

The quantum black hole as a gravitational hydrogen atom

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Nathan Rosen’s quantization approach to the gravitational collapse is applied in the simple case of a pressureless “*star of dust*” by finding the gravitational potential, the Schrödinger equation and the solution for the collapse’s energy levels. By applying the constraints for a Schwarzschild black hole (BH) and by using the concept of *BH effective state*, previously introduced by one of the authors (CC), the BH quantum gravitational potential, Schrödinger equation and the BH energy spectrum are found. Remarkably, such an energy spectrum is in agreement (in its absolute value) with the one which was conjectured by Bekenstein in 1974 and consistent with other ones in the literature. This approach also allows us to find an interesting quantum representation of the Schwarzschild BH ground state at the Planck scale. Moreover, two fundamental issues about black hole quantum physics are addressed by this model: the area quantization and the singularity resolution. As regards the former, a result similar to the one obtained by Bekenstein, but with a different coefficient, has been found. About the latter, it is shown that the traditional classical singularity in the core of the Schwarzschild BH is replaced, in a full quantum treatment, by a two-particle system where the two components strongly interact with each other via a quantum gravitational potential. The two-particle system seems to be non-singular from the quantum point of view and is analogous to the hydrogen atom because it consists of a “nucleus” and an “electron”.

1. Introduction

It is a general conviction that, in the search of a quantum gravity theory, a black hole should play a role similar to that of the hydrogen atom in quantum mechanics [1]. It should be a “theoretical laboratory” where one discusses and tries to understand conceptual problems and potential contradictions in the attempt to unify Einstein’s general theory of relativity with quantum mechanics. This analogy suggests that black holes should be

regular quantum systems with a discrete mass spectrum [1]. In this paper, the authors attempt to contribute to the above by finding the Schrödinger equation and the wave function of the Schwarzschild BH. The knowledge of such quantities could, in principle, also play a role in the solution of the famous BH information paradox [2] because here black holes seem to be well defined quantum mechanical systems, having ordered and discrete quantum spectra. This issue appears consistent with the unitarity of the underlying quantum gravity theory and with the idea that information should come out in BH evaporation.

A quantization approach proposed about 30 years ago by the historical collaborator of Einstein, Nathan Rosen [3], will be applied to the gravitational collapse in the simple case of a pressureless “*star of dust*”. Therefore, the gravitational potential, the Schrödinger equation and the solution for the collapse’s energy levels will be found. After that, the constraints for a BH will be applied and this will permit to find the BH quantum gravitational potential, Schrödinger equation and energy spectrum. Such an energy spectrum, in its absolute value, is in agreement with both the one conjectured by J. Bekenstein in 1974 [4] and that found by Maggiore’s description of black hole in terms of quantum membranes [5]. Rosen’s approach also allows us to find an interesting quantum representation of the Schwarzschild BH ground state at the Planck scale.

It is well-known that the canonical quantization of general relativity leads to the Wheeler-DeWitt equation introducing the so-called Superspace, an infinite-dimensional space of all possible 3-metrics; Rosen, instead, preferred to start his work from the classical cosmological equations using a simplified quantization scheme, reducing, at least formally, the cosmological Einstein-Friedman equations of general relativity to a quantum mechanical system; the Friedman equations can be then recast as a Schrödinger equation and the cosmological solutions can be read as eigensolutions of such a “cosmological Schrödinger equation”. In this way Rosen found that, in the case of a Universe filled with pressureless matter, the equation is like that for the s states of a hydrogen-like atom [3]. It is important to recall that quantization of FLRW universe date back at least to DeWitt’s famous 1967 paper [6], where one understands that a large number of particles is required in order to ensure the semiclassical behaviour.

Furthermore, we try to clarify two important issues such as Bekenstein area law and singularity resolution. With regard to the former, a result similar to that obtained by Bekenstein, but with a different coefficient, has been found. About the latter, it is shown that the traditional classical singularity in the core of the Schwarzschild BH is replaced, in a full quantum treatment,

by a two-particle system where the two components strongly interact with each other via a quantum gravitational potential. The two-particle system seems to be non-singular from the quantum point of view and is analogous to the hydrogen atom because it consists of a “core” and an “electron”.

Returning to the Rosen’s quantization approach, it has also been recently applied to a cosmological framework by one of the authors (FF) and collaborators in [7] and to the famous Hartle-Hawking initial state by both of the authors and I. Licata in [8].

For the sake of completeness, one stresses that this quantization approach is only applicable to homogeneous space-times where the Weyl tensor vanishes. As soon as one introduces inhomogeneities (for example in the Lemaître-Tolman-Bondi (LTB) models [9–11]), there exists open sets of initial data where the collapse ends to a BH absolutely similar to Oppenheimer-Snyder-Datt (OSD) collapse. Hence, the exterior space-time remains the same whereas the interior is absolutely different. A further generalization of the proposed approach, which could be the object of future works, should include also a cosmological term or other sources of dark energy. One also underlines that the Oppenheimer and Snyder gravitational collapse is not physical and can be considered a toy model. But here the key point is that, as it is well known from the historical paper of Oppenheimer and Snyder [13], the final state of this simplified gravitational collapse is the SBH, which, instead, has a fundamental role in quantum gravity. It is indeed a general conviction, arising from an idea of Bekenstein [1], that, in the search of a quantum gravity theory, the SBH should play a role similar to the hydrogen atom in quantum mechanics. Thus, despite non-physical, the Oppenheimer and Snyder gravitational collapse must be here considered as a tool which allows to understand a fundamental physical system, that is the SBH. In fact, it will be shown that, by setting the constraints for the formation of the SBH in the quantized Oppenheimer and Snyder gravitational collapse, one arrives to quantize the SBH, and this will be a remarkable, important result in the quantum gravity’s search.

2. Application of Rosen’s quantization approach to the gravitational collapse

Classically, the gravitational collapse in the simple case of a pressureless “*star of dust*” with uniform density is well known [12]. Historically, it was originally analysed in the famous paper of Oppenheimer and Snyder [13], while a different approach has been developed by Beckerdoff and Misner

[14]. Furthermore, a non-linear electrodynamics Lagrangian has been recently added in this collapse's framework by one of the authors (CC) and Herman J. Mosquera Cuesta in [15]. This different approach allows to find a way to remove the black hole singularity at the classical level. The traditional, classical framework of this kind of gravitational collapse is well known [12–14]. In the following we will follow [12]. In regard to the interior of the collapsing star, it is described by the well-known Friedmann-Lemaître-Robertson-Walker (FLRW) line-element with comoving hyper-spherical coordinates χ , θ , φ . Therefore, one writes down (hereafter Planck units will be used, i.e., $G = c = k_B = \hbar = \frac{1}{4\pi\epsilon_0} = 1$)

$$(2.1) \quad ds^2 = d\tau^2 + a^2(\tau)[(-d\chi^2 - \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2))],$$

where the origin of coordinates is set at the centre of the star, and the following relations hold:

$$(2.2) \quad \begin{aligned} a &= \frac{1}{2}a_m(1 + \cos\eta), \\ \tau &= \frac{1}{2}a_m(\eta + \sin\eta). \end{aligned}$$

The density is given by

$$(2.3) \quad \rho = \left(\frac{3a_m}{8\pi}\right) a^{-3} = \left(\frac{3}{8\pi a_m^2}\right) \left[\frac{1}{2}(1 + \cos\eta)\right]^{-3}.$$

Setting $\sin^2\chi$ one chooses the case of positive curvature, which corresponds to a gas sphere whose dynamics begins at rest with a finite radius and, in turn, it is the only one of interest. Thus, the choice $k = 1$ is made for dynamical reasons (the initial rate of change of density is null, that means “momentum of maximum expansion”), but the dynamics also depends on the field equations. As it has been stressed in the Introduction, for isotropic models, a cosmological term - or other sources of dark energy - can be in principle included in future works, in order to obtain a more realistic physical framework for the collapse.

In order to discuss the simplest model of a “*star of dust*”, that is the case of zero pressure, one sets the stress-energy tensor as

$$(2.4) \quad T = \rho u \otimes u,$$

where ρ is the density of the collapsing star and u the four-vector velocity of the matter. On the other hand, the external geometry is given by the

Schwarzschild line-element

$$(2.5) \quad ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - r^2 (\sin^2 \theta d\varphi^2 + d\theta^2) - \frac{dr^2}{1 - \frac{2M}{r}},$$

where M is the total mass of the collapsing star. Since there are no pressure gradients, which can deflect the particles motion, the particles on the surface of any ball of dust move along radial geodesics in the exterior Schwarzschild space-time. Considering a ball which begins at rest with finite radius (in terms of the Schwarzschild radial coordinate) $r = r_i$ at the (Schwarzschild) time $t = 0$, the geodesics motion of its surface is given by the following equations:

$$(2.6) \quad r = \frac{1}{2} r_i (1 + \cos \eta),$$

$$(2.7) \quad t = 2M \ln \left[\frac{\sqrt{\frac{r_i}{2M} - 1} + \tan\left(\frac{\eta}{2}\right)}{\sqrt{\frac{r_i}{2M} - 1} - \tan\left(\frac{\eta}{2}\right)} \right] + 2M \sqrt{\frac{r_i}{2M} - 1} \left[\eta + \left(\frac{r_i}{4M}\right) (\eta + \sin \eta) \right].$$

The proper time measured by a clock put on the surface of the collapsing star can be written as

$$(2.8) \quad \tau = \sqrt{\frac{r_i^3}{8M}} (\eta + \sin \eta).$$

The collapse begins at $r = r_i$, $\eta = \tau = t = 0$, and ends at the singularity $r = 0$, $\eta = \pi$ after a duration of proper time measured by the falling particles

$$(2.9) \quad \Delta\tau = \pi \sqrt{\frac{r_i^3}{8M}},$$

which coincidentally corresponds, as it is well known, to the interval of Newtonian time for free-fall collapse in Newtonian theory. Different from the cosmological case, where the solution is homogeneous and isotropic everywhere, here the internal homogeneity and isotropy of the FLRW line-element are broken at the star's surface, that is, at some radius $\chi = \chi_0$. At that surface, which is a 3-dimensional world tube enclosing the star's fluid, the interior FLRW geometry must smoothly match the exterior Schwarzschild geometry. One considers a range of χ given by $0 \leq \chi \leq \chi_0$, with $\chi_0 < \frac{\pi}{2}$ during the collapse. For the pressureless case the match is possible. The external

Schwarzschild solution indeed predicts a cycloidal relation for the star's circumference

$$(2.10) \quad \begin{aligned} C &= 2\pi r = 2\pi \left[\frac{1}{2} r_i (1 + \cos \eta) \right], \\ \tau &= \sqrt{\frac{r_i^3}{8M}} (\eta + \sin \eta). \end{aligned}$$

The interior FLRW predicts a similar cycloidal relation

$$(2.11) \quad \begin{aligned} C &= 2\pi r = 2\pi a \sin \chi_0 = \pi \sin \chi_0 a_m (1 + \cos \eta), \\ \tau &= \frac{1}{2} a_m (\eta + \sin \eta). \end{aligned}$$

Therefore, the two predictions agree perfectly for all time if and only if

$$(2.12) \quad \begin{aligned} r_i &= a_0 \sin \chi_0, \\ M &= \frac{1}{2} a_0 \sin^3 \chi_0, \end{aligned}$$

where r_i and a_0 represent the values of the Schwarzschild radial coordinate in Eq. (2.5) and of the scale factor in Eq. (2.1) at the beginning of the collapse, respectively. Thus, Eqs. (2.12) represent the requested match, while the Schwarzschild radial coordinate, in the case of the matching between the internal and external geometries, is

$$(2.13) \quad r = a \sin \chi_0.$$

The attentive reader notes that the initial conditions on the matching are the simplest possible that could be relaxed, still having a continuous matching without extra surface terms. In fact, taking the interior solution to be homogeneous requires very fine tuned initial conditions for the collapse and the dynamics of the edge. So, on one hand, further analyses for a better characterization of the initial conditions on the matching between the internal and external geometries could be the object of future works. On the other hand, despite the analysis of this paper is not as general as possible, one stresses that the BH quantization is one of the most important problems of modern theoretical physics which has not yet been solved. Thus, in order to attempt to solve such a fundamental problem, one must start from the simplest case rather than from more complicated ones. This is in complete analogy with the history of general relativity. In fact, the first solution of Einstein field equations was the Schwarzschild solution, but it was not a general, rotating solution which included cosmological term or other sources

of dark energy, as well the corresponding gravitational collapse developed by Oppenheimer and Snyder did not include a class of non-homogeneous models. Thus, as this is a new approach to the BH quantization, here one starts from the simplest conditions rather than from more complicated ones. So, the initial conditions on the matching that are applied here are exactly the ones proposed by Oppenheimer and Snyder in their paper on the gravitational collapse [13]. It is well known that the final result of the gravitational collapse studied by Oppenheimer and Snyder is the Schwarzschild BH [12].

In the following, the quantization approach derived by Rosen in [3] will be applied to the above case; some differences will be found, because here one analyses the case of a collapsing star, while Rosen analysed a closed homogeneous and isotropic universe. Let us start by rewriting the FLRW line-element (2.1) in spherical coordinates and comoving time as [3, 12]

$$(2.14) \quad ds^2 = d\tau^2 - a^2(\tau) \left(\frac{dr^2}{1-r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right).$$

The Einstein field equations

$$(2.15) \quad G_{\mu\nu} = -8\pi T_{\mu\nu}$$

gives the relations (we are assuming zero pressure)

$$(2.16) \quad \begin{aligned} \dot{a}^2 &= \frac{8}{3}\pi a^2 \rho - 1, \\ \ddot{a} &= -\frac{4}{3}\pi a \rho, \end{aligned}$$

with $\dot{a} = \frac{da}{d\tau}$. For consistency, one gets

$$(2.17) \quad \frac{d\rho}{da} = -\frac{3\rho}{a},$$

which, when integrated, gives

$$(2.18) \quad \rho = \frac{C}{a^3}.$$

In the case of a collapse C is determined by the initial conditions which predict a cycloidal relation for the star's circumference, see Eqs. (2.11) and

Section 32.4 in Ref. [12]. One gets

$$(2.19) \quad C = \frac{3a_0}{8\pi}.$$

This value is consistent with the one found by Rosen [3]. Thus, one rewrites Eq. (2.18) as

$$(2.20) \quad \rho = \frac{3a_0}{8\pi a^3}.$$

Multiplying the first of (2.16) by $M/2$ one gets

$$(2.21) \quad \frac{1}{2}M\dot{a}^2 - \frac{4}{3}\pi Ma^2\rho = -\frac{M}{2},$$

which seems like the energy equation for a particle in one-dimensional motion having coordinate a :

$$(2.22) \quad E = T + V,$$

where the kinetic energy is

$$(2.23) \quad T = \frac{1}{2}M\dot{a}^2,$$

and the potential energy is

$$(2.24) \quad V(a) = -\frac{4}{3}\pi Ma^2\rho.$$

Thus, the total energy is

$$(2.25) \quad E = -\frac{M}{2}.$$

From the second of Eqs. (2.16), one gets the equation of motion of this particle:

$$(2.26) \quad M\ddot{a} = -\frac{4}{3}M\pi a\rho.$$

The momentum of the particle is

$$(2.27) \quad P = M\dot{a},$$

with an associated Hamiltonian

$$(2.28) \quad \mathcal{H} = \frac{P^2}{2M} + V.$$

Till now, the problem has been discussed from the classical point of view. In order to discuss it from the quantum point of view, one needs to define a wave-function as

$$(2.29) \quad \Psi \equiv \Psi(a, \tau).$$

Thus, in correspondence of the classical equation (2.28), one gets the traditional Schrödinger equation

$$(2.30) \quad i \frac{\partial \Psi}{\partial \tau} = -\frac{1}{2M} \frac{\partial^2 \Psi}{\partial a^2} + V\Psi.$$

For a stationary state with energy E one obtains

$$(2.31) \quad \Psi = \Psi(a) \exp(-iE\tau),$$

and Eq. (2.29) becomes

$$(2.32) \quad -\frac{1}{2M} \frac{\partial^2 \Psi}{\partial a^2} + V\Psi = E\Psi.$$

Inserting Eq. (2.20) into Eq. (2.24) one obtains

$$(2.33) \quad V(a) = -\frac{Ma_0}{2a}.$$

Setting

$$(2.34) \quad \Psi = aX,$$

Eq. (2.32) becomes

$$(2.35) \quad -\frac{1}{2M} \left(\frac{\partial^2 X}{\partial a^2} + \frac{2}{a} \frac{\partial X}{\partial a} \right) + VX = EX.$$

With V given by Eq. (2.33), (2.35) is analogous to the Schrödinger equation in polar coordinates for the s states ($l = 0$) of a hydrogen-like atom [17]

in which the squared electron charge e^2 is replaced by $\frac{Ma_0}{2}$. Thus, for the bound states ($E < 0$) the energy spectrum is

$$(2.36) \quad E_n = -\frac{a_0^2 M^3}{8n^2},$$

where n is the principal quantum number. At this point, one inserts (2.25) into (2.36), obtaining the mass spectrum as

$$(2.37) \quad M_n = \frac{a_0^2 M_n^3}{4n^2} \Rightarrow M_n = \frac{2n}{a_0}.$$

On the other hand, by using Eq. (2.25) one finds the energy levels of the collapsing star as

$$(2.38) \quad E_n = -\frac{n}{a_0}.$$

In fact, Eq. (2.37) represents the spectrum of the total mass of the collapsing star, while Eq. (2.38) represents the energy spectrum of the collapsing star where the gravitational energy, which is given by Eq. (2.33), is included. The total energy of a quantum system with bound states is indeed negative.

What is the meaning of Eq. (2.38) and of its ground state? One sees that the Hamiltonian (2.28) governs the quantum mechanics of the gravitational collapse. Therefore, the square of the wave function (2.29) must be interpreted as the probability density of a single particle in a finite volume. Thus, the integral over the entire volume must be normalized to unity as

$$(2.39) \quad \int dx^3 |X|^2 = 1.$$

For stable quantum systems, this normalization must remain the same at all times of the collapse's evolution. As the wave function (2.29) obeys the Schrodinger equation (2.30), this is assured if and only if the Hamiltonian operator (2.28) is Hermitian [16]. In other words, the Hamiltonian operator (2.28) must satisfy for arbitrary wave functions X_1 and X_2 the equality [16]

$$(2.40) \quad \int dx^3 [HX_2]^* X_1 = \int dx^3 X_2^* HX_1.$$

One notes that both \vec{p} and a are Hermitian operators. Therefore, the Hamiltonian (2.28) is automatically a Hermitian operator because it is a sum of a

kinetic and a potential energy [16],

$$(2.41) \quad H = T + V.$$

This is always the case for non-relativistic particles in Cartesian-like coordinates and works also for the gravitational collapse under consideration. In this framework, the ground state of Eq. (2.38) represents the minimum energy which can collapse in an astrophysical scenario. We conclude that the gravitational collapse can be interpreted as a “perfect” quantum system.

It is also important to clarify the issue concerning the gravitational energy. It is well known that, in the framework of the general theory of relativity, the gravitational energy cannot be localized [12]. This is a consequence of Einstein’s equivalence principle (EEP) [12], which implies that one can always find in any given locality a reference’s frame (the local Lorentz reference’s frame) in which all local gravitational fields are null. No local gravitational fields means no local gravitational energy-momentum and, in turn, no stress-energy tensor for the gravitational field. In any case, this general situation admits an important exception [12], given by the case of a spherical star [12], which is exactly the case analysed in this paper. In fact, in this case the gravitational energy is localized not by mathematical conventions, but by the circumstance that transfer of energy is detectable by local measures, see Box 23.1 of [12] for details. Therefore, one can surely consider Eq. (2.33) as the gravitational potential energy of the collapsing star.

3. Black hole energy spectrum, ground state and singularity resolution

Thus, let us see what happens when the star is completely collapsed, i.e. when the star is a BH. One sees that, inserting $r_i = 2M = r_g$, where r_g is the gravitational radius (the Schwarzschild radius), in Eqs. (2.12), one obtains $\sin^2 \chi_0 = 1$. Therefore, as the range $\chi > \frac{\pi}{2}$ must be discarded [12], one concludes that it is $\chi_0 = \frac{\pi}{2}$, $r = a$ and $r_i = a_0 = 2M = r_g$ in Eqs. (2.12) and (2.13) for a BH. Then, Eqs. from (2.33) to (2.38) become, respectively,

$$(3.1) \quad V(r) = -\frac{M^2}{r},$$

$$(3.2) \quad \Psi = rX,$$

$$(3.3) \quad -\frac{1}{2M} \left(\frac{\partial^2 X}{\partial r^2} + \frac{2}{r} \frac{\partial X}{\partial r} \right) + VX = EX,$$

$$(3.4) \quad E_n = -\frac{r_g^2 M^3}{8n^2},$$

$$(3.5) \quad M_n = \sqrt{n},$$

$$(3.6) \quad E_n = -\sqrt{\frac{n}{4}}.$$

Eqs. (3.1), (3.3), (3.5) and (3.6) should be the exact gravitation potential energy, Schrödinger equation, mass spectrum and energy spectrum for the Schwarzschild BH interpreted as “gravitational hydrogen atom”, respectively. Actually, a further final correction is needed. To clarify this point, let us compare Eq. (3.1) with the analogous potential energy of an hydrogen atom, which is [17]

$$(3.7) \quad V(r) = -\frac{e^2}{r}.$$

Eqs. (3.1) and (3.7) are formally identical, but there is an important physical difference. In the case of Eq. (3.7) the electron’s charge is constant for all the energy levels of the hydrogen atom. Instead, in the case of Eq. (3.1), based on the emissions of Hawking quanta or on the absorptions of external particles, the BH mass changes during the jumps from one energy level to another. In fact, such a BH mass decreases for emissions and increases for absorptions. Therefore, one must also consider this dynamical behavior. One way to take into account this dynamical behavior is by introducing the *BH effective state* (see [18, 19] for details). Let us start from the emissions of Hawking quanta. If one neglects the above mentioned BH dynamical behavior, the probability of emission of Hawking quanta is the one originally found by Hawking, which represents a strictly thermal spectrum [20]

$$(3.8) \quad \Gamma \sim \exp\left(-\frac{\omega}{T_H}\right),$$

where ω is the energy-frequency of the emitted particle and $T_H \equiv \frac{1}{8\pi M}$ is the Hawking temperature. Taking into account the BH dynamical behavior, i.e.,

the BH contraction allowing a varying BH geometry, one gets the famous correction found by Parikh and Wilczek [21]:

$$(3.9) \quad \Gamma \sim \exp \left[-\frac{\omega}{T_H} \left(1 - \frac{\omega}{2M} \right) \right] \implies \Gamma = \alpha \exp \left[-\frac{\omega}{T_H} \left(1 - \frac{\omega}{2M} \right) \right],$$

where $\alpha \sim 1$ and the additional term $\frac{\omega}{2M}$ is present. By introducing the *effective temperature* [18, 19]

$$(3.10) \quad T_E(\omega) \equiv \frac{2M}{2M - \omega} T_H = \frac{1}{4\pi(2M - \omega)},$$

Eq. (3.9) can be rewritten in a Boltzmann-like form [18, 19], namely

$$(3.11) \quad \Gamma = \alpha \exp[-\beta_E(\omega)\omega] = \alpha \exp \left(-\frac{\omega}{T_E(\omega)} \right),$$

where $\exp[-\beta_E(\omega)\omega]$ is the *effective Boltzmann factor*, with [18, 19]

$$(3.12) \quad \beta_E(\omega) \equiv \frac{1}{T_E(\omega)}.$$

Therefore, the effective temperature replaces the Hawking temperature in the equation of the probability of emission as dynamical quantity. There are indeed various fields of science where one can take into account the deviation from the thermal spectrum of an emitting body by introducing an effective temperature which represents the temperature of a black body that would emit the same total amount of radiation [18, 19]. The effective temperature depends on the energy-frequency of the emitted radiation and the ratio $\frac{T_E(\omega)}{T_H} = \frac{2M}{2M - \omega}$ represents the deviation of the BH radiation spectrum from the strictly thermal feature due to the BH dynamical behavior [18, 19]. Besides, one can introduce other *effective quantities*. In particular, if M is the initial BH mass *before* the emission, and $M - \omega$ is the final BH mass *after* the emission, the *BH effective mass* and the *BH effective horizon* can be introduced as [18, 19]

$$(3.13) \quad M_E \equiv M - \frac{\omega}{2}, \quad r_E \equiv 2M_E.$$

They represent the BH mass and horizon *during* the BH contraction, i.e. *during* the emission of the particle [18, 19], respectively. These are average quantities. The variable r_E is indeed the average of the initial and final horizons while M_E is the average of the initial and final masses [18, 19]. In regard

to the effective temperature, it is the inverse of the average value of the inverses of the initial and final Hawking temperatures; *before* the emission we have $T_H^i = \frac{1}{8\pi M}$, *after* the emission $T_H^f = \frac{1}{8\pi(M-\omega)}$ [18, 19]. To show that the effective mass is indeed the correct quantity which characterizes the BH dynamical behavior, one can rely on Hawking's periodicity argument [22, 23]. One rewrites Eq. (3.12) as [24]

$$(3.14) \quad \beta_E(\omega) \equiv \frac{1}{T_E(\omega)} = \beta_H \left(1 - \frac{\omega}{2M}\right),$$

where $\beta_H \equiv \frac{1}{T_H}$. Following Hawking's arguments [22, 23], the Euclidean form of the metric is given by [24]

$$(3.15) \quad ds_e^2 = x^2 \left[\frac{d\tau}{4M \left(1 - \frac{\omega}{2M}\right)} \right]^2 + \left(\frac{r}{r_E} \right)^2 dx^2 + r^2 (\sin^2 \theta d\varphi^2 + d\theta^2).$$

This equation is regular at $x = 0$ and $r = r_E$. One also treats τ as an angular variable with period $\beta_E(\omega)$ [22–24]. Following [24], one replaces the quantity $\sum_i \beta_i \frac{\hbar^i}{M^{2i}}$ in [22] with $-\frac{\omega}{2M}$. Then, following the analysis presented in [22], one obtains [24]

$$(3.16) \quad ds_E^2 \equiv \left(1 - \frac{2M_E}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{2M_E}{r}} - r^2 (\sin^2 \theta d\varphi^2 + d\theta^2).$$

One can also show that r_E in Eq. (3.15) is the same as in Eq. (3.13).

Despite the above analysis has been realized for emissions of particles, one immediately argues by symmetry that the same analysis works also in the case of absorptions of external particles, which can be considered as emissions having opposite sign. Thus, the effective quantities (3.13) become

$$(3.17) \quad M_E \equiv M + \frac{\omega}{2}, \quad r_E \equiv 2M_E.$$

Now they represent the BH mass and horizon *during* the BH expansion, i.e., *during* the absorption of the particle, respectively. Hence, Eq. (3.16) implies that, in order to take the BH dynamical behavior into due account, one must replace the BH mass M with the BH effective mass M_E in Eqs. (3.1), (3.3), (3.4), (2.25), obtaining

$$(3.18) \quad V(r) = -\frac{M_E^2}{r},$$

$$(3.19) \quad -\frac{1}{2M_E} \left(\frac{\partial^2 X}{\partial r^2} + \frac{2}{r} \frac{\partial X}{\partial r} \right) + VX = EX,$$

$$(3.20) \quad E_n = -\frac{r_E^2 M_E^3}{8n^2},$$

$$(3.21) \quad E = -\frac{M_E}{2}.$$

From the quantum point of view, we want to obtain the energy eigenvalues as being absorptions starting from the BH formation, that is from the BH having null mass. This implies that we must replace $M \rightarrow 0$ and $\omega \rightarrow M$ in Eq. (3.17). Thus, we obtain

$$(3.22) \quad M_E \equiv \frac{M}{2}, \quad r_E \equiv 2M_E = M.$$

Following again [3], one inserts Eqs. (3.21) and (3.22) into Eq. (3.20), obtaining the BH mass spectrum as

$$(3.23) \quad M_n = 2\sqrt{n},$$

and by using Eq. (3.21) one finds the BH energy levels as

$$(3.24) \quad E_n = -\frac{1}{2}\sqrt{n}.$$

Remarkably, in its absolute value, this final result is consistent with the BH energy spectrum which was conjectured by Bekenstein in 1974 [4]. Bekenstein indeed obtained $E_n \sim \sqrt{n}$ by using the Bohr-Sommerfeld quantization condition because he argued that the Schwarzschild BH behaves as an adiabatic invariant. Besides, Maggiore [5] conjectured a quantum description of BH in terms of quantum membranes. He obtained the energy spectrum

$$(3.25) \quad E_n = \sqrt{\frac{A_0 n}{16\pi}}.$$

One sees that, in its absolute value, the result of Eq. (3.24) is consistent also with Maggiore's result. On the other hand, it should be noted that both Bekenstein and Maggiore used heuristic analyses, approximations and/or conjectures. Instead, Eq. (3.23) has been obtained through an exact quantization process. In addition, neither Bekenstein nor Maggiore realized that

the BH energy spectrum must have negative eigenvalues because the “gravitational hydrogen atom” is a quantum system composed by bound states.

Let us again consider the analogy between the potential energy of a hydrogen atom, given by Eq. (3.7), and the effective potential energy of the “gravitational hydrogen atom” given by Eq. (3.18). Eq. (3.7) represents the interaction between the nucleus of the hydrogen atom, having a charge e and the electron, having a charge $-e$. Eq. (3.18) represents the interaction between the nucleus of the “gravitational hydrogen atom”, i.e. the BH, having an effective, dynamical mass M_E , and another, mysterious, particle, i.e., the “electron” of the “gravitational hydrogen atom” having again an effective, dynamical mass M_E . Therefore, let us ask: what is the “electron” of the BH? An intriguing answer to this question has been given by one of the authors (CC), who recently developed a semi-classical Bohr-like approach to BH quantum physics where, for large values of the principal quantum number n , the BH quasi-normal modes (QNMs), “triggered” by emissions (Hawking radiation) and absorption of external particles, represent the “electron” which jumps from a level to another one; the absolute values of the QNMs frequencies represent the energy “shells” of the “gravitational hydrogen atom”. In this context, the QNM jumping from a level to another one has been indeed interpreted in terms of a particle quantized on a circle [18, 19], which is analogous to the electron travelling in circular orbits around the hydrogen nucleus, similar in structure to the solar system, of Bohr’s semi-classical model of the hydrogen atom [25, 26]. Therefore, the results in the present paper seem consistent with the above mentioned works [18, 19].

For the BH ground state ($n = 1$), from Eq. (3.23) one gets the mass as

$$(3.26) \quad M_1 = 2,$$

in Planck units. Thus, in standard units one gets $M_1 = 2m_P$, where m_P is the Planck mass, $m_P = 2,17645 \times 10^{-8}$ Kg. To this mass is associated a total negative energy arising from Eq. (3.24), which is

$$(3.27) \quad E_1 = -\frac{1}{2},$$

and a Schwarzschild radius

$$(3.28) \quad r_{g1} = 4.$$

Hence, this is the state having minimum mass and minimum energy (the energy of this state is minimum in absolute value; in its real value, being

negative, it is maximum). In other words, this ground state represents the smallest possible BH. In the case of Bohr’s semi-classical model of hydrogen atom, the Bohr radius, which represents the classical radius of the electron at the ground state, is [17]

$$(3.29) \quad \text{Bohr radius} = b_1 = \frac{1}{m_e e^2},$$

where m_e is the electron mass. To obtain the correspondent “Bohr radius” for the “gravitational hydrogen atom”, one needs to replace both the electron mass m_e and the charge e in Eq. (3.29) with the effective mass of the BH ground state, which is $\frac{M_1}{2} = 1$. Thus, now the “Bohr radius” becomes

$$(3.30) \quad b_1 = 1,$$

which in standard units reads $b_1 = l_P$, where $l_P = 1,61625 \times 10^{-35}$ m is the Planck length. Hence, we have found that the “Bohr radius” associated to the smallest possible black hole is equal to the Planck length. Following [3], the wave-function associated to the BH ground state is

$$(3.31) \quad \Psi_1 = 2b_1^{-\frac{3}{2}} r \exp\left(-\frac{r}{b_1}\right) = 2r \exp(-r),$$

where Ψ_1 is normalized as

$$(3.32) \quad \int_0^\infty \Psi_1^2 dr = 1.$$

The size of this BH is of the order of

$$(3.33) \quad \bar{r}_1 = \int_0^\infty \Psi_1^2 r dr = \frac{3}{2} b_1 = \frac{3}{2}.$$

The issue that the size of the BH ground state is, on average, shorter than the gravitational radius could appear surprising, but one recalls again that one interprets the “BH electron states” in terms of BH QNMs [18, 19]. Thus, the BH size which is, on average, shorter than the gravitational radius, seems consistent with the issue that the BH horizon oscillates with damped oscillations when the BH energy state jumps from a quantum level to another one through emissions of Hawking quanta and/or absorption of external particles.

This seems an interesting quantum representation of the Schwarzschild BH ground state at the Planck scale. This Schwarzschild BH ground state

represents the BH minimum energy level which is compatible with the generalized uncertainty principle (GUP) [27]. The GUP indeed prevents a BH from its total evaporation by stopping Hawking's evaporation process in exactly the same way that the usual uncertainty principle prevents the hydrogen atom from total collapse [27].

Now, let us discuss a fundamental issue. Can one say that the quantum BH expressed by the system of equations from (3.18) to (3.21) is non-singular? It seems that the correct answer is yes. It is well known that, in the classical general relativistic framework, in the internal geometry all time-like radial geodesics of the collapsing star terminate after a lapse of finite proper time in the termination point $r = 0$ and it is impossible to extend the internal space-time manifold beyond that termination point [12]. Thus, the point $r = 0$ represents a singularity based on the rigorous definition by Schmidt [28]. But what happens in the quantum framework that has been analysed in this paper is completely different. By inserting the constraints for a Schwarzschild BH in Rosen's quantization process applied to the gravitational collapse, it has been shown that the completely collapsed object has been split in a two-particle system where the two components strongly interact with each other through a quantum gravitational interaction. In concrete terms, the system that has been analysed is indeed formally equal to the well known system of two quantum particles having finite distance with the mutual attraction of the form $1/r$ [17]. These two particles are the "nucleus" and the "electron" of the "gravitational hydrogen atom". Thus, the key point is the meaning of the word "particle" in a quantum framework. Quantum particles remain in an uncertain, non-deterministic, smeared, probabilistic wave-particle orbital state [17]. Then, they cannot be localized in a particular "termination point where all time-like radial geodesics terminate". As it is well known, such a localization is also in contrast with the Heisenberg uncertainty principle (HUP). The HUP says indeed that either the location or the momentum of a quantum particle such as the BH "electron" can be known as precisely as desired, but as one of these quantities is specified more precisely, the value of the other becomes increasingly indeterminate. This is not simply a matter of observational difficulty, but rather a fundamental property of nature. This means that, within the tiny confines of the "gravitational atom", the "electron" cannot really be regarded as a "point-like particle" having a definite energy and location. Thus, it is somewhat misleading to talk about the BH "electron" "falling into" the BH "nucleus". In other words, the Schwarzschild radial coordinate cannot become equal to zero. The GUP makes even stronger this last statement: as we can notice

from its general expression [29]

$$(3.34) \quad \Delta x \Delta p \geq \frac{1}{2} \left[1 + \eta (\Delta p)^2 + \dots \right],$$

it implies a non-zero lower bound on the minimum value of the uncertainty on the particle's position (Δx) which is of order of the Planck length [29]. In other words, the GUP implies the existence of a minimal length in quantum gravity.

One notes also another important difference between the hydrogen atom of quantum mechanics [17] and the “gravitational hydrogen atom” discussed in this paper. In the standard hydrogen atom the nucleus and the electron are different particles. In the quantum BH analysed here they are equal particles instead, as one easily checks in the system of equations from (3.18) to (3.21). Thus, the “nucleus” and the “electron” can be mutually exchanged without varying the physical properties of the system. Hence, the quantum state of the two particles seems even more uncertain, more non-deterministic, more smeared and more probabilistic than the corresponding quantum states of the particles of the hydrogen atom. These quantum arguments seem to be strong arguments in favour of the non-singular behavior of the Schwarzschild BH in a quantum framework. Notice that the results in this paper are also in agreement with the general conviction that quantum gravity effects become fundamental in the presence of strong gravitational fields. In a certain sense, the results in this paper permit to “see into” the Schwarzschild BH. The authors hope to further deepen these fundamental issues in future works.

4. Area quantization

Bekenstein proposed that the area of the BH horizon is quantized in units of the Planck length in quantum gravity (let us remember that the Planck length is equal to one in Planck units) [4]. His result was that the Schwarzschild BH area quantum is $\Delta A = 8\pi$ [4].

In the Schwarzschild BH the *horizon area* A is related to the mass by the relation $A = 16\pi M^2$. Thus, a variation ΔM of the mass implies a variation

$$(4.1) \quad \Delta A = 32\pi M \Delta M$$

of the area. Let us consider a BH which is excited at the level n . The corresponding BH mass is given by Eq. (3.23), that is

$$(4.2) \quad M_n = 2\sqrt{n}.$$

Now, let us assume that a neighboring particle is captured by the BH causing a transition from n to $n + 1$. Then, the variation of the BH mass is

$$(4.3) \quad M_{n+1} - M_n = \Delta M_{n \rightarrow n+1},$$

where

$$(4.4) \quad M_{n+1} = 2\sqrt{n+1}.$$

Therefore, using Eqs. (4.1) and (4.3) one gets

$$(4.5) \quad \Delta A_n \equiv 32\pi M_n \Delta M_{n \rightarrow n+1}.$$

Eq. (4.5) should give the area quantum of an excited BH when one considers an absorption from the level n to the level $n + 1$ in function of the principal quantum number n . But, let us consider the following problem. An emission from the level $n + 1$ to the level n is now possible due to the potential emission of a Hawking quantum. Then, the correspondent mass lost by the BH will be

$$(4.6) \quad M_{n+1} - M_n = -\Delta M_{n \rightarrow n+1} \equiv \Delta M_{n+1 \rightarrow n}.$$

Hence, the area quantum for the transition (4.6) should be

$$(4.7) \quad \Delta A_n \equiv 32\pi M_{n+1} \Delta M_{n+1 \rightarrow n},$$

and one gets the strange result that the absolute value of the area quantum for an emission from the level $n + 1$ to the level n is different from the absolute value of the area quantum for an absorption from the level n to the level $n + 1$ because it is $M_{n+1} \neq M_n$. One expects the area spectrum to be the same for absorption and emission instead. In order to resolve this inconsistency, one considers the *effective mass*, which has been introduced in Section 3, corresponding to the transitions between the two levels n and $n + 1$. In fact, the effective mass is the same for emission and absorption

$$(4.8) \quad M_{E(n, n+1)} \equiv \frac{1}{2} (M_n + M_{n+1}) = \sqrt{n} + \sqrt{n+1}.$$

By replacing M_{n+1} with $M_{E(n, n+1)}$ in equation (4.7) and M_n with $M_{E(n, n+1)}$ in Eq. (4.5) one obtains

$$(4.9) \quad \begin{aligned} \Delta A_{n+1} &\equiv 32\pi M_{E(n, n+1)} \Delta M_{n+1 \rightarrow n} && \text{emission} \\ \Delta A_n &\equiv 32\pi M_{E(n, n+1)} \Delta M_{n \rightarrow n+1} && \text{absorption} \end{aligned}$$

and now it is $|\Delta A_n| = |\Delta A_{n-1}|$. By using Eqs. (4.3) and (4.8) one finds

$$(4.10) \quad |\Delta A_n| = |\Delta A_{n+1}| = 64\pi,$$

which is similar to the original result found by Bekenstein in 1974 [4], but with a different coefficient. This is not surprising because there is no general consensus on the area quantum. In fact, in [30, 31] Hod considered the black hole QNMs like quantum levels for absorption of particles, obtaining a different numerical coefficient. On the other hand, the expression found by Hod

$$(4.11) \quad \Delta A = 4 \ln 3$$

is actually a special case of the one suggested by Mukhanov in [32], who proposed

$$(4.12) \quad \Delta A = 4 \ln k, \quad k = 2, 3, \dots$$

This can be found in [1, 33].

Thus, the approach in this paper seems consistent with the Bekenstein area law.

5. Discussion and conclusion remarks

Rosen's quantization approach has been applied to the gravitational collapse in the simple case of a pressureless "*star of dust*". In this way, the gravitational potential, the Schrödinger equation and the solution for the collapse's energy levels have been found. After that, by applying the constraints for a BH and by using the concept of BH effective state [18, 19], the analogous results and the energy spectrum have been found for the Schwarzschild BH. Remarkably, such an energy spectrum is consistent (in its absolute value) with both the one which was found by J. Bekenstein in 1974 [4] and that found by Maggiore in [5]. The discussed approach also allowed to find an interesting quantum representation of the Schwarzschild BH ground state at the Planck scale; in other words, the smallest BH has been found, by also showing that it has a mass of two Planck masses and a "Bohr radius" equal to the Planck length. Furthermore, two fundamental issues such as Bekenstein area law and singularity resolution have been discussed. Thus, despite the gravitational collapse analysed in this paper is the simplest possible, the analysis that it has been performed permitted to obtain important results in BH quantum physics.

Finally, for the sake of completeness, it is necessary to discuss how the results found in this paper are related to those appearing in the literature. The results of Bekenstein and Maggiore concerning the BH energy spectrum have been previously cited. In general, such an energy spectrum has been discussed and derived in many different ways, see for example [34, 35]. In the so-called reduced phase space quantization method [34, 35], the BH energy spectrum gets augmented by an additional zero-point energy; this becomes important if one attempts to address the ultimate fate of BH evaporation, but, otherwise, it can safely be ignored for macroscopic BHs for which the principal quantum number n will be extremely large. It is important to stress that the discreteness of the energy spectrum needs a drastic departure from the thermal behavior of the Hawking radiation spectrum. A popular way to realize this is through the popular tunnelling framework arising from the paper of Parikh and Wilczek [21]. In that case, the energy conservation forces the BH to contract during the emission of the particle; the horizon recedes from its original radius and becomes smaller at the end of the emission process [21]. Therefore, BHs do not exactly emit like perfect black bodies, see also the discussion on this issue in Section 3. Moreover, Loop Quantum Gravity predicts a discrete energy spectrum which indicates a physical Planck scale cutoff of the Hawking temperature law [36]. In the framework of String-Theory one can identify microscopic BHs with long chains living on the worldvolume of two dual Euclidean brane pairs [37]. This leads to a discrete Bekenstein-like energy spectrum for the Schwarzschild black hole [38]. The Bekenstein energy spectrum is present also in canonical quantization schemes [39]. This approach yields a BH picture that is shown to be equivalent to a collection of oscillators whose density of levels is corresponding to that of the statistical bootstrap model [39].

In a series of interesting papers [40–42], Stojkovic and collaborators wrote down the Schrödinger equation for a collapsing object and showed by explicit calculations that quantum mechanics is perhaps able to remove the singularity at the BH center (in various space-time slicings). This is consistent with our analysis. In [40–42] it is indeed proved (among other things) that the wave function of the collapsing object is non-singular at the center even when the radius of the collapsing object (classically) reaches zero. Moreover, in [42], they also considered charged BHs.

Another interesting approach to the area quantization, based on graph theory, has been proposed by Davidson in [43]. In such a paper, the Bekenstein-Hawking area entropy formula is obtained, being automatically accompanied by a proper logarithmic term (a subleading correction), and

the size of the horizon unit area is fixed [43]. Curiously, Davidson also found a hydrogen-like spectrum in a totally different contest [44].

Thus, it seems that the results of this paper are consistent with the previous literature on BH quantum physics.

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References

- [1] J. D. Bekenstein, *Quantum black holes as atoms*, In: T. Piran, R. Ruffini, (eds.), *Proceedings of the Eight Marcel Grossmann Meeting*, World Scientific Publishing, Singapore, pp. 92–111 (1999).
- [2] S. W. Hawking, *Breakdown of predictability in gravitational collapse*, *Phys. Rev. D* **14**, 2460 (1976).
- [3] N. Rosen, *Quantum mechanics of a miniuniverse*, *Int. Journ. Theor. Phys.*, **32**, 1435 (1993).
- [4] J. Bekenstein, *The quantum mass spectrum of the Kerr black hole*, *Lett. Nuovo Cimento*, **11**, 467 (1974).
- [5] M. Maggiore, *Black holes as quantum membranes*, *Nucl. Phys. B* **429**, 205 (1994).
- [6] B. S. DeWitt, *Quantum theory of gravity. I. the canonical theory*, *Phys. Rev.* **160**, 1113 (1967).
- [7] A. Feoli, E. Benedetto, M. Capriolo and F. Feleppa, *Is Dark Energy due to an excited quantum state of the Universe?* *Eur. Phys. J. Plus* **132**, 211 (2017).
- [8] F. Feleppa, I. Licata, C. Corda, *Hartle-Hawking boundary conditions as Nucleation by de Sitter Vacuum*, *Phys. Dark Un.* **26**, 100381 (2019).
- [9] G. Lemaître, *The expanding universe*, *Ann. Soc. Sci. Bruxelles*, **A53**, 51 (1933).
- [10] R. C. Tolman, *Effect of inhomogeneity on cosmological models*, *Proc. Natl. Acad. Sci.* **20** (3), 169 (1934).

- [11] H. Bondi, *Spherically symmetrical models in general relativity*, MNRAS **107** (5-6), 410 (1947).
- [12] C. W. Misner, K. S. Thorne, J. A. Wheeler, *Gravitation*, In: W. H. Feeman and Co., New York (1973).
- [13] J. R. Oppenheimer and H. Snyder, *On continued gravitational contraction*, Phys. Rev. **56**, 455 (1939).
- [14] D. L. Beckerdooff, C. W. Misner, *Terminal Configurations of Stellar Evolution*, Princeton University (1962).
- [15] C. Corda, H. J. Mosquera Cuesta, *Removing black-hole singularities with nonlinear electrodynamics*, Mod. Phys. Lett. A **25**, 2423 (2010).
- [16] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets*, 5th edition, World Scientific, Singapore (2009).
- [17] A. Messiah, *Quantum Mechanics, Vol. 1*, North-Holland, Amsterdam (1961).
- [18] C. Corda, *Precise model of Hawking radiation from the tunnelling mechanism*, Class. Quantum Grav. **32**, 195007 (2015).
- [19] C. Corda, *Quasi-normal modes: The “electrons” of black holes as “gravitational atoms”? Implications for the black hole information puzzle*, Adv. High En. Phys. **2015**, 867601 (2015).
- [20] S. W. Hawking, *Particle creation by black holes*, Commun. Math. Phys. **43**, 199 (1975).
- [21] M. K. Parikh and F. Wilczek, *Hawking radiation as tunneling*, Phys. Rev. Lett. **85**, 5042 (2000).
- [22] R. Banerjee, B. R. Majhi, *Quantum tunneling and trace anomaly*, Phys. Lett. B **674**, 218 (2009).
- [23] S. W. Hawking, *The path integral approach to quantum gravity*, In: General Relativity: An Einstein Centenary Survey, (eds.), S. W. Hawking and W. Israel, Cambridge University Press (1979).
- [24] C. Corda, *Non-strictly black body spectrum from the tunnelling mechanism*, Ann. Phys., **337**, 49 (2013).
- [25] N. Bohr, *I. On the constitution of atoms and molecules*, Phil. Magaz. **6**, 26, 151, 1–25 (1913).

- [26] N. Bohr, *XXXVII. On the constitution of atoms and molecules*, *Phil. Mag.* **6**, 26, 153, 476–502, (1913).
- [27] R. J. Adler, P. Chen, D. I. Santiago, *The generalized uncertainty principle and black hole remnants*, *Gen. Rel. Grav.* **33**, 2101–2108 (2001).
- [28] B. G. Schmidt, *A new definition of singular points in general relativity*, *Gen. Rel. Grav.* **1**, 269 (1971).
- [29] A. Kempf, G. Mangano, R. B. Mann, *Hilbert space representation of the minimal length uncertainty relation*, *Phys. Rev. D* **52**, 1108 (1995).
- [30] S. Hod, *Bohr's correspondence principle and the area spectrum of quantum black holes*, *Phys. Rev. Lett.* **81**, 4293 (1998).
- [31] S. Hod, *Gravitation, the quantum and Bohr's correspondence principle*, *Gen. Rel. Grav.* **31**, 1639 (1999).
- [32] V. Mukhanov, *Are black holes quantized?* *JETP Lett.* **44**, 63 (1986).
- [33] J. D. Bekenstein and V. Mukhanov, *Spectroscopy of the quantum black hole*, *Phys. Lett. B* **360**, 7 (1995).
- [34] S. Das, P. Ramadevi, U.A. Yajnik, *Black hole area quantization*, *Mod. Phys. Lett. A* **17**, 993 (2002).
- [35] A. Barvinski, S. Das, G. Kunstatter, *Quantum mechanics of charged black holes*, *Phys. Lett. B* **517**, 415 (2001).
- [36] C. Röken, *On the nature of black holes in loop quantum gravity*, *Class. Quantum Grav.* **30**, 015005 (2013).
- [37] A. Krause, *Dual brane pairs, chains and the Bekenstein–Hawking entropy*, *Int. J. Mod. Phys. A* **20**, 4055 (2005).
- [38] A. Krause, *Bekenstein-spectrum, temperature and specific heat of black holes from chains*, *Int. J. Mod. Phys. A* **20**, 2813 (2005).
- [39] C. Vaz, *Canonical quantization and the statistical entropy of the Schwarzschild black hole*, *Phys. Rev. D* **61**, 064017 (2000).
- [40] A. Saini and D. Stojkovic, *Nonlocal (but also nonsingular) physics at the last stages of gravitational collapse*, *Phys. Rev. D* **89**, 044003 (2014).
- [41] E. Greenwood and D. Stojkovic, *Quantum gravitational collapse: non-singularity and non-locality*, *JHEP* **0806**, 042 (2008).

- [42] J. E. Wang, E. Greenwood and D. Stojkovic, *Schrodinger formalism, black hole horizons, and singularity behavior*, Phys. Rev. D **80**, 124027 (2009).
- [43] A. Davidson, *From Planck area to graph theory: Topologically distinct black hole microstates*, Phys. Rev. D **100**, 081502 (2019).
- [44] A. Davidson, *Hydrogen-like spectrum of spontaneously created brane universes with de-Sitter ground state*, Phys. Lett. B **780**, 29 (2018).

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