

Cluster transformations, the tetrahedron equation and three-dimensional gauge theories

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We define three families of quivers in which the braid relations of the symmetric group S_n are realized by mutations and automorphisms. A sequence of eight braid moves on a reduced word for the longest element of S_4 yields three trivial cluster transformations with 8, 32 and 32 mutations. For each of these cluster transformations, a unitary operator representing a single braid move in a quantum mechanical system solves the tetrahedron equation. The solutions thus obtained are constructed from the noncompact quantum dilogarithm and can be identified with the partition functions of three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories on a squashed three-sphere.

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1. Introduction

The Zamolodchikov tetrahedron equation [30] is a fundamental relation for integrability of quantum field theories in $2 + 1$ spacetime dimensions and of statistical mechanical models on three-dimensional lattices, much in the same way as its lower-dimensional analog, the Yang–Baxter equation, is a fundamental relation in integrable $(1 + 1)$ -dimensional quantum field theories and two-dimensional lattice models. Compared to the Yang–Baxter equation, however, our understanding of the tetrahedron equation is still limited despite its obvious importance and relatively long history.

In this work we hope to shed some light on the tetrahedron equation by uncovering its connections to quantum cluster algebras and three-dimensional supersymmetric gauge theories.

1.1. The Yang–Baxter equation and the tetrahedron equation

Graphically, the Yang–Baxter equation is represented as an equality between two configurations of three intersecting lines in a plane. The tetrahedron equation is likewise represented as an equality between two configurations of four intersecting planes in a three-dimensional space. See Figure 1. Combinatorially, the Yang–Baxter equation and the tetrahedron equation can be understood in terms of the reduced expressions for the longest elements of the symmetric groups S_3 and S_4 , respectively. Basic notions regarding the symmetric groups are recalled in section 3.1.

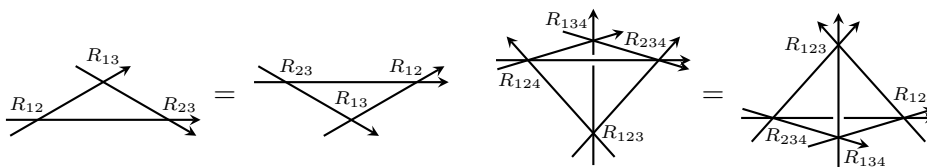


Figure 1: Graphical representations of the Yang–Baxter equation (left) and the tetrahedron equation (right).

A solution of the Yang–Baxter equation is associated with adjacent transpositions. For concreteness let us consider a solution of vertex type. This is a set R of linear operators

$$(1) \quad R_{ab} : V_a \otimes V_b \rightarrow V_b \otimes V_a,$$

where V_a , $a \in \{1, 2, 3\}$, are vector spaces and $a < b$. (The operator R_{ab} is often denoted by \check{R}_{ab} in the literature.) In the graphical representation, V_a

is the a th line and R_{ab} is the crossing of the a th and b th lines. Corresponding to the two adjacent transpositions s_1, s_2 of S_3 , the R-matrix R can act on a tensor product $V_a \otimes V_b \otimes V_c$ of V_1, V_2, V_3 in two ways, either by $R_{ab} \otimes \text{id}_{V_c}$ or $\text{id}_{V_a} \otimes R_{bc}$. (The identity maps are often suppressed in the notation.) The longest element of S_3 has two reduced expressions $s_1s_2s_1$ and $s_2s_1s_2$, and the Yang–Baxter equation

$$(2) \quad R_{23}R_{13}R_{12} = R_{12}R_{13}R_{23}$$

reflects the equality $s_1s_2s_1 = s_2s_1s_2$ satisfied by the two reduced expressions as elements of S_3 .

The tetrahedron equation lifts the above structure to one higher level. Its solution is associated with braid moves

$$(3) \quad s_a s_{a+1} s_a \rightarrow s_{a+1} s_a s_{a+1},$$

as opposed to adjacent transpositions. A vertex-type solution is a set R of linear operators

$$(4) \quad R_{abc}: V_{ab} \otimes V_{bc} \otimes V_{ac} \rightarrow V_{ac} \otimes V_{bc} \otimes V_{ab},$$

where $a, b, c \in \{1, 2, 3, 4\}$ and $a < b < c$. Graphically, the vector space V_{ab} is represented by the intersection of the a th and b th planes and R_{abc} is represented by the intersection of the a th, b th and c th planes. The tetrahedron equation

$$(5) \quad R_{234}R_{134}R_{124}R_{123} = R_{123}R_{124}R_{134}R_{234}$$

corresponds to an equivalence between two sequences of braid moves on reduced expressions for the longest element of S_4 modulo far commutativity, which takes $s_1s_2s_3s_1s_2s_1 \sim s_1s_2s_1s_3s_2s_1$ to $s_3s_2s_1s_3s_2s_3 \sim s_3s_2s_3s_1s_2s_3$.

The relation between the graphical and combinatorial interpretations of the Yang–Baxter equation and the tetrahedron equation can be made manifest by use of wiring diagrams. A wiring diagram on n wires is a diagrammatic representation of a word for a permutation in S_n . The wiring diagrams for the reduced words 121 and 212 for the longest element of S_3 , corresponding to the reduced expressions $s_1s_2s_1$ and $s_2s_1s_2$, are

$$(6) \quad 121 = \overline{\text{---}} \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} \text{---} \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \text{---} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array}, \quad 212 = \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array}.$$

The equality between these wiring diagrams is topologically equivalent to the graphical representation of the Yang–Baxter equation. To relate the two

interpretations of the tetrahedron equation, for each side of the equation we vertically juxtapose the relevant sequence of wiring diagrams and think of these wiring diagrams as “time slices” of the surfaces swept out by four moving wires [5]. The four surfaces correspond to the four planes in the graphical representation, as illustrated in Figure 2. From the figure we see the correspondence

- segments of wires \leftrightarrow regions on planes ,
- (7) intersections of wires \leftrightarrow intersections of planes ,
- regions in wiring diagrams \leftrightarrow regions in the three-dimensional space .

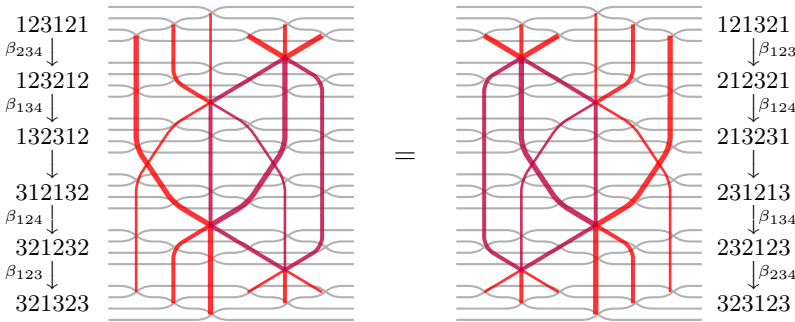


Figure 2: The tetrahedron equation arises from two sequences of braid moves with the isotopic start and end wiring diagrams on four wires. Each wiring diagram represents a slice of four surfaces that bound (after flattened to planes) a tetrahedron. The six intersection curves of these surfaces are drawn in color. Each braid move β_{abc} changes the local configuration of the surfaces and acts downward. The corresponding R-matrix R_{abc} acts upward.

A solution of the tetrahedron equation obtained by Kapranov and Voevodsky [21] realizes the symmetric group structure with representations of the quantized coordinate ring $A_q(A_3)$ on q -oscillator Fock spaces. This is the same solution as the one discovered by Bazhanov and Sergeev [2], as pointed out in [20], and is expected to arise from a brane configuration in M-theory [26].

1.2. Summary

In this paper we will construct three solutions of the tetrahedron equation with a help of quantum cluster transformations and explain how these solutions arise from three-dimensional supersymmetric gauge theories. Let us summarize the main ideas.

To each word for a permutation in S_n we assign three cluster seeds, or equivalently three quivers, which we call the triangle, square and butterfly quivers. Thus we obtain three families of quivers assigned to the words for the permutations in S_n . In each of these families, braid moves are realized as transformations of quivers composed of mutations and automorphisms relabeling vertices (Proposition 3.5). For the triangle quivers a braid move is a single mutation followed by an automorphism, whereas for the other two families a braid move involves four mutations.

The theory of quantum cluster varieties [8, 9] tells us how to represent such quiver transformations in quantum mechanics. For every seed Σ , there is a corresponding quantum mechanical system whose observables are generated by pairs of variables indexed by the vertices of Σ and satisfy commutation relations determined by the way in which the vertices are connected by arrows. The Hilbert space of states \mathcal{H}_Σ of the system is the space of wavefunctions of half of these variables. A composition \mathbf{c} of mutations and automorphisms that transforms Σ to another seed Σ' induces an isomorphism $\mathbf{K}_\mathbf{c}: \mathcal{H}_{\Sigma'} \xrightarrow{\sim} \mathcal{H}_\Sigma$ between the associated quantum mechanical systems, which is constructed from the noncompact quantum dilogarithm. The intertwiner $\mathbf{K}_\mathbf{c}$ transforms the observables by conjugation, the transformation known as the quantum cluster transformation \mathbf{c}^q . A crucial property of $\mathbf{K}_\mathbf{c}$ is that if $\Sigma' = \Sigma$ and \mathbf{c}^q is the identity map, then $\mathbf{K}_\mathbf{c}$ itself is the identity map (Propositions 2.4 and 2.5).

The two sequences of braid moves that lead to the tetrahedron equation can be concatenated (with one of them reversed) to form a loop of braid moves from one reduced expression for the longest element of S_4 to itself. By construction, on each of the three quivers assigned to this reduced expression the corresponding quiver transformation \mathbf{c} acts trivially. A key observation made in this paper is that \mathbf{c}^q is also trivial (Proposition 3.7).

Therefore, $\mathbf{K}_\mathbf{c}$ equals the identity map. This equality can be rewritten as the tetrahedron equation solved by the intertwiner corresponding to a single braid move. In this way we obtain three solutions of the tetrahedron equation, associated with the triangle, square and butterfly quivers. They define local Boltzmann weights for three-dimensional statistical mechanical lattice models with continuous spin variables.

The triangle, square and butterfly quivers have appeared in connection with gauge theories and the Yang–Baxter equation [3, 25, 29]. In that context, these quivers describe supersymmetric gauge theories with four supercharges, and the Yang–Baxter equation is interpreted as an infrared duality [27]. Computing physical quantities that are invariant under these dualities,

such as supersymmetric indices, one obtains solutions of the Yang–Baxter equation.

The solutions of the tetrahedron equation constructed in this paper also admit gauge theory interpretations. In section 5 we show that for any composition \mathbf{c} of mutations and automorphisms from a quiver Σ to another quiver Σ' , the intertwiner $\mathbf{K}_{\mathbf{c}}$ can be identified with the partition function of a three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory on a squashed three-sphere. A similar observation was made for a closely related operator in [24], and we adapt their computation to $\mathbf{K}_{\mathbf{c}}$.

1.3. Relations to other works

In [28], Yamazaki derived an equation of the form

$$(8) \quad L_{23}(\tau_{23})L_{13}(\tau_{13})L_{12}(\tau_{12}) = L_{12}(\tau'_{12})L_{13}(\tau'_{13})L_{23}(\tau'_{23})$$

as an equality between the sphere partition functions of two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric gauge theories described by the butterfly quivers assigned to the reduced expressions $s_1s_2s_1$ and $s_2s_1s_2$. This equation is almost the Yang–Baxter equation but not quite since L_{ab} depends on a set of variables τ_{ab} . In the gauge theory language, these variables are Fayet–Iliopoulos parameters and known to transform under quiver mutations in the same way as classical \mathcal{X} -variables do [1]. Yamazaki’s equation can be understood as a classical limit of the tetrahedron equation of type $RLLL = LLLR$:

$$(9) \quad L_{234}L_{134}L_{124}R_{123} = R_{123}L_{124}L_{134}L_{234},$$

where R_{123} is the R-matrix for the butterfly quiver constructed in this paper, which transforms quantum \mathcal{X} -variables by conjugation.

A classical limit of the RLLL relation (9) with different L has been obtained in [12], also from the butterfly quiver. The L-operator considered in [12] is a classical limit of the L-operator constructed in [2], which satisfies the RLLL relation with a q -oscillator-valued solution of the tetrahedron equation [2, 21]. This fact suggests that there is a close relation between the R-matrix for the butterfly quiver and the R-matrix of [2, 21].

It appears that the three-dimensional gauge theory corresponding to the intertwiner $\mathbf{K}_{\mathbf{c}}: \mathcal{H}_{\Sigma'} \rightarrow \mathcal{H}_{\Sigma}$ is (an infrared description of) a domain wall that separates two different parameter configurations of a four-dimensional $\mathcal{N} = 2$ supersymmetric field theory, for which the BPS spectra are encoded

in Σ and Σ' , respectively. At least it is akin to such domain walls which arise from pairs of M5-branes compactified on three-manifolds with boundary [4, 7, 23], and it has been argued that in the infrared on the Coulomb branch, the relevant sector of the four-dimensional theory on S_b^3 is, essentially, captured by the quantum mechanical systems assigned to Σ and Σ' . There are, however, differences between the construction of those domain walls and our construction, namely we use twice as many cluster coordinates and twice as many quantum dilogarithms.

2. Cluster transformations

In this section we recall relevant definitions and facts about cluster transformations. We will mainly follow the conventions of [8, 9]. The main result of this section is Proposition 2.5 which provides, for each trivial cluster transformation, an identity satisfied by a product of noncompact quantum dilogarithms. This identity will play a crucial role in the construction of solutions of the tetrahedron equation in section 4.

2.1. Cluster varieties

A *cluster seed* (or simply *seed*) Σ is a pair (I, ε) of a finite set I and a skew-symmetric integer matrix $\varepsilon = (\varepsilon_{ij})_{i,j \in I}$, called the *exchange matrix* of the seed. We will identify a seed with a *quiver*, a directed graph consisting of vertices connected by arrows. The quiver corresponding to a seed $\Sigma = (I, \varepsilon)$ has $|I|$ vertices, labeled by elements of I , and ε_{ij} arrows $j \rightarrow i$ between vertices $i, j \in I$ if $\varepsilon_{ij} > 0$.

A seed $\Sigma' = (I, \varepsilon')$ is said to be obtained from a seed $\Sigma = (I, \varepsilon)$ by the *mutation* $\mu_k: \Sigma \rightarrow \Sigma'$ in the direction of $k \in I$ if

$$(10) \quad \varepsilon'_{ij} = \begin{cases} -\varepsilon_{ij} & \text{if } i = k \text{ or } j = k; \\ \varepsilon_{ij} + \frac{1}{2}(|\varepsilon_{ik}|\varepsilon_{kj} + \varepsilon_{ik}|\varepsilon_{kj}|) & \text{otherwise.} \end{cases}$$

The quiver corresponding to Σ' is obtained from that corresponding to Σ by the following procedure:

- 1) For each pair of arrows $i \rightarrow k$ and $k \rightarrow j$, draw an arrow $i \rightarrow j$.
- 2) Reverse the directions of all arrows incident to the vertex k .
- 3) Delete pairs of arrows $i \rightarrow j$ and $j \rightarrow i$ (“2-cycles”).

To a seed $\Sigma = (I, \varepsilon)$ we assign three algebraic tori. Let \mathbb{C}^\times be the multiplicative group of complex numbers. The *seed \mathcal{A} -torus* \mathcal{A}_Σ is the algebraic torus $(\mathbb{C}^\times)^I$, and the standard coordinates $A := (A_i)_{i \in I}$ of \mathcal{A}_Σ are referred to as the *cluster \mathcal{A} -variables* (or simply *\mathcal{A} -variables*). The *seed \mathcal{D} -torus* \mathcal{D}_Σ is the torus $(\mathbb{C}^\times)^I \times (\mathbb{C}^\times)^I$, equipped with the *cluster \mathcal{D} -variables* $(X, B) := (X_i, B_i)_{i \in I}$. The *seed \mathcal{X} -torus* \mathcal{X}_Σ is the torus $(\mathbb{C}^\times)^I$ obtained from \mathcal{D}_Σ by projection to the first factor and equipped with the *cluster \mathcal{X} -variables* $X = (X_i)_{i \in I}$.

A mutation $\mu_k: \Sigma \rightarrow \Sigma'$ induces birational maps $\mu_k: \mathcal{A}_\Sigma \rightarrow \mathcal{A}_{\Sigma'}$, $\mu_k: \mathcal{X}_\Sigma \rightarrow \mathcal{X}_{\Sigma'}$ and $\mu_k: \mathcal{D}_\Sigma \rightarrow \mathcal{D}_{\Sigma'}$, all denoted by the same symbol μ_k . On the cluster variables the pullback of μ_k acts by

$$(11) \quad \mu_k^*(A'_i) := \begin{cases} A_k^{-1} \left(\prod_{j \in I | \varepsilon_{kj} > 0} A_j^{\varepsilon_{kj}} + \prod_{j \in I | \varepsilon_{kj} < 0} A_j^{-\varepsilon_{kj}} \right) & \text{if } i = k; \\ A_i & \text{if } i \neq k, \end{cases}$$

$$(12) \quad \mu_k^*(X'_i) := \begin{cases} X_k^{-1} & \text{if } i = k; \\ X_i (1 + X_k^{\text{sgn}(-\varepsilon_{ik})})^{-\varepsilon_{ik}} & \text{if } i \neq k, \end{cases}$$

$$(13) \quad \mu_k^*(B'_i) := \begin{cases} (X_k \prod_{j | \varepsilon_{kj} > 0} B_j^{\varepsilon_{kj}} + \prod_{j | \varepsilon_{kj} < 0} B_j^{-\varepsilon_{kj}}) / B_k (1 + X_k) & \text{if } i = k; \\ B_i & \text{if } i \neq k. \end{cases}$$

Here $\text{sgn}(a) = +1$ for $a \geq 0$ and -1 for $a < 0$.

Mutations are involutive: applying μ_k twice leaves a seed invariant and $\mu_k^* \circ \mu_k$ is the identity map on each of the seed tori.

For a permutation $\alpha: I \rightarrow I$, a seed $\Sigma' = (I, \varepsilon')$ is said to be obtained from a seed Σ by the *automorphism* $\alpha: \Sigma \rightarrow \Sigma'$ if

$$(14) \quad \varepsilon'_{\alpha(i)\alpha(j)} = \varepsilon_{ij}.$$

On the seed tori α acts by relabeling coordinates:

$$(15) \quad \alpha^*(A'_{\alpha(i)}) = A_i, \quad \alpha^*(X'_{\alpha(i)}) = X_i, \quad \alpha^*(B'_{\alpha(i)}) = B_i.$$

A *cluster transformation* $\mathbf{c}: \Sigma \rightarrow \Sigma'$ is a composition of mutations and automorphisms that takes a seed Σ to a seed Σ' . We write

$$(16) \quad \Sigma \xrightarrow{\mathbf{c}} \Sigma'$$

to mean that Σ' is obtained from Σ by a cluster transformation \mathbf{c} . A cluster transformation $\mathbf{c}: \Sigma \rightarrow \Sigma'$ induces birational maps from the seed tori assigned to Σ to those assigned to Σ' , which are the compositions of the birational maps corresponding to the mutations and automorphisms consisting of \mathbf{c} .

Definition 2.1. A cluster transformation $\mathbf{c}: \Sigma \rightarrow \Sigma$ is said to be *trivial* if it acts on \mathcal{A}_Σ , \mathcal{X}_Σ and \mathcal{D}_Σ by the identity maps.

Remark 2.2. In fact, \mathbf{c} is trivial if it acts trivially on one of \mathcal{A}_Σ , \mathcal{X}_Σ and \mathcal{D}_Σ . If \mathbf{c} acts trivially on \mathcal{X}_Σ or \mathcal{D}_Σ , then it acts on the tropical \mathcal{X} -variables trivially and hence is trivial by Theorem 2.3. Suppose \mathbf{c} acts on \mathcal{A}_Σ trivially. For each vertex $i \in I$, add another vertex and connect it to the vertex i by an arrow; the resulting seed $\tilde{\Sigma} = (\tilde{I}, \tilde{\varepsilon})$ has $\det \tilde{\varepsilon} = \pm 1$. Theorem 4.3 of [22] (applied to the case in which the semifield \mathbb{P} is trivial) shows that the cluster transformation $\tilde{\mathbf{c}}$ on $\tilde{\Sigma}$ corresponding to \mathbf{c} leaves $\tilde{\Sigma}$ invariant and acts on $\mathcal{A}_{\tilde{\Sigma}}$ trivially. Then, $\tilde{\mathbf{c}}$ also acts on $\mathcal{X}_{\tilde{\Sigma}}$ trivially because the isomorphism $\tilde{p}_{\tilde{\Sigma}}: \mathcal{A}_{\tilde{\Sigma}} \rightarrow \mathcal{X}_{\tilde{\Sigma}}$ given by $p_{\tilde{\Sigma}}^* X_k = \prod_{i \in I} A_i^{\tilde{\varepsilon}_{ki}}$ commutes with $\tilde{\mathbf{c}}$. Since $\tilde{\mathbf{c}}$ restricts to \mathbf{c} on \mathcal{X}_Σ , the latter acts on \mathcal{X}_Σ trivially.

Let $\mathbf{c}: \Sigma \rightarrow \Sigma'$ be a cluster transformation from a seed $\Sigma = (I, \varepsilon)$ to a seed $\Sigma' = (I, \varepsilon')$. By definition there is a decomposition

$$(17) \quad \mathbf{c} = \mathbf{c}[N] \circ \dots \circ \mathbf{c}[2] \circ \mathbf{c}[1],$$

where $\mathbf{c}[t]$ is either a mutation or an automorphism. This decomposition defines a collection of seeds $\Sigma[t] = (I, \varepsilon[t])$, $t = 1, 2, \dots, N + 1$, such that

$$(18) \quad \Sigma =: \Sigma[1] \xrightarrow{\mathbf{c}[1]} \Sigma[2] \xrightarrow{\mathbf{c}[2]} \dots \xrightarrow{\mathbf{c}[N]} \Sigma[N + 1] := \Sigma'.$$

The cluster transformation \mathbf{c} is trivial if and only if $\varepsilon[N + 1] = \varepsilon$, $A[N + 1] = A$, $X[N + 1] = X$ and $B[N + 1] = B$.

Gluing the seed \mathcal{A} -tori $\mathcal{A}_{\Sigma'}$ assigned to all seeds Σ' related to Σ by cluster transformations, we obtain the *cluster \mathcal{A} -variety* $\mathcal{A}_{|\Sigma|}$. Similarly, the *cluster \mathcal{X} -variety* $\mathcal{X}_{|\Sigma|}$ and the *cluster \mathcal{D} -variety* $\mathcal{D}_{|\Sigma|}$ are constructed from the seed \mathcal{X} -tori $\mathcal{X}_{\Sigma'}$ and the seed \mathcal{D} -tori $\mathcal{D}_{\Sigma'}$ assigned to all seeds Σ' related to Σ by cluster transformations. The Poisson structure on \mathcal{D}_Σ given by

$$(19) \quad \{X_i, X_j\} = \varepsilon_{ij} X_i X_j, \quad \{X_i, B_j\} = \delta_{ij} X_i B_j, \quad \{B_i, B_j\} = 0$$

is invariant under cluster transformations and defines a Poisson structure on $\mathcal{D}_{|\Sigma|}$ and $\mathcal{X}_{|\Sigma|}$.

2.2. Tropical \mathcal{X} -variables

Sometimes it is convenient to introduce the cluster variables A, X and B as formal variables assigned to a seed Σ . In this context, the pairs (Σ, A) , (Σ, X) and $(\Sigma, (X, B))$ are referred to as an \mathcal{A} -seed, an \mathcal{X} -seed and a \mathcal{D} -seed, respectively, and a cluster transformation $\mathbf{c}: \Sigma \rightarrow \Sigma'$ is interpreted as providing relations between variables A, X, B assigned to Σ and variables A', X', B' assigned to Σ' via formulas (11), (12), (13) and (15). For each t in a decomposition (17) of \mathbf{c} , the variable $A_i[t]$ (pulled back by $c[t-1] \circ \dots \circ c[2] \circ c[1]$) is an element of the *universal semifield* $\mathbb{P}_{\text{univ}}(A)$, which is the set of nonzero rational functions in the initial variables $A = (A_i)_{i \in I}$ that can be written as subtraction-free expressions, endowed with addition and multiplication. Similarly, $X_i[t] \in \mathbb{P}_{\text{univ}}(X)$ and $B_i[t] \in \mathbb{P}_{\text{univ}}((X, B))$.

For an I -tuple of variables $u = (u_i)_{i \in I}$, the *tropical semifield* $\mathbb{P}_{\text{trop}}(u)$ is defined as the abelian multiplicative group freely generated by u , endowed with the addition \oplus given by

$$(20) \quad \prod_{i \in I} u_i^{a_i} \oplus \prod_{i \in I} u_i^{b_i} := \prod_{i \in I} u_i^{\min(a_i, b_i)}.$$

There is a semifield homomorphism $\pi_{\text{trop}}: \mathbb{P}_{\text{univ}}(u) \rightarrow \mathbb{P}_{\text{trop}}(u)$, known as the *tropicalization map*, such that

$$(21) \quad \pi_{\text{trop}}(u_i) = u_i, \quad i \in I, \quad \pi_{\text{trop}}(c) = 1, \quad c \in \mathbb{Q}_{>0}.$$

The tropicalization of \mathcal{X} -variables are called *tropical \mathcal{X} -variables*.

For any $t \in \{1, 2, \dots, L\}$, we have

$$(22) \quad \mathbf{X}_i[t] := \pi_{\text{trop}}(X_i[t]) = \prod_{j \in I} \mathbf{X}_j^{a_{ij}[t]}$$

with either $a_{ij}[t] \geq 0$ for all $j \in I$ or $a_{ij}[t] \leq 0$ for all $j \in I$ (“sign coherence”). The *tropical sign* $\epsilon(\mathbf{X}_i[t])$ of $\mathbf{X}_i[t]$ is defined by

$$(23) \quad \epsilon(\mathbf{X}_i[t]) := \begin{cases} +1 & \text{if } a_{ij}[t] \geq 0 \text{ for all } j \in I; \\ -1 & \text{if } a_{ij}[t] \leq 0 \text{ for all } j \in I. \end{cases}$$

If $\mathbf{c}[t]$ is a mutation in the direction of $k \in I$, then

$$(24) \quad \mathbf{X}_i[t+1] = \begin{cases} \mathbf{X}_k[t]^{-1} & \text{if } i = k; \\ \mathbf{X}_i[t] \mathbf{X}_k[t]^{\lceil \epsilon(\mathbf{X}_k[t]) \epsilon_{ik} \rceil_+} & \text{if } i \neq k, \end{cases}$$

where $[a]_+ = a$ for $a \geq 0$ and 0 for $a < 0$.

The following theorem provides considerable simplification of the criterion for a cluster transformation to be trivial:

Theorem 2.3 ([15, Theorem 5.1]). *A cluster transformation $\mathbf{c}: \Sigma \rightarrow \Sigma$ from a seed $\Sigma = (I, \varepsilon)$ to itself is trivial if and only if $\pi_{\text{trop}}(\mathbf{c}^*(X_i)) = \pi_{\text{trop}}(X_i)$ for all $i \in I$.*

2.3. Quantum cluster varieties

There is canonical deformation quantization of the algebra of regular functions on $\mathcal{X}_{|\Sigma|}$ and that on $\mathcal{D}_{|\Sigma|}$, which leads to the notions of quantum cluster \mathcal{X} -variety and quantum cluster \mathcal{D} -variety.

Let q be a formal parameter. For a seed $\Sigma = (I, \varepsilon)$, the *quantum torus algebra* \mathbf{D}_Σ^q is the algebra over $\mathbb{Z}[q, q^{-1}]$ generated by variables $(X^q, B^q) = (X_i^q, B_i^q)_{i \in I}$ subject to the relations

$$(25) \quad q^{-\varepsilon_{ij}} X_i^q X_j^q = q^{-\varepsilon_{ji}} X_j^q X_i^q, \quad q^{-\delta_{ij}} X_i^q B_j^q = q^{\delta_{ij}} B_j^q X_i^q, \quad B_i^q B_j^q = B_j^q B_i^q.$$

We introduce an algebra over \mathbb{C} generated by a formal parameter \hbar and variables $x^\hbar = (x_i^\hbar)_{i \in I}$, $b^\hbar = (b_i^\hbar)_{i \in I}$ such that

$$(26) \quad [x_i^\hbar, x_j^\hbar] = 2\pi i \hbar \varepsilon_{ij}, \quad [x_i^\hbar, b_j^\hbar] = 2\pi i \hbar \delta_{ij}, \quad [b_i^\hbar, b_j^\hbar] = 0.$$

Then,

$$(27) \quad q := \exp(\pi i \hbar), \quad X_i^q := \exp(x_i^\hbar), \quad B_i^q := \exp(b_i^\hbar)$$

gives an embedding of \mathbf{D}_Σ^q into this algebra.

We also introduce

$$(28) \quad \tilde{x}_i^\hbar := x_i^\hbar + \sum_{j \in I} \varepsilon_{ij} b_j^\hbar$$

and set

$$(29) \quad \tilde{X}_i^q := \exp(\tilde{x}_i^\hbar) = X_i^q \prod_{j \in I} (B_j^q)^{\varepsilon_{ij}}.$$

These variables satisfy the relations

$$(30) \quad [\tilde{x}_i^\hbar, \tilde{x}_j^\hbar] = -2\pi i \hbar \varepsilon_{ij}, \quad [\tilde{x}_i^\hbar, b_j^\hbar] = 2\pi i \hbar \delta_{ij}, \quad [x_i^\hbar, \tilde{x}_j^\hbar] = 0$$

and

$$(31) \quad q^{\varepsilon_{ij}} \tilde{X}_i^q \tilde{X}_j^q = q^{\varepsilon_{ji}} \tilde{X}_j^q \tilde{X}_i^q, \quad q^{-\delta_{ij}} \tilde{X}_i^q B_j^q = q^{\delta_{ij}} B_j^q \tilde{X}_i^q, \quad X_i^q \tilde{X}_j^q = X_j^q \tilde{X}_i^q.$$

Let \mathbb{D}_Σ^q be the skew-field of fractions of \mathbf{D}_Σ^q . For a cluster transformation $\mathbf{c}: \Sigma \rightarrow \Sigma'$, we define the *quantum cluster transformation*

$$(32) \quad \mathbf{c}^q: \mathbb{D}_{\Sigma'}^q \rightarrow \mathbb{D}_\Sigma^q$$

as follows. (Note the direction; \mathbf{c}^q quantizes the pullback by $\mathbf{c}: \mathcal{D}_\Sigma \rightarrow \mathcal{D}_{\Sigma'}$.) For $\mathbf{c} = \alpha$, the map $\mathbf{c}^q = \alpha^q$ is given by

$$(33) \quad \alpha^q(X_{\alpha(i)}'^q) = X_i^q, \quad \alpha^q(B_{\alpha(i)}'^q) = B_i^q.$$

For $\mathbf{c} = \mu_k$, the map $\mathbf{c}^q = \mu_k^q$ is a composition of two maps:

$$(34) \quad \mu_k^q := \mu_k^{\sharp(+)} \circ \mu_k'^{(+)} = \mu_k^{\sharp(-)} \circ \mu_k'^{(-)}.$$

The fact that μ_k^q admits two decompositions will be important. In general, \mathbf{c}^q is given by the composition $\mathbf{c}^q[1] \circ \mathbf{c}^q[2] \circ \dots \circ \mathbf{c}^q[N]$ of quantum cluster transformations corresponding to a decomposition of \mathbf{c} into mutations and automorphisms.

The “automorphism part” $\mu_k^{\sharp(\epsilon)}: \mathbb{D}_\Sigma^q \rightarrow \mathbb{D}_\Sigma^q$ of μ_k^q is defined by

$$(35) \quad \mu_k^{\sharp(\epsilon)} := \text{Ad}_{\Psi^q((X_k^q)^\epsilon) \Psi^q((\tilde{X}_k^q)^\epsilon)^{-\epsilon}}, \quad \epsilon = \pm,$$

where

$$(36) \quad \Psi^q(x) := \prod_{k=1}^\infty (1 + q^{2k-1}x)^{-1}$$

is the *quantum dilogarithm*. Using the difference equation

$$(37) \quad \Psi^q(q^2x) = (1 + qx)\Psi^q(x),$$

one can show that $\mu_k^{\sharp(\epsilon)}(X_i'^q)$ and $\mu_k^{\sharp(\epsilon)}(B_i'^q)$ belong to \mathbb{D}_Σ^q [8, Lemma 3.2].

The “monomial part” $\mu_k^{(\epsilon)}: \mathbf{D}_{\Sigma'}^q \rightarrow \mathbf{D}_{\Sigma}^q$ is given by

$$(38) \quad \mu_k^{(\epsilon)}(X_i^{\hbar}) := \exp(m_k^{(\epsilon)}(x_i^{\hbar})), \quad \mu_k^{(\epsilon)}(B_i^{\hbar}) := \exp(m_k^{(\epsilon)}(b_i^{\hbar})),$$

where

$$(39) \quad m_k^{(\epsilon)}(x_i^{\hbar}) := \begin{cases} -x_k^{\hbar} & \text{if } i = k; \\ x_i^{\hbar} + [\epsilon\varepsilon_{ik}]_+ x_k^{\hbar} & \text{if } i \neq k, \end{cases}$$

$$(40) \quad m_k^{(\epsilon)}(b_i^{\hbar}) := \begin{cases} -b_k^{\hbar} + \sum_{j \in I} [-\epsilon\varepsilon_{kj}]_+ b_j^{\hbar} & \text{if } i = k; \\ b_i^{\hbar} & \text{if } i \neq k. \end{cases}$$

The transformations of x^{\hbar} and b^{\hbar} are dual to each other: if we write $m_k^{(\epsilon)}(x_i^{\hbar}) = \sum_{j \in I} (M_k^{(\epsilon)})_{ij} x_j^{\hbar}$ using a matrix $M_k^{(\epsilon)}$, then $m_k^{(\epsilon)}(b_i^{\hbar}) = \sum_{j \in I} b_j^{\hbar} ((M_k^{(\epsilon)})^{-1})_{ji}$.

For $q = 1$, the formula for \mathbf{c}^q reduces to that for the action of \mathbf{c} on \mathcal{D}_{Σ} . In particular, if the quantum cluster transformation \mathbf{c}^q induced from $\mathbf{c}: \Sigma \rightarrow \Sigma$ is the identity map, then \mathbf{c} also acts trivially on \mathcal{D}_{Σ} and hence on the tropicalization of \mathcal{X}_{Σ} , which implies that \mathbf{c} is a trivial cluster transformation by Theorem 2.3. It turns out that the converse is also true:

Proposition 2.4. *A cluster transformation $\mathbf{c}: \Sigma \rightarrow \Sigma$ is trivial if and only if $\mathbf{c}^q = \text{id}_{\mathbb{D}_{\Sigma}^q}$.*

This proposition can be proved from Proposition 5.21 of [8] and Theorem 4.3 of [22]. The proposition also follows from Proposition 2.5.

2.4. Representations of quantum cluster varieties

From now on we take \hbar to be a positive real number. For a seed $\Sigma = (I, \varepsilon)$, let

$$(41) \quad \mathcal{H}_{\Sigma} := L^2(\mathcal{A}_{\Sigma}^+)$$

be the Hilbert space of square-integrable complex functions on the set of positive real points $\mathcal{A}_{\Sigma}^+ \cong \mathbb{R}^I$ of \mathcal{A}_{Σ} . This is the Hilbert space of states in quantum mechanics of a particle moving in \mathcal{A}_{Σ}^+ . Coordinates of \mathcal{A}_{Σ}^+ are given by $a_i := \log A_i, i \in I$.

The differential operators

$$(42) \quad \hat{x}_i = \pi i \hbar \frac{\partial}{\partial a_i} - \sum_{j \in I} \varepsilon_{ij} a_j, \quad \hat{b}_i = 2a_i$$

on functions in \mathcal{H}_Σ satisfy the commutation relations (26), and their exponentials

$$(43) \quad \widehat{X}_i := \exp(\hat{x}_i), \quad \widehat{B}_i := \exp(\hat{b}_i)$$

satisfy relations (25). The operators

$$(44) \quad \hat{\tilde{x}}_i = \pi i \hbar \frac{\partial}{\partial a_i} + \sum_{j \in I} \varepsilon_{ij} a_j, \quad \widehat{\tilde{X}}_i := \exp(\hat{\tilde{x}}_i).$$

satisfy relations (30) and (31).

Let \mathbb{L}_Σ be the space of Laurent polynomials in the quantum variables (X^q, B^q) assigned to Σ such that for any cluster transformation $\mathbf{c}' : \Sigma' \rightarrow \Sigma$, the quantum cluster transformation $\mathbf{c}'^q : \mathbb{D}_\Sigma^q \rightarrow \mathbb{D}_{\Sigma'}^q$ maps them to Laurent polynomials in (X'^q, B'^q) . The operators corresponding to the elements of \mathbb{L}_Σ preserve a certain subspace \mathcal{S}_Σ of rapidly decreasing functions in \mathcal{H}_Σ , and they provide a representation of \mathbb{L}_Σ on \mathcal{S}_Σ .

A cluster transformation $\mathbf{c} : \Sigma \rightarrow \Sigma'$ gives rise to an isomorphism of algebras $\mathbf{c}^q : \mathbb{L}_{\Sigma'} \rightarrow \mathbb{L}_\Sigma$. The representation of \mathbb{L}_Σ on \mathcal{S}_Σ and that of $\mathbb{L}_{\Sigma'}$ on $\mathcal{S}_{\Sigma'}$ are intertwined by a unitary operator $\mathbf{K}_\mathbf{c} : \mathcal{H}_{\Sigma'} \rightarrow \mathcal{H}_\Sigma$:

$$(45) \quad \mathbf{K}_\mathbf{c} \widehat{A} \mathbf{K}_\mathbf{c}^{-1} = \widehat{\mathbf{c}^q(A)}, \quad A \in \mathbb{L}_{\Sigma'}.$$

Here \widehat{A} denotes the operator corresponding to A .

For $\mathbf{c} = \alpha$, the intertwiner is simply

$$(46) \quad \mathbf{K}_\alpha := \alpha^*,$$

the pullback by α considered as a transformation on \mathbb{R}^I :

$$(47) \quad \alpha : (a_i)_{i \in I} \mapsto (a'_i)_{i \in I} = (a_{\alpha^{-1}(i)})_{i \in I}.$$

The intertwiner \mathbf{K}_{μ_k} for a mutation μ_k decomposes as

$$(48) \quad \mathbf{K}_{\mu_k} := \mathbf{K}_{\mu_k}^{\sharp(+)} \mathbf{K}_{\mu_k}^{\prime(+)} = \mathbf{K}_{\mu_k}^{\sharp(-)} \mathbf{K}_{\mu_k}^{\prime(-)},$$

corresponding to the decompositions (34). The fact that \mathbf{K}_{μ_k} admits these two decompositions was observed by Kim [18]. We will show the equality of the two decompositions in section 5.

The monomial part $\mathbf{K}_{\mu_k}^{l(\epsilon)} : \mathcal{H}_{\Sigma'} \rightarrow \mathcal{H}_{\Sigma}$ is the pullback by the transformation

$$(49) \quad (a_i)_{i \in I} \mapsto (a'_i)_{i \in I}, \quad a'_i = \begin{cases} -a_k + \sum_{j \in I} [-\epsilon \varepsilon_{kj}]_+ a_j & \text{if } i = k; \\ a_i & \text{if } i \neq k. \end{cases}$$

The automorphism part $\mathbf{K}_{\mu_k}^{\sharp(\epsilon)} : \mathcal{H}_{\Sigma} \rightarrow \mathcal{H}_{\Sigma}$ is given by

$$(50) \quad \mathbf{K}_{\mu_k}^{\sharp(\epsilon)} = \Phi^{\hbar}(\epsilon \hat{x}_k)^{\epsilon} \Phi^{\hbar}(\epsilon \hat{x}_k)^{-\epsilon}.$$

The function $\Phi^{\tau}(z)$, $\tau \in \mathbb{C}$, is the *noncompact quantum dilogarithm*. It is defined for $|\operatorname{Im} z| < \pi(1 + \operatorname{Re} \tau)$ by the integral

$$(51) \quad \Phi^{\tau}(z) := \exp\left(-\frac{1}{4} \int_{\mathbb{R}+i0} \frac{e^{-i\omega z}}{\sinh(\pi\omega) \sinh(\pi\tau\omega)} \frac{d\omega}{\omega}\right),$$

where the contour along the real axis going from $-\infty$ to ∞ and bypassing the origin from above, and is analytically continued to the entire complex plane. The unitarity of \mathbf{K}_{μ_k} follows from the property that

$$(52) \quad \overline{\Phi^{\tau}(z)} = \Phi^{\tau}(\bar{z})^{-1}$$

if τ is a positive real number or a pure phase.

For $\operatorname{Im} \tau > 0$, we have

$$(53) \quad \Phi^{\tau}(z) = \frac{\Psi^q(e^z)}{\Psi^{1/q^{\vee}}(e^{z/\tau})}, \quad q^{\vee} := e^{\pi i/\tau},$$

and hence $\Phi^{\tau}(z)$ satisfies the difference equations

$$(54) \quad \Phi^{\tau}(z + 2\pi i\tau) = (1 + qe^z)\Phi^{\tau}(z),$$

$$(55) \quad \Phi^{\tau}(z + 2\pi i) = (1 + q^{\vee}e^{z/\tau})\Phi^{\tau}(z).$$

From the first of these equations we see that conjugation by $\mathbf{K}_{\mu_k}^{\sharp(\epsilon)}$ acts as $\mu_k^{\sharp(\epsilon)}$ on \widehat{X}_i and \widehat{B}_i .

2.5. Quantum dilogarithm identities

Let $\mathbf{c}: \Sigma \rightarrow \Sigma$ be a trivial cluster transformation. By Proposition 2.4, the quantum cluster transformation \mathbf{c}^q is also trivial. The dual variables X^{q^\vee}, B^{q^\vee} , defined by

$$(56) \quad X^{q^\vee} := \exp(x^{\hbar^\vee}), \quad B^{q^\vee} := \exp(b^{\hbar^\vee}), \quad x^{\hbar^\vee} := \hbar^{-1}x^\hbar, \quad b^{\hbar^\vee} := \hbar^{-1}b^\hbar,$$

commute with X^q, B^q and satisfy relations (25) with q replaced by $q^\vee = \exp(\pi i/\hbar)$. The difference equation (55) shows that conjugation by \mathbf{K}_{μ_k} acts on $\widehat{X}^\vee, \widehat{B}^\vee$ as $\mu_k^{q^\vee}$, whereas conjugation by \mathbf{K}_α acts in the same way on $\widehat{X}^\vee, \widehat{B}^\vee$ and on \widehat{X}, \widehat{B} . It follows that $\mathbf{K}_\mathbf{c}$ commutes with $\widehat{X}^\vee, \widehat{B}^\vee$ since it commutes with \widehat{X}, \widehat{B} . The fact that $\mathbf{K}_\mathbf{c}$ commutes with both sets of variables implies that $\mathbf{K}_\mathbf{c} = \lambda_\mathbf{c} \text{id}_{\mathcal{H}_\Sigma}$ for some complex number $\lambda_\mathbf{c}$ with $|\lambda_\mathbf{c}| = 1$ [8, Theorem 5.4].

In fact, $\lambda_\mathbf{c} = 1$, as was pointed out for important specific cluster transformations by Kim [18].

Proposition 2.5. $\mathbf{K}_\mathbf{c} = \text{id}_{\mathcal{H}_\Sigma}$ for a trivial cluster transformation $\mathbf{c}: \Sigma \rightarrow \Sigma$.

We will demonstrate this proposition using quantum dilogarithm identities proved in [19], which we now explain.

Consider a decomposition of \mathbf{c} into mutations and automorphisms. The positions of the automorphisms in the decomposition can be moved by the relation

$$(57) \quad \mu_k \circ \alpha = \alpha \circ \mu_{\alpha^{-1}(k)},$$

hence we can decompose \mathbf{c} into the form

$$(58) \quad \mathbf{c}: \Sigma =: \Sigma[1] \xrightarrow{\mu_{k[1]}} \Sigma[2] \xrightarrow{\mu_{k[2]}} \dots \xrightarrow{\mu_{k[L]}} \Sigma[L+1] \xrightarrow{\alpha} \Sigma.$$

Let $\mathbf{X}_i[t] := \pi_{\text{trop}}(X_i[t])$ be the tropicalization of the variable $X_i[t]$ of $\mathcal{X}_{\Sigma[t]}$. For each t , the relation between $\mathbf{X}[t]$ and $\mathbf{X} = \mathbf{X}[1]$ defines an I -tuple of integers $\gamma[t] = (\gamma_i[t])_{i \in I} \in \mathbb{Z}^I$ and a sign $\epsilon[t] \in \{\pm\}$ by

$$(59) \quad \mathbf{X}_{k[t]}[t] =: \prod_{i \in I} \mathbf{X}_i^{\gamma_i[t]}, \quad \epsilon[t] := \epsilon(\mathbf{X}_{k[t]}[t]) = \begin{cases} +1 & \gamma_i[t] \geq 0 \text{ for all } i \in I; \\ -1 & \gamma_i[t] \leq 0 \text{ for all } i \in I. \end{cases}$$

The noncompact quantum dilogarithm satisfies the following identity:

Theorem 2.6 ([19, Theorem 4.5], [17, Theorem 5.16]).

$$(60) \quad \Phi^{\hbar}(\epsilon[1]\gamma[1] \cdot \hat{x})^{\epsilon[1]} \Phi^{\hbar}(\epsilon[2]\gamma[2] \cdot \hat{x})^{\epsilon[2]} \dots \Phi^{\hbar}(\epsilon[L]\gamma[L] \cdot \hat{x})^{\epsilon[L]} = \text{id}_{\mathcal{H}_{\Sigma}} .$$

The operator $\gamma[t] \cdot \hat{x} := \sum_{i \in I} \gamma_i[t] \cdot \hat{x}_i$ appearing in the above identity can be understood as follows. Comparing the transformation (24) of $\mathbf{X}_i[t]$ under μ_k and the definition (39) of $m_k^{(\epsilon)}$, we see that $x_i^{\hbar}[t+1]$ transforms under $m_k^{(\epsilon[t])}$ in the same way as $\log \mathbf{X}_i[t+1]$ does under $\mu_k[t]$. Thus we have

$$(61) \quad \gamma[t] \cdot x^{\hbar} = \mu_{k[1]}^{\prime(\epsilon[1])} \circ \mu_{k[2]}^{\prime(\epsilon[2])} \circ \dots \circ \mu_{k[t-1]}^{\prime(\epsilon[t-1])} (x_{k[t]}^{\hbar}[t]),$$

and $\gamma[t] \cdot \hat{x}$ is the corresponding operator obtained from $\hat{x}_{k[t]}[t]$ by conjugation by $\mathbf{K}_{\mu_{k[1]}}^{\prime(\epsilon[1])} \mathbf{K}_{\mu_{k[2]}}^{\prime(\epsilon[2])} \dots \mathbf{K}_{\mu_{k[t-1]}}^{\prime(\epsilon[t-1])}$.

Proof of Proposition 2.5. Since $\overline{\Phi^{\hbar}(z)} = \Phi^{\hbar}(\bar{z})^{-1}$ and $\overline{\hat{x}_i} = -\hat{x}_i$, taking the complex conjugate of identity (60) and conjugating the resulting identity with the map $a \mapsto -a$, we obtain

$$(62) \quad \begin{aligned} & \Phi^{\hbar}(\epsilon[1]\gamma[1] \cdot \hat{\hat{x}})^{-\epsilon[1]} \Phi^{\hbar}(\epsilon[2]\gamma[2] \cdot \hat{\hat{x}})^{-\epsilon[2]} \dots \Phi^{\hbar}(\epsilon[L]\gamma[L] \cdot \hat{\hat{x}})^{-\epsilon[L]} \\ & = \text{id}_{\mathcal{H}_{\Sigma}} . \end{aligned}$$

Furthermore, a_i transforms under $\mathbf{K}_{\mu_{k[t]}}^{\prime(\epsilon[t])}$ in the same way as b_i^{\hbar} does under $m_k^{(\epsilon[t])}$, and the latter transforms in the dual manner to x_i^{\hbar} and hence to the transformation of $\log \mathbf{X}_i[t]$ under $\mu_k[t]$. Since \mathbf{c} acts on \mathbf{X}_i trivially by Theorem 2.3, we have

$$(63) \quad \mathbf{K}_{\mu_{k[1]}}^{\prime(\epsilon[1])} \mathbf{K}_{\mu_{k[2]}}^{\prime(\epsilon[2])} \dots \mathbf{K}_{\mu_{k[L]}}^{\prime(\epsilon[L])} \mathbf{K}_{\alpha} = \text{id}_{\mathcal{H}_{\Sigma}} .$$

Therefore,

$$(64) \quad \begin{aligned} \mathbf{K}_{\mathbf{c}} &= \mathbf{K}_{\mu_{k[1]}}^{\#(\epsilon[1])} \mathbf{K}_{\mu_{k[1]}}^{\prime(\epsilon[1])} \mathbf{K}_{\mu_{k[2]}}^{\#(\epsilon[2])} \mathbf{K}_{\mu_{k[2]}}^{\prime(\epsilon[2])} \dots \mathbf{K}_{\mu_{k[L]}}^{\#(\epsilon[L])} \mathbf{K}_{\mu_{k[L]}}^{\prime(\epsilon[L])} \mathbf{K}_{\alpha} \\ &= \Phi^{\hbar}(\epsilon[1]\gamma[1] \cdot \hat{x})^{\epsilon[1]} \Phi^{\hbar}(\epsilon[2]\gamma[2] \cdot \hat{x})^{\epsilon[2]} \dots \Phi^{\hbar}(\epsilon[L]\gamma[L] \cdot \hat{x})^{\epsilon[L]} \\ &\quad \cdot \Phi^{\hbar}(\epsilon[1]\gamma[1] \cdot \hat{\hat{x}})^{-\epsilon[1]} \Phi^{\hbar}(\epsilon[2]\gamma[2] \cdot \hat{\hat{x}})^{-\epsilon[2]} \dots \Phi^{\hbar}(\epsilon[L]\gamma[L] \cdot \hat{\hat{x}})^{-\epsilon[L]} \\ &\quad \cdot \mathbf{K}_{\mu_{k[1]}}^{\prime(\epsilon[1])} \mathbf{K}_{\mu_{k[2]}}^{\prime(\epsilon[2])} \dots \mathbf{K}_{\mu_{k[L]}}^{\prime(\epsilon[L])} \mathbf{K}_{\alpha} \end{aligned}$$

is the identity map. □

3. Trivial cluster transformations from the longest element of S_4

In this section we introduce the three families of quivers described in section 1, which are assigned to the words for permutations in the symmetric group S_n . A loop of braid moves on reduced expressions for the longest element of S_4 yields trivial cluster transformations acting on the seed tori of relevant quivers.

3.1. Symmetric groups and wiring diagrams

The *symmetric group* S_n is the group of permutations of $\{1, 2, \dots, n\}$. It is generated by the *adjacent transpositions* $\{s_a\}_{a=1}^{n-1}$ satisfying the relations

$$(65) \quad s_a^2 = 1,$$

$$(66) \quad s_a s_b = s_b s_a \quad \text{for } |a - b| \geq 2 \quad (\text{far commutativity}),$$

$$(67) \quad s_a s_{a+1} s_a = s_{a+1} s_a s_{a+1} \quad (\text{braid relation}).$$

A *word* for a permutation $s \in S_n$ is a finite string $a_1 a_2 \dots a_k$ of elements of $\{1, 2, \dots, n\}$ such that $s = s_{a_1} s_{a_2} \dots s_{a_k}$. The *length* $l(s)$ of s is the minimal number such that $s = s_{a_1} s_{a_2} \dots s_{a_{l(s)}}$ for some $a_1, a_2, \dots, a_{l(s)} \in \{1, 2, \dots, n\}$. The expression $s_{a_1} s_{a_2} \dots s_{a_{l(s)}}$ and the string $a_1 a_2 \dots a_{l(s)}$ are called a *reduced expression* for s and a *reduced word* for s , respectively. A reduced expression for a permutation can be transformed to any other reduced expression for the same permutation by a sequence of far commutativity and braid relations (Tits' lemma). The longest element of S_n is the order reversing permutation $a \mapsto n - a + 1$ and its length is $n(n - 1)/2$.

To the end of constructing quivers corresponding to words for permutations in S_n , we represent words diagrammatically.

Definition 3.1. A *wiring diagram on n wires* is a union of n continuous paths, called *wires*, inside the vertical strip $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1\}$ such that

- 1) the wires start from distinct points on the left boundary ($x = 0$) and end at distinct points on the right boundary ($x = 1$);
- 2) no three wires intersect at a point; and
- 3) no two intersections of wires take place at the same horizontal position (x -coordinate).

Two wiring diagrams are identified if they are related by an isotopy that preserves the horizontal ordering of the intersections of wires.

The words for permutations in S_n are in one-to-one correspondence with the wiring diagrams on n wires. To represent a word $a_1 a_2 \dots a_k$ by a wiring diagram, we move rightward in the positive x -direction from the left boundary of the vertical strip toward the right boundary, and let wires intersect according to the letters appearing in the reduced word. Thus, we first let the a_1 th wire and the $(a_1 + 1)$ st wire intersect, counted from bottom to top in the ascending order of the y -coordinates of the wires, and next let the a_2 th wire and the $(a_2 + 1)$ st intersect, again counted from bottom to top but at a horizontal position right to the first intersection, and so on. Conversely, given a wiring diagram on n wires, the corresponding word for a permutation can be written down.

Definition 3.2. A wiring diagram is said to be *reduced* if no two wires intersect more than once.

The reduced words are in one-to-one correspondence with the reduced wiring diagrams. For example, the reduced word 123121 for the longest element of S_4 is represented by the reduced wiring diagram



As can be seen from this example, a wiring diagram for a reduced word of the longest element of S_n has the property that each pair of wires intersect exactly once.

Given a wiring diagram on n wires, we name the wires 1, 2, ..., n from bottom to top according to their positions on the left boundary of the vertical strip. We need to label the segments and intersections of wires. The segments of wire a is labeled, from left to right, a_1, a_2, \dots, a_n . The intersection of wires a and b with $a < b$ is labeled ab .

A wiring diagram divides the vertical strip into *chambers*, i.e., connected components of the complement of the wires, and we also need labels for them. We label the chambers with subsets of $\{1, 2, \dots, n\}$ as follows. First, we label the chamber extending to $y = -\infty$ the empty set \emptyset . Starting from the chamber \emptyset , we can go to any other chamber by crossing some number of wires. If we reach a new chamber from a chamber C by crossing wire a , then we label that chamber $C \cup \{a\}$.

Figure 3 illustrates our labeling scheme with the wiring diagram for the reduced word 121 for the longest element of S_3 .

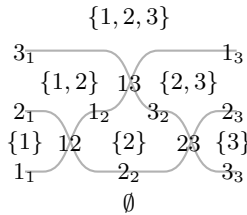
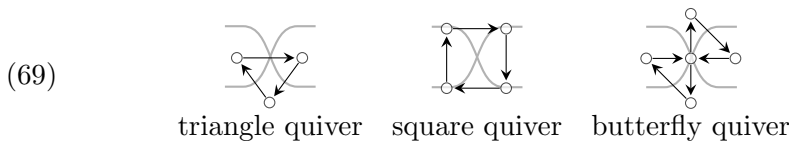


Figure 3: Labeling of the segments, intersections and chambers of the wiring diagram for the reduced word 121 for the longest element of S_3 .

3.2. Quivers assigned to wiring diagrams

Definition 3.3. The *triangle quiver*, the *square quiver* and the *butterfly quiver* assigned to a word for a permutation of S_n is constructed as follows:

- 1) Around each intersection of wires of the corresponding wiring diagram, place vertices and connect them with arrows according to the rule



- 2) Label each vertex with the name of the segment, intersection or chamber where the vertex is placed, and identify vertices with the same labels.
- 3) Delete 2-cycles formed in the previous step.
- 4) For the triangle quiver and the butterfly quiver, add a vertex for each subset of $\{1, 2, \dots, n\}$ if not present already. This vertex is disconnected from any other vertices.

The introduction of disconnected vertices in the last step is a technicality that can be avoided if we allow either mutations or automorphisms to rename the elements of the index set for a quiver. Figure 4 shows the three quivers assigned to the reduced word 123121 for the longest element of S_4 .

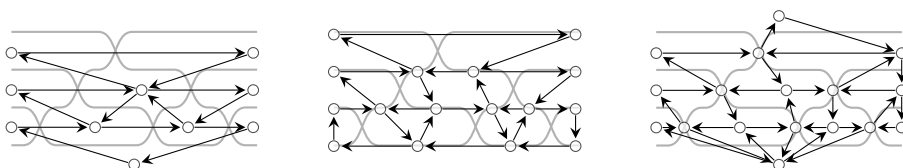
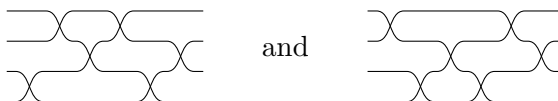


Figure 4: The triangle, square and butterfly quivers assigned to the reduced word 123121 for the longest element of S_4 . The vertex labels and the disconnected vertices are not shown.

The three quivers assigned to a wiring diagram depend only on the isotopy class of the wiring diagram. For example, to the two wiring diagrams



the same quivers are assigned. Therefore, these quivers are really assigned to an *equivalence class* of words with respect to far commutativity (66). Any two equivalence classes of reduced words for the same permutation can be obtained from one another by a sequence of braid relations, which translate to a sequence of local moves in the wiring diagrams.

Definition 3.4. The *braid move* β_{abc} on a wiring diagram is the local transformation

$$(70) \quad \begin{array}{c} c \\ b \\ a \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \diagdown \\ \diagup \\ \diagdown \\ \diagup \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \rightarrow \begin{array}{c} c \\ b \\ a \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \diagup \\ \diagdown \\ \diagup \\ \diagdown \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array}$$

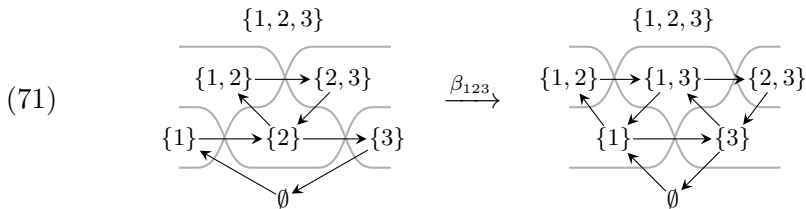
on wires a, b, c .

The braid move β_{abc} on a wiring diagram transforms the assigned quivers. A key fact is that the transformations induced by a braid move can be expressed as compositions of mutations and automorphisms:

Proposition 3.5. *The braid move β_{abc} induces a cluster transformation on the triangle, square and butterfly quivers assigned to a wiring diagram.*

Proof. Since braid moves are local transformations, it is sufficient to check the statement for wiring diagrams on three wires. (Although some arrows can be canceled when more wires are added, the action of μ_k depends only on $\varepsilon_{ki}, i \in I$, and for the mutations relevant to the braid move these integers remain unchanged.)

For the triangle quiver, β_{123} acts on the reduced word 121 as

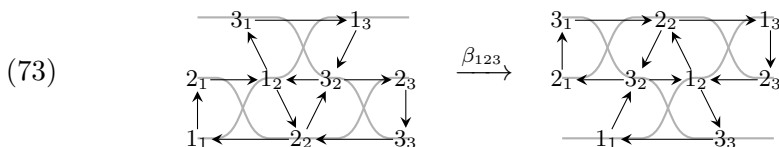


This transformation is the mutation $\mu_{\{2\}}$ at $\{2\}$, followed by relabeling of the vertex $\{2\}$ to $\{1,3\}$:

(72)
$$\beta_{123} = \alpha_{\{2\},\{1,3\}} \circ \mu_{\{2\}} \quad (\text{triangle quiver}).$$

Here $\alpha_{i,j}$ denotes the automorphism given by the permutation interchanging $i, j \in I$. Note that the braid move changes the index set for the connected vertices. To deal with this complication we have enlarged the quiver by additional disconnected vertices.

For the square quiver, β_{123} acts as

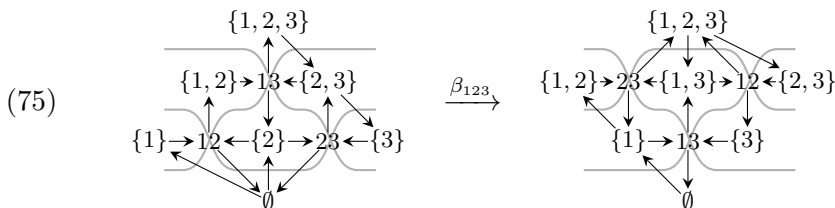


This transformation can be written as a composition of four mutations and one automorphism:

(74)
$$\beta_{123} = \alpha_{1_2,3_2} \circ \mu_{2_2} \circ \mu_{1_2} \circ \mu_{3_2} \circ \mu_{2_2} \quad (\text{square quiver}).$$

See Figure 5.

For the butterfly quiver, we have



This transformation can be written as

(76)
$$\begin{aligned} \beta_{123} = & \alpha_{\{2\},\{1,3\}} \circ \alpha_{\{2\},13} \circ \alpha_{12,23} \\ & \circ \mu_{13} \circ \mu_{23} \circ \mu_{12} \circ \mu_{\{2\}} \quad (\text{butterfly quiver}). \end{aligned}$$

□

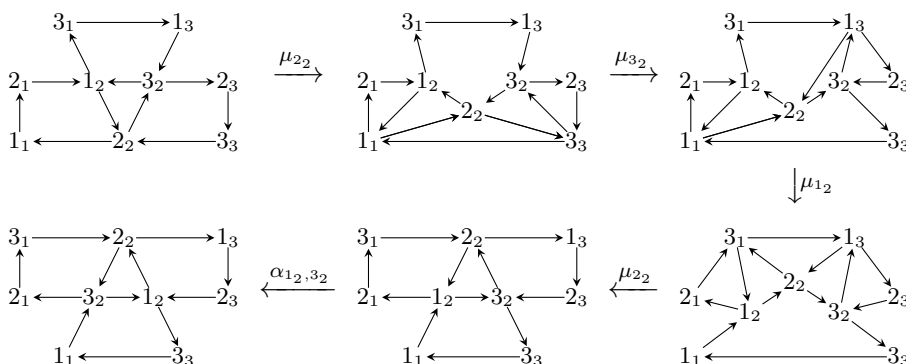


Figure 5: The braid move β_{123} for the square quiver is a composition of four mutations and one automorphism.

Remark 3.6. In the context of supersymmetric gauge theories and the Yang–Baxter equation, the triangle quiver appeared in [25], the square quiver in [3, 25] and the butterfly quiver in [29]. The triangle quiver has also appeared in relation to cluster algebras before, e.g. in [11]. A connection between the butterfly quiver and cluster algebras was pointed out in [28].

3.3. Trivial cluster transformations

Take the sequence of braid moves on the left-hand side of Figure 2 and concatenate it with the inverse of the sequence of braid moves on the right-hand side. This creates a sequence of transformations on reduced words for the longest element of S_4 which starts with 123121 and ends with 121321. Let $\Sigma_{a_1 a_2 a_3 a_4 a_5 a_6}$ be any of the triangle, square and butterfly quivers assigned to the reduced word $a_1 a_2 a_3 a_4 a_5 a_6$ for this element. By Proposition 3.5, the sequence of transformations in question induces the loop of cluster transformations

$$\begin{aligned}
 & \Sigma_{123121} \xrightarrow{\beta_{234}} \Sigma_{123212} \xrightarrow{\beta_{134}} \Sigma_{132312} = \Sigma_{312132} \xrightarrow{\beta_{124}} \Sigma_{321232} \xrightarrow{\beta_{123}} \Sigma_{321323} \\
 (77) \quad & \parallel \hspace{15em} \parallel \\
 & \Sigma_{121321} \xleftarrow{\beta_{123}^{-1}} \Sigma_{212321} \xleftarrow{\beta_{124}^{-1}} \Sigma_{213231} = \Sigma_{231213} \xleftarrow{\beta_{134}^{-1}} \Sigma_{232123} \xleftarrow{\beta_{234}^{-1}} \Sigma_{323123}
 \end{aligned}$$

This loop of cluster transformation turns out to be trivial:

Proposition 3.7. *For the triangle, square and butterfly quivers assigned to the reduced word 123121 for the longest element of S_4 , the cluster transformation*

$$(78) \quad \beta_{123}^{-1} \circ \beta_{124}^{-1} \circ \beta_{134}^{-1} \circ \beta_{234}^{-1} \circ \beta_{123} \circ \beta_{124} \circ \beta_{134} \circ \beta_{234}$$

is trivial.

The proposition can be proved by Theorem 2.3 and straightforward calculations. The proof is given in Appendix A.

4. Solutions of the tetrahedron equation from trivial cluster transformations

Now we discuss the solutions of the tetrahedron equation arising from the three trivial cluster transformations obtained in section 3.

4.1. The trivial cluster transformations and the tetrahedron equation

The triviality of the cluster transformation (78) can be expressed as the equality

$$(79) \quad \beta_{123} \circ \beta_{124} \circ \beta_{134} \circ \beta_{234} = \beta_{234} \circ \beta_{134} \circ \beta_{124} \circ \beta_{123}$$

between two cluster transformations from the seed tori for $\Sigma_{123121} = \Sigma_{121321}$ to those for $\Sigma_{321323} = \Sigma_{323123}$. This equation takes the form of the tetrahedron equation.

By Proposition 2.4 the corresponding quantum cluster transformations for the same quivers also satisfy the tetrahedron equation

$$(80) \quad \beta_{234}^q \circ \beta_{134}^q \circ \beta_{124}^q \circ \beta_{123}^q = \beta_{123}^q \circ \beta_{124}^q \circ \beta_{134}^q \circ \beta_{234}^q.$$

Furthermore, Proposition 2.5 and the tetrahedron equation (79) imply that the unitary operator

$$(81) \quad R_{abc} = \mathbf{K}_{\beta_{abc}}$$

solves the tetrahedron equation (5). This is an equality between operators from the Hilbert space $\mathcal{H}_{\Sigma_{123121}}$ to the Hilbert space $\mathcal{H}_{\Sigma_{321323}}$.

Remark 4.1. For the triangle quiver, the tropical sign sequence $(\epsilon[t])_{t=1}^8$ is given by $(+, +, +, +, -, -, -, -)$ and one can use simpler quantum dilogarithm identities (Eqs. (1.8) and (1.10) of [19]) to construct solutions of the tetrahedron equation.

4.2. R-matrices and three-dimensional integrable lattice models

The R-matrix R_{123} may be regarded as the local Boltzmann weight of a three-dimensional statistical lattice model with continuous spin variables. Let us explain this point taking the square quiver case as an example.

Consider the cluster transformation $\beta_{123}: \Sigma_{121} \rightarrow \Sigma_{212}$ between the square quivers assigned to the reduced words 121 and 212 for the longest element of S_3 . The vertices of the square quivers are placed on the segments of wires, hence

$$(82) \quad I = \{1_1, 1_2, 1_3, 2_1, 2_2, 2_3, 3_1, 3_2, 3_3\}$$

is the set labeling the vertices of Σ_{121} and Σ_{212} . The quantum variables (X^q, B^q) assigned to Σ_{121} and (X'^q, B'^q) assigned to Σ_{212} have representations on the Hilbert spaces $\mathcal{H}_{\Sigma_{121}} = L^2(\mathbb{R}^I)$ and the Hilbert space $\mathcal{H}_{\Sigma_{212}} = L^2(\mathbb{R}^I)$, respectively. Let $a = (a_i)_{i \in I}$ and $a' = (a'_i)_{i \in I}$ be the standard coordinates for functions in $\mathcal{H}_{\Sigma_{121}}$ and $\mathcal{H}_{\Sigma_{212}}$. Then, the R-matrix $R_{123}: \mathcal{H}_{\Sigma_{212}} \rightarrow \mathcal{H}_{\Sigma_{121}}$ can be expressed as an integral operator:

$$(83) \quad (R_{123}f)(a) = \int_{\mathbb{R}^I} da' S_{123}(a, a')f(a').$$

Here $da' := \prod_{i \in I} da'_i$. Since $\mathbf{K}_{\beta_{123}}(a'_i) = a_i$ unless β_{123} contains the mutation μ_i or an automorphism acting nontrivially on i , the kernel $S_{123}(a, a')$ takes the form

$$(84) \quad S_{123}(a, a') = S \begin{bmatrix} a_{1_1} & a_{1_2} & a_{1_3} & a'_{1_2} \\ a_{2_1} & a_{2_2} & a_{2_3} & a'_{2_2} \\ a_{3_1} & a_{3_2} & a_{3_3} & a'_{3_2} \end{bmatrix} \prod_{i \in I \setminus \{1_2, 2_2, 3_2\}} \delta(a_i - a'_i),$$

where S is a function of the indicated variables and δ is the delta function.

In the lattice model, R_{123} resides at an intersection of three planes constituting part of a lattice, as shown in Figure 6. Each vertex of Σ_{121} and Σ_{212} corresponds to one quadrant of one of the three planes, separated from the other quadrants by the intersections with the other two planes. To construct a statistical lattice model, for each vertex $i \in I$ we place an \mathbb{R} -valued

spin variable a_i or a'_i on the corresponding region, depending on whether the vertex is from Σ_{121} or Σ_{212} , and identify $a_i = a'_i$ for $i \neq 1_2, 2_2, 3_2$. Given a configuration of the 12 spin variables thus prepared, we define the local “energy”¹ E for that configuration by $S = \exp(-E/k_B T)$, where k_B is the Boltzmann constant and T is the temperature.

The tetrahedron equation (5) translates to the equation

$$\begin{aligned}
 & \int_{\mathbb{R}^4} da'_{1_3} da'_{2_3} da'_{3_3} da'_{4_3} S \begin{bmatrix} a_{2_2} & a_{2_3} & a_{2_4} & a'_{2_3} \\ a_{3_2} & a_{3_3} & a_{3_4} & a'_{3_3} \\ a_{4_2} & a_{4_3} & a_{4_4} & a'_{4_3} \end{bmatrix} S \begin{bmatrix} a_{1_2} & a_{1_3} & a_{1_4} & a'_{1_3} \\ a_{3_1} & a_{3_2} & a'_{3_3} & a''_{3_2} \\ a_{4_1} & a_{4_2} & a'_{4_3} & a''_{4_2} \end{bmatrix} \\
 & \quad \times S \begin{bmatrix} a_{1_1} & a_{1_2} & a'_{1_3} & a''_{1_2} \\ a_{2_1} & a_{2_2} & a'_{2_3} & a''_{2_2} \\ a'_{4_2} & a'_{4_3} & a_{4_4} & a''_{4_3} \end{bmatrix} S \begin{bmatrix} a''_{1_2} & a'_{1_3} & a_{1_4} & a''_{1_3} \\ a''_{2_2} & a'_{2_3} & a_{2_4} & a''_{2_3} \\ a''_{3_2} & a'_{3_3} & a_{3_4} & a''_{3_3} \end{bmatrix} \\
 (85) \quad & = \int_{\mathbb{R}^4} da'_{1_2} da'_{2_2} da'_{3_2} da'_{4_2} S \begin{bmatrix} a_{1_1} & a_{1_2} & a_{1_3} & a'_{1_2} \\ a_{2_1} & a_{2_2} & a_{2_3} & a'_{2_2} \\ a_{3_1} & a_{3_2} & a_{3_3} & a'_{3_2} \end{bmatrix} S \begin{bmatrix} a'_{1_2} & a_{1_3} & a_{1_4} & a''_{1_3} \\ a'_{2_2} & a_{2_3} & a_{2_4} & a''_{2_3} \\ a_{4_1} & a_{4_2} & a_{4_3} & a'_{4_2} \end{bmatrix} \\
 & \quad \times S \begin{bmatrix} a_{1_1} & a'_{1_2} & a''_{1_3} & a''_{1_2} \\ a'_{3_2} & a_{3_3} & a_{3_4} & a''_{3_3} \\ a'_{4_2} & a_{4_3} & a_{4_4} & a''_{4_3} \end{bmatrix} S \begin{bmatrix} a_{2_1} & a'_{2_2} & a''_{2_3} & a''_{2_2} \\ a_{3_1} & a'_{3_2} & a'_{3_3} & a''_{3_2} \\ a_{4_1} & a'_{4_2} & a'_{4_3} & a''_{4_2} \end{bmatrix}.
 \end{aligned}$$

This is the form of the tetrahedron equation appropriate for the local Boltzmann weight of a three-dimensional lattice model with spin variables placed on the faces of the lattice [14, 21].

Remark 4.2. The Boltzmann weights for the lattice models corresponding to the triangle and butterfly quivers can be determined in the same fashion. The triangle quiver model has spin variables placed inside the regions bounded by the planes making up the cubic lattice, and the butterfly quiver model has spin variables in those regions as well as the edges of the lattice. The tetrahedron equation for the former can be found in [14, 21].

The fact that the local Boltzmann weight solves the tetrahedron equation is closely related to the integrability of the lattice model.

In Figure 6, let us make the direction common to the first and second planes periodic and replace the third plane with a stack of n parallel planes. The resulting configuration of planes can be thought of as defining a local Boltzmann weight W for a two-dimensional lattice model, which is obtained

¹The quantity E is not real in the present case and cannot be interpreted as a physical energy.

from the three-dimensional lattice model by “dimensional reduction” on the periodic direction. Schematically, we can write W as $W = \text{Tr}(S^n)$, where each factor of S represents one of the n parallel planes. The tetrahedron equation implies that W satisfies the Yang–Baxter equation.

For a solution of the Yang–Baxter equation to define a two-dimensional integrable lattice model, it must also depend on a continuous parameter, called a spectral parameter. A spectral parameter can be introduced to W by “twisting” of the periodic boundary condition. Let

$$(86) \quad q_{12'} := a'_{1_2} - a_{2_3}, \quad q_{12} := a_{1_1} - a_{2_2}.$$

We can regard $q_{12'}$ and q_{12} as “charges” assigned to the edges $12'$ and 12 of the lattice. As we will see shortly, these charges are conserved: the local Boltzmann weight vanishes unless $q_{12} = q_{12'}$. As a result, $W(z) = \text{Tr}(z^{q_{12}} S^n)$ defines a solution of the Yang–Baxter equation with spectral parameter z . The integrability of the corresponding two-dimensional lattice model implies the integrability of the original three-dimensional lattice model.

Remark 4.3. A similar dimensional reduction for the solution of [2, 21] was considered in [2], where it was shown that the resulting solution of the Yang–Baxter equation is a trigonometric R-matrix associated with the direct sum of symmetric tensor representations of \mathfrak{sl}_n . In [26], this reduction was interpreted in string theory as a duality transformation to a brane configuration studied in [6, 16].

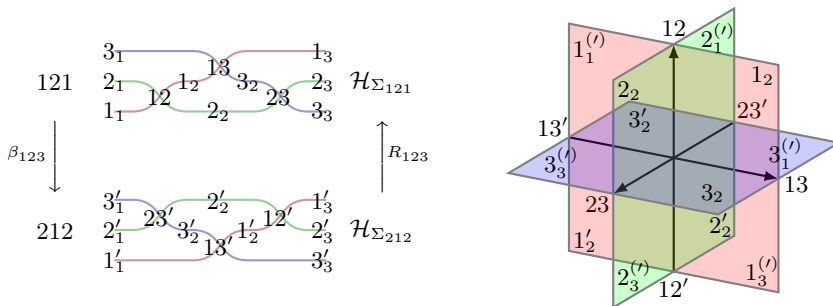


Figure 6: The braid move $\beta_{123}: \Sigma_{121} \rightarrow \Sigma_{212}$ and the corresponding three-dimensional lattice. The vertices i_a and i'_a are identified for $i = 1, 2, 3$ and $a = 1, 3$.

4.3. R-matrix for the square quiver

Let us calculate an explicit form of the function S for the square quiver. The R-matrix

$$(87) \quad R_{123} = \mathbf{K}_{\mu_{2_2}}^{\#(\epsilon[1])} \mathbf{K}'_{\mu_{2_2}}(\epsilon[1]) \mathbf{K}_{\mu_{3_2}}^{\#(\epsilon[2])} \mathbf{K}'_{\mu_{3_2}}(\epsilon[2]) \mathbf{K}_{\mu_{1_2}}^{\#(\epsilon[3])} \mathbf{K}'_{\mu_{1_2}}(\epsilon[3]) \mathbf{K}_{\mu_{2_2}}^{\#(\epsilon[4])} \mathbf{K}'_{\mu_{2_2}}(\epsilon[4]) \mathbf{K}_{\alpha_{1_2,3_2}}$$

is independent of the choice of signs $(\epsilon[1], \epsilon[2], \epsilon[3], \epsilon[4]) \in \{\pm\}^4$, and different choices lead to different expressions for R_{123} . Let us take them to be the tropical signs, which are all $+$. Then,

$$(88) \quad \begin{aligned} R_{123} &= \Phi^{\hbar}(\gamma[1] \cdot \hat{x}) \Phi^{\hbar}(\gamma[2] \cdot \hat{x}) \Phi^{\hbar}(\gamma[3] \cdot \hat{x}) \Phi^{\hbar}(\gamma[4] \cdot \hat{x}) \\ &\quad \cdot \Phi^{\hbar}(\gamma[1] \cdot \hat{\hat{x}})^{-1} \Phi^{\hbar}(\gamma[2] \cdot \hat{\hat{x}})^{-1} \Phi^{\hbar}(\gamma[3] \cdot \hat{\hat{x}})^{-1} \Phi^{\hbar}(\gamma[4] \cdot \hat{\hat{x}})^{-1} \\ &\quad \cdot \mathbf{K}'_{\mu_{2_2}}(+)\mathbf{K}'_{\mu_{3_2}}(+)\mathbf{K}'_{\mu_{1_2}}(+)\mathbf{K}'_{\mu_{2_2}}(+)\mathbf{K}_{\alpha_{1_2,3_2}} \end{aligned}$$

The transformations of the tropical \mathcal{X} -variables is listed in Table 1. According to the table we have

$$(89) \quad \gamma[1] \cdot \hat{x} = \hat{x}_{2_2}, \quad \gamma[2] \cdot \hat{x} = \hat{x}_{2_2} + \hat{x}_{3_2}, \quad \gamma[3] \cdot \hat{x} = \hat{x}_{1_2}, \quad \gamma[4] \cdot \hat{x} = \hat{x}_{3_2}.$$

Expressing the relation between $\hat{x}[1]$ and $\hat{x}[6]$ in the matrix form and taking the inverse transpose matrix, we find that $\mathbf{K}'_{\mu_{2_2}}(+)\mathbf{K}'_{\mu_{3_2}}(+)\mathbf{K}'_{\mu_{1_2}}(+)\mathbf{K}'_{\mu_{2_2}}(+)\mathbf{K}_{\alpha_{1_2,3_2}}$ maps

$$(90) \quad a'_{1_2} \mapsto a_{1_1} - a_{2_2} + a_{2_3}, \quad a'_{2_2} \mapsto a_{2_3} + a_{3_1} - a_{3_2}, \quad a'_{3_2} \mapsto a_{1_1} - a_{1_2} + a_{3_1}.$$

Thus we obtain

$$(91) \quad \begin{aligned} S \begin{bmatrix} a_{1_1} & a_{1_2} & a_{1_3} & a'_{1_2} \\ a_{2_1} & a_{2_2} & a_{2_3} & a'_{2_2} \\ a_{3_1} & a_{3_2} & a_{3_3} & a'_{3_2} \end{bmatrix} &= \frac{\Phi^{\hbar}(\hat{x}_{2_2})\Phi^{\hbar}(\hat{x}_{2_2} + \hat{x}_{3_2})\Phi^{\hbar}(\hat{x}_{1_2})\Phi^{\hbar}(\hat{x}_{3_2})}{\Phi^{\hbar}(\hat{\hat{x}}_{2_2})\Phi^{\hbar}(\hat{\hat{x}}_{2_2} + \hat{\hat{x}}_{3_2})\Phi^{\hbar}(\hat{\hat{x}}_{1_2})\Phi^{\hbar}(\hat{\hat{x}}_{3_2})} \\ &\quad \times \delta(a_{1_1} - a_{2_2} - a'_{1_2} + a_{2_3})\delta(a_{2_3} - a_{3_2} - a'_{2_2} + a_{3_1}) \\ &\quad \times \delta(a_{3_1} - a_{1_2} - a'_{3_2} + a_{1_1}). \end{aligned}$$

5. Cluster transformations and three-dimensional gauge theories

Terashima and Yamazaki [24] observed that the sequence of quantum dilogarithms (60) can be interpreted as the partition function of a three-dimensional

t	$X_{1_1}[t]$	$X_{1_2}[t]$	$X_{1_3}[t]$	$X_{2_1}[t]$	$X_{2_2}[t]$	$X_{2_3}[t]$	$X_{3_1}[t]$	$X_{3_2}[t]$	$X_{3_3}[t]$
1	X_{1_1}	X_{1_2}	X_{1_3}	X_{2_1}	X_{2_2}	X_{2_3}	X_{3_1}	X_{3_2}	X_{3_3}
2	$X_{1_1}X_{2_2}$	X_{1_2}	X_{1_3}	X_{2_1}	$X_{2_2}^{-1}$	X_{2_3}	X_{3_1}	$X_{2_2}X_{3_2}$	X_{3_3}
3	$X_{1_1}X_{2_2}$	X_{1_2}	X_{1_3}	X_{2_1}	X_{3_2}	$X_{2_2}X_{2_3}X_{3_2}$	X_{3_1}	$X_{2_2}^{-1}X_{3_2}^{-1}$	X_{3_3}
4	$X_{1_1}X_{1_2}X_{2_2}$	$X_{1_2}^{-1}$	X_{1_3}	X_{2_1}	X_{3_2}	$X_{2_2}X_{2_3}X_{3_2}$	$X_{1_2}X_{3_1}$	$X_{2_2}^{-1}X_{3_2}^{-1}$	X_{3_3}
5	$X_{1_1}X_{1_2}X_{2_2}$	$X_{1_2}^{-1}$	X_{1_3}	X_{2_1}	$X_{3_2}^{-1}$	$X_{2_2}X_{2_3}X_{3_2}$	$X_{1_2}X_{3_1}X_{3_2}$	$X_{2_2}^{-1}$	X_{3_3}
6	$X_{1_1}X_{1_2}X_{2_2}$	$X_{2_2}^{-1}$	X_{1_3}	X_{2_1}	$X_{3_2}^{-1}$	$X_{2_2}X_{2_3}X_{3_2}$	$X_{1_2}X_{3_1}X_{3_2}$	$X_{1_2}^{-1}$	X_{3_3}

Table 1: The transformation of the tropical \mathcal{X} -variables under the cluster transformation $\Sigma[1] \xrightarrow{\mu_{2_2}} \Sigma[2] \xrightarrow{\mu_{3_2}} \Sigma[3] \xrightarrow{\mu_{1_2}} \Sigma[4] \xrightarrow{\mu_{2_2}} \Sigma[5] \xrightarrow{\alpha_{1_2,3_2}} \Sigma[6]$ from the square quiver $\Sigma[1] := \Sigma_{121}$ to $\Sigma[6] := \Sigma_{212}$.

$\mathcal{N} = 2$ supersymmetric gauge theory formulated on the squashed three-sphere

$$(92) \quad S_b^3 := \{(z_1, z_2) \in \mathbb{C}^2 \mid b|z_1|^2 + b^{-1}|z_2|^2 = 1\},$$

where the squashing parameter $b \in \mathbb{R}_{>0}$ is related to the Planck constant \hbar as

$$(93) \quad \hbar = b^2.$$

In this section we rewrite the intertwiner $\mathbf{K}_{\mathbf{c}}: \mathcal{H}_{\Sigma'} \rightarrow \mathcal{H}_{\Sigma}$ for a cluster transformation $\mathbf{c}: \Sigma \rightarrow \Sigma'$ as the partition function of a three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory on S_b^3 . This result shows that the R-matrices obtained from trivial cluster transformations in section 4 can be identified with S_b^3 partition functions.

Let

$$(94) \quad \mathbf{c}: \Sigma =: \Sigma[1] \xrightarrow{\mu_{k[1]}} \Sigma[2] \xrightarrow{\mu_{k[2]}} \Sigma[3] \cdots \xrightarrow{\mu_{k[L]}} \Sigma[L+1] \xrightarrow{\alpha} \Sigma[L+2] := \Sigma'$$

be a decomposition of \mathbf{c} into L mutations and one automorphism. Following [19], for each Hilbert space $\mathcal{H}_{\Sigma[t]}$ we introduce a set of position and momentum operators $(\hat{a}_i[t], \hat{p}_i[t])_{i \in I}$, a basis of position eigenstates $\{|a[t]\}\}_{a \in \mathbb{R}^I}$ and a basis of momentum eigenstates $\{|p[t]\}\}_{p \in \mathbb{R}^I}$ such that

$$(95) \quad \hat{a}_i[t]|a[t]\rangle = a_i[t]|a[t]\rangle, \quad \langle a[t]|a'[t]\rangle = \delta(a[t] - a'[t]),$$

$$(96) \quad \hat{p}_i[t]|p[t]\rangle = p_i[t]|p[t]\rangle, \quad \langle p[t]|p'[t]\rangle = \delta(p[t] - p'[t])$$

and

$$(97) \quad \langle a[t]|p[t]\rangle = e^{2\pi i p[t] \cdot a[t]}, \quad \langle a[t]|\hat{p}_i[t]|f\rangle = -\frac{i}{2\pi} \frac{\partial}{\partial a_i[t]} \langle a[t]|f\rangle, \quad |f\rangle \in \mathcal{H}_{\Sigma[t]}.$$

The completeness relations read

$$(98) \quad \int_{\mathbb{R}^I} da[t]|a[t]\rangle\langle a[t]| = \int_{\mathbb{R}^I} dp[t]|p[t]\rangle\langle p[t]| = 1.$$

Inserting the completeness relations for the position eigenstates into the expression $\mathbf{K}_c = \mathbf{K}_{\mu_{k[1]}} \mathbf{K}_{\mu_{k[2]}} \cdots \mathbf{K}_{\mu_{k[L]}} \mathbf{K}_\alpha$, we can write the matrix elements of \mathbf{K}_c as

$$(99) \quad \begin{aligned} & \langle a[1]|\mathbf{K}_c|a[L+2]\rangle \\ &= \int \left(\prod_{t=1}^L da[t+1] \langle a[t]|\mathbf{K}_{\mu_{k[t]}}|a[t+1]\rangle \right) \langle a[L+1]|\mathbf{K}_\alpha|a[L+2]\rangle. \end{aligned}$$

The last factor is a product of delta functions:

$$(100) \quad \langle a[L+1]|\mathbf{K}_\alpha|a[L+2]\rangle = \prod_{i \in I} \delta(a_i[L+1] - a_{\alpha(i)}[L+2]).$$

Let us calculate the remaining factors.

Using the completeness relations we can write

$$(101) \quad \begin{aligned} & \langle a[t]|\mathbf{K}_{\mu_{k[t]}}|a[t+1]\rangle \\ &= \int dp[t] d\tilde{a}[t] \langle a[t]|\Phi^{\hbar}(\epsilon[t]\hat{x}_{k[t]}[t])^{\epsilon[t]}|p[t]\rangle \langle p[t]|\Phi^{\hbar}(\epsilon[t]\hat{\tilde{x}}_{k[t]}[t])^{-\epsilon[t]}|\tilde{a}[t]\rangle \\ & \quad \times \langle \tilde{a}[t]|\mathbf{K}'_{\mu_{k[t]}}(\epsilon[t])|a[t+1]\rangle \\ &= \int dp[t] d\tilde{a}[t] \frac{\Phi^{\hbar}(\epsilon[t](-2\pi^2 b^2 p_{k[t]}[t] - \sum_{j \in I} \epsilon_{k[t]j}[t] a_j[t]))^{\epsilon[t]}}{\Phi^{\hbar}(\epsilon[t](-2\pi^2 b^2 p_{k[t]}[t] + \sum_{j \in I} \epsilon_{k[t]j}[t] \tilde{a}_j[t]))^{\epsilon[t]}} \\ & \quad \times e^{2\pi i p[t] \cdot (a[t] - \tilde{a}[t])} \prod_{i \in I \setminus \{k[t]\}} \delta(\tilde{a}_i[t] - a_i[t+1]) \\ & \quad \times \delta\left(-\tilde{a}_{k[t]}[t] + \sum_{j \in I} [-\epsilon[t] \epsilon_{k[t]j}]_+ \tilde{a}_j[t] - a_{k[t]}[t+1]\right), \end{aligned}$$

where we have chosen a sign $\epsilon[t] \in \{\pm\}$. (Note that $\hat{p}_{k[t]}$ and $\sum_{j \in I} \epsilon_{k[t]j}[t] \hat{a}_j$ commute.) Integrating over $p_i[t]$ for $i \neq k[t]$ yields $\delta(a_i[t] - \tilde{a}_i[t])$. Then, integrating over $\tilde{a}[t]$ gives

$$(102) \quad \langle a[t] | \mathbf{K}_{\mu_{k[t]}} | a[t+1] \rangle = \frac{1}{\pi b} \int dx[t] \left(\frac{\Phi^{\hbar}(2\pi b(x[t] - \epsilon[t]u[t]))}{\Phi^{\hbar}(2\pi b(x[t] + \epsilon[t]u[t]))} \right)^{\epsilon[t]} \times e^{2\pi i w[t]x[t]} \prod_{i \in I \setminus \{k[t]\}} \delta(a_i[t] - a_i[t+1]),$$

where

$$(103) \quad x[t] := -\pi b \epsilon[t] p_{k[t]}[t],$$

$$(104) \quad u[t] := \frac{1}{2\pi b} \sum_{j \in I} \epsilon_{k[t]j}[t] a_j[t],$$

$$(105) \quad w[t] := -\frac{\epsilon[t]}{\pi b} \left(a_{k[t]}[t] + a_{k[t]}[t+1] - \sum_{j \in I} [-\epsilon[t] \epsilon_{k[t]j}[t]]_+ a_j[t] \right).$$

The integration over $x[t]$ can be performed with the formula [10]²

$$(106) \quad \int_{\mathbb{R}} \frac{\Phi^{\hbar}(2\pi b(x-u))}{\Phi^{\hbar}(2\pi b(x+u))} e^{2\pi i w x} dx = s_b \left(2u - \frac{i}{2} Q \right) s_b \left(w + \frac{i}{2} Q \right) s_b \left(-2u - w + \frac{i}{2} Q \right).$$

Here

$$(107) \quad Q = b + b^{-1}$$

and

$$(108) \quad s_b(z) := e^{-\pi i z^2 / 2 + \pi i (2 - Q^2) / 24} \Phi^{b^2} (2\pi b z)^{-1} = s_b(-z)^{-1}.$$

²This formula is valid for $\text{Im}(-u + iQ/2) > 0$ and $\text{Im}(-2u) < \text{Im}(w) < 0$. To satisfy these conditions we can give $a_j[t]$ an imaginary part $i\epsilon_{k[t]j}[t]c[t] + i\delta_{kj}d[t]$ with some constants $c[t], d[t]$. In the corresponding three-dimensional gauge theory this operation amounts to shifting the R-charges of the three kinds of chiral multiplets to positive values that sum to 2.

We obtain

$$\begin{aligned}
 & \langle a[t] | \mathbf{K}_{\mu_{k[t]}} | a[t+1] \rangle \\
 &= \frac{1}{\pi b} Z_{\mu_{k[t]}}(\sigma_{k[t]}[t], \sigma_{k[t]}[t+1]; (\sigma_i[t])_{i \in I \setminus \{k[t]\}}) \\
 (109) \quad & \times \prod_{i \in I \setminus \{k[t]\}} \delta(a_i[t] - a_i[t+1]),
 \end{aligned}$$

with

$$(110) \quad \sigma_i[t] := \frac{a_i[t]}{\pi b}$$

and

$$\begin{aligned}
 & Z_{\mu_{k[t]}}(\sigma_{k[t]}[t], \sigma_{k[t]}[t+1]; (\sigma_i[t])_{i \in I \setminus \{k[t]\}}) \\
 &:= s_b \left(\sum_{j \in I} \varepsilon_{k[t]j}[t] \sigma_j[t] - \frac{i}{2} Q \right) \\
 & \quad \times s_b \left(\sigma_{k[t]}[t] + \sigma_{k[t]}[t+1] - \sum_{j \in I} [\varepsilon_{k[t]j}[t]]_+ \sigma_j[t] + \frac{i}{2} Q \right) \\
 (111) \quad & \quad \times s_b \left(-\sigma_{k[t]}[t] - \sigma_{k[t]}[t+1] + \sum_{j \in I} [-\varepsilon_{k[t]j}[t]]_+ \sigma_j[t] + \frac{i}{2} Q \right).
 \end{aligned}$$

This is independent of the choice of the sign $\epsilon[t]$, as claimed before.

From this calculation we find that the matrix elements of $\mathbf{K}_{\mathbf{c}}$ are given by

$$\begin{aligned}
 & \langle a[1] | \mathbf{K}_{\mathbf{c}} | a[L+2] \rangle \\
 &= \int \left(\prod_{t=1}^L d\sigma_{k[t]}[t+1] Z_{\mu_{k[t]}}(\sigma_{k[t]}[t], \sigma_{k[t]}[t+1]; (\sigma_i[t])_{i \in I \setminus \{k[t]\}}) \right) \\
 (112) \quad & \times \prod_{i \in I} \delta(a_i[L+1] - a_{\alpha(i)}[L+2]),
 \end{aligned}$$

where the variables are understood to satisfy the relation

$$(113) \quad \sigma_i[t] = \sigma_i[t+1], \quad i \neq k[t], \quad t = 1, 2, \dots, L.$$

Let

$$(114) \quad K := \bigcup_{t=1}^L \{k[t]\}$$

and write the product of delta functions in the above expression as

$$(115) \quad \prod_{i \in I} \delta(a_i[L+1] - a_{\alpha(i)}[L+2]) \\ = \prod_{k \in K} \delta(a_k[L+1] - a_{\alpha(k)}[L+2]) \prod_{l \in I \setminus K} \delta(a_l[L+1] - a_{\alpha(l)}[L+2]).$$

For each $k \in K$, let t_k be the largest t such that $k = k[t]$. The integration over $\sigma_k[t_k + 1]$ sets

$$(116) \quad \sigma_k[t_k + 1] = \sigma_{\alpha(k)}[L+2], \quad k \in K.$$

By the relations (113) and (116), each variable $a_i[t]$ is now equal to either $a_i[1]$, $a_{\alpha(i)}[L+2]$ or one of the remaining integration variables. Finally, we arrive at the expression

$$(117) \quad \langle a[1] | \mathbf{K}_c | a[L+2] \rangle \\ = \int \left(\prod_{s \in \{1, 2, \dots, L\} \setminus \bigcup_{k \in K} \{t_k\}} d\sigma_{k[s]}[s+1] \right. \\ \left. \times \prod_{t=1}^L Z_{\mu_{k[t]}}(\sigma_{k[t]}[t], \sigma_{k[t]}[t+1]; (\sigma_i[t])_{i \in I \setminus \{k[t]\}}) \right) \\ \times (\pi b)^{-|K|} \prod_{l \in I \setminus K} \delta(a_l[1] - a_{\alpha(l)}[L+2]).$$

The first line of the right-hand side of the matrix element (117) coincides with the integral formula for the partition function of a three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory on S_b^3 [13]. This theory has

- abelian symmetry groups $U(1)_{\sigma_i[t]}$, $i \in I$, $t \in \{1, 2, \dots, L+2\}$, with the identification (113) and (116), among which $U(1)_{\sigma_{k[s]}[s+1]}$, $s \in \{1, 2, \dots, L\} \setminus \bigcup_{k \in K} \{t_k\}$, are gauged;
- a vector multiplet for each gauge group $U(1)_{\sigma_{k[s]}[s+1]}$;

- a background (i.e. nondynamical) vector multiplet for each global symmetry group $U(1)_{\sigma_i[t]}$ whose real scalar component is $\sigma_i[t]$;
- a chiral multiplet for each $t \in \{1, 2, \dots, L\}$ with $U(1)_{\sigma_j[t]}$ -charge $\varepsilon_{k[t]j}[t]$ and R-charge 2;
- a chiral multiplet for each $t \in \{1, 2, \dots, L\}$ with $U(1)_{\sigma_{k[t]}[t]}$ -charge +1, $U(1)_{\sigma_{k[t]}[t+1]}$ -charge +1, $U(1)_{\sigma_j[t]}$ -charge $-\varepsilon_{k[t]j}[t]_+$ for $j \neq k[t]$ and R-charge 0;
- a chiral multiplet for each $t \in \{1, 2, \dots, L\}$ with $U(1)_{\sigma_{k[t]}[t]}$ -charge -1, $U(1)_{\sigma_{k[t]}[t+1]}$ -charge -1, $U(1)_{\sigma_j[t]}$ -charge $[-\varepsilon_{k[t]j}[t]]_+$ for $j \neq k[t]$ and R-charge 0; and
- zero Chern–Simons levels and zero Fayet–Iliopoulos parameters.

The field content is almost that of an $\mathcal{N} = 4$ supersymmetric gauge theory and is compatible with the cubic superpotential required for such a theory, but the $U(1)_{\sigma_j[t]}$ -charge assignment does not allow the fields to form $\mathcal{N} = 4$ supermultiplets.

Acknowledgments. We are grateful to Hyun Kyu Kim and Mauricio Romo for illuminating discussions and to Dylan Allegretti for helpful comments.

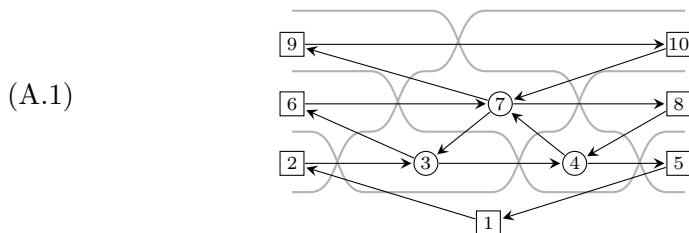
Appendix A. Proof of Proposition 3.7

In this appendix we give a proof of Proposition 3.7. For the ease of presentation we will employ different naming conventions for the vertices of quivers.

The proof is based on calculations. For each of the triangle, square and butterfly quivers assigned to Σ_{123121} , we will decompose the cluster transformation (78) into a sequence of mutations followed by automorphisms. Then, for each step of the decomposition, we will list the tropical \mathcal{X} -variables that are transformed nontrivially and their relations to the initial tropical \mathcal{X} -variables $(X_i)_{i \in I}$. We will find that if X_i is ever transformed, then the last entry in which a variable $X_i[t]$ of any t appears (emphasized in bold letters) is always a relation $X_i[t] = X_i$. Therefore, the cluster transformation acts on the tropical \mathcal{X} -variables trivially, and Proposition 3.7 follows from Theorem 2.3.

A.1. Triangle quiver

There are 10 connected vertices in the triangle quivers assigned to reduced words for the longest element of S_4 . For the triangle quivers assigned to 123121, we label the vertices as



The cluster transformation (78) has a decomposition

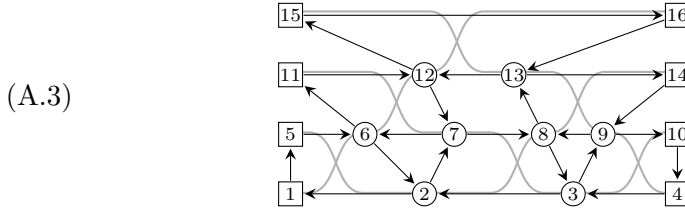
$$(A.2) \quad \beta_{123}^{-1} \circ \beta_{124}^{-1} \circ \beta_{134}^{-1} \circ \beta_{234}^{-1} \circ \beta_{123} \circ \beta_{124} \circ \beta_{134} \circ \beta_{234} \\ = \alpha_{4,7} \circ \alpha_{3,7} \circ \mu_7 \circ \mu_4 \circ \mu_3 \circ \mu_7 \circ \mu_4 \circ \mu_3 \circ \mu_7 \circ \mu_4 .$$

Under this sequence of mutations and automorphisms the tropical \mathcal{X} -variables transform as follows:

$\mu_4: \mathbf{X}_4[2] = 1/\mathbf{X}_4$ $\mathbf{X}_5[2] = \mathbf{X}_4\mathbf{X}_5$ $\mathbf{X}_7[2] = \mathbf{X}_4\mathbf{X}_7$	$\mu_3: \mathbf{X}_3[7] = \mathbf{X}_3\mathbf{X}_4$ $\mathbf{X}_5[7] = \mathbf{X}_5$ $\mathbf{X}_7[7] = 1/\mathbf{X}_3$
$\mu_7: \mathbf{X}_4[3] = \mathbf{X}_7$ $\mathbf{X}_7[3] = 1/\mathbf{X}_4\mathbf{X}_7$ $\mathbf{X}_9[3] = \mathbf{X}_4\mathbf{X}_7\mathbf{X}_9$	$\mu_4: \mathbf{X}_4[8] = \mathbf{X}_7$ $\mathbf{X}_8[8] = \mathbf{X}_8$ $\mathbf{X}_9[8] = \mathbf{X}_9$
$\mu_3: \mathbf{X}_3[4] = 1/\mathbf{X}_3$ $\mathbf{X}_5[4] = \mathbf{X}_3\mathbf{X}_4\mathbf{X}_5$ $\mathbf{X}_6[4] = \mathbf{X}_3\mathbf{X}_6$	$\mu_7: \mathbf{X}_3[9] = \mathbf{X}_4$ $\mathbf{X}_6[9] = \mathbf{X}_6$ $\mathbf{X}_7[9] = \mathbf{X}_3$
$\mu_4: \mathbf{X}_4[5] = 1/\mathbf{X}_7$ $\mathbf{X}_7[5] = 1/\mathbf{X}_4$ $\mathbf{X}_8[5] = \mathbf{X}_7\mathbf{X}_8$	$\alpha_{3,7}: \mathbf{X}_3[10] = \mathbf{X}_3$ $\mathbf{X}_7[11] = \mathbf{X}_4$
$\mu_7: \mathbf{X}_3[6] = 1/\mathbf{X}_3\mathbf{X}_4$ $\mathbf{X}_7[6] = \mathbf{X}_4$ $\mathbf{X}_9[6] = \mathbf{X}_7\mathbf{X}_9$	$\alpha_{4,7}: \mathbf{X}_4[11] = \mathbf{X}_4$ $\mathbf{X}_7[11] = \mathbf{X}_7$

A.2. Square quiver

We label the 16 connected vertices of the square quiver assigned to the reduced word 123121 for the longest element of S_4 as



Under the cluster transformation

$$\begin{aligned}
 & \beta_{123}^{-1} \circ \beta_{124}^{-1} \circ \beta_{134}^{-1} \circ \beta_{234}^{-1} \circ \beta_{123} \circ \beta_{124} \circ \beta_{134} \circ \beta_{234} \\
 &= \alpha_{8,9} \circ \alpha_{6,7} \circ \alpha_{3,13} \circ \alpha_{2,12} \circ \mu_{12} \circ \mu_6 \circ \mu_7 \circ \mu_{12} \circ \mu_9 \circ \mu_3 \circ \mu_2 \circ \mu_9 \\
 & \quad \circ \mu_{13} \circ \mu_8 \circ \mu_6 \circ \mu_{13} \circ \mu_7 \circ \mu_2 \circ \mu_{12} \circ \mu_7 \circ \mu_9 \circ \mu_{13} \circ \mu_3 \circ \mu_9 \\
 \text{(A.4)} \quad & \quad \circ \mu_2 \circ \mu_6 \circ \mu_8 \circ \mu_2 \circ \mu_7 \circ \mu_{12} \circ \mu_{13} \circ \mu_7 \circ \mu_3 \circ \mu_8 \circ \mu_9 \circ \mu_3
 \end{aligned}$$

the tropical \mathcal{X} -variables transform as

$ \begin{aligned} \mu_3: \quad & \mathcal{X}_2[2] = \mathcal{X}_2\mathcal{X}_3 \\ & \mathcal{X}_3[2] = 1/\mathcal{X}_3 \\ & \mathcal{X}_9[2] = \mathcal{X}_3\mathcal{X}_9 \\ \\ \mu_9: \quad & \mathcal{X}_3[3] = \mathcal{X}_9 \\ & \mathcal{X}_9[3] = 1/\mathcal{X}_3\mathcal{X}_9 \\ & \mathcal{X}_{10}[3] = \mathcal{X}_3\mathcal{X}_9\mathcal{X}_{10} \\ \\ \mu_8: \quad & \mathcal{X}_2[4] = \mathcal{X}_2\mathcal{X}_3\mathcal{X}_8 \\ & \mathcal{X}_8[4] = 1/\mathcal{X}_8 \\ & \mathcal{X}_{13}[4] = \mathcal{X}_8\mathcal{X}_{13} \\ \\ \mu_3: \quad & \mathcal{X}_3[5] = 1/\mathcal{X}_9 \\ & \mathcal{X}_9[5] = 1/\mathcal{X}_3 \\ & \mathcal{X}_{13}[5] = \mathcal{X}_8\mathcal{X}_9\mathcal{X}_{13} \\ \\ \mu_7: \quad & \mathcal{X}_6[6] = \mathcal{X}_6\mathcal{X}_7 \\ & \mathcal{X}_7[6] = 1/\mathcal{X}_7 \\ & \mathcal{X}_{13}[6] = \mathcal{X}_7\mathcal{X}_8\mathcal{X}_9\mathcal{X}_{13} \end{aligned} $	$ \begin{aligned} \mu_{13}: \quad & \mathcal{X}_3[7] = \mathcal{X}_7\mathcal{X}_8\mathcal{X}_{13} \\ & \mathcal{X}_7[7] = \mathcal{X}_8\mathcal{X}_9\mathcal{X}_{13} \\ & \mathcal{X}_{13}[7] = 1/\mathcal{X}_7\mathcal{X}_8\mathcal{X}_9\mathcal{X}_{13} \\ \\ \mu_{12}: \quad & \mathcal{X}_6[8] = \mathcal{X}_6\mathcal{X}_7\mathcal{X}_{12} \\ & \mathcal{X}_{12}[8] = 1/\mathcal{X}_{12} \\ & \mathcal{X}_{15}[8] = \mathcal{X}_{12}\mathcal{X}_{15} \\ \\ \mu_7: \quad & \mathcal{X}_7[9] = 1/\mathcal{X}_8\mathcal{X}_9\mathcal{X}_{13} \\ & \mathcal{X}_{13}[9] = 1/\mathcal{X}_7 \\ & \mathcal{X}_{15}[9] = \mathcal{X}_8\mathcal{X}_9\mathcal{X}_{12}\mathcal{X}_{13}\mathcal{X}_{15} \\ \\ \mu_2: \quad & \mathcal{X}_1[10] = \mathcal{X}_1\mathcal{X}_2\mathcal{X}_3\mathcal{X}_8 \\ & \mathcal{X}_2[10] = 1/\mathcal{X}_2\mathcal{X}_3\mathcal{X}_8 \\ & \mathcal{X}_8[10] = \mathcal{X}_2\mathcal{X}_3 \\ \\ \mu_8: \quad & \mathcal{X}_2[11] = 1/\mathcal{X}_8 \\ & \mathcal{X}_8[11] = 1/\mathcal{X}_2\mathcal{X}_3 \\ & \mathcal{X}_9[11] = \mathcal{X}_2 \end{aligned} $
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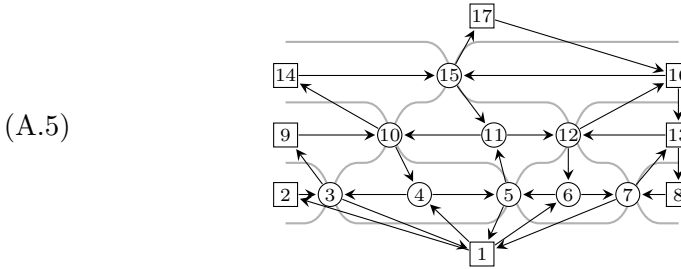
$$\begin{array}{ll}
\mu_6: X_1[12] = X_1X_2X_3X_6X_7X_8X_{12} & \mu_6: X_1[23] = X_1X_2X_3X_6 \\
X_6[12] = 1/X_6X_7X_{12} & X_2[23] = 1/X_{12} \\
X_{12}[12] = X_6X_7 & X_6[23] = X_7X_8X_{12} \\
\\
\mu_2: X_2[13] = X_8 & \mu_8: X_8[24] = X_3X_9 \\
X_6[13] = 1/X_6X_7X_8X_{12} & \mathbf{X}_{10}[24] = \mathbf{X}_{10} \\
X_{13}[13] = 1/X_7X_8 & X_{13}[24] = 1/X_3 \\
\\
\mu_9: X_3[14] = X_2X_7X_8X_{13} & \mu_{13}: X_1[25] = X_1X_2X_6 \\
X_8[14] = 1/X_3 & X_8[25] = X_9 \\
X_9[14] = 1/X_2 & X_{13}[25] = X_3 \\
\\
\mu_3: X_3[15] = 1/X_2X_7X_8X_{13} & \mu_9: X_3[26] = 1/X_2X_7X_8X_{13} \\
X_9[15] = X_7X_8X_{13} & X_9[26] = X_{13} \\
X_{14}[15] = X_2X_7X_8X_{13}X_{14} & X_{15}[26] = X_{12}X_{15} \\
\\
\mu_{13}: X_2[16] = 1/X_7 & \mu_2: X_2[27] = X_{12} \\
X_9[16] = X_{13} & X_6[27] = X_7X_8 \\
X_{13}[16] = X_7X_8 & \mathbf{X}_{15}[27] = \mathbf{X}_{15} \\
\\
\mu_9: X_3[17] = 1/X_2X_7X_8 & \mu_3: X_3[28] = X_2X_7X_8X_{13} \\
X_7[17] = 1/X_8X_9 & X_9[28] = 1/X_2X_7X_8 \\
X_9[17] = 1/X_{13} & \mathbf{X}_{14}[28] = \mathbf{X}_{14} \\
\\
\mu_7: X_2[18] = 1/X_7X_8X_9 & \mu_9: X_3[29] = X_{13} \\
X_7[18] = X_8X_9 & X_6[29] = 1/X_2 \\
X_{15}[18] = X_{12}X_{13}X_{15} & X_9[29] = X_2X_7X_8 \\
\\
\mu_{12}: X_7[19] = X_6X_7X_8X_9 & \mu_{12}: X_6[30] = 1/X_2X_7 \\
X_{11}[19] = X_6X_7X_{11} & X_{11}[30] = X_6X_{11} \\
X_{12}[19] = 1/X_6X_7 & X_{12}[30] = X_7 \\
\\
\mu_2: X_2[20] = X_7X_8X_9 & \mu_7: X_1[31] = X_1X_2 \\
X_7[20] = X_6 & X_7[31] = X_6 \\
X_{13}[20] = 1/X_9 & \mathbf{X}_{11}[31] = \mathbf{X}_{11} \\
\\
\mu_7: X_6[21] = 1/X_7X_8X_{12} & \mu_6: X_6[32] = X_2X_7 \\
X_7[21] = 1/X_6 & X_9[32] = X_8 \\
X_{12}[21] = 1/X_7 & X_{12}[32] = 1/X_2 \\
\\
\mu_{13}: X_2[22] = X_7X_8 & \mu_{12}: \mathbf{X}_1[33] = \mathbf{X}_1 \\
X_8[22] = 1/X_3X_9 & X_6[33] = X_7 \\
X_{13}[22] = X_9 & X_{12}[33] = X_2 \\
\\
\alpha_{2,12}: \mathbf{X}_2[34] = \mathbf{X}_2 & \\
\mathbf{X}_{12}[34] = \mathbf{X}_{12} &
\end{array}$$

$$\alpha_{3,13}: \begin{aligned} \mathbf{X}_3[35] &= \mathbf{X}_3 \\ \mathbf{X}_{13}[35] &= \mathbf{X}_{13} \end{aligned} \qquad \alpha_{8,9}: \begin{aligned} \mathbf{X}_8[37] &= \mathbf{X}_8 \\ \mathbf{X}_9[37] &= \mathbf{X}_9 \end{aligned}$$

$$\alpha_{6,7}: \begin{aligned} \mathbf{X}_6[36] &= \mathbf{X}_6 \\ \mathbf{X}_7[36] &= \mathbf{X}_7 \end{aligned}$$

A.3. Butterfly quiver

For the butterfly quiver assigned to 123121, we label the vertices as



Under the cluster transformation

$$\begin{aligned} & \beta_{123}^{-1} \circ \beta_{124}^{-1} \circ \beta_{134}^{-1} \circ \beta_{234}^{-1} \circ \beta_{123} \circ \beta_{124} \circ \beta_{134} \circ \beta_{234} \\ &= \alpha_{7,12} \circ \alpha_{5,15} \alpha_{4,6} \circ \alpha_{3,10} \circ \mu_6 \circ \mu_{10} \circ \mu_{15} \circ \mu_3 \circ \mu_{11} \circ \mu_{15} \circ \mu_7 \circ \mu_5 \\ & \quad \circ \mu_4 \circ \mu_6 \circ \mu_{12} \circ \mu_{11} \circ \mu_3 \circ \mu_{10} \circ \mu_6 \circ \mu_{15} \circ \mu_5 \circ \mu_7 \circ \mu_6 \circ \mu_{12} \\ (A.6) \quad & \quad \circ \mu_{11} \circ \mu_6 \circ \mu_3 \circ \mu_4 \circ \mu_{15} \circ \mu_5 \circ \mu_{10} \circ \mu_{11} \circ \mu_{12} \circ \mu_7 \circ \mu_5 \circ \mu_6 \end{aligned}$$

the tropical \mathcal{X} -variables transform as

$$\begin{aligned} \mu_6: \quad & \mathbf{X}_5[2] = \mathbf{X}_5 \mathbf{X}_6 \\ & \mathbf{X}_6[2] = 1/\mathbf{X}_6 \\ & \mathbf{X}_7[2] = \mathbf{X}_6 \mathbf{X}_7 \\ \mu_{11}: \quad & \mathbf{X}_5[6] = \mathbf{X}_{11} \\ & \mathbf{X}_{10}[6] = \mathbf{X}_5 \mathbf{X}_6 \mathbf{X}_{10} \mathbf{X}_{11} \\ & \mathbf{X}_{11}[6] = 1/\mathbf{X}_5 \mathbf{X}_6 \mathbf{X}_{11} \\ \mu_5: \quad & \mathbf{X}_5[3] = 1/\mathbf{X}_5 \mathbf{X}_6 \\ & \mathbf{X}_6[3] = \mathbf{X}_5 \\ & \mathbf{X}_{11}[3] = \mathbf{X}_5 \mathbf{X}_6 \mathbf{X}_{11} \\ \mu_{10}: \quad & \mathbf{X}_{10}[7] = 1/\mathbf{X}_5 \mathbf{X}_6 \mathbf{X}_{10} \mathbf{X}_{11} \\ & \mathbf{X}_{11}[7] = \mathbf{X}_{10} \\ & \mathbf{X}_{14}[7] = \mathbf{X}_5 \mathbf{X}_6 \mathbf{X}_{10} \mathbf{X}_{11} \mathbf{X}_{14} \\ \mu_7: \quad & \mathbf{X}_6[4] = \mathbf{X}_5 \mathbf{X}_6 \mathbf{X}_7 \\ & \mathbf{X}_7[4] = 1/\mathbf{X}_6 \mathbf{X}_7 \\ & \mathbf{X}_{13}[4] = \mathbf{X}_6 \mathbf{X}_7 \mathbf{X}_{13} \\ \mu_5: \quad & \mathbf{X}_5[8] = 1/\mathbf{X}_{11} \\ & \mathbf{X}_{11}[8] = \mathbf{X}_{10} \mathbf{X}_{11} \\ & \mathbf{X}_{16}[8] = \mathbf{X}_{11} \mathbf{X}_{12} \mathbf{X}_{16} \\ \mu_{12}: \quad & \mathbf{X}_6[5] = \mathbf{X}_5 \mathbf{X}_6 \mathbf{X}_7 \mathbf{X}_{12} \\ & \mathbf{X}_{12}[5] = 1/\mathbf{X}_{12} \\ & \mathbf{X}_{16}[5] = \mathbf{X}_{12} \mathbf{X}_{16} \\ \mu_{15}: \quad & \mathbf{X}_{11}[9] = \mathbf{X}_{10} \mathbf{X}_{11} \mathbf{X}_{15} \\ & \mathbf{X}_{15}[9] = 1/\mathbf{X}_{15} \\ & \mathbf{X}_{17}[9] = \mathbf{X}_{15} \mathbf{X}_{17} \end{aligned}$$

μ_4 : $X_3[10] = X_3X_4$	μ_3 : $X_3[21] = X_3X_4X_5X_6$
$X_4[10] = 1/X_4$	$X_6[21] = 1/X_3X_4X_5X_6X_{10}$
$X_6[10] = X_4X_5X_6X_7X_{12}$	$X_{10}[21] = 1/X_3X_4$
μ_3 : $X_3[11] = 1/X_3X_4$	μ_{11} : $X_4[22] = X_3X_4X_5X_6X_7X_{10}$
$X_4[11] = X_3$	$X_5[22] = 1/X_{15}$
$X_9[11] = X_3X_4X_9$	$X_{11}[22] = X_{11}X_{12}X_{15}$
μ_6 : $X_4[12] = X_3X_4X_5X_6X_7X_{12}$	μ_{12} : $X_4[23] = X_3X_4X_5X_{10}$
$X_6[12] = 1/X_4X_5X_6X_7X_{12}$	$X_{12}[23] = X_6X_7$
$X_{12}[12] = X_4X_5X_6X_7$	$X_{13}[23] = X_{13}$
μ_{11} : $X_4[13] = X_3X_4X_5X_6X_7X_{10}X_{11}X_{12}X_{13}$	μ_6 : $X_3[24] = 1/X_{10}$
$X_{11}[13] = 1/X_{10}X_{11}X_{15}$	$X_4[24] = 1/X_6$
$X_{15}[13] = X_{10}X_{11}$	$X_6[24] = X_3X_4X_5X_6X_{10}$
μ_{12} : $X_6[14] = 1/X_{12}$	μ_4 : $X_4[25] = X_6$
$X_7[14] = X_4X_5$	$X_6[25] = X_3X_4X_5X_{10}$
$X_{12}[14] = 1/X_4X_5X_6X_7$	$X_{12}[25] = X_7$
μ_6 : $X_5[15] = 1/X_{11}X_{12}$	μ_5 : $X_5[26] = X_{15}$
$X_6[15] = X_{12}$	$X_{11}[26] = X_{11}X_{12}$
$X_{11}[15] = 1/X_{10}X_{11}X_{12}X_{15}$	$X_{17}[26] = X_{17}$
μ_7 : $X_7[16] = 1/X_4X_5$	μ_7 : $X_7[27] = X_4X_5X_{11}X_{12}$
$X_{12}[16] = 1/X_6X_7$	$X_{11}[27] = 1/X_4X_5$
$X_{16}[16] = X_4X_5X_{11}X_{12}X_{16}$	$X_{16}[27] = X_{16}$
μ_5 : $X_5[17] = X_{11}X_{12}$	μ_{15} : $X_{11}[28] = 1/X_4X_5X_{11}$
$X_6[17] = 1/X_{11}$	$X_{14}[28] = X_{10}X_{14}$
$X_7[17] = 1/X_4X_5X_{11}X_{12}$	$X_{15}[28] = X_{11}$
μ_{15} : $X_6[18] = X_{10}$	μ_{11} : $X_7[29] = X_{12}$
$X_{10}[18] = 1/X_5X_6$	$X_{11}[29] = X_4X_5X_{11}$
$X_{15}[18] = 1/X_{10}X_{11}$	$X_{15}[29] = 1/X_4X_5$
μ_6 : $X_6[19] = 1/X_{10}$	μ_3 : $X_3[30] = X_{10}$
$X_{11}[19] = 1/X_{11}X_{12}X_{15}$	$X_6[30] = X_3X_4X_5$
$X_{15}[19] = 1/X_{11}$	$X_{14}[30] = X_{14}$
μ_{10} : $X_3[20] = 1/X_3X_4X_5X_6$	μ_{15} : $X_6[31] = X_3$
$X_{10}[20] = X_5X_6$	$X_{11}[31] = X_{11}$
$X_{14}[20] = X_{10}X_{11}X_{14}$	$X_{15}[31] = X_4X_5$

$$\begin{array}{ll}
\mu_{10}: X_6[32] = 1/X_4 & \alpha_{4,6}: \mathbf{X}_4[35] = \mathbf{X}_4 \\
\mathbf{X}_9[32] = \mathbf{X}_9 & \mathbf{X}_6[35] = \mathbf{X}_6 \\
X_{10}[32] = X_3X_4 & \\
\mu_6: X_6[33] = X_4 & \alpha_{5,15}: \mathbf{X}_5[36] = \mathbf{X}_5 \\
X_{10}[33] = X_3 & \mathbf{X}_{15}[36] = \mathbf{X}_{15} \\
X_{15}[33] = X_5 & \alpha_{7,12}: \mathbf{X}_7[37] = \mathbf{X}_7 \\
\alpha_{3,10}: \mathbf{X}_3[34] = \mathbf{X}_3 & \mathbf{X}_{12}[37] = \mathbf{X}_{12} \\
\mathbf{X}_{10}[34] = \mathbf{X}_{10} &
\end{array}$$

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