ADV. THEOR. MATH. PHYS.
Volume 26, Number 10, 3801–3894, 2022

On $\mathcal{M}$-theory dual of large-N thermal QCD-like theories up to $\mathcal{O}(R^4)$ and G-structure classification of underlying non-supersymmetric geometries

Vikas Yadav and Aalok Misra

Dedicated by one of the authors (AM) to the memory of his father, his "rock"

Construction of a top-down holographic dual of thermal QCD-like theories (equivalence class of theories which are UV-conformal, IR-confining and have fundamental quarks) *at intermediate 't Hooft coupling* and the G-structure (torsion classes) classification of the underlying geometries (in the Infra Red (IR)/non-conformal sector in particular) of the *non-supersymmetric* string/$\mathcal{M}$-theory duals, have been missing in the literature. We take the first important steps in this direction by studying the $\mathcal{M}$ theory dual of large-N thermal QCD-like theories at intermediate gauge and 't Hooft couplings and obtaining the $\mathcal{O}(l_p^6)$ corrections arising from the $\mathcal{O}(R^4)$ terms to the "MQGP" background ($\mathcal{M}$-theory dual of large-N thermal QCD-like theories at intermediate gauge/string coupling, but large 't Hooft coupling) of [1]. The main Physics lesson learnt is that there is a competition between non-conformal IR enhancement and Planckian and large-N suppression and going to orders beyond the $\mathcal{O}(l_p^6)$ is necessitated if the IR enhancement wins out. The main lesson learnt in Math is in the context of the differential geometry (G-structure classification) of the internal manifolds relevant to the string/$\mathcal{M}$-theory duals of large-N thermal QCD-like theories, wherein we obtain for the first time inclusive of the $\mathcal{O}(R^4)$ corrections in the Infra-Red (IR), the $SU(3)$-structure torsion classes of the type IIA mirror of [2] (making contact en route with Siegel theta functions related to appropriate hyperelliptic curves, as well as the Kiepert's algorithm of solving quintics), and the $G_2/SU(4)/Spin(7)$-structure torsion classes of the seven- and eight-folds associated with its $\mathcal{M}$ theory uplift.

1. Introduction

The study of the dynamics of non-Abelian gauge theories at finite temperatures is essential to studying various physical processes, like electroweak and hadronic matter and various other phenomena. No effective theory developed over the years has given a suitable explanation for the intermediate coupling regime. The results for the intermediate coupling regime have been obtained by extrapolating the results obtained via perturbation theory. In recent years, gauge/gravity duality has provided a simple, classical computational tool for understanding the strongly coupled systems and overcome the theoretical limitations in study of non-Abelian gauge theories. In its simplest form for maximally supersymmetric $SU(N_c)$ Yang-Mills theory ($\mathcal{N} = 4$ SYM), in the $N_c \to \infty$ limit, the gauge/gravity duality provides a tool for analysing its properties in the large 't Hooft coupling limit. The gauge/gravity duality also allows us to study the corrections to the infinite coupling limit. These corrections appear as higher order derivative corrections on the gravity side. The effect of these corrections to the action are incorporated in the background metric and fluxes, perturbatively by considering perturbations of the equations of motion. However, other than higher-derivative corrections quartic in the Weyl tensor, or of the Gauss-Bonnet type, in $AdS_5 \times S^5$, dual to supersymmetric thermal Super Yang-Mills [3], there is little known about top-down string theory duals at intermediate 't Hooft coupling of thermal QCD-like theories. In this work, we address precisely this issue. We include terms quartic in the eleven-dimensional Riemann curvature R in the eleven-dimensional supergravity action that appear as $\mathcal{O}(l_p^6)$(l_p being the 11D Planckian length)-corrections in the $\mathcal{M}$-theory dual of large-N thermal QCD-like theories (equivalence class of theories which are UV-conformal, IR-confining and have fundamental quarks).

Physics motivation behind this work: Significance of higher order derivative corrections is not just only related to corrections to the infinite coupling limit. They also serve as the leading quantum gravity corrections to the $\mathcal{M}$-Theory action to study the compactifications of $\mathcal{M}$-Theory on compact eight-dimensional manifolds. The study of warped compactification of $\mathcal{M}$-Theory on eight-dimensional compact manifolds is very interesting. Conceptually, on one hand this compactification allows for the study of three-dimensional effective theories with small amounts of supersymmetry. On the other hand it allows us to study the lifting of three-dimensional theories to four space-time dimensions for a certain class of eight-dimensional manifolds using $\mathcal{M}$-Theory to F-theory limit. In the past, vacua for warped compactifications of $\mathcal{M}$-Theory on compact eight-dimensional manifolds have been studied by including the higher derivative terms to the action. The leading quantum gravity corrections to $\mathcal{M}$-Theory actions are fourth order, R^4, and third order, R^3G^2, in the eleven-dimensional Riemann curvature R , where G is the field strength of $\mathcal{M}$-Theory three form. The terms of $\mathcal{O}(R^4)$ have been used in past [8], while terms involving third order have been recently analyzed in [9]. The construction of a top-down holographic dual of thermal Quantum Chromodynamics (QCD) at intermediate 't Hooft coupling has been missing in the literature. This work takes important steps to fill this gap by studying the $\mathcal{M}$ theory dual of large-N thermal QCD-like theories at high temperatures, at intermediate gauge and 't Hooft couplings by obtaining the $\mathcal{O}(l_p^6)$ corrections to the $\mathcal{M}$-Theory uplift of [2] as constructed in [1].

Mathematics motivation behind this work: The study of the differential geometry of fluxed compatifications involving non-Kähler six-folds in Heterotic string theory via the study of $SU(3)$-structure torsion classes, was initiated in [10]; $SU(3)$ and G_2-structure torsions classes of respectively six- and seven-folds in respectively type II and M-theory flux compactifications was extensively studied in [11–14]. $SU(3)$- and G_2-structure torsion classes of type IIB/A holographic dual of thermal QCD-like theories and their $\mathcal{M}$-theory uplift in the intermediate/large "r" (the radial coordinate in the gravity dual which corrsponds to the energy on the gauge theory side), i.e., Ultra-Violet (UV)-Infra-Red (IR) interpolating region/UV region were obtained in the second reference in [15] and [16]. In this work, for the first time, we classify the underlying six-, seven- and eight-dimensional geometries at small r, i.e., the IR, inclusive of the aforementioned $\mathcal{O}(l_p^6)$-corrections in the $D = 11$ supergravity action, as regards their $SU(3), G_2, SU(4), Spin(7)$-torsion classes (note these corrections vanish in the very large-r limit, i.e., the deep UV, wherein G-structure approaches G-holonomy), both in the SYZ type IIA mirror of the type IIB holographic dual constructed in [2] of thermal QCD-like theories, as well as its $\mathcal{M}$-theory uplift.

The following are the main results of this work.

- The $\mathcal{M}$-theory dual of thermal QCD-like theories inclusive of $\mathcal{O}(l_p^6)$-corrections, was obtained.
- **Proposition**:
  1) The non-Kähler warped six-fold M_6, obtained as a cone over a compact five-fold M_5, that appears in the type IIA background corresponding to the the $\mathcal{M}$-theory uplift of thermal QCD-like theories at high temperatures, in the neighborhood of the Ouyang embedding (32) of the type IIB flavor $D7$-branes [20] (that figure in the

type IIB string dual of thermal QCD [2]) effected by working in the neighborhood of small $\theta_{1,2}$ (such as (33)), in the MQGP limit (1) and *inclusive of the $\mathcal{O}(l_p^6)$ corrections* (l_p being the 11D Planckian length),

 – is a non-complex manifold (though the deviation from $W_{1,2}^{SU(3)} = 0$ being N-suppressed),
 – $W_4^{SU(3)} \sim W_5^{SU(3)}$ (upon comparison with [14], interpreted as "almost" supersymmetric [in the large-N limit]).

2) The G_2-structure torsion classes of the seven-fold M_7 (part of the eleven-fold $M_{11}(x^0, x^{1,2,3}, r, \theta_1, \theta_2, \phi_1, \phi_2, \psi, x^{10})$ which is a warped product of $S^1 \times_w \mathbb{R}^3$ and a cone over $\mathcal{M}$-theory-S^1-fibration over M_5: $p_1^2(M_{11}) = p_2(M_{11}) = 0$, p_a being the a-th Pontryagin class, and solves the $D = 11$ supergravity equations of motion (20)) are: $W_{M_7}^{G_2} = W_{14}^{G_2} \oplus W_{27}^{G_2}$.
3) Inclusive of an S^1-valued x^0 at finite temperature, referred to henceforth as the thermal circle, the $SU(4)/Spin(7)$-structure torsion classes of $M_8(r, \theta_{1,2}, \phi_{1,2}, \psi, x^{10}, x^0)$ are $W_{M_8}^{SU(4)/Spin(7)} = W_2^{SU(4)} \oplus W_3^{SU(4)} \oplus W_5^{SU(4)}/W_1^{Spin(7)} \oplus W_2^{Spin(7)}$.

Organization of the remainder of the paper

The remainder of the paper is organized as follows. Section **2** is a short review of the type IIB string theoretic dual of large-N thermal QCD-like theories as obtained in [2], as well as its Strominger-Yau-Zaslow type IIA mirror and the $\mathcal{M}$ theory uplift of the same as constructed in [1]. Section **3** begins with a summary of the $\mathcal{O}(R^4)$ terms in $D = 11$ supergravity that are considered in the remainder of the paper. The $\mathcal{O}(l_p^6)$(l_p being the $D = 11$ Planck length) corrections to the $\mathcal{M}$-Theory uplift in the "MQGP" limit as obtained in [1], near the $\psi = 2n\pi, n = 0, 1, 2$-branches are consequently obtained in **3.1** and for $\psi \neq 2n\pi, n = 0, 1, 2$ in **3.2**. There are three main lemmas in **3.1** pertaining to working in the neighborhood of $\psi = 2n\pi$-branches. The first is on comparing the large-N behaviors of two $\mathcal{O}(R^4 l_p^6)$ terms in the $D = 11$ supergravity action; the second is on the $\mathcal{M}$-theory metric inclusive of $\mathcal{O}(l_p^6)$ corrections, and the third is on the consistency of setting the $\mathcal{O}(l_p^6)$ corrections to the $\mathcal{M}$-theory three-form potential, to zero. Subsection

3.2 has an analogous lemma working in the neighbhorhood of $\psi \neq 2n\pi$-coordinate patch. Section **4** discusses the major Physics lessons learnt. Section **5** through four sub-sections, discusses the $SU(3)/G_2/SU(4), Spin(7)$-structure torsion classes in **5.1/5.2/5.3/5.4**. Section **5** has five main lemmas. The first, in **5.1**, is on the type IIA metric components along the compact directions. The second, also in **5.1**, is on the underlying type IIA internal six-fold being non-complex and yet satisfying a relation of [14] for supersymmetric compactification. The third, in **5.2**, is on the evaluation of the G_2-structure torsion classes of the relevant seven-fold which is a cone over a six-fold that is itself an $\mathcal{M}$-theory circle fibration over a compact five-fold. Inclusive of a "thermal circle", the fourth, in **5.3**, is on the $SU(4)$-structure torsion classes of the underlying eight-fold. Finally, the fifth is on the evaluation of $Spin(7)$-structure torsion classes of the aforementioned eight-fold. The nine lemmas together imply the proposition stated in Section **1**. Section **6** is a summary of the results obtained in the paper and a summary of the applications of the same to Physics as obtained in [4], [5]. There are four supplementary appendices - a long appendix **A** on the equations of motion for the metric perturbations (f_{MN}) and their explicit solutions obtained inclusive of the aforementioned $\mathcal{O}(R^4)$ terms in the IR, both near the $\psi = 2n\pi, n = 0, 1, 2$-branches in **A.1** leading up to **3.1**, and near the $\psi \neq 2n\pi, n = 0, 1, 2$ coordinate patches in **A.2** leading up to **3.2**. Appendix **B** has a step-by-step discussion of the Kiepert's algorithm for diagonalizing the $M_5(\theta_{1,2}, \phi_{1,2}, \psi)$ metric leading to the evaluation of G-structure torsion classes for $M_6(r, \theta_{1,2}, \phi_{1,2}, \psi)$, $M_6(r, \theta_{1,2}, \phi_{1,2}, \psi) \times_w S^1(x^{10})$ and $S^1(x^0) \times_w \left(M_6(r, \theta_{1,2}, \phi_{1,2}, \psi) \times_w S^1(x^{10})\right)$. Appendix **C** lists out the non-trivial "structure constants" of the algebra of the fufnbeings/sechsbeins in section **4**. Appendix **D** gives some calculational details relevant to showing that one can, up to $\mathcal{O}(l_p^6)$-corrections, consistently set the corrections at the same order in the $\mathcal{M}$-theory three-form potential, to zero. Finally appendix **E** gives details of the G_2 structure torsion classes $W_{1,7}$.

2. String/$\mathcal{M}$-theory dual of thermal QCD - a quick review of (and results related to) [1, 2]

In this section, we provide a short review of the UV complete type IIB holographic dual - *the only one we are aware of* - of large-N thermal QCD-like theories constructed in [2], its Strominger-Yau-Zaslow (SYZ) type IIA mirror at intermediate string coupling and its subsequent $\mathcal{M}$-Theory uplift constructed in [1, 16], as well as a summary of results in applications of the same to the study of transport coefficients and glueball-meson phenomenology.

We begin with the UV-complete type IIB holographic dual of large-N thermal QCD-like theories as constructed in [2] which built up on the zero-temperature Klebanov-Witten model [17], the non-conformal Klebanov-Tseytlin model [18], its IR completion as given in the Klebanov-Strassler model [19] and Ouyang's [20] inclusion of flavor in the same, as well as the non-zero temperature/non-extremal version of [21] (wherein the non-extremality function and the ten-dimensional warp factor simultaneously vanished at the horizon radius), [22] (which was valid only at large temperatures) and [23, 24] (which addressed the IR), in the absence of flavors. The authors of [2] considered N $D3$-branes placed at the tip of a six-dimensional conifold, M $D5$-branes wrapping the vanishing S^2 and M $\overline{D5}$-branes distributed along the resolved S^2 and placed at the anti-podal points relative to the M $D5$-branes. Denoting the average $D5/\overline{D5}$ separation by $\mathcal{R}_{D5/\overline{D5}}$, roughly speaking, $r > \mathcal{R}_{D5/\overline{D5}}$, would correspond to the UV. The N_f flavor $D7$-branes (holomorphically embedded via Ouyang embedding [20] in the resolved conifold geometry) are present in the UV, the IR-UV interpolating region and dip into the (confining) IR (without touching the $D3$-branes; the shortest $D3-D7$ string corresponding to the lightest quark). In addition, N_f $\overline{D7}$-branes are also present in the UV and the UV-IR interpolating region but not the IR, for the reason given below. In the UV, there is $SU(N+M)\times SU(N+M)$ color symmetry and $SU(N_f)\times SU(N_f)$ flavor symmetry. As one goes from $r > \mathcal{R}_{D5/\overline{D5}}$ to $r < \mathcal{R}_{D5/\overline{D5}}$, there occurs a partial Higgsing of $SU(N+M)\times SU(N+M)$ to $SU(N+M)\times SU(N)$ because in the IR, i.e., at energies less than $\mathcal{R}_{D5/\overline{D5}}$, the $\overline{D5}$-branes are integrated out resulting in the reduction of the rank of one of the product gauge groups (which is $SU(N+\text{number of } D5-\text{branes})\times SU(N+\text{number of } \overline{D5}-\text{branes})$). Similarly, the $\overline{D5}$-branes are "integrated in" in the UV, resulting in the conformal Klebanov-Witten-like $SU(M+N)\times SU(M+N)$ product color gauge group [17]. The gauge couplings, $g_{SU(N+M)}$ and $g_{SU(N)}$, were shown in [19] to flow oppositely with the flux of the NS-NS B through the vanishing S^2 being the obstruction to obtaining conformality which is why M $\overline{D5}$-branes were included in [2] to cancel the net $D5$-brane charge in the UV. Also, as the number N_f of the flavor $D7$-branes enters the RG flow of the gauge couplings via the dilaton, their contribution therefore needs to be canceled by N_f $\overline{D7}$-branes. The RG flow equations for the gauge coupling $g_{SU(N+M)}$ - corresponding to the relatively higher rank gauge group - can be used to show that the same flows towards strong coupling, and the relatively lower rank $SU(N)$ gauge coupling flows towards weak coupling. One can show that the strongly coupled $SU(N+M)$ is Seiberg-like dual to weakly coupled $SU(N-(M-N_f))$. Under a Seiberg-like duality cascade[1] all the N $D3$-branes are cascaded away with a finite M left at the end in the IR. One will thus be left with a strongly coupled IR-confining $SU(M)$ gauge theory the finite temperature version of which is what was looked at in [2]. So, at the end of the Seiberg-like duality cascade in the IR, the number of colors N_c gets identified with M, which in the 'MQGP' limit can be tuned to equal the value in QCD, i.e., 3. Now, N_c can be written as the sum of the effective number $N_{\rm eff}(r)$ of $D3$-branes and the effective number $M_{\rm eff}$ of the fractional $D3$-branes: $N_c = N_{\rm eff}(r) + M_{\rm eff}(r)$; $N_{\rm eff}(r)$ is defined via $\tilde{F}_5 \equiv dC_4 + B_2 \wedge F_3 = \mathcal{F}_5 + *\mathcal{F}_5$ where $\mathcal{F}_5 \equiv N_{\rm eff}\mathrm{Vol}(\text{Base of Resolved Warped Deformed Conifold})$, and $M_{\rm eff}$ is defined via $M_{\rm eff} = \int_{S^3} \tilde{F}_3 (= F_3 - \tau H_3)$ (the S^3 being dual to $e_\psi \wedge (\sin\theta_1 d\theta_1 \wedge d\phi_1 - B_1 \sin\theta_2 \wedge d\phi_2)$, wherein B_1 is an asymmetry factor defined in [2], and $e_\psi \equiv d\psi + \cos\,\theta_1\, d\phi_1 + \cos\,\theta_2\, d\phi_2$). (See [2, 25] for details.). The finite temperature on the gauge/brane side is effected in [2] in the gravitational dual via a black hole in the latter. Turning on of the temperature (in addition to requiring a finite separation between the M $D5$-branes and M $\overline{D5}$-branes so as to provide a natural energy scale to demarcate the UV) corresponds in the gravitational dual to having a non-trivial resolution parameter of the conifold. IR confinement on the brane/gauge theory side, like the KS model [19], corresponds to having a non-trivial deformation (in addition to the aforementioned resolution) of the conifold geometry in the gravitational dual. The gravity dual is hence given by a resolved warped deformed conifold wherein the $D3$-branes and the $D5$-branes are replaced by fluxes in the IR, and the back-reactions are included in the 10D warp factor as well as fluxes. Hence, the type IIB model of [2] is an ideal holographic dual of thermal QCD-like theories because: (i) it is UV conformal (with the Landau poles being absent), (ii) it is IR confining, (iii) the quarks transform in the fundamental representation of flavor and color groups, and (iv) it is defined for the entire range of temperature - both low (i.e., $T < T_c$ corresponding to a vanishing horizon radius in the gravitational dual) and high (i.e., $T > T_c$ corresponding to non-vanishing horizon radius in the gravitational dual).

[1]Even though the Seiberg duality (cascade) is applicable for supersymmetric theories, for non-supersymmetric theories such as the holographic type IIB string theory dual of [2], the same is effected via a radial rescaling: $r \rightarrow e^{-\frac{2\pi}{3g_s M_{\rm eff}}} r$ [20] under an RG flow from the UV to the IR.

Now, we give a brief review the type IIA Stominger-Yau-Zaslow (SYZ) mirror [26] of [2] and its $\mathcal{M}$-Theory uplift at intermediate gauge coupling, as constructed in [1]. Now, to construct a holographic dual of thermal QCD-like theories, one would have to consider intermediate gauge coupling (as well as finite number of colors) – dubbed as the 'MQGP limit' defined in [1] as follows:

$$g_s \stackrel{<}{\sim} 1, M, N_f \equiv \mathcal{O}(1),\ N \gg 1,\ \frac{g_s M^2}{N} \ll 1. \tag{1}$$

From the perspective of gauge-gravity duality, this therefore requires looking at the strong-coupling/non-perturbative limit of string theory - $\mathcal{M}$ theory.

The $\mathcal{M}$-Theory uplift of the type IIB holographic dual of [2] was constructed in [1] by working out the SYZ type IIA mirror of [2] implemented via a triple T duality along a delocalized special Lagrangian (sLag) T^3 – which could be identified with the T^2-invariant sLag of [27] with a large base $\mathcal{B}(r, \theta_1, \theta_2)$ [15, 16] [2] Let us explain the basic idea. Consider the aforementioned N D3-branes oriented along $x^{0,1,2,3}$ at the tip of conifold and the M $D5$-branes parallel to these $D3$-branes as well as wrapping the vanishing $S^2(\theta_1, \phi_1)$. A single T-dual along ψ yields N $D4$-branes wrapping the ψ circle and M $D4$-branes straddling a pair of orthogonal $NS5$-branes. This pair of $NS5$-branes correspond to the vanishing $S^2(\theta_1, \phi_1)$ and the blown-up $S^2(\theta_2, \phi_2)$ with a non-zero resolution parameter a - the radius of the blown-up $S^2(\theta_2, \phi_2)$. Now, two more T-dualities along ϕ_1 and ϕ_2, convert the aforementioned pair of orthogonal $NS5$-branes into a pair of orthogonal Taub-NUT spaces, the N $D4$-branes into N color $D6$-branes and the M straddling $D4$-branes also to color $D6$-branes. Similarly, in the presence of the aforementioned N_f flavor $D7$-branes, oriented parallel to the $D3$-branes and "wrapping" a non-compact four-cycle $\Sigma^{(4)}(r, \psi, \theta_1, \phi_1)$,

[2]Consider $D5$-branes wrapping the resolved S^2 of a resolved conifold geometry as in [28], which, globally, breaks SUSY [29]. As in [11], to begin with, a delocalized SYZ mirror is constructed wherein the pair of S^2s are replaced by a pair of T^2s, and the correct T-duality coordinates are identified. Then, when uplifting the mirror to $\mathcal{M}$ theory, it was found that a G_2-structure can be chosen that is in fact, free, of the aforementioned delocalization. For the delocalized SYZ mirror of the resolved warped deformed conifold uplifted to $\mathcal{M}$-Theory with G_2 in [1], the idea is precisely the same. Also, as shown in the second reference of [15] and [16], the type IIB/IIA $SU(3)$ structure torsion classes in the MQGP limit and in the UV/UV-IR interpolating region (and as will be shown in **Sec. 4** of this paper, also in the IR and inclusive of $\mathcal{O}(l_p^6)$ corrections), satisfy the same relationships as satisfied by corresponding supersymmetric conifold geometries [14].

upon T-dualization yield N_f flavor $D6$-branes "wrapping" a non-compact three-cycle $\Sigma^{(3)}(r, \theta_1, \phi_2)$. An uplift to $\mathcal{M}$-Theory will convert the $D6$-branes to KK monopoles, which are variants of Taub-NUT spaces. All the branes are hence converted into geometry and fluxes, and one ends up with $\mathcal{M}$-Theory on a G_2-structure manifold. Similarly, one may perform identical three T-dualities on the gravity dual on the type IIB side, which is a resolved warped-deformed conifold with fluxes, to obtain another G_2 structure manifold, yielding the $\mathcal{M}$-Theory uplift of [1, 16].

To the best of our knowledge, [1] is the only holographic $\mathcal{M}$-Theory dual of thermal QCD that is able to:

- yield a deconfinement temperature T_c from a Hawking-Page phase transition at vanishing baryon chemical potential consistent with the very recent lattice QCD results in the heavy quark [30] limit
- yield a conformal anomaly variation with temperature compatible with the very recent lattice results at high ($T > T_c$) *and* low ($T < T_c$) temperatures [30]
- Condensed Matter Physics: inclusive of the non-conformal corrections, obtain:
  1) a lattice-compatible shear-viscosity-to-entropy-density ratio (first reference in [15])
  2) temperature variation of a variety of transport coefficients including the bulk-viscosity-to-shear-viscosity ratio, diffusion coefficient, speed of sound (the last reference in [15]), electrical and thermal conductivity and the Wiedemann-Franz law (first reference in [15]);
- Particle Phenomenology: obtain:
  1) lattice compatible glueball spectroscopy [31]
  2) meson spectroscopy (first reference of [32])
  3) glueball-to-meson decay widths (second reference of [32])
- Mathematics: provide, for the first time, an $SU(3)$-structure (for type IIB (second reference of [15])/IIA [16] holographic dual) and G_2-structure [16] torsion classes of the six- and seven-folds in the UV-IR interpolating region/UV, relevant to type string/$\mathcal{M}$-Theory holographic duals of thermal QCD-like theories at high temperatures.

3. $\mathcal{O}(l_p^6)$ corrections to the background of [1] in the MQGP limit

In this section, we discuss how the equations of motion (EOMs) starting from $D = 11$ supergravity action inclusive of the $\mathcal{O}(R^4)$ terms in the same (which provide the $\mathcal{O}(l_p^6)$ corrections to the leading order terms in the action), are obtained and how the same are solved. The actual EOMs are given in Appendix **A** - EOMs in **A.1** obtained near the $\psi = 2n\pi, n = 0, 1, 2$ coordinate patches (wherein $G^{\mathcal{M}}_{rM}, M \neq r$ and $G^{\mathcal{M}}_{x^{10}N}, N \neq x^{10}$ vanish) and EOMs in **A.2** obtained away from the same. The solutions of the EOMs are similarly split across subsections **3.1** and **3.2**.

Let us begin with a discussion on the $\mathcal{O}(R^4)$-corrections to the $\mathcal{N} = 1, D = 11$ supergravity action. There are two ways of understanding the origin of these corrections. One is in the context of the effects of D-instantons in IIB supergravity/string theory via the four-graviton scattering amplitude [33]. The other is $D = 10$ supersymmetry [34]. Let us discuss both in some detail.

- Let us first look at interactions that are induced at leading order in an instanton background in both, the supergravity and the string descriptions, including a one-instanton correction to the tree-level and one loop R^4 terms [33]. The bosonic zero modes are parameterised by the coordinates corresponding to the position of the D-instanton. The fermionic zero modes are generated by the broken supersymmetries. The physical closed-string states can be expressed in terms of a light-cone scalar superfield $\Phi(x, \theta)$, $\theta^{a(=1,\dots,8)}$ being an $\mathbf{8}_s$ $SO(8)$ spinor. The 16-component (indexed by A) broken supersymmetry chiral spinor can be decomposed under $SO(8)$ into $\eta^a, \dot{\eta}^{\dot{a}}$. The Grassmann parameters are fermionic supermoduli corresponding to zero modes of λ - the dilatino - and must be integrated over together with the bosonic zero modes, y^μ. The simplest open-string world-sheet that arises in a D-brane process is the disk diagram. An instanton carrying some zero modes corresponds, at lowest order, to a disk world-sheet with open-string states attached to the boundary. An instanton carrying some zero modes corresponds, at lowest order, to a disk world-sheet with open-string states attached to the boundary. The one-instanton terms in the supergravity effective action can be deduced by considering on-shell amplitudes in the instanton background. The integration over the fermionic moduli absorbs the sixteen independent fermionic zero modes. The authors of [33] considered a contact term proportional to λ^{16} arising in IIB supergravity from the nonlocal Green function which at long distances looks like a momentum-independent term in the S-matrix with sixteen external on-shell dilatinos:

$$e^{2\pi i W_1}\epsilon^{A_1\cdots A_{16}}\lambda^{A_1}\ldots\lambda^{A_{16}}.$$

The same result is also obtained in string theory from diagrams with sixteen disconnected disks with a single dilatino vertex operator and a single open-string fermion state attached to each one. The overall factor of $e^{2\pi i W_1}$, which is characteristic of the stringy D-instanton , is evaluated at $\chi = \Re W_1 = 0$ in the string calculation. Consider now amplitudes with four external gravitons. The leading term in supergravity is again one in which each graviton is associated with four fermionic zero modes. Integration over y^μ generates a nonlocal four-graviton interaction. In the corresponding string calculation the world-sheet consists of four disconnected disks to each of which is attached a single closed-string graviton vertex and four fermionic open-string vertices. Writing the polarization tensor as $\zeta^{\mu_r\nu_r} = \zeta^{(\mu_r}\tilde{\zeta}^{\nu_r)}$, and evaluating the fermionic integrals in a special frame as described in [33], in terms of its $SO(8)$ components, one obtains the following result for $\langle h\rangle_4$ [3] :

$$\langle h\rangle_4 = -\frac{1}{2}\eta_a\gamma^{ij}_{ab}\eta_b\ \dot{\eta}_{\dot{a}}\gamma^{mn}_{\dot{a}\dot{b}}\dot{\eta}_{\dot{b}}R_{ijmn} \tag{3}$$

where: $\gamma^{ij}_{ab} = \frac{1}{2}\gamma^{[i}_{a\dot{a}}\gamma^{j]}_{\dot{a}b}, \gamma^i$ being the generators of Cl_8 Clifford algebra and $R_{ijmn} \equiv k_i k_m \zeta_{(j}\tilde{\zeta}_{n)}$ is the linearized curvature. The result contains

[3]In both, string theory and supergravity, the four-graviton scattering result is given as an integral of the product of four factors of "$\langle h\rangle_4$" defined as the tadpole associated with the disk with four fermion zero modes coupled to the graviton and the self dual fourth-rank antisymmetric tensor:

$$\langle h\rangle_4 = \bar{\epsilon}_0\gamma^{\rho\mu\tau}\epsilon_0\ \bar{\epsilon}_0\gamma^{\lambda\nu\tau}\epsilon_0\ \zeta_{\mu\nu}k_\rho k_\lambda, \tag{2}$$

ϵ_0 corresponding to the broken supersymmetry - the only covariant combination of four ϵ_0's, two physical momenta and the physical polarization tensor.

two parity-conserving terms [4]

$$(6)\quad A_4(\{\zeta_h^{(r)}\}) = Ce^{2i\pi W_1}\int d^{10}y e^{i\sum_r k_r\cdot y} \times\left(t^{i_1j_1\cdots i_4j_4}t_{m_1n_1\cdots m_4n_4} - \frac{1}{4}\epsilon^{i_1j_1\cdots i_4j_4}\epsilon_{m_1n_1\cdots m_4n_4}\right) \times R^{m_1n_1}_{i_1j_1}R^{m_2n_2}_{i_2j_2}R^{m_3n_3}_{i_3j_3}R^{m_4n_4}_{i_4j_4}.$$

- In [34], it is shown that the eleven-dimensional $\mathcal{O}(R^4)$ corrections have an independent motivation based on supersymmetry in ten dimensions. This was shown to follow from its relation to the term $C^{(3)}\wedge X_8$ in the $\mathcal{M}$-theory effective action which is known to arise from a variety of arguments, e.g. anomaly cancellation [35]. The expression X_8 is the eight-form in the curvatures that is inherited from the term in type IIA superstring theory [36] which is given by

$$(7)\quad -\int d^{10}xB\wedge X_8 = -\frac{1}{2}\int d^{10}x\sqrt{-g^{A(10)}}\epsilon_{10}BX_8,$$

where

$$(8)\quad X_8 = \frac{1}{192}\left(\mathrm{tr}\ R^4 - \frac{1}{4}(\mathrm{tr}\ R^2)^2\right).$$

There are two independent ten-dimensional $N=1$ super-invariants which contain an odd-parity term ([37] and previous authors):

$$(9)\quad I_3 = t_8\mathrm{tr}\ R^4 - \frac{1}{4}\epsilon_{10}B\mathrm{tr}\ R^4$$

[4]The integral over the dotted and undotted spinors in the four-graviton scattering amplitude factorizes and can be evaluated by using,

$$(4)\quad \int d^8\eta^a\eta^{a_1}\cdots\eta^{a_8} = \epsilon^{a_1\cdots a_8}, \qquad \int d^8\dot{\eta}^{\dot{a}}\dot{\eta}^{\dot{a}_1}\cdots\dot{\eta}^{\dot{a}_8} = \epsilon^{\dot{a}_1\cdots\dot{a}_8}.$$

Substituting into the four-graviton scattering amplitude the following tensors appear

$$(5)\quad \begin{aligned}\epsilon_{a_1a_2\cdots a_8}\gamma^{i_1j_1}_{a_1a_2}\cdots\gamma^{i_4j_4}_{a_7a_8} &= t^{i_1j_1\cdots i_4j_4} = t_8^{i_1j_1\cdots i_4j_4} + \frac{1}{2}\epsilon^{i_1j_1\cdots i_4j_4}\\ \epsilon_{\dot{a}_1\dot{a}_2\cdots\dot{a}_8}\gamma^{i_1j_1}_{\dot{a}_1\dot{a}_2}\cdots\gamma^{i_4j_4}_{\dot{a}_7\dot{a}_8} &= t_8^{i_1j_1\cdots i_4j_4} = t^{i_1j_1\cdots i_4j_4} - \frac{1}{2}\epsilon^{i_1j_1\cdots i_4j_4},\end{aligned}$$

t_8 symbol defined in (13).

and:

$$I_4 = t_8(\text{tr } R^2)^2 - \frac{1}{4}\epsilon_{10}B(\text{tr } R^2)^2. \tag{10}$$

Using the fact that

$$t_8 t_8 R^4 = 24 t_8 \text{tr}(R^4) - 6 t_8(\text{tr } R^2)^2, \tag{11}$$

it follows that the particular linear combination,

$$I_3 - \frac{1}{4}I_4 = \frac{1}{24}t_8 t_8 R^4 - 48\epsilon_{10}B\ X_8 \tag{12}$$

contains both the ten-form $B \wedge X_8$ and $t_8 t_8 R^4$. The R refers to the curvature two-form, ϵ_{10} is the ten-dimensional Levi-Civita symbol and the t_8 symbol is defined as follows:

(13)

$$\begin{aligned} t_8^{N_1\ldots N_8} = \frac{1}{16}\Big(&-2\big(G^{N_1N_3}G^{N_2N_4}G^{N_5N_7}G^{N_6N_8} + G^{N_1N_5}G^{N_2N_6}G^{N_3N_7}G^{N_4N_8} \\ &+ G^{N_1N_7}G^{N_2N_8}G^{N_3N_5}G^{N_4N_6}\big) \\ &+ 8\big(G^{N_2N_3}G^{N_4N_5}G^{N_6N_7}G^{N_8N_1} + G^{N_2N_5}G^{N_6N_3}G^{N_4N_7}G^{N_8N_1} \\ &+ G^{N_2N_5}G^{N_6N_7}G^{N_8N_3}G^{N_4N_1}\big) \\ &- (N_1 \leftrightarrow N_2) - (N_3 \leftrightarrow N_4) - (N_5 \leftrightarrow N_6) - (N_7 \leftrightarrow N_8)\Big), \end{aligned}$$

wherein $G^{M_1M_2}$ is the metric inverse.

The $\mathcal{N} = 1, D = 11$ supergravity action inclusive of $\mathcal{O}(l_p^6)$ terms, is hence given by:

$$\begin{aligned} \mathcal{S}_{D=11} = \frac{1}{2\kappa_{11}^2}\Bigg[&\int_{M_{11}}\sqrt{G}R + \int_{\partial M_{11}}\sqrt{h}K \\ &- \frac{1}{2}\int_{M_{11}}\sqrt{G}G_4^2 - \frac{1}{6}\int_{M_{11}} C_3 \wedge G_4 \wedge G_4 \\ &+ \frac{(4\pi\kappa_{11}^2)^{\frac{2}{3}}}{(2\pi)^4 3^2.2^{13}}\left(\int_{\mathcal{M}} d^{11}x\sqrt{G^{\mathcal{M}}}\left(J_0 - \frac{1}{2}E_8\right)\right. \\ &\left.+ 3^2.2^{13}\int C_3 \wedge X_8 + \int t_8 t_8 G^2 R^3 + \cdot\right)\Bigg] - \mathcal{S}^{\text{ct}}, \end{aligned} \tag{14}$$

where:

$$
\begin{aligned}
(15) \qquad J_0 &= 3\cdot 2^8 (R^{HMNK} R_{PMNQ} R_H{}^{RSP} R^Q{}_{RSK} \\
&\quad + \frac{1}{2} R^{HKMN} R_{PQMN} R_H{}^{RSP} R^Q{}_{RSK}) \\
E_8 &= \frac{1}{3!} \epsilon^{ABCM_1N_1\ldots M_4N_4} \epsilon_{ABCM_1'N_1'\ldots M_4'N_4'} R^{M_1'N_1'}{}_{M_1N_1} \ldots R^{M_4'N_4'}{}_{M_4N_4}, \\
t_8 t_8 G^2 R^3 &= t_8^{M_1\ldots M_8} t^8_{N_1\ldots . N_8} G_{M_1}{}^{N_1PQ} G_{M_2}{}^{N_2}{}_{PQ} R_{M_3M_4}{}^{N_3N_4} R_{M_5M_6}{}^{N_5N_6} R_{M_7M_8}{}^{N_7N_8}; \\
\kappa_{11}^2 &= \frac{(2\pi)^8 l_p^9}{2};
\end{aligned}
$$

κ_{11}^2 being related to the eleven-dimensional Newtonian coupling constant, and $G = dC$ with C being the $\mathcal{M}$-theory three-form potential with the four-form G being the associated four-form field strength.

In the spirit of completion of the 1-loop $\mathcal{O}(R^4)$ in the presence of NS-NS B in type IIA compatible with T duality, and hence defining the torsionful spin connection, $\Omega_\pm \equiv \Omega \pm \frac{1}{2}\mathcal{H}, \mathcal{H}^{ab} = \mathcal{H}^{ab}_\mu dx^\mu$, and $\overline{X_8} \equiv \frac{X_8(R(\Omega_+)) + X_8(R(\Omega_-))}{2}$, where $R(\Omega_+) = R(\Omega) + \frac{1}{2} d\mathcal{H} + \frac{1}{4}\mathcal{H}\wedge\mathcal{H}$, the ten dimensional $\overline{X}_8$ shifts by an exact form [9] [5]:

$$
\begin{aligned}
(16) \quad \overline{X}_8 = \frac{1}{192(2\pi)^4} \Bigg[& \left(\mathrm{tr}R^4 - \frac{1}{4}(\mathrm{tr}R^2)^2 \right) \\
& + d \left(\frac{1}{2} \mathrm{tr}\left(\mathcal{H}\nabla\mathcal{H}R^2 + \mathcal{H}R\nabla\mathcal{H}R + \mathcal{H}R^2\nabla\mathcal{H} \right) \right. \\
& \quad - \frac{1}{8}\left(\mathrm{tr}R^2\,\mathrm{tr}\mathcal{H}\nabla\mathcal{H} + 2\mathrm{tr}\mathcal{H}R\,\mathrm{tr}R\nabla\mathcal{H} \right) \\
& \frac{1}{16}\mathrm{tr}\left(2\mathcal{H}^3(\nabla\mathcal{H}R + R\nabla\mathcal{H}) + \mathcal{H}R\mathcal{H}^2\nabla\mathcal{H} + \mathcal{H}\nabla\mathcal{H}\mathcal{H}^2 R \right) \\
& - \frac{1}{2}\left(\mathrm{tr}\mathcal{H}\nabla\mathcal{H}\,\mathrm{tr}R\mathcal{H}^2 + \mathrm{tr}R\nabla\mathcal{H}\,\mathrm{tr}\mathcal{H}^3 + \mathrm{tr}\nabla\mathcal{H}\mathcal{H}^2\,\mathrm{tr}\mathcal{H}R \right) \\
& \quad + \frac{1}{32}\mathrm{tr}\nabla\mathcal{H}\mathcal{H}^5 - \frac{1}{192}\mathrm{tr}\nabla\mathcal{H}\mathcal{H}^2\,\mathrm{tr}\mathcal{H}^3 \\
& \quad \left. + \frac{1}{16}\mathrm{tr}\mathcal{H}(\nabla\mathcal{H})^3 - \frac{1}{64}\mathrm{tr}\mathcal{H}\nabla\mathcal{H}\,\mathrm{tr}(\nabla\mathcal{H})^2 \right) \Bigg].
\end{aligned}
$$

[5] To be consistent with the notation of the rest of the paper, we have dropped the ˆ over eleven-dimensional objects in (17); when wedged with C it will be understood that the objects like the metric, curvature, etc. are eleven-dimensional and when wedged with B, ten dimensional.

Defining the $O(1,10)$-valued one-form $\mathcal{G}^{abc} \equiv 4G_{\mu\nu\rho\lambda}dx^\mu e^{a\nu}e^{b\rho}e^{c\lambda}$, the $\mathcal{M}$-theory uplift of the first two lines of (16) of type IIA, yields [9] [6]:

$$
\begin{aligned}
(17)\quad B_2 \wedge \overline{X}_8 \longrightarrow \frac{1}{192(2\pi)^4}\Big[&C \wedge \left(\mathrm{tr}R^4 - \frac{1}{4}(\mathrm{tr}R^2)^2\right) \\
&+ G \wedge \Big(\frac{1}{4}\left(R^{ab}R^{bc}\mathcal{G}^{cde}\nabla\mathcal{G}^{dae} + 2R^{ab}\mathcal{G}^{bce}R^{cd}\nabla\mathcal{G}^{dae} + R^{ab}R^{bc}\nabla\mathcal{G}^{cde}\mathcal{G}^{dae}\right) \\
&- \frac{1}{24}\left(\mathrm{tr}R^2 \wedge \mathcal{G}^{abe}\nabla\mathcal{G}^{bae} + 6R^{ab}\mathcal{G}^{bae}R^{cd}\mathcal{G}^{dce}\right) + \cdots\Big)\Big].
\end{aligned}
$$

In this paper, we restrict ourselves only to the first line in (17). Given that the same was shown to vanish [1], perhaps to be T-duality invariant, the sum of the terms in the second and third lines of (17) too yield zero. We have not proven the same.

The action in (14) is holographically renormalizable by construction of appropriate counter terms $\mathcal{S}^{\mathrm{ct}}$. This is seen as follows. It can be shown [5] that the bulk on-shell $D=11$ supergravity action inclusive of $\mathcal{O}(R^4)$-corrections is given by:

$$
\begin{aligned}
(18)\quad S^{\mathrm{on-shell}}_{D=11} = -\frac{1}{2}\Bigg[&-2S^{(0)}_{\mathrm{EH}} + 2S^{(0)}_{\mathrm{GHY}} + \beta\Bigg(\frac{20}{11}S_{\mathrm{EH}} - 2\int_{M_{11}}\sqrt{-g^{(1)}}R^{(0)} \\
&+ 2S_{\mathrm{GHY}} - \frac{2}{11}\int_{M_{11}}\sqrt{-g^{(0)}}g^{MN}_{(0)}\frac{\delta J_0}{\delta g^{MN}_{(0)}}\Bigg)\Bigg].
\end{aligned}
$$

The UV divergences of the various terms in (18) are summarized below:

$$
\begin{aligned}
&\int_{M_{11}}\sqrt{-g}R\Big|_{\mathrm{UV-divergent}},\ \int_{\partial M_{11}}\sqrt{-h}K\Big|_{\mathrm{UV-divergent}} \sim r^4_{\mathrm{UV}}\log r_{\mathrm{UV}}, \\
(19)\quad &\int_{M_{11}}\sqrt{-g}g^{MN}\frac{\delta J_0}{\delta g^{MN}}\Big|_{\mathrm{UV-divergent}} \sim \frac{r^4_{\mathrm{UV}}}{\log r_{\mathrm{UV}}}.
\end{aligned}
$$

[6]Strictly speaking, (17) is valid when M_{11} is a trivial S^1 fibration over an M_{10} and $G_{\mu\nu\rho x^{10}} \neq 0, G_{\mu\nu\rho\lambda} = 0$. We, near the $\psi = 2n\pi$-coordinate patches, have M_{11} as a warped product of the $\mathcal{M}$-theory circle and M_{10}, which for a delocalized (IR-valued in this paper) value of r can be thought of as a trivial circle fibration. The $G_{\mu\nu\rho\lambda}$ arising from $A^{\mathrm{IIA}} \wedge H^{\mathrm{IIA}}$, via $\int G \wedge *G$, results in a UV-divergent contribution which is canceled off by an appropriate boundary flux term [51].

It can be shown [5] that an appropriate linear combination of the boundary terms: $\int_{\partial M_{11}} \sqrt{-h} K \Big|_{r=r_{\rm UV}}$ and $\int_{\partial M_{11}} \sqrt{-h} h^{mn} \frac{\partial J_0}{\partial h^{mn}} \Big|_{r=r_{\rm UV}}$ serves as the appropriate counter terms to cancel the UV divergences (19) [7].

The EOMS are:

$$
\begin{aligned}
& R_{MN} - \frac{1}{2} g_{MN} \mathcal{R} - \frac{1}{12} \left(G_{MPQR} G_N^{\ PQR} - \frac{g_{MN}}{8} G_{PQRS} G^{PQRS} \right) \\
& \quad = -\beta \left[\frac{g_{MN}}{2} \left(J_0 - \frac{1}{2} E_8 \right) + \frac{\delta}{\delta g^{MN}} \left(J_0 - \frac{1}{2} E_8 \right) \right], \\
& d * G = \frac{1}{2} G \wedge G + 3^2 2^{13} (2\pi)^4 \beta X_8,
\end{aligned} \tag{20}
$$

where [38]:

$$
\beta \equiv \frac{\left(2\pi^2\right)^{\frac{1}{3}} \left(\kappa_{11}^2\right)^{\frac{2}{3}}}{(2\pi)^4 3^2 2^{12}} \sim l_p^6, \tag{21}
$$

$R_{MNPQ}, R_{MN}, \mathcal{R}$ in (14)/(20) being respectively the elven-dimensional Riemann curvature tensor, Ricci tensor and the Ricci scalar.

Now, one sees that if one makes an ansatz:

$$
\begin{aligned}
g_{MN} &= g^{(0)}_{MN} + \beta g^{(1)}_{MN}, \\
C_{MNP} &= C^{(0)}_{MNP} + \beta C^{(1)}_{MNP},
\end{aligned} \tag{22}
$$

then symbolically, one obtains:

$$
\begin{aligned}
& \beta \partial \left(\sqrt{-g} \partial C^{(1)} \right) + \beta \partial \left[\left(\sqrt{-g} \right)^{(1)} \partial C^{(0)} \right] + \beta \epsilon_{11} \partial C^{(0)} \partial C^{(1)} \\
& \qquad = \mathcal{O}(\beta^2) \sim 0 [\text{up to } \mathcal{O}(\beta)].
\end{aligned} \tag{23}
$$

One can see that one can find a consistent set of solutions to (23) wherein $C^{(1)}_{MNP} = 0$ up to $\mathcal{O}(\beta)$. This will be shown after (44). Assuming that one

[7] For consistency, one needs to impose the following relationship between the UV-valued effective number of flavor $D7$-branes of the parent type IIB dual, $N_f^{\rm UV}$ and $\log r_{\rm UV}$: $N_f^{\rm UV} = \frac{(\log r_{\rm UV})^{\frac{15}{2}}}{\log N}$.

can do so, henceforth we will define:

$$\delta g_{MN} = \beta g^{(1)}_{MN} = G^{\rm MQGP}_{MN} f_{MN}(r), \tag{24}$$

no summation implied. The first equation in (20) will be denoted by EOM_{MN} in Appendix A. Appendix A has all the EOMs listed. The discussion in the same is divided into two sub-sections: the EOMs and their solutions for f_{MN}s are worked out for the $\psi = 0, 2\pi, 4\pi$-branches in **A.1** and for $\psi = \psi_0 \neq 2n\pi, n = 0, 1, 2$ in **A.2**.

One can show:

$$\begin{aligned}
\delta J_0 \xrightarrow{\text{MQGP, IR}} & \, 3 \times 2^8 \delta R^{HMNK} R_H^{\ \ RSP} \Bigg(R_{PQNK} R^Q_{\ \ RSM} + R_{PSQK} R^Q_{\ \ MNR} \\
& + 2 \left[R_{PMNQ} R^Q_{\ \ RSK} + R_{PNMQ} R^Q_{\ \ SRK} \right] \Bigg) \\
& \equiv 3 \times 2^8 \delta R^{HMNK} \chi_{HMNK} \\
= -\delta g_{\tilde{M}\tilde{N}} \Bigg[& g^{M\tilde{N}} R^{H\tilde{N}NK} \chi_{HMNK} + g^{N\tilde{N}} R^{HM\tilde{M}K} \chi_{HMNK} \\
& + g^{K\tilde{M}} R^{HMN\tilde{N}} \chi_{HMNK} + \frac{1}{2} \Bigg(g^{H\tilde{N}} [D_{K_1}, D_{N_1}] \chi_H^{\ \tilde{M}N_1K_1} \\
& + g^{H\tilde{N}} D_{M_1} D_{N_1} \chi_H^{\ M_1[\tilde{N}_1\tilde{M}]} - g^{H\tilde{H}} D_{\tilde{H}} D_{N_1} \chi_H^{\ \tilde{N}[N_1\tilde{M}]} \Bigg) \Bigg],
\end{aligned} \tag{25}$$

where:

$$\begin{aligned}
\chi_{HMNK} \equiv R_H^{\ \ RSP} \Big[& R_{PQNK} R^Q_{\ \ RSM} + R_{PSQK} R^Q_{\ \ MNR} \\
& + 2 \left(R_{PMNQ} R^Q_{\ \ RSK} + R_{PNMQ} R^Q_{\ \ SRK} \right) \Big].
\end{aligned} \tag{26}$$

Further:

$$
\begin{aligned}
(27)\quad \delta E_8 \sim & -\frac{2}{3}\delta g_{\tilde{M}\tilde{N}} g^{N_1'\tilde{N}} \epsilon^{ABCM_1N_1,,,M_4N_4}\epsilon_{ABCM_1'N_1'...M_4'N_4'} \\
& \times R^{M_1'\tilde{M}}{}_{M_1N_1} R^{M_2'N_2'}{}_{M_2N_2} R^{M_3'N_3'}{}_{M_2N_2} R^{M_4'N_4'}{}_{M_4N_4} \\
& + \frac{\delta g_{\tilde{M}\tilde{N}}}{3}\Bigg[2\epsilon^{ABCM_1\tilde{N},,,M_4N_4}\epsilon_{ABCM_1'N_1'...M_4'N_4'} g^{N_1'\tilde{N}_1} g^{M_1'\tilde{M}} \\
& \times D_{\tilde{N}_1} D_{M_1}\left(R^{M_2'N_2'}{}_{M_2N_2} R^{M_3'N_3'}{}_{M_2N_2} R^{M_4'N_4'}{}_{M_4N_4}\right) \\
& + \epsilon^{ABCM_1N_1,,,M_4N_4}\epsilon_{ABCM_1'N_1'...M_4'N_4'} g^{N_1'\tilde{N}_1} g^{M_1'\tilde{M}} \\
& \times [D_{\tilde{N}_1}, D_{M_1}]\left(R^{M_2'N_2'}{}_{M_2N_2} R^{M_3'N_3'}_{M_2N_2} R^{M_4'N_4'}{}_{M_4N_4}\right) \\
& - 2\epsilon^{ABCM_1\tilde{M},,,M_4N_4}\epsilon_{ABCM_1'N_1'...M_4'N_4'} g^{N_1'\tilde{N}} g^{M_1'\tilde{L}} \\
& \times D_{\tilde{L}_1} D_{M_1}\left(R^{M_2'N_2'}{}_{M_2N_2} R^{M_3'N_3'}{}_{M_2N_2} R^{M_4'N_4'}{}_{M_4N_4}\right)\Bigg],
\end{aligned}
$$

where, e.g., [39]

$$
\begin{aligned}
(28)\quad & \epsilon^{ABCM_1M_2,,,M_8}\epsilon_{ABCM_1'M_2'...M_8'} R^{M_1'M_2'}{}_{M_1M_2} R^{M_3'M_4'}{}_{M_3M_4} R^{M_5'M_6'}_{M_5M_6} R^{M_7'M_8'}{}_{M_7M_8} \\
& = -3!8!\delta^{M_1}_{N_1]}...\delta^{M_8}_{M_8']} R^{M_1'M_2'}{}_{M_1M_2} R^{M_3'M_4'}{}_{M_3M_4} R^{M_5'M_6'}_{M_5M_6} R^{M_7'M_8'}{}_{M_7M_8}.
\end{aligned}
$$

Writing: $T_{MN} \equiv G_M{}^{PQR}G_{NPQR} - \frac{g_{MN}}{8}G^2$, the $\mathcal{O}(l_p^6)$ "perturbations" $T^{(1)}_{MN}$ therein will be given by:

$$
(29)\qquad T^{(1)}_{MN} = \mathcal{T}^{(1)}_{MN} + \mathcal{T}^{(2)}_{MN} - \frac{g_{MN}}{2}\delta g^{PP'}\mathcal{T}^{(3)}_{PP'},
$$

where:

$$
\begin{aligned}
& \mathcal{T}^{(1)}_{MN} \equiv 3\delta g^{PP'} g^{QQ'} G_{MPQR}G_{NP'Q'R'} \equiv \delta g^{PP'}\mathcal{C}_{MNPP'}, \\
& \mathcal{T}^{(2)}_{MN} \equiv -\frac{\delta g_{MN}}{8} G_{PQRS}G^{PQRS} \equiv -\frac{\delta g_{MN}}{8}G^2, \\
(30)\quad & \mathcal{T}^{(3)}_{PQ} \equiv g^{QQ'}g^{RR'}g^{SS'}G_{PQRS}G_{P'Q'R'S'}.
\end{aligned}
$$

In the IR (i.e. small-r limit), the various EOMs, denoted by EOM_{MN} henceforth, corresponding to perturbation of the first equation of (20) up to $\mathcal{O}(\beta)$, and their solutions, have been obtained in appendix **A**: near the $\psi = 2n\pi, n = 0, 1, 2$-patches in **A.1**, and near the $\psi \neq 2n\pi, n = 0, 1, 2$-patch (wherein, unlike $\psi = 2n\pi, n = 0, 1, 2$-patches, some $G^{\mathcal{M}}_{rM}, M \neq r$ and

$G^{\mathcal{M}}_{x^{10}N}, N \neq x^{10}$ components are non-zero) in **A.2**. The EOMs are obtained by expanding the coefficients of $f^{(n)}_{MN}, n = 0, 1, 2$ near $r = r_h$ and retaining the LO terms in the powers of $(r - r_h)$ in the same, and then performing a large-N-large-$|\log r_h|$-$\log N$ expansion. It is shown in **A.1** that near the $\psi = 2n\pi, n = 0, 1, 2$-patches the EOMs reduce to fifteen independent EOMs and four consistency checks, and in **A.2** in $\psi = \psi_0$ (arbitrary but different from $2n\pi, n = 0, 1, 2$)-branches to seven independent EOMs and one consistency check equation. In the following pair of subsections - **3.1** and **3.2** - we present the final results for the $\mathcal{M}$-theory metric components up to $\mathcal{O}(\beta)$.

3.1. Near $\psi = 2n\pi, n = 0, 1, 2$-coordinate patches and near $r = r_h$

In this sub-section, we will obtain the EOMs and their solutions, in the IR, near the $\psi = 0, 2\pi, 4\pi$-coordinate patches for the $\mathcal{M}$ theory black hole solution dual to thermal QCD-like theories at high temperature:

$$
(31) \quad ds^2_{11} = e^{-\frac{2\phi^{\rm IIA}}{3}}\left[\frac{1}{\sqrt{h(r,\theta_{1,2})}}\left(-g(r)dt^2 + \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2\right) + \sqrt{h(r,\theta_{1,2})}\left(\frac{dr^2}{g(r)} + ds^2_{\rm IIA}(r,\theta_{1,2},\phi_{1,2},\psi)\right)\right] + e^{\frac{4\phi^{\rm IIA}}{3}}\left(dx^{11} + A^{F_1^{\rm IIB}+F_3^{\rm IIB}+F_5^{\rm IIB}}_{\rm IIA}\right)^2,
$$

where $A^{F^{\rm IIB}_{i=1,3,5}}_{\rm IIA}$ are the type IIA RR 1-forms obtained from the triple T/SYZ-dual of the type IIB $F^{\rm IIB}_{1,3,5}$ fluxes in the type IIB holographic dual of [2], and $g(r) = 1 - \frac{r_h^4}{r^4}$. For simplicity, we will be restricting to the Ouyang embedding:

$$
(32) \qquad \left(r^6 + 9a^2r^4\right)^{\frac{1}{4}} e^{\frac{i}{2}(\psi-\phi_1-\phi_2)} \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} = \mu,
$$

μ being the Ouyang embedding parameter assuming $|\mu| \ll r^{\frac{3}{2}}$, effected, e.g., by working in the neighborhood of:

$$
(33) \qquad \theta_1 = \frac{\alpha_{\theta_1}}{N^{\frac{1}{5}}}, \theta_2 = \frac{\alpha_{\theta_2}}{N^{\frac{3}{10}}}; \alpha_{\theta_{1,2}} \equiv \mathcal{O}(1)
$$

(wherein an explicit $SU(3)$-structure for the type IIB dual of [2] and its delocalized SYZ type IIA mirror [1], and an explicit G_2-structure for its

$\mathcal{M}$-Theory uplift [1] was worked out in [16]). Note, using (C31) - (C33) and arguments similar to the ones given in [11], one can show that our results are independent of any delocalization in $\theta_{1,2}$. The EOMs, corresponding to the $\mathcal{O}(l_p^6)$ variation of (20) (via the substitution of (22) into (20)) will be labelled as EOM_{MN} in this section and appendix **A**.
This brings us to the first main lemma of this paper:
Lemma 1: In the neighborhood of the Ouyang embedding of flavor $D7$-branes [20] (that figure in the type IIB string dual of thermal QCD-like theories at high temperatures[2]) effected by working in the neighborhood of small $\theta_{1,2}$ (assuming a vanishingly small Ouyang embedding parameter), in the MQGP limit (1), $\lim_{N\to\infty}\frac{E_8}{J_0}=0, \lim_{N\to\infty}\frac{t_8t_8G^2R^3}{E_8}=0$.

Proof:

- One can show that the leading-order-in-N contribution to J_0 is given by:

$$J_0=\frac{1}{2}R^{\phi_2r\theta_1r}R_{r\psi\theta_1r}R_{\phi_2}{}^{r\phi_1r}R^{\psi}_{r\phi_1r}-R^{\phi_2r\theta_1r}R_{\phi_1r\theta_1r}R_{\phi_2}{}^{r\phi_1r}R^{\theta_1}{}_{r\theta_1r},\tag{34}$$

where, e.g., near (33),

$$\begin{aligned}
&R^{\phi_2r\theta_1r}\sim\frac{\left(\frac{1}{N}\right)^{5/4}(9a^2+r^2)(r^4-{r_h}^4)\alpha_{\theta_1}^3\sum_{n_1,n_2,n_3:n_1+n_2+n_3=6}a^{2n_1}r^{2n_2}r_h^{2n_3}}{{g_s}^{5/4}{N_f}^2r^4(r^2-3a^2)^3(6a^2+r^2)^3\log^3(N)\alpha_{\theta_2}^2}\\
&R_{\phi_1r\theta_1r}\sim-\frac{{g_s}^{7/4}MN^{11/20}{N_f}^{5/3}\log^{\frac{5}{3}}(N)\log(r)\sum_{n_1,n_2,n_3:n_1+n_2+n_3=6}a^{2n_1}r^{2n_2}r_h^{2n_3}}{r^4(r^2-3a^2)(6a^2+r^2)(9a^2+r^2)(r^4-{r_h}^4)\alpha_{\theta_1}\alpha_{\theta_2}^2}\\
&R_{r\psi\theta_1r}\sim-\frac{{g_s}^{7/4}MN^{3/20}{N_f}^{5/3}\log^{\frac{5}{3}}(N)\log(r)\alpha_{\theta_1}\sum_{n_1,n_2,n_3:n_1+n_2+n_3=6}a^{2n_1}r^{2n_2}r_h^{2n_3}}{r^4(r^2-3a^2)(6a^2+r^2)(9a^2+r^2)(r^4-{r_h}^4)\alpha_{\theta_2}^2}\\
&R_{\phi_2}{}^{r\phi_1r}\sim-\frac{a^4\left(\frac{1}{N}\right)^{21/20}\left(9a^2+r^2\right)^2\left(\frac{1}{\log(N)}\right)^{4/3}\left(r^4-{r_h}^4\right)^2\alpha_{\theta_2}\Sigma_1}{g_s{N_f}^{4/3}r^6\left(r^2-3a^2\right)^2\left(6a^2+r^2\right)^2\alpha_{\theta_1}^2}\\
&R^{\theta_1}{}_{r\theta_1r}\sim\frac{\sum_{n_1,n_2,n_3:n_1+n_2+n_3=6}a^{2n_1}r^{2n_2}r_h^{2n_3}}{r^2\left(r^2-3a^2\right)^2\left(6a^2+r^2\right)\left(9a^2+r^2\right)\left(r^4-{r_h}^4\right)},\\
&R^{\psi}{}_{r\phi_1r}\sim-\frac{N^{2/5}\sum_{n_1,n_2,n_3:n_1+n_2+n_3=6}a^{2n_1}r^{2n_2}r_h^{2n_3}}{r^2\left(r^2-3a^2\right)^2\left(6a^2+r^2\right)\left(9a^2+r^2\right)\left(r^4-{r_h}^4\right)\alpha_{\theta_1}^2},
\end{aligned}\tag{35}$$

where Σ_1 is defined in (A2), and $\sum_{n_1,n_2,n_3:n_1+n_2+n_3=6}a^{2n_1}r^{2n_2}r_h^{2n_3}=-81a^8r^4+243a^8{r_h}^4+27a^6r^6-36a^6r^2{r_h}^4+15a^4r^8-27a^4r^4{r_h}^4+a^2r^{10}-2a^2r^6{r_h}^4$. In (35) and henceforth, $r/a/r_h$ in fact would imply $\frac{r/a/r_h}{\mathcal{R}_{D5/\overline{D5}}}$ (see the last reference in [15]). Substituting (35) into (34), one therefore obtains:

(36)
$$J_0 \sim \frac{a^{10}\left(\frac{1}{\log N}\right)^{8/3} M\left(\frac{1}{N}\right)^{7/4}(9a^2+r^2)(r^4-{r_h}^4)\log(r)\Sigma_1 \sum_{n_1,n_2,n_3:n_1+n_2+n_3=6} a^{2n_1} r^{2n_2} r_h^{2n_3}}{\sqrt{g_s}{N_f}^{5/3} r^{16}(3a^2-r^2)^8(6a^2+r^2)^7 \alpha_{\theta_2}^3} \sim \left(\frac{1}{N}\right)^{7/4}.$$

For an arbitrary but small $\theta_{1,2}$, one can show that:

(37)
$$J_0 \sim \frac{a^{10}\left(\frac{1}{\log N}\right)^{8/3} M\left(\frac{1}{N}\right)^{29/20}(9a^2+r^2)(r^4-{r_h}^4)\log(r)\left(19683\sqrt{6}\sin^6\theta_1+6642\sin^2\theta_2\sin^3\theta_1-40\sqrt{6}\sin^4\theta_2\right)}{\sqrt{g_s}{N_f}^{5/3} r^{16}(3a^2-r^2)^8(6a^2+r^2)^7\sin^3\theta_2} \times \sum_{n_1,n_2,n_3:n_1+n_2+n_3=6} a^{2n_1} r^{2n_2} r_h^{2n_3}.$$

- For evaluating the contribution of E_8, (28), one notes that one needs to pick out eight of the eleven space-time indices (and anti-symmetrize appropriately). Let us consider $R^{M_1N_1}{}_{M_1N_1} R^{M_2N_2}{}_{M_2N_2} R^{M_3N_3}{}_{M_3N_3} R^{M_4N_4}{}_{M_4N_4}$ which will be one of the kinds of terms one will obtain using (28). After a very long and careful computation, one can then show that for arbitrary small $\theta_{1,2}$ and not just restricted to (33), the above contributes a $\frac{1}{N^2}$ via the following most dominant term in the MQGP limit:

(38)
$$E_8 \ni R^{tx^1}{}_{tx^1} R^{x^2x^3}{}_{x^2x^3} R^{r\theta_1}{}_{r\theta_1}\left(R^{\psi x^{10}}{}_{\psi x^{10}} + R^{\phi_1\psi}{}_{\phi_1\psi} + R^{yz}{}_{\phi_2\psi}\right)\left(\sim \mathcal{O}\left(\frac{1}{N^2}\right)\right).$$

  A similar computation for the other types of summands in (28) yield a similar N dependence. Consequently, $\frac{E_8}{J_0} \sim \frac{1}{N^\alpha}, \alpha > 0$.

- Summing first w.r.t. $M_{3,4}, N_{3,4}$ in (15), one obtains

(39)
$$\frac{\chi_1(r,\langle\theta_{1,2}\rangle)}{N^{31/20}} G_{M_1}^{\ \ N_1MN} G_{M_2}^{\ \ N_2}{}_{MN} R_{M_5M_6}^{\ \ N_5N_6} R_{M_7M_8}^{\ \ N_7N_8} t_{N_1N_2x^0\theta_2N_5N_6N_7N_8} t^{M_1M_2x^0\phi_2M_5M_6M_7M_8}$$

  as the LO term in N. Summing w.r.t. $M_{5,6}, N_{5,6}$, one obtains

(40)
$$\frac{\chi_2(r;\langle\theta_{1,2}\rangle)}{N^{31/10}} G_{M_1}^{N_1MN} G_{M_2}^{\ \ N_2}{}_{MN} R_{M_7M_8}^{\ \ N_7N_8} t_{N_1N_2x^0\theta_2N_7N_8} t^{M_1M_2x^0\phi_2M_7M_8}$$

up to LO in N. Finally, summing w.r.t. $M_{7,8}, N_{7,8}$ one obtains:

$$\frac{\chi_3(r, \langle\theta_{1,2}\rangle)}{N^{93/20}} G_{M_1}^{\ \ N_1 M N} G_{M_2}^{\ \ N_2}{}_{MN} t_{N_1 N_2 x^0 \theta_2 x^0 \theta_2 x^0 \theta_2} t^{M_1 M_2 x^0 \phi_2 x^0 \phi_2 x^0 \phi_2}, \tag{41}$$

and restricted to (32)∩(33) one obtains:

$$t_8 t_8 G^2 R^3\big|_{(32)\cap(33)} \sim \frac{\chi_4(r; \langle\theta_{1,2}\rangle}{N^{111/20}}, \tag{42}$$

which is large-N suppressed as compared to the J_0 and E_8.

Hence, (36)-(37) along with (42), one proves Lemma 1, and obtains the following hierarchy:

$$t_8^2 G^2 R^3 < E_8 < J_0. \tag{43}$$

Henceforth, $E_8, t_8^2 G^2 R^3$ (and their variations) will be disregarded as compared to J_0 (and its variation) in the MQGP limit.

We now come to the second lemma of this paper:

Lemma 2: The $\mathcal{O}(\beta)$-corrected $\mathcal{M}$-theory metric of [1] in the MQGP limit near the $\psi = 2n\pi, n = 0, 1, 2$-branches up to $\mathcal{O}((r - r_h)^2)$ [and up to $\mathcal{O}((r - r_h)^3)$ for some of the off-diagonal components along the delocalized $T^3(x, y, z)$] - the components which do not receive an $\mathcal{O}(\beta)$ corrections, are not listed in (44) - is given below:

$$G_{tt} = G_{tt}^{\rm MQGP}\left[1 + \frac{1}{4}\frac{4b^8\left(9b^2+1\right)^3\left(4374b^6+1035b^4+9b^2-4\right)\beta M\left(\frac{1}{N}\right)^{9/4}\Sigma_1\left(6a^2+{r_h}^2\right)\log(r_h)}{27\pi\left(18b^4-3b^2-1\right)^5\log N^2 N_f {r_h}^2\alpha_{\theta_2}^3\left(9a^2+{r_h}^2\right)}(r-r_h)^2\right]$$

$$G_{x^{1,2,3}x^{1,2,3}} = G_{x^{1,2,3}x^{1,2,3}}^{\rm MQGP}\left[1 - \frac{1}{4}\frac{4b^8\left(9b^2+1\right)^4\left(39b^2-4\right)M\left(\frac{1}{N}\right)^{9/4}\beta\left(6a^2+{r_h}^2\right)\log(r_h)\Sigma_1}{9\pi\left(3b^2-1\right)^5\left(6b^2+1\right)^4\log N^2 N_f {r_h}^2\left(9a^2+{r_h}^2\right)\alpha_{\theta_2}^3}(r-r_h)^2\right]$$

$$G_{rr} = G_{rr}^{\rm MQGP}\left[1 + \left(-\frac{2\left(9b^2+1\right)^4 b^{10} M\left(6a^2+{r_h}^2\right)\left((r-r_h)^2+{r_h}^2\right)\Sigma_1}{3\pi\left(-18b^4+3b^2+1\right)^4\log N N^{8/15} N_f\left(-27a^4+6a^2{r_h}^2+{r_h}^4\right)\alpha_{\theta_2}^3}\right.\right.$$
$$\left.\left. + {C_{zz}}^{(1)} - 2{C_{\theta_1 z}}^{(1)} + 2{C_{\theta_1 x}}^{(1)}\right)\beta\right]$$

$$G_{\theta_1 x} = G_{\theta_1 x}^{\rm MQGP}\left[1 + \left(-\frac{\left(9b^2+1\right)^4 b^{10} M\left(6a^2+{r_h}^2\right)\left((r-r_h)^2+{r_h}^2\right)\Sigma_1}{3\pi\left(-18b^4+3b^2+1\right)^4\log N N^{8/15} N_f\left(-27a^4+6a^2{r_h}^2+{r_h}^4\right)\alpha_{\theta_2}^3} + {C_{\theta_1 x}}^{(1)}\right)\beta\right]$$

$$G_{\theta_1 z} = G_{\theta_1 z}^{\rm MQGP}\left[1 + \left(\frac{16\left(9b^2+1\right)^4 b^{12}\left(\frac{(r-r_h)^3}{{r_h}^3}+1\right)\Sigma_1}{243\sqrt{2}\pi^3\left(1-3b^2\right)^{10}\left(6b^2+1\right)^8 {g_s}^{9/4}\log N^4 N^{7/6} {N_f}^3\left(-27a^4 r_h+6a^2{r_h}^3+{r_h}^5\right)\alpha_{\theta_1}^7\alpha_{\theta_2}^6} + C_{\theta_1 z}^{(1)}\right)\beta\right]$$

$$
(44)
$$

$$
\begin{aligned}
G_{\theta_2 x} &= G^{\rm MQGP}_{\theta_2 x}\left[1+\left(\frac{16\left(9b^2+1\right)^4 b^{12}\left(\frac{(r-r_h)^3}{{r_h}^3}+1\right)\Sigma_1}{243\sqrt{2}\pi^3\left(1-3b^2\right)^{10}\left(6b^2+1\right)^8{g_s}^{9/4}\log N^4 N^{7/6}{N_f}^3\left(-27a^4 r_h+6a^2{r_h}^3+{r_h}^5\right)\alpha_{\theta_1}^7\alpha_{\theta_2}^6}+C^{((1)}_{\theta_2 x}\right)\beta\right]\\
G_{\theta_2 y} &= G^{\rm MQGP}_{\theta_2 y}\left[1+\frac{3b^{10}\left(9b^2+1\right)^4 M\beta\left(6a^2+{r_h}^2\right)\left(1-\frac{(r-r_h)^2}{{r_h}^2}\right)\log(r_h)\Sigma_1}{\pi\left(3b^2-1\right)^5\left(6b^2+1\right)^4\log N^2 N^{7/5}N_f\left(9a^2+{r_h}^2\right)\alpha_{\theta_2}^3}\right]\\
G_{\theta_2 z} &= G^{\rm MQGP}_{\theta_2 z}\left[1+\left(\frac{3\left(9b^2+1\right)^4 b^{10}M\left(6a^2+{r_h}^2\right)\left(1-\frac{(r-r_h)^2}{{r_h}^2}\right)\log(r_h)\Sigma_1}{\pi\left(3b^2-1\right)^5\left(6b^2+1\right)^4\log N^2 N^{7/6}N_f\left(9a^2+{r_h}^2\right)\alpha_{\theta_2}^3}+{C_{\theta_2 z}}^{(1)}\right)\beta\right]\\
G_{xy} &= G^{\rm MQGP}_{xy}\left[1+\left(\frac{3\left(9b^2+1\right)^4 b^{10}M\left(6a^2+{r_h}^2\right)\left(\frac{(r-r_h)^2}{{r_h}^2}+1\right)\log(r_h)\alpha_{\theta_2}^3\Sigma_1}{\pi\left(3b^2-1\right)^5\left(6b^2+1\right)^4\log N^2 N^{21/20}N_f\left(9a^2+{r_h}^2\right)\alpha_{\theta_{2l}}^6}+C^{(1)}_{xy}\right)\beta\right]\\
G_{xz} &= G^{\rm MQGP}_{xz}\left[1+\frac{18b^{10}\left(9b^2+1\right)^4 M\left(6a^2+{r_h}^2\right)\left(\frac{(r-r_h)^2}{{r_h}^2}+1\right)\log^3(r_h)\Sigma_1}{\pi\left(3b^2-1\right)^5\left(6b^2+1\right)^4\log N^4 N^{5/4}N_f\left(9a^2+{r_h}^2\right)\alpha_{\theta_2}^3}\right]\\
G_{yy} &= G^{\rm MQGP}_{yy}\left[1-\frac{3b^{10}\left(9b^2+1\right)^4 M\left(\frac{1}{N}\right)^{7/4}\left(6a^2+{r_h}^2\right)\log(r_h)\Sigma_1\left(\frac{(r-r_h)^2}{r_h^2}+1\right)}{\pi\left(3b^2-1\right)^5\left(6b^2+1\right)^4\log N^2 N_f{r_h}^2\left(9a^2+{r_h}^2\right)\alpha_{\theta_2}^3}\right]\\
G_{yz} &= G^{\rm MQGP}_{yz}\left[1+\left(\frac{64\left(9b^2+1\right)^8 b^{22}M\left(\frac{1}{N}\right)^{29/12}\left(6a^2+{r_h}^2\right)\left(\frac{(r-r_h)^3}{{r_h}^3}+1\right)\log(r_h)\Sigma_3}{27\pi^4\left(3b^2-1\right)^{15}\left(6b^2+1\right)^{12}{g_s}^{9/4}\log N^6{N_f}^4{r_h}^3\left({r_h}^2-3a^2\right)\left(9a^2+{r_h}^2\right)^2\alpha_{\theta_1}^7\alpha_{\theta_2}^9}+C^{(1)}_{yz}\right)\beta\right]\\
G_{zz} &= G^{\rm MQGP}_{zz}\left[1+\left(C^{(1)}_{zz}-\frac{b^{10}\left(9b^2+1\right)^4 M\left({r_h}^2-\frac{(r-r_h)^3}{r_h}\right)\log(r_h)\Sigma_1}{27\pi^{3/2}\left(3b^2-1\right)^5\left(6b^2+1\right)^4\sqrt{g_s}\log N^2 N^{23/20}N_f\alpha_{\theta_2}^5}\right)\beta\right]\\
G_{x^{10}x^{10}} &= G^{\rm MQGP}_{x^{10}x^{10}}\left[1-\frac{27b^{10}\left(9b^2+1\right)^4 M\left(\frac{1}{N}\right)^{5/4}\beta\left(6a^2+{r_h}^2\right)\left(1-\frac{(r-r_h)^2}{{r_h}^2}\right)\log^3(r_h)\Sigma_1}{\pi\left(3b^2-1\right)^5\left(6b^2+1\right)^4\log N^4 N_f{r_h}^2\left(9a^2+{r_h}^2\right)\alpha_{\theta_2}^3}\right],
\end{aligned}
$$

where $\Sigma_{1,3}$ are defined in (A2), and $G^{\rm MQGP}_{MN}$ are the $\mathcal{M}$ theory metric components in the MQGP limit at $\mathcal{O}(\beta^0)$ [32]. The explicit dependence on $\theta_{1,2}$ of the $\mathcal{M}$-theory metric components up to $\mathcal{O}(\beta)$, using (33), is effected by the replacemements: $\alpha_{\theta_1}\rightarrow N^{\frac{1}{5}}\sin\theta_1,\ \ \alpha_{\theta_2}\rightarrow N^{\frac{3}{10}}\sin\theta_2$ in (44). Also, see footnote 5.

We now present the third lemma of this paper:

Lemma 3: $C^{(1)}_{MNP}=0$ up to $\mathcal{O}(\beta)$ is a consistent solution of (20).

Proof: The eleven-fold M_{11} in the $\mathcal{M}$ theory uplift as obtained in [1] is a warped product of $S^1(x^0)\times\mathbb{R}_{\rm conformal}$ and $M_7(r,\theta_{1,2},\phi_{1,2},\psi,x^{10})$, the latter being a cone over $M_6(\theta_{1,2},\phi_{1,2},\psi,x^{10})$ where $M_6(\theta_{1,2},\phi_{1,2},\psi,x^{10})$ has the following nested fibration structure:

$$
\begin{array}{ccc}
\mathcal{M}_6(\theta_{1,2},\phi_{1,2},\psi,x_{10}) & \longleftarrow & S^1(x^{10})\\
\downarrow & & \\
\mathcal{M}_5(\theta_{1,2},\phi_{1,2},\psi) & \longleftarrow & \mathcal{M}_3(\phi_1,\phi_2,\psi)\ \ .\\
\downarrow & & \\
\mathcal{B}_2(\theta_1,\theta_2) & &
\end{array}
$$

As shown in [1], $p_1^2(M_{11})=p_2(M_{11})=0$ up to $\mathcal{O}(\beta^0)$ where p_a is the a-th Pontryagin class of M_{11}. This hence implies that $X_8=0$ up to $\mathcal{O}(\beta^0)$.

Now, (23) implies:

(45)
$$\sum_{N,P\in\{t,x^{1,2,3},r,\theta_{1,2},\phi_{1,2},\psi,x^{10}\}} \beta\partial_M\left(\sqrt{-g^{(0)}}g^{(0)NP}g^{(0)}_{NP}f_{NP}\partial^{[M}C^{M_1M_2M_3]}_{(0)}\right)\sim 0,$$

where the "(0)" implies the $\mathcal{O}(\beta^0)$-terms of [1, 16, 32] and the $\sim$ implies equality up to $\mathcal{O}(\beta^2)$ corrections. For simplicity we work near the $\psi = 2n\pi, n = 0, 1, 2$-branches (resulting in the decoupling of $M_5(t, x^{1,2,3}, r)$ and $M_6(\theta_{1,2}, \phi_{1,2}, \psi, x^{10})$ and $g^{(0)}_{MN}$ being diagonal for $M = r, x^{10}$ [16]) restricted to the Ouyang embedding (effected by the delocalized limit wherein one works in the neighborhood of $\theta_{10} = \frac{\alpha_{\theta_1}}{N^{\frac{1}{5}}}, \theta_{20} = \frac{\alpha_{\theta_2}}{N^{\frac{3}{10}}}$ (see footnote 5) wherein, as also mentioned in **3.1**, an explicit $SU(3)$-structure for the type IIB dual as well as its delocalized Strominger-Yau-Zaslow (SYZ) type IIA mirror as string theory duals of large-N thermal QCD-like theories, and an explicit G_2-structure for its $\mathcal{M}$-theory uplift [1], was worked out in [16]; using (C31) - (C33) and arguments similar to the ones given in [11], one can show that our results are independent of any delocalization in $\theta_{1,2}$). Using the non-zero components of $C_{MNP} : C_{\theta_{1,2}\ \phi_{1,2}/\psi\ x^{10}}$ [1], one can show that (45) implies:

(46)
$$\sum_{N,P\in\{t,x^{1,2,3},r,\theta_{1,2},\phi_{1,2},\psi,x^{10}\}}\beta\partial_r\left(\sqrt{-g^{(0)}}g^{(0)NP}g^{(0)}_{NP}f_{NP}g^{rr}_{(0)}\partial_r C^{M_1M_2x^{10}}_{(0)}\right)\delta^{M_3}_{x^{10}}\sim 0,$$

where $M_1, M_2 = \theta_{1,2}, \phi_{1,2}, \psi$ or precisely $\theta_{1,2}, x, y, z$ where the delocalized $T^3(x, y, z)$ coordinates are defined near $r = r_0 \in$IR as [1] [8]:

(48)
$$\begin{aligned} x &= \sqrt{h_2}\,[h(r_0,\theta_{10,20})]^{\frac{1}{4}}\sin\theta_{10}\ r_0\phi_1,\\ y &= \sqrt{h_4}\,[h(r_0,\theta_{10,20})]^{\frac{1}{4}}\sin\theta_{20}\ r_0\phi_2,\\ z &= \sqrt{h_1}\,[h(r_0,\theta_{10,20})]^{\frac{1}{4}}\ r_0\psi, \end{aligned}$$

[8]As explained in [40], the T^3-valued (x, y, z) are defined via:

(47)
$$\begin{aligned} \phi_1 &= \phi_{10} + \frac{x}{\sqrt{h_2}\,[h(r_0,\theta_{10,20})]^{\frac{1}{4}}\sin\theta_{10}\ r_0},\\ \phi_2 &= \phi_{20} + \frac{y}{\sqrt{h_4}\,[h(r_0,\theta_{10,20})]^{\frac{1}{4}}\sin\theta_{20}\ r_0}\\ \psi &= \psi_0 + \frac{z}{\sqrt{h_1}\,[h(r_0,\theta_{10,20})]^{\frac{1}{4}}\ r_0}, \end{aligned}$$

and one works up to linear order in (x, y, z). Up to linear order in r, i.e., in the IR, it can be shown [13] that $\theta_{10,20}$ can be promoted to global coordinates $\theta_{1,2}$ in all the results in the paper.

h being the delocalized warp factor [2]:

(49)
$$h(r_0,\theta_{10,20}) = \frac{L^4}{r_0^4}\left[1+\frac{3g_sM_{\rm eff}^2}{2\pi N}{\rm log}r_0\left\{1+\frac{3g_sN_f^{\rm eff}}{2\pi}\left({\rm log}r_0+\frac{1}{2}\right)+\frac{g_sN_f^{\rm eff}}{4\pi}{\rm log}\left({\rm sin}\frac{\theta_{10}}{2}{\rm sin}\frac{\theta_{20}}{2}\right)\right\}\right],$$

wherein $M_{\rm eff}$ was defined in Section **2** and $N_f^{\rm eff}$ is defined via the type IIB axion $C_0 = \frac{N_f^{\rm eff}}{4\pi}(\psi-\phi_1-\phi_2)$ (by standard monodromy arguments); the squashing factors are defined below [2]:

$$h_1 = \frac{1}{9}+\mathcal{O}\left(\frac{g_sM^2}{N}\right),\ h_2 = \frac{1}{6}+\mathcal{O}\left(\frac{g_sM^2}{N}\right),\ h_4 = h_2+\frac{4a^2}{r_0^2}, \tag{50}$$

(a being the radius of the blown-up S^2).

One immediately notes from (46) that (45) is identically satisfied for $M_{1,2,3} \in x^{0,1,2,3}, r$. The set of ${}^5C_2$ equations (46) for $M_{1,2} \in \theta_{1,2}, x, y, z$, and $M_3 = x^{10}$ are considered in Appendix **D** where one sees that in the IR: $r = \chi r_h, \chi = \mathcal{O}(1)$ [and a(the resolution parameter)$= \left(b+\mathcal{O}\left(\frac{g_sM^2}{N}\right)\right)r_h$ [15]], all ten of these equations substituting in the solutions for f_{MN} from **3.1**, reduce to:

$$\begin{aligned}\beta N_f^{\alpha_f}(\log N)^{\alpha_{\log N}}\Bigg(&N^{\alpha_1}\mathcal{F}_{\theta_1 x}^{M_1M_2}(b,\chi,\alpha_{\theta_{1,2}},r_h)C_{\theta_1 x}^{(1)}\\ &+N^{\alpha_2}\sum_{(M,N)=(z,z),(\theta_1,z),(\theta_2,z)}\mathcal{F}_{MN}^{M_1M_2}(b,\chi,\alpha_{\theta_{1,2}},r_h)C_{MN}^{(1)}\Bigg)=0,\end{aligned} \tag{51}$$

where $\alpha_f = 2,3; \alpha_{\log N} = 1,2; \alpha_1 > \alpha_2$ and $C_{MN}^{(1)}$ are the constants of integration appearing in the solutions (44) to the $\mathcal{O}(\beta)$-corrections to the $\mathcal{M}$ theory metric components of [1, 16, 32] (44). Hence, up $\mathcal{O}(\beta)$ and LO in N, (45) is identically satisfied if:

$$C_{\theta_1 x}^{(1)} = 0 \text{ in } (44), \tag{52}$$

and up to $\mathcal{O}(\beta)$ and NLO in N (assuming as in (A52), $b \sim \frac{1}{\sqrt{3}}$), additionally:

$$\begin{aligned}&\left\{\mathcal{F}_{\theta_1 z,\theta_2 z,zz}^{M_1M_2}=0\right\},\\ &\text{implying :}\\ &C_{zz}^{(1)} = 2C_{\theta_1 z}^{(1)},\ C_{\theta_2 z}^{(1)} = 0 \text{ in } (44).\end{aligned} \tag{53}$$

One therefore sees that one can consistently set $C_{MNP}^{(1)} = 0$ up to $\mathcal{O}(\beta)$.

3.2. Near $\psi \neq 2n\pi, n = 0, 1, 2$, and Near $r = r_h$

In this sub-section we will be looking at the EOMs and their solutions near the $\psi = \psi_0 \neq 2n\pi, n = 0, 1, 2$-branches (wherein some $G^{\mathcal{M}}_{rM}, M \neq r$ and $G^{\mathcal{M}}_{x^{10}N}, N \neq x^{10}$ components are non-zero) near $r = r_h$.

One can show that leading-order-in-N contribution to J_0 for small $\theta_{1,2}$ (i.e. corresponding to the Ouyang embedding of type IIB $D7$-branes' embedding [20] for vanishing small $\mu_{\rm Ouyang}$) is given by:

$$J_0 \sim \frac{1}{2} R^{x^{10}t\theta_2 t} R_{t\theta_1\theta_2 t} R_{x^{10}}{}^{tx^{10}t} R^{\theta_1}{}_{tx^{10}t}, \tag{54}$$

where:

(55)

$$\begin{aligned}
R^{x^{10}t\theta_2 t} &\sim \frac{a^2 N^{\frac{17}{20}} \sin\theta_1 \sin^2\theta_2 \left(9a^2 r^4 + 9a^2 {r_h}^4 + r^6 + r^2 {r_h}^4\right)}{(g_s - 1){g_s}^{5/4} \log N^2 M {N_f}^4 \sin\left(\frac{\psi_0}{2}\right) r \left(6a^2 + r^2\right)\left(r^4 - {r_h}^4\right)^2 \log(r)}\\
R_{t\theta_1\theta_2 t} &\sim \frac{{g_s}^{5/2}\left(\frac{1}{\log N}\right)^{4/3} M^2 {N_f}^{8/3} \sin^2\left(\frac{\psi_0}{2}\right)(9a^2+r^2)(r^4-{r_h}^4)(r^4+{r_h}^4)\log(r)}{N^{3/2} r^6 \sin^3\theta_1 \sin^3\theta_2 (6a^2+r^2)}\\
R_{x^{10}}{}^{tx^{10}t} &\sim -\frac{\left(\frac{1}{\log N}\right)^{4/3} \sin^4\left(\frac{\psi_0}{2}\right)\left(9a^2 + r^2\right)\left(r^4 + {r_h}^4\right)}{{N_f}^{4/3} \sin^4\phi_{20} r^2 \sin^4\theta_2 \left(6a^2 + r^2\right)\left(r^4 - {r_h}^4\right)\log(r)}\\
R^{\theta_1}{}_{tx^{10}t} &\sim \frac{\left(\frac{1}{N}\right)^{3/4} \sin\left(\frac{\psi_0}{2}\right) \sin^2\theta_1 \left(9a^2 + r^2\right)\left(r^4 - {r_h}^4\right)\left(r^4 + {r_h}^4\right)}{{g_s}^{11/4} M {N_f}^2 \sin^2\phi_{20} r^6 \sin\theta_2 \left(6a^2 + r^2\right)\log^2(r)}.
\end{aligned}$$

yields:

(56)

$$J_0 \sim -\frac{a^2\left(\frac{1}{\log N}\right)^{14/3} \sin^6\left(\frac{\psi_0}{2}\right)(9a^2+r^2)^4(r^4+{r_h}^4)^4}{N^{7/5}(g_s-1){g_s}^{3/2}{N_f}^{14/3} \sin^6\phi_{20} r^{15} \sin^6\theta_2 (6a^2+r^2)^4 (r^4-{r_h}^4)\log^3(r)}.$$

We now arrive at the fourth lemma of this paper:

Lemma 4: The following is the final result as regards the $\mathcal{O}(\beta)$-corrected $\mathcal{M}$-theory metric of [1] in the MQGP limit in the $\psi \neq 2n\pi, n = 0, 1, 2$-branches, e.g., near (33), up to $\mathcal{O}((r - r_h)^2)$ - the components which do

not receive an $\mathcal{O}(\beta)$ corrections, are not listed in (57):
(57)

$$
\begin{aligned}
G_{\theta_1\theta_2} &= G^{\rm MQGP}_{\theta_1\theta_2}\left[1+\left(\frac{\kappa_{\theta_1\theta_2}\kappa_b^2\sin^2\left(\frac{\psi_0}{2}\right){r_h}^2\left(\frac{1}{N}\right)^{2\alpha+\frac{3}{2}}}{\sqrt{g_s}{N_f}^6\sin^2\phi_{20}(r-r_h)^2\alpha_{\theta_2}^2}+{C_{\theta_1\theta_2}}^{(1)}\right)\beta\right]\\
G_{yy} &= G^{\rm MQGP}_{yy}\Bigg[1+\Bigg(-\frac{256N^{3/5}\sin^2\phi_{20}{r_h}^4\alpha_{\theta_2}^2}{9(g_s-1)\left({r_h}^2-3a^2\right)^2\log^2(N)\alpha_{\theta_1}^6\log^2\left(9a^2{r_h}^4+{r_h}^6\right)}\\
&\times\Bigg\{\frac{\kappa_{yy}N^{3/10}\sin^2\left(\frac{\psi_0}{2}\right)\left(9a^2+{r_h}^2\right)\left(\log(r_h)-1\right)\left(\left(9a^2+{r_h}^2\right)\log\left(9a^2{r_h}^4+{r_h}^6\right)-8\left(6a^2+{r_h}^2\right)\log(r_h)\right)^2}{\sqrt{g_s}{N_f}^6\sin^2\phi_{20}{r_h}^2\left(6a^2+{r_h}^2\right)^3(r-r_h)^2\log^4(r_h)\alpha_{\theta_2}^2\log^8\left(9a^2{r_h}^4+{r_h}^6\right)}\\
&+{C_{\theta_1\theta_2}}^{(1)}\Bigg\}\Bigg)\beta\Bigg]\\
G_{\theta_1y} &= G^{\rm MQGP}_{\theta_1y}\left[1+\frac{\kappa_{\theta_1y}\beta g_s\kappa_bM16\sin^4\left(\frac{\psi_0}{2}\right)r_h\beta N^{-\alpha}}{{N_f}^5\sin^2\phi_{20}(r-r_h)^3\log^{\frac{9}{2}}(r_h)\alpha_{\theta_1}^4\alpha_{\theta_2}^3}\right]\\
G_{\theta_1z} &= G^{\rm MQGP}_{\theta_1z}\left[1+\frac{\kappa_{\theta_1z}\beta {g_s}^{15/2}\kappa_bM\sin^4\left(\frac{\psi_0}{2}\right)r_h\beta N^{-\alpha}}{{g_s}^{13/2}{N_f}^5\sin^2\phi_{20}(r-r_h)^3\log^{\frac{9}{2}}(r_h)\alpha_{\theta_1}^4\alpha_{\theta_2}^3}\right]\\
G_{xy} &= G^{\rm MQGP}_{xy}[1+\frac{\kappa_{xy}\beta\sqrt{g_s}(3g_s-4)\kappa_b\left(\frac{1}{N}\right)^{2/5}\sin^4\left(\frac{\psi_0}{2}\right)\beta N^{-\alpha}}{(g_s-1)^2{N_f}^6\sin^2\phi_{20}\log(N)(r-r_h)\log^{\frac{11}{2}}(r_h)\alpha_{\theta_1}\alpha_{\theta_2}^2}]\\
G_{yz} &= G^{\rm MQGP}_{yz}\left[1+\frac{\kappa_{yz}\beta g_s\kappa_b^2\log NM\sin^2\left(\frac{\psi_0}{2}\right){r_h}^3\beta N^{-2\alpha-\frac{11}{10}}\log^2(r_h)}{(g_s-1){N_f}^5\sin^2\phi_{20}(r-r_h)^3\alpha_{\theta_1}^3\alpha_{\theta_2}^3}\right]\\
G_{xz} &= G^{\rm MQGP}_{xz}\left[1+\left(C^{(1)}_{xz}-\frac{\kappa_{xz}\tilde{\Sigma}_2\sqrt{2}\pi^{11/2}\sqrt{g_s}(3g_s-4)\kappa_b\sin^4\left(\frac{\psi_0}{2}\right)\beta N^{-\alpha-\frac{2}{5}}}{(g_s-1)^2{N_f}^6\sin^2\phi_{20}(r-r_h)\log^{\frac{13}{2}}(r_h)\alpha_{\theta_1}\alpha_{\theta_2}^2}\right)\beta\right],
\end{aligned}
$$

where $\kappa_{\theta_1\theta_2,\ \theta_1y,\ \theta_1z,\ xy,\ xz,\ yz} \ll 1,\ \kappa_{yy}\sim\mathcal{O}(1)$ and $\tilde{\Sigma_2}$ is defined in (A80) and $\alpha\in\mathbb{Z}^+$ appearing via (A52). Analogous to working near the $\psi=2n\pi$-coordinate patches, the explicit dependence on $\theta_{1,2}$ of the $\mathcal{M}$-theory metric components up to $\mathcal{O}(\beta)$, using (33), is effected by the replacemements: $\alpha_{\theta_1}\to N^{\frac{1}{5}}\sin\theta_1,\ \alpha_{\theta_2}\to N^{\frac{3}{10}}\sin\theta_2$ in (57). Also, see footnote 5. The Physics implication of (57) is similar to (58) arising from (44).

4. Physics Lessons Learnt - IR-enhancement large-N/Planckian-suppresion competition and when $\mathcal{O}(l_p^6)$ is (not) enough

Based on the results of this paper and its applications as discussed in detail in [4], [5], we now discuss the Physics lessons learnt as a consequence of working out the $\mathcal{O}(R^4)/\mathcal{O}(l_p^6)$ corrections to the $\mathcal{M}$-theory dual of large-N thermal QCD-like theories.

The main Physics-related take-away of Section **3**, e.g. from (44), can be abstracted from the following table:

S. No.	$G^{\mathcal{M}}_{MN}$	IR Enhancement Factor $\frac{(\log\mathcal{R}_h)^m}{\mathcal{R}_h^n}, m, n \in \mathbb{Z}^+$ in the $\mathcal{O}(R^4)$ Correction	N Suppression Factor in the $\mathcal{O}(R^4)$ Correction
1	$G^{\mathcal{M}}_{\mathbb{R}^{1,3}}$	$\log\mathcal{R}_h$	$N^{-\frac{9}{4}}$
2	$G^{\mathcal{M}}_{rr,\theta_1 x}$	1	$N^{-\frac{8}{15}}$
3	$G^{\mathcal{M}}_{\theta_1 z,\theta_2 x}$	$\mathcal{R}_h^{-5}$	$N^{-\frac{7}{6}}$
4	$G^{\mathcal{M}}_{\theta_2 y}$	$\log\mathcal{R}_h$	$N^{-\frac{7}{5}}$
5	$G^{\mathcal{M}}_{\theta_2 z}$	$\log\mathcal{R}_h$	$N^{-\frac{7}{6}}$
6	$G^{\mathcal{M}}_{xy}$	$\log\mathcal{R}_h$	$N^{-\frac{21}{20}}$
7	$G^{\mathcal{M}}_{xz}$	$(\log\mathcal{R}_h)^3$	$N^{-\frac{5}{4}}$
8	$G^{\mathcal{M}}_{yy}$	$\log\mathcal{R}_h$	$N^{-\frac{7}{4}}$
9	$G^{\mathcal{M}}_{yz}$	$\frac{\log\mathcal{R}_h}{\mathcal{R}_h^7}$	$N^{-\frac{29}{12}}$
10	$G^{\mathcal{M}}_{zz}$	$\log\mathcal{R}_h$	$N^{-\frac{23}{20}}$
11	$G^{\mathcal{M}}_{x^{10}x^{10}}$	$\frac{\log\mathcal{R}_h^3}{\mathcal{R}_h^2}$	$N^{-\frac{5}{4}}$

Table 1: IR Enhancement vs. large-N Suppression in $\mathcal{O}(R^4)$-Corrections in the M-theory Metric in the $\psi = 2n\pi, n = 0, 1, 2$ Patches; $\mathcal{R}_h \equiv \frac{r_h}{\mathcal{R}_{D5/\overline{D5}}} \ll 1$, $\mathcal{R}_{D5/\overline{D5}}$ being the $D5 - \overline{D5}$ separation

One notes that in the IR: $r = \chi r_h, \chi \equiv \mathcal{O}(1)$, and up to $\mathcal{O}(\beta)$:

$$f_{MN} \sim \beta \frac{(\log\mathcal{R}_h)^m}{\mathcal{R}_h^n N^{\beta_N}},\ m \in \{0, 1, 3\},\ n \in \{0, 2, 5, 7\},\ \beta_N > 0. \tag{58}$$

Now, $|\mathcal{R}_h| \ll 1$. As estimated in [41], $|\log\mathcal{R}_h| \sim N^{\frac{1}{3}}$, implying there is a competition between Planckian and large-N suppression and infra-red enhancement arising from $m, n \neq 0$ in (58). One could choose a heirarchy: $\beta \sim e^{-\gamma_\beta N^{\gamma_N}}, \gamma_\beta, \gamma_N > 0 : \gamma_\beta N^{\gamma_N} > 7N^{\frac{1}{3}} + \left(\frac{m}{3} - \beta_N\right)\log N$ (ensuring that the IR-enhancement does not overpower Planckian suppression - we took the $\mathcal{O}(\beta)$ correction to $G^{\mathcal{M}}_{yz}$, which had the largest IR enhancement, to set a lower bound on $\gamma_{\beta,N}$/Planckian suppression). If $\gamma_\beta N^{\gamma_N} \sim 7N^{\frac{1}{3}}$, then one will be required to go to a higher order in β. This hence answers the question, when one can truncate at $\mathcal{O}(\beta)$.

5. Differential geometry or (IR) G-structure torsion classes of non-supersymmetric string/$\mathcal{M}$-theory duals including the $\mathcal{O}(R^4)$ corrections

The use of G-structure torsion classes is a very useful tool for classifying, specially non-Kähler geometries. A complete classification of the $SU(n)$ structures relevant to non-supersymmetric string vacua, does not exist [53]. In the literature, in the context of $SU(3)$-structure manifolds, classes of maximally symmetric non-supersymmetric vacua that break supersymmetry in a controllable way, have been constructed, e.g. [54] wherein the first vacuum of this type was obtained by compactifying type IIB/F-theory with $O3$ planes on conformally CY manifolds ($SU(3)$-structure manifolds with $W_1 = W_2 = W_3 = 0$ and $3W_4 = 2W_5$); type II vacua of this type were studied in [55] and classified using calibrations in [56], and similar solutions in heterotic string theory were obtained in [57] - see [58] for G_2 structures relevant to non-supersymmetric vacua in heterotic($\mathcal{M}$-)SUGRA.

A classification of $SU(3)/G_2/Spin(7)/Spin(4)$ structures relevant to non-supersymmetric (UV-complete) string theoretic dual of large-N thermal QCD-like theories, and its $\mathcal{M}$-theory uplift, has been missing in the literature. This is what we aim at achieving in this section.

Using the results for Ricci scalar of $M_6(r,\theta_1,\theta_2,\phi_1,\phi_2,\psi)$, $M_7(r,\theta_1,\theta_2,\phi_1,\phi_2,\psi,x^{10})$, $M_8(x^0,r,\theta_1,\theta_2,\phi_1,\phi_2,\psi,x^{10})$ that figure in the string/$\mathcal{M}$-theory dual of large-N thermal QCD-like theories in this work, in terms of the:

1) $SU(3)$**-structure torsion classes** [59], it is observed:

$$(59)\quad \begin{aligned} & R(M_6(r,\theta_1,\theta_2,\phi_1,\phi_2,\psi)) \\ & \quad = 15|W_1|^2 - |W_2|^2 - |W_3|^2 + 8\langle W_5, W_4\rangle - 2|W_4|^2 + 4d*(W_4+W_5) \\ & \quad \neq 0 \end{aligned}$$

($\langle,\rangle$ denoting Mukai pairing),

2) G_2**-structure torsion classes** [60], it is observed::

$$(60)\quad \begin{aligned} & R(M_7(r,\theta_1,\theta_2,\phi_1,\phi_2,\psi,x^{10})) \\ & \quad = 12\delta W_7 + \frac{21}{8}W_1^2 + 30|W_7|^2 - \frac{1}{2}|W_{14}|^2 - \frac{1}{2}|W_{27}|^2, \end{aligned}$$

3) *Spin*(7)**-structure torsion classes** [61], it is observed:

$$R(M_8(x^0, r, \theta_1, \theta_2, \phi_1, \phi_2, \psi, x^{10})) = \frac{49}{18}||\theta||^2 - \frac{1}{12}||T||^2 + \frac{7}{2}\delta\theta, \tag{61}$$

where: $\theta = \frac{1}{7} * (\delta\Phi \wedge \Phi)$, $T = -\delta\Phi - \frac{7}{6} * (\theta \wedge \Phi)$, Φ being the *Spin*(7) fundamental four-form. Note, that the eight-fold $M_8(x^0, r, \theta_1, \theta_2, \phi_1, \phi_2, \psi, x^{10})$ admits a *Spin*(7) structure if $p_1^2(M_8) - 4p_2(M_8) + 8\chi(M_8) = 0$ [61], $p_a(M_8)$ being the a-th Pontryagin class of M_8. Given that M_8 could be thought of as elliptic/$T^2(x^0, x^{10})$ fibration over $M_6(r, \theta_1, \theta_2, \phi_1, \phi_2, \psi)$, using the Kunneth formula one sees that $\chi(M_8) = \chi(T^2)\chi(M_6) = 0$. In the delocalized limit, also modifying the arguments of [1] (which showed $X_8 = 0$ as $p_1^2(M_{11} = \mathbb{R}^3 \times M_8) = p_2(M_{11}) = 0$), one can show that the $p_1^2(M_8) = p_2(M_8) = 0$.

In this section, we will derive in the IR near the $\psi = 2n\pi, n = 0, 1, 2$-branches the non-zero $SU(3)$-structure torsion classes of the six-fold relevant to the type IIA mirror, the G_2-structure torsion classes of the seven-fold and the $SU(4)$- and *Spin*(7)-structure torsion classes of the eight-fold relevant to the $\mathcal{M}$-Theory uplift of the type IIA mirror. As in Section **3**, for simplicity, we work near the Ouyang embedding (assuming a very small Ouyang embedding parameter). But as mentioned later, using (C31) - (C33), based on arguments of [11], one can see that the results of Table 1 in Section **5**, will still remain valid for arbitrary $\theta_{1,2}$. For arbitrary ψ, using the results of subsection **3.2**, it is expected that the results of Table 1 in Section **5** will go through, though the co-frames will be considerably modified. We postpone this discussion to a later work.

5.1. *SU*(3)-Structure Torsion Classes of the Type IIA Mirror

Generically for $SU(n > 2)$-structures, the intrinsic torsion decomposes into five torsion classes $W_{i=1,\ldots,5}$ [62], i.e.,

$$T \in \Lambda^1 \otimes su(n)^{\perp} = \bigoplus_{i=1}^{5} W_i. \tag{62}$$

The adjoint representation 15 of $SO(6)$ decomposes under $SU(3)$ as $15 = 1 + 8 + 3 + \bar{3}$. Thus, $su(3)^{\perp} \sim 1 \oplus 3 \oplus \bar{3}$, and:

$$\begin{aligned}
&T \in \Lambda^1 \otimes su(3)^{\perp} = (3 \oplus \bar{3}) \otimes (1 \oplus 3 \oplus \bar{3})\\
&= (1 \oplus 1) \oplus (8 \oplus 8) \oplus (6 \oplus \bar{6}) \oplus (3 \oplus \bar{3}) \oplus (3 \oplus \bar{3})'\\
&\equiv W_1 \oplus W_2 \oplus W_3 \oplus W_4 \oplus W_5.
\end{aligned}$$

The $SU(3)$ structure torsion classes can be defined in terms of J, Ω, dJ, $d\Omega$ and the contraction operator $\lrcorner : \Lambda^k T^\star \otimes \Lambda^n T^\star \rightarrow \Lambda^{n-k} T^\star$, J being given by:

$$J = e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6$$

(the metric being understood to be given in terms of the coframes as: $ds_6^2 = \sum_{i=1}^{6} (de^a)^2$), and the (3,0)-form Ω being given by

$$\begin{aligned}
&\Omega = (e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6).\\
&W_1 = W_1^+ + W_1^- \ with:\\
&d\Omega_+ \wedge J = \Omega_+ \wedge dJ = W_1^+ J \wedge J \wedge J,\\
&d\Omega_- \wedge J = \Omega_- \wedge dJ = W_1^- J \wedge J \wedge J;\\
&(d\Omega_+)^{(2,2)} = W_1^+ J \wedge J + W_2^+ \wedge J,\\
&(d\Omega_-)^{(2,2)} = W_1^- J \wedge J + W_2^- \wedge J;\\
&W_3 = dJ^{(2,1)} - [J \wedge W_4]^{(2,1)},\\
&W_4 = \frac{1}{2} J \lrcorner dJ,\\
&W_5 = \frac{1}{2} \Omega_+ \lrcorner d\Omega_+.
\end{aligned} \tag{63}$$

We now proceed to work out the coframes $\{e^a\}$. This brings us to the next lemma:

Lemma 5: The non-zero components of the type IIA metric near $\psi = 0, 2\pi, 4\pi$ coordinate patch, e.g. near (33), obtained from the $\mathcal{M}$ theory metric

of Sec. **3** inclusive of the $\mathcal{O}(R^4)$ corrections are given by:
(64)

$$G^{\rm IIA}_{\theta_1 x} = -\frac{{g_s}^{7/4}\log N M N^{11/20} N_f\left(r^2-3a^2\right)\log(r)}{3\sqrt{2}\pi^{5/4}r^2\alpha_{\theta_1}\alpha_{\theta_2}^2} - \frac{\beta\mathcal{C}_{\theta_1 x}{}^{(1)}{g_s}^{7/4}\log N M N^{11/20}N_f\left({r_h}^2-3a^2\right)\log(r_h)}{3\sqrt{2}\pi^{5/4}{r_h}^2\alpha_{\theta_1}\alpha_{\theta_2}^2}$$

$$G^{\rm IIA}_{\theta_1 z} = \frac{{g_s}^{7/4}\log N M N^{3/20}N_f\alpha_{\theta_1}\left(r^2-3a^2\right)\log(r)}{2\sqrt{2}\pi^{5/4}r^2\alpha_{\theta_2}^2} + \frac{\beta\mathcal{C}_{\theta_1 z}{}^{(1)}{g_s}^{7/4}\log N M N^{3/20}N_f\alpha_{\theta_1}\left(r^2-3a^2\right)\log(r)}{2\sqrt{2}\pi^{5/4}r^2\alpha_{\theta_2}^2}$$

$$G^{\rm IIA}_{\theta_2 x} = \frac{{g_s}^{7/4}M N^{13/20}N_f\log(r)\left(36a^2\log(r)+r\right)}{3\sqrt{2}\pi^{5/4}r\alpha_{\theta_2}^3} + \frac{\beta\mathcal{C}_{\theta_2 x}{}^{(1)}{g_s}^{7/4}M N^{13/20}N_f\log(r)\left(36a^2\log(r)+r\right)}{3\sqrt{2}\pi^{5/4}r\alpha_{\theta_2}^3}$$

$$G^{\rm IIA}_{\theta_2 y} = \frac{\sqrt{2}\sqrt[4]{\pi}\sqrt[4]{g_s}N^{7/20}\alpha_{\theta_2}}{9\alpha_{\theta_1}^2} - \frac{\beta\mathcal{C}_{\theta_2 y}(1){g_s}^{7/4}M\sqrt[4]{N}\sqrt{\frac{1}{{N_f}^{4/3}}}{N_f}^{5/3}\alpha_{\theta_1}^2\log(r)\left(36a^2\log(r)+r\right)}{2\sqrt{2}\pi^{5/4}r\alpha_{\theta_2}^3}$$

$$G^{\rm IIA}_{\theta_2 z} = -\frac{{g_s}^{7/4}M\sqrt[4]{N}N_f\alpha_{\theta_1}^2\log(r)\left(36a^2\log(r)+r\right)}{2\sqrt{2}\pi^{5/4}r\alpha_{\theta_2}^3} - \frac{\beta\mathcal{C}_{\theta\psi}{}^{(1)}{g_s}^{7/4}M\sqrt[4]{N}N_f\alpha_{\theta_1}^2\log(r)\left(36a^2\log(r)+r\right)}{2\sqrt{2}\pi^{5/4}r\alpha_{\theta_2}^3}$$

$$G^{\rm IIA}_{xx} = 1 - \frac{27b^{10}\left(6b^2+1\right)\left(9b^2+1\right)^3\beta M\left(\frac{1}{N}\right)^{5/4}\left(19683\sqrt{6}\alpha_{\theta_1}^6+6642\alpha_{\theta_2}^2\alpha_{\theta_1}^3-40\sqrt{6}\alpha_{\theta_2}^4\right)\log^3(r_h)}{2\pi\left(3b^2-1\right)^5N_f{r_h}^2\alpha_{\theta_2}^3\left(6b^2\log N+\log N\right)^4}$$

$$G^{\rm IIA}_{yy} = 1 + \frac{27\left(9b^2+1\right)^3\beta b^{10}M\left(\frac{1}{N}\right)^{5/4}\left(-19683\sqrt{6}\alpha_{\theta_1}^6-6642\alpha_{\theta_2}^2\alpha_{\theta_1}^3+40\sqrt{6}\alpha_{\theta_2}^4\right)\log^3(r_h)}{2\pi\left(3b^2-1\right)^5\left(6b^2+1\right)^3\log N^4N_f{r_h}^2\alpha_{\theta_2}^3}$$

$$G^{\rm IIA}_{zz} = \frac{2N^{3/5}}{27\alpha_{\theta_2}^2} + \frac{2\beta C_{zz}^{(1)}N^{3/5}}{27\alpha_{\theta_2}^2}$$

$$G^{\rm IIA}_{xy} = \frac{2\sqrt{\frac{2}{3}}N^{7/10}}{9\alpha_{\theta_1}^2\alpha_{\theta_2}} + \frac{2\sqrt{\frac{2}{3}}\beta\mathcal{C}_{\phi_1\phi_2}{}^{(1)}N^{7/10}}{9\alpha_{\theta_1}^2\alpha_{\theta_2}}$$

$$G^{\rm IIA}_{xz} = -\frac{4N}{81\alpha_{\theta_1}^2\alpha_{\theta_2}^2} + \frac{2b^{10}\left(9b^2+1\right)^3\beta M\sqrt[4]{\frac{1}{N}}\left(19683\sqrt{6}\alpha_{\theta_1}^6+6642\alpha_{\theta_2}^2\alpha_{\theta_1}^3-40\sqrt{6}\alpha_{\theta_2}^4\right)\log^3(r_h)}{3\pi\left(3b^2-1\right)^5\left(6b^2+1\right)^3\log N^4N_f{r_h}^2\alpha_{\theta_1}^2\alpha_{\theta_2}^5}$$

$$G^{\rm IIA}_{yz} = -\frac{\sqrt{\frac{2}{3}}N^{3/10}}{3\alpha_{\theta_2}} - \frac{\sqrt{\frac{2}{3}}\beta\mathcal{C}_{\phi_2\psi}{}^{(1)}N^{3/10}}{3\alpha_{\theta_2}}.$$

To work out the co-frames corresponding to (64), one diagonalizes $G_{mn}(r,\theta_{1,2},\phi_{1,2},\psi)$ or equivalently $G_{\tilde{m}\tilde{n}}(\theta_{1,2},\phi_{1,2},\psi)$ for which one needs to solve the following secular equation - a quintic:

$$P(x) \equiv x^5 + Ax^4 + Bx^3 + Cx^2 + Fx + G = 0, \quad (65)$$

where:

$$A = -\frac{2N^{3/5}(\beta\mathcal{C}_{zz}^{(1)}+1)}{27\alpha_{\theta_2}^2},$$
$$B = -\frac{16N^2}{6561\alpha_{\theta_1}^4\alpha_{\theta_2}^4},$$
$$C = \frac{16N^2(\beta\mathcal{C}_{zz}^{(1)}-2\beta{\mathcal{C}_{yz}}^{(1)})}{6561\alpha_{\theta_1}^4\alpha_{\theta_2}^4}$$

$$
\begin{aligned}
F &= \frac{32\sqrt{\pi}\sqrt{g_s}N^{27/10}}{531441\alpha_{\theta_1}^8\alpha_{\theta_2}^2},\\
G &= -\frac{2{g_s}^4\log N^2 M^2 N^{12/5}{N_f}^2}{19683\pi^2 r^4 {r_h}^4\alpha_{\theta_1}^6\alpha_{\theta_2}^4}\\
&\quad\times\Big({r_h}^2\left(3a^2-r^2\right)\log(r)(\beta({\mathcal{C}_{zz}}^{(1)}-2{\mathcal{C}_{\theta_1 z}}^{(1)})-1)\\
&\quad+\beta\mathcal{C}_{\theta_1 x}^{(1)}r^2\left(3a^2-{r_h}^2\right)(\beta{\mathcal{C}_{zz}}^{(1)}+1)\log(r_h)\Big)\\
&\quad\times\Big(\beta{\mathcal{C}_{\theta_1 x}}^{(1)}r^2\left(3a^2-{r_h}^2\right)\log(r_h)+{r_h}^2\left(3a^2-r^2\right)\log(r)\Big) \qquad (66)
\end{aligned}
$$

Using Umemura's result [63] on expressing the roots of an algebraic polynomial of degree n in terms of Siegel theta functions of genus $g(>1)=[(n+2)/2]$: $\theta\left[\begin{array}{c}\mu\\ \nu\end{array}\right](z,\Omega)$ for $\mu,\nu\in\mathbf{R}^g, z\in\mathbf{C}^g$ and Ω being a complex symmetric $g\times g$ period matrix of the hyperelliptic curve $Y^2=P(\mathcal{Z})$ with $Im(\Omega)>0$, and defined as follows:

$$
\theta\left[\begin{array}{c}\mu\\ \nu\end{array}\right](z,\Omega)=\sum_{n\in\mathbf{Z}^g}e^{i\pi(n+\mu)^T\Omega(n+\mu)+2i\pi(n+\mu)^T(z+\nu)}.
$$

Hence for a quintic, one needs to use Siegel theta functions of genus three. The period matrix Ω will be defined as follows:

$$
\Omega_{ij}=\sigma^{ik}\rho_{kj}
$$

where

$$
\sigma_{ij}\equiv\oint_{A_j}d\mathcal{Z}\frac{\mathcal{Z}^{i-1}}{\sqrt{\mathcal{Z}(\mathcal{Z}-1)P(\mathcal{Z})}}
$$

and

$$
\rho_{ij}\equiv\oint_{B_j}\frac{\mathcal{Z}^{i-1}}{\sqrt{\mathcal{Z}(\mathcal{Z}-1)(\mathcal{Z}-2)P(\mathcal{Z})}},
$$

$\{A_i\}$ and $\{B_i\}$ being a canonical basis of cycles satisfying: $A_i\cdot A_j=B_i\cdot B_j=0$ and $A_i\cdot B_j=\delta_{ij}$; σ^{ij} are normalization constants determined by: $\sigma^{ik}\sigma_{kj}=\delta^i_j$. Umemura's result then is that a root:

$$
\frac{1}{2\left(\theta\left[\begin{array}{ccc}\frac{1}{2}&0&0\\ 0&0&0\end{array}\right](0,\Omega)\right)^4\left(\theta\left[\begin{array}{ccc}\frac{1}{2}&\frac{1}{2}&0\\ 0&0&0\end{array}\right](0,\Omega)\right)^4}
$$

$$\times\left[\left(\theta\left[\begin{array}{ccc}\frac{1}{2} & 0 & 0\\ 0 & 0 & 0\end{array}\right](0,\Omega)\right)^4\left(\theta\left[\begin{array}{ccc}\frac{1}{2} & \frac{1}{2} & 0\\ 0 & 0 & 0\end{array}\right](0,\Omega)\right)^4\right.$$

$$+\left(\theta\left[\begin{array}{ccc}0 & 0 & 0\\ 0 & 0 & 0\end{array}\right](0,\Omega)\right)^4\left(\theta\left[\begin{array}{ccc}0 & \frac{1}{2} & 0\\ 0 & 0 & 0\end{array}\right](0,\Omega)\right)^4$$

$$\left.-\left(\theta\left[\begin{array}{ccc}0 & 0 & 0\\ \frac{1}{2} & 0 & 0\end{array}\right](0,\Omega)\right)^4\left(\theta\left[\begin{array}{ccc}0 & \frac{1}{2} & 0\\ \frac{1}{2} & 0 & 0\end{array}\right](0,\Omega)\right)^4\right].$$

However, using the results of [64], one can express the roots of a quintic in terms of derivatives of genus-two Siegel theta functions as follows:

$$x_0=\left[\frac{\sigma_{22}\frac{d}{dz_1}\theta\left[\begin{array}{cc}\frac{1}{2} & \frac{1}{2}\\ 0 & \frac{1}{2}\end{array}\right]((z_1,z_2),\Omega)-\sigma_{21}\frac{d}{dz_2}\theta\left[\begin{array}{cc}\frac{1}{2} & \frac{1}{2}\\ 0 & \frac{1}{2}\end{array}\right]((z_1,z_2),\Omega)}{\xi\theta\left[\begin{array}{cc}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{array}\right]((z_1,z_2),\Omega)\theta\left[\begin{array}{cc}\frac{1}{2} & 0\\ 0 & 0\end{array}\right]((z_1,z_2),\Omega)\theta\left[\begin{array}{cc}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{array}\right]((z_1,z_2),\Omega)\theta\left[\begin{array}{cc}0 & \frac{1}{2}\\ 0 & 0\end{array}\right]((z_1,z_2),\Omega)}\right]_{z_1=z_2=0},$$

$$x_1=\left[\frac{\sigma_{22}\frac{d}{dz_1}\theta\left[\begin{array}{cc}0 & \frac{1}{2}\\ 0 & \frac{1}{2}\end{array}\right]((z_1,z_2),\Omega)-\sigma_{21}\frac{d}{dz_2}\theta\left[\begin{array}{cc}0 & \frac{1}{2}\\ 0 & \frac{1}{2}\end{array}\right]((z_1,z_2),\Omega)}{\xi\theta\left[\begin{array}{cc}0 & 0\\ 0 & \frac{1}{2}\end{array}\right]((z_1,z_2),\Omega)\theta\left[\begin{array}{cc}0 & 0\\ 0 & 0\end{array}\right]((z_1,z_2),\Omega)\theta\left[\begin{array}{cc}\frac{1}{2} & \frac{1}{2}\\ 0 & 0\end{array}\right]((z_1,z_2),\Omega)\theta\left[\begin{array}{cc}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{array}\right]((z_1,z_2),\Omega)}\right]_{z_1=z_2=0},$$

$$x_2=\left[\frac{\sigma_{22}\frac{d}{dz_1}\theta\left[\begin{array}{cc}0 & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{array}\right]((z_1,z_2),\Omega)-\sigma_{21}\frac{d}{dz_2}\theta\left[\begin{array}{cc}0 & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{array}\right]((z_1,z_2),\Omega)}{\xi\theta\left[\begin{array}{cc}0 & 0\\ \frac{1}{2} & \frac{1}{2}\end{array}\right]((z_1,z_2),\Omega)\theta\left[\begin{array}{cc}0 & 0\\ 0 & 0\end{array}\right]((z_1,z_2),\Omega)\theta\left[\begin{array}{cc}\frac{1}{2} & \frac{1}{2}\\ 0 & 0\end{array}\right]((z_1,z_2),\Omega)\theta\left[\begin{array}{cc}0 & \frac{1}{2}\\ 0 & 0\end{array}\right]((z_1,z_2),\Omega)}\right]_{z_1=z_2=0},$$

$$x_3=\left[\frac{\sigma_{22}\frac{d}{dz_1}\theta\left[\begin{array}{cc}\frac{1}{2} & 0\\ \frac{1}{2} & \frac{1}{2}\end{array}\right]((z_1,z_2),\Omega)-\sigma_{21}\frac{d}{dz_2}\theta\left[\begin{array}{cc}\frac{1}{2} & 0\\ \frac{1}{2} & \frac{1}{2}\end{array}\right]((z_1,z_2),\Omega)}{\xi\theta\left[\begin{array}{cc}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{array}\right]((z_1,z_2),\Omega)\theta\left[\begin{array}{cc}0 & 0\\ 0 & 0\end{array}\right]((z_1,z_2),\Omega)\theta\left[\begin{array}{cc}0 & 0\\ \frac{1}{2} & 0\end{array}\right]((z_1,z_2),\Omega)\theta\left[\begin{array}{cc}\frac{1}{2} & 0\\ 0 & 0\end{array}\right]((z_1,z_2),\Omega)}\right]_{z_1=z_2=0},$$

$$x_4=\left[\frac{\sigma_{22}\frac{d}{dz_1}\theta\left[\begin{array}{cc}\frac{1}{2} & 0\\ \frac{1}{2} & 0\end{array}\right]((z_1,z_2),\Omega)-\sigma_{21}\frac{d}{dz_2}\theta\left[\begin{array}{cc}\frac{1}{2} & 0\\ \frac{1}{2} & 0\end{array}\right]((z_1,z_2),\Omega)}{\xi\theta\left[\begin{array}{cc}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{array}\right]((z_1,z_2),\Omega)\theta\left[\begin{array}{cc}0 & 0\\ 0 & \frac{1}{2}\end{array}\right]((z_1,z_2),\Omega)\theta\left[\begin{array}{cc}0 & 0\\ \frac{1}{2} & \frac{1}{2}\end{array}\right]((z_1,z_2),\Omega)\theta\left[\begin{array}{cc}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{array}\right]((z_1,z_2),\Omega)}\right]_{z_1=z_2=0},$$

where:

$$\xi\equiv-A\pi^2\sum_{m=1}^5\left[\frac{\sigma_{22}\frac{d}{dz_1}\theta[\eta_m]((z_1,z_2),\Omega)-\sigma_{21}\frac{d}{dz_2}\theta[\eta_m]((z_1,z_2),\Omega)}{\frac{d}{dz_1}\theta[\eta_m]((z_1,z_2),\Omega)\frac{d}{dz_2}\theta[\eta_6]((z_1,z_2),\Omega)-\frac{d}{dz_2}\theta[\eta_m]((z_1,z_2),\Omega)\frac{d}{dz_1}\theta[\eta_6]((z_1,z_2),\Omega)}\right]_{z_1=z_2=0},$$

$$[\eta_1]\equiv\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ 0 & \frac{1}{2}\end{bmatrix},\quad [\eta_2]\equiv\begin{bmatrix}0 & \frac{1}{2}\\ 0 & \frac{1}{2}\end{bmatrix},\quad [\eta_3]\equiv\begin{bmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix},$$
$$[\eta_4]\equiv\begin{bmatrix}\frac{1}{2} & 0\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix},\quad [\eta_5]\equiv\begin{bmatrix}\frac{1}{2} & 0\\ \frac{1}{2} & 0\end{bmatrix},\quad [\eta_6]\equiv\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & 0\end{bmatrix}.$$

The symmetric period matrix corresponding to the hyperelliptic curve $w^2 = P(z)$ is given by:

$$\begin{pmatrix}\Omega_{11} & \Omega_{12}\\ \Omega_{12} & \Omega_{22}\end{pmatrix} = \frac{1}{\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21}}\begin{pmatrix}\sigma_{22} & -\sigma_{12}\\ -\sigma_{21} & \sigma_{11}\end{pmatrix}\begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix},$$

where $\sigma_{ij} = \int_{\mathcal{Z}_* A_j}\frac{\mathcal{Z}^{i-1}d\mathcal{Z}}{\sqrt{P(\mathcal{Z})}}$ and $\rho_{ij} = \int_{z_* B_j}\frac{\mathcal{Z}^{i-1}d\mathcal{Z}}{\sqrt{P(\mathcal{Z})}}$ where $\mathcal{Z}$ maps the A_i and B_j cycles to the $\mathcal{Z}-$plane. However, both results are not amenable to actual calculations due to the non-trivial period matrix computations.

The quintic (65) is solved using the Kiepert's algorithm described very nicely in the wonderful book [65]. The details of the same are given in Appendix **B**. Utilising the results of Appendix **B**, we will now work out the $SU(3)$ torsion classes of $M_6(r,\theta_{1,2},\phi_{1,2},\psi)$ and prove the following lemma:

Lemma 6: In the neighborhood of $(\theta_{10}=\frac{\alpha_{\theta_1}}{N^{\frac{1}{5}}},\theta_{20}=\frac{\alpha_{\theta_{20}}}{N^{\frac{3}{10}}},\psi=2n\pi), n= 0,1,2$, the $SU(3)$-structure torsion classes $W^{M_6}_{i=1,2,3,4,5}\neq 0$ (implying M_6 is a non-complex manifolds) with $W_4\sim W_5$.

Proof: In the neighborhood of $(\theta_{10}=\frac{\alpha_{\theta_1}}{N^{\frac{1}{5}}},\theta_{20}=\frac{\alpha_{\theta_{20}}}{N^{\frac{3}{10}}},\psi=2n\pi), n=0,1,2$, in the MQGP limit (1), inverting the co-frames of Appendix **C** :

$$d\theta_{i=1/2} = \sum_{a=2}^{6}\Theta_{ia}e^a,$$
$$dx = \sum_{a=2}^{6}\mathcal{X}_a e^a,$$
$$dy = \sum_{a=2}^{6}\mathcal{Y}_a e^a,$$
$$dz = \sum_{a=2}^{6}\mathcal{Z}_a e^a, \tag{67}$$

and (C31)-(C33):

$$e^a = e^{a\theta_1}(r)d\theta_1 + e^{a\theta_2}(r)d\theta_2 + e^{ax}(r)dx + e^{ay}(r)dy + e^{az}(r)dz, \tag{68}$$

and defining:

$$e^1 = \sqrt{G^{\mathcal{M}}_{rr}}dr, \tag{69}$$

one notes that:

$$de^a = \Omega_{ab} e^1 \wedge e^b, \tag{70}$$

where the "structure constants" Ω_{ab} are defined as under:

$$\Omega_{ab} \equiv \frac{\left(e^{a\theta_1}\ '(r)\Theta_{1b} + e^{a\theta_2}\ '(r)\Theta_{2b} + e^{ax}\ '(r)\mathcal{X}_b + e^{ay}\ '(r)\mathcal{Y}_b + e^{az}\ '(r)\mathcal{Z}_b\right)}{\sqrt{G^{\mathcal{M}}_{rr}}}. \tag{71}$$

The components of Ω_{ab}s after a small-β large-N small-a expansion are given in (D1). The two-form associated with the almost complex structure is given by:

$$J = e^{12} + e^{34} + e^{56}, \tag{72}$$

and the nowhere vanishing $(3,0)-$form Ω is given by:

$$\Omega = \left(e^1 + ie^2\right) \wedge \left(e^3 + ie^4\right) \wedge \left(e^5 + ie^6\right) \equiv \Omega_+ + i\Omega_-, \tag{73}$$

where $e^{a_1 \dots a_p} \equiv e^{a_1} \wedge \dots . e^{a_p}$. The five $SU(3)$-structure torsion classes are denoted by $W_{1,2,3,4,5}$.

One sees:

$$
\begin{aligned}
dJ &= \Omega_{32}e^{124} - \Omega_{42}e^{312} + \Omega_{52}e^{126} - \Omega_{62}e^{512} + (\Omega_{35} + \Omega_{62})\, e^{154} \\
&\quad + (\Omega_{36} - \Omega_{54})\, e^{146} + (\Omega_{33} + \Omega_{44})\, e^{134} + (\Omega_{45} + \Omega_{63})\, e^{315} \\
&\quad + (-\Omega_{46} - \Omega_{53})\, e^{316} + (\Omega_{55} + \Omega_{66})\, e^{156}, \\
d\Omega_+ &= -\Big(\Omega_{23}e^{1345} + \Omega_{26}e^{1645} + \Omega_{55}e^{2415} + \Omega_{22}e^{1236} + \Omega_{24}e^{1436} \\
&\quad + \Omega_{25}e_{1536} + (\Omega_{22}\Omega_{44} + \Omega_{55})\, e^{1245} + (\Omega_{35} - \Omega_{46})\, e^{2165} \\
&\quad + (\Omega_{34} + \Omega_{56})\, e_{2416} - (\Omega_{33} + \Omega_{66})\, e^{2136} + (\Omega_{53} - \Omega_{64})\, e^{2413} \\
&\quad - (\Omega_{43} + \Omega_{65})\, e^{2135}\Big), \\
d\Omega_- &= \Omega_{24}e^{1435} + \Omega_{26}e^{1635} - \Omega_{23}e^{1346} - \Omega_{25}e^{1546} - \Omega_{66}e^{2416} \\
&\quad + (\Omega_{22} + \Omega_{33} + \Omega_{55})\, e^{1235} + (-\Omega_{34} + \Omega_{65})\, e_{2145} \\
&\quad + (-\Omega_{36} - \Omega_{45})\, e^{2165} + (\Omega_{54} + \Omega_{63})\, e^{2314} \\
&\quad + (\Omega_{56} - \Omega_{43})\, e^{2316} - (\Omega_{22} + \Omega_{44})\, e^{1246},
\end{aligned}
\tag{74}
$$

implying:

$$
\begin{aligned}
2W_4 = J \lrcorner dJ &= \Omega_{32}e^{4} - \Omega_{42}e^{3} + \Omega_{52}e^{6} - \Omega_{62}e^{5} \\
&\quad + (\Omega_{33} + \Omega_{44})\, e^{1} + (\Omega_{55} + \Omega_{66})\, e^{1}.
\end{aligned}
\tag{75}
$$

Now, substituting (D1) - (D3), one sees that the $\mathcal{O}(l_p^0)$ terms in Ω_{62} goes like $\left(12.5 - 43.6\frac{a^2}{r^2}\right)\sqrt{1 - \frac{r_h^4}{r^4}}$ which assuming $a = r_h\left(0.6 + \frac{g_sM^2}{N}\right)$ [31], vanishes for $r \sim 1.25r_h$. Similarly, the $\mathcal{O}(l_p^0)$ term in Ω_{24} can be proportional to the $\mathcal{O}(l_p^0)$ term in $-\Omega_{42}$, i.e., $\mathcal{O}(1)\frac{a^2}{r^2}$ for $r \sim 0.5\sqrt{4 + \mathcal{O}(1)}a$. Thus:

$$
W_4 \approx \frac{\Omega_{32}e^{4} + \Omega_{52}e^{6} + \Omega_{66}e^{1}}{2}.
\tag{76}
$$

$$
\begin{aligned}
2W_5 = \Omega_+ \lrcorner d\Omega_+ &= \Omega_{23}e^{4} - \Omega_{25}e^{6} + \left(\Omega_{[43]} + \Omega_{[65]}\right) e^{2} + \Omega_{24}e^{3} \\
&\quad + (2\Omega_{22} + \Omega_{33} + \Omega_{44} + 2\Omega_{55} + \Omega_{66})\, e^{1}.
\end{aligned}
\tag{77}
$$

Now, one sees that the $\mathcal{O}(l_p^0)$ terms in $\left(\Omega_{[43]} + \Omega_{[65]}\right) e^2$ for the aforementioned IR-valued r would vanish for $\frac{\alpha_{\theta_1}}{\alpha_{\theta_2}} \sim \frac{1.3}{\alpha_r g_s^{\frac{7}{8}}\sqrt{MN_f}}$, where $|\log r| = \alpha_r N^{\frac{1}{3}}$ [41]. Also, from (D1) - (D3), one sees that: $2\Omega_{22} + \Omega_{33} + \Omega_{44} + 2\Omega_{55} + \Omega_{66} \approx$

Ω_{66}. Thus:

$$W_5 \sim \frac{\Omega_{23}e^4 - \Omega_{25}e^6 + \Omega_{66}e^1}{2} \xrightarrow{|\Omega_{23,25}|<|\Omega_{66}|} \frac{\Omega_{66}e^1}{2} \sim W_4. \tag{78}$$

This mimics supersymmetric [14, 66] IIA mirror though for a non-complex manifold - see below. As:

$$\Omega_+ \wedge dJ = -\left(\Omega_{(46)} + \Omega_{(53)}\right)\frac{J^3}{6} \equiv W_1^+ J^3, \tag{79}$$

implying:

$$W_1^+ = -\frac{\left(\Omega_{(46)} + \Omega_{(53)}\right)}{6}. \tag{80}$$

Similarly, as:

$$\Omega_- \wedge dJ = \left(\Omega_{(36)} - \Omega_{(45)}\right)\frac{J^3}{6} \equiv W_1^- J^3, \tag{81}$$

implying:

$$W_1^- = \frac{\left(\Omega_{(36)} - \Omega_{(45)}\right)}{6}. \tag{82}$$

Also, using the notation: $E^{i_1} \wedge ....E^{i_p} \wedge \bar{E}^{j_1} \wedge ....\bar{E}^{j_q} \equiv E^{i_1....i_p\bar{j}_1....\bar{j}_q}$, one notes that:

$$\begin{aligned}(dJ)^{(2,1)} = &\frac{1}{4}\left(\Omega_{32} - i\Omega_{42}\right)E^{1\bar{1}2} + \frac{1}{4}\left(\Omega_{52} - \Omega_{62}\right)\\ &+ \frac{1}{8}\left(-2\Omega_{45} + 2\Omega_{63} + \Omega_{46} + \Omega_{53}\right)E^{21\bar{3}}\\ &+ \frac{1}{8}\left(-\Omega_{45} + \Omega_{63} + i\Omega_{46} + i\Omega_{53}\right)E^{\bar{2}13}\\ &+ \frac{i}{4}\left(\Omega_{55} + \Omega_{66}\right)E^{3\bar{3}1}.\end{aligned} \tag{83}$$

Therefore:

$$(84)\quad \begin{aligned} W_3 &= (dJ)^{(2,1)} - (J\wedge W_4)^{(2,1)} \\ &= \left(\frac{\Omega_{32}}{4} - \frac{\Omega_{32}}{4} + i\frac{\Omega_{42}}{2}\right)E^{1\bar{1}2} \\ &\quad + \left[\frac{1}{4}(\Omega_{52}-\Omega_{62}) - \frac{1}{2}(\Omega_{52}-i\Omega_{62})\right]E^{1\bar{1}3} \\ &\quad + \frac{1}{8}(-2\Omega_{45}+2\Omega_{63}+\Omega_{46}+\Omega_{53})E^{21\bar{3}} \\ &\quad + \frac{1}{8}(-\Omega_{45}+\Omega_{63}+i\Omega_{46}+i\Omega_{53})E^{\bar{2}13} - \frac{1}{2}(\Omega_{52}-i\Omega_{62})E^{2\bar{2}3} \\ &\quad - i(\Omega_{33}+\Omega_{44}+\Omega_{55}+\Omega_{66})E^{2\bar{2}1} + (i\Omega_{42}-\Omega_{32})E^{3\bar{3}2} \\ &\quad + \left[\frac{i}{4}(\Omega_{55}+\Omega_{66}) - \frac{i}{2}(\Omega_{33}+\Omega_{44}+\Omega_{55}+\Omega_{66})\right]E^{3\bar{3}1}. \end{aligned}$$

To determine $W_2^{\pm}$, one notes that:

(85)

$$\begin{aligned} -(d\Omega_+)^{2,2} &= \left(\frac{i\Omega_{23}-\Omega_{24}}{8}\right)E^{2\bar{2}\bar{1}3} + \left(\frac{i\Omega_{23}+\Omega_{24}}{8}\right)E^{2\bar{2}1\bar{3}} \\ &\quad - \left(\frac{\Omega_{26}+i\Omega_{25}}{8}\right)E^{3\bar{3}1\bar{2}} + \left(\frac{\Omega_{26}-i\Omega_{25}}{8}\right)E^{3\bar{3}\bar{1}2} \\ &\quad + \left(\frac{-2\Omega_{22}+\Omega_{33}-\Omega_{44}+\Omega_{66}}{8} + \frac{i}{8}\left(\Omega_{(34)}+\Omega_{(56)}\right)\right)E^{1\bar{1}2\bar{3}} \\ &\quad + \left(\frac{2\Omega_{22}-\Omega_{33}+\Omega_{44}-\Omega_{66}}{8} - \frac{i}{8}\left(\Omega_{[34]}+\Omega_{[56]}\right)\right)E^{1\bar{1}\bar{2}3} \\ &\quad - \frac{(\Omega_{35}-\Omega_{46})}{4}E^{1\bar{1}3\bar{3}} + \frac{1}{4}(\Omega_{53}-\Omega_{64})E^{1\bar{1}2\bar{2}} \\ &= -\left(W_1^+J^2 + W_2^+\wedge J\right), \end{aligned}$$

and this implies:

$$(86)\quad \begin{aligned} W_2^+ &= \alpha_1 E^{1\bar{1}} + \beta_1 E^{2\bar{2}} + \gamma_1 E^{3\bar{3}} + \alpha_2 E^{1\bar{2}} + \beta_2 E^{2\bar{3}} \\ &\quad + \gamma_2 E^{1\bar{3}} + \alpha_3 E^{\bar{1}2} + \beta_3 E^{\bar{2}3} + \gamma_3 E^{\bar{1}3}, \end{aligned}$$

where:

$$\begin{aligned}
\alpha_1 &= -i\left[\frac{W_1^+ + \Omega_{53} - \Omega_{64}}{2}\right] + \frac{i}{4}\left(\Omega_{[53]} - \Omega_{[64]} + W_1^+\right)\\
\beta_1 &= -\frac{i}{4}\left(\Omega_{[53]} - \Omega_{[64]} + W_1^+\right)\\
\gamma_1 &= -\frac{i}{4}\left(W_1^+ - \Omega_{[53]} + \Omega_{[64]}\right),\\
\alpha_2 &= -\frac{\Omega_{26} - i\Omega_{25}}{8},\\
\gamma_2 &= \frac{\Omega_{24} + i\Omega_{23}}{8}\\
\beta_2 &= \frac{-2\Omega_{22} + \Omega_{33} - \Omega_{44} + \Omega_{66}}{8} + \frac{i}{8}\left(\Omega_{[56]} + \Omega_{[34]}\right),\\
\alpha_3 &= \frac{\Omega_{26} - i\Omega_{25}}{8},\\
\beta_3 &= \frac{2\Omega_{22} - 3\Omega_{33} + \Omega_{44} - \Omega_{66}}{8} + \frac{i}{8}\left(\Omega_{[43]} + \Omega_{[65]}\right).
\end{aligned} \tag{87}$$

5.2. G_2-Structure Torsion Classes of the Seven-Fold in the $\mathcal{M}$-Theory Uplift

Given that the adjoint of $SO(7)$ decomposes under G_2 as $\mathbf{21} \to \mathbf{7} \oplus \mathbf{14}$ where $\mathbf{14}$ is the adjoint representation of G_2, one obtains:

$$T \in \Lambda^1 \otimes g_2^\perp = W_1 \oplus W_{14} \oplus W_{27} \oplus W_7. \tag{88}$$

We now present the seventh lemma:

Lemma 7: In the neighborhood of $(\theta_{10} = \frac{\alpha_{\theta_1}}{N^{\frac{1}{5}}}, \theta_{20} = \frac{\alpha_{\theta_{20}}}{N^{\frac{3}{10}}}, \psi = 2n\pi), n = 0, 1, 2$, the G_2-structure torsion classes of M_7 - a cone over a six-fold which is an $\mathcal{M}$-theory S^1-fibration over a compact five-fold $M_5(\theta_1, \theta_2, \phi_1, \phi_2, \psi)$ - are given by: $W_{M_7}^{G_2} = W_{14} \oplus W_{27}$.

Proof: Now, near the $\psi = 0, 2\pi, 4\pi$-branches, the $\mathcal{M}$-Theory coframe $e^7 = \sqrt{G^{\mathcal{M}}_{x^{10}x^{10}}}dx^{10}$. Further, the three-form Φ corresponding to a G_2 structure is given by [67]:

$$\begin{aligned}
\Phi &= e^{-\Phi^{\rm IIA}} f_{abc} e^{abc} + e^{-\frac{2}{3}\Phi^{\rm IIA}} J \wedge dx^{10}\\
&= e^{-\Phi^{\rm IIA}}\left(e^{135} - e^{146} - e^{236} - e^{245}\right) + \frac{e^{-\frac{2}{3}\Phi^{\rm IIA}}}{\sqrt{G^{\mathcal{M}}_{x^{10}x^{10}}}}\left(e^{3417} + e^{5617}\right)
\end{aligned} \tag{89}$$

(90)

$$d\Phi = -\frac{e^{-\Phi^{\rm IIA}}\Phi'_{\rm IIA}}{\sqrt{G^{\mathcal{M}}_{rr}}}\left(-e^{1236}-e^{1245}\right) - \frac{2}{3}\frac{e^{-\frac{2}{3}\Phi^{\rm IIA}}\Phi'_{\rm IIA}}{\sqrt{G^{\mathcal{M}}_{rr}G^{\mathcal{M}}_{x^{10}x^{10}}}}J\wedge e^{17}$$

$$= e^{-\Phi^{IIA}}\Bigg(-\Omega_{24}e^{1436} - \Omega_{25}e^{1536} - \Omega_{66}e^{2316} - \Omega_{23}e^{1345} - \Omega_{26}e^{1645}$$

$$- \left(\Omega_{33}+\Omega_{22}\right)e^{1236} - \left(\Omega_{22}+\Omega_{44}+\Omega_{55}\right)e^{1245} + \left(\Omega_{35}-\Omega_{46}\right)e^{2156}$$

$$+ \left(\Omega_{53}-\Omega_{64}\right)e^{2314} - \left(\Omega_{43}+\Omega_{65}\right)e^{2315} + \left(\Omega_{34}+\Omega_{56}\right)e^{2146}\Bigg)$$

$$+ \frac{e^{-\frac{2}{3}\Phi^{\rm IIA}}}{\sqrt{G^{\mathcal{M}}_{x^{10}x^{10}}}}\Bigg(\Omega_{32}e^{1247} - \Omega_{42}e^{3127} + \Omega_{52}e^{1267} - \Omega_{62}e^{5127} + \left(\Omega_{35}+\Omega_{64}\right)e^{1547}$$

$$+ \left(\Omega_{36}-\Omega_{54}\right)e^{1467} + \left(\Omega_{33}+\Omega_{44}\right)e^{1347} + \left(\Omega_{45}+\Omega_{63}\right)e^{3157}$$

$$- \left(\Omega_{46}+\Omega_{53}\right)e^{3167} + \left(\Omega_{55}+\Omega_{66}\right)e^{1567}\Bigg)$$

$$= 4W_1 *_7 \Phi - 3W_7\wedge\Phi - *_7 W_{27}.$$

Similarly:

(91)

$$d *_7 \Phi = -\frac{e^{-\Phi^{\rm IIA}}\Phi'_{\rm IIA}}{\sqrt{G^{\mathcal{M}}_{rr}}}\left(e^{12467}-e^{12357}\right) - \frac{2}{3}\frac{e^{-\frac{2}{3}\Phi^{\rm IIA}}\Phi'_{\rm IIA}}{\sqrt{G^{\mathcal{M}}_{rr}}}e^{13456}$$

$$= e^{-\Phi^{\rm IIA}}\Bigg(\Omega_{23}e^{13467} + \Omega_{25}e^{15467} - \Omega_{43}e^{21367} - \Omega_{44}e^{21467} - \Omega_{45}e^{21567}$$

$$+ \Omega_{22}e^{12467} + \Omega_{63}e^{24137} + \Omega_{65}e^{24157} + \Omega_{66}e^{24167} - \Omega_{22}e^{12357} - \Omega_{24}e^{14357}$$

$$- \Omega_{26}e^{16357} + \Omega_{33}e^{21357} + \Omega_{34}e^{21457} + \Omega_{36}e^{21657} - \Omega_{54}e^{23147}$$

$$- \Omega_{55}e^{23157} - \Omega_{56}e^{23167} + \frac{e^{24617}-e^{23517}}{2\sqrt{G^{\mathcal{M}}_{rr}G^{\mathcal{M}}_{x^{10}x^{10}}}}\Bigg)$$

$$+ e^{-\frac{2}{3}\Phi^{\rm IIA}}\Bigg(\Omega_{32}e^{12456} + \Omega_{33}e^{13456} - \Omega_{42}e^{31256} - \Omega_{44}e^{31456}$$

$$+ \Omega_{52}e^{34126} + \Omega_{55}e^{34156} - \Omega_{62}e^{34512} - \Omega_{66}e^{34516}\Bigg)$$

$$= -4W_7\wedge *_7\Phi - 2 *_7 W_{14}.$$

One hence obtains (see App. **E** for details):

$$
\begin{aligned}
&W_1 = W_7 = 0,\\
&W_{27} = - *_7 d\Phi,\\
&W_{14} = -\frac{1}{2} *_7 d *_7 \Phi.
\end{aligned}
\tag{92}
$$

5.3. $SU(4)$-Structure Torsion Classes of the Eight-Fold in the $\mathcal{M}$-Theory Uplift

The $SU(4)$-structure torsion classes are given by [68]:

$$
\begin{aligned}
\Lambda^1 \otimes su(4)^\perp &= (\mathbf{4} \oplus \bar{\mathbf{4}}) \otimes (\mathbf{1} \oplus \mathbf{6} \oplus \bar{\mathbf{6}})\\
&= (\mathbf{4} \oplus \bar{\mathbf{4}}) \oplus \left(\mathbf{20} \oplus \overline{\mathbf{20}}\right) \oplus \left(\mathbf{20} \oplus \overline{\mathbf{20}}\right) \oplus (\mathbf{4} \oplus \bar{\mathbf{4}}) \oplus (\mathbf{4} \oplus \bar{\mathbf{4}})\\
&= W_1 \oplus W_2 \oplus W_3 \oplus W_4 \oplus W_5,
\end{aligned}
\tag{93}
$$

where:

$$
\begin{aligned}
dJ_4 &= W_1 \lrcorner \bar{\Omega}_4 + W_3 + W_4 \wedge J_4 + \text{c.c.}\\
d\Omega_4 &= \frac{8i}{3} W_1 \wedge J_4^2 + W_2 \wedge J_4 + \bar{W}_5 \wedge \Omega_4.
\end{aligned}
\tag{94}
$$

We are now set to present the eighth lemma:

Lemma 8: In the neighborhood of $(\theta_{10} = \frac{\alpha_{\theta_1}}{N^{\frac{1}{5}}}, \theta_{20} = \frac{\alpha_{\theta_{20}}}{N^{\frac{3}{10}}}, \psi = 2n\pi), n = 0, 1, 2$, the $SU(4)$-structure torsion classes of $M_8(r, \theta_{1,2}, \phi_{1,2}, \psi, x^{10}, x^0)$ are $W_{M_8}^{SU(4)} = W_2^{SU(4)} \oplus W_3^{SU(4)} \oplus W_5^{SU(4)}$.

Proof: Near the $\psi = 0, 2\pi, 4\pi$-branches, defining:

$$
\begin{aligned}
e^7 &= \sqrt{G^{\mathcal{M}}_{x^{10}x^{10}}} dx^{10},\\
e^0 &= \sqrt{G^{\mathcal{M}}_{00}} dx^0,
\end{aligned}
\tag{95}
$$

using which construct $E^4 = e^7 + ie^0$. Defining:

(96)

$$\zeta_{12a} \equiv \frac{\left[\left(e^{1\theta_1}{}' + ie^{2\theta_1}{}'\right)\Theta_{1a} + \left(e^{1\theta_2}{}' + ie^{2\theta_2}{}'\right)\Theta_{2a} + \left(e^{1x}{}' + ie^{2x}{}'\right)\mathcal{X}_a + \left(e^{1y}{}' + ie^{2y}{}'\right)\mathcal{Y}_a + \left(e^{1z}{}' + ie^{2z}{}'\right)\mathcal{Z}_a\right]}{\sqrt{G^{\mathcal{M}}_{rr}}},$$

$$\zeta_{34a} \equiv \frac{\left[\left(e^{3\theta_1}{}' + ie^{4\theta_1}{}'\right)\Theta_{13} + \left(e^{3\theta_2}{}' + ie_{4\theta_2}{}'\right)\Theta_{2a} + \left(e^{3x}{}' + ie^{4x}{}'\right)\mathcal{X}_a + \left(e^{3y}{}' + ie^{4y}{}'\right)\mathcal{Y}_a + \left(e^{3z}{}' + ie^{4z}{}'\right)\mathcal{Z}_a\right]}{\sqrt{G^{\mathcal{M}}_{rr}}},$$

$$\zeta_{56a} \equiv \frac{\left[\left(e^{5\theta_1}{}' + ie^{2\theta_1}{}'\right)\Theta_{1a} + \left(e^{5\theta_2}{}' + ie^{6\theta_2}{}'\right)\Theta_{2a} + \left(e^{5x}{}' + ie^{6x}{}'\right)\mathcal{X}_a + \left(e^{5y}{}' + ie^{6y}{}'\right)\mathcal{Y}_a + \left(e^{5z}{}' + ie^{6z}{}'\right)\mathcal{Z}_a\right]}{\sqrt{G^{\mathcal{M}}_{rr}}},$$

where $a = 2, ..., 6$, one obtains:

$$\begin{aligned} dE^1 = {} & \frac{i}{2}\frac{\zeta_{122}}{\sqrt{G^{\mathcal{M}}_{rr}}}E^1 \wedge \bar{E}^1 + \frac{\zeta_{12(3-i4)}}{\sqrt{G^{\mathcal{M}}_{rr}}}E^1 \wedge E^2 + \frac{\zeta_{12(3+i4)}}{\sqrt{G^{\mathcal{M}}_{rr}}}\bar{E}^1 \wedge \bar{E}^2 \\ & + \frac{\zeta_{12(3-i4)}}{\sqrt{G^{\mathcal{M}}_{rr}}}E^1 \wedge E^3 + \frac{\zeta_{12(3+i4)}}{\sqrt{G^{\mathcal{M}}_{rr}}}\bar{E}^1 \wedge \bar{E}^3 + + \frac{\zeta_{12(3+i4)}}{\sqrt{G^{\mathcal{M}}_{rr}}}E^1 \wedge \bar{E}^2 \\ & + \frac{\zeta_{12(3-i4)}}{\sqrt{G^{\mathcal{M}}_{rr}}}\bar{E}^1 \wedge E^2 + \frac{\zeta_{12(5+i6)}}{\sqrt{G^{\mathcal{M}}_{rr}}}\bar{E}^1 \wedge E^3 + \frac{\zeta_{12(5-i6)}}{\sqrt{G^{\mathcal{M}}_{rr}}}E^1 \wedge \bar{E}^3, \end{aligned} \tag{97}$$

etc., where, e.g., $\zeta_{12(3\pm i4)} \equiv \zeta_{123} \pm i\zeta_{124}$. One obtains:

$$\begin{aligned} & W_1 = W_4 = 0, \\ & W_3 + \bar{W}_3 = dJ. \end{aligned} \tag{98}$$

Writing:

$$\Omega_4 = \Omega_3 \wedge E^4, \tag{99}$$

one obtains:

$$
\begin{aligned}
d\Omega_4 = {} & \frac{i}{2}\frac{\zeta_{122}}{\sqrt{G^{\mathcal{M}}_{rr}}}E^{1\bar{1}234} + \frac{i}{4}\frac{\zeta_{12(3+i4)}}{\sqrt{G^{\mathcal{M}}_{rr}}}E^{\bar{1}\bar{2}234} + \frac{\zeta_{12(3+i4)}}{4\sqrt{G^{\mathcal{M}}_{rr}}}E^{\bar{1}\bar{3}234} \\
& + \frac{\zeta_{12(3+i4)}}{4\sqrt{G^{\mathcal{M}}_{rr}}}E^{1\bar{2}234} + \frac{\zeta_{12(5-i6)}}{4\sqrt{G^{\mathcal{M}}_{rr}}}E^{1\bar{3}234} \\
& - \left(\frac{i}{4}\frac{\zeta_{34(3+i4)}}{\sqrt{G^{\mathcal{M}}_{rr}}}E^{1\bar{1}\bar{2}34} + \frac{\zeta_{23(3+i4)}}{4\sqrt{G^{\mathcal{M}}_{rr}}}E^{1\bar{1}\bar{3}34} + \frac{\zeta_{34(3-i4)}}{4\sqrt{G^{\mathcal{M}}_{rr}}}E^{1\bar{1}234}\right) \\
& + \frac{\zeta_{564(3+i4)}}{\sqrt{G^{\mathcal{M}}_{rr}}}E^{12\bar{1}\bar{2}34} + \frac{\zeta_{56(3+i4)}}{4\sqrt{G^{\mathcal{M}}_{rr}}}E^{12\bar{1}\bar{3}4} + \frac{\zeta_{56(5+i6)}}{4\sqrt{G^{\mathcal{M}}_{rr}}}E^{12\bar{1}34} \\
& + \left(\frac{{G^{\mathcal{M}}_{x^{10}x^{10}}}'}{8\sqrt{G^{\mathcal{M}}_{x^{10}x^{10}}G^{\mathcal{M}}_{rr}}} - i\frac{{G^{\mathcal{M}}_{00}}'}{8\sqrt{G^{\mathcal{M}}_{00}G^{\mathcal{M}}_{rr}}}\right)\left(E^1 + \bar{E}^{\bar{1}}\right)\wedge\Omega_4 \\
& - \left(\frac{{G^{\mathcal{M}}_{x^{10}x^{10}}}'}{8\sqrt{G^{\mathcal{M}}_{x^{10}x^{10}}G^{\mathcal{M}}_{rr}}} + i\frac{{G^{\mathcal{M}}_{00}}'}{8\sqrt{G^{\mathcal{M}}_{00}G^{\mathcal{M}}_{rr}}}\right)\Omega_3\wedge E^{\bar{1}\bar{4}} \\
& - \left(\frac{{G^{\mathcal{M}}_{x^{10}x^{10}}}'}{8\sqrt{G^{\mathcal{M}}_{x^{10}x^{10}}G^{\mathcal{M}}_{rr}}} + i\frac{{G^{\mathcal{M}}_{00}}'}{8\sqrt{G^{\mathcal{M}}_{00}G^{\mathcal{M}}_{rr}}}\right)\Omega_3\wedge E^{1\bar{4}} \\
= {} & W_2\wedge J_4 + W_5\wedge\Omega_4,
\end{aligned} \tag{100}
$$

implying:

$$
\begin{aligned}
W_5 = {} & \left(\frac{{G^{\mathcal{M}}_{x^{10}x^{10}}}'}{8\sqrt{G^{\mathcal{M}}_{x^{10}x^{10}}G^{\mathcal{M}}_{rr}}} - i\frac{{G^{\mathcal{M}}_{00}}'}{8\sqrt{G^{\mathcal{M}}_{00}G^{\mathcal{M}}_{rr}}}\right)E^1 \\
& + \left(\frac{{G^{\mathcal{M}}_{x^{10}x^{10}}}'}{8\sqrt{G^{\mathcal{M}}_{x^{10}x^{10}}G^{\mathcal{M}}_{rr}}} - i\frac{{G^{\mathcal{M}}_{00}}'}{8\sqrt{G^{\mathcal{M}}_{00}G^{\mathcal{M}}_{rr}}} + \frac{i}{2}\frac{\zeta_{122}}{\sqrt{G^{\mathcal{M}}_{rr}}}\right)E^{\bar{1}} \\
& + \frac{\zeta_{12(3+i4)}}{4\sqrt{G^{\mathcal{M}}_{rr}}}E^{\bar{2}} + \frac{\zeta_{12(5-i6)}}{4\sqrt{G^{\mathcal{M}}_{rr}}}E^{\bar{3}}; \\
W_2 = {} & -i\frac{\zeta_{12(3+i4)}}{4\sqrt{G^{\mathcal{M}}_{rr}}}E^{134} + \frac{\zeta_{12(3+i4)}}{4\sqrt{G^{\mathcal{M}}_{rr}}}E^{\bar{1}24} - i\frac{\zeta_{34(3+i4)}}{4\sqrt{G^{\mathcal{M}}_{rr}}}E^{\bar{2}34} \\
& - \frac{\zeta_{56(3+i4)}}{4\sqrt{G^{\mathcal{M}}_{rr}}}E^{2\bar{3}4} + \frac{\alpha_1}{4\sqrt{G^{\mathcal{M}}_{rr}}}E^{3\bar{3}4} + \frac{\alpha_1}{4\sqrt{G^{\mathcal{M}}_{rr}}}E^{2\bar{2}4} \\
& + \frac{\alpha_1}{4\sqrt{G^{\mathcal{M}}_{rr}}}E^{1\bar{1}4},
\end{aligned} \tag{101}
$$

where:

$$\alpha_1 = i\left(\zeta_{23(3+i4)} + \zeta_{56(3+i4)}\right) = -\alpha_2,$$
$$\alpha_3 = -i\left(\zeta_{56(3+i4)} - \zeta_{23(3+i4)}\right). \tag{102}$$

5.4. *Spin*(7)-Structure Torsion Classes of the Eight-Fold in the $\mathcal{M}$-Theory Uplift

The $Spin(7)$-torsion classes are given by:

$$\Lambda^1 \otimes Spin(7)^{\perp} = (\mathbf{8} \oplus \mathbf{7}) \otimes \mathbf{7} = \mathbf{8} \oplus \mathbf{48} = W_1 \oplus W_2, \tag{103}$$

where:

$$W_1 = \Psi \lrcorner d\Psi, \tag{104}$$

where Ψ is a $Spin(7)$-invariant self-dual four-form:

$$\Psi = e^{1234} + e^{1256} + e^{1278} + e^{3456} + e^{3478} + e^{5678} + e^{1357} - e^{1268} - e^{1458} - e^{1467} - e^{2358} - e^{2367} - e^{2457} + e^{2468}. \tag{105}$$

We now present the final lemma:

Lemma 9: In the neighborhood of $(\theta_{10} = \frac{\alpha_{\theta_1}}{N^{\frac{1}{5}}}, \theta_{20} = \frac{\alpha_{\theta_{20}}}{N^{\frac{3}{10}}}, \psi = 2n\pi), n = 0, 1, 2,$ the $Spin(7)$-torsion classes are given by: $W^{Spin(7)}_{M_8} = W_1^{Spin(7)} \oplus W_2^{Spin(7)}$.

Proof: Hence:

$$\begin{aligned} d\Psi = \sum_{a=2}^{6}\Big[&\Omega_{3a}\left(e^{1a456} + e^{1a478} + e^{21a58} + e^{21a67}\right) \\ &- \Omega_{4a}\left(e^{21a68} + e^{31a56} + e^{31a78} - e^{21a57}\right) \\ &+ \Omega_{5a}\left(e^{341a6} + e^{1a678} - e^{231a8} - e^{241a7}\right) \\ &- \Omega_{6a}\left(e^{3451a} + e^{51a78} + e^{231a7} - e^{241a8}\right) \\ &+ \Omega_{7a}\left(e^{341a8} + e^{561a8} + e^{2361a}\right) \\ &+ \Omega_{2a}\left(-e^{1a358} - e^{1a367} - e^{1a457} + e^{1a468}\right)\Big] + \frac{{G^{\mathcal{M}}_{x^{10}x^{10}}}'}{2\sqrt{G^{\mathcal{M}}_{x^{10}x^{10}} G^{\mathcal{M}}_{rr}}} e^{24517} \\ &- \frac{{G^{\mathcal{M}}_{x^0x^0}}'}{2\sqrt{G^{\mathcal{M}}_{x^0x^0} G^{\mathcal{M}}_{rr}}}\left(e^{34718} + e^{56718} - e^{23518} + e^{24618}\right), \end{aligned} \tag{106}$$

implying $W_1 = \sum_{a=1}^{8} \Lambda_a \left(\{\Omega_{bc}\}\right) e^a$ and non-trivial W_2.

Now, W_2 is identified with the space of three-forms: $\Lambda^3_{48} = \{\gamma \in \Lambda^{\cdot} \gamma \wedge \Psi = 0\}$ [69]. From (105), one sees that such a three-form will be given by:

$$(107) \quad \gamma = \sum_{a,n\neq 1:21 \text{ components}} \alpha_{[ab]} e^{1ab} + \sum_{c,d\neq 1,2:15 \text{ components}} \beta_{[cd]} e^{2cd} + \sum_{e,f\neq 1,2,3:10 \text{ components}} \kappa_{[ef]} e^{3ef} + \sum_{g,h\neq 1,2,3,5:3 \text{ componentsi}} \omega_{[gh]} e^{5gh},$$

along with one constraint on the 49 $\alpha_{[ab]}, \beta_{[cd]}, \kappa_{[ef]}, \omega_{[gh]}$ coefficients in (107).

6. Summary and future directions

Finite (gauge/'t Hooft) coupling top-down non-conformal holography is a largely unexplored territory in the field of gauge-gravity duality. The only Ultra Violet-complete top-down holographic dual of thermal QCD-like theories that we are aware of, was proposed in [2]. Later, the type IIA mirror of the same at intermediate gauge/string coupling was constructed and the $\mathcal{M}$ theory uplift of the same were constructed in [1, 16]. Other than higher-derivative corrections quartic in the Weyl tensor, or of the Gauss-Bonnet type, in $AdS_5 \times S^5$, dual to supersymmetric thermal Super Yang-Mills [3], there is little known about top-down string theory duals at intermediate 't Hooft coupling of thermal QCD-like theories. This paper fills this gap by working out the $\mathcal{M}$ theory dual of thermal QCD-like theories at intermediate 't Hooft coupling in the IR.

The following is a summary of the important results obtained in this work.

1) We work out the $\mathcal{O}(l_p^6)$ corrections to the $\mathcal{M}$-Theory metric worked out in [1, 16] arising from the $\mathcal{O}(R^4)$ terms in $D = 11$ supergravity. We realize that in the MQGP limit of [1], the contribution from the J_0 (and its variation) dominate over the contribution from E_8 and its variation as a consequence of which E_8 has been disregarded. The computations have been partitioned into two portions - one near the $\psi = 2n\pi, n = 0, 1, 2$ patches and the other away from the same (wherein there is no decoupling of the radial direction, the six angles and the $\mathcal{M}$-Theory circle).

2) We also note that there is a close connection between finite ('t Hooft) coupling effects in the IR and non-conformality (which being effected via the effective number of the fractional $D3$-branes, vanishes in the UV) as almost all corrections to the $\mathcal{M}$-Theory metric components of [1, 16] in the IR arising from the aforementioned $\mathcal{O}(R^4)$ terms in $D = 11$ supergravity action, vanish when the number of fractional $D3$-branes is set to zero.

3) The importance of the higher derivative corrections arises from the competition between the non-conformal Infra-Red enhancement $\frac{(\log \mathcal{R}_h)^m}{\mathcal{R}_h^n}, \mathcal{R}_h \equiv \frac{r_h}{\mathcal{R}_{D5/\overline{D5}}}, m = 0, 1, 2, 3, n = 0$, and the Planckian and large-N suppression $\frac{l_p^6}{N^{\beta_N}}, \beta_N > 0$ in the $\mathcal{O}(l_p^6)$ corrections to the $\mathcal{M}$-theory dual [1, 16] of thermal QCD-like theories. As $|\log \mathcal{R}_h| \sim N^{\frac{1}{3}}$ [41], for appropriate values of N, it may turn out that this correction may become of $\mathcal{O}(1)$, and thereby very significant. This would also then imply that one will need to consider higher order corrections beyond $\mathcal{O}(l_p^6)$.

4)
- On the mathematical side, using Lemmas 1 - 9 of sections **4** and **5**, the main result of this work, in addition to providing for the first time the $\mathcal{O}(l_p^6)$-corrections to the $\mathcal{M}$-theory dual of thermal QCD-like theories of [1], is Proposition 1 stated in Section **1**. We work out the fundamental two-form and the nowhere vanishing holomorphic three-form of the six-fold obtained by an $\mathcal{M}$-Theory circle reduction of the $\mathcal{M}$-Theory dual obtained. This enabled us to work out the $SU(3)$-structure torsion classes of the aforementioned six-fold $M_6(r, \theta_{1,2}, \phi_{1,2}, \psi)$ relevant to the type IIA SYZ mirror, the G_2-structure torsion classes of the seven-fold $M_7(r, \theta_{1,2}, \phi_{1,2}, \psi, x^{10})$ as well as the $SU(4)$-structure and $Spin(7)$-structure torsion classes of the eight-fold $M_8(x^0, r, \theta_{1,2}, \phi_{1,2}, \psi, x^{10})$ relevant to the $\mathcal{M}$-Theory uplift. Table 1 summarizes the G-structure torsion classes' results. Table 2 summarizes the G-structure torsion classes' results.
- Along the Ouyang embedding (for very small [modulus of the] Ouyang embedding parameter) effected, e.g., near the $\psi = 2n\pi, n = 0, 1, 2$-patches in the MQGP limit, the large-base of the delocalized T^2-invariant sLag-fibration relevant to constructing the delocalized SYZ type IIA mirror in [1, 16] of the type IIB dual of thermal QCD-like theories in [2] manifests itself in the $\mathcal{O}(R^4)$ corrections to the co-frames that diagonalize the mirror six-fold metric in the following sense. It is only the constant of integration $\mathcal{C}_{zz}^{(1)}$

S. No.	Manifold	G-Structure	Non-Trivial Torsion Classes
1.	$M_6(r, \theta_1, \theta_2, \phi_1, \phi_2, \psi)$	$SU(3)$	$T^{\rm IIA}_{SU(3)} = W_1 \oplus W_2 \oplus W_3 \oplus W_4 \oplus W_5 : W_4 \sim W_5$
2.	$M_7(r, \theta_1, \theta_2, \phi_1, \phi_2, \psi, x^{10})$	G_2	$T^{\mathcal{M}}_{G_2} = W_{14} \oplus W_{27}$
3.	$M_8(x^0, r, \theta_1, \theta_2, \phi_1, \phi_2, \psi, x^{10})$	$SU(4)$	$T^{\mathcal{M}}_{SU(4)} = W_2 \oplus W_3 \oplus W_5$
4.	$M_8(x^0, r, \theta_1, \theta_2, \phi_1, \phi_2, \psi, x^{10})$	$Spin(7)$	$T^{\mathcal{M}}_{Spin(7)} = W_1 \oplus W_2$

Table 2: IR G-Structure Classification of Six-/Seven-/Eight-Folds in the type IIA/$\mathcal{M}$-Theory Duals of Thermal QCD-Like Theories (at High Temperatures)

appearing in the solution to the f_{zz} EOM corresponding to the (delocalized version of the) $U(1)$-fiber $S^1(\psi)$ - part of the (delocalized version of) $T^3(\psi, \phi_1, \phi_2)$ orthogonal to the aforementioned large base $B_3(r, \theta_1, \theta_2)$ - that determines in the MQGP limit, the aforementioned l_p^6 corrections in the IR to the MQGP results of the co-frames and hence G-structure torsion classes.

5) **Brief summary of published) applications of the results of this paper**: *We had decided to first work out applications of the results obtained in this paper to a variety of issues in Physics also including comparison (for some of the issues) with experiments/phenomenological data available, and after successfully doing so in [4], [5], submit an abridged version of the original version of this work (that was posted on the arXiv last year, arXiv:2004.07259[hep-th], cross-listed with math.dg), to ATMP.*

- As an application of the results of this paper modified to a thermal $\mathcal{M}$-theory dual of thermal QCD-like theories at low temperatures, we now summarize in the context of $\mathcal{M}\chi PT$, the main result of [4] (involving both the authors):
  - $\mathcal{O}(R^4)$*-large-*N *connection*: In the context of low energy coupling constants (LECs) of the $SU(3)$ χPT Lagrangian in the chiral limit at $\mathcal{O}(p^4)$, as shown in detail in [4] (and briefly explained in Section **4**) as an application of the $\mathcal{O}(R^4)$ corrections to the $\mathcal{M}$-theory uplift of large-N thermal QCD-like theories, matching the values of the one-loop renormalized coupling constants up to $\mathcal{O}(p^4)$ with experimental/lattice results shows that there is an underlying connection between large-N suppression and higher derivative corrections.
  - $\mathcal{M}\chi PT$ *and Flavor Memory*: As shown in [4] (involving both the authors), matching the phenomenological value of the

1-loop renormalized coupling constant corresponding to the $\mathcal{O}(p^4)$ $SU(3)$ χPT Lagrangian term "$\left(\nabla_\mu U^\dagger \nabla^\mu U\right)^2$", with the value obtained from the type IIA dual of thermal QCD-like theories inclusive of the aforementioned $\mathcal{O}(R^4)$ corrections, required the $\mathcal{O}(R^4)$ corrections arising from the contributions arising from the corrections to the metric along the compact S^3 part of the non-compact four-cycle "wrapped" by the flavor $D7$-branes of the parent type IIB theory, to have a definite sign (negative). The thermal supergravity background dual to type IIB (solitonic) $D3$-branes at low temperatures, includes $\mathbb{R}^2 \times S^1(\frac{1}{M_{\rm KK}})$. By taking the $M_{\rm KK} \to 0$ limit (to recover a boundary four-dimensional QCD-like theory after compactifying on the base of a G_2-structure cone), remarkably, via a delicate cancelation between some of the aforementioned contributions arising from the $\mathcal{O}(R^4)$ metric corrections with a resultant contribution solely along the vanishing S^2 (with the abovementioned S^3, an S^1 fibration over the vanishing S^2) of the parent type IIB suviving, we *derive* and hence verify the MχPT requirement of the sign. We also referred to this as "Flavor Memory" in [5] (involving both the authors).

- As an application of the results of this paper as well as the same modified to $\mathcal{M}$-theory duals of thermal QCD-like theories at respectively high and low temperatures, we now summarize in the context of obtaining *Deconfinement temperature*, the main result of [5] (involving both the authors): :
  - *UV-IR Mixing and Flavor Memory*: Performing a semiclassical computation [50] in [5] (involving both the authors), by matching the actions at the deconfinement temperature of the $\mathcal{M}$-theory uplifts of the thermal and black-hole backgrounds at the UV cut-off, it was shown that one obtains a relationship in the IR between the $\mathcal{O}(R^4)$ corrections to the $\mathcal{M}$-theory metric along the $\mathcal{M}$-theory circle in the thermal background and the $\mathcal{O}(R^4)$ correction to a specific combination of the $\mathcal{M}$-theory metric components along the compact part of the four-cycle "wrapped" by the flavor $D7$-branes of the parent type IIB (warped resolved deformed) conifold geometry - the latter referred to as "Flavor Memory" in the context of $\mathcal{M}\chi$PT above.
  - *Non-Renormalization of T_c*: We further showed in [5] that the LO result for T_c also holds even after inclusion of the $\mathcal{O}(R^4)$

corrections. The dominant contribution from the $\mathcal{O}(R^4)$ terms in the large-N limit arises from the $t_8 t_8 R^4$ terms, which from a type IIB perspective in the zero-instanton sector, correspond to the tree-level contribution at $\mathcal{O}\left((\alpha')^3\right)$ as well as one-loop contribution to four-graviton scattering amplitude and obtained from integration of the fermionic zero modes. As from the type IIB perspective, the $SL(2,\mathbb{Z})$ completion of these R^4 terms [33] suggests that they are not renormalized perturbatively beyond one loop in the zero-instanton sector, this therefore suggests the non-renormalization of T_c at all loops in $\mathcal{M}$-theory at $\mathcal{O}(R^4)$.

- *T_c from Entanglement Entropy*: With an obvious generalization of [6] to $\mathcal{M}$-theory, the entanglement entropy between two regions by dividing one of the spatial coordinates of the thermal $\mathcal{M}$-theory background into a segment of finite length l and its complement, was also calculated in [5]. Like [6], there are two RT surfaces - connected and disconnected. There is a critical value of l - denoted by l_{crit} - such that if $l < l_{crit}$, corresponding to the confined phase then it is the connected surface that dominates the entanglement entropy, and if $l > l_{crit}$ corresponding to the deconfined phase then it is the disconnected surface that dominates the entanglement entropy. This is interpreted as confinement-deconfinement phase transition.
- *Non-Renormalization of T_c from Entanglement Entropy*: Remarkably, when evaluating the deconfinement temperature from an entanglement entropy computation in the thermal gravity dual, due to an exact and delicate cancelation between the $\mathcal{O}(R^4)$ corrections from a subset of the abovementioned metric components, one sees that there are consequently no corrections to T_c at quartic order in the curvature supporting the conjecture made in on the basis of a semiclassical computation.

6) **Future directions**:

- *Math*:
  - **Almost Contact Metric Structure, Contact Structure and $SU(3)/SU(2)$ structure from G_2 structure**: Using the G_2 structure seven-fold M_7 of the $\mathcal{M}$-theory uplift of large-N thermal QCD-like theories inclusive of $\mathcal{O}(R^4)$ corrections as obtained in this work, equipped with a positive form φ and the G_2 metric g_{G_2}, it can be shown that the same is equipped with

an Almost Contact Metric Structure (ACMS) (J, R, σ, g_{G_2}) [71, 72], R being a unit vector field with J being a vector-valued one-form on M_7: $J^i_j = -\varphi^i_{jk} R^k$ and σ being a one-form: $\sigma_i = g^{G_2}_{ij} R^j$. It will be very interesting to explicitly construct the R and hence J and σ, and verify if the ACMS so obtained is also a contact structure [75]. Using results of [73], it will also be extremely interesting to explicitly obtain an embedding of $SU(3)$ and $SU(2)$ structures in G_2 structure and using the results of [74], an Almost Contact 3-Structure (AC3S) [75].

– For simplicity, we worked out the aforementioned G-Structures near the $\psi = 2n\pi, n = 0, 1, 2$-branches restricted to small-parameter Ouyang embedding, but as mentioned towards the beginning of Section **4**, using (C31) - (C33) and ideas of [11], the results of Table 1 are independent of angular delocalization in $\theta_{1,2}$. As regards independence of the results of Table 1 of ψ-delocalization, using the results of Section **3**, one sees that the secular equation needed to be solved to diagonalize the $M_7(r, \theta_{1,2}, \phi_{1,2}, \psi, x^{10})$ will be a septic $\mathcal{P}_7 = 0$. Hence, one needs to use Siegel theta functions of genus four. The period matrix Ω will be defined as follows:

$$\Omega_{ij} = \sigma^{ik} \rho_{kj}$$

where

$$\sigma_{ij} \equiv \oint_{A_j} d\mathcal{Z} \frac{\mathcal{Z}^{i-1}}{\sqrt{\mathcal{Z}(\mathcal{Z}-1)\mathcal{P}_7(\mathcal{Z})}}$$

and

$$\rho_{ij} \equiv \oint_{B_j} \frac{\mathcal{Z}^{i-1}}{\sqrt{\mathcal{Z}(\mathcal{Z}-1)(\mathcal{Z}-2)\mathcal{P}_7(\mathcal{Z})}},$$

$\{A_i\}$ and $\{B_i\}$ being a canonical basis of cycles satisfying: $A_i \cdot A_j = B_i \cdot B_j = 0$ and $A_i \cdot B_j = \delta_{ij}$; σ^{ij} are normalization constants determined by: $\sigma^{ik}\sigma_{kj} = \delta^i_j$. Umemura's result then is that a root is given by:

$$\frac{1}{2\left(\theta\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}(0,\Omega)\right)^4 \left(\theta\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}(0,\Omega)\right)^4}$$

$$\times\Bigg[\left(\theta\left[\begin{array}{cccc}\frac{1}{2} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right](0,\Omega)\right)^4\left(\theta\left[\begin{array}{cccc}\frac{1}{2} & \frac{1}{2} & 0 & 0\\ 0 & 0 & 0 & \end{array}\right](0,\Omega)\right)^4$$

$$+\left(\theta\left[\begin{array}{cccc}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right](0,\Omega)\right)^4\left(\theta\left[\begin{array}{cccc}0 & \frac{1}{2} & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right](0,\Omega)\right)^4$$

$$-\left(\theta\left[\begin{array}{cccc}0 & 0 & 0 & 0\\ \frac{1}{2} & 0 & 0 & 0\end{array}\right](0,\Omega)\right)^4\left(\theta\left[\begin{array}{cccc}0 & \frac{1}{2} & 0 & 0\\ \frac{1}{2} & 0 & 0 & 0\end{array}\right](0,\Omega)\right)^4\Bigg].$$

However, it will again be a non-trivial task to evaluate the period integrals. Alternatives will be deferred to a later work.

- *Physics*: In the context of intermediate 't Hooft coupling top-down holography, there is no known literature on applying gauge-gravity duality techniques to studying the perturbative regime of thermal QCD-like theories so as to be able to explain, e.g., low-frequency peaks expected to occur in spectral functions associated with transport coefficients, from $\mathcal{M}$ theory. In higher dimensional (Gauss-Bonnet or quartic in the Weyl tensor) holography, in the past couple of years using previously known results, it has been shown by the (Leiden-)MIT-Oxford collaboration [3] that one obtains low frequency peaks in correlation/spectral functions of energy momentum tensor, per unit frequency, obtained from the dissipative (i.e. purely imaginary) quasi-normal modes. As an extremely crucial application of the results of our paper, for the first time, spectral/correlation functions involving the energy momentum tensor with the inclusion of the $\mathcal{O}(l_p^6)$ corrections in the $\mathcal{M}$ theory (uplift) metric of [1] can be evaluated and hence one would be able to make direct connection between previous results in perturbative thermal QCD-like theories (e.g., [70]) as well as QCD plasma in RHIC experiments, and $\mathcal{M}$ theory. Further, the temperature dependence of the speed of sound, the attenuation constant and bulk viscosity can also be obtained from its solution, as well as the $\mathcal{O}(l_p^6)$ and the non-conformal corrections to the conformal results thereof. One could see if one could reproduce the known weak-coupling result from $\mathcal{M}$ theory that the ratio of the bulk and shear viscosities goes like the square of the deviation of the square of the speed of sound from its conformal value (the last reference in [15]). Generically, the dissipative quasi-normal modes in the spectral functions at low frequencies can be investigated to study the existence of peaks at low frequencies in transport coefficients, thus making direct contact with perturbative QCD results as well as (QCD plasma in) RHIC experiments.

Acknowledgements

VY is supported by a Senior Research Fellowship (SRF) from the University Grants Commission, Govt. of India. AM was supported by IIT Roorkee and the Council of Scientific and Industrial Research via a grant CSR-1477-PHY. He would also like to thank the theory group at McGill University, and Keshav Dasgupta in particular, for the wonderful hospitality and support during various stages of the work.

Appendix A. Equations of motion

This appendix discusses the form of the equations of motion (EOMs) satisfied by f_{MN}s inclusive of the $\mathcal{O}(l_p^6 R^4)$ corrections to the $\mathcal{M}$-Theory uplift of [1] as well as their solutions. **A.1** works out the same near the $\psi = 2n\pi, n = 0, 1, 2$-patches, and **A.2** near the $\psi \neq 2n\pi, n = 0, 1, 2$-patch. The EOMs have been obtained expanding the coefficients of $f_{MN}^{(p)}, p = 0, 1, 2$ near $r = r_h$ and retaining the LO terms in the powers of $(r - r_h)$ in the same, and then performing a large-N-large-$|\log r_h|$-$\log N$ expansion the resulting LO terms are written out. We should keep in mind that near the $\psi = \psi_0 \neq 2n\pi, n = 0, 1, 2$-patch, some $G^{\mathcal{M}}_{rM}, M \neq r$ and $G^{\mathcal{M}}_{x^{10}N}, N \neq x^{10}$ components are non-zero, making this exercise much more non-trivial.

As the EOMs are too long, they have not been explicitly typed but their forms have been written out. The solutions of the EOMs are discussed in detail.

A.1. EOMs for f_{MN} and Their Solutions Near $\psi = 2n\pi, n = 0, 1, 2$-Branches and Near $r = r_h$

Working in the IR, the EOMs near the $\psi = 0$ 2π 4π-branches near $r = r_h$ as described in Section **5.2**, can be written as follows:

$$\text{(A1)}\quad \text{EOM}_{MN}: \sum_{p=0}^{2}\sum_{i=0}^{2} a_{MN}^{(p,i)}\left(r_h, a, N, M, N_f, g_s, \alpha_{\theta_{1,2}}\right)(r - r_h)^i f_{MN}^{(p)}(r) + \beta \mathcal{F}_{MN}\left(r_h, a, N, M, N_f, g_s, \alpha_{\theta_{1,2}}\right)(r - r_h)^{\alpha_{MN}^{\rm LO}} = 0,$$

where M, N run over the $D = 11$ coordinates, $f_{MN}^{(p)} \equiv \frac{d^p f_{MN}}{dr^p}, p = 0, 1, 2,$ $\alpha_{MN}^{\rm LO} = 0, 1, 2, 3$ denotes the leading order (LO) terms in powers of $r - r_h$ in the IR when the $\mathcal{O}(\beta)$-terms are expanded in a Taylor series about $r = r_h$, and $\mathcal{F}_{\theta_1\theta_2} = \mathcal{F}_{\theta_2\theta_2} = 0$ (i.e. EOM$_{\theta_1\theta_2}$ and EOM$_{\theta_2\theta_2}$ are homogeneous up to $\mathcal{O}(\beta)$).

In the EOMs in this appendix and their solutions in Section **3**:

(A2)

$$
\begin{aligned}
\Sigma_1 &\equiv 19683\sqrt{6}\alpha_{\theta_1}^6 + 6642\alpha_{\theta_2}^2\alpha_{\theta_1}^3 - 40\sqrt{6}\alpha_{\theta_2}^4 \\
&\stackrel{\rm Global}{\longrightarrow} N^{\frac{6}{5}}\left(19683\sqrt{6}\sin^6\theta_1 + 6642\sin^2\theta_2\sin^3\theta_1 - 40\sqrt{6}\sin^4\theta_2\right), \\
\Sigma_2 &\equiv \Big(387420489\sqrt{2}\alpha_{\theta_1}^{12} + 87156324\sqrt{3}\alpha_{\theta_2}^2\alpha_{\theta_1}^9 + 5778054\sqrt{2}\alpha_{\theta_2}^4\alpha_{\theta_1}^6 \\
&\quad - 177120\sqrt{3}\alpha_{\theta_2}^6\alpha_{\theta_1}^3 + 1600\sqrt{2}\alpha_{\theta_2}^8\Big) \\
&\stackrel{\rm Global}{\longrightarrow} N^{\frac{12}{5}}\Big(387420489\sqrt{2}\sin_{\theta_1}^{12} + 87156324\sqrt{3}\sin_{\theta_2}^2\sin^9\theta_1 \\
&\quad + 5778054\sqrt{2}\sin^4\theta_2\sin^6\theta_1 - 177120\sqrt{3}\sin^6\theta\sin_{\theta_1}^3 + 1600\sqrt{2}\sin^8\theta_2\Big).
\end{aligned}
$$

The following EOMs' solutions will be obtained assuming $f_{\theta_1 y}''(r) = 0$. One can show that one hence ends up 15 independent EOMs and four that serve as consistency checks. We now discuss all below.

(i) EOM$_{tt}$:

(A3)

$$
\begin{aligned}
&\frac{4\left(9b^2+1\right)^3\left(4374b^6+1035b^4+9b^2-4\right)\beta b^8 M\left(\frac{1}{N}\right)^{9/4}\Sigma_1\left(6a^2+{r_h}^2\right)\log(r_h)}{27\pi\left(18b^4-3b^2-1\right)^5\log N^2 N_f {r_h}^2\alpha_{\theta_2}^3\left(9a^2+{r_h}^2\right)} \\
&-\frac{6\left({r_h}^2-2a^2\right)f_t(r)}{r_h\left({r_h}^2-3a^2\right)(r-r_h)} \\
&-\frac{32\sqrt{2}\left(9b^2+1\right)^4\beta b^{12}\left(\frac{1}{N}\right)^{3/20}\Sigma_1(r-r_h)}{81\pi^3\left(1-3b^2\right)^{10}\left(6b^2+1\right)^8 {g_s}^{9/4}\log N^4 N^{61/60}{N_f}^3{r_h}^4\alpha_{\theta_1}^7\alpha_{\theta_2}^6\left(-27a^4+6a^2{r_h}^2+{r_h}^4\right)} \\
&+2{f_t}''(r) = 0,
\end{aligned}
$$

where Σ_1 is defined in (A2).

As, the solution to the differential equation:

$$
2{f_t}''(r) + \frac{\Gamma_{f_{t1}}f_t(r)}{r-r_h} + \Gamma_{f_{t2}}(r-r_h) + \Gamma_{f_{t3}} = 0, \tag{A4}
$$

is given by:

(A5)

$$\frac{1}{2}\Bigg[-\frac{1}{\Gamma_{f_{t1}}}\Bigg\{(r-r_h)^2\Bigg(\Gamma_{f_{t1}}\Gamma_{f_{t2}}(r-r_h)I_1\left(\sqrt{2}\sqrt{\Gamma_{f_{t1}}(r_h-r)}\right)G_{1,3}^{2,1}\left(\frac{\sqrt{\Gamma_{f_{t1}}(r_h-r)}}{\sqrt{2}},\frac{1}{2}\middle|\begin{array}{c}-\frac{3}{2}\\ -\frac{1}{2},\frac{1}{2},-\frac{5}{2}\end{array}\right)$$
$$+\Gamma_{f_{t1}}\Gamma_{f_{t3}}I_1\left(\sqrt{2}\sqrt{\Gamma_{f_{t1}}(r_h-r)}\right)G_{1,3}^{2,1}\left(\frac{\sqrt{\Gamma_{f_{t1}}(r_h-r)}}{\sqrt{2}},\frac{1}{2}\middle|\begin{array}{c}-\frac{1}{2}\\ -\frac{1}{2},\frac{1}{2},-\frac{3}{2}\end{array}\right)$$
$$+K_1\left(\sqrt{2}\sqrt{\Gamma_{f_{t1}}(r_h-r)}\right)\Bigg[\sqrt{2}\sqrt{\Gamma_{f_{t1}}(r_h-r)}\left(2\Gamma_{f_{t2}}I_4\left(\sqrt{2}\sqrt{\Gamma_{f_{t1}}(r_h-r)}\right)-\Gamma_{f_{t1}}\Gamma_{f_{t3}}\,{}_0\tilde{F}_1\left(;3;\frac{1}{2}\Gamma_{f_{t1}}(r_h-r)\right)\right)$$
$$+8\Gamma_{f_{t2}}I_3\left(\sqrt{2}\sqrt{\Gamma_{f_{t1}}(r_h-r)}\right)\Bigg]\Bigg)\Bigg\}$$
$$+\sqrt{2}c_1\sqrt{\Gamma_{f_{t1}}(r_h-r)}I_1\left(\sqrt{2}\sqrt{\Gamma_{f_{t1}}(r_h-r)}\right)-\sqrt{2}c_2\sqrt{\Gamma_{f_{t1}}(r_h-r)}K_1\left(\sqrt{2}\sqrt{\Gamma_{f_{t1}}(r_h-r)}\right)\Bigg].$$

To prevent the occurency of a (logarithmic) singularity at $r=r_h$, one sets: $c_2=0$ which yields:

(A6)
$$f_t(r)=\frac{1}{4}\Gamma_{f_{t3}}(r-r_h)^2+\mathcal{O}\left((r-r_h)^2\right),$$

where:

(A7)
$$\Gamma_{f_{t3}}\equiv\frac{4b^8\left(9b^2+1\right)^3\left(4374b^6+1035b^4+9b^2-4\right)\beta M\left(\frac{1}{N}\right)^{9/4}\Sigma_1\left(6a^2+{r_h}^2\right)\log(r_h)}{27\pi(18b^4-3b^2-1)^5\log N^2 N_f {r_h}^2\alpha_{\theta_2}^3(9a^2+{r_h}^2)},$$

where Σ_1 is defined in (A2).

(ii) $\mathrm{EOM}_{x^1x^1}$:

(A8)
$$-\frac{6r_h\left(57a^4+14a^2{r_h}^2+{r_h}^4\right)f(r)}{\left({r_h}^2-3a^2\right)\left(6a^2+{r_h}^2\right)\left(9a^2+{r_h}^2\right)(r-r_h)}+2f''(r)$$
$$-\frac{4\left(9b^2+1\right)^4\left(39b^2-4\right)\beta b^8 M\left(\frac{1}{N}\right)^{9/4}\Sigma_1\left(6a^2+{r_h}^2\right)\log(r_h)}{9\pi\left(3b^2-1\right)^5\left(6b^2+1\right)^4\log N^2 N_f {r_h}^2\alpha_{\theta_2}^3\left(9a^2+{r_h}^2\right)}$$
$$-\frac{32\sqrt{2}\left(9b^2+1\right)^4\beta b^{12}\left(\frac{1}{N}\right)^{3/20}\Sigma_1(r-r_h)}{81\pi^3\left(1-3b^2\right)^{10}\left(6b^2+1\right)^8{g_s}^{9/4}\log N^4 N^{61/60}{N_f}^3{r_h}^4\alpha_{\theta_1}^7\alpha_{\theta_2}^6\left(-27a^4+6a^2{r_h}^2+{r_h}^4\right)}=0.$$

This yields:

(A9)
$$f(r) = \frac{1}{4}\gamma_{f_2}(r - r_h)^2 + \mathcal{O}\left((r - r_h)^3\right),$$

where:

(A10)
$$\gamma_{f_2} \equiv -\frac{4b^8\left(9b^2+1\right)^4\left(39b^2-4\right)M\left(\frac{1}{N}\right)^{9/4}\beta\left(6a^2+{r_h}^2\right)\log(r_h)\Sigma_1}{9\pi\left(3b^2-1\right)^5\left(6b^2+1\right)^4\log N^2 N_f {r_h}^2\left(9a^2+{r_h}^2\right)\alpha_{\theta_2}^3}.$$

(iii) EOM$_{\theta_1 x}$:

(A11)
$$-3f_{\theta_1 z}''(r) + 2f_{\theta_1 x}''(r) - 3f_{\theta_2 y}''(r) - \frac{4\left(9b^2+1\right)^4\beta b^{10}M\sqrt[5]{\frac{1}{N}}\Sigma_1\left(6a^2+{r_h}^2\right)}{3\pi\left(-18b^4+3b^2+1\right)^4\log N\sqrt[3]{N}N_f\alpha_{\theta_2}^3\left({r_h}^2-3a^2\right)\left(9a^2+{r_h}^2\right)} - \frac{32\sqrt{2}\left(9b^2+1\right)^4\beta b^{12}\left(\frac{1}{N}\right)^{3/20}\Sigma_1(r-r_h)}{81\pi^3\left(1-3b^2\right)^{10}\left(6b^2+1\right)^8 {g_s}^{9/4}\log N^4 N^{61/60}{N_f}^3{r_h}^4\alpha_{\theta_1}^7\alpha_{\theta_2}^6\left(-27a^4+6a^2{r_h}^2+{r_h}^4\right)} = 0.$$

Choosing the two constants of integration obtained by solving (A11) in such a way that the Neumann b.c. at $r = r_h : f'_{\theta_1 x}(r = r_h) = 0$, one obtains:

(A12)
$$f_{\theta_1 x}(r) = \left(-\frac{\left(9b^2+1\right)^4 b^{10}M\left(6a^2+{r_h}^2\right)\left((r-r_h)^2+{r_h}^2\right)\Sigma_1}{3\pi\left(-18b^4+3b^2+1\right)^4\log N N^{8/15}N_f\left(-27a^4+6a^2{r_h}^2+{r_h}^4\right)\alpha_{\theta_2}^3} + {C_{\theta_1 x}}^{(1)}\right)\beta + \mathcal{O}(r-r_h)^3$$

(iv) EOM$_{\theta_1 y}$:

(A13)
$$-6f''(r) + 2f_{zz}''(r) - 2f_{x^{10}x^{10}}''(r) - 3f_{\theta_1 z}''(r) - 3f_{\theta_2 y}''(r) - 3f_{xz}''(r) - 2f_t''(r) \frac{32\sqrt{2}\left(9b^2+1\right)^4\beta b^{12}\left(\frac{1}{N}\right)^{3/20}\Sigma_1(r-r_h)}{81\pi^3\left(3b^2-1\right)^{10}\left(6b^2+1\right)^8 {g_s}^{9/4}\log N^4 N^{61/60}{N_f}^3{r_h}^4\alpha_{\theta_1}^7\alpha_{\theta_2}^6\left({r_h}^2-3a^2\right)\left(9a^2+{r_h}^2\right)} = 0.$$

The equation (A13) can be shown to be equivalent to a decoupled second order EOM for f_{xz}. Then, expanding the solution around the horizon and requiring the constant of integration $C_{xz}^{(1)}$ appearing in the $\mathcal{O}(r - r_h)^0$ term to satisfy:

(A14)
$$-\frac{32\left(9b^2+1\right)^4 b^{12}\beta\left(19683\sqrt{3}\alpha_{\theta_1}^6+3321\sqrt{2}\alpha_{\theta_2}^2\alpha_{\theta_1}^3-40\sqrt{3}\alpha_{\theta_2}^4\right)}{729\pi^3\left(1-3b^2\right)^{10}\left(6b^2+1\right)^8 {g_s}^{9/4}\log N^4 N^{7/6}{N_f}^3\left(-27a^4 r_h+6a^2{r_h}^3+{r_h}^5\right)\alpha_{\theta_1}^7\alpha_{\theta_2}^6} - \frac{4\left(9b^2+1\right)^4 b^{10}M{r_h}^2\beta\log(r_h)\Sigma_1}{81\pi^{3/2}\left(3b^2-1\right)^5\left(6b^2+1\right)^4\sqrt{g_s}\log N^2 N^{23/20}N_f\alpha_{\theta_2}^5} + C_{xz}^{(1)} = 0,$$

one obtains:

(A15)
$$f_{xz}(r) = \frac{18b^{10}\left(9b^2+1\right)^4 M\beta\left(6a^2+{r_h}^2\right)\left(\frac{(r-r_h)^2}{{r_h}^2}+1\right)\log^3(r_h)\Sigma_1}{\pi\left(3b^2-1\right)^5\left(6b^2+1\right)^4\log N^4 N^{5/4} N_f\left(9a^2+{r_h}^2\right)\alpha_{\theta_2}^3}.$$

(v) $\text{EOM}_{\theta_1 z}$:
(A16)
$$2{f_{\theta_1 z}}''(r) - \frac{32\sqrt{2}b^{12}(9b^2+1)^4\beta\left(\frac{1}{N}\right)^{3/20}\Sigma_1(r-r_h)}{81\pi^3(1-3b^2)^{10}(6b^2+1)^8{g_s}^{9/4}\log N^4 N^{61/60}{N_f}^3{r_h}^4\alpha_{\theta_1}^7\alpha_{\theta_2}^6(-27a^4+6a^2{r_h}^2+{r_h}^4)} = 0.$$

Choosing the two constants of integration obtained by solving (A16) in such a way that the Neumann b.c. at $r = r_h : f'_{\theta_1 z}(r = r_h) = 0$, one obtains:

(A17)
$$f_{\theta_1 z}(r) = \left(\frac{16(9b^2+1)^4 b^{12}\left(\frac{1}{N}\right)^{3/20}\left(\frac{(r-r_h)^3}{{r_h}^3}+1\right)\left(19683\sqrt{3}\alpha_{\theta_1}^6+3321\sqrt{2}\alpha_{\theta_2}^2\alpha_{\theta_1}^3-40\sqrt{3}\alpha_{\theta_2}^4\right)}{243\pi^3(1-3b^2)^{10}(6b^2+1)^8{g_s}^{9/4}\log N^4 N^{61/60}{N_f}^3(-27a^4 r_h+6a^2{r_h}^3+{r_h}^5)\alpha_{\theta_1}^7\alpha_{\theta_2}^6} + C_{\theta_1 z}^{(1)}\right)\beta + \mathcal{O}(r-r_h)^3.$$

(vi) $\text{EOM}_{\theta_2 x}$:
(A18)
$$2{f_{\theta_2 x}}''(r) - \frac{32\sqrt{2}b^{12}(9b^2+1)^4\beta\left(\frac{1}{N}\right)^{3/20}\Sigma_1(r-r_h)}{81\pi^3(1-3b^2)^{10}(6b^2+1)^8{g_s}^{9/4}\log N^4 N^{61/60}{N_f}^3{r_h}^4\alpha_{\theta_1}^7\alpha_{\theta_2}^6(-27a^4+6a^2{r_h}^2+{r_h}^4)} = 0.$$

(A19)
$$f_{\theta_2 x}(r) = \left(\frac{16(9b^2+1)^4 b^{12}\left(\frac{1}{N}\right)^{3/20}\left(\frac{(r-r_h)^3}{{r_h}^3}+1\right)\left(19683\sqrt{3}\alpha_{\theta_1}^6+3321\sqrt{2}\alpha_{\theta_2}^2\alpha_{\theta_1}^3-40\sqrt{3}\alpha_{\theta_2}^4\right)}{243\pi^3(1-3b^2)^{10}(6b^2+1)^8{g_s}^{9/4}\log N^4 N^{61/60}{N_f}^3(-27a^4 r_h+6a^2{r_h}^3+{r_h}^5)\alpha_{\theta_1}^7\alpha_{\theta_2}^6} + C_{\theta_2 x}^{((1)}\right)\beta$$

(vii) $\text{EOM}_{\theta_2 y}$:
(A20)
$$2{f_{\theta_2 y}}''(r) + \frac{12\left(9b^2+1\right)^4\beta b^{10}M\left(\frac{1}{N}\right)^{7/5}\Sigma_1\left(6a^2+{r_h}^2\right)\log(r_h)}{\pi\left(3b^2-1\right)^5\left(6b^2+1\right)^4\log N^2 N_f{r_h}^2\alpha_{\theta_2}^3\left(9a^2+{r_h}^2\right)} - \frac{32\sqrt{2}\left(9b^2+1\right)^4\beta b^{12}\left(\frac{1}{N}\right)^{3/20}\Sigma_1(r-r_h)}{81\pi^3\left(1-3b^2\right)^{10}\left(6b^2+1\right)^8{g_s}^{9/4}\log N^4 N^{61/60}{N_f}^3{r_h}^4\alpha_{\theta_1}^7\alpha_{\theta_2}^6\left(-27a^4+6a^2{r_h}^2+{r_h}^4\right)} = 0.$$

Choosing the two constants of integration obtained by solving (A20) in such a way that the Neumann b.c. at $r = r_h : f'_{\theta_2 y}(r = r_h) = 0$, and requiring the constant of integration $C_{\theta_2 y}^{(1)}$ that figures in the $\mathcal{O}(r-r_h)^0$-term to satisfy:
(A21)
$$\frac{16(9b^2+1)^4 b^{12}\left(19683\sqrt{3}\alpha_{\theta_1}^6+3321\sqrt{2}\alpha_{\theta_2}^2\alpha_{\theta_1}^3-40\sqrt{3}\alpha_{\theta_2}^4\right)}{243\pi^3(1-3b^2)^{10}(6b^2+1)^8{g_s}^{9/4}\log N^4 N^{7/6}{N_f}^3(-27a^4 r_h+6a^2{r_h}^3+{r_h}^5)\alpha_{\theta_1}^7\alpha_{\theta_2}^6} + C_{\theta_2 y}^{(1)} = 0,$$

one obtains:

(A22)
$$f_{\theta_2 y} = \frac{3b^{10}(9b^2+1)^4 M\beta(6a^2+{r_h}^2)\left(1-\frac{(r-r_h)^2}{{r_h}^2}\right)\log(r_h)\Sigma_1}{\pi(3b^2-1)^5(6b^2+1)^4\log N^2 N^{7/5}N_f(9a^2+{r_h}^2)\alpha_{\theta_2}^3} + \mathcal{O}\left((r-r_h)^3\right).$$

(viii) $\text{EOM}_{\theta_2 z}$

(A23)
$$\frac{12\left(9b^2+1\right)^4\beta b^{10}M\sqrt{\frac{1}{N}}\Sigma_1\left(6a^2+{r_h}^2\right)\log(r_h)}{\pi\left(3b^2-1\right)^5\left(6b^2+1\right)^4\log N^2 N^{2/3}N_f {r_h}^2\alpha_{\theta_2}^3\left(9a^2+{r_h}^2\right)} - \frac{32\sqrt{2}\left(9b^2+1\right)^4\beta b^{12}\left(\frac{1}{N}\right)^{3/20}\Sigma_1(r-r_h)}{81\pi^3\left(1-3b^2\right)^{10}\left(6b^2+1\right)^8 {g_s}^{9/4}\log N^4 N^{61/60}{N_f}^3{r_h}^4\alpha_{\theta_1}^7\alpha_{\theta_2}^6\left(-27a^4+6a^2{r_h}^2+{r_h}^4\right)} + 2{f_{\theta_2 z}}''(r) = 0.$$

Choosing the two constants of integration obtained by solving (A23) in such a way that the Neumann b.c. at $r = r_h : f'_{\theta_2 z}(r = r_h) = 0$, one obtains:

(A24)
$$f_{\theta_2 z} = \left(\frac{3(9b^2+1)^4 b^{10}M(6a^2+{r_h}^2)\left(1-\frac{(r-r_h)^2}{{r_h}^2}\right)\log(r_h)\left(19683\sqrt{6}\alpha_{\theta_1}^6+6642\alpha_{\theta_2}^2\alpha_{\theta_1}^3-40\sqrt{6}\alpha_{\theta_2}^4\right)}{\pi(3b^2-1)^5(6b^2+1)^4\log N^2 N^{7/6}N_f(9a^2+{r_h}^2)\alpha_{\theta_2}^3} + {C_{\theta_2 z}}^{(1)}\right)\beta.$$

(ix) EOM_{xx}:

(A25)
$$f_{zz}(r) - 2f_{\theta_1 z}(r) + 2f_{\theta_1\phi_1}(r) - f_r(r) + \frac{81\left(9b^2+1\right)^4\beta b^{10}M\left(\frac{1}{N}\right)^{53/20}\alpha_{\theta_1}^4\left(19683\sqrt{6}\alpha_{\theta_1}^6+6642\alpha_{\theta_2}^2\alpha_{\theta_1}^3-40\sqrt{6}\alpha_{\theta_2}^4\right)\left({r_h}^2-3a^2\right)^2\left(6a^2+{r_h}^2\right)\log(r_h)}{16\pi\left(3b^2-1\right)^5\log N^2 N_f\left(6ab^2+a\right)^4\alpha_{\theta_2}\left(9a^2+{r_h}^2\right)} = 0.$$

Substituting (A12), (A17) and (A39) into (A25), one obtains:

$$f_r(r) = \left(-\frac{2\left(9b^2+1\right)^4 b^{10}M\left(6a^2+{r_h}^2\right)\left((r-r_h)^2+{r_h}^2\right)\Sigma_1}{3\pi\left(-18b^4+3b^2+1\right)^4\log N N^{8/15}N_f\left(-27a^4+6a^2{r_h}^2+{r_h}^4\right)\alpha_{\theta_2}^3} + {C_{zz}}^{(1)} - 2{C_{\theta_1 z}}^{(1)} + 2{C_{\theta_1 x}}^{(1)}\right)\beta + \mathcal{O}(r-r_h)^3. \quad \text{(A26)}$$

(x) EOM$_{xy}$:

(A27)

$$2f_{xy}''(r) + \frac{12\left(9b^2+1\right)^4 \beta b^{10} M \left(\frac{1}{N}\right)^{21/20} \Sigma_1 \left(6a^2+r_h{}^2\right)\log(r_h)}{\pi\left(3b^2-1\right)^5\left(6b^2+1\right)^4 \log N^2 N_f r_h{}^2 \alpha_{\theta_2}^3 \left(9a^2+r_h{}^2\right)}$$
$$- \frac{32\sqrt{2}\left(9b^2+1\right)^4 \beta b^{12}\left(\frac{1}{N}\right)^{3/20}\Sigma_1(r-r_h)}{81\pi^3\left(1-3b^2\right)^{10}\left(6b^2+1\right)^8 g_s{}^{9/4}\log N^4 N^{61/60} N_f{}^3 r_h{}^4 \alpha_{\theta_1}^7 \alpha_{\theta_2}^6\left(-27a^4+6a^2 r_h{}^2+r_h{}^4\right)} = 0.$$

Choosing the two constants of integration obtained by solving (A28) in such a way that the Neumann b.c. at $r = r_h : f'_{xy}(r = r_h) = 0$, one obtains:

(A28)

$$f_{xy}(r) = \left(\frac{3(9b^2+1)^4 b^{10} M(6a^2+r_h{}^2)\left(\frac{(r-r_h)^2}{r_h{}^2}+1\right)\log(r_h)\alpha_{\theta_2}^3\Sigma_1}{\pi(3b^2-1)^5(6b^2+1)^4\log N^2 N^{21/20} N_f (9a^2+r_h{}^2)\alpha_{\theta_{2l}}^6} + C_{xy}^{(1)}\right)\beta + \mathcal{O}\left(r-r_h\right)^3.$$

(xi) EOM $_{xz}$:

(A29)

$$-8f_t''(r) + \frac{24\left(9b^2+1\right)^4 \beta b^{10} M\left(\frac{1}{N}\right)^{3/4}\Sigma_1\left(9a^2+r_h{}^2\right)\log(r_h)}{\pi^{3/2}\left(3b^2-1\right)^5\left(6b^2+1\right)^4\sqrt{g_s}\log N^2 N_f \alpha_{\theta_1}^2\alpha_{\theta_2}^5}$$
$$- \frac{64\sqrt{2}\left(9b^2+1\right)^4 \beta b^{12}\left(\frac{1}{N}\right)^{3/20}\Sigma_1(r-r_h)}{81\pi^3\left(1-3b^2\right)^{10}\left(6b^2+1\right)^8 g_s{}^{9/4}\log N^4 N^{61/60} N_f{}^3 r_h{}^4 \alpha_{\theta_1}^7 \alpha_{\theta_2}^6\left(-27a^4+6a^2 r_h{}^2+r_h{}^4\right)} = 0.$$

The solution is given as under:

(A30)
$$f_t(r) = \frac{1}{16}\gamma_{f_{t2}}(r-r_h)^2 + \mathcal{O}\left((r-r_h)^3\right),$$

where:

(A31)
$$\gamma_{f_{t2}} \equiv \frac{24b^{10}\left(9b^2+1\right)^4\beta M\left(\frac{1}{N}\right)^{3/4}\Sigma_1\left(9a^2+r_h{}^2\right)\log(r_h)}{\pi^{3/2}\left(3b^2-1\right)^5\left(6b^2+1\right)^4\sqrt{g_s}\log N^2 N_f\alpha_{\theta_1}^2\alpha_{\theta_2}^5}.$$

Consistency with (A7) requires (as in [32] wherein $r_h \sim N^{-\alpha}, \alpha > 0$):

(A32)
$$r_h = \frac{\sqrt[8]{\pi}\sqrt[4]{4374b^6+1035b^4+9b^2-4}\sqrt[8]{g_s}\left(\frac{1}{N}\right)^{3/8}\sqrt{\alpha_{\theta_1}\alpha_{\theta_2}}}{3\sqrt[4]{2}\sqrt{b}\left(9b^2+1\right)^{3/4}}.$$

Note $4374b^6 + 1035b^4 + 9b^2 - 4 > 0$ for b given as in (A52).

(xii) EOM$_{yy}$:

(A33)

$$2f_{\phi_2\phi_2}''(r) + \frac{12\left(9b^2+1\right)^4\beta b^{10}M\left(\frac{1}{N}\right)^{7/4}\Sigma_1\left(6a^2+{r_h}^2\right)\log(r_h)}{\pi\left(3b^2-1\right)^5\left(6b^2+1\right)^4\log N^2 N_f {r_h}^2\alpha_{\theta_2}^3\left(9a^2+{r_h}^2\right)} - \frac{32\sqrt{2}\left(9b^2+1\right)^4\beta b^{12}\left(\frac{1}{N}\right)^{3/20}\Sigma_1(r-r_h)}{81\pi^3\left(1-3b^2\right)^{10}\left(6b^2+1\right)^8{g_s}^{9/4}\log N^4 N^{61/60}{N_f}^3{r_h}^4\alpha_{\theta_1}^7\alpha_{\theta_2}^6\left(-27a^4+6a^2{r_h}^2+{r_h}^4\right)} = 0$$

Choosing the two constants of integration obtained by solving (A33) in such a way that the Neumann b.c. at $r = r_h : f'_{yy}(r = r_h) = 0$, and choosing the constant of integration $C_{yy}^{(1)}$ appearing in the $\mathcal{O}(r-r_h)^0$-term to satisfy:

(A34)

$$\frac{16(9b^2+1)^4 b^{12}\left(19683\sqrt{3}\alpha_{\theta_1}^6+3321\sqrt{2}\alpha_{\theta_2}^2\alpha_{\theta_1}^3-40\sqrt{3}\alpha_{\theta_2}^4\right)}{243\pi^3(1-3b^2)^{10}(6b^2+1)^8{g_s}^{9/4}\log N^4 N^{7/6}{N_f}^3(-27a^4 r_h+6a^2{r_h}^3+{r_h}^5)\alpha_{\theta_1}^7\alpha_{\theta_2}^6} + C_{yy}^{(1)} = 0,$$

one obtains:

(A35)

$$f_{yy}(r) = -\frac{3b^{10}(9b^2+1)^4 M\left(\frac{1}{N}\right)^{7/4}\beta(6a^2+{r_h}^2)\log(r_h)\Sigma_1\left(\frac{(r-r_h)^2}{h^2 r^2}+1\right)}{\pi(3b^2-1)^5(6b^2+1)^4\log N^2 N_f {r_h}^2(9a^2+{r_h}^2)\alpha_{\theta_2}^3} + \mathcal{O}\left((r-r_h)^3\right).$$

(xiii) EOM$_{yz}$:

(A36)

$$2f_{\phi_2\psi}''(r) - \frac{128\sqrt{2}b^{22}(9b^2+1)^8\beta^2 M\left(\frac{1}{N}\right)^{3/5}{\Sigma_1}^2(6a^2+{r_h}^2)(r-r_h)\log(r_h)}{27\pi^4(3b^2-1)^{15}(6b^2+1)^{12}{g_s}^{9/4}\log N^6 N^{109/60}{N_f}^4{r_h}^6\alpha_{\theta_1}^7\alpha_{\theta_2}^9({r_h}^2-3a^2)(9a^2+{r_h}^2)^2} = 0.$$

Choosing the two constants of integration obtained by solving (A40) in such a way that the Neumann b.c. at $r = r_h : f'_{x^{10}x^{10}}(r = r_h) = 0$, one obtains:

(A37)

$$f_{yz}(r) = \left(\frac{64\left(9b^2+1\right)^8 b^{22}M\left(\frac{1}{N}\right)^{3/5}\left(6a^2+{r_h}^2\right)\left(\frac{(r-r_h)^3}{{r_h}^3}+1\right)\log(r_h)}{27\pi^4\left(3b^2-1\right)^{15}\left(6b^2+1\right)^{12}{g_s}^{9/4}\log N^6 N^{109/60}{N_f}^4{r_h}^3\left({r_h}^2-3a^2\right)\left(9a^2+{r_h}^2\right)^2\alpha_{\theta_1}^7\alpha_{\theta_2}^9} \times\left(387420489\sqrt{2}\alpha_{\theta_1}^{12}+87156324\sqrt{3}\alpha_{\theta_2}^2\alpha_{\theta_1}^9+5778054\sqrt{2}\alpha_{\theta_2}^4\alpha_{\theta_1}^6-177120\sqrt{3}\alpha_{\theta_2}^6\alpha_{\theta_1}^3+1600\sqrt{2}\alpha_{\theta_2}^8\right)+C_{yz}^{(1)}\right)\beta + \mathcal{O}(r-r_h)^3.$$

(xiv) EOM$_{zz}$:

(A38)

$$2f_{zz}''(r) - \frac{32\sqrt{2}\left(9b^2+1\right)^4\beta b^{12}\left(\frac{1}{N}\right)^{3/20}\Sigma_1(r-r_h)}{81\pi^3\left(1-3b^2\right)^{10}\left(6b^2+1\right)^8{g_s}^{9/4}\log N^4 N^{61/60}{N_f}^3{r_h}^4\alpha_{\theta_1}^7\alpha_{\theta_2}^6\left(-27a^4+6a^2{r_h}^2+{r_h}^4\right)} + \frac{4\left(9b^2+1\right)^4\beta b^{10}M\left(\frac{1}{N}\right)^{23/20}\Sigma_1(r-r_h)\log(r_h)}{9\pi^{3/2}\left(3b^2-1\right)^5\left(6b^2+1\right)^4\sqrt{g_s}\log N^2 N_f r_h\alpha_{\theta_2}^5} = 0.$$

Choosing the two constants of integration obtained by solving (A38) in such a way that the Neumann b.c. at $r = r_h : f'_{zz}(r = r_h) = 0$, one obtains:

(A39)

$$f_{zz}(r) = \left(C^{(1)}_{zz} - \frac{b^{10}(9b^2+1)^4 M\left({r_h}^2 - \frac{(r-r_h)^3}{r_h}\right)\log(r_h)\Sigma_1}{27\pi^{3/2}(3b^2-1)^5(6b^2+1)^4\sqrt{g_s}\log N^2 N^{23/20} N_f \alpha^5_{\theta_2}} \right)\beta + \mathcal{O}(r-r_h)^3.$$

(xv) $\text{EOM}_{x^{10}x^{10}}$:

(A40)

$$\frac{4\left(9b^2+1\right)^3 \beta b^8 M\left(\frac{1}{N}\right)^{5/4}\Sigma_1\left(6a^2+{r_h}^2\right)\left(9b^4(27\log N+16)+3b^2(9\log N-8)-8\right)\log^3(r_h)}{\pi\left(3b^2-1\right)^5\left(6b^2+1\right)^4\log N^5 N_f {r_h}^2\alpha^3_{\theta_2}\left(9a^2+{r_h}^2\right)}$$
$$-\frac{4\left(9b^2+1\right)^4\beta b^{10}M\left(\frac{1}{N}\right)^{23/20}\Sigma_1(r-r_h)\log(r_h)}{9\pi^{3/2}\left(3b^2-1\right)^5\left(6b^2+1\right)^4\sqrt{g_s}(\log N)^2 N_f r_h \alpha^5_{\theta_2}} + 2{f_{x^{10}x^{10}}}''(r) = 0.$$

Choosing the two constants of integration obtained by solving (A40) in such a way that the Neumann b.c. at $r = r_h : f'_{x^{10}x^{10}}(r = r_h) = 0$, and requiring the constant of integration $C^{(1)}_{x^{10}x^{10}}$ appearing in the $\mathcal{O}(r-r_h)^0$ to satisfy:

$$\frac{\left(9b^2+1\right)^4 b^{10} M {r_h}^2\beta\log(r_h)\Sigma_1}{27\pi^{3/2}\left(3b^2-1\right)^5\left(6b^2+1\right)^4\sqrt{g_s}\log N^2 N^{23/20} N_f\alpha^5_{\theta_2}} + C^{(1)}_{x^{10}x^{10}} = 0, \quad \text{(A41)}$$

one obtains:

(A42)

$$f_{x^{10}x^{10}} = -\frac{27b^{10}(9b^2+1)^4 M\left(\frac{1}{N}\right)^{5/4}\beta(6a^2+{r_h}^2)\left(1-\frac{(r-r_h)^2}{{r_h}^2}\right)\log^3(r_h)\Sigma_1}{\pi(3b^2-1)^5(6b^2+1)^4\log N^4 N_f {r_h}^2(9a^2+{r_h}^2)\alpha^3_{\theta_2}} + \mathcal{O}(r-r_h)^3.$$

The remaining EOMS provide consistency checks and are listed below:

- EOM_{rr}:

$$\frac{3\alpha_h\left(9b^2+1\right)^3\beta b^{10}M\Sigma_1}{\pi\left(3b^2-1\right)^5\left(6b^2+1\right)^3\log N^2 N^{11/12} N_f {r_h}^2\alpha^3_{\theta_2}} - \frac{{f_{\theta_1 z}}''(r)}{4} - \frac{{f_{\theta_2 y}}''(r)}{4} - \frac{{f_{xz}}''(r)}{4} = 0. \quad \text{(A43)}$$

- $\text{EOM}_{\theta_1\theta_1}$

(A44)

$$\frac{f_{zz}(r)}{2} - f_{yz}(r) + \frac{f_{yy}(r)}{2}$$
$$-\frac{32\sqrt{2}\sqrt[4]{\pi}\left(9b^2+1\right)^3\beta b^{12}\left(\frac{1}{N}\right)^{7/10}\left(-19683\alpha^6_{\theta_1}+216\sqrt{6}\alpha^2_{\theta_2}\alpha^3_{\theta_1}+530\alpha^4_{\theta_2}\right)\Sigma_1\left(6a^2 r_h+{r_h}^3\right)(r-r_h)^2}{14348907\left(1-3b^2\right)^4 {g_s}^{7/4}\log N N^{4/5}\alpha^8_{\theta_1}\left(9a^2+{r_h}^2\right)\left(6b^2 r_h+r_h\right)^3\left(108b^2 N_f {r_h}^2+N_f\right)^2} = 0.$$

• $\text{EOM}_{\theta_1\theta_2}$

(A45)
$$-\frac{441N^{3/10}\left(2{r_h}^2\alpha_{\theta_1}^3 f_{x^{10}x^{10}}(r)+{r_h}^2\alpha_{\theta_1}^3 f_{\theta_2 y}(r)\right)}{512\alpha_{\theta_2}^3\left({r_h}^2-3a^2\right)\log(r_h)} - \frac{3\sqrt{\frac{3}{2}}{g_s}^{3/2}M\sqrt[10]{N}N_f r_h\left(9a^2+{r_h}^2\right)\left(108b^2{r_h}^2+1\right)^2 f_r(r)(r-r_h)}{\pi^{3/2}\alpha_{\theta_1}^3\left(-18a^4+3a^2{r_h}^2+{r_h}^4\right)}=0.$$

• $\text{EOM}_{\theta_2\theta_2}$:

(A46)
$$f_{zz}(r)-f_{x^{10}x^{10}}(r)-2f_{\theta_1 z}(r)-f_r(r)=0.$$

One can show that by requiring:

(A47)
$$C_{zz}^{(1)}-2C_{\theta_1 z}^{(1)}+2C_{\theta_1 x}^{(1)}=0,$$
$$C_{zz}^{(1)}-2C_{yz}=0,$$
$$|\Sigma_1|\ll 1,$$
$$\frac{2b^{10}\left(9b^2+1\right)^4 M{r_h}^2\beta\left(6a^2+{r_h}^2\right)\Sigma_1}{3\pi\left(-18b^4+3b^2+1\right)^4\log N N^{8/15}N_f\left(-27a^4+6a^2{r_h}^2+{r_h}^4\right)\alpha_{\theta_2}^3}-2{C_{\theta_1 x}}^{(1)}=0,$$

(A43)-(A46) will automatically be satisfied.

A.2. $\psi\neq 2n\pi, n=0,1,2$ near $r=r_h$

Working in the IR, the EOMs near $r=r_h$ and up to LO in N, can be written as follows:

(A48)
$$\text{EOM}_{MN}:\ \sum_{p=0}^{2}\sum_{i=0}^{2} b_{MN}^{(p,i)}\left(r_h,a,N,M,N_f,g_s,\alpha_{\theta_{1,2}}\right)(r-r_h)^i f_{MN}^{(p)}(r) + \beta\frac{\mathcal{H}_{MN}\left(r_h,a,N,M,N_f,g_s,\alpha_{\theta_{1,2}}\right)}{(r-r_h)^{\gamma_{MN}^{\text{LO}}}}=0,$$

where as in **A.1**, M,N run over the $D=11$ coordinates, $f_{MN}^{(p)}\equiv\frac{d^p f_{MN}}{dr^p}, p=0,1,2$, $\gamma_{MN}^{\text{LO}}=1,2$ denotes the leading order (LO) terms in powers of $r-r_h$ in the IR when the $\mathcal{O}(\beta)$-terms are Laurent-expanded about $r=r_h$.

One can show that a set of ten linearly independent EOMs for the $\mathcal{O}(l_p^6R^4)$ corrections to the MQGP metric, with the simplifying assumtion

$f_{\theta_1\theta_1} = f_{\theta_1 x^{10}} = f_{x^{10}x^{10}} = 0$, reduce to the following set of seven equations and one that serves as a consistency check.

(a) EOM_{tt}:

$$\alpha_{tt}^{f'_{\theta_1\theta_2}}(r-r_h)^2 f'_{\theta_1\theta_2}(r) + \frac{\alpha_{tt}^{\beta}\beta}{r-r_h} = 0, \tag{A49}$$

where:

(A50)

$$\alpha_{tt}^{f'_{\theta_1\theta_2}} \equiv \frac{12\times 4a^2\left(\frac{1}{N}\right)^{2/5}\sin^2\left(\frac{\psi_0}{2}\right)\left(9a^2+{r_h}^2\right)\log(r_h)}{\pi(g_s-1)g_s\sin^2\phi_{20}\left(6a^2+{r_h}^2\right)\alpha_{\theta_2}^2}$$

$$\alpha_{tt}^{\beta} \equiv \frac{8192\times 16\pi^{9/2}a^2\sqrt[10]{\frac{1}{N}}\sin^4\left(\frac{\psi_0}{2}\right)\beta\left(9a^2+{r_h}^2\right)^2\left(\log(r_h)-1\right)\left(\left(9a^2+{r_h}^2\right)\log\left(9a^2{r_h}^4+{r_h}^6\right)-8\left(6a^2+{r_h}^2\right)\log(r_h)\right)^2}{729\times 16(g_s-1){g_s}^{3/2}{N_f}^6\sin^4\phi_{20}{r_h}^2\left(6a^2+{r_h}^2\right)^4\log^3(r_h)\alpha_{\theta_2}^4\log^8\left(9a^2{r_h}^4+{r_h}^6\right)}.$$

whose solution is given by:

(A51)

$$f_{\theta_1\theta_2}(r) = \left(\frac{\upsilon_{\theta_1\theta_2}N^{3/10}\sin^2\left(\frac{\psi_0}{2}\right)\left(9a^2+{r_h}^2\right)\left(\log(r_h)-1\right)\left(\left(9a^2+{r_h}^2\right)\log\left(9a^2{r_h}^4+{r_h}^6\right)-8\left(6a^2+{r_h}^2\right)\log(r_h)\right)^2}{\sqrt{g_s}{N_f}^6\sin^2\phi_{20}{r_h}^2\left(6a^2+{r_h}^2\right)^3(r-r_h)^2\log^4(r_h)\alpha_{\theta_2}^2\log^8\left(9a^2{r_h}^4+{r_h}^6\right)} + {C_{\theta_1\theta_2}}^{(1)}\right)\beta$$

$$= \left(\frac{\tilde{\upsilon}_{\theta_1\theta_2}\left(1-3b^2\right)^2\left(9b^2+1\right)N^{3/10}\sin^2\left(\frac{\psi_0}{2}\right)\beta}{\left(6b^2+1\right)^3\sqrt{g_s}{N_f}^6\sin^2\phi_{20}{r_h}^2(r-r_h)^2\log^9(r_h)\alpha_{\theta_2}^2} + {C_{\theta_1\theta_2}}^{(1)}\right)\beta + \mathcal{O}\left(\frac{1}{N^{7/10}}\right),$$

where $\upsilon_{\theta_1\theta_2} \sim \mathcal{O}(1),\ \tilde{\upsilon}_{\theta_1\theta_2} \ll 1$.

Assuming:

$$b = \frac{1}{\sqrt{3}} - \kappa_b r_h^2\left(\log r_h\right)^{\frac{9}{2}} N^{-\frac{9}{10}-\alpha}, \tag{A52}$$

one obtains:

$$f_{\theta_1\theta_2}(r) = \left(\frac{\tilde{\tilde{\upsilon}}_{\theta_1\theta_2}\kappa_b^2\sin^2\left(\frac{\psi_0}{2}\right){r_h}^2\left(\frac{1}{N}\right)^{2\alpha+\frac{3}{2}}}{\sqrt{g_s}{N_f}^6\sin^2\phi_{20}(r-r_h)^2\alpha_{\theta_2}^2} + {C_{\theta_1\theta_2}}^{(1)}\right)\beta, \tag{A53}$$

where $\tilde{\tilde{\upsilon}}_{\theta_1\theta_2} \ll 1$.

(b) EOM_{ry}

$$\alpha_{ry}^{\theta_1\theta_2} f_{\theta_1\theta_2}(r) + \alpha_{ry}^{yy} f_{yy}(r) = 0, \tag{A54}$$

where:
(A55)

$$\alpha_{ry}^{\theta_1\theta_2} \equiv \frac{7\pi^{17/4}\left(108a^2+r_h\right)\alpha_{\theta_1}^4\alpha_{\theta_2}\log^4\left(9a^2{r_h}^4+{r_h}^6\right)}{768\sqrt{3}(g_s-1)^2{g_s}^{19/4}M^3\left(\frac{1}{N}\right)^{3/20}{N_f}^3\sin^4\phi_{20}{r_h}^2\log^3(r_h)},$$

$$\alpha_{ry}^{yy} \equiv \frac{7\sqrt{3}\pi^{17/4}\log N^2\left(\frac{1}{N}\right)^{9/20}\left(108a^2+r_h\right)\left({r_h}^2-3a^2\right)^2(2\log(r_h)+1)^2\alpha_{\theta_1}^{10}\log^6\left(9a^2{r_h}^4+{r_h}^6\right)}{65536(g_s-1){g_s}^{19/4}M^3{N_f}^3\sin^6\phi_{20}{r_h}^6\log^5(r_h)\alpha_{\theta_2}},$$

and one obtains:

(A56)

$$f_{yy}(r) = -\frac{256N^{3/5}\sin^2\phi_{20}{r_h}^4\alpha_{\theta_2}^2}{9(g_s-1)\left({r_h}^2-3a^2\right)^2\log^2(N)\alpha_{\theta_1}^6\log^2\left(9a^2{r_h}^4+{r_h}^6\right)}$$
$$\times\left(\frac{\upsilon_{yy}N^{3/10}\sin^2\left(\frac{\psi_0}{2}\right)\left(9a^2+{r_h}^2\right)(\log(r_h)-1)\left(\left(9a^2+{r_h}^2\right)\log\left(9a^2{r_h}^4+{r_h}^6\right)-8\left(6a^2+{r_h}^2\right)\log(r_h)\right)^2}{\sqrt{g_s}{N_f}^6\sin^2\phi_{20}{r_h}^2\left(6a^2+{r_h}^2\right)^3(r-r_h)^2\log^4(r_h)\alpha_{\theta_2}^2\log^8\left(9a^2{r_h}^4+{r_h}^6\right)}+{C_{\theta_1\theta_2}}^{(1)}\right)\beta,$$

where $\upsilon_{yy}\sim\mathcal{O}(1)$.

Even though $f_{yy}(r)$ is numerically suppressed as the same is $\mathcal{O}\left(10^{-7}\right)$ apart from an $\mathcal{O}\left(l_p^6\right)$-suppression - the latter of course common to most f_{MN}s - $f_{yy}(r)$, near $r=r_h$ for $\mathcal{O}(1)$ ${C_{\theta_1\theta_2}}^{(1)}$, goes like $\frac{\frac{N^{\frac{9}{10}}}{r_h^2\log^{11}r_h}}{(r-r_h)^2}$. To ensure f_{yy} remains finite one has to forego the assumption that ${C_{\theta_1\theta_2}}^{(1)}$ is $\mathcal{O}(1)$. Around a chosen (ψ_0,ϕ_{20}), writing $r=r_h+\epsilon_r, \epsilon<<r_h$ close to the horizon, by assuming ${C_{\theta_1\theta_2}}^{(1)}={C_{\theta_1\theta_2}}^{(1)}(\psi_0,\phi_{20})$:

$$\text{(A57)}\quad \frac{\delta_{\theta_1\theta_2}\left(9a^2+{r_h}^2\right)\left({r_h}^2-3a^2\right)^2\sin^2\left(\frac{\psi_0}{2}\right)}{{\epsilon_r}^3\sqrt{g_s}\left(\frac{1}{N}\right)^{3/10}{N_f}^6{r_h}^2\left(6a^2+{r_h}^2\right)^3\log^9(r_h)\alpha_{\theta_2}^2\sin^2\phi_{20}} + {C_{\theta_1\theta_2}}^{(1)}(\psi_0,\phi_{20})=0,$$

(wherein $\delta_{\theta_1\theta_2}\ll 1$) which would imply one can consistently set $f_{yy}(r)=0$ up to $\mathcal{O}(\beta)$. The idea is that for every chosen value of (ψ_0,ϕ_{20}), once upgraded to a local uplift, using the ideas similar to [11], one can show that the same will correspond to a G_2 structure.

(c) $\mathrm{EOM}_{x^1x^1}$

$$\text{(A58)}\qquad \frac{\alpha_{tt}^{\beta}\beta}{4\left(1-\frac{r}{r_h}\right)(r-r_h)}+\alpha_{x^1x^1}^{\theta_1\phi_2}f_{\theta_1y}(r)(r-r_h)=0,$$

where:

$$\alpha^{\theta_1\phi_2}_{x^1x^1} \equiv -\frac{\sqrt{\frac{3\pi}{2}}\left({r_h}^2-3a^2\right)\left(9a^2+{r_h}^2\right)\alpha_{\theta_1}^4\log^2\left(9a^2{r_h}^4+{r_h}^6\right)}{32(g_s-1){g_s}^{5/2}MNN_f\sin^2\phi_{20}\left(6a^2+{r_h}^2\right)\log(r_h)\alpha_{\theta_2}}, \tag{A59}$$

and obtain:

(A60)

$$f_{\theta_1y}(r) = -\frac{\tilde{\upsilon}_{\theta_1y}a^2\beta g_sMN^{9/10}16\sin^4\left(\frac{\psi_0}{2}\right)\beta\left(9a^2+{r_h}^2\right)\left(\log(r_h)-1\right)\left(\left(9a^2+{r_h}^2\right)\log\left(9a^2{r_h}^4+{r_h}^6\right)-8\left(6a^2+{r_h}^2\right)\log(r_h)\right)^2}{{N_f}^5\sin^2\phi_{20}\left(6a^2+{r_h}^2\right)^3\left({r_h}^3-3a^2r_h\right)(r-r_h)^3\log^2(r_h)\alpha_{\theta_1}^4\alpha_{\theta_2}^3\log^{10}\left(9a^2{r_h}^4+{r_h}^6\right)}$$
$$= \frac{\upsilon_{\theta_1y}b^2\left(3b^2-1\right)\left(9b^2+1\right)\beta g_sMN^{9/10}\sin^4\left(\frac{\psi_0}{2}\right)\beta}{\left(6b^2+1\right)^3{N_f}^5\sin^2\phi_{20}r_h(r-r_h)^3\log^9(r_h)\alpha_{\theta_1}^4\alpha_{\theta_2}^3},$$

where $\tilde{\upsilon}_{\theta_1y} \sim \mathcal{O}(100), \upsilon_{\theta_1y} \ll 1$, yielding:

$$f_{\theta_1y}(r) = \frac{\tilde{\tilde{\upsilon}}_{\theta_1y}\sqrt{2}\pi^4\beta g_s\kappa_bM16\sin^4\left(\frac{\psi_0}{2}\right)r_h\beta N^{-\alpha}}{{N_f}^5\sin^2\phi_{20}(r-r_h)^3\log^{\frac{9}{2}}(r_h)\alpha_{\theta_1}^4\alpha_{\theta_2}^3}. \tag{A61}$$

where $\tilde{\tilde{\upsilon}}_{\theta_1y} \ll 1$.

(d) EOM_{θ_1z}

$$\alpha^{\theta_1\phi_2}_{yx^{10}}f_{\theta_1y}(r)+\alpha^{\theta_1z}_{yx^{10}}f_{\theta_1z}(r)+\alpha^{yy}_{yx^{10}}f_{yy}(r)=0, \tag{A62}$$

where:

(A63)

$$\alpha^{\theta_1\phi_2}_{yx^{10}} \equiv -\frac{\upsilon_{\theta_1z}\log N\left(\frac{1}{N}\right)^{3/10}\left({r_h}^2-3a^2\right)\left(\log(r_h)+1\right)\alpha_{\theta_1}^9\log^4\left(9a^2{r_h}^4+{r_h}^6\right)}{{g_s}^{13/2}(g_s-1)M^4{N_f}^5\sin^4\phi_{20}2\sin\left(\frac{\psi_0}{2}\right){r_h}^2\log^5(r_h)}$$
$$\alpha^{\theta_1z}_{yx^{10}} = -\alpha^{\theta_1\phi_2}_{yx^{10}} = \alpha^{yy}_{yx^{10}},$$

and obtain:

(A64)

$$f_{\theta_1z}(r) = \frac{\upsilon_{\theta_1z}\log N\left(\frac{1}{N}\right)^{3/10}\left({r_h}^2-3a^2\right)\left(2\log(r_h)+1\right)\alpha_{\theta_1}^9\log^4\left(9a^2{r_h}^4+{r_h}^6\right)}{(g_s-1){g_s}^{13/2}M^4{N_f}^5\sin^4\phi_{20}2\sin\left(\frac{\psi_0}{2}\right){r_h}^2\log^5(r_h)},$$

$\upsilon_{\theta_1z} \sim \mathcal{O}(1)$, yielding:

(A65)
$$f_{\theta_1 z}(r) = \frac{\tilde{\upsilon}_{\theta_1 z} \times 16\sqrt{2}\pi^4 \beta {g_s}^{15/2} \kappa_b M \sin^4\left(\frac{\psi_0}{2}\right) r_h \beta N^{-\alpha}}{{g_s}^{13/2} {N_f}^5 \sin^2\phi_{20} (r-r_h)^3 \log^{\frac{9}{2}}(r_h) \alpha_{\theta_1}^4 \alpha_{\theta_2}^3}$$

where $\tilde{\upsilon}_{\theta_1 z} \ll 1$.

(e1) EOM_{xy}

(A66)
$$\frac{\alpha_{tt}^{\beta}\beta}{R_{\frac{tt}{r\phi_1}}(r-r_h)^2} + \alpha_{r\phi_1}^{f''_{xy}}(r-r_h) f''_{xy}(r) + \alpha_{r\phi_1}^{f'_{xy}}(r-r_h) f'_{xy}(r) = 0,$$

where:

(A67)
$$R_{\frac{tt}{r\phi_1}} \equiv \frac{3\left(\frac{1}{N}\right)^{3/5} r_h \left(9a^2+{r_h}^2\right) \alpha_{\theta_2} \log^3\left(9a^2{r_h}^4+{r_h}^6\right)}{160\sqrt{\pi}\sqrt{g_s}\sin\phi_{10}\sin\left(\frac{\psi_0}{2}\right)\alpha_{\theta_1}\left(24(g_s-1)^2\left(6a^2+{r_h}^2\right)\log(r_h) + (2g_s-3)\left(9a^2+{r_h}^2\right)\log\left(9a^2{r_h}^4+{r_h}^6\right)\right)},$$

$$\alpha_{r\phi_1}^{f''_{xy}} \equiv \frac{5(g_s-1)\sin\phi_1 2\sin\left(\frac{\psi_0}{2}\right) r_h \left(9a^2+{r_h}^2\right)}{729\sqrt{6\pi}{g_s}^{3/2}N_f\sin^2\phi_{20}\left(6a^2+{r_h}^2\right)\log(r_h)\alpha_{\theta_1}\alpha_{\theta_2}^3\log^2\left(9a^2{r_h}^4+{r_h}^6\right)}\Bigg(112(g_s-1)g_sN_f\psi^2\log(r_h)\left(81\alpha_{\theta_1}^3+5\sqrt{6}\alpha_{\theta_2}^2\right)$$
$$-\frac{1}{{r_h}^2}\Bigg\{243\alpha_{\theta_1}^3\log^2\left(9a^2{r_h}^4+{r_h}^6\right)$$
$$\times\Bigg[\log(r_h)\left(6a^2(g_s\log NN_f-4\pi)+4g_sN_f\left({r_h}^2-3a^2\right)\log\left(\frac{1}{4}\alpha_{\theta_1}\alpha_{\theta_2}\right)+{r_h}^2(8\pi-g_s(2\log N+3)N_f)\right)$$
$$+g_sN_f\left(3a^2-{r_h}^2\right)\left(\log N-2\log\left(\frac{1}{4}\alpha_{\theta_1}\alpha_{\theta_2}\right)\right)+18g_sN_f\left({r_h}^2-3a^2(6r_h+1)\right)\log^2(r_h)\Bigg]\Bigg\}\Bigg),$$

which yields:

(A68)
$$f_{xy}(r) = \frac{e^{-\frac{\alpha_{r\phi_1}^{f'_{xy}} r}{\alpha_{r\phi_1}^{f''_{xy}}}}\left(\alpha_{tt}^{\beta}\alpha_{r\phi_1}^{f'_{xy}}{}^2\beta(r-r_h)e^{\frac{\alpha_{r\phi_1}^{f'_{xy}} r_h}{\alpha_{r\phi_1}^{f''_{xy}}}}Ei\left(\frac{\alpha_{r\phi_1}^{f'_{xy}}(r-r_h)}{\alpha_{r\phi_1}^{f''_{xy}}}\right) - \alpha_{tt}^{\beta}\alpha_{r\phi_1}^{f''_{xy}}\alpha_{r\phi_1}^{f'_{xy}}\beta e^{\frac{\alpha_{r\phi_1}^{f'_{xy}} r}{\alpha_{r\phi_1}^{f''_{xy}}}} + 2\alpha_{r\phi_1}^{f''_{xy}}{}^3 c_1^{(89)} R_{\frac{tt}{r\phi_1}}(r_h-r)\right)}{2\alpha_{r\phi_1}^{f''_{xy}}{}^2\alpha_{r\phi_1}^{f'_{xy}} R_{\frac{tt}{r\phi_1}}(r-r_h)} + c_2^{(89)}$$
$$= -\frac{\alpha_{tt}^{\beta}\beta}{2\left(\alpha_{r\phi_1}^{f''_{xy}}R_{\frac{tt}{r\phi_1}}\right)(r-r_h)} + \left(\frac{\alpha_{tt}^{\beta}\alpha_{r\phi_1}^{f'_{xy}}\beta\log(r-r_h)}{2\alpha_{r\phi_1}^{f''_{xy}}{}^2 R_{\frac{tt}{r\phi_1}}} + \frac{\gamma\alpha_{tt}^{\beta}\alpha_{r\phi_1}^{f'_{xy}}\beta}{2\alpha_{r\phi_1}^{f''_{xy}}{}^2 R_{\frac{tt}{r\phi_1}}} - \frac{\alpha_{tt}^{\beta}\alpha_{r\phi_1}^{f'_{xy}}\beta\log\left(\frac{\alpha_{r\phi_1}^{f''_{xy}}}{\alpha_{r\phi_1}^{f'_{xy}}}\right)}{4\alpha_{r\phi_1}^{f''_{xy}}{}^2 R_{\frac{tt}{r\phi_1}}} + \frac{\alpha_{tt}^{\beta}\alpha_{r\phi_1}^{f'_{xy}}\beta\log\left(\frac{\alpha_{r\phi_1}^{f'_{xy}}}{\alpha_{r\phi_1}^{f''_{xy}}}\right)}{4\alpha_{r\phi_1}^{f''_{xy}}{}^2 R_{\frac{tt}{r\phi_1}}}\right.$$
$$\left.-\frac{\alpha_{r\phi_1}^{f''_{xy}} c_1 e^{-\frac{\alpha_{r\phi_1}^{f'_{xy}} r_h}{\alpha_{r\phi_1}^{f''_{xy}}}}}{\alpha_{r\phi_1}^{f'_{xy}}} + c_2\right) + O(r-r_h) \sim \frac{b^2\left(3b^2-1\right)\beta\sqrt{N}\sin^4\left(\frac{\psi_0}{2}\right)\beta\left(3b^2\left(8{g_s}^2-10g_s-1\right)+4{g_s}^2-6g_s+1\right)}{\left(6b^2+1\right)^3(g_s-1)^2\sqrt{g_s}\log N{N_f}^6\sin^2\phi_{20}{r_h}^2(r-r_h)\log^{10}(r_h)\alpha_{\theta_1}\alpha_{\theta_2}^2},$$

implying:

(A69) $$f_{xy}(r) = \frac{\upsilon_{xy}\beta\sqrt{g_s}(3g_s-4)\kappa_b\left(\frac{1}{N}\right)^{2/5}\sin^4\left(\frac{\psi_0}{2}\right)\beta N^{-\alpha}}{(g_s-1)^2 N_f{}^6\sin^2\phi_{20}\log(N)(r-r_h)\log^{\frac{11}{2}}(r_h)\alpha_{\theta_1}\alpha_{\theta_2}^2},$$

where $\upsilon_{xy} \ll 1$.

(e2) $\text{EOM}_{\theta_1 x}$ (consistency)

(A70) $$\alpha_{\theta_1 x}^{f'_{xy}}\ f'_{xy}(r) + \frac{\alpha_{\theta_1 x}^{\beta}\beta}{(r-r_h)^2} = 0,$$

where:

(A71)

$$\alpha_{\theta_1 x}^{f'_{xy}} \equiv -\frac{4g_s{}^{5/4}\log N M N_f\sin^2\left(\frac{\psi_0}{2}\right)\left(r_h{}^2-3a^2\right)\left(9a^2+r_h{}^2\right)\left(2\log(r_h)+1\right)}{36\sqrt{2}\pi^{7/4}\left(\frac{1}{N}\right)^{13/20}\sin^2\phi_{20}r_h{}^2\left(6a^2+r_h{}^2\right)\log(r_h)\alpha_{\theta_1}\alpha_{\theta_2}^4}$$

$$\alpha_{\theta_1 x}^{\beta} \equiv \frac{a^2 g_s{}^{3/4}MN^{23/20}\sin^6\left(\frac{\psi_0}{2}\right)\beta\left(9a^2+r_h{}^2\right)^2\left(\log(r_h)-1\right)\left(\left(9a^2+r_h{}^2\right)\log\left(9a^2r_h{}^4+r_h{}^6\right)-8\left(6a^2+r_h{}^2\right)\log(r_h)\right)^2}{2187\sqrt{3}(g_s-1)N_f{}^5\sin^4\phi_{20}r_h{}^3\left(6a^2+r_h{}^2\right)^4\log^2(r_h)\alpha_{\theta_1}^2\alpha_{\theta_2}^6\log^{10}\left(9a^2r_h{}^4+r_h{}^6\right)},$$

which obtains as its LHS:

(A72) $$\frac{\upsilon_{\theta_1 x}b^2\left(1-3b^2\right)^2\left(9b^2+1\right)^2\beta g_s{}^{3/4}MN^{23/20}\sin\left(\frac{\psi_0}{2}\right)\beta}{(g_s-1)N_f{}^5 r_h\left(6b^2+1\right)^4\sin^4\phi_{20}(r-r_h)^2\log^9(r_h)\alpha_{\theta_1}^2\alpha_{\theta_2}^6}$$
$$= \frac{\tilde{\upsilon}_{\theta_1 x}\beta g_s{}^{3/4}\kappa_b^2 M\sin^6\left(\frac{\psi_0}{2}\right)r_h{}^3\beta\left(\frac{1}{N}\right)^{2\alpha+\frac{13}{20}}}{(g_s-1)N_f{}^5\sin^4\phi_{20}(r-r_h)^2\alpha_{\theta_1}^2\alpha_{\theta_2}^6},$$

where $\upsilon_{\theta_1 x}, \tilde{\upsilon}_{\theta_1 x} \ll 1$ that in the MQGP limit, is vanishingly small.

(f) $\text{EOM}_{r\theta_1}$

(A73) $$a_{r\theta_1}^{\theta_1 y}f_{\theta_1 y}(r) + a_{r\theta_1}^{xy}f_{xy}(r) + a_{r\theta_1}^{yz}f_{yz}(r) + \frac{a_{r\theta_1}^{\beta}\beta}{(r-r_h)^2} = 0,$$

where:

(A74)

$$a_{r\theta_1}^{\theta_1 y} \sim -\frac{\log N\left(108a^2+r_h\right)\left(r_h{}^2-3a^2\right)\left(2\log(r_h)+1\right)\alpha_{\theta_1}^7\log^4\left(9a^2r_h{}^4+r_h{}^6\right)}{(g_s-1)g_s{}^3M^2\sqrt[5]{\frac{1}{N}}N_f{}^2\sin^4\phi_{20}r_h{}^4\log^3(r_h)\alpha_{\theta_2}^2},$$

$$a_{r\theta_1}^{xy} \sim \frac{\sin^2\left(\frac{\psi_0}{2}\right)\left(108a^2+r_h\right)\alpha_{\theta_1}^6\log^2\left(9a^2r_h{}^4+r_h{}^6\right)}{g_s{}^3M^2\left(\frac{1}{N}\right)^{2/5}N_f{}^2\sin^4\phi_{20}r_h{}^2\log^2(r_h)\alpha_{\theta_2}^2},$$

$$a_{r\theta_1}^{yz} \sim -\frac{\sin^2\left(\frac{\psi_0}{2}\right)\left(108a^2+r_h\right)\alpha_{\theta_1}^6\log^2\left(9a^2r_h{}^4+r_h{}^6\right)}{g_s{}^3M^2\left(\frac{1}{N}\right)^{2/5}N_f{}^2\sin^4\phi_{20}r_h{}^2\log^2(r_h)\alpha_{\theta_2}^2},$$

$$a_{r\theta_1}^{\beta} \sim -\frac{a^2g_s{}^{3/4}MN^{13/20}\sin\phi_{10}\sin^5\left(\frac{\psi_0}{2}\right)\beta\left(9a^2+r_h{}^2\right)^2\left(\log(r_h)-1\right)\left(\left(9a^2+r_h{}^2\right)\log\left(9a^2r_h{}^4+r_h{}^6\right)-8\left(6a^2+r_h{}^2\right)\log(r_h)\right)^2}{(g_s-1)N_f{}^5\sin^4\phi_{20}r_h{}^3\left(6a^2+r_h{}^2\right)^4(r-r_h)^2\log^2(r_h)\alpha_{\theta_1}\alpha_{\theta_2}^5\log^{10}\left(9a^2r_h{}^4+r_h{}^6\right)},$$

that yields:

$$f_{yz}(r) = \frac{\upsilon_{yz} b^2 \left(1-3b^2\right)^2 \left(9b^2+1\right) \beta g_s \log N M N^{7/10} \sin^2\left(\frac{\psi_0}{2}\right)\beta}{\left(6b^2+1\right)^3 (g_s-1) {N_f}^5 \sin^2\phi_{20} r_h (r-r_h)^3 \log^7(r_h) \alpha_{\theta_1}^3 \alpha_{\theta_2}^3} \tag{A75}$$

where $\upsilon_{yz} \ll 1$, implying:

$$f_{yz}(r) = \frac{\tilde{\upsilon}_{yz} \beta g_s \kappa_b^2 \log N M \sin^2\left(\frac{\psi_0}{2}\right) {r_h}^3 \beta N^{-2\alpha-\frac{11}{10}} \log^2(r_h)}{(g_s-1){N_f}^5 \sin^2\phi_{20} (r-r_h)^3 \alpha_{\theta_1}^3 \alpha_{\theta_2}^3}. \tag{A76}$$

where $\tilde{\upsilon}_{yz} \ll 1$.

(g) EOM_{xx}

$$a_{xx}^{f'_{xy}} f'_{xy}(r) + a_{xx}^{f'_{xz}} f'_{xz}(r) = 0, \tag{A77}$$

where:

(A78)

$$a_{xx}^{f'_{xy}} \sim -\frac{\left(2\pi^3 \alpha_{\theta_1}^2 \alpha_{\theta_2}^4 \log^2\left(9a^2{r_h}^4+{r_h}^6\right) + 360(g_s-1){g_s}^3 M^2 {N_f}^2 \sin^2\left(\frac{\psi_0}{2}\right)\log^2(r_h)\alpha_{\theta_2}^2 + 243\times 4\sqrt{6}(g_s-1){g_s}^3 M^2 {N_f}^2 \sin^2\left(\frac{\psi_0}{2}\right)\log^2(r_h)\alpha_{\theta_1}^3\right)}{(g_s-1){g_s}^{7/2} M^2 \left(\frac{1}{N}\right)^{7/10} {N_f}^2 \sin^2\left(\frac{\psi_0}{2}\right)\log^2(r_h)\alpha_{\theta_1}^6 \alpha_{\theta_2}^2 \log\left(9a^2{r_h}^4+{r_h}^6\right)},$$

$$a_{xx}^{f'_{xz}} \equiv -\frac{r_h\left(9a^2+{r_h}^2\right)\alpha_{\theta_2}^2}{(g_s-1){g_s}^{7/2} M^2 \left(\frac{1}{N}\right)^{7/10} {N_f}^2 \sin^2\phi_{20}\psi^2\left(6a^2+{r_h}^2\right)\log^2(r_h)\alpha_{\theta_1}^6 \log^3\left(9a^2{r_h}^4+{r_h}^6\right)}$$
$$\times\left(\frac{\kappa_{xx}^{(1)}(g_s-1){g_s}^3 M^2 {N_f}^2 \sin^2\phi_{20}\sin^2\left(\frac{\psi_0}{2}\right)\left(6a^2+{r_h}^2\right)\log^2(r_h)\left(4\alpha_{\theta_2}^2 - 27\sqrt{6}\alpha_{\theta_1}^3\right)\log^2\left(9a^2{r_h}^4+{r_h}^6\right)}{\left(9a^2 r_h+{r_h}^3\right)\alpha_{\theta_2}^4}\right.$$
$$+\frac{\kappa_{xx}^{(2)}(g_s-1)^2{g_s}^3 M^2 {N_f}^2 \sin^6\left(\frac{\psi_0}{2}\right)\log(r_h)\alpha_{\theta_1}^6 \log\left(9a^2{r_h}^4+{r_h}^6\right)}{r_h\alpha_{\theta_2}^6} + \frac{\kappa_{xx}^{(3)}\times 64(g_s-1)^3{g_s}^3 M^2 {N_f}^2 \sin^6\left(\frac{\psi_0}{2}\right)\left(6a^2+{r_h}^2\right)\log^2(r_h)\alpha_{\theta_1}^6}{\left(9a^2 r_h+{r_h}^3\right)\alpha_{\theta_2}^6}$$
$$\left.-\frac{14\pi^3\sin^2\phi_{20}\left(6a^2+{r_h}^2\right)\alpha_{\theta_1}^2\log^4\left(9a^2{r_h}^4+{r_h}^6\right)}{9a^2 r_h+{r_h}^3}\right),$$

which yields:

(A79)

$$f_{xz}(r) = \left(C_{xz}^{(1)} - \frac{\upsilon_{xz}\tilde{\Sigma}_2\sqrt{2}\pi^{11/2}\sqrt{g_s}(3g_s-4)\kappa_b \sin^4\left(\frac{\psi_0}{2}\right)\beta N^{-\alpha-\frac{2}{5}}}{(g_s-1)^2{N_f}^6\sin^2\phi_{20}(r-r_h)\log^{\frac{13}{2}}(r_h)\alpha_{\theta_1}\alpha_{\theta_2}^2}\right)\beta.$$

where $\upsilon_{xz} \ll 1$:

(A80)

$$\tilde{\Sigma}_2 \equiv \frac{\left(40(g_s-1){g_s}^3 M^2 {N_f}^2 \sin^2\left(\frac{\psi_0}{2}\right)\alpha_{\theta_2}^2 + 108\sqrt{6}(g_s-1){g_s}^3 M^2 {N_f}^2 \sin^2\left(\frac{\psi_0}{2}\right)\alpha_{\theta_1}^3 + 8\pi^3\alpha_{\theta_1}^2\alpha_{\theta_2}^4\right)}{\left(-4(g_s-1){g_s}^3 M^2 {N_f}^2\psi^2\alpha_{\theta_2}^2 + 108\sqrt{6}(g_s-1){g_s}^3 M^2 {N_f}^2 \sin^2\left(\frac{\psi_0}{2}\right)\alpha_{\theta_1}^3 + 8\pi^3\alpha_{\theta_1}^2\alpha_{\theta_2}^4\right)}$$

Appendix B. The Kiepert's algorithm for solving the quintic (65) and diagonalization of the SYZ type IIA mirror inclusive of $\mathcal{O}(R^4)$ corrections

In this appendix, we give the details pertaining to solving the quintic (65) to help in obtaining the G-structure torsion classes of six-, seven- and eight-folds in Section **4**. This appendix is based on techniques and results summarized in [65], and laid out as a five-step algorithm in this appendix.

- **Step 1** Consider the Tschirnhausen transformation to convert general quintic (65) to the principal quintic:

$$z^5 + 5az^2 + 5bz + c = 0, \tag{C1}$$

where:

$$z = x^2 - ux + v. \tag{C2}$$

In (C2) u is determined by:

$$2A^4 + u\left(4A^3 - 13AB + 15P\right) + u^2\left(2A^2 - 5B\right) - 8A^2B + 10AP + 3B^2 - 10F = 0, \tag{C3}$$

whose root, e.g., near (33) that we work with is:

$$u = \frac{4iN}{27\sqrt{15}\alpha_{\theta_1}^2\alpha_{\theta_2}^2} + \frac{13\beta C_{zz}^{(1)} N^{3/5}}{135\alpha_{\theta_2}^2}. \tag{C4}$$

The global small-$\theta_{1,2}$-uplift of (C4) will be: $u = \frac{i{\kappa_u^{\beta}}^0}{\sin^2\theta_1 \sin^2\theta_2} + \frac{\kappa_u^{\beta} C_{zz}^{(1)}}{\sin^2\theta_2}$. Using (C4), v is given by:

$$\begin{aligned} v &= \frac{-Au - A^2 + 2B}{5} \\ &= -\frac{32N^2}{32805\alpha_{\theta_1}^4\alpha_{\theta_2}^4} + \frac{8i\beta C_{zz}^{(1)} N^{8/5}}{3645\sqrt{15}\alpha_{\theta_1}^2\alpha_{\theta_2}^4}. \end{aligned} \tag{C5}$$

The global small-$\theta_{1,2}$-uplift of (C5) will be: $-\frac{\kappa_v^{\beta^0}}{\sin^4\theta_1 \sin^4\theta_2} + \frac{\kappa_v^{\beta} i\beta C_{zz}^{(1)}}{\sin\theta_1^2 \sin^4\theta_2}$.
The constants a, b and c in (C1) are given by:

$$\begin{aligned}
a &= \frac{1}{5}\big(F\left(3Au+2B+4u^2\right) - P\left(Au^2+Bu+P+u^3\right) \\
&\quad - G(2A+5u) - 10v^3\big) \\
b &= \frac{1}{5}\big(-10av - G\left(4Au^2+3Bu+P+5u^3\right) \\
&\quad + F\left(Bu^2+F+Pu+u^4+4u^3\right) - 5v^4\big) \\
c &= -F\left(u^5+Au^4+Bu^3+Cu^2+Fu+G\right) - v^5 - 5av^2 - 5bv.
\end{aligned} \tag{C6}$$

It should be noted that the vanishingly small numerical pre-factors appearing in (C6) are compensated by very large powers of N.

- **Step 2** To transform the principal quintic to the Brioschi quintic:

$$y^5 - 10Zy^3 + 45Z^2y - Z^2 = 0, \tag{C7}$$

via the Tschirnhausen transformation:

$$z_k = \frac{\lambda + \mu y_k}{\frac{y_k^2}{Z} - 3}, \tag{C8}$$

λ in (C8) is determined by the quadratic:

$$\begin{aligned}
&\lambda^2\left(a^4+abc-b^3\right) - \lambda\left(11a^3b - ac^2 + 2b^2c\right) \\
&\qquad - 27a^3c + 64a^2b^2 - bc^2 = 0.
\end{aligned} \tag{C9}$$

Defining:

$$\begin{aligned}
&f \equiv uv\left(u^{10} + 11u^5v^5 - v^{10}\right) \\
&T \equiv u^{30} + 522u^{25}v^5 - 10005u^{20}v^{10} - 10005u^{10}v^{20} - 522u^5v^{25} + v^{30} \\
&Z \equiv \frac{f^5}{T^2},
\end{aligned} \tag{C10}$$

one determines:

$$\mu \equiv \sqrt{\frac{\lambda b + c}{Za}}. \tag{C11}$$

- **Step 3**: We now discuss the transformation of the Brioschi quintic to the Jacobi sextic:

$$s^6 - 10fs^3 + Hs + 5f^2 = 0, \tag{C12}$$

where:

$$H \equiv -u^{20} + 228u^{15}v^5 - 494u^{10}v^{10} - 228u^5v^{15} - v^{20}.. \tag{C13}$$

Defining:

$$\begin{aligned}
\Delta &\equiv \frac{1}{Z}\\
g_2 &\equiv \frac{\left(\frac{1-1728Z}{Z^2}\right)^{\frac{1}{3}}}{12}\\
g_3 &\equiv \sqrt{\frac{g_2^3 - \Delta}{27}},
\end{aligned} \tag{C14}$$

one solves the cubic:

$$x^3 - \frac{g_2}{4}x - \frac{g_3}{4} = 0. \tag{C15}$$

The roots of (C15), e.g., near (33) are given by:

$$\mathcal{E}^i = \kappa_{\mathcal{E}^i,\ \mathbb{C}}\ {\beta^0} N^{5/3}\sqrt[3]{\frac{1}{\alpha_{\theta_1}^{10}\alpha_{\theta_2}^{10}}} + \frac{\kappa_{\mathcal{E}^i,\ \mathbb{C}}\ {\beta}\sqrt{\beta}\sqrt{C_{zz}^{(1)}}N^{22/15}}{\sqrt[3]{\alpha_{\theta_1}^7\alpha_{\theta_2}^{10}}}, \tag{C16}$$

where $i = 1,2,3$ and $|\kappa_{\mathcal{E}^i,\mathbb{C}}\ {\beta^0/\beta}| \ll 1$. The global small-$\theta_{1,2}$-uplift of (C16) is $\mathcal{E}^i = \tilde{\kappa}_{\mathcal{E}^i,\ \mathbb{C}}\ {\beta^0}\sqrt[3]{\frac{1}{\sin^{10}\theta_1\sin^{10}\theta_2}} + \frac{\tilde{\kappa}_{\mathcal{E}^i,\ \mathbb{C}}\ {\beta}\sqrt{\beta}\sqrt{C_{zz}^{(1)}}}{\sqrt[3]{\sin^7\theta_1\sin^{10}\theta_2}}$ Defining, $L \equiv \frac{\sqrt[4]{\mathcal{E}^1-\mathcal{E}^3}-\sqrt[4]{\mathcal{E}^1-\mathcal{E}^2}}{\sqrt[4]{\mathcal{E}^1-\mathcal{E}^2}+\sqrt[4]{\mathcal{E}^1-\mathcal{E}^3}}$, e.g., near (33), $L = -1 + (2.2 + 0.4i)\sqrt[8]{\beta}\sqrt[4]{C_{zz}^{(1)}}\sqrt[20]{\frac{1}{N}\sqrt[4]{\alpha_{\theta_1}}}$, whose global small-$\theta_{1,2}$-uplift will be: $L = -1 + \kappa_{L,\ \mathbb{C}}\sqrt[8]{\beta}\sqrt[4]{C_{zz}^{(1)}}\sqrt[4]{\sin\theta_1}$.

The Jacobi nome q is defined as:

$$q = \sum_{j=0}^{\infty} q_j\left(\frac{L}{2}\right)^{4j+1}, \tag{C17}$$

wherein e.g., near (33):

$$
\begin{aligned}
& q_0 = 1 \\
& q_1 = 2 \\
& q_2 = 15 \\
& q_3 = 150 \\
& q_4 = 1707 \\
& q_5 = 20,910 \\
& q_6 = 268,616 \\
& q_7 = 3,567,400 \\
& q_8 = 48,555,069 \\
& q_9 = 673,458,874 \\
& q_{10} = 9,481,557,398 \\
& q_{11} = 135,119,529,972 \\
& q_{12} = 1,944,997,539,623 \\
& q_{13} = 28,235,172,753,886 \\
& ....
\end{aligned}
\tag{C18}
$$

The value of q, e.g., near (33), appears to converge to a form:

$$
q = -0.7 + \sqrt[4]{\beta}\sqrt{C_{zz}^{(1)}}\mathcal{C}_q \sqrt[10]{\frac{1}{N}}\sqrt{\alpha_{\theta_1}}, \tag{C19}
$$

whose global small-$\theta_{1,2}$-uplift will be: $q = -0.7 + \sqrt[4]{\beta}\sqrt{C_{zz}^{(1)}}\mathcal{C}_q\sqrt{\sin_{\theta_1}}$
The roots, e.g., near (33), are given by:
1)

$$
\sqrt{s_\infty} = \frac{\sqrt{5}}{\Delta^{\frac{1}{6}}}\frac{\sum_{j:-\infty}^{\infty}(-)^j q^{\frac{5(6j+1)^2}{12}}}{\sum_{j:-\infty}^{\infty}(-)^j q^{\frac{(6j+1)^2}{12}}}; \tag{C20}
$$

appears to converge to:

$$
\sqrt{S_\infty} = \frac{(11.4 + 18031i)\left(\alpha_{\theta_1}\alpha_{\theta_2}\right)^{10/3}}{N^{5/3}} - \frac{(160.2 + 266468i)\sqrt[4]{\beta}\sqrt{C_{zz}^{(1)}}\mathcal{C}_q\alpha_{\theta_1}^{23/6}\alpha_{\theta_2}^{10/3}}{N^{53/30}}, \tag{C21}
$$

whose global small-$\theta_{1,2}$-uplift will be: $\kappa^{\beta^0}_{S_\infty}\left(\sin\theta_1\sin\theta_2\right)^{10/3} - \kappa^{\beta}_{S_\infty}\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q\sin^{23/6}\theta_1\sin^{10/3}\theta_2$.

2)

$$\sqrt{S_k} = -\frac{1}{\Delta^{\frac{1}{6}}}\frac{\sum_{j:-\infty}^{\infty}(-)^j\varepsilon^{k(6j+1)^2}q^{\frac{(6j+1)^2}{60}}}{\sum_{j:-\infty}^{\infty}(-)^j q^{\frac{(6j+1)^2}{12}}}, \tag{C22}$$

e.g., near (33), yielding:

$$S_i = \frac{\kappa^{\beta^0}_{S_i,\ \mathbb{C}}\left(\alpha_{\theta_1}\alpha_{\theta_2}\right)^{10/3}}{N^{5/3}} + \frac{\kappa^{\beta}_{S_i,\ \mathbb{C}}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q\sqrt[4]{\beta}\alpha_{\theta_2}^{10/3}\alpha_{\theta_1}^{23/6}}{N^{53/30}}, \tag{C23}$$

$i = 0, 1, ..., 8$, whose global small-$\theta_{1,2}$-uplift will be: $S_i = -\kappa^{\beta^0}_{S_i}\left(\sin\theta_1\sin\theta_2\right)^{10/3} + \kappa^{\beta}_{S_i}\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q\sin^{23/6}\theta_1\sin^{10/3}\theta_2$. It turns out that seven of the nine S_is, have $|\kappa^{\beta^0,\beta}_{S_i,\ \mathbb{C}}| \gg 1$ and the remaining two have moduli much less than unity.

The roots of the Jacobi sextic are related to those of the Brioschi quintic via:

$$y_k = \frac{1}{\sqrt{5}}\left(S_\infty - S_k\right)\left(S_{k+2} - S_{k+3}\right)\left(S_{k+4} - S_{k+1}\right), \tag{C24}$$

yielding, e.g., near (33),

$$y_i = \frac{\kappa^{\beta^0}_{y_i,\ \mathbb{C}}\alpha_{\theta_1}^5\alpha_{\theta_2}^5}{N^{5/2}} - \frac{\kappa^{\beta}_{y_i,\ \mathbb{C}}\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q\alpha_{\theta_1}^{11/2}\alpha_{\theta_2}^5}{N^{13/5}}, \tag{C25}$$

$i = 0, 1, ..., 5$ and $|\kappa^{\beta^0,\beta}_{y_i,\ \mathbb{C}}| \gg 1$. It should be noted that the large numerical factors in the numerators of (C23) and (C25) are balanced off by the large powers of N - numerically taken to be $10^2 - 10^3$ - in the denominators of the same. The global small-$\theta_{1,2}$-uplift of (C25) will be: $y_i = \tilde{\kappa}^{\beta^0}_{y_i,\ \mathbb{C}}\sin^5\theta_1\sin^5\theta_2 - \kappa^{\beta}_{y_i,\ \mathbb{C}}\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q\sin^{11/2}\theta_1\sin^5\theta_2$.

- **Step 4**: Hence, the roots of the principal quintic using $z_k = \frac{\lambda+\mu y_k}{\frac{y_k^2}{Z}-3}$ can be obtained. One notes that the same involve vanishing small numerical prefactors accompanied by very large powers of N.

- **Step 5**: Now, finally the roots x_k of the original quintic are given by:

(C26)
$$x_k = \frac{-(zk-v)\left(Au^2+Bu+C+u^3\right)-(A+2u)(z_k-v)^2-G}{(z_k-v)\left(2Av+B+3u^2\right)+Au^3+Bu^2+F+Cu+u^4+(z_k-v)^2}.$$

The above, e.g., near (33), yields the six roots, five of which are:

(C27)
$$x_{i\neq 1} = \frac{0.07N^{3/5}}{\alpha_{\theta_2}^2} + \frac{\kappa_{x_i}^{\beta^0} r^6 \alpha_{\theta_2}^{61}\alpha_{\theta_1}^{64}}{{g_s}^6 \log N^3 M^3 N^{311/10} {N_f}^3 \log^3(r)} + \frac{\kappa_{x_i}^{\beta}\sqrt{C_{zz}^{(1)}}\mathcal{C}_q N^{9/10}\sqrt[4]{\beta}}{\alpha_{\theta_2}^2\alpha_{\theta_1}^{3/2}},$$

and,

(C28)
$$x_1 = \frac{0.07N^{3/5}}{\alpha_{\theta_2}^2} + \frac{\kappa_{x_1}^{\beta^0} r^6 \alpha_{\theta_2}^{61}\alpha_{\theta_1}^{64}}{{g_s}^6 \log N^3 M^3 N^{311/10} {N_f}^3 \log^3(r)} - \frac{\kappa_{x_1}^{\beta}\sqrt{C_{zz}^{(1)}}\mathcal{C}_q \sqrt{N}\sqrt[4]{\beta}\sqrt{\alpha_{\theta_1}}}{\alpha_{\theta_2}^2},$$

wherein $|\kappa_{x_k}^{\beta^0}| \gg 1, |\kappa_{x_k}^{\beta}| \ll 1$ where $\kappa_{x_i}^{\beta^0} \approx \frac{10^{1-3}}{\left(\kappa_{x_k}^{\beta}\right)^2}, k = 0, 1, ..., 4,$ whose global small-$\theta_{1,2}$-uplift are given as under:

(C29)
$$x_i = \frac{\tilde{\kappa}_{x_i}^{\beta^0}}{\sin^2\theta_2} + \frac{\tilde{\kappa}_{x_i}^{\beta^0} r^6 \sin_{\theta_2}^{61}\sin^{64}\theta_1}{{g_s}^6 \log N^3 M^3 {N_f}^3 \log^3(r)} + \frac{\tilde{\kappa}_{x_i}^{\beta}\sqrt{C_{zz}^{(1)}}\mathcal{C}_q\sqrt[4]{\beta}}{\sin^2\theta_2\sin^{3/2}\theta_1}.$$

Strictly speaking, one ought to also consider the $\sqrt{\beta}, \beta^{\frac{3}{4}}, \beta$ terms in $x_{0,1,2,3,4}$. Their forms however is extremely cumbersome. To capture the essence of the results that one gets if one were to actually do so, in the following what is being assumed is that one is working near the type IIB Ouyang's $D7$-brane embedding coordinate patches effected via delocalization parameters $\alpha_{\theta_{1,2}} \sim \mathcal{O}(1)$ and small values of $\theta_{1,2}$, and setting $N \sim 10^2, g_s \sim 0.1, M = N_f = 3$, and r in the IR estimated as $r\,(\in \text{IR}) \sim N^{-\frac{f_r}{3}}$ with $f_r \sim 1$ [4], [5]. One can then show, e.g., in x_0 that by working in the neighborhood of $(\theta_{10} = \frac{\alpha_{\theta_1}}{N^{\frac{1}{5}}}, \theta_{20} = \frac{\alpha_{\theta_{20}}}{N^{\frac{3}{10}}}, \psi = 2n\pi), n = 0, 1, 2,$ one obtains the following β-dependent terms: $\mathcal{O}(10^{-14})\sqrt[4]{\beta} + \mathcal{O}(10^{-13})\sqrt{\beta} + \mathcal{O}(10^{-10})\beta^{\frac{3}{4}} + \mathcal{O}(1)\beta$. Hence, by choosing $\beta \sim \mathcal{O}(10^{-19})$, one sees that the most dominant terms are the $\beta^{\frac{1}{4}}$ and the β terms which are both of the same order. This is hence the reason why we will work with corrections in the co-frames up to $\mathcal{O}(\beta^{\frac{1}{4}})$ and as explained above, this will capture the essence of the exact calculation up to $\mathcal{O}(\beta)$. Further one notes that in (C29), the very large numerical factors in one of the two $\mathcal{O}(\beta^0)$ terms in the each of the five x_i's are compensated by very large N-suppression factors in the denominators.

Denoting the matrix with entries u_{ij} ($i = 1, ..., 5$ indexing the eigenvector and $j = 1, ..., 5$ indexing the column vector element of the ith eigenvector u_i) and embedding the same in a 6×6 matrix $\mathcal{U}$ with $\mathcal{U}_{ab}, a/b = r, i$ then:

(C30)
$$\sum_{a=1}^{6} (e^a)^2 = \begin{pmatrix} dr & d\theta_1 & d\theta_2 & dx & dy & dz \end{pmatrix} \times \mathcal{U} \begin{pmatrix} G_{rr} & 0 & 0 & 0 & 0 & 0 \\ 0 & x_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_4 \end{pmatrix} \mathcal{U}^T \begin{pmatrix} dr \\ d\theta_1 \\ d\theta_2 \\ dx \\ dy \\ dz \end{pmatrix}.$$

The co-frames are hence given by:

(C31)
$$e^1 = \sqrt{G_{rr}} dr,$$

$$\begin{aligned} e^2 = &\sqrt{\kappa^2_{1;\beta^0} \csc^2(\theta_2) + \kappa^2_{2,\beta^0} \frac{r^6 \sin^{61}(\theta_2) \sin^{64}(\theta_1)}{{g_s}^6 \log N^3 M^3 {N_f}^3 \log^3(r)} + \kappa^2_{1;\beta} \frac{\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q}{\sin^{\frac{7}{2}}(\theta_2)}} \\ &\times \Bigg[\frac{d\theta_1 \left(\kappa^2_{\theta_1,1;\beta^0} \frac{{g_s}^{7/4} \log N M N_f (0.25a^2 - 0.06r^2) \sin(\theta_1)\csc(\theta_2)\log(r)}{r^2} + \kappa^2_{\theta_1;2;\beta} \frac{\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q {g_s}^{7/4} \log N M N_f (1.8a^2 + 2.5r^2)\csc(\theta_2)\log(r)}{r^2\sqrt{\sin(\theta_1)}} \right)}{\sqrt[4]{N}} \\ &+ d\theta_2 \Bigg(\kappa^2_{\theta_2,1;\beta^0} \frac{{g_s}^{7/4} M N_f \sin^2(\theta_1)\csc^2(\theta_2)\log(r)\left(3a^2\log(r) + 0.08r\right)}{\sqrt[4]{N} r} \\ &\quad + \kappa^2_{\theta_2;2;\beta} \frac{\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q {g_s}^{21/4} M^3 {N_f}^3 \sin^{\frac{9}{2}}(\theta_1)\csc^6(\theta_2)\log^3(r)\left(a^2r^2\log(r) + r^3\right)}{N^{3/4} r^3} \Bigg) \\ &+ dx \left(\kappa^2_{x,1;\beta^0} \sin^2(\theta_1)\csc(\theta_2) + \kappa^2_{x;2;\beta} \frac{\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q {g_s}^{7/2} M^2 {N_f}^2 \log^2(r) \sin^{\frac{9}{2}}(\theta_1)\csc^5(\theta_2)\left(3.6a^2 r\log(r) + 4.8r^2\right)}{\sqrt{N} r^2} \right) \\ &+ dy \left(1 - \kappa^2_{y,1;\beta^0} \frac{{g_s}^{7/2} M^2 {N_f}^2 \sin^4(\theta_1)\csc^4(\theta_2)\log^2(r)\left(3a^2\log(r) + 0.08r\right)^2}{\sqrt{N} r^2} - \kappa^2_{y;1;\beta} \frac{\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q}{\sin^{\frac{3}{2}}(\theta_1)} \right) \\ &+ dz \left(\kappa^2_{z,1;\beta^0} \sin(\theta_2) - \kappa^2_{z;1;\beta} \frac{\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q \sin(\theta_2)}{\sin^{\frac{3}{2}}(\theta_1)} \right) \Bigg] \end{aligned}$$

$$e^3 = \sqrt{\kappa^3_{1;\beta^0}\csc^2(\theta_2) + \kappa^3_{2;\beta^0}\frac{r^6\sin^{61}(\theta_2)\sin^{64}(\theta_1)}{g_s{}^6\log N^3M^3N^{311/10}N_f{}^3\log^3(r)} - \kappa^3_{1;\beta}\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q\sqrt{\sin(\theta_1)}\csc^2(\theta_2)}$$

$$\Bigg[\frac{d\theta_1\left(\kappa^3_{\theta_1;1;\beta^0}\frac{4g_s{}^{7/4}\log NMN_f(0.2a^2-0.1r^2)\sin(\theta_1)\csc(\theta_2)\log(r)}{N^{3/20}r^2} + \kappa^3_{\theta_1;1;\beta}\frac{16.\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q g_s{}^{7/4}\log NMN_f(1.56a^2-0.522r^2)\csc(\theta_2)\log(r)\sin^{\frac{3}{2}}(\theta_1)}{r^2}\right)}{\sqrt[4]{N}}$$

$$+\frac{d\theta_2\left(\kappa^3_{\theta_2;1;\beta^0}\frac{g_s{}^{7/4}MN_f\sin^2(\theta_1)\csc^2(\theta_2)\log(r)(0.0005a^2\log(r)+0.000014r)}{\sqrt[20]{N}r} + \kappa^3_{\theta_2;1;\beta}\frac{16\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q g_s{}^{7/4}MN_f\sin^{\frac{5}{2}}(\theta_1)\csc^2(\theta_2)\log(r)(2.7a^2\log(r)+0.081r)}{r}\right)}{\sqrt[4]{N}}$$

$$+ dy\left(1 - \kappa^3_{y;1;\beta^0}\frac{\frac{g_s{}^{7/2}M^2N_f{}^2\sin^4(\theta_1)\csc^4(\theta_2)\log^2(r)(4.32a^2\log(r)+0.064r)}{r} - \kappa^3_{y;1;\beta}\frac{16.\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q g_s{}^{7/2}M^2N_f{}^2\sin^{\frac{9}{2}}(\theta_1)\csc^4(\theta_2)\log^2(r)(-7a^2\log(r)-0.1r)}{\sqrt[5]{N}r}}{\sqrt{N}}\right)$$

$$+ dx\left(\kappa^3_{x;1;\beta^0}\sin^2(\theta_1)\csc(\theta_2) - \kappa^3_{x;1;\beta}\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q\sin^{\frac{5}{2}}(\theta_1)\csc(\theta_2)\right) + dz\left(\kappa^3_{z;1;\beta^0}\sin\theta_2 + \kappa^3_{z;1;\beta}\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q\sqrt{\sin\theta_1}\sin\theta_2\right)\Bigg]$$

(C32)

$$e^4 = \sqrt{\kappa^4_{1;\beta^0}\csc^2(\theta_2) - \kappa^4_{2;\beta^0}\frac{r^6\sin^{61}(\theta_2)\sin^{64}(\theta_1)}{g_s{}^6\log N^3M^3N_f{}^3\log^3(r)} - \kappa^4_{1;\beta}\frac{\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q}{\sin^{\frac{7}{2}}(\theta_2)}}$$

$$\times\Bigg[d\theta_1\sqrt[4]{N}\left(\kappa^4_{\theta_1;1;\beta^0}\frac{\left(0.006a^2r^2 - 0.002r^4\right)\csc(\theta_1)}{g_s{}^{7/4}\log NMN_f\left(0.02a^4 - 0.01a^2r^2 + 0.002r^4\right)\log(r)} + \kappa^4_{\theta_1;1;\beta}\frac{\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q\csc^3(\theta_2)}{g_s{}^{15/4}\log N^3M^2N_f{}^2\sqrt{\sin(\theta_1)}\log^2(r)}\right)$$

$$+ d\theta_2\left(\kappa^4_{\theta_2;1;\beta^0}\frac{\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q\sqrt[4]{N}\csc^2(\theta_2)}{g_s{}^{15/4}\log N^2M^2N_f{}^2\sin^{\frac{3}{2}}(\theta_1)\log^2(r)}\right.$$

$$\left. - \kappa^4_{\theta_2;2;\beta^0}\frac{r^{10}\sin^{\frac{127}{2}}(\theta_1)\sin^{59}(\theta_2)}{\sqrt{C^{(1)}_{zz}}\mathcal{C}_q g_s{}^{25/4}\log N^3M^3\sqrt[4]{N}N_f{}^3\left(0.022a^4 - 0.015a^2r^2 + 0.002r^4\right)\log^3(r)}\right)$$

$$+ dx\left(\kappa^4_{x;1;\beta^0}\frac{\left(0.033a^4 - 0.02a^2r^2 + 0.004r^4\right)\sin^2(\theta_1)}{0.022a^4 - 0.015a^2r^2 + 0.002r^4} - \kappa^4_{x;1;\beta}\frac{\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q\sqrt{N}}{g_s{}^{7/2}\log N^2M^2N_f{}^2\sin^{\frac{3}{2}}(\theta_1)\log^2(r)}\right)$$

$$+ dy\left(\kappa^4_{y;1;\beta^0}\frac{\pi^{5/2}r^9\sin^{57}(\theta_2)\sin^{66}(\theta_1)\left(9.54a^2\log(r) + 0.265r\right)}{g_s{}^8\log N^5M^4N_f{}^4\left(0.022a^4 - 0.015a^2r^2 + 0.002r^4\right)\log^4(r)} + \kappa^4_{y;1;\beta}\frac{\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q\sqrt{\sin(\theta_1)}\csc^4(\theta_2)}{g_s{}^2\log N^2MN_f\log(r)}\right)$$

$$+ dz\left(1 - \sqrt{N}\left(\kappa^4_{z;1;\beta^0}\frac{r^{20}\sin^{118}(\theta_2)\sin^{128}(\theta_1)}{g_s{}^{39/2}\log N^{10}M^{10}N_f{}^{10}\left(0.022a^4 - 0.015a^2r^2 + 0.002r^4\right)^2\log^{10}(r)} + \kappa^4_{z;1;\beta}\frac{\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q\sqrt{\sin(\theta_1)}}{g_s{}^{7/2}\log N^2M^2N_f{}^2\log^2(r)}\right)\right)\Bigg]$$

$$e^5 = \sqrt{\kappa^5_{1;\beta^0}\csc^2(\theta_2) + \kappa^5_{2;\beta^0}\frac{r^6\sin^{61}(\theta_2)\sin^{64}(\theta_1)}{g_s{}^6\log N^3M^3N_f{}^3\log^3(r)} + \kappa^5_{1;\beta}\frac{\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q}{\sin^{\frac{7}{2}}(\theta_2)}}$$

$$\times\Bigg[d\theta_1\left(-\kappa^5_{\theta_1;1;\beta^0}\frac{g_s{}^{7/4}\log NMN_f\left(0.00002r^2 - 0.00004a^2\right)\sin(\theta_1)\csc(\theta_2)\log(r)}{\sqrt[4]{N}r^2}\right.$$

$$\left. + \kappa^5_{\theta_1;1;\beta}\frac{\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q g_s{}^{21/4}\log NM^3N_f{}^3\left(0.592r^2 - 2.37a^2\right)\sin^{\frac{7}{2}}(\theta_1)\csc^5(\theta_2)\log^3(r)\left(0.00049a^2\log(r) + 0.000014r\right)^2}{N^{3/4}r^4}\right)$$

$$+ d\theta_2\left(-\kappa^5_{\theta_2;1;\beta^0}\frac{g_s{}^{7/4}MN_f\sin^2(\theta_1)\csc^2(\theta_2)\log(r)\left(-0.00049a^2\log(r) - 0.000014r\right)}{\sqrt[4]{N}r}\right.$$

$$\left. + \kappa^5_{\theta_2;1;\beta}\frac{\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q g_s{}^{21/4}M^3N_f{}^3\sin^{\frac{9}{2}}(\theta_1)\csc^6(\theta_2)\log^3(r)\left(-1.89a^2\log(r) - 0.0197r\right)\left(0.00049a^2\log(r) + 0.000014r\right)^2}{N^{3/4}r^3}\right)$$

$$+ dx\left(\kappa^5_{x;1;\beta}\frac{\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q g_s{}^{7/2}M^2N_f{}^2\sin^{\frac{9}{2}}(\theta_1)\csc^5(\theta_2)\log^2(r)\left(0.00049a^2\log(r) + 0.000014r\right)^2}{N^{2/5}r^2} + \kappa^5_{x;1;\beta^0}\sin^2(\theta_1)\csc(\theta_2)\right)$$

$$+ dy\left(1 - \frac{1}{\sqrt{N}}\left\{\kappa^5_{y;1;\beta^0}\frac{g_s{}^{7/2}M^2N_f{}^2\sin^4(\theta_1)\csc^4(\theta_2)\log^2(r)\left(0.00049a^2\log(r) + 0.000014r\right)^2}{r^2}\right.\right.$$

$$\left.\left. + \kappa^5_{y;1;\beta}\frac{\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q g_s{}^{7/2}M^2\sqrt[5]{N}N_f{}^2\sin^{\frac{5}{2}}(\theta_1)\csc^4(\theta_2)\log^2(r)\left(0.00049a^2\log(r) + 0.000014r\right)^2}{r^2}\right\}\right) + dz\left(\kappa^5_{z;1;\beta^0}\sin(\theta_2) + \kappa^5_{z;1;\beta}\frac{\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q}{\sqrt{\sin(\theta_2)}}\right)\Bigg]$$

(C33)

$$
\begin{aligned}
e^6 = &\sqrt{\kappa^6_{1;\beta^0}\csc^2(\theta_2) + \kappa^6_{2;\beta^0}\frac{r^6\sin^{61}(\theta_2)\sin^{64}(\theta_1)}{{g_s}^6\log N^3M^3{N_f}^3\log^3(r)} - \kappa^6_{1;\beta}\frac{\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q}{\sin^{\frac{7}{2}}(\theta_2)}}\\
&\times\Bigg[d\theta_1\left(\kappa^6_{\theta_1;1;\beta^0}\frac{{g_s}^{7/4}\log NMN_f\left(34.9a^2-11.6r^2\right)\sin(\theta_1)\csc(\theta_2)\log(r)}{\sqrt[4]{N}r^2} + \kappa^6_{\theta_1;1;\beta}\frac{\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q{g_s}^{7/4}\log NMN_f\left(0.064r^2-5.8a^2\right)\csc(\theta_2)\log(r)}{\sqrt[4]{N}r^2\sqrt{\sin(\theta_1)}}\right)\\
&+\frac{d\theta_2\left(\kappa^6_{\theta_2;1;\beta^0}\frac{{g_s}^{7/4}MN_f\sin^2(\theta_1)\csc^2(\theta_2)\log(r)(0.5a^2\log(r)+0.014r)}{r} + \kappa^6_{\theta_2;1;\beta}\frac{\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q{g_s}^{7/4}MN_f\sqrt{\sin(\theta_1)}\csc^2(\theta_2)\log(r)(2.9a^2\log(r)+0.1r)}{r}\right)}{\sqrt[4]{N}}\\
&+dy\Bigg(1-\kappa^6_{y;1;\beta^0}\frac{{g_s}^{7/2}M^2{N_f}^2\sin^4(\theta_1)\csc^4(\theta_2)\log^2(r)\left(15.625a^4\log^2(r)+0.875a^2r\log(r)+0.01225r^2\right)}{\sqrt{N}r^2}\\
&+\kappa^6_{y;1;\beta}\frac{\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q{g_s}^{7/2}M^2{N_f}^2\sin^{\frac{5}{2}}(\theta_1)\csc^4(\theta_2)\log^2(r)\left(-a^4\log^2(r)-0.1a^2r\log(r)\right)}{\sqrt{N}r^2}\Bigg)\\
&+dx\left(\kappa^6_{x;1;\beta^0}\sin^2(\theta_1)\csc(\theta_2) - \kappa^6_{x;1;\beta}\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q\sqrt{\sin(\theta_1)}\csc(\theta_2)\right) + dz\left(\kappa^6_{z;1;\beta^0}\sin(\theta_2) + \kappa^6_{z;1;\beta}\frac{\sqrt[4]{\beta}\sqrt{C^{(1)}_{zz}}\mathcal{C}_q\sin(\theta_2)}{\sin^{\frac{3}{2}}(\theta_1)}\right)\Bigg].
\end{aligned}
$$

In (C31)–(C33), $\kappa^{a=1,\ldots,6}_{\theta_{1,2}/x/y/z;1;\beta} \ll 1$. Except for e^4, however, all the rest have an IR-enhancement factor involving some power of $\log r$ appearing in the contributions picked up from the $\mathcal{O}(R^4)$ terms. Further, these contributions also receive near-Ouyang-embedding enhancements around small $\theta_{1,2}$ - which also provide the most dominant contributions to all the terms of the action. Also, $\kappa^a_{1;\beta^0} \gg 1$ but are accompanied by IR-suppression factors involving exponents of r along with near-Ouyang-embedding enhancements around small $\theta_{1,2}$.

Now, (C31)–(C33) can be inverted - in Section **3**, for simplicity, one restricts to the Ouyang embedding $\left(r^6+9a^2r^4\right)^{\frac{1}{4}}e^{\frac{i}{2}(\psi-\phi_1-\phi_2)}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} = \mu$, μ being the Ouyang embedding parameter assuming $|\mu| \ll r^{\frac{3}{2}}$, effected, e.g., by working near the $\theta_1 = \frac{\alpha_{\theta_1}}{N^{\frac{1}{5}}}, \theta_2 = \frac{\alpha_{\theta_2}}{N^{\frac{3}{10}}}$-coordinate patch (wherein an explicit $SU(3)$-structure for the type IIB dual of [2] and its delocalized SYZ type IIA mirror [1], and an explicit G_2-structure for its $\mathcal{M}$-Theory uplift [1] was worked out in [16]).

Appendix C. Ω_{ab}s appearing in (71)

The components of the "structure constants" of (71) Ω_{ab}s after a small-β large-N small-a expansion are given as under:

(D1)

$$\Omega_{22} = \frac{\left(\frac{1}{|\log r|}\right)^{4/3}\sqrt[4]{\frac{1}{N}}\left(\frac{32.9a^2}{r^2}+9.4\right)}{g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}} + \frac{\omega_{22}\sqrt[3]{\frac{1}{|\log r|}}\sqrt{\mathcal{C}_{zz}^{(1)}}\mathcal{C}_q {g_s}^{3/2}M\sqrt[4]{N}N_f\sqrt[4]{\beta}\alpha_{\theta_1}^{5/2}}{r\alpha_{\theta_2}^5 g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}},$$

$$\Omega_{23} = \frac{\left(\frac{1}{|\log r|}\right)^{4/3}\sqrt[4]{\frac{1}{N}}\left(\frac{23.6a^2}{r^2}+15.7\right)}{g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}} + \frac{\omega_{23}\sqrt[3]{\frac{1}{|\log r|}}\sqrt{\mathcal{C}_{zz}^{(1)}}\mathcal{C}_q {g_s}^{3/2}M\sqrt[4]{N}N_f\sqrt[4]{\beta}\alpha_{\theta_1}^{5/2}}{\alpha_{\theta_2}^5 g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}},$$

$$\Omega_{24} = \frac{|\log r|^{2/3}{g_s}^{7/2}\log N^2M^2\left(\frac{1}{N}\right)^{17/20}{N_f}^2\alpha_{\theta_1}^2\left(3.97-\frac{4a^2}{r^2}\right)}{\alpha_{\theta_2}g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}}$$
$$+\frac{\omega_{24}|\log r|^{2/3}\sqrt{\mathcal{C}_{zz}^{(1)}}\mathcal{C}_q{g_s}^{7/2}\log N^2M^2\left(\frac{1}{N}\right)^{11/20}{N_f}^2\sqrt[4]{\beta}\sqrt{\alpha_{\theta_1}}}{\alpha_{\theta_2}g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}},$$

$$\Omega_{25} = -\frac{\left(\frac{1}{|\log r|}\right)^{4/3}\sqrt[4]{\frac{1}{N}}\left(\frac{36.6a^2}{r^2}+24.4\right)}{g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}} + \frac{\omega_{25}\sqrt[3]{\frac{1}{|\log r|}}\sqrt{\mathcal{C}_{zz}^{(1)}}\mathcal{C}_q {g_s}^{3/2}M\sqrt[4]{N}N_f\sqrt[4]{\beta}\alpha_{\theta_1}^{5/2}}{\alpha_{\theta_2}^5 g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}},$$

$$\Omega_{26} = -\frac{|\log r|^{2/3}{g_s}^{7/2}M^2\left(\frac{1}{N}\right)^{7/20}{N_f}^2\left(\frac{0.2a^2}{r^2}+0.1\right)\alpha_{\theta_1}^4}{\alpha_{\theta_2}^4 g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}}$$
$$+\frac{\omega_{26}|\log r|^{2/3}\sqrt{\mathcal{C}_{zz}^{(1)}}\mathcal{C}_q{g_s}^{7/2}M^2\sqrt[20]{\frac{1}{N}}{N_f}^2\sqrt[4]{\beta}\alpha_{\theta_1}^{5/2}}{g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}};$$

$$\Omega_{32} = \frac{\left(\frac{1}{|\log r|}\right)^{4/3}\sqrt[4]{\frac{1}{N}}\left(\frac{43.3a^2}{r^2}+12.4\right)}{g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}} + \frac{\omega_{32}\sqrt[3]{\frac{1}{|\log r|}}\sqrt{\mathcal{C}_{zz}^{(1)}}\mathcal{C}_q {g_s}^{3/2}M\sqrt[4]{N}N_f\sqrt[4]{\beta}\alpha_{\theta_1}^{5/2}}{\alpha_{\theta_2}^5 g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}},$$

$$\Omega_{33} = \frac{\left(\frac{1}{|\log r|}\right)^{4/3}\sqrt[4]{\frac{1}{N}}\left(\frac{31a^2}{r^2}+20.7\right)}{g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}} + \frac{\omega_{33}\sqrt[3]{\frac{1}{|\log r|}}\sqrt{\mathcal{C}_{zz}^{(1)}}\mathcal{C}_q {g_s}^{3/2}M\sqrt[4]{N}N_f\sqrt[4]{\beta}\alpha_{\theta_1}^{5/2}}{\alpha_{\theta_2}^5 g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}},$$

$$\Omega_{34} = \frac{|\log r|^{2/3}{g_s}^{7/2}\log N^2M^2\left(\frac{1}{N}\right)^{17/20}{N_f}^2\left(\frac{28.1a^2}{r^2}+5.2\right)\alpha_{\theta_1}^2}{g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}\alpha_{\theta_2}}\sqrt{1-\frac{r_h^4}{r^4}}$$
$$-\frac{\omega_{34}\sqrt[3]{\frac{1}{|\log r|}}\sqrt{\mathcal{C}_{zz}^{(1)}}\mathcal{C}_q{g_s}^{3/2}M\sqrt[20]{\frac{1}{N}}N_f\sqrt[4]{\beta}\alpha_{\theta_1}^{5/2}}{\alpha_{\theta_2}^4 g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}},$$

$$\Omega_{35} = -\frac{\left(\frac{1}{|\log r|}\right)^{4/3}\sqrt[4]{\frac{1}{N}}\left(\frac{48.1a^2}{r^2}+32.1\right)}{g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}} - \frac{\omega_{35}\sqrt[3]{\frac{1}{|\log r|}}\sqrt{\mathcal{C}_{zz}^{(1)}}\mathcal{C}_q {g_s}^{3/2}M\sqrt[4]{N}N_f\sqrt[4]{\beta}\alpha_{\theta_1}^{5/2}}{\alpha_{\theta_2}^5 g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}};$$

(D2)

$$\Omega_{42} = -\frac{0.02a^2 g_s{}^{7/4}\log NM\sqrt[10]{\frac{1}{N}}N_f\alpha_{\theta_1}}{\sqrt[3]{|\log r|}r^2\alpha_{\theta_2}^2 g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}} - \frac{\omega_{42}\sqrt{\mathcal{C}_{zz}^{(1)}}\mathcal{C}_q N^{17/20}\sqrt[4]{\beta}\sqrt{\alpha_{\theta_1}}}{|\log r|^2 g_s{}^2\log N^2 MN_f\alpha_{\theta_2}^4 g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}}$$

$$\Omega_{43} = -\frac{\left(\frac{1}{|\log r|}\right)^{4/3}\sqrt[20]{N}\left(\frac{0.49a^2}{r^2}+0.49\right)}{\alpha_{\theta_2}g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}}$$

$$\Omega_{45} = -\frac{0.2a^2\sqrt[3]{\frac{1}{|\log r|}}g_s{}^{7/4}\log NM\left(\frac{1}{N}\right)^{7/10}N_f\alpha_{\theta_1}}{r^2 g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}} + \frac{\omega_{45}\left(\frac{1}{|\log r|}\right)^{7/3}\sqrt{\mathcal{C}_{zz}^{(1)}}\mathcal{C}_q N^{11/20}\sqrt[4]{\beta}\sqrt{\alpha_{\theta_1}}}{g_s{}^2\log N^2 MN_f r\alpha_{\theta_2}^3 g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}}$$

$$\Omega_{46} = \frac{\left(\frac{1}{|\log r|}\right)^{4/3}\sqrt[20]{N}\left(0.29-\frac{0.44a^2}{r^2}\right)}{\alpha_{\theta_2}g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}} + \frac{\omega_{46}\sqrt{\mathcal{C}_{zz}^{(1)}}\mathcal{C}_q N^{17/20}\sqrt[4]{\beta}\sqrt{\alpha_{\theta_1}}}{|\log r|^{7/3}g_s{}^{7/3}\log N^2 MN_f\alpha_{\theta_2}^4}\sqrt{1-\frac{r_h^4}{r^4}};$$

$$\Omega_{52} = \frac{\left(\frac{1}{|\log r|}\right)^{4/3}\sqrt[4]{\frac{1}{N}}\left(11.4-\frac{40.1a^2}{r^2}\right)}{\sqrt{\frac{\sqrt{g_s}N_f{}^{2/3}r^4}{r^4-r_h{}^4}}} + \frac{\sqrt[3]{\frac{1}{|\log r|}}\sqrt{\mathcal{C}_{zz}^{(1)}}\mathcal{C}_q g_s{}^{3/2}M\sqrt[4]{N}N_f\sqrt[4]{\beta}\alpha_{\theta_1}^{5/2}}{10^{13}\alpha_{\theta_2}^5 g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}},$$

$$\Omega_{53} = \frac{\left(\frac{1}{|\log r|}\right)^{4/3}\sqrt[4]{\frac{1}{N}}\left(\frac{28.7a^2}{r^2}+19.1\right)}{g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}} + \omega_{53}\sqrt[3]{\frac{1}{|\log r|}}\sqrt{\mathcal{C}_{zz}^{(1)}}\mathcal{C}_q g_s{}^{19/12}M\sqrt[4]{N}N_f{}^{2/3}\sqrt[4]{\beta}\alpha_{\theta_1}^{5/2}\sqrt{1-\frac{r_h^4}{r^4}}$$

$$\Omega_{54} = \frac{|\log r|^{2/3}g_s{}^{7/2}\log N^2 M^2\left(\frac{1}{N}\right)^{17/20}N_f{}^2\left(\frac{4.8a^2}{r^2}+4.8\right)\alpha_{\theta_1}^2\sqrt{\frac{1}{\alpha_{\theta_2}^2}}}{r g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}}$$
$$-\frac{\omega_{54}\sqrt[3]{\frac{1}{|\log r|}}\sqrt{\mathcal{C}_{zz}^{(1)}}\mathcal{C}_q g_s{}^{3/2}M\sqrt[20]{\frac{1}{N}}N_f\sqrt[4]{\beta}\alpha_{\theta_1}^{5/2}}{r\alpha_{\theta_2}^4 g_s^{1/4}N_f^{1/3}}\sqrt{1-\frac{r_h^4}{r^4}}$$

$$\Omega_{55} = -\frac{\left(\frac{1}{|\log r|}\right)^{4/3}\sqrt[4]{\frac{1}{N}}\left(\frac{44.5a^2}{r^2}+29.7\right)}{g_s^{1/4}N_f^{1/3}}\sqrt{1-\frac{r_h^4}{r^4}} - \frac{\omega_{55}\sqrt[3]{\frac{1}{|\log r|}}\sqrt{\mathcal{C}_{zz}^{(1)}}\mathcal{C}_q g_s{}^{3/2}M\sqrt[4]{N}N_f\sqrt[4]{\beta}\alpha_{\theta_1}^{5/2}}{r\alpha_{\theta_2}^5 g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}}$$

$$\Omega_{56} = -\frac{|\log r|^{2/3}g_s{}^{7/2}M^2\left(\frac{1}{N}\right)^{7/20}N_f{}^2\left(\frac{0.27a^2}{r^2}+0.18\right)\alpha_{\theta_1}^4}{\alpha_{\theta_2}^4 g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}} + \frac{\omega_{56}\sqrt[3]{\frac{1}{|\log r|}}\sqrt{\mathcal{C}_{zz}^{(1)}}\mathcal{C}_q g_s{}^{3/2}M\sqrt[4]{N}N_f\sqrt[4]{\beta}\alpha_{\theta_1}^{5/2}}{\alpha_{\theta_2}^5 g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}};$$

(D3)

$$\Omega_{62} = \frac{\left(\frac{1}{|\log r|}\right)^{4/3}\sqrt[4]{\frac{1}{N}}\left(12.5-\frac{43.6a^2}{r^2}\right)}{g_s^{1/4}N_f^{1/3}}\sqrt{1-\frac{r_h^4}{r^4}} + \frac{\omega_{62}\sqrt[3]{\frac{1}{|\log r|}}\sqrt{\mathcal{C}_{zz}^{(1)}}\mathcal{C}_q g_s{}^{3/2}M\sqrt[4]{N}N_f\sqrt[4]{\beta}\alpha_{\theta_1}^{5/2}}{r\alpha_{\theta_2}^5 g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}}$$

$$\Omega_{63} = \frac{\left(\frac{1}{|\log r|}\right)^{4/3}\sqrt[4]{\frac{1}{N}}\left(\frac{31.3a^2}{r^2}+20.8\right)}{g_s^{1/4}N_f^{1/3}}\sqrt{1-\frac{r_h^4}{r^4}} + \frac{\omega_{63}\sqrt[3]{\frac{1}{|\log r|}}\sqrt{\mathcal{C}_{zz}^{(1)}}\mathcal{C}_q g_s{}^{3/2}M\sqrt[4]{N}N_f\sqrt[4]{\beta}\alpha_{\theta_1}^{5/2}}{r\alpha_{\theta_2}^5 g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}}$$

$$\Omega_{64} = \frac{|\log r|^{2/3}g_s{}^{7/2}\log N^2 M^2\left(\frac{1}{N}\right)^{17/20}N_f{}^2\alpha_{\theta_1}^2\sqrt{\frac{1}{\alpha_{\theta_2}^2}}\left(5.3-\frac{5.2a^2}{r^2}\right)}{g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}}$$
$$-\frac{\omega_{64}\sqrt[3]{\frac{1}{|\log r|}}\sqrt{\mathcal{C}_{zz}^{(1)}}\mathcal{C}_q g_s{}^{3/2}M\sqrt[20]{\frac{1}{N}}N_f\sqrt[4]{\beta}\alpha_{\theta_1}^{5/2}}{\alpha_{\theta_2}^4 g_s^{1/4}N_f^{1/3}}\sqrt{1-\frac{r_h^4}{r^4}}$$

$$\Omega_{65} = -\frac{\left(\frac{1}{|\log r|}\right)^{4/3}\sqrt[4]{\frac{1}{N}}\left(\frac{48.5a^2}{r^2}+32.3\right)}{g_s^{1/4}N_f^{1/3}}\sqrt{1-\frac{r_h^4}{r^4}} - \frac{\omega_{65}\sqrt[3]{\frac{1}{|\log r|}}\sqrt{\mathcal{C}_{zz}^{(1)}}\mathcal{C}_q g_s{}^{3/2}M\sqrt[4]{N}N_f\sqrt[4]{\beta}\alpha_{\theta_1}^{5/2}}{r\alpha_{\theta_2}^5 g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}}$$

$$\Omega_{66} = -\frac{|\log r|^{2/3}g_s{}^{7/2}M^2\left(\frac{1}{N}\right)^{7/20}N_f{}^2\left(\frac{0.3a^2}{r^2}+0.2\right)\alpha_{\theta_1}^4}{\alpha_{\theta_2}^4 g_s^{1/4}N_f^{1/3}}\sqrt{1-\frac{r_h^4}{r^4}} + \frac{\omega_{66}\sqrt[3]{\frac{1}{|\log r|}}\sqrt{\mathcal{C}_{zz}^{(1)}}\mathcal{C}_q g_s{}^{3/2}M\sqrt[4]{N}N_f\sqrt[4]{\beta}\alpha_{\theta_1}^{5/2}}{r\alpha_{\theta_2}^5 g_s^{\frac{1}{4}}N_f^{\frac{1}{3}}}\sqrt{1-\frac{r_h^4}{r^4}}.$$

wherein the numerical constants $\omega_{ab} \ll 1$; it turns out that in $\Omega_{4a}, a = 2,3,4,5,6$, these very small numerical pre-factors arising in the higher derivative contribution to the "structure constants", are maximally compensated by the highest positive powers of N (amongst the other $\Omega'_{ab}s, a \neq 4$).

Appendix D. Equation (46) relevant to proving result 3

In this appendix, we will show that the equation (46):

$$\sum_{N,P\in\{t,x^{1,2,3},r,\theta_{1,2},\phi_{1,2},\psi,x^{10}\}} \beta\partial_r\left(\sqrt{-g^{(0)}}g^{(0)NP}g^{(0)}_{NP}f_{NP}g^{rr}_{(0)}\partial_r C^{M_1M_2x^{10}}_{(0)}\right)\delta^{M_3}_{x^{10}} = 0,$$

is satisfied by setting the constant of integration $C^{(1)}_{\theta_1 x}$, to zero up to LO in β and N; up to $\mathcal{O}(\beta)$ and NLO in N, one would additionally require $C^{(1)}_{\theta_1 z} - 2C^{(1)}_{zz} = C^{(1)}_{\theta_2 z} = 0$.

- $(M_1, M_2) = (\theta_1, \theta_2)$:

  One can show that near $r = \chi r_h, \chi = \mathcal{O}(1)$, (46) reduces to:

(E1)
$$\begin{aligned}&\beta\frac{1}{4\pi^{19/4}{g_s}^{5/4}\chi^5\alpha_{\theta_1}^8\alpha_{\theta_2}^4}\Bigg\{27\log N\left(\frac{1}{N}\right)^{3/20}{N_f}^2{r_h}^3\Bigg(\left(\frac{1}{N}\right)^{2/5}\left(-162b^4+9b^2\left(\chi^6+\chi^2\right)+4\chi^8\right)\\&\times\left(0.01\alpha_{\theta_2}^7({C_{zz}}^{(1)}-2C^{(1)}_{\theta_1 z})+0.09C^{(1)}_{\theta_2 z}{g_s}^{3/2}\log r_h M N_f\alpha_{\theta_1}^8\right)-\sqrt{3}\pi^{3/2}b^4C^{(1)}_{\theta_1 x}\alpha_{\theta_1}^4\alpha_{\theta_2}^3\Bigg)\Bigg\}\\&\sim\beta\frac{\log N r_h^3}{N^{\frac{3}{20}}}\frac{N_f^2}{g_s^{\frac{5}{4}}}C^{(1)}_{\theta_1 x},\end{aligned}$$

  where the "$\sim$" in (E1) and henceforth implies equality up to NLO-in-N terms. Therefore by setting:

(E2)
$$C^{(1)}_{\theta_1 x} = 0,$$

  (46) is satisfied in the IR [9].

- $(M_1, M_2) = (\theta_1, x)$:

[9] If one wishes to also consider the NLO-in-N term in (E1), one sees that one needs to impose the additional constraints: $0.01\alpha_{\theta_2}^7({C_{zz}}^{(1)} - 2C^{(1)}_{\theta_1 z}) + 0.09C^{(1)}_{\theta_2 z}{g_s}^{3/2}\log r_h M N_f\alpha_{\theta_1}^8 = 0$ - see (53).

Working in the IR, i.e., $r = \chi r_h, \chi = \mathcal{O}(1)$, one an show that (46) yields:

(E3)

$$\beta\frac{N_f^2 r_h^3}{g_s}\left(\mathcal{F}^{\theta_1 x}_{\theta_1 x}(b,\chi,\alpha_{\theta_{1,2}},r_h)C^{(1)}_{\theta_1 x}\frac{\log N}{N^{\frac{7}{10}}}+\frac{1}{N^{\frac{11}{10}}}\sum_{(M,N)=(z,z),(\theta_1,z),(\theta_2,z)}\mathcal{F}^{\theta_1 x}_{MN}(b,\chi,\alpha_{\theta_{1,2}})C^{(1)}_{MN}\right)$$
$$\sim\beta\frac{N_f^2 r_h^3}{g_s}\mathcal{F}^{\theta_1 x}_{\theta_1 x}(b,\chi,\alpha_{\theta_{1,2}},r_h)C^{(1)}_{\theta_1 x}\frac{\log N}{N^{\frac{7}{10}}},$$

[wherein the notation used is $\mathcal{F}^{M_1 M_2}_{MN}, (M,N) \in \{(\theta_1,x),(\theta_1,z),(\theta_2,z),(z,z)\}$], which again would vanish at (E2).

We will, for the remaining eight equations, given only the equivalents of (E3) below (assuming one is working in the IR, i.e., $r = \chi r_h, \chi = \mathcal{O}(1)$)

- $(M_1, M_2) = (\theta_1, y)$:
  One obtains:

(E4)

$$\beta\frac{N_f^2 r_h^3 \log N}{g_s}\left(N^{\frac{2}{5}}\mathcal{F}^{\theta_1 y}_{\theta_1 x}(b,\chi,\alpha_{\theta_{1,2}},r_h)C^{(1)}_{\theta_1 x}+\sum_{(M,N)=(z,z),(\theta_1,z),(\theta_2,z)}\mathcal{F}^{\theta_1 y}_{MN}(b,\chi,\alpha_{\theta_{1,2}},r_h)C^{(1)}_{MN}\right)$$
$$\sim\beta\frac{N_f^2 r_h^3 N^{\frac{2}{5}}}{g_s}\mathcal{F}^{\theta_1 y}_{\theta_1 x}(b,\chi,\alpha_{\theta_{1,2}},r_h)C^{(1)}_{\theta_1 x},$$

which again would vanish at (E2).

- $(M_1, M_2) = (\theta_1, z)$
  One obtains:

(E5)

$$\beta\frac{N_f^2 r_h^3 \log N}{g_s}\left(N^{\frac{1}{10}}\mathcal{F}^{\theta_1 z}_{\theta_1 x}(b,\chi,\alpha_{\theta_{1,2}},r_h)C^{(1)}_{\theta_1 x}+\frac{1}{N^{\frac{3}{10}}}\sum_{(M,N)=(z,z),(\theta_1,z),(\theta_2,z)}\mathcal{F}^{\theta_1 z}_{MN}(b,\chi,\alpha_{\theta_{1,2}},r_h)C^{(1)}_{MN}\right)$$
$$\sim\beta\frac{N_f^2 r_h^3 N^{\frac{1}{10}}}{g_s}\mathcal{F}^{\theta_1 z}_{\theta_1 x}(b,\chi,\alpha_{\theta_{1,2}},r_h)C^{(1)}_{\theta_1 x},$$

which again would vanish at (E2).

- $(M_1, M_2) = (\theta_2, x)$
  One obtains:

(E6)

$$\beta M N_f^3\sqrt{g_s}\left(\log N\right)^2\left(\frac{1}{N^{\frac{7}{5}}}\mathcal{F}^{\theta_2 x}_{\theta_1 x}(b,\chi,\alpha_{\theta_{1,2}},r_h)C^{(1)}_{\theta_1 x}+\frac{1}{N^{\frac{9}{5}}}\sum_{(M,N)=(z,z),(\theta_1,z),(\theta_2,z)}\mathcal{F}^{\theta_2 x}_{MN}(b,\chi,\alpha_{\theta_{1,2}},r_h)C^{(1)}_{MN}\right)$$
$$\sim\beta M N_f^3\sqrt{g_s}\left(\log N\right)^2\frac{1}{N^{\frac{7}{5}}}\mathcal{F}^{\theta_2 x}_{\theta_1 x}(b,\chi,\alpha_{\theta_{1,2}},r_h)C^{(1)}_{\theta_1 x},$$

which again would vanish at (E2).

- $(M_1, M_2) = (\theta_2, y)$
  One obtains:

(E7)

$$\beta M N_f^3 \sqrt{g_s} (\log N)^2 \left(\frac{1}{N^{\frac{3}{10}}} \mathcal{F}^{\theta_2 y}_{\theta_1 x}(b, \chi, \alpha_{\theta_{1,2}}, r_h) C^{(1)}_{\theta_1 x} + \frac{1}{N^{\frac{7}{10}}} \sum_{(M,N)=(z,z),(\theta_1,z),(\theta_2,z)} \mathcal{F}^{\theta_2 x}_{MN}(b, \chi, \alpha_{\theta_{1,2}}, r_h) C^{(1)}_{MN} \right)$$
$$\sim \beta M N_f^3 \sqrt{g_s} (\log N)^2 \frac{1}{N^{\frac{3}{10}}} \mathcal{F}^{\theta_2 y}_{\theta_1 x}(b, \chi, \alpha_{\theta_{1,2}}, r_h) C^{(1)}_{\theta_1 x},$$

which again would vanish at (E2).

- $(M_1, M_2) = (\theta_2, z)$
  One obtains:

(E8)

$$\beta M N_f^3 \sqrt{g_s} r_h^3 (\log N)^2 \left(\frac{1}{N^{\frac{3}{5}}} \mathcal{F}^{\theta_2 z}_{\theta_1 x}(b, \chi, \alpha_{\theta_{1,2}}, r_h) C^{(1)}_{\theta_1 x} + \frac{1}{N} \sum_{(M,N)=(z,z),(\theta_1,z),(\theta_2,z)} \mathcal{F}^{\theta_2 z}_{MN}(b, \chi, \alpha_{\theta_{1,2}}, r_h) C^{(1)}_{MN} \right)$$
$$\sim \beta M N_f^3 \sqrt{g_s} r_h^3 (\log N)^2 \frac{1}{N^{\frac{3}{5}}} \mathcal{F}^{\theta_2 z}_{\theta_1 x}(b, \chi, \alpha_{\theta_{1,2}}, r_h) C^{(1)}_{\theta_1 x},$$

which again would vanish at (E2).

- $(M_1, M_2) = (x, y)$
  One obtains:

(E9)

$$\beta M N_f^3 {g_s}^{\frac{9}{4}} (\log N)^2 \left(\frac{1}{N^{\frac{5}{4}}} \mathcal{F}^{xy}_{\theta_1 x}(b, \chi, \alpha_{\theta_{1,2}}, r_h) C^{(1)}_{\theta_1 x} + \frac{1}{N^{\frac{33}{20}}} \sum_{(M,N)=(z,z),(\theta_1,z),(\theta_2,z)} \mathcal{F}^{xy}_{MN}(b, \chi, \alpha_{\theta_{1,2}}, r_h) C^{(1)}_{MN} \right)$$
$$\sim \beta M N_f^3 {g_s}^{\frac{9}{4}} (\log N)^2 \frac{1}{N^{\frac{5}{4}}} \mathcal{F}^{xy}_{\theta_1 x}(b, \chi, \alpha_{\theta_{1,2}}, r_h) C^{(1)}_{\theta_1 x},$$

which again would vanish at (E2).

- $(M_1, M_2) = (x, z)$
  One obtains:

(E10)

$$\beta M N_f^3 (\log N)^2 \left(\frac{1}{N^{\frac{23}{20}}} \mathcal{F}^{xz}_{\theta_1 x}(b, \chi, \alpha_{\theta_{1,2}}, r_h) C^{(1)}_{\theta_1 x} + \frac{1}{N^{\frac{31}{20}}} \sum_{(M,N)=(z,z),(\theta_1,z),(\theta_2,z)} \mathcal{F}^{xy}_{MN}(b, \chi, \alpha_{\theta_{1,2}}, r_h) C^{(1)}_{MN} \right)$$
$$\sim \beta M N_f^3 (\log N)^2 \frac{1}{N^{\frac{23}{20}}} \mathcal{F}^{xz}_{\theta_1 x}(b, \chi, \alpha_{\theta_{1,2}}, r_h) C^{(1)}_{\theta_1 x},$$

which again would vanish at (E2).

- $(M_1, M_2) = (y, z)$

One obtains:

(E11)

$$\beta M N_f^3 g_s^{\frac{3}{4}} (\log N)^2 \left(\frac{1}{N^{\frac{1}{20}}} \mathcal{F}^{yz}_{\theta_1 x}(b, \chi, \alpha_{\theta_{1,2}}, r_h) C^{(1)}_{\theta_1 x} + \frac{1}{N^{\frac{9}{20}}} \sum_{(M,N)=(z,z),(\theta_1,z),(\theta_2,z)} \mathcal{F}^{yz}_{MN}(b, \chi, \alpha_{\theta_{1,2}}, r_h) C^{(1)}_{MN} \right)$$
$$\sim \beta M N_f^3 g_s^{\frac{3}{4}} (\log N)^2 \frac{1}{N^{\frac{1}{20}}} \mathcal{F}^{yz}_{\theta_1 x}(b, \chi, \alpha_{\theta_{1,2}}, r_h) C^{(1)}_{\theta_1 x},$$

which again would vanish at (E2).

Appendix E. Details of G_2 structure of $M_7(r, \theta_{1,2}, \phi_{1,2}, \psi, x^{10})|_{\text{Ouyang}-\text{embedding[parent type IIB]}\cap|\mu_{\text{Ouyang}}|\ll 1}$

The details of the evaluation of $\tau_{0,1}$ in the neighborhood of the Ouyang embedding of the flavor $D7$-branes in the parent type IIB theory assuming a infinitesimal modulus of the Ouyang embedding parameter μ_{Ouyang}, are provided below.

The four intrinsic G_2-structure torsion classes are then given by [76]

$$\begin{aligned} W_1 &= \frac{1}{7} d\Phi \lrcorner *_7 \Phi, \\ W_7 &= -\frac{1}{12} d\Phi \lrcorner \Phi = \frac{1}{12} \Phi \wedge *_7 d\Phi, \\ W_{14} &= \frac{1}{2} \left(d *_7 \Phi \lrcorner \Phi - *_7 d *_7 \Phi \right) - 2 W_7 \lrcorner \Phi = - *_7 d *_7 \Phi + 4 W_7 \lrcorner \Phi, \\ W_{27} &= *_7 d\Phi - W_1 \Phi + 3 W_7 \lrcorner *_7 \Phi. \end{aligned} \tag{E1}$$

E.1. $W_7(M_7(r, \theta_{1,2}, \phi_{1,2}, \psi, x^{10})$

Utilizing, $W_7 = *_7 \left(\Phi \wedge *_7 d\Phi \right)$, let us first evaluate $\Phi \wedge *_7 d\Phi$. One sees that,

(E2)

$$\Phi \wedge *_7 d\Phi = e^{-2\Phi^{\text{IIA}}} \big(\Omega_{25} e^{135247} - \Omega_{23} e^{135267} - (\Omega_{43} + \Omega_{65}) e^{135467} - \Omega_{24} e^{135257} - \Omega_{26} e^{146237}$$
$$+ (\Omega_{34} + \Omega_{56}) e^{146357} - (\Omega_{66} + \Omega_{22} + \Omega_{33}) e^{236457} - (\Omega_{22} + \Omega_{44} + \Omega_{55}) e^{245367} \big)$$
$$- \frac{e^{-\frac{5}{3}\Phi^{\text{IIA}}}}{\sqrt{G^{\mathcal{M}}_{x^{10}x^{10}}}} \left(-(\Omega_{45} + \Omega_{63}) e^{135246} - (\Omega_{36} + \Omega_{54}) e^{146235} \right) - \frac{4}{3} \frac{e^{-\frac{5}{3}\Phi^{\text{IIA}}}}{\sqrt{\left(G^{\mathcal{M}}_{x^{10}x^{10}} \right)^2 G^{\mathcal{M}}_{rr}}} e^{234567}$$
$$+ \frac{e^{-\frac{4}{3}\Phi^{\text{IIA}}}}{G^{\mathcal{M}}_{x^{10}x^{10}}} \Big((\Omega_{33} + \Omega_{44}) e^{347256} - (\Omega_{55} + \Omega_{66}) e^{567234} + \Omega_{32} e^{127356} + \Omega_{42} e^{127456} - \Omega_{52} e^{127345}$$
$$+ \Omega_{62} e^{127346} \Big)$$

From appendix **C**, the most dominant Ω_{ij}s are given as under:
(E3)

$$\Omega_{34} = \frac{|\log r|^{2/3} {g_s}^{7/2} \log N^2 M^2 \left(\frac{1}{N}\right)^{17/20} {N_f}^2 \left(\frac{28.1a^2}{r^2} + 5.2\right) \alpha_{\theta_1}^2}{g_s^{\frac{1}{4}} N_f^{\frac{1}{3}} \alpha_{\theta_2}} \sqrt{1 - \frac{r_h^4}{r^4}}$$
$$- \frac{\omega_{34} \sqrt[3]{\frac{1}{|\log r|}} \sqrt{\mathcal{C}_{zz}^{(1)}} \mathcal{C}_q {g_s}^{3/2} M \sqrt[20]{\frac{1}{N}} N_f \sqrt[4]{\beta} \alpha_{\theta_1}^{5/2}}{\alpha_{\theta_2}^4 g_s^{\frac{1}{4}} N_f^{\frac{1}{3}}} \sqrt{1 - \frac{r_h^4}{r^4}},$$

$$\Omega_{36} = -\frac{|\log r|^{2/3} gs^{13/4} M^2 \left(\frac{1}{N}\right)^{7/20} Nf^{5/3} \alpha_{\theta_1}^4 \sqrt{\frac{1}{\alpha_{\theta_2}^2}} \sqrt{1 - \frac{rh^4}{r^4}} \left(0.2 - \frac{0.3a^2}{r^2}\right)}{\alpha_{\theta_2}^3}$$
$$+ \frac{\omega_{36} \sqrt[3]{\frac{1}{|\log r|}} \sqrt[4]{\beta} \sqrt{\mathcal{C}_{zz}^{(1)}} \mathcal{C}_q gs^{5/4} M \sqrt[4]{N} Nf^{2/3} \alpha_{\theta_1}^{5/2} \sqrt{1 - \frac{rh^4}{r^4}}}{\alpha_{\theta_2}^5};$$

$$\Omega_{54} = \frac{|\log r|^{2/3} {g_s}^{7/2} \log N^2 M^2 \left(\frac{1}{N}\right)^{17/20} {N_f}^2 \left(\frac{4.8a^2}{r^2} + 4.8\right) \alpha_{\theta_1}^2 \sqrt{\frac{1}{\alpha_{\theta_2}^2}}}{r g_s^{\frac{1}{4}} N_f^{\frac{1}{3}}} \sqrt{1 - \frac{r_h^4}{r^4}}$$
$$- \frac{\omega_{54} \sqrt[3]{\frac{1}{|\log r|}} \sqrt{\mathcal{C}_{zz}^{(1)}} \mathcal{C}_q {g_s}^{3/2} M \sqrt[20]{\frac{1}{N}} N_f \sqrt[4]{\beta} \alpha_{\theta_1}^{5/2}}{r \alpha_{\theta_2}^4 g_s^{1/4} N_f^{1/3}} \sqrt{1 - \frac{r_h^4}{r^4}}$$

$$\Omega_{56} = -\frac{|\log r|^{2/3} {g_s}^{7/2} M^2 \left(\frac{1}{N}\right)^{7/20} {N_f}^2 \left(\frac{0.27a^2}{r^2} + 0.18\right) \alpha_{\theta_1}^4}{\alpha_{\theta_2}^4 g_s^{\frac{1}{4}} N_f^{\frac{1}{3}}} \sqrt{1 - \frac{r_h^4}{r^4}} +$$
$$\frac{\omega_{56} \sqrt[3]{\frac{1}{|\log r|}} \sqrt{\mathcal{C}_{zz}^{(1)}} \mathcal{C}_q {g_s}^{3/2} M \sqrt[4]{N} N_f \sqrt[4]{\beta} \alpha_{\theta_1}^{5/2}}{\alpha_{\theta_2}^5 g_s^{\frac{1}{4}} N_f^{\frac{1}{3}}} \sqrt{1 - \frac{r_h^4}{r^4}},$$

where $\omega_{ij} \ll 1$. The $\mathcal{M}$-theory metric components $G^{\mathcal{M}}_{x^{10}x^{10}}$, $G^{\mathcal{M}}_{rr}$ are given by:
(E4)

$$G^{\mathcal{M}}_{rr} = \left\{ \frac{\sqrt{g_s}\left(6a^2 + r^2\right)}{2\sqrt[3]{3}\sqrt[6]{\pi}\sqrt[3]{\mathcal{A}} r^2 \left(9a^2 + r^2\right)\left(1 - \frac{r_h^4}{r^4}\right)} \right\} + \left[3\mathcal{A}\sqrt{N} - \frac{96\pi a^2 g_s M^2 \sqrt{\frac{1}{N}} N_f (c_1 + c_2 \log(r_h))}{9a^2 + r^2} \right.$$
$$\left. + \frac{9}{32}\mathcal{A} g_s M^2 \sqrt{\frac{1}{N}} \left(-\frac{64a^2 r^2 (c_1 + c_2\log(r_h))}{\left(6a^2 + r^2\right)\left(9a^2 + r^2\right)} - \frac{\log(r)(Ag_s + 12 g_s N_f + 8\pi)}{\pi^2} \right)) \right]$$
$$+ \beta \left\{ \frac{3^{2/3}\mathcal{A}^{2/3}\sqrt{g_s}\sqrt{N} r^2 \left(6a^2 + r^2\right)\left(\mathcal{C}_{zz}^{bh} - 2\mathcal{C}_{\theta_1 z}^{bh} + 2\mathcal{C}_{\theta_1 x}^{bh}\right)}{2\sqrt[6]{\pi}\left(9a^2 + r^2\right)\left(r^4 - r_h^4\right)} \right\}$$

$$G^{\mathcal{M}}_{x_{10}x_{10}} = \left\{\frac{16\pi^{4/3}\left(64\pi a^2 g_s M^2 N_f(c_1+c_2\log(r_h)) + \mathcal{A}N\left(9a^2+r^2\right)\right)}{3\sqrt[3]{3}\mathcal{A}^{7/3}N\left(9a^2+r^2\right)}\right\}$$
$$-\beta\left[\left\{\frac{-19683\sqrt{6}\alpha_{\theta_1}^6 - 6642\alpha_{\theta_2}^2\alpha_{\theta_1}^3 + 40\sqrt{6}\alpha_{\theta_2}^4}{\mathcal{A}^{4/3}\left(3b^2-1\right)^5 N_f r_h^4 \alpha_{\theta_2}^3\left(9a^2+r_h^2\right)}\right\}\right.$$
$$\left.\left(\frac{48\ 3^{2/3}\sqrt[3]{\pi}b^{10}\left(9b^2+1\right)^4 M\left(\frac{1}{N}\right)^{5/4} r\left(6a^2+r_h^2\right)(r-2r_h)\log^3(r_h)}{\left(6b^2\log N+\log N\right)^4}\right)\right]$$

where using [77],
(E5)
$$\mathcal{A} \equiv 4\left[-N_f\log\left(\frac{4\sqrt{N}}{\left(9a^2r^4+r^6\right)^{\frac{1}{4}}\alpha_{\theta_1}\alpha_{\theta_2}}\right) + \frac{2\pi}{g_s}\right] \stackrel{r\in\mathrm{IR}}{\longrightarrow} N_f\log r \stackrel{r\in\mathrm{IR}}{\longrightarrow} N_f N^{\frac{1}{3}}.$$

Replacing $\alpha_{\theta_1}\to N^{\frac{1}{5}}\sin\theta_1$ and $\alpha_{\theta_2}\to N^{\frac{3}{10}}\sin\theta_2$:
(E6)
$$\Omega_{22}^{\beta^0}\sim\Omega_{23}^{\beta^0}\sim\Omega_{25^{\beta^0}}\sim\Omega_{32}^{\beta^0}\sim\Omega_{33}^{\beta^0}\sim\Omega_{35}^{\beta^0}\sim-\Omega_{42}^{\beta^0}\sim\Omega_{43}^{\beta^0}\sim\Omega_{45}^{\beta^0}\sim\Omega_{46}^{\beta^0}\sim\Omega_{52}^{\beta^0}\sim\Omega_{53}^{\beta^0}\sim-\Omega_{55}$$
$$\sim\Omega_{62}^{\beta^0}\sim\Omega_{63}^{\beta^0}\sim-\Omega_{65}^{\beta^0}\sim\frac{1}{N^{\frac{1}{4}}|\ln r|^{\frac{4}{3}}}\stackrel{r\in\mathrm{IR}}{\longrightarrow}\frac{1}{N_f^{\frac{4}{3}}N^{\frac{25}{36}}};$$
$$\Omega_{24}^{\beta^0}\sim-\Omega_{26}^{\beta^0}\sim-\Omega_{36}^{\beta^0}\sim\Omega_{56}^{\beta^0}\sim\Omega_{66}^{\beta^0}\sim\frac{|\log r|^{\frac{2}{3}}}{N^{\frac{3}{4}}}\stackrel{r\in\mathrm{IR}}{\longrightarrow}\frac{N_f^{\frac{2}{3}}}{N^{\frac{19}{36}}};$$
$$\Omega_{34}\sim\Omega_{54}^{\beta^0}\sim\frac{(\log N)^2|\log r|^{\frac{2}{3}}}{N^{\frac{3}{4}}}\stackrel{r\in\mathrm{IR}}{\longrightarrow}\frac{(\log N)^2 N_f^{\frac{2}{3}}}{N^{\frac{19}{36}}}.$$

We thus see:
(E7)
$$\Phi\wedge *_7 d\Phi(r\in\mathrm{IR}))\sim e^{-2\Phi^{\mathrm{IIA}}}\left(\Omega_{34}+\Omega_{56}\right)e^{146357}-\frac{e^{-\frac{5}{3}\Phi^{\mathrm{IIA}}}}{\sqrt{G^{\mathcal{M}}_{x^{10}x^{10}}}}\left(\Omega_{36}-\Omega_{54}\right)e^{146235}$$
$$\stackrel{r\in\mathrm{IR}}{\longrightarrow}\frac{(\log N)^2}{N^{\frac{3}{4}}|\log r|^{\frac{4}{3}}}\stackrel{r\in\mathrm{IR}}{\longrightarrow}\frac{1}{N^{\alpha_{W_7}}}e^{146357}+\frac{\alpha_{\Phi^{\mathrm{IIA}}}G_{x^{10}x^{10}}}{|\log r|}\left(\Omega_{36}-\Omega_{54}\right)e^{146235},$$

where $\alpha_{W_7}>1$. Now, from (E3), one can show that:

$$\left(\Omega_{36}-\Omega_{54}\right)^{\beta^0}=0, \quad \text{(E8)}$$

for $\forall r \sim \sqrt{\frac{3}{2}}a \in$IR - note that $r = \sqrt{3}a$ is the interface of the UV and IR-UV interpolating regions. Therefore,

(E9)
$$W_7\left[= *_7\left(\Phi \wedge *_7 d\Phi\right)\right]|_{\text{Ouyang-embedding[parent type IIB]}\cap|\mu_{\text{Ouyang}}|\ll 1}$$
$$= -\, d\Phi \lrcorner \Phi|_{\text{Ouyang-embedding[parent type IIB]}\cap|\mu_{\text{Ouyang}}|\ll 1}$$
$$= \mathcal{O}\left(\frac{1}{N^{\alpha>1}}\right) \sim 0 \left(\text{as work only up to } \mathcal{O}\left(\frac{1}{N}\right)\right),$$

as stated in (92).

E.2. $W_1(M_7(r, \theta_{1,2}, \phi_{1,2}, \psi, x^{10}))$

One can show that:
(E10)
$$W_1 = \frac{1}{7} d\Phi \lrcorner *_7 \Phi = \frac{2}{7} \frac{e^{-\frac{5}{3}\Phi^{\text{IIA}}}}{\sqrt{G^{\mathcal{M}}_{x^{10}x^{10}}}} \left(\Omega_{53} - \Omega_{64}\right) \xrightarrow{r\in\text{IR}} \sim \frac{1}{|\log r|} \left(\Omega_{53} - \Omega_{64}\right).$$

Using results of appendix **C**,
(E11)
$$(\Omega_{53} - \Omega_{64})^{\beta^0} = \frac{\sqrt{1 - \frac{{r_h}^4}{r^4}} \left(\left(\frac{1}{|\log r|}\right)^{4/3} \sqrt[4]{\frac{1}{N}} \left(\frac{28.7a^2}{r^2} + 19.1\right) - \frac{|\log r|^{2/3} {g_s}^{7/2} \log N^2 M^2 \left(\frac{1}{N}\right)^{17/20} {N_f}^2 \alpha_{\theta_1}^2 \left(5.3 - \frac{5.2a^2}{r^2}\right)}{\alpha_{\theta_2}}\right)}{\sqrt[4]{g_s} \sqrt[3]{N_f}}$$

and replacing $\alpha_{\theta_2} \to N^{\frac{3}{10}} \sin\theta_2$ to get the conjectured result $\forall \theta_{1,2}, \phi_{1,2}, \psi$ but in the neighborhood of the Ouyuang embedding in the parent type IIB theoryone can show
(E12)
$$(\Omega_{53} - \Omega_{64})^{\beta^0} = \frac{\left(5.2|\log r|^{5/3} {g_s}^{\frac{7}{2}} \log N^2 M^2 \left(\frac{1}{N}\right)^{3/5} {N_f}^2 \alpha_{\theta_1}^2 \left(a^2 - 1.02r^2\right) + 28.7 \sqrt[3]{\frac{1}{|\log r|}} \alpha_{\theta_2} \left(a^2 + 0.67r^2\right)\right)}{N^{\frac{1}{4}+\frac{3}{10}} (\sin\theta_2) \log r} \sqrt{1 - \frac{{r_h}^4}{r^4}}$$

One can further show that (guided by [77]) assuming $|\log r| \sim \alpha_{|\log r|_{\text{IR}}} N^{\frac{1}{3}}$, for
(E13)
$$r = \left[0.99 + \frac{0.095|\kappa_r|}{{\alpha_{|\log r|}}^{5/3} {g_s}^{7/2} \log N^2 M^2 {N_f}^2 \alpha_{\theta_1}^2} + \mathcal{O}\left(\left(\frac{1}{\log N}\right)^4\right)\right] a \in \text{IR},$$

$$W_1 \sim \frac{\sqrt{1-\frac{{r_h}^4}{r^4}}}{|\log r|^2 N^{\frac{11}{20}} \sin\theta_2} \left(\frac{|\kappa_r|}{N^{\frac{2}{25}}}\right) a^2 \xrightarrow{r\in\mathrm{IR}} \mathcal{O}\left(\frac{1}{N^{\alpha_{W_1}>1}}\right) a^2 \sim 0 \left(\text{as work only up to } \mathcal{O}\left(\frac{1}{N}\right)\right), \quad \text{(E14)}$$

for $\kappa_r < 0$.

References

[1] M. Dhuria and A. Misra, *Towards MQGP*, JHEP 1311 (2013) 001.

[2] M. Mia, K. Dasgupta, C. Gale and S. Jeon, *Five Easy Pieces: The Dynamics of Quarks in Strongly Coupled Plasmas*, Nucl. Phys. B **839**, 187 (2010).

[3] S. Grozdanov and A. O. Starinets, *Second-order transport, quasinormal modes and zero-viscosity limit in the Gauss-Bonnet holographic fluid*, JHEP **03**, 166 (2017); J. Casalderrey-Solana, S. Grozdanov and A. O. Starinets, *Transport Peak in the Thermal Spectral Function of $\mathcal{N}=4$ Supersymmetric Yang-Mills Plasma at Intermediate Coupling*, Phys. Rev. Lett. **121**, no. 19, 191603 (2018).

[4] V. Yadav, G. Yadav and A. Misra, *(Phenomenology/Lattice-Compatible) $SU(3)$ $M\chi PT$ HD up to $\mathcal{O}(p^4)$ and the $\mathcal{O}\left(R^4\right)$-Large-N Connection*, JHEP 08 (2021) 151.

[5] G. Yadav, V. Yadav and A. Misra, *McTEQ (M chiral perturbation theory-compatible deconfinement Temperature and Entanglement entropy up to terms Quartic in curvature) and FM (Flavor Memory)*, JHEP 10 (2021) 220.

[6] I. R. Klebanov, D. Kutasov and A. Murugan, *Entanglement as a probe of confinement*, Nucl. Phys. B **796**, 274 (2008).

[7] C. A. Ballon Bayona, H. Boschi-Filho and N. R. F. Braga, *Deep inelastic scattering from gauge string duality in the soft wall model*, JHEP **03**, 064 (2008).

[8] M. J. Duff, J. T. Liu and R. Minasian, *Eleven-dimensional origin of string-string duality: A One loop test*, Nucl. Phys. B 452, 261 (1995); M. B. Green and P. Vanhove, *D instantons, strings and $\mathcal{M}$ theory*, Phys. Lett. B 408, 122 (1997); M. B. Green, M. Gutperle and P. Vanhove, *One loop in eleven-dimensions*, Phys. Lett. B 409, 177 (1997); E. Kiritsis and

B. Pioline, *On R**4 threshold corrections in IIb string theory and (p, q) string instantons*, Nucl. Phys. B 508, 509 (1997); J. G. Russo and A. A. Tseytlin, *One loop four graviton amplitude in eleven-dimensional supergravity*, Nucl. Phys. B 508, 245 (1997); I. Antoniadis, S. Ferrara, R. Minasian and K. S. Narain, *R**4 couplings in M and type II theories on Calabi-Yau spaces*, Nucl. Phys. B 507, 571 (1997); A. A. Tseytlin, *R**4 terms in 11 dimensions and conformal anomaly of (2,0) theory*, Nucl. Phys. B 584, 233 (2000).

[9] J. T. Liu and R. Minasian, *Higher-derivative couplings in string theory: dualities and the B-field*, `arXiv:1304.3137 [hep-th]`.

[10] G. Lopes Cardoso, G. Curio, G. Dall'Agata, D. Lust, P. Manousselis and G. Zoupanos, *Non-Kahler string backgrounds and their five torsion classes*, Nucl. Phys. B **652**, 5-34 (2003).

[11] M. Becker, K. Dasgupta, A. Knauf and R. Tatar, *Geometric transitions, flops and nonKahler manifolds. I.*, Nucl. Phys. B **702**, 207 (2004) [hep-th/0403288].

[12] M. Becker, K. Dasgupta, S. H. Katz, A. Knauf and R. Tatar, *Geometric transitions, flops and non-Kahler manifolds. II.*, Nucl. Phys. B **738**, 124-183 (2006); F. Chen, K. Dasgupta, P. Franche, S. Katz and R. Tatar, *Supersymmetric Configurations, Geometric Transitions and New Non-Kahler Manifolds*, Nucl. Phys. B **852**, 553-591 (2011).

[13] K. Dasgupta, M. Grisaru, R. Gwyn, S. H. Katz, A. Knauf and R. Tatar, *Gauge-Gravity Dualities, Dipoles and New Non-Kahler Manifolds*, Nucl. Phys. B **755**, 21-78 (2006).

[14] A. Butti, M. Grana, R. Minasian, M. Petrini and A. zaffaroni, *The baryonic branch of Klebanov-Strassler solution: A supersymmetric family of SU(3) structure backgrounds*, JHEP 0503, 069 (2005).

[15] K. Sil and A. Misra, *New Insights into Properties of Large-N Holographic Thermal QCD at Finite Gauge Coupling at (the Non-Conformal/Next-to) Leading Order in N*, Eur. Phys. J. C **76**, no. 11, 618 (2016); M. Dhuria and A. Misra, *Transport Coefficients of Black MQGP M3-Branes*, Eur. Phys. J. C **75**, no. 1, 16 (2015).

[16] K. Sil and A. Misra, *On Aspects of Holographic Thermal QCD at Finite Coupling*, Nucl. Phys. B **910**, 754 (2016).

[17] Igor R. Klebanov and Edward Witten, *Superconformal Field Theory on Threebranes at a Calabi-Yau Singularity*, Nucl. Phys. B 536, 199 (1998).

[18] I.R. Klebanov and A. Tseytlin, *Gravity Duals of Supersymmetric $SU(N) \times SU(M+N)$ Gauge Theories*, `arXiv:hep-th/0002159`.

[19] I. R. Klebanov and M. J. Strassler, *Supergravity and a Confining Gauge Theory: Duality Cascades and χSB-Resolution of Naked Singularities*, JHEP 0008:052, (2000).

[20] P. Ouyang, *Holomorphic D7-Branes and Flavored N=1 Gauge Theories*, Nucl.Phys.B 699:207-225 (2004).

[21] A. Buchel, *Finite temperature resolution of the Klebanov-Tseytlin singularity*, Nucl. Phys. B **600**, 219 (2001).

[22] S. S. Gubser, C. P. Herzog, I. R. Klebanov and A. A. Tseytlin, *Restoration of chiral symmetry: A Supergravity perspective*, JHEP **0105**, 028 (2001); A. Buchel, C. P. Herzog, I. R. Klebanov, L. A. Pando Zayas and A. A. Tseytlin, *Nonextremal gravity duals for fractional D-3 branes on the conifold*, JHEP **0104**, 033 (2001).

[23] B. A. Burrington, J. T. Liu, L. A. Pando Zayas and D. Vaman, *Holographic duals of flavored N=1 super Yang-mills: Beyond the probe approximation*, JHEP **0502**, 022 (2005).

[24] M. Mahato, L. A. Pando Zayas and C. A. Terrero-Escalante, *Black Holes in Cascading Theories: Confinement/Deconfinement Transition and other Thermal Properties*, JHEP **0709**, 083 (2007).

[25] M. Mia, K. Dasgupta, C. Gale and S. Jeon, *Toward Large N Thermal QCD from Dual Gravity: The Heavy Quarkonium Potential*," Phys. Rev. D **82**, 026004 (2010).

[26] A. Strominger, S. T. Yau and E. Zaslow, *Mirror symmetry is T duality*, Nucl. Phys. B **479**, 243 (1996).

[27] M. Ionel and M. Min-OO, *Cohomogeneity One Special Lagrangian 3-Folds in the Deformed and the Resolved Conifolds*, Illinois Journal of Mathematics, Vol. 52, Number 3 (2008).

[28] L. A. Pando Zayas and A. A. Tseytlin, *3-branes on resolved conifold*, JHEP **0011**, 028 (2000).

[29] P. Franche, *Towards New Classes of Flux Compactifications*, `arXiv: 1303.6726 [hep-th]`.

[30] A. Misra and C. Gale, *The QCD Trace Anomaly at Strong Coupling from $\mathcal{M}$-Theory*, Eur. Phys. J. C **80**, no. 7, 620 (2020).

[31] K. Sil, V. Yadav and A. Misra, *Top-down holographic G-structure glueball spectroscopy at (N)LO in N and finite coupling*, Eur. Phys. J. C **77**, no. 6, 381 (2017).

[32] V. Yadav and A. Misra, *$\mathcal{M}$-Theory Exotic Scalar Glueball Decays to Mesons at Finite Coupling*, JHEP **1809**, 133 (2018).

[33] M. B. Green and M. Gutperle, *Effects of D instantons*, Nucl. Phys. B bf498, 195-227 (1997).

[34] M. B. Green and P. Vanhove, *D instantons, strings and $\mathcal{M}$ theory*, Phys. Lett. B bf408, 122-134 (1997).

[35] P. Horava and E. Witten, *Eleven-dimensional supergravity on a manifold with boundary*, Nucl. Phys. B **475**, 94-114 (1996).

[36] C. Vafa and E. Witten, *A One loop test of string duality*, Nucl. Phys. B bf447, 261-270 (1995).

[37] A. A. Tseytlin, *Heterotic type I superstring duality and low-energy effective actions*, Nucl. Phys. B bf467, 383-398 (1996).

[38] K. Becker and M. Becker, *Supersymmetry breaking, $\mathcal{M}$ theory and fluxes*, JHEP bf07, 038 (2001).

[39] A. A. Tseytlin, *R**4 terms in 11 dimensions and conformal anomaly of (2,0) theory*, Nucl. Phys. B bf584, 233-250 (2000).

[40] A. Knauf, *Geometric Transitions on non-Kaehler Manifolds*, Fortsch. Phys. **55**, 5-107 (2007).

[41] A. Czajka, K. Dasgupta, C. Gale, S. Jeon, A. Misra, M. Richard and K. Sil, *Bulk Viscosity at Extreme Limits: From Kinetic Theory to Strings*, JHEP **1907**, 145 (2019).

[42] V. Yadav and A. Misra, *M-Theory Exotic Scalar Glueball Decays to Mesons at Finite Coupling*, JHEP **09**, 133 (2018).

[43] M. Kruczenski, D. Mateos, R. C. Myers and D. J. Winters, *Towards a holographic dual of large N(c) QCD*, JHEP **05**, 041 (2004).

[44] M. Dhuria and A. Misra, *Transport Coefficients of Black MQGP M3-Branes*, Eur. Phys. J. C **75**, 16 (2015).

[45] R. Argurio, A. Armoni, M. Bertolini, F. Mignosa and P. Niro, *Vacuum structure of large N QCD_3 from holography*, JHEP **07**, 134 (2020).

[46] C. P. Herzog, *A Holographic Prediction of the Deconfinement Temperature*, Phys. Rev. Lett. **98**, 091601 (2007).

[47] J. Gasser and H. Leutwyler, *Chiral perturbation theory to one loop*, Annals of Physics 158, 142 (1984).

[48] V. Yadav, A. Misra and K. Sil, *Delocalized SYZ Mirrors and Confronting Top-Down $SU(3)$-Structure Holographic Meson Masses at Finite g and N_c with P(article) D(ata) G(roup) Values*, Eur. Phys. J. C bf77, no. 10, 656 (2017).

[49] M. Harada, S. Matsuzaki and K. Yamawaki, *Holographic QCD Integrated back to Hidden Local Symmetry*, Phys. Rev. D82:076010, 2010.

[50] E. Witten, *Anti-de Sitter space, thermal phase transition, and confinement in gauge theories*, Adv. Theor. Math. Phys. **2**, 505 (1998).

[51] K. Sil, V. Yadav and A. Misra, *Top-down holographic G-structure glueball spectroscopy at (N)LO in N and finite coupling*, Eur. Phys. J. C **77**, no. 6, 381 (2017).

[52] S. Ryu, T. Takayanagi, *Holographic Derivation of Entanglement Entropy from AdS/CFT*, Phys. Rev. Lett. **96**, 181602 (2006).

[53] M. Larfors, *Revisiting toric SU(3) structures*, Fortsch. Phys. **61**, 1031-1055 (2013).

[54] S. B. Giddings, S. Kachru and J. Polchinski, *Hierarchies from fluxes in string compactifications*, Phys. Rev. D **66**, 106006 (2002).

[55] P. G. Camara and M. Grana, No-scale supersymmetry breaking vacua and soft terms with torsion, JHEP 0802 (2008) 017.

[56] D. Lust, F. Marchesano, L. Martucci, and D. Tsimpis, *Generalized non-supersymmetric flux vacua*, JHEP 0811 (2008) 021.

[57] J. Held, D. Lust, F. Marchesano, and L. Martucci, *DWSB in heterotic flux compactifications*, JHEP 1006 (2010) 090.

[58] J. G. J. Held, *Non-supersymmetric flux compactifications of heterotic string- and M-theory*, Ph.D. thesis 2012, Max-Planck-Institut für Physik and the Ludwig-Maximilians-Universit, Munich.

[59] L. Bedulli and L. Vezzoni, *The Ricci tensor of SU(3)-manifolds*, Journal of Geometry and Physics 57 (Mar., 2007) 1125–1146.

[60] R. L. Bryant, *Some remarks on G(2)-structures*, arXiv:math/0305124 [math-dg].

[61] S. Ivanov, *Connection with torsion, parallel spinors and geometry of spin(7) manifolds*, arXiv:math/0111216 [math.DG].

[62] S. Chiossi and S. Salamon, *The Intrinsic torsion of SU(3) and G(2) structures*, Contribution to: International Conference on Differential Geometry held in honor of the 60th Birthday of A. M. Naveira.

[63] H. Umemura, *Resolution of algebraic equations by theta constants*, in D. Mumford, Tata Lectures on Theta II, Progress in Math. Vol. 43, Birkhäuser, 1984, 261-272.

[64] A. Zhivkov, *Resolution of degree≤6 algebraic equations by genus two theta constants*, Journal of Geometry and Symmetry in Physics, **11**, 77 (2008).

[65] *Beyond the Quartic Equation*, R. B. King, Birkhäuser (1996).

[66] F. Chen, K. Dasgupta, P. Franche and R. Tatar, *Toward the Gravity Dual of Heterotic Small Instantons*, Phys. Rev. D **83**, 046006 (2011).

[67] P. Kaste, R. Minasian, M. Petrini and A. Tomasiello, *Kaluza-Klein bundles and manifolds of exceptional holonomy*, JHEP **0209**, 033 (2002).

[68] J. P. Gauntlett, D. Martelli and D. Waldram, *Superstrings with intrinsic torsion*, Phys. Rev. D **69**, 086002 (2004); D. Prins and D. Tsimpis, *Type IIA supergravity and M -theory on manifolds with SU(4) structure*, Phys. Rev. D **89**, 064030 (2014).

[69] C. Puhle, *Spin(7) Manifolds with Parallel Torsion Form*, arXiv: 0807.4875 [math.DG].

[70] G. D. Moore and O. Saremi, *Bulk viscosity and spectral functions in QCD*, JHEP **09**, 015 (2008).

[71] M. F. Arikan, H. Cho and S. Salur, *Contact Structures on G2-Manifolds and Spin 7-Manifolds*, arXiv:1207.2046; *Existence of Compatible Contact Structures on G2-manifolds*, Asian J. Math. 17 (2013) 321; A. J. Todd, *An Almost Contact Structure on G2-Manifolds*, arXiv: 1501.06966.

[72] T. Friedrich, I. Kath, A. Moroianu and U. Semmelmann, *On nearly parallel G2-structures*, Journal of Geometry and Physics 23 (1997) 259.

[73] K. Behrndt, M. Cvetic and T. Liu, *Classification of supersymmetric flux vacua in M theory*, Nucl. Phys. B **749**, 25-68 (2006).

[74] X. de la Ossa, M. Larfors, M. Magill *Almost contact structures on manifolds with a G2 structure*, arXiv:2101.12605.

[75] A. Misra, G. Yadav, To appear.

[76] J. Held, Ph.D. thesis, *Non-Supersymmetric Flux Compacti cations of Heterotic String- and M-theory* , 2012.

[77] A. Czajka, K. Dasgupta, C. Gale, S. Jeon, A. Misra, M. Richard and K. Sil, *Bulk Viscosity at Extreme Limits: From Kinetic Theory to Strings*, JHEP bf07, 145 (2019).

Department of Physics
Indian Institute of Technology Roorkee
Roorkee 247667, India
E-mail address: vyadav@ph.iitr.ac.in
E-mail address: aalok.misra@ph.iitr.ac.in