e^{tl_z \cos(\theta_z-\psi)} \frac{dt}{2\pi} \\ &\leq \int_r^\infty e^{-tl_z C} \frac{dt}{2\pi} = \frac{1}{l_z C} e^{-rl_z C} \\ &\leq \frac{1}{AC} e^{-rAC} \ .\end{aligned}$$

These computations allow us to conclude that given $\epsilon > 0$ there is $r_0 > 0$ such that for every $r > r_0$ and for every $z \in K$ we have the estimate:

$$|g_r(z) - g_\infty(z)| \leq \frac{1}{AB} e^{-rAB} + \frac{1}{AC} e^{-rAC} < \epsilon \ .$$

Item 2 follows from the fact that the function $\dfrac{1}{\zeta(s^2+h)}$ is bounded on the angular contour κ_∞ for $|s| \to \infty$. □

Let us now denote the angle ψ described in Lemma 4.5 by $\psi(g)$. From the result in Proposition 4.6 we have the following remark

Remark. *We have*

1) *Proposition* 4.6 *implies that the function* g_∞ *is an anlytic function which extends* g*; that is* $g_\infty(t) = g(t)\ \forall t \in \mathbb{R}_+$.

2) *The sequences* $\{f_r\}_{r>0}$ *and* $\{g_r\}_{r>0}$ *are sequences of entire functions of increasing exponential type* r*. On the other hand, functions* f_∞ *and* g_∞ *from Proposition* 4.6 *are, generally speaking, neither entire nor of finite exponential type.*

3) *In principle the functions* g_∞ *and* f_∞ *depend on* ψ, *with* $\frac{3\pi}{4} < \psi \leq \psi(g)$*: for each angle* ψ *in* $]\frac{3\pi}{4}, \psi(g)]$ *and each* $r > 0$, *we obtain the finite angular contour* κ_r^ψ *(which is part of the infinite angular contour* κ^ψ*), the sequence of functions* $\{f_r^\psi\}_{r>0}$ *and* $\{g_r^\psi\}_{r>0}$, *and the limit functions* g_∞^ψ *and* f_∞^ψ. *We also note that for* $\psi_1 \leq \psi_2$ *in* $]\frac{3\pi}{4}, \psi(g)]$ *the functions* $g_\infty^{\psi_2}$ *and* $f_\infty^{\psi_2}$ *are analytic extensions of* $g_\infty^{\psi_1}$ *and* $f_\infty^{\psi_1}$ *respectively.*

Motivated by this remark and Proposition 4.6, we define the following nonempty set:

$$\mathcal{W}(g) := \left\{ f_\infty^\psi \ : \ \psi \in]\frac{3\pi}{4}, \psi(g)] \right\} .$$

Also, we denote by $\Omega_{\frac{3\pi}{4}}$ the reflexion of $D_{\frac{3\pi}{4}}$ with respect to the imaginary axis. We define the operator $\widetilde{\zeta}(\partial_t^2 + h)$ on $\mathcal{W}(g)$ as follows:

Definition 4.7. *Let* $f_\infty^\psi \in \mathcal{W}(g)$ *and let* $f_r^\psi \in Exp(\Omega_{\frac{3\pi}{4}})$ *be a family which satisfies Equation* (4.5) *and such that* $f_r^\psi \to f_\infty^\psi$ *in the the topology of uniform convergence on compact subsets of* $Dom(f_\infty^\psi)$. *Then,*

$$\widetilde{\zeta}(\partial_t^2 + h) f_\infty^\psi := \lim_{r\to\infty} \zeta(\partial_t^2 + h) f_r^\psi \ , \tag{4.10}$$

where the limit is also taken in the topology of uniform convergence on compact subsets of $Dom(f_\infty^\psi)$.

Because of [20, Theorem 25.1] the limit appearing in the right hand side of Equation (4.10) *does not depend* on the choice of the angle ψ. By the same reason the function f_∞^ψ *does not depend* on ψ. Thus we can use Definition 4.7 to interpret Equation (4.4) in the case in which the data $g \in \mathcal{L}_>(\mathbb{R}_+)$: we look, for a fixed ψ, a solution f_∞^ψ in the set $\mathcal{W}(g)$ to the following equation:

$$\widetilde{\zeta}(\partial_t^2 + h) f_\infty^\psi = g \ , \tag{4.11}$$

and we understand Equation (4.11) in the following limit sense:

$$\lim_{r\to\infty} \zeta(\partial_t^2 + h) f_r^\psi = \lim_{r\to\infty} g_r^\psi = g_\infty^\psi \ ,$$

where the sequences $\{f_r^\psi\}_{r>0}$ and $\{g_r^\psi\}_{r>0}$ are in $Exp(\Omega_{\frac{3\pi}{4}})$ and they are related as in Proposition 4.6. We recall once more that limit is taken under the topology of uniform convergence on compact subsets of $Dom(f_\infty^\psi)$, and that g_∞^ψ do not depend on the angle ψ (again because of [20, Theorem 25.1],).

Proposition 4.8. *Let us consider the particular angle $\psi = \psi(g)$ defined after Proposition* 4.6*. The solution to Equation* (4.11) *is the function $f_\infty^{\psi(g)} \in \mathcal{W}(g)$ given in Proposition* 4.6*.*

Proof. From Proposition 4.6, we recall that

$$f_\infty^{\psi(g)}(z) = \int_{\kappa^{\psi(g)}} e^{sz} \frac{\mathcal{L}(g)(s)}{\zeta(s^2+h)} \frac{ds}{2\pi i} ,$$

and that there exist a function $g_\infty^{\psi(g)}$ given by

$$g_\infty^{\psi(g)}(z) = \int_{\kappa^{\psi(g)}} e^{zs} \mathcal{L}(g)(s) \frac{ds}{2\pi i} .$$

on the domain $Dom(f_\infty^{\psi(g)})$. The analytic function $g_\infty^{\psi(g)}$ extends the function g defined in principle on $\mathbb{R}_+$.

Furthermore, there exist explicit sequences $\{f_r^{\psi(g)}\}_{r>0}$ and $\{g_r^{\psi(g)}\}_{r>0}$ in $Exp(\Omega_{\frac{3\pi}{4}})$ given by:

$$f_r^{\psi(g)}(z) = \int_{\kappa_r^{\psi(g)}} e^{sz} \frac{\mathcal{L}(g)(s)}{\zeta(s^2+h)} \frac{ds}{2\pi i} ,$$

and

$$g_r^{\psi(g)}(z) = \int_{\kappa_r^{\psi(g)}} e^{sz} \mathcal{L}(g)(s) \frac{ds}{2\pi i} .$$

These sequences, for each $r > 0$, satisfy the following truncated equations on $Exp(\Omega_{\frac{3\pi}{4}})$

$$\zeta(\partial_t^2 + h) f_r^{\psi(g)} = g_r^{\psi(g)} . \tag{4.12}$$

Furthermore, in Proposition 4.6 we proved that on compact subsets of $Dom(f_\infty^{\psi(g)})$, the following two uniform limits holds

a).

$$\lim_{r\to\infty} g_r^{\psi(g)}(z) = g_\infty^{\psi(g)}(z) ,$$

b).

$$\lim_{r\to\infty} f_r^{\psi(g)}(z) = f_\infty^{\psi(g)}(z) .$$

Therefore, taking limits in Equation (4.12) and using items a) and b), the following equality hold (on $Dom(f_\infty^{\psi(g)})$)

$$\lim_{r\to\infty} \zeta(\partial_t^2 + h) f_r^{\psi(g)}(z) = \lim_{r\to\infty} g_r^{\psi(g)}(z) = g_\infty^{\psi(g)}(z) \ .$$

That is, on $Dom(f_\infty^{\psi(g)})$ we have

$$\widetilde{\zeta}(\partial_t^2 + h) f_\infty^{\psi(g)} = g^{\psi(g)} \ .$$

In particular

$$\widetilde{\zeta}(\partial_t^2 + h) f_\infty^{\psi(g)}(t) = g(t) \ \ in \ \ \mathbb{R}_+ \ .$$

□

Appendix: Some Zeta-nonlocal scalar fields

4.3. Equations of motion

Following Dragovich's work [21], we show how to deduce several mathematical interesting nonlocal scalar field equations whose dynamics depends on the Riemann zeta function, Hurwitz-zeta function and also on a Dirichlet-Taylor series.

Recall that, given a prime number p, the Lagrangian formulation of the open $p-$adic string tachyon is

$$L_p = \frac{m_p^D}{g_p^2}\frac{p^2}{p-1}\Big(-\frac{1}{2}\phi p^{-\Box/(2m_p^2)}\phi + \frac{1}{p+1}\phi^{p+1}\Big) \ , \tag{4.13}$$

where $\Box$ is the D'Alembert operator defined by $\Box := -\partial_t^2 + \triangle_x$, in which $\triangle_x$ is the Laplace operator and we are using metric signature $(-,+,\cdots,+)$, following [21]. This Lagrangian is defined only formally; as we have shown here, the terms appearing therein are well-defined in the $1+0$ case, see also [15, 16, 30]. The equation of motion for (4.13) is

$$p^{-\Box/(2m_p^2)}\phi = \phi^p.$$

Dragovich has considered the model

$$L = \sum_{n=1}^{\infty} C_n L_n = \sum_{n=1}^{\infty} C_n \frac{m_n^D}{g_n^2}\frac{n^2}{n-1}\Big(-\frac{1}{2}\phi n^{-\Box/(2m_n^2)}\phi + \frac{1}{n+1}\phi^{n+1}\Big) \ ,$$

in which all lagrangians L_n given by (4.13) are taken into account. Explicit lagrangians L depend on the choices of the coefficients C_n. Some particular cases are considered bellow.

4.3.1. The Riemann zeta function as a symbol. This is the case in [21] and one of our main motivations. We recall once again that the Riemann zeta function is defined by (see for instant [32])

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \,, \qquad Re(s) > 1 \,.$$

It is analytic on its domain of definition and it has an analytic extension to the whole complex plane with the exception of the point $s = 1$, at which it has a simple pole with residue 1.

If we consider the explicit coefficient

$$C_n = \frac{n-1}{n^{2+h}} \,,$$

in which h is a real number, Dragovich's Lagrangian becomes

$$L_h = \frac{m^D}{g^2} \left(-\frac{1}{2} \phi \sum_{n=1}^{\infty} n^{-\Box/(2m_n^2)-h} \phi + \sum_{n=1}^{\infty} \frac{n^{-h}}{n+1} \phi^{n+1} \right) \,.$$

We write L_h in terms of the zeta function and, in order to avoid convergence issues, we replace the nonlinear term for an adequate analytic function $G(\phi)$. The Lagrangian L_h becomes:

$$L_h = \frac{m^D}{g^2} \left(-\frac{1}{2} \phi \zeta(\frac{\Box}{2m^2} + h) \phi + G(\phi) \right) \,.$$

The equation of motion is

$$\zeta(\frac{\Box}{2m^2} + h)\phi = g(\phi) \,,$$

in which $g = G'$.

4.3.2. Dirichlet zeta function as symbol. Let us consider χ a Dirichlet character modulo m and let us define

$$C_n = \frac{\chi(n)(n-1)}{n^{2+h}} \,.$$

We recall that a L-Dirichlet series is of the following form:

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$$

following Dragovich's approach, we can consider the Lagrangian

$$L_h=\frac{m^D}{g^2}\left(-\frac{1}{2}\phi L(\frac{\Box}{2m^2}+h,\chi)\phi+F(\varphi)\right)$$

and the corresponding equation of motion

$$L(\frac{\Box}{2m^2}+h,\chi)\phi=f(\phi)\ , \tag{4.14}$$

in which $f=F'$.

4.3.3. Almost periodic Dirichlet series as symbol. Let $\{a_n\}$ be a sequence of complex numbers. A Dirichlet series is a series of the form

$$F(s):=\sum_{n=1}^{\infty}\frac{a_n}{n^s}\ .$$

Then, for a given sequence $\{a_n\}$, if we consider the coefficients

$$C_n=\frac{a_n(n-1)}{n^{2+h}},$$

we arrive at the following Lagrangian and equation of motion:

$$L_h=\frac{m^D}{g^2}\left(-\frac{1}{2}\phi F(\frac{\Box}{2m^2}+h)\phi+D(\phi)\right)\ ,$$

$$F(\frac{\Box}{2m^2}+h)\phi=d(\phi)\ , \tag{4.15}$$

in which $d=D'$.

A particular case of this equation is the equation with dynamics depending on Dirichlet series with *almost periodic coefficients*: following [33], we consider a piecewise continuous, 1-periodic and L^2-function $f:\mathbb{R}\to\mathbb{C}$ with Fourier expansion $f(x)=\sum_{k=-\infty}^{\infty}b_k e^{2\pi ikx}$; the particular symbol of interest for equation (4.15) is the following *almost periodic Dirichlet series*:

$$F_\alpha(s):=\sum_{n=1}^{\infty}\frac{f(n\alpha)}{n^s}\ .$$

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