

# Fractional quantum Hall effect and M-theory

CUMRUN VAFA

We propose a unifying model for FQHE which on the one hand connects it to recent developments in string theory and on the other hand leads to new predictions for the principal series of experimentally observed FQH systems with filling fraction  $\nu = \frac{n}{2n \pm 1}$  as well as those with  $\nu = \frac{m}{m+2}$ . Our model relates these series to minimal unitary models of the Virasoro and superVirasoro algebra and is based on  $SL(2, \mathbf{C})$  Chern-Simons theory in Euclidean space or  $SL(2, \mathbf{R}) \times SL(2, \mathbf{R})$  Chern-Simons theory in Minkowski space. This theory, which has also been proposed as a soluble model for 2+1 dimensional quantum gravity, and its N=1 supersymmetric cousin, provide effective descriptions of FQHE. The principal series corresponds to quantized levels for the two  $SL(2, \mathbf{R})$ 's such that the diagonal  $SL(2, \mathbf{R})$  has level 1. The model predicts, contrary to standard lore, that for principal series of FQH systems the quasiholes possess non-abelian statistics. For the multi-layer case we propose that complex ADE Chern-Simons theories provide effective descriptions, where the rank of the ADE is mapped to the number of layers. Six dimensional  $(2, 0)$  ADE theories on the Riemann surface  $\Sigma$  provides a realization of FQH systems in M-theory. Moreover we propose that the q-deformed version of Chern-Simons theories are related to the anisotropic limit of FQH systems which splits the zeroes of the Laughlin wave function. Extensions of the model to 3+1 dimensions, which realize topological insulators with non-abelian topologically twisted Yang-Mills theory is pointed out.

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## 1. Introduction

Since its discovery [1] fractional quantum Hall effect has attracted a great deal of attention by both theorists and experimentalists. On the theory side, with the proposal of Laughlin [2] as well as development of other theoretical ideas such as hierarchy states of Haldane and Halperin [3, 4], and Jain's composite fermion theory [5], many of the observed filling fractions were explained, including the prediction of abelian anyonic statistics. Moreover, non-abelian statistics which was anticipated in [6] (see also [7]), was connected to FQHE in [8, 9] and further extensions were considered [10] (see also [11]). These constructions utilize Chern-Simons theory based on Witten's discovery of non-abelian braiding in these theories [12]. To date neither the abelian nor the non-abelian statistics has been fully verified experimentally.

In this brief note we propose a new model for FQHE which connects it on the one hand to recent developments in string theory and on the other hand leads to new predictions for the principal series of FQHE's with filling fraction  $\nu = \frac{n}{2n \pm 1}$ , as well as those with  $\nu = \frac{m}{m+2}$ . In particular our model predicts that for principal series of FQHE's, unlike the prediction of hierarchy states and composite fermion theory, the quasiholes possess non-abelian statistics, related to Fusion algebra of  $(2n, 2n \pm 1)$  unitary CFT minimal models. For the filling fraction  $\nu = \frac{m}{m+2}$  we obtain the fusion algebra of SCFT unitary minimal models  $(m, m+2)$ . Moreover the first in the CFT series correspond to Laughlin's  $\nu = \frac{1}{3}$  and the first in the SCFT series to Moore-Read state  $\nu = \frac{1}{2}$ . The higher values of  $n$  can be obtained by composite fermion model or hierarchy model and the ones with higher  $m$  can be obtained from Read-Rezayi states. However, our models predict a different fusion algebra, and thus a different statistics than expected from the corresponding constructions.<sup>1</sup> In particular if our model is correct it would predict that essentially all the observed cases of FQHE involve non-abelian statistics, which is potentially a welcome news for quantum computing [13]! For a nice review of constructions of non-abelian statistics in FQHE and connection to quantum computing see [19]. Moreover our model predicts, on a sharp edge, charged downstream and neutral upstream currents which distinguishes it for filling fraction  $\nu = n/(2n+1)$  from the standard model of FQH systems which predicts no upstream neutral currents. Moreover we predict the Hall conductivity for these edge modes which is different from

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<sup>1</sup>Except there are subtleties for the fusion algebra coefficients of Degenerate fields of Liouville theory, which we are currently studying [32].

the hierarchy or composite models. Compared to the usual constructions ours has the advantage of having essentially no adjustable parameters and for the single layer FQHE the assumption of unitarity picks these sequences in our construction.

Even though there is a relation between the fusion algebra of the quasi-holes and the minimal models, it is important to note that the relation to a CFT arises as descriptions on the 1+1 dimensional boundary of the sample, which is echoed in the bulk by the associated monodromy structure of excitations in the 2+1 dimensional bulk. There are different boundary conditions that we can have in our model. In one boundary condition, which we identify as a superconducting interface, we obtain the minimal model chiral blocks as effective description of the 1+1 edge modes which propagate only in one direction. Another boundary condition, which is the more standard one corresponding to a sharp edge, we get a different CFT which has the same block structure as the minimal model, but which leads to downstream charged currents and neutral upstream currents. The bulk theory does not depend on the choice of the boundary condition and in either case is given by Chern-Simons theory based on complex gauge group  $SL(2, \mathbf{C})$  studied in [20] (see also [21, 22] and related work [23]), and its supersymmetric cousin (for recent discussions of  $SL(2, \mathbf{C})$  Chern-Simons theory see [30, 31]). More precisely,  $SL(2, \mathbf{C})$  has a pair of levels  $l = (k, \sigma)$ , where  $k$  is an integer (specifying the level of  $SU(2) \subset SL(2, \mathbf{C})$ ) and  $\sigma$  is a real parameter<sup>2</sup>. For  $l = (\pm 1, 4n \pm 1)$  we obtain the principal series with filling fractions  $\nu = \frac{n}{2n \pm 1}$ . In the Minkowski signature this corresponds to  $SL(2, \mathbf{R}) \times SL(2, \mathbf{R})$  Chern-Simons theory and for  $|k| = 1$  the values for  $\sigma$  which yield the principal series of filling fractions are *exactly* the ones which would follow if we make the individual  $SO(2)$ 's in the  $SL(2, \mathbf{R})$ 's quantized with levels  $(-2n, 2n \pm 1)$ . It is natural to conjecture that the higher Jain series with filling fractions  $\nu = \frac{n}{2nk' \pm 1}$  correspond in our setup to the two  $SL(2, \mathbf{R})$  levels being given by  $(-2n, 2nk' \pm 1)$ , which would correspond to the level  $k = 2n(k' - 1) \pm 1$  for the diagonal  $SL(2, R)$ .

In [23] (p.75-76) Witten specifically suggests viewing minimal Virasoro models in 2d as holographic realization of 2+1 gravity. In our context this would suggest that the FQHE is holographically encoding 3d gravity and large enough Laughlin quasi-holes are actually black-holes! In this context the 1 + 1 dimensional edge theory on the boundary of the sample is holographic dual to the FQHE bulk interpreted as a gravitational theory. In other words we can realize holography in the lab! It is natural to connect

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<sup>2</sup>This is related to cosmological constant in the gravitational picture of the theory.

our model to the observations in [24] involving an emergent geometry in FQHE. Indeed the elements of Haldane's proposal, and in particular the appearance of  $SL(2, \mathbf{R}) \times SL(2, \mathbf{R})$  in his setup are in harmony with the picture proposed here [25].

These constructions were motivated by string theory which in turn leads to a proposal for the corresponding Hamiltonian. These involve compactification of 6d (2, 0) theories on a surface  $[\Sigma \times \mathbf{R}] \times S_{k, b^2}^3$ , and  $\Sigma$  is identified with the plane of the FQHE and  $\mathbf{R}$  with time,  $S_k^3$  is the lens space  $S^3/\mathbf{Z}_k$  and  $b^2$  is a squashing parameter<sup>3</sup> for  $S_k^3$ . There is an ADE classification for 6d (2, 0) theories and we identify the rank of the ADE with the number of layers in the FQHE. This leads to our *identification of the effective theory of FQH systems with complex ADE Chern-Simons theories with the rank of ADE corresponding to the number of layers*. The  $A_1$  case, corresponding to single layer leads in Euclidean description, to  $SL(2, \mathbf{C})$  Chern-Simons effective theory on  $\Sigma \times \mathbf{R}$ .

The organization of this paper is as follows: In section 2 we explain the heuristic motivation for this model and the connection with Liouville theory and 2d minimal CFT's. Also discussed there is the connection with non-abelian statistics for quasi-holes. In section 3 we propose a Hamiltonian whose improved Berry's connection is expected to yield the results outlined in section 2 for the monodromy of the quasihole in Liouville theory. In section 4 we sharpen our proposal by embedding it in string theory. Section 4.2 is a self-contained summary of the model and readers who are not interested in the motivation for our model can go directly to that section. The string theory perspective leads to reformulation of the effective theory of the (single layer) FQH systems in the bulk as complex  $SL(2, \mathbf{C})$  Chern-Simons theory and provides various generalizations of it, including predictions of new defects for FQHE, as well as potential applications to multi-layer and anisotropic FQH systems. We also explain the heterotic aspect of the edge states for our model as well as the different choices of boundary conditions. Moreover we comment on the possibility of lifting the construction to one higher dimension and potentially exciting realization of topological insulators with topologically twisted non-abelian gauge symmetry in 3+1 dimensions in the lab!

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<sup>3</sup>We will be interested in the limit where  $b^2$  is a negative rational number, so this is defined in the sense of analytic continuation.

## 2. Basic idea

Let us start with the Laughlin wave function [2] for electrons at positions  $z_i$  and quasi-holes at positions  $\zeta_a$  with filling fraction  $\nu = 1/m$ , given by

$$\psi(z_i, \zeta_a) = \prod_{i,a} (z_i - \zeta_a) \prod_{i < j} (z_i - z_j)^{\frac{1}{\nu}} \exp(-B \sum_i |z_i|^2)$$

This has proven to be a powerful model for FQHE (for a beautiful introduction to this subject see [14]). This problem has been mapped to the study of RCFT's [9], whose basic blocks satisfy the Verlinde algebra [15], and which has been elegantly systematized in [16] even though there is no full classification. Consider  $c = 1$  theory at radius  $R^2 = \nu$ , and consider the chiral vertex operators  $V(z_i) = \exp(i\phi(z_i)/\sqrt{\nu})$ , and  $W(\zeta_a) = \exp(i\sqrt{\nu}\phi(\zeta_a))$ . Then the holomorphic part of the wave function (i.e. dropping the B-field part) can be captured (up to prefactors depending only on  $\zeta_a$ ) by

$$\psi(z_i, \zeta_a) = \left\langle \prod_{i,a} V(z_i) W(\zeta_a) \right\rangle$$

Moreover to compute physical amplitudes one considers

$$\langle \mathcal{O} \rangle = \int d^2 z_i [\psi^*(z_i, \zeta_a) \mathcal{O} \psi(z_i, \zeta_a)]$$

One can also add a chemical potential  $\mu$  for the fermions and add to the action  $\mu \int d^2 z e^{i\phi/\sqrt{\nu}}$  for which the above term corresponds to the term  $\mu^N$  where  $N$  is the number of electrons. Moreover one imagines  $B$  field as being given by an additional smeared term by adding to the above  $\exp \int B\phi$ . However already there is a clash with conformal field theory paradigm: in conformal theories we usually do not integrate over the position of fields unless they have dimension 1. In the condensed matter context we are discussing, this is not strictly necessary (see [18] and references therein)<sup>4</sup> when we have a B-field. However, in the absence of B-field, as in studying superconductor phases where we can ignore the B-field, the dimensions should be 1. To make this natural for FQH system imagine a thin strip of material which on both sides is in contact with a superconducting material. With this in mind, it would be interesting to see what demanding marginality of this operator would imply about the possible bulk theories. Our strategy would be to first study implications that this would have in identifying the bulk

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<sup>4</sup>We thank N. Read for discussions on this point.

theory, and then using other boundary conditions such as the sharp edge one which would be interesting for charged edge currents. Later in section 3.3 we return to the question of introducing back the magnetic field and show how it can be incorporated in our setup.

Demanding that  $\exp(i\phi/\sqrt{\nu})$  have dimension 1 is rather significant. In the  $c = 1$  model it has dimension  $h = \frac{1}{2\nu} \neq 1$ . So how can this be rescued? It is natural to add a background term to the action  $Q'R\phi$  where  $R$  is the curvature (which disappears in flat space), in order to make the dimension of this field 1, without affecting the above realization of the wave function. In such a case a vertex operator  $\exp(\alpha\phi)$  will have dimension

$$h_\alpha = \frac{-1}{2}(\alpha(\alpha + Q'))$$

To make the dimension of  $\exp(i\phi/\sqrt{\nu})$  equal to 1, we need to take<sup>5</sup>

$$\frac{Q'}{\sqrt{2}} = Q = \frac{1}{b} + b = i(\sqrt{2\nu} - \frac{1}{\sqrt{2\nu}})$$

where

$$b = \frac{-i}{\sqrt{2\nu}}.$$

Moreover the central charge of the 2d chiral theory is

$$c = 1 + 6Q^2 = 1 - 3\frac{(2\nu - 1)^2}{\nu}$$

If we put all the ingredients together, we get the Liouville theory (see [33] and lectures in [34] for a review of Liouville theory)

$$S = \int d^2z \left[ \frac{1}{8\pi} \partial\phi\bar{\partial}\phi + iQ'R\phi + \mu \exp(i\phi/\sqrt{\nu}) \right]$$

Related ideas have recently been suggested independently in [35–37], but with a different motivation and without fixing the relation between  $Q$  and  $\nu$  and which reach rather different conclusions from the present paper<sup>6</sup>. So far we have assumed that  $\psi(z_i)$  is a single valued wave function which in Laughlin's case is related to  $1/\nu = m \in \mathbf{Z}$ . This is because electron's wave function

<sup>5</sup>We have chosen an unconventional normalization of Liouville field in order to make the dimension of the fields match the more familiar normalization for  $c = 1$  theory.

<sup>6</sup>We thank A. Abanov for pointing out these papers to us.

should be single valued. But now we overcome the condition that  $1/\nu \in \mathbf{Z}$  and propose the Liouville theory for all fractional values of  $\nu$  as giving us the overlaps  $\langle \psi | \psi \rangle$  (i.e. the overlap of the multi-particle wave function ignoring the B-field term). It is known in the context of Liouville that one can associate chiral blocks to the wave functions [38]. This gives us a prescription as to how to compute overlaps of multi-valued wave functions. Roughly speaking we can view them as wave function for suitable excitations which can have non-trivial statistics, and so in particular the wave functions do not have to be single valued. However this is not precise, because extracting physical quantities from these wave functions is more subtle than the usual prescription (as we review in the next section) and is not simply given by  $\int d^2z \psi^* \mathcal{O} \psi$ . In section 3 we will explain how this multi-valued wave function is related to a microscopic theory where we have well defined single valued wave functions.

Relaxing the assumption that  $1/\nu$  is integer, we consider FQHE with filling fraction  $\nu = \frac{n}{m}$ . If we evaluate the central charge of the Liouville theory for that case we get

$$c = 1 - 6 \frac{(2n - m)^2}{2nm}$$

which is the same as central charge of 2d CFT minimal model  $(2n, m)$  [39]. Note that  $m$  needs to be odd in this context as otherwise  $2n$  and  $m$  are not relatively prime! Moreover Liouville theory has a special set of fields (called ‘degenerate fields’) whose OPE [40] realize the operator algebra for the minimal model when restricted to  $-b^2$  a rational number<sup>7</sup>. Namely consider

$$\Phi_{r,s} = \exp \left[ i(r-1)\sqrt{\nu}\phi + (s-1)\frac{1}{2\sqrt{\nu}} \right]$$

for  $1 \leq r < 2n, 1 \leq s < m$ . Then these degenerate fields of Liouville realize the operator algebra of the  $(2n, m)$  minimal models (see [17] for a review)<sup>8</sup>

$$\Phi_{r_1, s_1} \times \Phi_{r_2, s_2} = \sum_{\substack{k=r_1+r_2-1 \\ l=s_1+s_2-1 \\ k=1+|r_1-r_2|, k+r_1+r_2+1=0 \pmod{2} \\ l=1+|s_1-s_2|, l+s_1+s_2+1=0 \pmod{2}}} \Phi_{k,l}$$

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<sup>7</sup>We thank Joerg Teschner for a discussion on this point.

<sup>8</sup>There are subtleties in this statement which arises because of analytic continuation of Liouville theory from  $b^2 > 0$  to  $b^2 < 0$ . See in particular [41–43].

Strictly speaking the above discussion is in the context of a superconducting boundary condition where the  $B$ -field is absent and the above algebra reflects the algebra of neutral currents in an interface with a superconductor. We use this to identify what the bulk theory is (i.e. Chern-Simons theory based on  $SL(2, \mathbf{C})$  gauge group). We postpone this discussion to section 4, where we connect it to string theoretic motivations for identifying the bulk. As we discuss there, the same bulk theory with a *different* boundary condition leads to FQH system with a sharp edge boundary, where the  $\Phi_{r,s}$  blocks are related to the downstream electric currents in the  $s$  label and neutral upstream currents labeled by  $r$ . In particular if one considers  $\Phi_{1,s}$ , these correspond to the usual  $s - 1$  quasi-hole states, which correspond to insertion of  $\prod (z_i - \zeta)^{s-1}$  in the wave function, and form a closed operator algebra generated by  $\Phi_{1,2}$ . Moreover, given that the quasi-hole operators correspond to these operators, we deduce that quasi-holes will possess the same non-abelian braiding properties as that of  $(2n, m)$  models. This is different from the hierarchy state or composite fermion model construction of these wave functions which predicts abelian statistics for quasiholes. In particular taking the minimal quasi-hole  $\Phi_{1,2}$  around another one (i.e. a  $2\pi$  rotation) is expected to lead, in our model, to two dimensional fusion channel and the two phases one picks up are given by the formulas for dimensions of blocks

$$[e^{2\pi i(h_{3,1}-2h_{2,1})}, e^{2\pi i(h_{1,1}-2h_{2,1})}] = [e^{2\pi i\nu}, e^{-2\pi i(3\nu)}] = [e^{2\pi i\frac{n}{m}}, e^{-6\pi i\frac{n}{m}}].$$

We can now further restrict the choices of  $m$ : Since the edge modes in the FQHE should have correlations which fall off rather than grow with distance we would need to restrict to theories which are unitary (see in particular the discussion in [18]). Applying this to when the FQH system is in an interface with superconductor, i.e. when the edge theory is given by the Liouville theory, we should get unitary 2d CFT's with central charge less than 1 because  $b^2 < 0$ , which fixes  $m = 2n \pm 1$  and corresponds to  $(2n, 2n \pm 1)$  minimal model and filling fractions

$$\nu = \frac{n}{2n \pm 1}$$

So we find that the 2d unitary minimal models map exactly to the two principal series of FQHE which have been experimentally observed! This is a remarkable check of our proposal which connects representations of minimal unitary 2d CFT's to principal series already observed in FQHE! For the original application of minimal unitary 2d CFT to critical phenomena see [44]. This is what we will obtain with a superconducting boundary condition. It



is also natural to ask what we get with the sharp edge boundary condition as is usually considered in the context of FQH system. This we will postpone to section 4, after we connect our model to  $SL(2, \mathbf{C})$  Chern-Simons theory. There we will find that the sharp edge 1+1 dimensional theory has a central charge given by  $(c_L, c_R) = (3 - \frac{6}{2n}, 3 - \frac{6}{2n \pm 1})$  and the blocks are mixed between left- and right-moving sectors.

The above series give us only the odd denominator filling fractions and it is natural to ask how one can obtain the even denominator ones as well. This turns out to have a natural answer: We simply extend the Liouville theory to  $\mathcal{N} = 1$  supersymmetric Liouville theory (see e.g. [45, 46]) whose action is given by

$$S = \int d^2z \left[ \frac{1}{8\pi} \partial\phi\bar{\partial}\phi + \frac{1}{2\pi} (\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi}) + iQR\phi + 2i\mu b^2 \bar{\psi}\psi e^{b\phi} + 2\pi b^2 \mu^2 e^{2b\phi} \right]$$

where  $Q = b + 1/b$ , With central charge  $\frac{2}{3}c = \hat{c} = 1 + 2Q^2$ . The case which corresponds to  $(m, n)$  SCFT minimal models is when  $b^2 = -\frac{n}{m}$  with  $n - m = 0 \pmod{2}$  and the unitary minimal series  $(m, m + 2)$  corresponds to  $n = m + 2$ . The  $\bar{\psi}\psi e^{b\phi}$  term has dimension one. Note that the chiral wave function this leads to in the free field realization is

$$\Psi = \prod_{i < j} (z_i - z_j)^{\frac{n}{m}} \text{Pf} \left[ \frac{1}{z_i - z_j} \right]$$

where Pf is the Pfaffian. The  $N = 1$  unitary series corresponds to

$$\Psi = \prod_{i < j} (z_i - z_j)^{\frac{m+2}{m}} \text{Pf} \left[ \frac{1}{z_i - z_j} \right]$$

where  $m = 2, 3, \dots$ . The first element of the series,  $m = 2$ , corresponds to the Moore-Read wave function [9]. The filling fractions we get for the unitary  $N = 1$  case are  $\nu = \frac{m}{m+2}$ . These values for filling fractions also arise in Read-Rezayi's construction [10] which generalizes the Moore-Read state, but these models (except possibly for the  $m = 2$  case) are distinct from ours. The bulk theory in this case is a supersymmetric version of  $SL(2, \mathbf{C})$ , and the superconducting boundary conditions lead to supersymmetric Liouville theory. As in the non-supersymmetric case, one can also consider the sharp edge boundary condition in these cases as well.

So far the relation we have found is rather suggestive. In the next section we propose a microscopic Hamiltonian motivated from string theory (see

section 4) whose Berry's connection, as we vary the position of quasiholes, is known to lead to the braiding properties for the Liouville amplitudes, in some limit. It is natural to expect a cousin of it also exists which gives the one for supersymmetric Liouville amplitudes.

### 3. A microscopic description

Here we give a Hamiltonian description which is relevant for the Liouville phase of the theory, i.e. the superconducting interface where the  $B$ -field is absent. We first start with a simple model hamiltonian for a single particle and then move on to the many particle case.

#### 3.1. Hamiltonian construction from $W(z)$

We first discuss a single particle toy model and then we generalize it to the case at hand. Consider a holomorphic function  $W(z)$ . To this we will associate a Hamiltonian for a particle which is a bi-spinor (i.e. has  $2 \times 2 = 4$  internal degrees of freedom) on the  $z$ -plane as follows:

$$H = \frac{1}{2}p^2 + \frac{1}{2}|\partial_z W(z)|^2 + \partial^2 W(z) \cdot \sigma^+ \otimes \sigma^- + \bar{\partial}^2 \bar{W}(\bar{z}) \cdot \sigma^- \otimes \sigma^+$$

Then for each critical point  $\partial W = 0$  we get a ground state for this theory, which turns out to have exactly zero energy (because this system secretly enjoys 4 units of supersymmetry—see [47]). It can also be written as<sup>9</sup>

$$H = \frac{1}{2}p^2 + \frac{1}{2}|\partial_z W(z)|^2 + [\partial^2 W(z) \cdot b_L^\dagger b_R + \bar{\partial}^2 \bar{W}(\bar{z}) \cdot b_R^\dagger b_L]$$

where  $b_{L,R}^\dagger, b_{L,R}$  form a pair of fermionic creation/annihilation operators with the non-vanishing anti-commutations being

$$\{b_L, b_L^\dagger\} = 1, \quad \{b_R, b_R^\dagger\} = 1$$

The non-abelian berry's connection [48, 49] for the degenerate ground states of this system as a function of parameters defining  $W$  satisfies a beautiful set of equations known as the  $tt^*$  geometry [50]. To compute this connection we need to compute overlap between the ground state wave functions  $\langle \psi_i | \psi_j \rangle$ .

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<sup>9</sup>In particular  $H = Q^2$  for a  $Q$  which the reader can easily identify.

Moreover the ground states can be labeled by chiral ring elements of  $W$ :

$$|\psi_j\rangle = \phi_j(z)|0\rangle$$

where the chiral ring is given by the monomials of  $z$  modulo setting  $\partial W = 0$ :

$$\phi_j(z) \in \mathcal{R}; \quad \mathcal{R} = \frac{\mathbf{C}[z]}{\partial W}$$

To compute this, it turns out to be useful to introduce a basis of states known as the D-brane states [52], which locally do not depend on the parameters and so do not vary as we change parameters:

$$\langle \psi_i | \psi_j \rangle = \sum_{\alpha} \langle \psi_i | D_{\alpha}^{+} \rangle \langle D_{\alpha}^{-} | \psi_j \rangle$$

$D_{\alpha}^{\pm}$  are identified with lines in  $z$  plane which when projected to  $W$ -plane using  $W(z)$ , correspond to straight lines in  $W$  plane emanating from the critical point and going to  $\text{Re}(W) = \pm\infty$ . In particular  $\alpha$  labels the critical points. Then it can be shown that in the asymmetric limit where we rescale  $\overline{W} \rightarrow \beta \overline{W}$  and set  $\beta \rightarrow 0$ , we obtain [52]

$$\langle D_{\alpha}^{-} | \psi_j \rangle = \int_{D_{\alpha}^{-}} dz \phi_j(z) e^{W(z)}$$

and we find

$$\begin{aligned} \langle \psi_i | \psi_j \rangle &= \sum_{\alpha} \left[ \int_{D_{\alpha}^{+}} d\bar{z} \phi_i(\bar{z}) e^{-\overline{W}(\bar{z})} \right] \left[ \int_{D_{\alpha}^{-}} dz \phi_j(z) e^{W(z)} \right] \\ &= \int d^2z \phi_i(\bar{z}) \phi_j(z) e^{W(z) - \overline{W}(\bar{z})} \end{aligned}$$

Where in the last equality we used the Riemann bilinear identity, which is somewhat of a formal step due to oscillatory nature of the integral. This result suggests that we can pretend *as if* the wave functions are holomorphic and given by  $\psi_j(z) \sim \phi_j(z)e^{W(z)}$ , except that the complex conjugate wave function is not given by the usual  $\phi_j(\bar{z})e^{\overline{W}(\bar{z})}$  but by  $\phi_j(\bar{z})e^{-\overline{W}(\bar{z})}$ . The oscillatory nature of the integral which makes it convergent is precisely due to this change in sign of  $\overline{W}$  making the exponent purely imaginary. We wish to emphasize that this is just an approximation to the actual wave function which has the usual definition of inner product one is familiar with in the context of quantum mechanics. One may ask in which limit is the computation of the inner product exact? Each one is exact if we set the other

$W$  to be small. So in the limit that we rescale  $W \rightarrow \beta W$  and  $\overline{W} \rightarrow \beta \overline{W}$  and send  $\beta \rightarrow 0$  this inner product becomes exact<sup>10</sup>, and in particular when  $W$  is quasi-homogeneous it is exact! We can view the  $D_\alpha^+$  as the analog of ‘conformal blocks’ for this theory. If we change the parameters of  $W$  and bring it back to itself, then the individual  $D_\alpha^+$  undergo a transformation to a linear combination [52] because as the D-branes defining the  $D_\alpha^+$  cross one another, we get a Stokes phenomenon. So when we come back to the original position the  $D_\alpha^+$  transform to a linear combination of the ones we had, and this gives a monodromy matrix<sup>11</sup>:

$$D_\alpha^+ \rightarrow M_\alpha^\beta D_\beta^+.$$

So the eigenvalues of the Berry’s connection around loops can be computed exactly in this limit. One may ask if there is any notion of  $tt^*$  connection which is independent of taking any limits. It turns out that  $tt^*$  geometry has an ‘improved connection’ [50]

$$\nabla_i = D_i + C_i$$

where  $D_i$  is the Berry’s connection and  $C_i$  is given by the action of  $\phi_i$  on the vacua (for a recent review and extension of  $tt^*$  geometry see [51]). Unlike the Berry’s connection,  $\nabla$  is flat for all parameters. Since it does not depend on any parameters the monodromy of this improved connection can be computed in this limit, which yields the above monodromy of the chiral wave functions.

### 3.2. The Hamiltonian

Now we come to the case of interest for us, and ask which Hamiltonian will give us the wave function associated to the Liouville theory, which is motivated from its connection with string theory, discussed in the next section. This Hamiltonian has indeed been studied [53]. Consider  $N$  quasi-particles each of which has in addition 4 degrees of freedom given by pairs of fermionic

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<sup>10</sup>This is the UV limit of the theory in the context of 2d versions of this theory.

<sup>11</sup>In the language of the supersymmetric quantum mechanics, the eigenvalues of the specific monodromy associated with  $W \rightarrow e^{2\pi i} W$  are given by  $\exp(2\pi i Q_R)$  where  $Q_R$  are the R-charges of the Ramond ground state.

creation/annihilation operators  $b_{L,R}^{i\dagger}, b_{L,R}^i$  where the only non-vanishing anti-commutators are

$$\{b_L^{i\dagger}, b_L^j\} = \delta^{ij}, \quad \{b_R^{i\dagger}, b_R^j\} = \delta^{ij}$$

and consider the Hamiltonian given by

$$H = \frac{1}{2} \sum_i [\mathbf{p}_i^2 + \mathcal{A}_z^i \mathcal{A}_{\bar{z}}^i] + \frac{1}{\nu} \sum_{\substack{i,j \\ i \neq j}} \left[ \frac{b_L^{i\dagger} b_R^j}{(z_i - z_j)^2} + \frac{b_R^{j\dagger} b_L^i}{(\bar{z}_i - \bar{z}_j)^2} \right]$$

where

$$\mathcal{A}_z^i = \frac{1}{\nu} \sum_{\substack{j \\ j \neq i}} \frac{1}{z_i - z_j}, \quad \mathcal{A}_{\bar{z}}^i = \frac{1}{\nu} \sum_{\substack{j \\ j \neq i}} \frac{1}{\bar{z}_i - \bar{z}_j}$$

which corresponds to taking  $W = \frac{1}{\nu} \sum_{i < j} \log(z_i - z_j)$ . Thus, we have a Hamiltonian involving two particle and three particle interactions<sup>12</sup>. Moreover we can introduce quasi-holes at  $\zeta_a$  by adding to  $W$

$$W \rightarrow W + \sum_{i,a} \log(z_i - \zeta_a)$$

The reason for labeling the terms  $\mathcal{A}_z^i$  by  $\mathcal{A}$ , usually reserved for gauge potential, is that if we consider  $\bar{\partial}_i \mathcal{A}_z^i$  we get

$$\bar{\partial}_i \mathcal{A}_z^i = \frac{2\pi i}{\nu} \sum_{\substack{j \\ j \neq i}} \delta(z_i - z_j).$$

This looks like a magnetic field and suggests that each particle has trapped  $1/\nu$  units of magnetic flux. This is somewhat reminiscent of composite fermions model [5].

However, this interpretation cannot be precise, because the form of the interactions above is not the usual form one expects for electromagnetic interactions: First of all, it does not contain  $\mathbf{p} \cdot \mathcal{A}$  terms. Moreover with the

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<sup>12</sup>Note that this Hamiltonian preserves Fermion number. Moreover one can show that the ground state has fermion number zero. Thus if one is interested only in the ground state, one can restrict the Hilbert space to the  $2^N \subset 2^{2N}$  dimensional subspace. Interestingly enough, this is the same as the dimension of a Hilbert space for  $N$  electrons, each of which has 2 spin states.

above definition of the gauge fields, the Field strength is actually zero, because  $\partial_z \mathcal{A}_{\bar{z}} = \partial_{\bar{z}} \mathcal{A}_z$ . One may be tempted to ‘gauge away’ the  $|\mathcal{A}|^2$  term by redefining the wave function, but then this introduces  $\mathbf{p} \cdot \mathcal{A}$  terms. Nevertheless it can be written in a form, familiar in the context of Dirac operator, which behaves like a magnetic field. Namely we can rewrite  $H$  as

$$H = Q^2$$

where

$$Q = [(b_L^{i\dagger} \mathbf{p}_z^i + b_L^i \mathbf{p}_{\bar{z}}^i) + (b_R^i \mathcal{A}_z^i + b_R^{i\dagger} \mathcal{A}_{\bar{z}}^i)]$$

If one can justify why the above Hamiltonian is a good description of fractional quantum Hall effect (for rational values of  $\nu$  and more specifically for  $\nu = \frac{n}{2n \pm 1}$ ) in the context of superconducting boundary conditions, we will have arrived at the conclusion of the previous section. In particular, this Hamiltonian, leads in the approximate sense we mentioned above, to the Liouville wave functions where the corresponding blocks are given by computing

$$B_\alpha(\zeta_a, \nu) = \int_{D_\alpha^-} \prod_i dz_i \exp(W) = \int_{D_\alpha^-} \prod_i dz_i \prod_{i,a} (z_i - \zeta_a) \prod_{i < j} (z_i - z_j)^{1/\nu}$$

These are known to compute the conformal blocks of Liouville theory (see [54, 55]). This in particular will undergo monodromy as is expected for Liouville conformal theory. For the 4-point quasi-holes we will get a 2 dimensional space, whose braiding eigenvalues we already discussed for the 2 channels of the fusion of  $\Phi_{1,2} \times \Phi_{1,2}$ . See in particular [56] for the explicit computation of this monodromy for the corresponding 4-pt function.<sup>13</sup>

### 3.3. Adding back the magnetic flux and connections with the standard approach to FQHE

The wave functions we obtain, using the Hamiltonians discussed above is very similar to the Laughlin type wave functions considered for FQHE, with one major difference: In those cases one has in addition a non-holomorphic

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<sup>13</sup>One may ask what is an effective Hamiltonian describing the dynamics of the quasi-holes? This gets related to the open string-wave function [64]. It is argued in [65] that this effective dynamics is captured by the Gaudin Hamiltonian (see also related discussions in [53, 56]).

term in the wave function given by

$$f(z_i) \exp(-B \sum_i |z_i|^2)$$

It is imperative for us that the wave functions be holomorphic, or at least meromorphic. How can we incorporate this in our setup? The most natural thing to do in our context is to consider, instead of a uniform  $B$  field, a lattice of fundamental units of  $B$  fluxes at lattice points  $\zeta_{ab}$ . In the usual description of the wave function this would have the effect of introducing

$$\prod_{i,ab} \frac{1}{(z_i - \zeta_{ab})}$$

This is reflected in our set up as follows: The Liouville theory has a conservation law for momenta at zero chemical potential, which is violated by the curvature term by  $(2g - 2)(2\nu - 1)$ . For simplicity let us consider the  $g = 1$  case, so we will not have to worry about this, i.e. consider the theory on the torus. In this case, putting  $N$  fermions in the theory will violate the charge by  $N$  units. To cancel this in the usual Liouville context one puts a charge at infinity. However, another way of doing this is to introduce fluxes localized at points which is accomplished by introducing  $\exp[-i\sqrt{\nu}\phi(\zeta_{ab})]$ . Let  $\Phi$  denote the total number of such flux quanta. Then Liouville conservation demands that

$$\nu\Phi = N.$$

This is the analog of the statement in the context of FQHE that for a given  $\nu$ , we have  $N/\Phi = \nu$ .

In the presence of such a term each particle picks up an additional term for  $W$  given by

$$\delta W = - \sum \log(z - \zeta_{ab})$$

and to find the allowed ground state configuration we need to study the critical points

$$\frac{d\delta W}{dz} = - \sum_{ab} \frac{1}{(z - \zeta_{ab})} = 0$$

which leads to  $\Phi - 1$  distinct critical points. We need to distribute the  $N$  states among these ground state choices. Since there are  $N = \nu\Phi$  such particles even if we put fermionic statistics for them, we would have  $\binom{\Phi-1}{\nu\Phi}$  ground states for the Hamiltonian, which is a large number of states. This is the analog of the problem which is faced in the standard approach to the

FQHE where there are  $\Phi$  lowest Landau levels, and one needs to fill only  $\nu\Phi$  of them. In the context of FQHE the question become which combination of these hugely degenerate states has the lowest energy, when one includes the electric repulsion between electrons. In the context of the supersymmetric Hamiltonian we have been discussing, the ground states are degenerate and this can be viewed as the analog of turning off the electric repulsion. Then the question in the present context becomes, which states among these hugely degenerate ground states of the supersymmetric system is ‘picked’ when we turn on interactions. It turns out that *there is* a distinguished state among the ground states of the supersymmetric Hamiltonian, and that is related to the fact that the operator state correspondence in this case maps the identity operator to a canonical ground state which is represented by the approximate holomorphic wave function of the Laughlin type we have already discussed. That this is the one which will have the lowest energy when repulsion is included is natural because the identity operator is the combination of the critical points of  $W$  where we take as spread out a combination of vacua as possible. *It is interesting that in the present context the degeneracy of the lowest Landau level, is mapped to the degeneracy of the ground states of a supersymmetric system.* It would be interesting to see if this can lead to insights into FQHE based on supersymmetry.

In the presence of the magnetic fields the monodromy properties of the defects becomes more complicated as the quasi-holes will have to also go around  $\zeta_{ab}$ . It would be interesting to work out the consequences of this. However, it is clear that in the present context if we create a region in the sample where  $B$ -flux is excluded, i.e. the  $\zeta_{ab}$  are not placed in this region, the monodromy properties of the quasi-holes we have discussed does not get modified and leads to that of the minimal model monodromies.

#### **4. Connections with string theory and identifying the bulk theory**

So far I have tried to motivate the discussion from the viewpoint of the FQHE. However, to make the proposal more precise and identify the bulk theory, I need to explain the main motivation for the present work. In section 4.1 I discuss the embedding in M-theory. This is somewhat technical and is discussed mainly to explain the motivation. Readers not familiar with string theory may wish to skip to section 4.2, which is largely independent, where I spell out the proposal for the bulk theory.



This work arose from the realization of an unexpected similarity between what one does in a special context in string theory and the structures involving FQHE.<sup>14</sup> It arose from the approach in [54] (building upon the earlier work [97, 98]) which relates the computation of supersymmetric amplitudes in these theories (the ‘refined topological string amplitudes’), to *ADE* matrix models and to chiral blocks of the corresponding Toda theory via the relation of matrix model to Toda theories. In particular the topological string amplitudes (which can be viewed as wave functions [99, 100]) correspond to the chiral blocks of Toda theory, from which the supersymmetric amplitudes [101, 102] are obtained by taking their squares and integrating them. The more natural limit from the viewpoint of topological string, unlike the geometric case requiring  $b^2 > 0$ , is also  $b^2 < 0$  (as we have been discussing in this paper) and  $b^2 = -1$  corresponds to the unrefined topological string. Matrix amplitudes are of the same form as the holomorphic part of the Laughlin wave function, which is what originally attracted my attention to a possible connection with FQHE.<sup>15</sup>

#### 4.1. Embedding FQHE in M-theory

There has been many attempts to connect FQHE to modern developments in string theory [58–63, 66]. Here we propose a connection between supersymmetric  $\mathcal{N} = 2$  gauge theories in 4-dimensions which arise from compactifications of 6-dimensional  $(2, 0)$  theories on a 2d Riemann surface (the ‘Gaiotto curve’) [67] and FQHE.

The  $(2, 0)$  theories are labeled by picking an ADE group. Moreover we need to consider the partition function of this theory on  $S_{k,b^2}^3$ , where  $b^2$  (sometimes written as  $b^2 = \epsilon_2/\epsilon_1$ ) denotes the squashing parameter of  $S_k^3$ , and  $S_k^3$  is the lens space  $L(k, 1) = S^3/\mathbf{Z}_k$ . In other words the worldvolume of the  $6d$   $(2, 0)$  theories is taken to be

$$\text{ADE } (2, 0) \text{ theory} \quad \text{on} \quad S_{k,b^2}^3 \times \mathbf{R} \times \Sigma$$

The spacetime worldvolume of the FQH system is identified with  $\mathbf{R} \times \Sigma$  where  $\mathbf{R}$  is taken as time and  $\Sigma$  as the space. We connect the rank of the corresponding *ADE* with the number of layers for FQHE. So the single layer

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<sup>14</sup>These ideas, and in particular the connection between FQHE and Liouville theory arising in Gaiotto theory were originally developed in discussions with Mina Aganagic and Sergei Gukov [57] to whom I am grateful.

<sup>15</sup>The potential connection between matrix models and Laughlin wave functions was pointed out to me by Shahin Sheikh-Jabbari in 2003.

case corresponds to the  $A_1$  theory. Moreover the filling fraction is given by  $\nu = -2\epsilon_2/\epsilon_1$ . We consider the Heegard decomposition of  $S_k^3$  to two solid tori (see [30, 90–93] for the discussion related to the present context) with suitable identification of boundary of solid tori depending on  $k$ . Moreover  $\epsilon_1, \epsilon_2$  can be viewed as the radii of the two circles of the middle torus. There are two natural circles in this geometry corresponding to the center of the two solid tori. Note that each of these circles is associated to one of the  $\epsilon_i$  circles which does not shrink at the center of the tori, and we denote them by  $S_{\epsilon_i}^1$ . Moreover this theory enjoys surface operators which is fixed by choosing a point  $\zeta_a \in \Sigma$  and a 2d plane of  $S_{k,b^2}^3 \times \mathbf{R}$ , taken to be  $S_{\epsilon_i}^1 \times \mathbf{R} \subset S_{k,b^2}^3 \times \mathbf{R}$ . From the view point of  $\Sigma \times \mathbf{R}$  they can be viewed as two distinct types of defects located at  $\zeta_a \in \Sigma$ . Picking the defects to be given by  $(r-1, s-1)$  copies of these two defects leads to the quasi-hole operators in the FQHE that we discussed, namely  $\Phi_{r,s}$ . The conjecture by [68]<sup>16</sup> and extensions by [69], adapted to this geometry [87], propose that the partition function of these theories for  $k=1$  case are given by Liouville theory and more generally by  $W_{ADE}$  Toda theories for the more general case which we identify with the ‘superconducting boundary condition’ in FQH systems. There are by now various derivations of these results [54, 86, 87, 89]. More precisely the effective theory on  $\Sigma \times \mathbf{R}$  is expected to be the complexified version of the ADE Chern-Simons theory with complexified level  $k + is$ , where  $k$  is a positive integer (labelling the level of  $SU(2) \subset SL(2, \mathbf{C})$ ). As was pointed out in [20] unitarity is consistent with  $s$  being either a real or purely imaginary number. We will be interested when  $s$  is purely imaginary and write it as  $s = -i\sigma$  with  $\sigma > 0$ . Interestingly enough the pure imaginary case was also the main interest in [20]. The Liouville theory arises for  $k=1$ , i.e. when we have  $S^3$ . The Liouville theory on boundary of the space (with signature 1+1, which can be interpreted as edge modes) arises in this construction by a specific boundary condition for fields [56]. For a discussion of  $A_1$  case see [95, 96] in the context of real  $SL(2, R)$  Chern-Simons theory and [30, 87] for the complex case. See also the discussion in the next section.

Our general setup naturally allows for the lens space extension given by  $k \geq 1$  and recently studied in [31, 94]. For more general  $k$  the corresponding conformal theory we get is roughly a mixture between Liouville theory and an extra parafermionic system which enjoys a  $Z_k$  symmetry. For general  $k$

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<sup>16</sup>More precisely these conjectures refer to  $S^4$  geometry, but one can connect it to  $S^3 \times \mathbf{R}$  geometry along the lines suggested in [86].

the  $b^2$  of Liouville is related to it by [31, 93, 94]<sup>17</sup>

$$\frac{2}{k - \sigma} = \frac{1 + b^2}{k}, \quad \frac{2}{k + \sigma} = \frac{1 + b^{-2}}{k}.$$

## 4.2. Complex Chern-Simons Theory as the effective theory of FQHE

The discussions up to now can be viewed as motivations for the statement to be made in this section, as we collect the various observations made in the previous sections to make our proposal precise. It is well known that FQH system in the IR can be described by a topological theory, Chern-Simons theories being the prototypical examples. We propose that:

*The effective IR theory describing a single layer FQH system is Chern-Simons theory based on  $SL(2, \mathbf{C})$  (as well as its supersymmetric version).*

More precisely we propose that the principal series with filling fraction  $\nu = n/(2n \pm 1)$  (i.e.  $b^2 = -(2n \pm 1)/2n$ ) are described by complex Chern-Simons theory  $SL(2, \mathbf{C})$  with

$$(k, \sigma) = (\pm 1, 4n \pm 1)$$

$$\frac{k + \sigma}{2} = 2n \pm 1, \quad -\frac{k - \sigma}{2} = 2n$$

(and similarly for supersymmetric version leading to the filling fractions  $\nu = m/(m + 2)$  which we leave to the interested reader). The action for the CS theory is given by [20]

$$\begin{aligned} S &= \frac{(k - \sigma)}{8\pi} \int_M \text{Tr}(\mathcal{A}d\mathcal{A} + \frac{2}{3}\mathcal{A}^3) + \frac{(k + \sigma)}{8\pi} \int_M \text{Tr}(\bar{\mathcal{A}}d\bar{\mathcal{A}} + \frac{2}{3}\bar{\mathcal{A}}^3) \\ &= \frac{-2n}{4\pi} \int_M \text{Tr}(\mathcal{A}d\mathcal{A} + \frac{2}{3}\mathcal{A}^3) + \frac{(2n \pm 1)}{4\pi} \int_M \text{Tr}(\bar{\mathcal{A}}d\bar{\mathcal{A}} + \frac{2}{3}\bar{\mathcal{A}}^3) \end{aligned}$$

where  $\mathcal{A}$  is a complex  $SL(2, \mathbf{C})$  connection. The action almost splits, between  $\mathcal{A}$  and  $\bar{\mathcal{A}}$ . However, the fact that one is complex conjugate of the other is what couples them in a non-trivial way. This is the formulation of the theory in Euclidean three dimensional space. In the physical context of the  $2 + 1$

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<sup>17</sup>I thank T. Dimofte and S. Gukov for discussions on this.

signature, instead of  $SL(2, \mathbf{C})$  we have  $SL(2, \mathbf{R}) \times SL(2, \mathbf{R})$ , where the two connections can be written as

$$\mathcal{A} = \omega - e \quad \bar{\mathcal{A}} = \omega + e$$

and  $(\omega, e)$  are independent, but on-shell they get related and interpreted in terms of the spin connection and vierbein respectively in the gravitational context. The coefficient  $k$  of Chern-Simons terms has to be integer: In that case we have  $\omega$  identified with  $SO(2) \subset SL(2, \mathbf{R})_{diagonal}$  and integrality of spin demands quantization of  $k$ . In gravitational context  $e$  is not quantized. In particular  $\sigma$  (related to cosmological constant) is not quantized. Nevertheless it is amusing that we are finding that the quantized values of  $\sigma$  which would have been natural if the two  $SL(2, R)$ 's have independently integral  $SO(2)$  charges leads *precisely* to the realized cases in FQHE! Changing  $\sigma$  in the context of FQHE can be viewed as changing the density of electrons (or equivalently the  $B$ -field) and so in a sense arbitrary values are also natural from the perspective of FQH applicationl. The fact that the levels differ by one unit simply reflects the fact that we have identified the principal series with  $|k| = 1$ , and we can of course consider other relative shifts between the two levels.

In FQHE chiral modes correspond to gapless edge modes (see [14] for a discussion and references), thus it is natural to consider this theory in the presence of a boundary. This will lead to different CFT's on the boundary with different values of  $(c_L, c_R)$ . One boundary condition [25], which we identify with sharp edge in the FQH system, corresponding to the polarization one usually chooses in quantization of it [20] and given by

$$\frac{\delta\Psi}{\delta A_{\bar{z}}} = 0 = \frac{\delta\Psi}{\delta \bar{A}_z},$$

gives

$$(c_L, c_R) = \left(3 - \frac{12}{k + \sigma}, 3 + \frac{12}{k - \sigma}\right) = \left(3 - \frac{6}{2n}, 3 - \frac{6}{2n \pm 1}\right)$$

The usual  $(c_L, c_R)$  for a compact group would have been [12]

$$(c_L, c_R) = \left(\frac{k \dim G}{k + h}, 0\right)$$

Note that this is a rather unusual situation where *the chiral blocks do not factorize* between left-moving and right-moving sector. This phenomenon was discovered in the beautiful paper [20] and is a key point for our model

of FQHE. This boundary condition we will identify with a sharp edge of the FQH system, which can support edge currents. In quantizing the Hilbert space there is one left- and one right-moving  $U(1)_{L,R}$  which are the Cartan of the two  $SL(2, \mathbf{R})$ 's. We identify the current of  $U(1)_R$  with electric current. The reason for this is that as was shown in eq.(5.3) of [20] (see also [88]) as far as the dependence of the partition function on the  $U(1)$  charges, the characters will include  $\theta(\tau, z; \bar{\tau}, \bar{z})$  functions of level  $(2n, 2n \pm 1)$  relative to  $\tau$  and  $\bar{\tau}$ . This implies that  $U(1)_R$  will have fractions  $\frac{r}{2n \pm 1}$  which is consistent with our description of the system as realizing FQHE with  $\nu = \frac{n}{2n \pm 1}$ . The fact that in the bulk they should reproduce the non-trivial braidings of  $SL(2, \mathbf{C})$  shows that they realize minimal model operator algebra, leading to non-abelian braidings that we have already mentioned. In particular as we take the basic quasi-hole around another one, we get two possible channels<sup>18</sup> instead of one channel expected in abelian models, with phases

$$\left( e^{\frac{2\pi i n}{2n \pm 1}}, e^{-\frac{6\pi i n}{2n \pm 1}} \right)$$

Let us consider  $(c_L, c_R)$  for the first few cases:

$$\nu = \frac{1}{3} : \quad (0, 1)$$

$$\nu = \frac{2}{3} : \quad \left( \frac{3}{2}, 1 \right)$$

$$\nu = \frac{2}{5} : \quad \left( \frac{3}{2}, \frac{9}{5} \right)$$

$$\nu = \frac{3}{5} : \quad \left( 2, \frac{9}{5} \right)$$

...

$$\nu \rightarrow \frac{1}{2} : \quad (3, 3)$$

Naively one would expect that the unitarity completely fixes the first one to be the usual  $c = 1$  and with the radius of the boson fixed by the level of  $\theta$  function to be the usual description of Laughlin's system of a chiral boson at radius  $R^2 = 1/3$ . However, even this case needs to be further studied: As pointed out in [20] for pure imaginary  $s$ , which is the case of interest for us, there is an 'exotic hermitian structure' on the Hilbert space. Nevertheless it was shown there, that at least in genus 1 case, which is the only case of

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<sup>18</sup>Except possibly for the  $\nu = 1/3$ , as discussed below.

relevance for our consideration in the  $1 + 1$  dimensional edge theory, there is in addition a *unitary structure* if  $0 < \left| \frac{\sigma - k}{\sigma + k} \right|^{\text{sgn}(k)} < 1$  and for our case this is given by

$$0 < \left| \frac{\sigma - k}{\sigma + k} \right|^{\text{sgn}(k)} = \left( \frac{2n}{2n \pm 1} \right)^{\pm 1} = (2\nu_{\pm})^{\pm 1} < 1$$

which is nicely satisfied. It is interesting that unitarity is related to properties of filling fraction for the principal series. It is clear that the unitarity structure for the problem at hand is rather special and one cannot simply borrow the technology familiar from the compact WZW models to the case at hand. In addition, familiar corrections such as  $k \rightarrow k + h$  in the compact case, do not apply to the non-compact case [20]. Moreover our model has, in addition to quasiholes, a rich structure of other modes with continuous wave-like modes in these blocks. We leave a detailed study of this non-trivial block structure and the corresponding character to a future work [25]. Regardless of the explicit structure of the left-right mixed blocks, it is true that the braiding properties of the defects are all determined by the bulk  $SL(2, \mathbf{C})$  theory and as we have discussed, as a part of that, the quasi-hole defects realize the operator algebra for  $(2n, 2n \pm 1)$  minimal models and the associated braiding structure.

One can ask why are the results we are obtaining so different from the conventional picture of the principal series? A hint comes from the fact that for  $\nu = 1/3$  our description seems to be particularly close to the usual one. In particular, the fact that we are getting a system with  $(c_L, c_R) = (0, 1)$  suggests that the wave function for the electrons are chiral and given by the original Laughlin wave-function which is holomorphic<sup>19</sup>. However, for higher values of  $(c_L, c_R)$ , a key feature of our model is that blocks are *not* purely holomorphic or anti-holomorphic. Therefore we expect that for higher  $(c_L, c_R)$  the electron wave function is *not purely holomorphic* and it should also have an anti-holomorphic dependence. We are currently studying details of this for our model. We thus believe that the assumption of holomorphic projection of the wave function, which is usually assumed in the conventional approach, is what sets it apart from ours, except for the  $\nu = 1/3$  case. It would be interesting to revisit the validity of this assumption in solving for the electron wave function in numerical analysis of FQH systems for other values of  $\nu$ .

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<sup>19</sup>The same is true for the Moore-Read state, where our construction would yield a  $c_L = 0$ .

There are other boundary conditions possible for this theory. In general, different boundary conditions can affect  $(c_L, c_R) \rightarrow (c_L - c, c_R - c)$  by some  $c$  which masses up some left-right chiral modes. Note that  $c_L - c_R$  is invariant (at least mod 24 due to gravitational anomaly [20]), and from the above formulas we obtain

$$c_L - c_R = \frac{6Q^2}{k}$$

where  $Q^2 = (b + \frac{1}{b})^2$ . Let us specialize to  $k = 1$  case. In that case we get

$$c_L - c_R = c_{Liouville} - 1$$

because  $c_{Liouville} = 1 + 6Q^2$ . So this suggests that we should be able to get Liouville theory by a suitable boundary condition, and indeed this is the case [56, 87]. Moreover, it also shows that there is a left-over right-moving piece with  $c_R = 1$ , which ends up being simply a free right-moving boson [87]<sup>20</sup>. Note that with this boundary condition, there is no interesting current. The reason for this identification is that the boundary condition leading up to Liouville theory, sets the upper-triangular part of  $SL(2, \mathbf{C})$  current  $\langle J^+ \rangle \neq 0$  which breaks the  $U(1)$  symmetries, i.e. higgses the  $U(1)$ 's, one of which we are identifying with the EM current. This can only happen if the EM is itself in a superconducting phase. We thus identify this boundary condition as placing the material next to a superconductor. We therefore predict that in these cases we should have a rich structure of neutral currents (for a discussion of placing FQH material next to a superconductor see e.g.[70]). There should be as many channels as the blocks of the minimal model  $(2n, 2n \pm 1)$ , without any electric currents.

This brings us full circle back to our original motivation: We connected the choices of filling fractions for principal series to the unitary minimal models of Virasoro and superVirasoro. The connection here is that the choice of boundary condition which this entails gets rid of right-moving electric modes and leads to a chiral theory with  $c < 1$ , which can be classified using unitarity. With the sharp edge boundary we get, as already mentioned, a related theory with  $(c_L, c_R) > (1, 1)$  (except for the very first one, which is

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<sup>20</sup>If the boundary of Riemann surface is an annulus, which corresponds to the  $S^4$  partition function [86], then we will have both a left-moving and a right-moving Liouville as well as a left-moving and a right-moving boson. We identify this as placing the FQH system in contact with two superconducting boundary components. This makes contact with AGT. Indeed it was already noted [68] that in addition to Liouville they needed an extra  $U(1)$  boson to get the correct partition function of 2 M5 branes on  $S^4$ , consistent with the extra  $c = 1$  system.

$(0, 1)$ ). It is then natural that with the sharp edge boundary we get filling fractions which are compatible with the unitarity of superconducting edge theory *in case we had put the sample next to a superconductor!*

In the sharp edge boundary, the electric current can be carried only in one direction in our model. The electric current is carried by the  $U(1)_R$  (called the ‘downstream’) and the left-moving direction (the ‘upstream’), is where the neutral mode propagates coming from our  $SL(2, \mathbf{R})_L$  modes. Our model therefore predicts that the electric current can *only* go in one direction along the edge (correlated with the magnetic field). It turns out that standard models that were originally constructed had electric currents going in both directions (for half of the principal series). These were not found in experiments. Later it was found how to remedy this and obtain electric currents which only go downstream, at least in some cases, by going to a new phase by taking into account disorder [71]. It is a nice feature of our model that it *predicts*<sup>21</sup> electric currents moving *only* downstream *and* neutral currents moving upstream in essentially all cases. The proliferation of neutral upstream currents is what our model predicts due to many left-moving modes. Recent experimental results [73, 74] confirm such upstream neutral currents which is difficult to reconcile with standard models of FQHE for filling fractions  $\nu = \frac{n}{2n+1}$  which has no upstream currents.

Moreover for the standard series with filling fraction  $\nu = \frac{n}{2n\pm 1}$  we have the left- and right- moving central charges which will have their imprint on thermal Hall conductivity [75], which (in fundamental units) is given by

$$\sigma_H = c_L - c_R = \frac{\mp 6}{2n(2n \pm 1)}$$

Note that  $R$  refers to downstream direction throughout this paper. Studying the thermal transport properties of the edge currents is another way to distinguish this model from other ones<sup>22</sup>. Moreover the experimental results in [73], where one measures left- and right-moving powers may be one way to experimentally measure  $(c_L, c_R)$ .

In our model we have two levels  $(k, \sigma)$ , and the quantization of  $k$  is required for the theory to be well defined, but  $\sigma$  is not, and changing it corresponds to changing the electron density. This allows us to move away from rational filling fractions and consider the transitions between Plateau regions in FQHE, which is a nice feature of our model, and can be potentially

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<sup>21</sup>For predictions of transverse response functions based on other models see in particular [72].

<sup>22</sup>We would like to thank T. Dumitrescu for discussions on this.



used to study universality properties in quantum Hall transitions. In fact the results of studies of  $SL(2, \mathbf{C})$  Chern-Simons knot invariants and their jumping phenomenon of the free energy as a function of  $b^2$  (which is related to inverse of the filling fraction) seems to be interpretable [25] as the relation between resistivity and the  $B$ -field in the FQHE (see in particular (fig. 4) of [77], where the vertical axis can be related to the B-field and the horizontal to the resistivity).

One can also study situations where the whole construction is lifted up one dimension. In fact, in a sense this is forced on us if we wish to also change the value of  $k$  continuously. This could arise only in a context where the FQH system is a boundary of a 3d material where the boundary theory cannot decouple from the 3 dimensional bulk theory at non-integral  $k$ , similar to what one encounters in the context of topological insulators or 2d Dirac/Weyl metals. Then we could ask what is the effective theory in the 3+1dimensional bulk theory which couples to  $SL(2, \mathbf{C})$  theory (or more generally complex  $ADE$  Chern-Simons theory) which fixes the issue related to non-integrality of  $k$  on the 2+1 dimensional boundary? This has been answered by Witten: The effective theory in this context would involve  $\mathcal{N} = 4$  topologically twisted  $ADE$  gauge symmetry in  $3 + 1$  dimensions (in this case  $A_1$ ), which can alternatively be viewed as  $ADE$  complexified gauge theories in  $3 + 1$  dimensions, whose boundary theory is known to give the corresponding 2+1 Chern-Simons theory [26, 27]. This would allow us to move away from  $k$  being an integer. Note that even though the underlying theory is supersymmetric, the topologically twisted version treats this as a BRST symmetry and the only manifestation of supersymmetry is in its topological properties. Moreover the choices of interesting 2+1 material which can be placed as an interface in this system translates to the choices of consistent boundary conditions for supersymmetric Yang-Mills theory, and this has been analyzed in [28], leading to a rich structure. A key role there is played by the Montonen-Olive  $SL(2, \mathbf{Z})$  symmetry [78] of  $\mathcal{N} = 4$  supersymmetric Yang-Mills theory for non-abelian gauge theories which leads to duality symmetries on the boundary.<sup>23</sup> It is interesting that our model of FQHE naturally suggests the *potential experimental realization of topologically twisted non-abelian Yang-Mills theories* as effective descriptions in the bulk of topological insulators! We are currently studying potential applications of this to condensed matter systems [29], which seems to also lead to

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<sup>23</sup>The abelian version of this at half-integral points of the Montonen-Olive torus is presumably related to dualities studied in the condensed matter literature.

beautiful block structures<sup>24</sup> on the  $2 + 1$  boundary, analogous to the  $1 + 1$  edge modes for the usual FQH systems.

It is natural to study  $k > 1$  systems as well and see whether they give interesting examples of filling fractions observed in experiments. For example filling fractions  $\nu = \frac{n}{(2nk' \pm 1)}$ , known as the Jain series, are anticipated in the composite fermions theory [5]. It is natural to conjecture that the higher Jain series correspond in our setup to the two  $SL(2, \mathbf{R})$  levels being given by  $(-2n, 2nk' \pm 1)$ , which would correspond to the level  $k = 2n(k' - 1) \pm 1$  for the diagonal  $SL(2, R)$ . In addition we can have more general values of  $k$  which would presumably give new series. This would be worth investigating. For  $k > 1$  the superconducting boundary conditions are expected to lead to paraLiouville theories [79] (see [80] for a discussion of para-Liouville theories). Similarly the supersymmetric case for  $k \geq 1$  would be interesting to study.

### 4.3. Multi-layer case

Let us briefly discuss the connection with Toda case. Most of our discussion before was in the context of  $A_1$  theory which gives the single layer theory.<sup>25</sup> We will restrict our attention to the non-supersymmetric case, though given what we found for the  $A_1$  case it is expected that the supersymmetric case of Toda theories are also interesting.<sup>26</sup> Our conjecture maps the multi-layer FQHE to  $ADE$  Toda system, when we put the sample in interface with a superconductor:

$$\int_{\Sigma} d^2 z \left[ \frac{1}{8\pi} \partial \vec{\phi} \cdot \bar{\partial} \vec{\phi} + iQ \vec{\rho} \cdot \vec{\phi} R + \sum_{j=1}^{r_{ADE}} \exp(b \vec{e}_j \cdot \vec{\phi}) \right]$$

where  $\vec{\phi}$  is an  $r$ -dimensional vector parameterizing the Cartan of  $ADE$ ,  $e_j$  are the simple roots,  $Q = b + \frac{1}{b}$ ,  $\vec{\rho}$  is half the sum of positive roots of  $ADE$  and the central charge of the Liouville theory is given by  $c = r + 12\vec{\rho} \cdot \vec{\rho} (b + \frac{1}{b})^2$ .

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<sup>24</sup>In the simplest cases this is related to sections of suitable bundles on Hitchin moduli space [103] on the boundary surface.

<sup>25</sup>As an interesting example of theoretical study of a bi-layer system see [82].

<sup>26</sup>From the expressions below it is natural to guess that the filling fraction matrix is given by  $\frac{2m}{m+2} C^{-1}$  for the supersymmetric case.

The corresponding holomorphic blocks will involve terms of the form

$$\prod_{i,j,a,b} (z_i^\alpha - z_j^\beta)^{-b^2 C_{\alpha\beta}}$$

where  $\alpha, \beta$  label the layers, and is as many as the rank of  $ADE$ . Moreover  $C_{\alpha\beta}$  is the Cartan matrix.<sup>27</sup> These theories possess  $W_{ADE}$  symmetry algebra (see [81] for the  $A_{N-1}$  case). For simplicity let us focus on the  $A_{N-1}$  case and specialize to the minimal model case (which has been found to lead to interesting structure in the AGT setup in [83–85]): This corresponds to  $W_N(p, p+1)$  models which map to  $b^2 = -\frac{2p\pm 1}{2p}$ , with central charge

$$c = (N-1) \left[ 1 - \frac{N(N+1)}{2p(2p\pm 1)} \right]$$

with  $p \geq N$ . This has the chiral wave function

$$\prod_{i,j,a,b} (z_i^\alpha - z_j^\beta)^{\frac{2p\pm 1}{2p} C_{\alpha\beta}}$$

which leads to the filling fraction matrix  $\nu$  given by

$$\nu = \frac{2p}{2p\pm 1} C^{-1}$$

with  $2p \geq N$ , where  $N-1$  is the number of layers and  $C^{-1}$  is the inverse of the Cartan matrix for  $A_{N-1}$ . The single layer case corresponds to  $C = 2$  which gives the filling fractions of the principal series  $\nu = \frac{2p}{2p\pm 1}$ . This formula for  $\nu$  is valid for all ADE Toda cases where  $C$  is the corresponding Cartan matrix. It would be interesting to connect this to experimental observations for multi-layer FQHE. Of course, just as in the case of the single layer, the sharp edge boundary condition will give a different theory, which differs from chiral Toda theories, but which will still have the same  $c_L - c_R$ .

#### 4.4. Punctures

Connections with string theory suggest that there is more one can do. In particular in the context of Gaiotto theories, we are led to put regular and irregular punctures on the Riemann surface. In the language of FQHE these

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<sup>27</sup>To make the wave function have no poles, we may have to choose a different basis for fields.

should be creating some ‘regular and irregular defects’ in the sample! It would be interesting to try and realize these. In the single-layer case the regular defects are equivalent to excizing a point  $w$  from the surface and considering the chiral wave function  $\prod_i \frac{1}{(z_i - w)^m}$ . The irregular ones are most naturally placed at the boundary of the space and involve a boundary term  $\exp[\oint W(z) \partial \phi]$  [54] where  $W = z^n$ . It would be interesting to find realizations of these ideas in the FQHE context. Moreover, the  $k > 1$  versions of these would be expected to also exist as in the  $A_1$  case.

#### 4.5. 5d systems and anisotropic FQHE

There is a 5d version of these supersymmetric theories which get connected to  $q$ -deformed versions of Toda theories. In these cases from the results in [104] one expects that the zeroes of the Laughlin wave function get split. In particular in this case the holomorphic part of the wave function for filling fraction  $\nu = 1/m$  instead of  $(z_1 - z_2)^m$  is given by

$$\prod_{i=1}^m [q^{i/2} \exp(x_1) - q^{-i/2} \exp(x_2)]$$

where  $z_{1,2} = \exp(x_{1,2})$  and  $q$  is an additional deformation parameter. It is conceivable that this is relevant for the anisotropic versions of FQHE. Moreover from the fact that  $z_i$  are replaced by their logs, it is suggestive that these are related to cylindrical geometries for the FQHE. These should have interesting physical realizations!

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JEFFERSON PHYSICAL LABORATORY, HARVARD UNIVERSITY  
CAMBRIDGE, MA 02138, USA

*E-mail address:* [vafa@physics.harvard.edu](mailto:vafa@physics.harvard.edu)