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On the construction of fuzzy spaces and modules over shift algebras

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We introduce shift algebras as certain crossed product algebras based on general function spaces and study properties, as well as the classification, of a particular class of modules depending on a set of matrix parameters. It turns out that the structure of these modules depends in a crucial way on the properties of the function spaces. Moreover, for a class of subalgebras related to compact manifolds, we provide a construction procedure for the corresponding fuzzy spaces, i.e. sequences of finite dimensional modules of increasing dimension as the deformation parameter tends to zero, as well as infinite dimensional modules related to fuzzy non-compact spaces.

1. Introduction

Over the past decades, noncommutative geometry has emerged as a crucial ingredient for physical theories describing the unification of quantum mechanics and general relativity. At small length scales, or high energies, there are good reasons to believe that space itself becomes noncommutative (see e.g. [12]). For instance, through the framework of spectral triples ([10]) the standard model of particles has been formulated in terms of noncommutative geometry as the Spectral Standard Model [9].

In parallel, the theory of “Fuzzy spaces” grew out of the quantization of Membrane theory: a theory of quantum gravity built on the idea of describing fundamental particles as two dimensional surfaces, rather than one dimensional objects as in String theory. To quantize the theory, one uses a regularization procedure where functions are replaced by sequences of matrices (of increasing dimension), enabling a straight-forward quantization (cf. [16]).

Parallel to this, in string theory the so-called IKKT matrix model emerged, which can be derived via a matrix regularization of the worldsheet action functional [17] or can be seen as a compactification of ten-dimensional super Yang-Mills theory to a point. Motivated by the IKKT model and other matrix models, more complicated fuzzy spaces, also in higher dimensions, were studied and used as physical models, such as emergent gravity ([11, 19, 21–23, 26]).

Hence, it becomes important to understand the geometry of such matrix regularizations, as well as finding explicit examples. Although there are general existence results (see e.g. [8]), showing that one may find matrix regularizations for arbitrary (quantizable) compact Kähler manifolds, it does not provide a deeper understanding on how geometrical and topological properties of the manifold are reflected in the regularization. Moreover, for a long time, the fuzzy sphere and the fuzzy torus were more or less the only known explicit examples. In [1], a one parameter class of surfaces was considered, interpolating between spheres and tori (including the singular point of topology change). For the first time, this allowed for the explicit study of how geometrical deformation and topology change affect the matrix regularization. It turned out the topology change implied a drop in the dimension of the matrix representation. By now there are many more examples of fuzzy spaces, as well as a broader understanding of their geometry (see e.g. [3, 4, 6, 20, 21, 23, 25].

In many cases, matrix regularizations arise as finite dimensional representations of a (noncommutative) algebra representing the noncommutative space. For instance, this is the case in [1], where representations of an algebra defined by cubic relations in the generators were considered. Fuzzy spaces with a dimension higher than two can be constructed from representations of finite dimensional Lie algebras, which are interpreted as fuzzy homogeneous spaces. For example, a fuzzy 4-hyperboloid is considered in [22] and a ”squashed” fuzzy hyperboloid is interpreted as space time in [24]. One of the main motivations for this paper is to provide a simple way of constructing representations of fuzzy spaces with a dimension higher than two.

In this paper, we study a class of *shift algebras* defined as twisted crossed product algebras via the action of $\mathbb{Z}^D$ on a function algebra $\mathcal{F}$ consisting of complex valued functions on $\mathbb{R}^D$. If $\mathcal{F}$ is chosen to be the algebra of continuous functions, then one recovers a crossed product algebra related to the noncommutative cylinder (cf. [27]) along with a particular choice of cocycle, in analogy with the algebra constructed in [5]. However, for other choices of $\mathcal{F}$, one obtains fundamentally different algebras such as the noncommutative torus, when $\mathcal{F}$ is chosen to be the algebra of constant functions. In particular, we are interested in classes of modules over shift algebras and a construction giving rise to fuzzy spaces; i.e. sequences of finite dimensional modules of increasing dimension as the deformation parameter $\hbar$ tends to zero.

The paper is organized as follows: In Section 2 we define shift algebras and present a few results on isomorphisms for different choices of parameters. Section 3 introduces a class of modules, depending on a set of (matrix) parameters, generalizing the construction in [5], and studies isomorphism classes with respect to different choices of parameters. Moreover, for a subclass of modules, we classify all simple modules in the case when the function algebra separates points. In Section 4 we study modules over a general class of subalgebras and present a construction procedure to generate fuzzy spaces related to compact manifolds as well as non-compact manifolds.

In Section 4.1 we consider one-dimensional shift subalgebras and motivate that these algebras relate to two-dimensional fuzzy spaces. We explicitly show that the fuzzy sphere, a fuzzy catenoid and the fuzzy plane can be reproduced with one-dimensional shift subalgebras. Section 4.2 provides examples illustrating the construction of fuzzy spaces with higher dimensional shift subalgebras. In particular, we construct shift subalgebras which can be interpreted as compact and non-compact four-dimensional level sets immersed into $\mathbb{R}^6$.

Section 5 is devoted to higher dimensional shift subalgebras related to Lie algebras. We show that any Lie algebra with a finite dimensional representation can be used to define elements in a shift subalgebra satisfying the commutation relations of the Lie algebra. The corresponding modules can be interpreted as fuzzy homogeneous spaces.

2. Shift algebras

Let us recall some basic facts about crossed product algebras. Thus, let $\mathcal{A}$ be a $*$-algebra and let G be a discrete group (of either finite or countably infinite cardinality). Given a group action $\alpha : G \to \mathrm{Aut}(\mathcal{A})$, one can form the crossed product algebra $\mathcal{A} \rtimes_\alpha G$. Moreover, given a normalized 2-cocyle $\omega : G \times G \to U(1)$, satisfying

$$\omega_{g,h}\omega_{gh,k} = \omega_{g,hk}\omega_{h,k} \qquad \omega_{g,e} = \omega_{e,g} = 1 \tag{1}$$

for $g, h, k \in G$, one can form the twisted crossed product algebra $\mathcal{A} \rtimes_{\alpha,\omega} G$. As a vector space, the twisted crossed product algebra $\mathcal{A} \rtimes_{\alpha,\omega} G$ is defined as the set of functions $G \to \mathcal{A}$ with compact support. When the group is discrete, one may identify this set with the group ring $\mathcal{A}[G]$ and every element $a \in \mathcal{A} \rtimes_{\alpha,\omega} G$ is written as a formal (finite) linear combination

$$a = \sum_{g\in G} a_g g \tag{2}$$

with $a_g \in \mathcal{A}$. Multiplication is then defined by

$$a \cdot b = \Big(\sum_{g\in G} a_g g\Big) \cdot \Big(\sum_{h\in G} b_h h\Big) = \sum_{g,h\in G} a_g \alpha_g(b_h)\omega_{g,h} gh \tag{3}$$

which may be written as

$$\begin{aligned} a \cdot b &= \sum_{g\in G}\Big[\sum_{h\in G} a_h \alpha_h\big(b_{h^{-1}g}\big)\omega_{h,h^{-1}g}\Big] g \\ &= \sum_{g\in G}\Big[\sum_{h\in G} a_{gh^{-1}} \alpha_{gh^{-1}}\big(b_h\big)\omega_{gh^{-1},h}\Big] g\,. \end{aligned}$$

Furthermore, the algebra $\mathcal{A} \rtimes_{\alpha,\omega} G$ has an involution defined by

$$a^* = \Big(\sum_{g\in G} a_g g\Big)^* = \sum_{g\in G} (\omega_{g^{-1},g})^{-1}\alpha_g(a^*_{g^{-1}}) g\,.$$

In the following, we will be interested in twisted crossed product algebras where $\mathcal{A}$ is a subalgebra $\mathcal{F}$ of the $*$-algebra $F(\mathbb{R}^D, \mathbb{C})$ of complex valued functions on $\mathbb{R}^D$, and the group G is chosen to be $\mathbb{Z}^D$. For $f \in F(\mathbb{R}^D, \mathbb{C})$ and $\hbar > 0$, define $S_{\hbar,k} : F(\mathbb{R}^D, \mathbb{C}) \to F(\mathbb{R}^D, \mathbb{C})$ as

$$(S_{\hbar,k} f)(u_1, \dots, u_D) = f(u_1, \dots, u_k + \hbar, \dots, u_D)\,,$$

and, for $\lambda \in \mathbb{R}$, set

$$(T_\lambda f)(u_1, \dots, u_D) = f(\lambda u_1, \dots, \lambda u_D)\,.$$

The operators $S_{\hbar,k}$ allow one to define a group action $S_\hbar : \mathbb{Z}^D \to \mathrm{Aut}(F(\mathbb{R}^D, \mathbb{C}))$ as

$$\begin{aligned}\big(S_\hbar(k)f\big)(u_1, \dots, u_D) &= \big(S_{\hbar,1}^{k_1} S_{\hbar,2}^{k_2} \cdots S_{\hbar,D}^{k_D} f\big)(u_1, \dots, u_D)\\ &= f(u_1 + k_1\hbar, u_2 + k_2\hbar, \dots, u_D + k_D\hbar)\end{aligned}$$

for $k = (k_1, \dots, k_D) \in \mathbb{Z}^D$, and it is easy to check that $S_\hbar(k)$ is a $*$-automorphism fulfilling $S_\hbar(k)S_\hbar(l) = S_\hbar(k+l)$ for $k, l \in \mathbb{Z}^D$. For notational convenience we will write $S_\hbar^k \equiv S_\hbar(k)$. Let us now introduce a 2-cocycle on $\mathbb{Z}^D$. For $k = (k_1, \dots, k_D), l = (l_1, \dots, l_D) \in \mathbb{Z}^D$ set

$$N(k,l) = \sum_{m=2}^{D} \sum_{n=1}^{m-1} k_m l_n\,, \tag{4}$$

with the convention that $N(k,l) = 0$ for $D = 1$, and for arbitrary $q \in \mathbb{C}$ with $|q| = 1$ define

$$\omega(k,l) = q^{N(k,l)}\,.$$

Since $N(k,l)$ is linear in both arguments, it follows immediately that ω fulfills the 2-cocycle condition (1). For later convenience, we extend the definition of N to $\mathbb{R}^D \times \mathbb{R}^D$ using (4) for $k, l \in \mathbb{R}^D$.

Definition 2.1. A $*$-subalgebra $\mathcal{F} \subseteq F(\mathbb{R}^D, \mathbb{C})$ is called *$\hbar$-invariant* if $S_\hbar^k f \in \mathcal{F}$ for all $k \in \mathbb{Z}^D$ and $f \in \mathcal{F}$.

In the following definition, we introduce type of algebras that will be studied throughout the paper.

Definition 2.2. Let $\hbar > 0$, $q \in \mathbb{C}$ with $|q| = 1$, and let $\mathcal{F}$ be a $\hbar$-invariant $*$-subalgebra of $F(\mathbb{R}^D, \mathbb{C})$. The *shift algebra* $\mathcal{A}_{\hbar,q}^D(\mathcal{F})$ is defined as

$$\mathcal{A}_{\hbar,q}^D(\mathcal{F}) = \mathcal{F} \rtimes_{S_\hbar,\omega} \mathbb{Z}^D\,.$$

Note that when the function space $\mathcal{F}$ consists of continuous functions, the construction above is similar to that of the noncommutative cylinders found in [27] and [5]. However, due to the freedom in choosing the function algebra $\mathcal{F}$, Definition 2.2 also contains compact manifolds, such as the noncommutative torus (when $\mathcal{F}$ is the algebra of constant functions).

Following the notation in (2), we write a generic element $f \in \mathcal{A}^D_{\hbar,q}(\mathcal{F})$ as

$$f = \sum_{k\in\mathbb{Z}^D} f_k U^k = \sum_{(k_1,\ldots,k_D)\in\mathbb{Z}^D} f_{k_1\cdots k_D} U_1^{k_1}\cdots U_D^{k_D}$$

with $U_1,\ldots,U_D$ being (multiplicative) generators of $\mathbb{Z}^D$. To simplify the notation, we shall often omit the explicit summation symbol and assume that any repeated index is implicitly summed over unless otherwise stated. The algebra product is given as

$$\Big(\sum_{k\in\mathbb{Z}^D} f_k U^k\Big)\Big(\sum_{l\in\mathbb{Z}^D} g_l U^l\Big) = \sum_{k,l\in\mathbb{Z}^D} q^{N(k,l)} f_k\big(S^k_\hbar g_l\big)U^{k+l}$$
$$= \sum_{k,n\in\mathbb{Z}^D} q^{N(k,n-k)} f_k(S^k_\hbar g_{n-k})U^n\,.$$

Note that in the case when $u_i \in \mathcal{F}$ (for $i = 1, 2, \ldots, D$) the above product is induced by the relations

$$U_jU_i = qU_iU_j \quad (\text{for } j > i) \qquad U_iu_j = u_jU_i \quad (\text{for } i \neq j)$$
$$u_iu_j = u_ju_i \qquad U_iu_i = (u_i + \hbar)U_i$$

for $i, j = 1, 2, \ldots, D$, perhaps providing a more intuitive understanding of the product and the terminology *shift algebra*. The involution is computed as

$$\Big(\sum_{k\in\mathbb{Z}^D} f_k U^k\Big)^* = \sum_{k\in\mathbb{Z}^D} q^{N(k,k)}\big(S^k_\hbar \bar{f}_{-k}\big)U^k\,, \tag{5}$$

where the bar denotes complex conjugation, implying that $U_i^* = U_i^{-1}$ and $f^* = \bar{f}$. Moreover, for convenience, we introduce $u = (u_1,\ldots,u_D)$ and for $f \in F(\mathbb{R}^D,\mathbb{C})$ we write $f(u) = f(u_1,\ldots,u_D)$.

The shift algebra $\mathcal{A}^D_{\hbar,q}(\mathcal{F})$ depends on a choice of the parameters $\hbar$ and q. In the following we show that if the function algebra $\mathcal{F}$ has certain properties, then the shift algebras for different choices of parameters are isomorphic. First, let us show that if $\mathcal{F}$ is scale invariant then the value of $\hbar$ is irrelevant.

Proposition 2.3. *Let $\mathcal{A}^D_{\hbar_1,q}(\mathcal{F})$ and $\mathcal{A}^D_{\hbar_2,q}(\mathcal{F})$ be shift algebras. If $T_\lambda f \in \mathcal{F}$ for all $f \in \mathcal{F}$ and $\lambda \in \mathbb{R}$ then $\mathcal{A}^D_{\hbar_1,q}(\mathcal{F}) \simeq \mathcal{A}^D_{\hbar_2,q}(\mathcal{F})$.*

Proof. The algebras $\mathcal{A}^D_{\hbar_1,q}$ and $\mathcal{A}^D_{\hbar_2,q}$ consist of the same underlying vector space, however with two different products which we denote by $\cdot_{\hbar_1}$ and $\cdot_{\hbar_2}$. Next, let us define a map $\phi : \mathcal{A}^D_{\hbar_1,q} \to \mathcal{A}^D_{\hbar_2,q}$ by setting

$$\phi\Big(\sum_{k\in\mathbb{Z}^D} f_k U^k\Big) = \sum_{k\in\mathbb{Z}^D} \big(T_{\hbar_1/\hbar_2} f_k\big) U^k\,,$$

which is clearly linear and invertible (and well-defined since $T_{\hbar_1/\hbar_2} f_k \in \mathcal{F}$ by assumption), with

$$\phi^{-1}\Big(\sum_{k\in\mathbb{Z}^D} f_k U^k\Big) = \sum_{k\in\mathbb{Z}^D} \big(T_{\hbar_2/\hbar_1} f_k\big) U^k\,.$$

Now, let us prove that ϕ is an algebra homomorphism. First, compute (implicitly assuming summation over k and l)

$$\begin{aligned}\phi\big(f_k U^k \cdot_{\hbar_1} g_l U^l\big) &= \phi\big(q^{N(k,l)} f_k (S^k_{\hbar_1} g_l) U^{k+l}\big)\\ &= q^{N(k,l)} (T_{\hbar_1/\hbar_2} f_k)\big(T_{\hbar_1/\hbar_2} S^k_{\hbar_1} g_l\big) U^{k+l}\,.\end{aligned}$$

Next, compute

$$\begin{aligned}\phi(f_k U^k) \cdot_{\hbar_2} \phi(g_l U^l) &= (T_{\hbar_1/\hbar_2} f_k) U^k \cdot_{\hbar_2} (T_{\hbar_1/\hbar_2} g_l) U^l\\ &= q^{N(k,l)} (T_{\hbar_1/\hbar_2} f_k)\big(S^k_{\hbar_2} T_{\hbar_1/\hbar_2} g_l\big) U^{k+l}\\ &= \phi\big(f_k U^k \cdot_{\hbar_1} g_l U^l\big)\,,\end{aligned}$$

by using that $S^k_{\hbar_2} T_{\hbar_1/\hbar_2} = T_{\hbar_1/\hbar_2} S^k_{\hbar_1}$. Furthermore, in a similar way, one can check that ϕ is a $*$-algebra homomorphism, showing that ϕ is indeed a $*$-algebra isomorphism. $\square$

Similarly, one can show that if $\mathcal{F}$ contains complex exponential functions, then the shift algebras are isomorphic for all values of q.

Proposition 2.4. *Let $\mathcal{A}_{\hbar,q}(\mathcal{F})$ be a shift algebra such that $e^{i\lambda\cdot u} f \in \mathcal{F}$ for all $f \in \mathcal{F}$ and $\lambda \in \mathbb{R}^D$. Then $\mathcal{A}^D_{\hbar,q}(\mathcal{F}) \simeq \mathcal{A}^D_{\hbar,1}(\mathcal{F})$.*

Proof. For $q = e^{i\hbar\tau}$ (with $\tau \in \mathbb{R}$) define $\phi_q : \mathcal{A}^D_{\hbar,q} \to \mathcal{A}^D_{\hbar,1}$ as

$$\phi_q(f_k U^k) = f_k e^{i\tau N(u,k)} U^k$$

Note that $\phi_q(f) \in \mathcal{A}^D_{\hbar,1}$ since, by assumption, $e^{i\lambda\cdot u} f_k \in \mathcal{F}$ for $\lambda \in \mathbb{R}^D$. Furthermore, ϕ_q is clearly invertible. Let us now show that ϕ_q is an algebra homomorphism. To this end, we denote the products in $\mathcal{A}^D_{\hbar,q}$ and $\mathcal{A}^D_{\hbar,1}$ by $\cdot_q$ and $\cdot_1$, respectively.

$$\begin{aligned}\phi_q\big(f_kU^k\cdot_q g_lU^l\big) &= \phi_q\big(q^{N(k,l)}f_k(S^k_\hbar g_l)U^{k+l}\big)\\ &= q^{N(k,l)}e^{i\tau N(u,k+l)}f_k(S^k_\hbar g_l)U^{k+l}\\ \phi_q(f_kU^k)\cdot_1\phi(g_lU^l) &= f_ke^{i\tau N(u,k)}(S^k_\hbar g_l)e^{i\tau N(u+k\hbar,l)}U^{k+l}\\ &= e^{i\hbar\tau N(k,l)}e^{i\tau N(u,k+l)}f_k(S^k_\hbar g_l)U^{k+l}\\ &= \phi_q\big(f_kU^k\cdot_q g_lU^l\big)\,.\end{aligned}$$

Now, let us show that ϕ_q is also a $*$-homomorphism, denoting the involutions by $*_q$ and $*_1$, respectively:

$$\begin{aligned}\phi_q\big((f_kU^k)^{*_q}\big) &= \phi_q\big(q^{N(k,k)}S^k_\hbar(\bar f_{-k})U^k\big)\\ &= q^{N(k,k)}e^{i\tau N(u,k)}S^k_\hbar(\bar f_{-k})U^k\\ \phi_q(f_kU^k)^{*_1} &= \big(f_ke^{i\tau N(u,k)}U^k\big)^{*_1} = S^k_\hbar(\bar f_{-k})e^{i\tau N(u+k\hbar,k)}U^k\\ &= e^{i\hbar\tau N(k,k)}e^{i\tau N(u,k)}S^k_\hbar(\bar f_{-k})U^k\\ &= \phi_q\big((f_kU^k)^{*_q}\big)\,,\end{aligned}$$

and we conclude that ϕ_q is a $*$-algebra isomorphism. □

A particular example for which the above result does not apply is when $\mathcal{F}$ is chosen to be the $*$-subalgebra of $F(\mathbb{R}^D,\mathbb{C})$ consisting of constant functions. For $D=2$ one then recovers the algebra of the noncommutative torus, for which it is well-known that there is a family of pairwise non-isomorphic algebras for different choices of q.

3. Modules over shift algebras

Let us now introduce a class of left $\mathcal{A}^D_{\hbar,q}(\mathcal{F})$-modules. These modules provide a generalization of the modules constructed in [5] for the noncommutative cylinder, and there are several novel results which may be applied to that case as well. As vector spaces, these modules consist of subspaces of $F(\mathbb{R}^D\times\mathbb{Z}^D,\mathbb{C})$ (the space of complex valued functions on $\mathbb{R}^D\times\mathbb{Z}^D$). However, one needs a few assumptions on $\mathcal{S}$ which we present in the following definition.

Definition 3.1. Let $\mathcal{F}$ be a $*$-subalgebra of $F(\mathbb{R}^D,\mathbb{C})$. A subspace $\mathcal{S}\subseteq F(\mathbb{R}^D\times\mathbb{Z}^D,\mathbb{C})$ is called *$\mathcal{F}$-invariant* if $\xi(x,k)\in\mathcal{S}$ implies that $f(x)\xi(x,k)\in\mathcal{S}$, $\xi(x+\lambda,k+r)\in\mathcal{S}$, and $e^{i\lambda\cdot x}\xi(x,k)\in\mathcal{S}$ for all $\lambda\in\mathbb{R}^D$, $r\in\mathbb{Z}^D$ and $f\in\mathcal{F}$.

Depending on the context (and the choice of $\mathcal{F}$), one can for instance let $\mathcal{S}$ be the space of continuous functions, or simply let $\mathcal{S} = F(\mathbb{R}^D \times \mathbb{Z}^D, \mathbb{C})$. In the next result, a class of left $\mathcal{A}^D_{\hbar,q}(\mathcal{F})$-modules is introduced, parametrized by a set of real and integer matrices.

Proposition 3.2. *Let $\mathcal{A}^D_{\hbar,q}(\mathcal{F})$ be a shift algebra and let $\mathcal{S} \subseteq F(\mathbb{R}^D \times \mathbb{Z}^D)$ be a $\mathcal{F}$-invariant subspace. If $\Lambda_0, \Lambda_1, E \in \mathrm{Mat}_D(\mathbb{R})$, $R \in \mathrm{Mat}_D(\mathbb{Z})$ and $\delta \in \mathbb{R}^D$ such that $\Lambda_0 E + \Lambda_1 R = \mathbb{1}$, then $\mathcal{S}$ is a left $\mathcal{A}^D_{\hbar,q}(\mathcal{F})$-module with the module structure defined by*

$$(f\xi)(x,k) = \sum_{n\in\mathbb{Z}^D} q^{N(\Phi(x,k,\delta),n)} f_n\big(\Phi(x,k,\delta)\hbar\big)\xi(x+En, k+Rn)$$

for $f = f_k U^k \in \mathcal{A}^D_{\hbar,q}(\mathcal{F})$, $\xi \in \mathcal{S}$ and $\Phi(x,k,\delta) = \Lambda_0 x + \Lambda_1 k + \delta$.

Proof. The action is clearly linear, and it remains to show that

$$\big((f_n U^n \cdot g_m U^m)\xi\big)(x,k) = \big(f_n U^n(g_m U^m \xi)\big)(x,k)\,.$$

One computes

$$\begin{aligned}
&\big((f_n U^n \cdot g_m U^m)\xi\big)(x,k)\\
&\quad = \big((q^{N(n,m)} f_n (S^n_\hbar g_m) U^{n+m})\xi\big)(x,k)\\
&\quad = q^{N(n,m)} q^{N(\Phi(x,k,\delta),n+m)} f_n(\Phi(x,k,\delta)\hbar)(S^n_\hbar g_m)\\
&\qquad \times (\Phi(x,k,\delta)\hbar) \times \xi\big(x + E(n+m), k + R(n+m)\big)\\
&\quad = q^{N(n,m)} q^{N(\Phi(x,k,\delta),n+m)} f_n(\Phi(x,k,\delta)\hbar)\\
&\qquad \times g_m\big(\Phi(x,k,\delta)\hbar + n\hbar\big) \times \xi\big(x + E(n+m), k + R(n+m)\big)\,.
\end{aligned}$$

Next, one computes

$$\begin{aligned}
&\big(f_n U^n(g_m U^m \xi)\big)(x,k)\\
&\quad = q^{N(\Phi(x,k,\delta),n)} f_n(\Phi(x,k,\delta)\hbar)\big((g_m U^m)\xi\big)(x+En, k+Rn)\\
&\quad = q^{N(\Phi(x,k,\delta),n)} f_n(\Phi(x,k,\delta)\hbar) q^{N(\Phi(x+En,k+Rn,\delta),m)}\\
&\qquad \times g_m\big(\Phi(x+En, k+Rn, \delta)\hbar\big)\xi\big(x + E(n+m), k + R(n+m)\big)\\
&\quad = q^{N(n,m)} q^{N(\Phi(x,k,\delta),n)} q^{N(\Phi(x,k,\delta),m)} f_n(\Phi(x,k,\delta)\hbar)\\
&\qquad \times g_m\big(\Phi(x,k,\delta)\hbar + n\hbar\big)\xi\big(x + E(n+m), k + R(n+m)\big)\\
&\quad = \big((f_n U^n \cdot g_m U^m)\xi\big)(x,k)
\end{aligned}$$

by using that $\Phi(x+En, k+Rn, \delta) = \Phi(x,k,\delta) + n$, since $\Lambda_0 E + \Lambda_1 R = \mathbb{1}$. □

A left $\mathcal{A}^D_{\hbar,q}(\mathcal{F})$-module defined as in Proposition 3.2 will be denoted by $\mathcal{S}^\delta_\hbar(\Lambda_0, E, \Lambda_1, R)$, or simply by $\mathcal{S}^\delta_\hbar$ (tacitly assuming a choice of $\Lambda_0, E, \Lambda_1, R$). Note that for $\Lambda_1 = R = 0$ and $\Lambda_0 E = \mathbb{1}$, one effectively obtains a representation acting on (a subalgebra of) $F(\mathbb{R}^D, \mathbb{C})$ as

$$(6) \qquad \big(f\xi\big)(x) = \sum_{n\in\mathbb{Z}^D} q^{N(\Lambda_0 x+\delta,n)} f_n\big((\Lambda_0 x + \delta)\hbar\big)\xi(x + En)\,.$$

Consequently, if $\mathcal{F}$ has the property that $e^{i\lambda\cdot u}f \in \mathcal{F}$ for all $f \in \mathcal{F}$ and $\lambda \in \mathbb{R}^D$, then one may choose (with a slight abuse of notation) $\mathcal{S} = \mathcal{F}$ to obtain a representation of $\mathcal{A}^D_{\hbar,q}(\mathcal{F})$. Note that, in this case, Proposition 2.4 implies that $\mathcal{A}^D_{\hbar,q}(\mathcal{F}) \simeq \mathcal{A}^D_{\hbar,1}(\mathcal{F})$. Similarly, for $\Lambda_0 = E = 0$ and $\Lambda_1 R = \mathbb{1}$, one obtains a representation on $F(\mathbb{Z}^D, \mathbb{C})$ given by

$$(7) \qquad \big(f\xi\big)(k) = \sum_{n\in\mathbb{Z}^D} q^{N(\Lambda_1 k+\delta,n)} f_n\big((\Lambda_1 k + \delta)\hbar\big)\xi(k + Rn)\,.$$

In particular, for $\Lambda_1 = R = \mathbb{1}$ one obtains

$$(8) \qquad \big(f\xi\big)(k) = \sum_{n\in\mathbb{Z}^D} q^{N(k+\delta,n)} f_n\big((k + \delta)\hbar\big)\xi(k + n)\,.$$

A $\mathcal{A}^D_{\hbar,q}(\mathcal{F})$-module of the form $\mathcal{S}^\delta_\hbar(\Lambda_0, E, \Lambda_1, R)$ is defined by matrices $\Lambda_0, \Lambda_1, E \in \mathrm{Mat}_D(\mathbb{R})$ and $R \in \mathrm{Mat}_D(\mathbb{Z})$ together with $\delta \in \mathbb{R}^D$. In the following, we try to understand how these parameters affect the isomorphism class of the module. The first result in this direction provides an isomorphism between modules of the type $\mathcal{S}^\delta_\hbar(\Lambda_0, E, \Lambda_1, R)$ for different $\delta \in \mathbb{R}^D$. Note that, in the following, when comparing modules of the form $\mathcal{S}^\delta_\hbar(\Lambda_0, E, \Lambda_1, R)$ for different choices of parameters, we will (unless otherwise stated) assume that the underlying $\mathcal{F}$-invariant subspace $\mathcal{S}$ is the same.

Proposition 3.3. *If $\delta, \delta' \in \mathbb{R}^D$ and $R \in \mathrm{Mat}_D(\mathbb{Z})$, such that $R(\delta - \delta') \in \mathbb{Z}^D$, then $\mathcal{S}^\delta_\hbar(\Lambda_0, E, \Lambda_1, R) \simeq \mathcal{S}^{\delta'}_\hbar(\Lambda_0, E, \Lambda_1, R)$.*

Proof. Define $\phi : \mathcal{S}^\delta_\hbar(\Lambda_0, E, \Lambda_1, R) \to \mathcal{S}^{\delta'}_\hbar(\Lambda_0, E, \Lambda_1, R)$ as

$$\phi(\xi)(x, k) = \xi\big(x + E(\delta' - \delta), k + R(\delta' - \delta)\big)$$

for $\xi \in \mathcal{S}$, which is well-defined since $R(\delta' - \delta) \in \mathbb{Z}^D$. Then ϕ is clearly an invertible linear map, and it remains to prove that it is a homomorphism. Thus, one computes

$$\phi\big((f_n U^n)\xi\big)(x, k) = q^{N(\Phi(x+E(\delta'-\delta),k+R(\delta'-\delta),\delta),n)}$$

$$\begin{aligned}
&\times f_n\big(\Phi(x+E(\delta'-\delta),k+R(\delta'-\delta),\delta)\hbar\big)\\
&\times \xi\big(x+E(n+\delta'-\delta),k+R(n+\delta'-\delta)\big)\\
&= q^{N(\Phi(x,k,\delta'),n)} f_n\big(\Phi(x,k,\delta')\hbar\big)\\
&\times \xi\big(x+E(n+\delta'-\delta),k+R(n+\delta'-\delta)\big)\\
&= \big((f_nU^n)\phi(\xi)\big)(x,k)\,,
\end{aligned}$$

by using that

$$\begin{aligned}
\Phi(x&+E(\delta'-\delta),k+R(\delta'-\delta),\delta)\\
&=\Lambda_0x+\Lambda_1k+(\Lambda_0E+\Lambda_1R)(\delta'-\delta)+\delta\\
&=\Lambda_0x+\Lambda_1k+\delta'-\delta+\delta=\Phi(x,k,\delta')
\end{aligned}$$

since $\Lambda_0E+\Lambda_1R=\mathbb{1}$.

We conclude that $\mathcal{S}^{\delta}_{\hbar}(\Lambda_0,E,\Lambda_1,R)\simeq\mathcal{S}^{\delta'}_{\hbar}(\Lambda_0,E,\Lambda_1,R)$. □

In particular, if $\delta-\delta'\in\mathbb{Z}^D$ then Proposition 3.3 implies that

$$\mathcal{S}^{\delta}_{\hbar}(\Lambda_0,E,\Lambda_1,R)\simeq\mathcal{S}^{\delta'}_{\hbar}(\Lambda_0,E,\Lambda_1,R)$$

for any choice of Λ_0,E,Λ_1,R such that $\Lambda_0E+\Lambda_1R=\mathbb{1}$. Hence, up to module isomorphism, one only needs to consider $\delta\in[0,1)^D$. Moreover, Proposition 3.3 also implies that

$$\mathcal{S}^{\delta}_{\hbar}(\Lambda_0,E,\Lambda_1,0)\simeq\mathcal{S}^{\delta'}_{\hbar}(\Lambda_0,E,\Lambda_1,0)$$

for arbitrary $\delta,\delta'\in\mathbb{R}^D$. Having considered module isomorphisms related to δ, let us now focus on the matrices defining $\mathcal{S}^{\delta}_{\hbar}(\Lambda_0,E,\Lambda_1,R)$, and derive a sufficient condition for isomorphic modules.

Lemma 3.4. *Let $\mathcal{S}^{\delta}_{\hbar}(\Lambda_0,E,\Lambda_1,R)$ and $\mathcal{S}^{\delta}_{\hbar}(\tilde\Lambda_0,\tilde E,\tilde\Lambda_1,\tilde R)$ be left $\mathcal{A}^D_{\hbar,q}(\mathcal{F})$-modules. If there exist $A\in\mathrm{GL}_D(\mathbb{R})$ and $B\in\mathrm{GL}_D(\mathbb{Z})$ such that*

$$\Lambda_0A=\tilde\Lambda_0\qquad A\tilde E=E \tag{9}$$

$$\Lambda_1B=\tilde\Lambda_1\qquad B\tilde R=R\,, \tag{10}$$

then $\Lambda_0E=\tilde\Lambda_0\tilde E$, $\Lambda_1R=\tilde\Lambda_1\tilde R$ and $\mathcal{S}^{\delta}_{\hbar}(\Lambda_0,E,\Lambda_1,R)\simeq\mathcal{S}^{\delta}_{\hbar}(\tilde\Lambda_0,\tilde E,\tilde\Lambda_1,\tilde R)$.

Proof. To distinguish the two different module structures on $\mathcal{S}$, let us denote the action of $f\in\mathcal{A}^D_{\hbar,q}(\mathcal{F})$ as $\rho(f)\xi$ and $\tilde\rho(f)\xi$ on $\mathcal{S}^{\delta}_{\hbar}(\Lambda_0,E,\Lambda_1,R)$ and

$\mathcal{S}^\delta_\hbar(\tilde{\Lambda}_0,\tilde{E},\tilde{\Lambda}_1,\tilde{R})$, respectively. Define a linear map $\phi:\mathcal{S}^\delta_\hbar(\Lambda_0,E,\Lambda_1,R)\to\mathcal{S}^\delta_\hbar(\tilde{\Lambda}_0,\tilde{E},\tilde{\Lambda}_1,\tilde{R})$ by

$$\phi(\xi)(x,k)=\xi(Ax,Bk)$$

for $\xi\in\mathcal{S}$; since $A\in\mathrm{GL}_D(\mathbb{R})$ and $B\in\mathrm{GL}_D(\mathbb{Z})$, ϕ is invertible. Let us show that (9) and (10) imply that ϕ is a left module homomorphism. First, writing

$$\Phi(x,k,\delta)=\Lambda_0x+\Lambda_1k+\delta \quad \textit{and} \quad \tilde{\Phi}(x,k,\delta)=\tilde{\Lambda}_0x+\tilde{\Lambda}_1k+\delta$$

for $f=f_nU^n\in\mathcal{A}^D_{\hbar,q}(\mathcal{F})$, one notes that

$$\Phi(Ax,Bk,\delta)=\Lambda_0Ax+\Lambda_1Bk+\delta=\tilde{\Lambda}_0x+\tilde{\Lambda}_1k+\delta=\tilde{\Phi}(x,k,\delta)$$

by using (9) and (10). To show that ϕ is a homomorphism, one computes

$$\begin{aligned}\phi(\rho(f)\xi)(x,k)&=q^{N(\Phi(Ax,Bk,\delta),n)}f_n\big(\Phi(Ax,Bk,\delta)\hbar\big)\xi(Ax+En,Bk+Rn)\\&=q^{N(\tilde{\Phi}(x,k,\delta),n)}f_n(\tilde{\Phi}(x,k,\delta)\hbar)\xi(Ax+A\tilde{E}n,Bk+B\tilde{R}n)\end{aligned}$$

and

$$\begin{aligned}\big(\tilde{\rho}(f)\phi(\xi)\big)(x,k)&=q^{N(\tilde{\Phi}(x,k,\delta),n)}f_n(\tilde{\Phi}(x,k,\delta)\hbar)\xi(Ax-A\tilde{E}n,Bk-B\tilde{R}n)\\&=\phi(\rho(f)\xi)(x,k)\,,\end{aligned}$$

by again using (9) and (10). Hence, ϕ is a module isomorphism. Moreover, multiplying $A\tilde{E}=E$ from the left by Λ_0 and using that $\Lambda_0A=\tilde{\Lambda}_0$ gives $\tilde{\Lambda}_0\tilde{E}=\Lambda_0E$. Similarly, (10) implies that $\Lambda_1R=\tilde{\Lambda}_1\tilde{R}$. □

Let us now use the above result to show that under certain regularity conditions of the parameters defining the module, a sufficient condition for $\mathcal{S}^\delta_\hbar(\Lambda_0,E,\Lambda_1,R)$ and $\mathcal{S}^\delta_\hbar(\tilde{\Lambda}_0,\tilde{E},\tilde{\Lambda}_1,\tilde{R})$ to be isomorphic, is that $\Lambda_0E=\tilde{\Lambda}_0\tilde{E}$ (or, equivalently, $\Lambda_1R=\tilde{\Lambda}_1\tilde{R}$). More precisely, we start by introducing the following compatibility conditions.

Definition 3.5. Let $\Lambda_0,\Lambda_1,E\in\mathrm{Mat}_D(\mathbb{R})$ and $R\in\mathrm{Mat}_D(\mathbb{Z})$. The tuple $(\Lambda_0,E,\Lambda_1,R)$ is called *regular* if Λ_0 or E is invertible and $R\in\mathrm{GL}_D(\mathbb{Z})$. Moreover, two tuples (M_1,M_2,M_3,M_4) and $(\tilde{M}_1,\tilde{M}_2,\tilde{M}_3,\tilde{M}_4)$ are called *compatible* if

$$M_i \text{ invertible} \quad\Leftrightarrow\quad \tilde{M}_i \text{ invertible}$$

for $i=1,2,3,4$.

Proposition 3.6. *Let* $\mathcal{S}_\hbar^\delta(\Lambda_0, E, \Lambda_1, R)$ *and* $\mathcal{S}_\hbar^\delta(\tilde{\Lambda}_0, \tilde{E}, \tilde{\Lambda}_1, \tilde{R})$ *be left* $\mathcal{A}_{\hbar,q}^D(\mathcal{F})$ *-modules such that* $(\Lambda_0, E, \Lambda_1, R)$ *and* $(\tilde{\Lambda}_0, \tilde{E}, \tilde{\Lambda}_1, \tilde{R})$ *are regular and compatible. If* $\Lambda_0 E = \tilde{\Lambda}_0 \tilde{E}$ *then*

$$\mathcal{S}_\hbar^\delta(\Lambda_0, E, \Lambda_1, R) \simeq \mathcal{S}_\hbar^\delta(\tilde{\Lambda}_0, \tilde{E}, \tilde{\Lambda}_1, \tilde{R}) .$$

Proof. Assume that $(\Lambda_0, E, \Lambda_1, R)$ and $(\tilde{\Lambda}_0, \tilde{E}, \tilde{\Lambda}_1, \tilde{R})$ are regular and compatible. If $\Lambda_0, \tilde{\Lambda}_0$ are invertible then one sets $A = \Lambda_0^{-1}\tilde{\Lambda}_0$, and if $E, \tilde{E}$ are compatible then one sets $A = E\tilde{E}^{-1}$. Since $\Lambda_0 E = \tilde{\Lambda}_0 \tilde{E}$ these choices of A satisfy (9) in Lemma 3.4. Note that, since $\Lambda_0 E + \Lambda_1 R = \tilde{\Lambda}_0 \tilde{E} + \tilde{\Lambda}_1 \tilde{R} = \mathbb{1}$ and $\Lambda_0 E = \tilde{\Lambda}_0 \tilde{E}$, it follows that $\Lambda_1 R = \tilde{\Lambda}_1 R$. Since $R, \tilde{R}$ are invertible, one can set $B = R\tilde{R}^{-1}$, which is seen to satisfy (10) by using that $\Lambda_1 R = \tilde{\Lambda}_1 \tilde{R}$. Lemma 3.4 then implies that $\mathcal{S}_\hbar^\delta(\Lambda_0, E, \Lambda_1, R) \simeq \mathcal{S}_\hbar^\delta(\tilde{\Lambda}_0, \tilde{E}, \tilde{\Lambda}_1, \tilde{R})$. □

Recall that in the cases where $\Lambda_0 = E = 0$ or $\Lambda_1 = R = 0$ one effectively obtains representations on functions on $\mathbb{R}^D$ and $\mathbb{Z}^D$, respectively (cf. equations (6) and (7)). Let us now examine these two cases in detail. The next results implies that for $\Lambda_1 = R = 0$, there is only one equivalence class of isomorphic modules.

Proposition 3.7. *Let* $\mathcal{S}_\hbar^\delta(\Lambda_0, E, 0, 0)$ *and* $\mathcal{S}_\hbar^{\delta'}(\tilde{\Lambda}_0, \tilde{E}, 0, 0)$ *be left* $\mathcal{A}_{\hbar,q}^D(\mathcal{F})$*-modules defined as in Proposition 3.2. Then* $\mathcal{S}_\hbar^\delta(\Lambda_0, E, 0, 0) \simeq \mathcal{S}_\hbar^{\delta'}(\tilde{\Lambda}_0, \tilde{E}, 0, 0)$*.*

Proof. It follows directly from Proposition 3.3 that

$$\mathcal{S}_\hbar^{\delta'}(\tilde{\Lambda}_0, \tilde{E}, 0, 0) \simeq \mathcal{S}_\hbar^\delta(\tilde{\Lambda}_0, \tilde{E}, 0, 0)$$

(since $\tilde{R} = 0$). Next, let us show that $\mathcal{S}_\hbar^\delta(\Lambda_0, E, 0, 0) \simeq \mathcal{S}_\hbar^\delta(\tilde{\Lambda}_0, \tilde{E}, 0, 0)$. Since $\Lambda_0 E + \Lambda_1 R = \Lambda_0 E = \mathbb{1}$ and $\tilde{\Lambda}_0 \tilde{E} + \tilde{\Lambda}_1 \tilde{R} = \tilde{\Lambda}_0 \tilde{E} = \mathbb{1}$, it follows that $\Lambda_0, \tilde{\Lambda}_0$ are invertible. Setting $A = \Lambda_0^{-1}\tilde{\Lambda}_0$ and $B = \mathbb{1}$ one can use Lemma 3.4, to conclude that $\mathcal{S}_\hbar^\delta(\Lambda_0, E, 0, 0)$ and $\mathcal{S}_\hbar^\delta(\tilde{\Lambda}_0, \tilde{E}, 0, 0)$ are isomorphic which, together with the previous argument, implies that $\mathcal{S}_\hbar^\delta(\Lambda_0, E, 0, 0) \simeq \mathcal{S}_\hbar^{\delta'}(\tilde{\Lambda}_0, \tilde{E}, 0, 0)$. □

Thus, for these modules one may always choose $\Lambda_0 = E = \mathbb{1}$ and $\delta = 0$, giving

$$\big(f\xi\big)(x) = \sum_{n\in\mathbb{Z}^D} q^{N(x,n)} f_n(x\hbar)\xi(x + n\hbar) . \tag{11}$$

Let us now consider modules of the form $\mathcal{S}_\hbar^\delta(0, 0, \Lambda_1, R)$. We note that specifying such a module amounts to choosing $\delta \in \mathbb{R}^D$ and a matrix $R \in$

$\mathrm{Mat}_D(\mathbb{Z})$ with $\det R \neq 0$, giving the module $\mathcal{S}_\hbar^\delta(0,0,R^{-1},R)$ (clearly satisfying $\Lambda_1 R = R^{-1}R = \mathbb{1}$). Moreover, as previously discussed, in the case when $\Lambda_0 = E = 0$ one may consider $\mathcal{S}$ to be a subset of complex valued functions on $\mathbb{Z}^D$, and in the following, we will consider the case when $\mathcal{S}$ consists of all *compactly supported* such functions. To emphasize this particular setup, we introduce the notation

$$\mathcal{S}_\hbar(R,\delta) = \mathcal{S}_\hbar^\delta(0,0,R^{-1},R)\,.$$

Furthermore, we let $\overline{\mathrm{Mat}}_D(\mathbb{Z}) \subseteq \mathrm{Mat}_D(\mathbb{Z})$ denote the set of integer $(D\times D)$-matrices with nonzero determinant. Note that a basis for (compactly supported) complex valued functions on $\mathbb{Z}^D$ is given by $\{|k\rangle\}_{k\in\mathbb{Z}^D}$ defined as

$$|k\rangle\,(n) = \begin{cases} 1 \text{ if } k=n \\ 0 \text{ if } k\neq n \end{cases}$$

giving for $f = f_n U^n$

$$f\,|k\rangle = \sum_{n\in\mathbb{Z}^D} q^{N(R^{-1}k-n+\delta,n)} f_n\big((R^{-1}k-n+\delta)\hbar\big)\,|k-Rn\rangle\ . \tag{12}$$

Let us begin by deriving sufficient conditions for isomorphisms of the modules $\mathcal{S}_\hbar(R,\delta)$.

Proposition 3.8. *Let $R,\tilde{R}\in\overline{\mathrm{Mat}}_D(\mathbb{Z})$ and $\delta,\tilde{\delta}\in\mathbb{R}^D$. If*

$$R\tilde{R}^{-1}\in \mathrm{GL}_D(\mathbb{Z}) \quad \text{and} \quad R(\delta-\tilde{\delta})\in\mathbb{Z}^D$$

then $\mathcal{S}_\hbar(R,\delta)\simeq\mathcal{S}_\hbar(\tilde{R},\tilde{\delta})$.

Proof. Under the assumption that $R\tilde{R}^{-1}\in\mathrm{GL}_D(\mathbb{Z})$, it follows directly from Lemma 3.4, with $A=\mathbb{1}$ and $B=R\tilde{R}^{-1}$, that $\mathcal{S}_\hbar(R,\tilde{\delta})\simeq\mathcal{S}_\hbar(\tilde{R},\tilde{\delta})$. Furthermore, since $R(\delta-\tilde{\delta})\in\mathbb{Z}^D$, Proposition 3.3 implies that $\mathcal{S}_\hbar(R,\tilde{\delta})\simeq\mathcal{S}_\hbar(R,\delta)$. □

Next, let us show that the conditions in Proposition 3.8 are also necessary if the function algebra separates points. Recall that a subalgebra $\mathcal{F}\subseteq F(\mathbb{R}^D,\mathbb{C})$ *separates points* if for any two distinct points $x,y\in\mathbb{R}^D$, there exists $f\in\mathcal{F}$ such that $f(x)\neq f(y)$.

Proposition 3.9. *Let $\mathcal{A}^D_{\hbar,q}(\mathcal{F})$ be a shift algebra such that $\mathcal{F}$ separates points. Then $\mathcal{S}_\hbar(R,\delta) \simeq \mathcal{S}_\hbar(\tilde{R},\delta)$ if and only if*

$$R\tilde{R}^{-1} \in \mathrm{GL}_D(\mathbb{Z}) \text{ and } R(\delta-\delta') \in \mathbb{Z}^D\,.$$

Proof. The sufficiency of the conditions given follows immediately from Proposition 3.8. Now, to show the necessity of the conditions, assume that

$$\phi : \mathcal{S}_\hbar(R,\delta) \to \mathcal{S}_\hbar(\tilde{R},\tilde{\delta})$$

is a module isomorphism. With respect to the basis $\{|k\rangle\}_{k\in\mathbb{Z}^D}$, let us write

$$\phi(|k\rangle) = \sum_{l\in\mathbb{Z}^D} \phi(k,l)\,|l\rangle$$

with $\phi(k,l) \in \mathbb{C}$. Note that, since ϕ is a vector space isomorphism, for each $k \in \mathbb{Z}^D$ there exist $k_1, k_2 \in \mathbb{Z}^D$ such that $\phi(k_1,k) \neq 0$ and $\phi(k,k_2) \neq 0$. Furthermore, since ϕ is a module homomorphism one has $\phi(f\,|k\rangle) = f\phi(|k\rangle)$ for all $f \in \mathcal{F}$, giving

$$\begin{aligned}
&\sum_{l\in\mathbb{Z}^D} f\big((R^{-1}k+\delta)\hbar\big)\phi(k,l)\,|l\rangle = \sum_{l\in\mathbb{Z}^D} f\big((\tilde{R}^{-1}l+\tilde{\delta})\hbar\big)\phi(k,l)\,|l\rangle \\
\Leftrightarrow\quad &\Big(f\big((R^{-1}k+\delta)\hbar\big) - f\big((\tilde{R}^{-1}l+\tilde{\delta})\hbar\big)\Big)\phi(k,l) = 0
\end{aligned}$$

for all $k, l \in \mathbb{Z}^D$ and $f \in \mathcal{F}$. Thus, for each $k \in \mathbb{Z}^D$ it follows that there exist k_1, k_2 (as introduced above) such that

$$\begin{aligned}
f\big((R^{-1}k+\delta)\hbar\big) &= f\big((\tilde{R}^{-1}k_2+\tilde{\delta})\hbar\big)\\
f\big((R^{-1}k_1+\delta)\hbar\big) &= f\big((\tilde{R}^{-1}k+\tilde{\delta})\hbar\big)
\end{aligned}$$

for all $f \in \mathcal{F}$. Since the algebra $\mathcal{F}$ separates points, it follows that

$$\begin{aligned}
R^{-1}k+\delta &= \tilde{R}^{-1}k_2+\tilde{\delta}\\
R^{-1}k_1+\delta &= \tilde{R}^{-1}k+\tilde{\delta}\,.
\end{aligned}$$

Multiplying the equations above from the left with $\tilde{R}$ and R, respectively, one obtains

$$\tilde{R}R^{-1}k + \tilde{R}(\delta-\tilde{\delta}) = k_2 \in \mathbb{Z}^D \tag{13}$$

$$R\tilde{R}^{-1}k + R(\tilde{\delta}-\delta) = k_1 \in \mathbb{Z}^D \tag{14}$$

for all $k \in \mathbb{Z}^D$. First let us show that $R(\tilde{\delta} - \delta) \in \mathbb{Z}^D$. If $R(\tilde{\delta} - \delta) \notin \mathbb{Z}^D$ then it follows from (14) that $R\tilde{R}^{-1}k \notin \mathbb{Z}^D$ for all $k \in \mathbb{Z}^D$ which contradicts the fact that $R\tilde{R}^{-1} \in \mathrm{Mat}_D(\mathbb{Q})$. Hence, $R(\tilde{\delta} - \delta) \in \mathbb{Z}^D$. It then follows from (14) that $R\tilde{R}^{-1}k \in \mathbb{Z}^D$ for all $k \in \mathbb{Z}^D$, which implies that $R\tilde{R}^{-1} \in \mathrm{Mat}_D(\mathbb{Z})$. A similar argument using (13) shows that $\tilde{R}R^{-1} \in \mathrm{Mat}_D(\mathbb{Z})$ from which we conclude that $R\tilde{R}^{-1} \in \mathrm{GL}_D(\mathbb{Z})$. □

The above result provides us with a necessary and sufficient condition for modules of the form $\mathcal{S}_\hbar(R, \delta)$ to be isomorphic. In the following, we shall obtain a more concrete description of the equivalence classes of such modules, by finding representatives for each equivalence class in a systematic way. Specifying a module $\mathcal{S}_\hbar(R, \delta)$ amounts to choosing $R \in \overline{\mathrm{Mat}}_D(\mathbb{Z})$ and $\delta \in \mathbb{R}^D$ implying that the set of all such modules can be parametrized by $\overline{\mathrm{Mat}}_D(\mathbb{Z}) \times \mathbb{R}^D$. Let us now introduce a group action α on $\overline{\mathrm{Mat}}_D(\mathbb{Z}) \times \mathbb{R}^D$ that will induce equivalence classes corresponding to isomorphism classes of modules. Namely, for the direct product of the groups $\mathrm{GL}_D(\mathbb{Z})$ (with multiplicative group structure) and $\mathbb{Z}^D$ (with additive group structure), one sets for $U \in \mathrm{GL}_D(\mathbb{Z})$ and $n \in \mathbb{Z}^D$

$$\alpha_{U,n}(R, \delta) = (UR, \delta + R^{-1}n)$$

for $(R, \delta) \in \overline{\mathrm{Mat}}_D(\mathbb{Z}) \times \mathbb{R}^D$. One may readily check that this is indeed a (left) group action. With $\tilde{R} = UR$ and $\tilde{\delta} = \delta + R^{-1}n$, Proposition 3.9 immediately implies that

$$\mathcal{S}_\hbar(R, \delta) \simeq \mathcal{S}_\hbar\big(\alpha_{U,n}(R, \delta)\big) = \mathcal{S}_\hbar(UR, \delta + R^{-1}n)\,,$$

showing that the isomorphism class of the module $\mathcal{S}_\hbar(R, \delta)$ is preserved by the group action. Hence, pairs (R, δ) in the same orbit of the action correspond to isomorphic modules. The group action α induces an equivalence relation $\sim_\alpha$ on $\overline{\mathrm{Mat}}_D(\mathbb{Z}) \times \mathbb{R}^D$, and we denote the set of orbits by $\overline{\mathrm{Mat}}_D(\mathbb{Z}) \times \mathbb{R}^D / \sim_\alpha$.

Proposition 3.10. *Let $\mathcal{A}_{\hbar,q}(\mathcal{F})$ be a shift algebra such that $\mathcal{F}$ separates points of $\mathbb{R}^D$. Then $\mathcal{S}_\hbar(R, \delta) \simeq \mathcal{S}_\hbar(\tilde{R}, \tilde{\delta})$ if and only if there exists $U \in \mathrm{GL}_D(\mathbb{Z})$ and $n \in \mathbb{Z}^D$ such that $\tilde{R} = UR$ and $\tilde{\delta} = \delta + R^{-1}n$.*

Proof. As already noted, if $\tilde{R} = UR$ and $\tilde{\delta} = \delta + R^{-1}n$ then it follows from Proposition 3.9 that $\mathcal{S}_\hbar(R, \delta) \simeq \mathcal{S}_\hbar(\tilde{R}, \tilde{\delta})$. Now, assume that $\mathcal{S}_\hbar(R, \delta) \simeq \mathcal{S}_\hbar(\tilde{R}, \tilde{\delta})$. Proposition 3.9 implies that $R\tilde{R}^{-1} \in \mathrm{GL}_D(\mathbb{Z})$, which implies that there exists $U \in \mathrm{GL}_D(\mathbb{Z})$ such that $R\tilde{R}^{-1} = U^{-1}$, giving $\tilde{R} = UR$. Furthermore, Proposition 3.9 implies that there exists $n \in \mathbb{Z}^D$ such that $R(\delta - \tilde{\delta}) = -n$, giving $\tilde{\delta} = \delta + R^{-1}n$. □

Let $M(\mathcal{S}_\hbar)$ denote the set of isomorphism classes of modules of the type $\mathcal{S}_\hbar(R,\delta)$, and let $[\mathcal{S}_\hbar(R,\delta)]$ denote the equivalence class of $\mathcal{S}_\hbar(R,\delta)$ in $M(\mathcal{S}_\hbar)$. Proposition 3.10 implies that the map $\iota : \overline{\mathrm{Mat}}_D(\mathbb{Z}) \times \mathbb{R}^D / \sim_\alpha \to M(\mathcal{S}_\hbar)$ given by

$$\iota\big([(R,\delta)]\big) = [\mathcal{S}_\hbar(R,\delta)]$$

is well-defined and, moreover, a bijection. In view of this bijection, is it possible to find a natural representative (R,δ) for each isomorphism class in $M(\mathcal{S}_\hbar)$? Let us recall the Hermite normal form of an integer matrix with nonzero determinant (see e.g. [18]).

Definition 3.11. A matrix $H \in \overline{\mathrm{Mat}}_D(\mathbb{Z})$ is said to be in *Hermite normal form* if

1) H is an upper triangular matrix,
2) $H_{ii} > 0$ for $1 \leq i \leq D$,
3) $0 \leq H_{ji} < H_{ii}$ for $1 \leq j < i \leq D$.

It is well-known that for any matrix $M \in \overline{\mathrm{Mat}}_D(\mathbb{Z})$ there exists a matrix $U \in \mathrm{GL}_D(\mathbb{Z})$ and a *unique* matrix $H \in \overline{\mathrm{Mat}}_D(\mathbb{Z})$ in Hermite normal form such that $M = UH$. Combining this result with Proposition 3.10, we conclude that every equivalence class in $\overline{\mathrm{Mat}}_D(\mathbb{Z}) \times \mathbb{R}^D / \sim_\alpha$ has a representative (R,δ) with R in Hermite normal form. More precisely, we formulate this as follows.

Proposition 3.12. *Let*

$$HR_D = \{(H,\delta) \in \overline{\mathrm{Mat}}_D(\mathbb{Z}) \times \mathbb{R}^D : \delta \in [0,1)^D \\ \text{and } H \text{ is in Hermite normal form}\}.$$

The map $\hat{\iota} : HR_D \to M(\mathcal{S}_\hbar)$ *given by* $\hat{\iota}(H,\delta) = [\mathcal{S}_\hbar(H, H^{-1}\delta)]$ *is a bijection.*

Proof. Let us first show that $\hat{\iota}$ is surjective. Let $[\mathcal{S}_\hbar(R,\delta)] \in M(\mathcal{S}_\hbar)$, and let H be the Hermite normal form of R implying that there exists $U \in \mathrm{GL}_D(\mathbb{Z})$ such that $R = UH$. It follows from Proposition 3.10 that $[\mathcal{S}_\hbar(R,\delta)]=$

$[\mathcal{S}_\hbar(H,\delta)]$. Next, let δ_0 be the fractional part of $H\delta$; i.e. the unique $\delta_0 \in [0,1)^D$ such that $H\delta = n + \delta_0$ for some $n \in \mathbb{Z}^D$. Since

$$\delta = H^{-1}\delta_0 + H^{-1}n\,,$$

Proposition 3.10 gives that $[\mathcal{S}_\hbar(H,\delta)] = [\mathcal{S}_\hbar(H, H^{-1}\delta_0)]$, implying $\hat{\imath}(H,\delta_0) = [\mathcal{S}_\hbar(R,\delta)]$. Next, let us show that $\hat{\imath}$ is injective. To this end we assume that $\hat{\imath}(H_1,\delta_1) = \hat{\imath}(H_2,\delta_2)$, which is equivalent to $\mathcal{S}_\hbar(H_1, H^{-1}\delta_1) \simeq \mathcal{S}_\hbar(H_2, H^{-1}\delta_2)$. From Proposition 3.10 it follows that there exists $U \in \mathrm{GL}_D(\mathbb{Z})$ such that $H_2 = UH_1$ and $n \in \mathbb{Z}^D$ such that $H_2^{-1}\delta_2 = H_1^{-1}\delta_1 + H_1^{-1}n$. Firstly, since the Hermite normal form is unique, it follows that $H_1 = H_2$ giving that $H_1^{-1}\delta_2 = H_1^{-1}\delta_1 + H_1^{-1}n$ which is equivalent to $\delta_2 = \delta_1 + n$. Secondly, since $\delta_1, \delta_2 \in [0,1)^D$, it follows that $n = 0$ and $\delta_1 = \delta_2$. We conclude that $H_1 = H_2$ and $\delta_1 = \delta_2$, which implies that $\hat{\imath}$ is injective. □

Thus, the above result tells us that to every equivalence class of modules, one may assign a unique matrix in Hermite normal form, together with a unique element of $[0,1)^D$. Conversely, it also gives a concrete way to determine whether the modules $\mathcal{S}_\hbar(R,\delta)$ and $\mathcal{S}_\hbar(\tilde{R},\tilde{\delta})$ are isomorphic or not, by comparing the Hermite normal forms H and $\tilde{H}$, of R and $\tilde{R}$ respectively, as well as the fractional parts of $H\delta$ and $\tilde{H}\tilde{\delta}$. As an illustration, let us apply Proposition 3.12 to a particular example. Namely, let us consider modules $\mathcal{S}_\hbar(R,\delta)$ such that R is diagonal; i.e.

$$R = \mathrm{diag}(p_1, p_2, \ldots, p_D)$$

with $0 \neq p_i \in \mathbb{Z}$ for $i = 1,\ldots,D$. Note that, since R is diagonal, it is already in Hermite normal form. First, we note that in order for $\mathcal{S}_\hbar(R,\delta) \simeq \mathcal{S}_\hbar(\tilde{R},\tilde{\delta})$, with $\tilde{R} = \mathrm{diag}(q_1,\ldots,q_D)$, it is necessary that $p_i = q_i$ for $i = 1,\ldots,D$ by the uniqueness of the Hermite normal form. For fixed diagonal R, the isomorphism classes are parametrized by $\delta = R^{-1}\delta_0$ for $\delta_0 \in [0,1)^D$, giving

$$\delta \in [0, 1/p_1) \times [0, 1/p_2) \times \cdots \times [0, 1/p_D)\,.$$

Moreover, for arbitrary $R \in \mathrm{GL}_D(\mathbb{Z})$ we note that $\mathcal{S}_\hbar(R,\delta) \simeq \mathcal{S}_\hbar(\mathbb{1},\delta)$ since the Hermite normal form of an invertible matrix is the identity matrix. Thus, the isomorphism classes of such modules can be represented by $\mathcal{S}_\hbar(\mathbb{1},\delta)$ with $\delta \in [0,1)^D$.

Now, having a more or less complete understanding of the isomorphism classes at hand, let us study when the module $\mathcal{S}_\hbar(R,\delta)$ is simple.

Proposition 3.13. *Let $\mathcal{A}_{\hbar,q}(\mathcal{F})$ be a shift algebra such that $\mathcal{F}$ separates points of $\mathbb{R}^D$. Then the module $\mathcal{S}_\hbar(R,\delta)$ is simple if and only if $R \in \mathrm{GL}_D(\mathbb{Z})$.*

Proof. A module is simple if and only if every element of the module is cyclic. Let us start by showing that if $R \notin \mathrm{GL}_D(\mathbb{Z})$ then there exists an element that is not cyclic. If $R \notin \mathrm{GL}_D(\mathbb{Z})$ then there exists $l \in \mathbb{Z}^D$ such that $Rn \neq l$ for all $n \in \mathbb{Z}^D$ (i.e. R is not surjective). Hence, the vector $|0\rangle$ is not cyclic since

$$\sum_{n\in\mathbb{Z}} f_n U^n |0\rangle = \sum_{n\in\mathbb{Z}} q^{N(\delta,n)} f_n(\delta\hbar) |-Rn\rangle \neq |l\rangle$$

for any choice of $f_n \in \mathcal{F}$. The above argument shows that if $\mathcal{S}_\hbar(R,\delta)$ is simple, then $R \in \mathrm{GL}_D(\mathbb{Z})$.

Next, assume that $R \in \mathrm{GL}_D(\mathbb{Z})$ and let

$$v = \sum_{k\in\mathbb{Z}^D} v_k |k\rangle$$

be an arbitrary element of $\mathcal{S}_\hbar(R,\delta)$. Assuming v to be non-zero, there exists $k_0 \in \mathbb{Z}^D$ such that $v_{k_0} \neq 0$. Since $R \in \mathrm{GL}_D(\mathbb{Z})$ one can form

$$\begin{aligned} &fU^{R^{-1}(k_0-l)}v \\ &\quad = \sum_{k\in\mathbb{Z}^D} q^{N(R^{-1}k+\delta,R^{-1}(k_0-l))} f\big((R^{-1}k+\delta)\hbar\big) v_k |k-k_0+l\rangle . \end{aligned} \tag{15}$$

for arbitrary $f \in \mathcal{F}$ and $l \in \mathbb{Z}^D$. Since $\mathcal{F}$ separates points, for every finite set $I \subseteq \mathbb{R}^D$ and $\mathbb{R}^D \ni x_0 \notin I$ there exists $f \in \mathcal{F}$ such that $f(x) = 0$ for all $x \in I$ and $f(x_0) = 1$. Hence, there exists a function $f_0 \in \mathcal{F}$ such that

$$f_0\big((R^{-1}k_0+\delta)\hbar\big) = 1 \quad \text{and} \quad f_0\big((R^{-1}k+\delta)\hbar\big) = 0$$

for all $k \neq k_0$ such that $v_k \neq 0$ (which, by the compact support of v, is a finite set). From (15) it follows that

$$\frac{1}{v_{k_0}} q^{-N(R^{-1}k_0+\delta,R^{-1}(k_0-l))} f_0 U^{R^{-1}(k_0-l)} v = |l\rangle ,$$

implying that each $v \in \mathcal{S}_\hbar(R,\delta)$ is cyclic and, hence, that $\mathcal{S}_\hbar(R,\delta)$ is a simple module. □

Thus, combined with the previous remark that $\mathcal{S}_\hbar(R,\delta) \simeq \mathcal{S}_\hbar(\mathbb{1},\delta)$ for $R \in \mathrm{GL}_D(\mathbb{Z})$, all simple modules can be represented by $\mathcal{S}_\hbar(\mathbb{1},\delta)$ for $\delta \in [0,1)$.

Let us now show that for arbitrary $R \in \overline{\mathrm{Mat}}_D(\mathbb{Z})$, $\mathcal{S}_\hbar(R, \mathbb{1})$ is in general a direct sum of such modules.

For $R \in \overline{\mathrm{Mat}}_D(\mathbb{Z})$ we define an action α_R of $\mathbb{Z}^D$ on $\mathbb{Z}^D$:

$$\alpha_R(n) \cdot m = m + Rn\,,$$

and it is easy to check that this is indeed a group action on $\mathbb{Z}^D$. Hence, α_R splits $\mathbb{Z}^D$ into N_R orbits. The integer N_R is approximately equal to the number of integer lattice points in the parallelotope spanned by the column vectors of R, and bounded from above by the determinant of R. In terms of the action α_R, one can formulate the decomposition of an arbitrary module in the following way.

Proposition 3.14. *Let $R \in \overline{\mathrm{Mat}}_D(\mathbb{Z})$ and $\delta \in \mathbb{R}^D$. Then*

$$\mathcal{S}_\hbar(R, \delta) \simeq \bigoplus_{k=1}^{N_R} \mathcal{S}_\hbar(\mathbb{1}, \delta)\,.$$

Proof. Let $\hat{n}_1, \ldots \hat{n}_{N_R} \in \mathbb{Z}^D$ be elements of the disjoint orbits of the group action α_R, respectively, and define for $\delta_k = R^{-1}\hat{n}_k + \delta$

$$\phi : \bigoplus_{k=1}^{N_R} \mathcal{S}_\hbar(\mathbb{1}, \delta_k) \to \mathcal{S}_\hbar(R, \delta)$$

by

$$\phi : \bigoplus_{k=1}^{N_R} \sum_{m \in \mathbb{Z}^D} v_{k,m} \left|m\right\rangle \mapsto \sum_{k=1}^{N_R} \sum_{m \in \mathbb{Z}^D} v_{k,m} \left|\hat{n}_k + Rm\right\rangle$$

where $v_{k,m} \in \mathbb{C}$ for $1 \le k \le N_R$ and $m \in \mathbb{Z}^D$. Let us now show that ϕ is a module homomorphism. One computes

$$\begin{aligned}
&\phi\Big(f_n U^n \cdot \bigoplus_{k=1}^{N_R} \sum_{m \in \mathbb{Z}^D} v_{k,m} \left|m\right\rangle \Big) \\
&\qquad = \phi\Big(\bigoplus_{k=1}^{N_R} \sum_{m \in \mathbb{Z}^D} q^{N(m+\delta_k, n)} f_n\big((m+\delta_k)\hbar\big) v_{k,m} \left|m - n\right\rangle \Big) \\
&\qquad = \sum_{k=1}^{N_R} \sum_{m \in \mathbb{Z}^D} q^{N(m+\delta_k, n)} f_n\big((m+\delta_k)\hbar\big) v_{k,m} \left|\hat{n}_k + R(m-n)\right\rangle\,,
\end{aligned}$$

and notes that

$$f_n U^n \cdot \phi\Big(\bigoplus_{k=1}^{N_R} \sum_{m\in\mathbb{Z}^D} v_{k,m}\,|m\rangle\Big) = f_n U^n \cdot \sum_{k=1}^{N_R}\sum_{m\in\mathbb{Z}^D} v_{k,m}\,|\hat{n}_k + Rm\rangle$$

$$= \sum_{k=1}^{N_R}\sum_{m\in\mathbb{Z}^D} q^{N(R^{-1}\hat{n}_k+m+\delta,n)} f_n\big((R^{-1}\hat{n}_k + m + \delta)\hbar\big) v_{k,m} \times |\hat{n}_k + R(m-n)\rangle$$

$$= \phi\Big(f_n U^n \cdot \bigoplus_{k=1}^{N_R}\sum_{m\in\mathbb{Z}^D} v_{k,m}\,|m\rangle\Big)$$

since $\delta_k = R^{-1}\hat{n}_k + \delta$. Hence, ϕ is a left module homomorphism. Next, let us show that ϕ is surjective. Let $m \in \mathbb{Z}^D$ and let $k_0 \in \{1,\ldots,N_R\}$ be such that $\hat{n}_{k_0}$ is in the same orbit as m, implying that there exists $m_0 \in \mathbb{Z}^D$ such that $m = \hat{n}_{k_0} + Rm_0$. Then

$$\phi\big(\underbrace{0\oplus\cdots\oplus 0\oplus |m_0\rangle}_{k_0}\oplus 0\oplus\cdots\oplus 0\big) = |\hat{n}_{k_0} + Rm_0\rangle = |m\rangle$$

implying that ϕ is surjective. Now, to show that ϕ is injective one assumes that

$$0 = \phi\Big(\bigoplus_{k=1}^{N_R}\sum_{m\in\mathbb{Z}^D} v_{k,m}\,|m\rangle\Big) = \sum_{k=1}^{N_R}\sum_{m\in\mathbb{Z}^D} v_{k,m}\,|\hat{n}_k + Rm\rangle\ .$$

Since the orbits of α_R are disjoint, this implies that

$$\sum_{m\in\mathbb{Z}^D} v_{k,m}\,|\hat{n}_k + Rm\rangle = 0$$

for $k = 1,\ldots,N_R$. Moreover, since $\det R \neq 0$, it follows that $v_{k,m} = 0$ for $k = 1,\ldots,N_R$ and $m \in \mathbb{Z}^D$, giving

$$\bigoplus_{k=1}^{N_R}\sum_{m\in\mathbb{Z}^D} v_{k,m}\,|m\rangle = 0\,.$$

Thus, we can conclude that ϕ is a module isomorphism implying that

$$\mathcal{S}_\hbar(R,\delta) \simeq \bigoplus_{k=1}^{N_R} \mathcal{S}_\hbar(\mathbb{1},\delta_k)\,.$$

Finally, we note that $R(\delta_k - \delta) = \hat{n}_k \in \mathbb{Z}^D$ which implies (by Proposition 3.8) that $\mathcal{S}_\hbar(\mathbb{1}, \delta_k) \simeq \mathcal{S}_\hbar(\mathbb{1}, \delta)$; hence, we have shown that $\mathcal{S}_\hbar(R, \delta) \simeq \mathcal{S}_\hbar(\mathbb{1}, \delta) \oplus \cdots \oplus \mathcal{S}_\hbar(\mathbb{1}, \delta)$. □

Proposition 3.14 shows that the basic building block of modules of the type $\mathcal{S}_\hbar(R, \delta)$ is given by $\mathcal{S}_\hbar(\mathbb{1}, \delta)$. Moreover, it is natural to ask whether or not these modules are free, or projective? The next pair of results show that, under the hypothesis that $\mathcal{F}$ separates points, $\mathcal{S}_\hbar(R, \delta)$ is never a free module, and only projective if the function algebra contains a δ-like function.

Proposition 3.15. *Let $\mathcal{A}_{\hbar,q}(\mathcal{F})$ be a shift algebra such that $\mathcal{F}$ separates points of $\mathbb{R}^D$. Then the module $\mathcal{S}_\hbar(R, \delta)$ is not free.*

Proof. For an arbitrary set of elements $v_1, \ldots, v_N \in \mathcal{S}_\hbar(R, \delta)$ write

$$v_i = \sum_{k\mathbb{Z}^D} v_i^k \,|k\rangle$$

and set

$$I_i = \{k \in \mathbb{Z}^D : v_i^k \neq 0\}\,.$$

Note that the I_i are finite sets. Since $\mathcal{F}$ separates points, one can find non-zero $f_i \in \mathcal{F}$ such that $f_i\big((R^{-1}k + \delta)\hbar\big) = 0$ for $k \in I_i$, implying that

$$f_i v_i = \sum_{k\in\mathbb{Z}^D} f_i\big((R^{-1}k + \delta)\hbar\big) v_i^k \,|k\rangle = 0$$

and, consequently

$$\sum_{i=1}^N f_i v_i = 0\,.$$

Hence $v_1, \ldots, v_N$ do not form a basis of $\mathcal{S}_\hbar(R, \delta)$. Since the set of vectors was arbitrary, we conclude that $\mathcal{S}_\hbar(R, \delta)$ is not free. □

Proposition 3.16. *Let $\mathcal{A}_{\hbar,q}(\mathcal{F})$ be a shift algebra such that $\mathcal{F}$ separates points of $\mathbb{R}^D$. Then the module $\mathcal{S}_\hbar(R, \delta)$ is a finitely generated projective module if and only if there exists $p_0 \in \mathcal{F}$ such that*

$$p_0(u) = \begin{cases} 1 & \textit{if } u = \hbar\delta \\ 0 & \textit{if } u \neq \hbar\delta \end{cases}.$$

Proof. We proceed by showing the statement for $R = \mathbb{1}$. It then follows from Proposition 3.14 that the same statement holds for arbitrary R, since a direct sum of modules is projective if and only if each factor is projective.

First, assume that $p_0 \in \mathcal{F}$. Let us construct a map $\phi : \mathcal{A}p_0 \to \mathcal{S}_\hbar(\mathbb{1}, \delta)$ by setting

$$\phi(f_n U^n p_0) = f_n U^n |0\rangle \ .$$

To show that ϕ is well-defined, one needs to prove that if $f_n U^n p_0 = 0$ then $f_n U^n |0\rangle = 0$. Assuming $f_n U^n p_0 = 0$ one finds that

$$\begin{aligned} f_n(u)U^n p_0(u) = 0 \quad &\Rightarrow \quad f_n(u)p_0(u + n\hbar)U^n = 0 \\ &\Rightarrow \quad f_n(u)p_0(u + n\hbar) = 0 \end{aligned}$$

for all $n \in \mathbb{Z}^D$. By the definition of p_0 it follows that $f_n\big((\delta - n)\hbar\big) = 0$ for all $n \in \mathbb{Z}^D$. Thus, one obtains

$$f_n U^n |0\rangle = q^{N(\delta-n,n)} f_n\big((\delta - n)\hbar\big) |{-n}\rangle = 0$$

showing that ϕ is indeed well-defined; moreover, ϕ is clearly a left module homomorphism. To show that ϕ is injective, one assumes that $\phi(f_n U^n p_0)$=0, giving

$$q^{N(\delta-n,n)} f_n\big((\delta - n)\hbar\big) |{-n}\rangle = 0 \quad \Rightarrow \quad f_n\big((\delta - n)\hbar\big) = 0$$

for all $n \in \mathbb{Z}^D$. It follows that

$$f_n(u)U^n p_0(u) = f_n(u)p_0(u + n\hbar)U^n = 0$$

since $p_0(u + n\hbar) \neq 0$ only if $u = (\delta - n)\hbar$. Hence, ϕ is injective. To prove that ϕ is surjective one simply notes that

$$\phi\big(q^{N(n+\delta,n)} U^{-n} p_0\big) = q^{N(n+\delta,n)} q^{N(n+\delta,-n)} |n\rangle = |n\rangle$$

implying that ϕ is surjective. Thus, if $p_0 \in \mathcal{F}$ then $\mathcal{S}_\hbar(\mathbb{1}, \delta)$ is isomorphic to $\mathcal{A}p_0$, showing that $\mathcal{S}_\hbar(\mathbb{1}, \delta)$ is a finitely generated projective module.

Next, let us assume that $\mathcal{S}_\hbar(\mathbb{1}, \delta)$ is a finitely generated projective module, and show that $p_0 \in \mathcal{F}$. Thus, we assume that there exists a left module isomorphism $\phi : \mathcal{S}_\hbar(\mathbb{1}, \delta) \to \mathcal{A}^N p$ for some $N \geq 1$ and $p \in \mathrm{Mat}_N(\mathcal{A})$. Since ϕ is an isomorphism, there exists a non-zero $m_0 \in \mathcal{A}^N p$ such that $\phi(|0\rangle) = m_0$.

Let $\{e_i\}_{i=1}^N$ be a basis of $\mathcal{A}^N$, and write $m_0 = m_0^i e_i$ with $m_0^i = (m_0^i)_n U^n$ for $i = 1, \ldots, N$. Furthermore, since ϕ is a module homomorphism, one obtains

$$f(\hbar\delta)m_0 = \phi\big(f(\hbar\delta)\,|0\rangle\,\big) = \phi(f\,|0\rangle) = f\phi(|0\rangle) = fm_0$$

for all $f \in \mathcal{F}$, implying that

$$\big(f(u) - f(\hbar\delta)\big)(m_0^i)_n(u) = 0$$

for $f \in \mathcal{F}$, $i = 1, \ldots, N$ and $n \in \mathbb{Z}^D$. Since $m_0 \neq 0$ there exist $i_0 \in \{1, \ldots, N\}$ and $n_0 \in \mathbb{Z}^D$ such that $\tilde{m} = (m_0^{i_0})_{n_0} \neq 0$. Since $\mathcal{F}$ separates points one can, for every $\hbar\delta \neq u_0 \in \mathbb{R}^D$, find $f_0 \in \mathcal{F}$ such that $f_0(u_0) - f_0(\hbar\delta) \neq 0$, implying that $\tilde{m}(u) = 0$ for every $u \neq \hbar\delta$. Since $\tilde{m} \neq 0$, one necessarily has $\tilde{m}(\hbar\delta) \neq 0$. Thus, one can define $p_0(u) = \tilde{m}(u)/\tilde{m}(\hbar\delta)$, proving that $p_0 \in \mathcal{F}$. □

For instance, if $\mathcal{F}$ consists of continuous functions, Proposition 3.16 implies that $\mathcal{S}_\hbar(R,\delta)$ is not a finitely generated projective module. In fact, for such function algebras, these modules are in some sense far from being projective. This is made precise in the following result.

Proposition 3.17. *Let $\mathcal{A}_{\hbar,q}(\mathcal{F})$ be a shift algebra. The module $\mathcal{S}_\hbar(R,\delta)$ is a torsion module if and only if there exists $f \in \mathcal{F}$ and $k_0 \in \mathbb{Z}^D$ such that $fg \neq 0$ for all nonzero $g \in \mathcal{F}$, and $f((k_0+\delta)\hbar) = 0$.*

Proof. The module $\mathcal{S}_\hbar(R,\delta)$ is a torsion module if for every $v \in \mathcal{S}_\hbar(R,\delta)$ there exists a non zero divisor $f_v \in \mathcal{A}_{\hbar,q}(\mathcal{F})$ such that $f_v v = 0$. Now, let $f \in \mathcal{F}$ be a function fulfilling the assumptions of Proposition 3.17. Defining

$$f_k(u) = (S_\hbar^{k_0-k}f)(u) = f\big(u + (k_0-k)\hbar\big)$$

(which is clearly in $\mathcal{F}$ since $\mathcal{F}$ is assumed to be $\hbar$-invariant) it follows that $f_k((k+\delta)\hbar) = 0$. Consequently, for any finite set $I \subseteq \mathbb{Z}^D$ the function

$$f_I = \prod_{k\in I} f_k$$

fulfills $f_I((k+\delta)\hbar) = 0$ for all $k \in I$. We note that f_I is not a zero divisor as an element of $\mathcal{A}_{\hbar,q}(\mathcal{F})$ since $f_I g \neq 0$ for all $0 \neq g \in \mathcal{F}$. Thus given arbitrary $v \in \mathcal{S}_\hbar(R,\delta)$ with

$$v = \sum_{k\in\mathbb{Z}^D} v_k\,|k\rangle$$

we let $I = \{k \in \mathbb{Z}^D : v_k \neq 0\}$ (which is a finite set) and find that

$$f_I v = \sum_{k\in I} v_k f_I\big((k+\delta)\hbar\big)\,|k\rangle = 0\,.$$

Since $f_I((k+\delta)\hbar) = 0$ for all $k \in I$. Since v was chosen arbitrarily, we conclude that $\mathcal{S}_\hbar(R,\delta)$ is a torsion module.

Conversely, assume that $\mathcal{S}_\hbar(R,\delta)$ is a torsion module. From Proposition 3.14 it then follows that $\mathcal{S}_\hbar(\mathbb{1},\delta)$ is also a torsion module. In particular, for each $k \in \mathbb{Z}^D$, there exists a non zero divisor $f_k = f_{kn}U^n \in \mathcal{A}_{\hbar,q}(\mathcal{F})$ such that

$$f_k\,|k\rangle = \sum_{n\in\mathbb{Z}^D} q^{N(k-n+\delta,n)} f_{kn}\big((k-n+\delta)\hbar\big)\,|k-n\rangle = 0\,,$$

implying that

$$f_{kn}\big((k-n+\delta)\hbar\big) = 0$$

for all $n, k \in \mathbb{Z}^D$. Now, since f_k is assumed to be a non zero divisor there exists $n_0 \in \mathbb{Z}^D$ such that $f_{kn_0}g \neq 0$ for all $0 \neq g \in \mathcal{F}$. Hence, $f_{kn_0} \in \mathcal{F}$ has the desired properties, which concludes the proof. □

3.1. Bimodules

In the previous section we have studied the structure of a class of left $\mathcal{A}^D_{\hbar,q}(\mathcal{F})$-modules. In the following, we shall give these modules the structure of both a right module and a bimodule. Moreover, we will show that when $\Lambda_0 \in \mathrm{GL}_D(\mathbb{C})$ and $R \in \mathrm{GL}_D(\mathbb{Z})$, given certain conditions on the function space $\mathcal{S}$, the module $\mathcal{S}^\delta_\hbar(\Lambda_0, E, \Lambda_1, R)$ is a free module of rank 1.

Let us start by introducing a class of right $\mathcal{A}^D_{\hbar,q}(\mathcal{F})$-modules in close analogy with the left modules in the previous section.

Proposition 3.18. *Let $\mathcal{A}_{\hbar,q}(\mathcal{F})$ be a shift algebra and let $\mathcal{S} \subseteq F(\mathbb{R}^D \times \mathbb{Z}^D)$ be a $\mathcal{F}$-invariant subspace. If $\Gamma_0, \Gamma_1, F \in \mathrm{Mat}_D(\mathbb{R})$, $P \in \mathrm{Mat}_D(\mathbb{Z})$ and $\delta \in \mathbb{R}^D$ such that $\Gamma_0 E + \Gamma_1 P = \mathbb{1}$, then $\mathcal{S}$ is a right $\mathcal{A}_{\hbar,q}(\mathcal{F})$-module with*

$$(16)\qquad (\xi f)(x,k) = \sum_{n\in\mathbb{Z}^D} q^{N(\Psi(x,k,\delta-n),n)} f_n\big(\Psi(x,k,\delta-n)\hbar\big)\xi\big(x-Fn, k-Pn\big)$$

for $f = f_n U^n \in \mathcal{A}^D_{\hbar,q}$, $\xi \in \mathcal{S}$ and $\Psi(x,k,\delta) = \Gamma_0 x + \Gamma_1 k + \delta$.

Proof. To show that (16) defines a right module action, one needs to check that $\xi(fg) = (\xi f)g$ for $f, g \in \mathcal{A}_{\hbar,q}$ and $\xi \in \mathcal{S}$. One computes

$$\begin{aligned}
&\big(\xi(fg)\big)(x,k)\\
&\quad= \big(\xi(q^{N(n,m)} f_n(S_\hbar^n g_m)U^{m+n})\big)(x,k)\\
&\quad= q^{N(n,m)} q^{N(\Psi(x,k,\delta-n-m),n+m)} f_n(\Psi(x,k,\delta-n-m)\hbar)\\
&\qquad\times g_m(\Psi(x,k,\delta-m)\hbar)\xi\big(x-F(m+n),k-P(n+m)\big)\\
&\quad= q^{-N(m,n)-N(n,n)-N(m,m)} q^{N(\Psi(x,k,\delta),m+n)} f_n(\Psi(x,k,\delta-n-m)\hbar)\\
&\qquad\times g_m(\Psi(x,k,\delta-m)\hbar)\xi\big(x-F(m+n),k-P(n+m)\big)\,,
\end{aligned}$$

as well as

$$\begin{aligned}
&\big((\xi f)g\big)(x,k)\\
&\quad= q^{N(\Psi(x,k,\delta-m),m)} g_m\big(\Psi(x,k,\delta-m)\hbar\big)(\xi f)(x-Fm,k-Pm)\\
&\quad= q^{N(\Psi(x,k,\delta-m),m)} q^{N(\Psi(x-Fm,k-Pm,\delta-n),n)} g_m\big(\Psi(x,k,\delta-m)\hbar\big)\\
&\qquad\times f_n\big(\Psi(x-Fm,k-Pm,\delta-n)\hbar\big)\xi\big(x-F(m+n),k-P(m+n)\big)\,.
\end{aligned}$$

Using that $\Gamma_0 F + \Gamma_1 P = \mathbb{1}$, giving

$$\Psi(x-Fm,k-Pm,\delta) = \Psi(x,k,\delta-m)\,,$$

one obtains

$$\begin{aligned}
&\big((\xi f)g\big)(x,k)\\
&\quad= q^{-N(m,n)-N(n,n)-N(m,m)} q^{N(\Psi(x,k,\delta),m+n)} g_m\big(\Psi(x,k,\delta-m)\hbar\big)\\
&\qquad\times f_n\big(\Psi(x,k,\delta-n-m)\hbar\big)\xi\big(x-F(m+n),k-P(m+n)\big)\\
&\quad= \big(\xi(fg)\big)(x,k)\,,
\end{aligned}$$

and we conclude that (16) is a right module action on $\mathcal{S}$. □

Thus, an $\mathcal{F}$-invariant subspace $\mathcal{S}$ carries both the structure of a left module (as given in Proposition 3.2) and the structure of a right module, as given above. The next result shows that if the corresponding matrices defining the module structure are compatible, then $\mathcal{S}$ is a bimodule. More precisely, we formulate it as follows.

Proposition 3.19. *Let $\mathcal{A}^D_{\hbar,q}(\mathcal{F})$ and $\mathcal{A}^D_{\hbar',q'}(\mathcal{F}')$ be shift algebras and let $\mathcal{S} \subseteq F(\mathbb{R}^D \times \mathbb{Z}^D)$ be a subspace which is both $\mathcal{F}$-invariant and $\mathcal{F}'$-invariant.*

Furthermore, let $\Lambda_0, \Lambda_1, E, \Gamma_0, \Gamma_1, F \in \mathrm{Mat}_D(\mathbb{R})$, $R, P \in \mathrm{Mat}_D(\mathbb{Z})$ *and* $\delta, \epsilon \in \mathbb{R}^D$ *such that*

$$\Lambda_0 E + \Lambda_1 R = \mathbb{1} \qquad \Gamma_0 F + \Gamma_1 P = \mathbb{1} \tag{17}$$

$$\Lambda_0 F + \Lambda_1 P = 0 \qquad \Gamma_0 E + \Gamma_1 R = 0\,. \tag{18}$$

Then $\mathcal{S}$ *is a* $\mathcal{A}^D_{\hbar,q}(\mathcal{F})$-$\mathcal{A}^D_{\hbar',q'}(\mathcal{F}')$*-bimodule with*

$$(f\xi)(x,k) = \sum_{n\in\mathbb{Z}^D} q^{N(\Phi(x,k,\delta),n)} f_n\big(\Phi(x,k,\delta)\hbar\big)\xi(x+En,k+Rn) \tag{19}$$

$$(\xi g)(x,k) = \sum_{n\in\mathbb{Z}^D} (q')^{N(\Psi(x,k,\epsilon-n),n)} g_n\big(\Psi(x,k,\epsilon-n)\hbar'\big)\xi\big(x-Fn,k-Pn\big) \tag{20}$$

for $f = f_n U^n \in \mathcal{A}^D_{\hbar,q}$, $g = g_n U^n \in \mathcal{A}^D_{\hbar',q'}(\mathcal{F}')$, $\xi \in \mathcal{S}$, $\Phi(x,k,\delta) = \Lambda_0 x + \Lambda_1 k + \delta$ *and* $\Psi(x,k,\epsilon) = \Gamma_0 x + \Gamma_1 k + \epsilon$.

Proof. We conclude from Proposition 3.2 and Proposition 3.18 that $\mathcal{S}$ is a left $\mathcal{A}^D_{\hbar,q}(\mathcal{F})$-module and a right $\mathcal{A}^D_{\hbar',q'}(\mathcal{F}')$-module. Thus, it remains to show that $f(\xi g) = (f\xi)g$. One computes

$$\begin{aligned}
&((f\xi)g)(x,k)\\
&\quad= (q')^{N(\Psi(x,k,\epsilon-m),m)} g_m\big(\Psi(x,k,\epsilon-m)\hbar'\big)(f\xi)(x-Fm,k-Pm)\\
&\quad= q^{N(\Phi(x-Fm,k-Pm,\delta),n)}(q')^{N(\Psi(x,k,\epsilon-m),m)} g_m\big(\Psi(x,k,\epsilon-m)\hbar'\big)\\
&\qquad\times f_n\big(\Phi(x-Fm,k-Pm,\delta)\hbar\big)\xi(x-Fm+En,k-Pm+Rn)\\
&\quad= q^{N(\Phi(x,k,\delta),n)}(q')^{N(\Psi(x,k,\epsilon-m),m)} g_m\big(\Psi(x,k,\epsilon-m)\hbar'\big)\\
&\qquad\times f_n\big(\Phi(x,k,\delta)\hbar\big)\xi(x-Fm+En,k-Pm+Rn)
\end{aligned}$$

by using that $\Lambda_0 F + \Lambda_1 P = 0$. On the other hand, one obtains

$$\begin{aligned}
&\big(f(\xi g)\big)(x,k)\\
&\quad= q^{N(\Phi(x,k,\delta),n)} f_n\big(\Phi(x,k,\delta)\hbar\big)(\xi g)(x+En,k+Rn)\\
&\quad= q^{N(\Phi(x,k,\delta),n)}(q')^{N(\Psi(x+En,k+Rn,\epsilon-m),m)} f_n\big(\Phi(x,k,\delta)\hbar\big)\\
&\qquad\times g_m\big(\Psi(x+En,k+Rn,\epsilon-m)\hbar'\big)\xi(x-Fm+En,k-Pm+Rn)\\
&\quad= q^{N(\Phi(x,k,\delta),n)}(q')^{N(\Psi(x,k,\epsilon-m),m)} f_n\big(\Phi(x,k,\delta)\hbar\big)\\
&\qquad\times g_m\big(\Psi(x,k,\epsilon-m)\hbar'\big)\xi(x-Fm+En,k-Pm+Rn)
\end{aligned}$$

$$= ((f\xi)g)\,(x,k)$$

by using that $\Gamma_0 E + \Gamma_1 R = 0$. We conclude that $\mathcal{S}$ is a $\mathcal{A}^D_{\hbar,q}(\mathcal{F})$-$\mathcal{A}^D_{\hbar',q'}(\mathcal{F}')$-bimodule. □

An immediate question arising from Proposition 3.19 is whether or not one can find matrices satisfying conditions (17) and (18)? The next result gives an explicit solution to these equations in the case when $\Lambda_0, \Gamma_0 \in \mathrm{GL}_D(\mathbb{C})$ and $R, P \in \mathrm{GL}_D(\mathbb{Z})$.

Lemma 3.20. *Let $\Lambda_0, \Gamma_0 \in \mathrm{GL}_D(\mathbb{C})$, $\Lambda_1, \Gamma_1, E, F \in \mathrm{Mat}_D(\mathbb{C})$ and $R, P \in \mathrm{GL}_D(\mathbb{Z})$. Then*

$$\begin{aligned} &\Lambda_0 E + \Lambda_1 R = \mathbb{1} \qquad &&\Gamma_0 F + \Gamma_1 P = \mathbb{1}\\ &\Lambda_0 F + \Lambda_1 P = 0 &&\Gamma_0 E + \Gamma_1 R = 0 \end{aligned}$$

is equivalent to

$$\begin{aligned} &\Gamma_0 = -P^{-1} R \Lambda_0 \qquad &&\Gamma_1 = P^{-1}(\mathbb{1} - R\Lambda_1)\\ &E = \Lambda_0^{-1}(\mathbb{1} - \Lambda_1 R) &&F = -\Lambda_0^{-1}\Lambda_1 P\,. \end{aligned}$$

Moreover, if $P = -R$ then the above system is equivalent to

$$\begin{aligned} &\Gamma_0 = \Lambda_0 \qquad &&\Gamma_1 = \Lambda_1 - R^{-1}\\ &E = \Lambda_0^{-1}(\mathbb{1} - \Lambda_1 R) &&F = \Lambda_0^{-1}\Lambda_1 R\,. \end{aligned}$$

Proof. It follows immediately from

$$\Lambda_0 E + \Lambda_1 R = \mathbb{1} \quad \text{and} \quad \Lambda_0 F + \Lambda_1 P = 0$$

that

$$E = \Lambda_0^{-1}(\mathbb{1} - \Lambda_1 R) \quad \text{and} \quad F = -\Lambda_0^{-1}\Lambda_1 P \tag{21}$$

since $\Lambda_0 \in \mathrm{GL}_D(\mathbb{C})$. Inserting these equations into

$$\Gamma_0 F + \Gamma_1 P = \mathbb{1} \quad \text{and} \quad \Gamma_0 E + \Gamma_1 R = 0$$

gives

$$\Gamma_1 = P^{-1} + \Gamma_0\Lambda_0^{-1}\Lambda_1 \quad \text{and} \quad \Gamma_0\Lambda_0^{-1}(\mathbb{1} - \Lambda_1 R) + \Gamma_1 R = 0$$

which are equivalent to

$$\Gamma_1 = P^{-1}(\mathbb{1} - R\Lambda_1) \quad \text{and} \quad \Gamma_0 = -P^{-1}R\Lambda_0 \tag{22}$$

proving the first part of the statement. Finally, equations (21) and (22) give

$$\begin{aligned} \Gamma_0 &= \Lambda_0 & \Gamma_1 &= \Lambda_1 - R^{-1} \\ E &= \Lambda_0^{-1}(\mathbb{1} - \Lambda_1 R) & F &= \Lambda_0^{-1}\Lambda_1 R\,. \end{aligned}$$

when setting $P = -R$. □

Thus, with the help of Lemma 3.20 one can easily construct matrices satisfying (17) and (18); e.g. for $\Lambda_0 = \mathbb{1}$, $R = -P = \mathbb{1}$ and $\Lambda_1 = 2 \cdot \mathbb{1}$, one finds that

$$\begin{aligned} (\Lambda_0, E, \Lambda_1, R) &= (\mathbb{1}, -\mathbb{1}, 2 \cdot \mathbb{1}, \mathbb{1}) \\ (\Gamma_0, F, \Gamma_1, P) &= (\mathbb{1}, 2 \cdot \mathbb{1}, \mathbb{1}, -\mathbb{1}) \end{aligned}$$

define a bimodule structure on $\mathcal{S}$. In Section 3 we studied the special cases when $\Lambda_0 = E = 0$ or $\Lambda_1 = R = 0$ in detail. In the context of bimodules, we note that there are no solutions of (17) and (18) with $\Lambda_0 = \Gamma_0 = 0$ or $\Lambda_1 = \Gamma_1 = 0$. As previously mentioned, when $\Lambda_0 \in \mathrm{GL}_D(\mathbb{C})$ and $R \in \mathrm{GL}_D(\mathbb{Z})$, the module $\mathcal{S}_\hbar^\delta(\Lambda_0, E, \Lambda_1, R)$ turns out to be a free module of rank 1. As a first step in proving this statement, let us construct a module homomorphism $\phi : \mathcal{A}_{\hbar,q}^D(\mathcal{F}) \to \mathcal{S}_\hbar^\delta(\Lambda_0, E, \Lambda_1, R)$.

Proposition 3.21. *Let $\mathcal{A}_{\hbar,q}^D(\mathcal{F})$ be a shift algebra and let $\mathcal{S} \subseteq F(\mathbb{R}^D \times \mathbb{Z}^D)$ be a $\mathcal{F}$-invariant subspace. If $R \in \mathrm{GL}_D(\mathbb{Z})$ and $f(\Phi(x,k,\delta)\hbar) \in \mathcal{S}$, for all $f \in \mathcal{F}$ and $\delta \in \mathbb{R}^D$, then the map $\phi : \mathcal{A}_{\hbar,q}^D(\mathcal{F}) \to \mathcal{S}_\hbar^\delta(\Lambda_0, E, \Lambda_1, R)$, defined as*

$$\phi(f_n U^n)(x,k) = q^{-N(\Phi(x,k,\delta),R^{-1}k)} f_{-R^{-1}k}\big(\Phi(x,k,\delta)\hbar\big), \tag{23}$$

is a left module homomorphism. Moreover, if $\mathcal{S}$ is a $\mathcal{A}_{\hbar,q}^D(\mathcal{F})$-$\mathcal{A}_{\hbar,q}^D(\mathcal{F})$ -bimodule, as defined in Proposition 3.19, with $P = -R$ and $\epsilon = \delta$, then ϕ is a bimodule homomorphism.

Proof. Let us start by showing that ϕ is a left module homomorphism. To this end we compute

$$\phi\big(f \cdot g\big)(x,k) = \sum_{n,l \in \mathbb{Z}^D} \phi\big(q^{N(l,n-l)} f_l (S_\hbar^l g_{n-l}) U^n\big)(x,k)$$

$$= \sum_{l\in\mathbb{Z}^D} q^{-N(\Phi(x,k,\delta),R^{-1}k)} q^{N(l,-R^{-1}k-l)} \times f_l(\Phi(x,k,\delta)\hbar) g_{-R^{-1}k-l}(\Phi(x,k,\delta)\hbar + l\hbar)\,,$$

as well as

$$\begin{aligned}
\big(f\cdot\phi(g)\big)(x,k) &= \sum_{l\in\mathbb{Z}^D} q^{N(\Phi(x,k,\delta),l)} f_l\big(\Phi(x,k,\delta)\hbar\big)\phi(g)(x+El,k+Rl)\\
&= \sum_{l\in\mathbb{Z}^D} q^{N(\Phi(x,k,\delta),l)} f_l\big(\Phi(x,k,\delta)\hbar\big) q^{-N(\Phi(x+El,k+Rl,\delta),R^{-1}(k+Rl))}\\
&\quad\times g_{-R^{-1}(k+Rl)}\big(\Phi(x+El,k+Rl,\delta)\hbar\big)\\
&= \sum_{l\in\mathbb{Z}^D} q^{-N(\Phi(x,k,\delta),R^{-1}k)} q^{-N(l,R^{-1}k+l)}\\
&\quad\times f_l\big(\Phi(x,k,\delta)\hbar\big) g_{-R^{-1}k-l}\big(\Phi(x,k,\delta)\hbar + l\hbar\big)\\
&= \big(f\cdot\phi(g)\big)(x,k)
\end{aligned}$$

by using that $\Phi(x+El,k+Rl,\delta) = \Phi(x,k,\delta)+l$. Hence, ϕ is a left module homomorphism. Now, assume that $\mathcal{S}$ is a bimodule as defined in Proposition 3.19 with $R\in \mathrm{GL}_D(\mathbb{Z})$, $P=-R$ and $\epsilon=\delta$. In this case, it follows from Lemma 3.20 that

$$\Psi(x,k,\delta) = \Gamma_0 x + \Gamma_1 k + \delta = \Lambda_0 x + \Lambda_1 k - R^{-1}k + \delta = \Phi(x,k,\delta) - R^{-1}k\,.$$

Let us now show that ϕ is also a right module homomorphism. One computes

$$\begin{aligned}
\phi(g\cdot f)(x,k) &= \sum_{n,l\in\mathbb{Z}^D} \phi(q^{N(l,n-l)} g_l (S_\hbar^l f_{n-l}) U^n)(x,k)\\
&= \sum_{l\in\mathbb{Z}^D} q^{-N(\Phi(x,k,\delta),R^{-1}k)} q^{-N(l,R^{-1}k+l)} g_l\big(\Phi(x,k,\delta)\hbar\big)\\
&\quad\times f_{-R^{-1}k-l}\big(\Phi(x,k,\delta)\hbar + l\hbar\big)
\end{aligned}$$

as well as

$$\begin{aligned}
\big(\phi(g)\cdot f\big)(x,k) &= \sum_{n\in\mathbb{Z}^D} q^{N(\Psi(x,k,\delta-n),n)} f_n(\Psi(x,k,\delta-n)\hbar)\\
&\quad\times \phi(g)(x-Fn,k-Pn)\\
&= \sum_{n\in\mathbb{Z}} q^{N(\Psi(x,k,\delta-n),n)} q^{-N(\Phi(x-Fn,k-Pn,\delta),R^{-1}(k-Pn))}
\end{aligned}$$

$$
\begin{aligned}
&\times f_n(\Psi(x,k,\delta-n)\hbar)g_{-R^{-1}(k-Pn)}\\
&\times \big(\Phi(x-Fn,k-Pn,\delta)\hbar\big)
\end{aligned}
$$

which, by using that $\Lambda_0 F + \Lambda_1 P = 0$ and $P = -R$, becomes

$$
\begin{aligned}
&\sum_{n\in\mathbb{Z}} q^{N(\Psi(x,k,\delta-n),n)} q^{-N(\Phi(x,k,\delta),R^{-1}k+n)}\\
&\qquad \times f_n(\Psi(x,k,\delta-n)\hbar)g_{-R^{-1}k-n}\big(\Phi(x,k,\delta)\hbar\big)\,.
\end{aligned}
$$

By changing the summation index to $l = -R^{-1}k - n$, and using that $\Psi(x,k,\delta) + R^{-1}k = \Phi(x,k,\delta)$, one obtains

$$
\begin{aligned}
&\sum_{l\in\mathbb{Z}^D} q^{N(\Phi(x,k,\delta)+l,-R^{-1}k-l)} q^{-N(\Phi(x,k,\delta),-l)}\\
&\qquad \times f_{-R^{-1}k-l}(\Phi(x,k,\delta)\hbar + l\hbar)g_l\big(\Phi(x,k,\delta)\hbar\big)\\
&= \sum_{l\in\mathbb{Z}^D} q^{N(\Phi(x,k,\delta),-R^{-1}k)} q^{N(l,-R^{-1}k-l)}\\
&\qquad \times f_{-R^{-1}k-l}(\Phi(x,k,\delta)\hbar + l\hbar)g_l\big(\Phi(x,k,\delta)\hbar\big)\\
&= \phi(g\cdot f)(x,k)\,,
\end{aligned}
$$

showing that ϕ is indeed a right module homomorphism under the above assumptions. □

In principle, one can now show that if $\Lambda_0 \in \mathrm{GL}_D(\mathbb{C})$, then the homomorphism in Proposition 3.21 is in fact an isomorphism. However, there are a few technical assumptions that one needs in order for the homomorphism to be surjective. This statement is made precise in the next result.

Proposition 3.22. *Let $\mathcal{S}$ be a $\mathcal{F}$-invariant subspace such that*

1) *ξ has compact support for all $\xi \in \mathcal{S}$,*
2) *$\xi(Au + \lambda, n) \in \mathcal{F}$ for all $\xi \in \mathcal{S}$, $A \in \mathrm{Mat}_D(\mathbb{C})$, $\lambda \in \mathbb{R}^D$ and $n \in \mathbb{Z}^D$,*
3) *$f(\Phi(x,k,\delta)\hbar) \in \mathcal{S}$, for all $f \in \mathcal{F}$ and $\delta \in \mathbb{R}^D$.*

If $\Lambda_0 \in \mathrm{GL}_D(\mathbb{C})$ then $\mathcal{S}^\delta_\hbar(\Lambda_0, E, \Lambda_1, R) \simeq \mathcal{A}^D_{\hbar,q}(\mathcal{F})$ as a left module. Furthermore, if $\mathcal{S}$ is a $\mathcal{A}^D_{\hbar,q}(\mathcal{F})$-$\mathcal{A}^D_{\hbar,q}(\mathcal{F})$-bimodule, as defined in Proposition 3.19, with $P = -R$ and $\epsilon = \delta$, then $\mathcal{S}^\delta_\hbar(\Lambda_0, E, \Lambda_1, R) \simeq \mathcal{A}^D_{\hbar,q}(\mathcal{F})$ as a bimodule.

Proof. Let us prove the statements by showing that the homomorphism in Proposition 3.21 is a left module (resp. bimodule) isomorphism under the given assumptions. Since Proposition 3.21 shows that ϕ is a module homomorphism, it remains to prove that ϕ is invertible.

For $\Lambda_0 \in \mathrm{GL}_D(\mathbb{C})$ one can explicitly construct the inverse of ϕ as

$$\phi^{-1}(\xi)(u) = \sum_{n\in\mathbb{Z}^D} q^{-\frac{1}{\hbar}N(u,n)}\xi\big(\Lambda_0^{-1}(u/\hbar + \Lambda_1 Rn - \delta), -Rn\big)U^n\,,$$

and we note that the sum is finite since ξ has compact support, by assumption. Moreover, the above function is clearly in $\mathcal{F}$ since one assumes that $\xi(Au+\lambda, n) \in \mathcal{F}$ for all $\xi \in \mathcal{S}$, $A \in \mathrm{Mat}_D(\mathbb{C})$, $\lambda \in \mathbb{R}^D$ and $n \in \mathbb{Z}^D$. □

Note that the hypotheses of Proposition 3.22 are fulfilled if, for instance, $\mathcal{F}$ and $\mathcal{S}$ consists of all functions with compact support on their respective domains.

4. Representations of shift subalgebras

From this section on we are interested in certain subalgebras of $\mathcal{A}^D_{\hbar,q}(\mathcal{F})$, and their representations in order to construct fuzzy analogues of a large class of level sets. We are particularly interested in finite dimensional representations for which the vector space dimension increases as the parameter $\hbar$ for the algebra decreases. Such sequences of representations can be associated with a level set, i.e. a subset of the form

$$\{(x_1,\ldots,x_n) \in \mathbb{R}^n : f(x_1,\ldots,x_n) = 0\}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a real valued function. We will also consider sequences of infinite dimensional representations with a countable basis, where the parameter $\hbar$ decreases.

For example, in the case $D = 1$, one can associate algebra elements $\sum f_n(u)U^n$ with functions $\sum_n f_n(u)e^{in\varphi}$. Then, we shall consider subalgebras generated by two elements of the form $f(u)U$ and $f(u)U^{-1}$ and the classical limit of such a subalgebra is a parametrized surface defined by

$$\begin{aligned} x &= f(u)\cos\varphi \\ y &= f(u)\sin\varphi \\ z &= u\,. \end{aligned} \tag{24}$$

Since the coordinates x and y fulfill

$$x^2 + y^2 = f(z)^2\,, \tag{25}$$

such equations describe an infinitely long cylinder immersed into $\mathbb{R}^3$, when $f(u)$ is real. We will see that for functions $f(u)$, which become 0 at some points and where the level set pinches off, it is possible to construct representations, which restrict to an interval of $\mathbb{R}$, where $f(u) \geq 0$ for $u_1 \leq u \leq u_2$ (where possibly $u_1 = -\infty$ and/or $u_2 = +\infty$). The classical limit of the construction is an immersion of a finite cylinder or an infinite half-cylinder in $\mathbb{R}^3$. Let us now start by defining the subalgebras we are interested in. As a standing assumption, we assume that $\mathcal{F}$ separates points of $\mathbb{R}^D$.

Definition 4.1. A $*$-subalgebra $\mathcal{A} \subseteq \mathcal{A}^D_{\hbar,q}(\mathcal{F})$ such that $fa \in \mathcal{A}$ for every $f \in \mathcal{F}$ and $a \in \mathcal{A}$ is called *shift subalgebra*.

In what follows, recall that $\{|n\rangle\}_{n\in\mathbb{Z}^D}$ is a basis of $\mathcal{S}_\hbar(\mathbb{1},\delta)$, which is also a (left) module for every subalgebra of $\mathcal{A}^D_{\hbar,q}(\mathcal{F})$, and that the action of an algebra element is given as

$$f_n U^n |k\rangle = \sum_{n\in\mathbb{Z}^D} q^{N(k-n+\delta,n)} f_n\big((k-n+\delta)\hbar\big) |k-n\rangle \ . \tag{26}$$

For a function $f \in \mathcal{F}$ and the generators U^n this reduces to

$$f|k\rangle = f\big((k+\delta)\hbar\big)|k\rangle \quad \text{and} \quad U^n|k\rangle = q^{N(k-n+\delta,n)}|k-n\rangle \ .$$

In the case when $\mathcal{F}$ separates points, one can show that cyclic modules, generated by a shift subalgebra acting on $|k\rangle$, are simple.

Proposition 4.2. *Assume that $\mathcal{F}$ separates points and let $\mathcal{A}$ be a shift subalgebra of $\mathcal{A}^D_{\hbar,q}(\mathcal{F})$. For $k \in \mathbb{Z}^D$, let $\mathcal{S}^\delta_\hbar(|k\rangle) \subseteq \mathcal{S}_\hbar(\mathbb{1},\delta)$ denote the $\mathcal{A}$-module generated by $|k\rangle$. Then $\mathcal{S}^\delta_\hbar(|k\rangle)$ is a simple $\mathcal{A}$-module.*

Proof. Let us show that $\mathcal{S}^\delta_\hbar(|k\rangle)$ is a simple module by showing that every vector is cyclic. We achieve this by showing that for every $v \in \mathcal{S}^\delta_\hbar(|k\rangle)$, there exists $a \in \mathcal{A}$ such that $av = |k\rangle$.

To this end, let $v \in \mathcal{S}^\delta_\hbar(|k\rangle)$ be an arbitrary (nonzero) element. Since the module is generated by $|k\rangle$, there exists $a_v \in \mathcal{A}$ such that $v = a_v|k\rangle$. Writing

$$a_v = \sum_{n\in I} a_{v,n} U^n$$

with $I \subseteq \mathbb{Z}^D$ such that $|I| < \infty$, one obtains

$$v = \sum_{n\in I} q^{N(k-n+\delta,n)} a_{v,n}\big((k-n+\delta)\hbar\big)|k-n\rangle \, ,$$

and, since $v \neq 0$, there exists $n_0 \in I$ such that $a_{v,n_0}\big((k-n_0+\delta)\hbar\big) \neq 0$. Now, since $\mathcal{F}$ is assumed to separate points, there exist $p_1, p_2 \in \mathcal{F}$ such that

$$p_1\big((k-m+\delta)\hbar\big) = \begin{cases} 1 & \text{if } m \in I \text{ and } m = n_0 \\ 0 & \text{if } m \in I \text{ and } m \neq n_0, \end{cases}$$
$$p_2\big((k-n_0+m+\delta)\hbar\big) = \begin{cases} 1 & \text{if } m \in I \text{ and } m = n_0 \\ 0 & \text{if } m \in I \text{ and } m \neq n_0\,, \end{cases}$$

giving

$$p_1 v = q^{N(k-n_0+\delta,n_0)} a_{v,n_0}\big((k-n_0+\delta)\hbar\big)\,|k-n_0\rangle\ .$$

Since $\mathcal{A}$ is a $*$-subalgebra, it follows that $a_v^* \in \mathcal{A}$ and

$$\begin{aligned} a_v^* p_1 v &= q^{N(k-n_0+\delta,n_0)} a_{v,n_0}\big((k-n_0+\delta)\hbar\big) \\ &\quad\times \sum_{n\in I} q^{-N(k-n_0+n+\delta,n)} \overline{a_{v,n}\big((k-n_0+\delta)\hbar\big)}\,|k-n_0+n\rangle \\ p_2 a_v^* p_1 v &= q^{-N(n_0,n_0)} |a_{v,n_0}\big((k-n_0+\delta)\hbar\big)|^2\,|k\rangle\ . \end{aligned}$$

Thus, setting

$$a = q^{N(n_0,n_0)} |a_{v,n_0}\big((k-n_0+\delta)\hbar\big)|^{-2} p_2 a_v^* p_1\,,$$

it follows that $av = |k\rangle$. Hence, we have shown that every $v \in \mathcal{S}_\hbar^\delta(|k\rangle)$ is cyclic and, consequently, that $\mathcal{S}_\hbar^\delta(|k\rangle)$ is a simple module. □

Recall that if the function algebra $\mathcal{F}$ separates points, then Proposition 3.15 implies that the module $\mathcal{S}_\hbar^\delta(|k\rangle)$ is not free. Furthermore, Proposition 3.16 implies that in the case when $\mathcal{F}$ separates points, the module $\mathcal{S}_\hbar^\delta(|k\rangle)$ is projective, if and only if there exists $p_0 \in \mathcal{F}$ such that

$$p_0(u) = \begin{cases} 1 & \text{if } u = \hbar\delta \\ 0 & \text{if } u \neq \hbar\delta\,. \end{cases}$$

In the case of shift subalgebras generated from a finite set of algebra elements, we shall see that for particular choices, modules defined in Proposition 4.2 might be finite dimensional. Moreover, by carefully choosing $\hbar$ in relation to the dimension of the representation, one can construct fuzzy spaces, i.e. sequences of finite dimensional representations of increasing dimension as $\hbar$ tends to zero.

4.1. Shift subalgebras for $D = 1$

For $f \in \mathcal{F}$, we let $\mathcal{A}_\hbar(\mathcal{F}, f)$ denote the shift subalgebra generated by

$$A_+ = f\left(u - \tfrac{1}{2}\hbar\right) U^{-1} \quad \text{and} \quad A_- = f\left(u + \tfrac{1}{2}\hbar\right) U \,.$$

The generators A_+ and A_- satisfy

$$[A_+, A_-] = f(u - \tfrac{1}{2}\hbar)^2 - f(u + \tfrac{1}{2}\hbar)^2 \tag{27}$$

$$A_+A_- + A_-A_+ = f(u - \tfrac{1}{2}\hbar)^2 + f(u + \tfrac{1}{2}\hbar)^2, \tag{28}$$

and the action of A_+ and A_- is given by

$$\begin{aligned} A_+ \left|n\right\rangle &= f\big((n + \tfrac{1}{2} + \delta)\hbar\big) \left|n + 1\right\rangle \quad \text{and} \\ A_- \left|n\right\rangle &= f\big((n - \tfrac{1}{2} + \delta)\hbar\big) \left|n - 1\right\rangle \,. \end{aligned} \tag{29}$$

Proposition 4.3. *For $f \in \mathcal{F}$ and $k \in \mathbb{Z}$, let $\mathcal{S}^\delta_\hbar(|k\rangle)$ denote the simple left $\mathcal{A}_\hbar(\mathcal{F}, f)$-module defined in Proposition 4.2.*

1) *If $f\big((N + \frac{1}{2} + \delta)\hbar\big) = 0$ for some $N \in \mathbb{Z}$ and $f\big((n + \frac{1}{2} + \delta)\hbar\big) \neq 0$ for all integers $n < N$ then*

$$\mathcal{S}^\delta_\hbar(|N\rangle) = \Big\{ \sum_{n \leq N} v_n \left|n\right\rangle : v_n \in \mathbb{C} \text{ and } |n : v_n \neq 0| < \infty \Big\} .$$

2) *If $f\big((M - \frac{1}{2} + \delta)\hbar\big) = 0$, for some $M \in \mathbb{Z}$, and $f\big((n - \frac{1}{2} + \delta)\hbar\big) \neq 0$ for all integers $n > M$ then*

$$\mathcal{S}^\delta_\hbar(|M\rangle) = \Big\{ \sum_{n \geq M} v_n \left|n\right\rangle : v_n \in \mathbb{C} \text{ and } |n : v_n \neq 0| < \infty \Big\} .$$

3) *If $f\big((M - \frac{1}{2} + \delta)\hbar\big) = 0$ and $f\big((N + \frac{1}{2} + \delta)\hbar\big) = 0$, for some $M, N \in \mathbb{Z}$ with $M < N$, and $f\big((n + \frac{1}{2} + \delta)\hbar\big) \neq 0$ for all integers $M \leq n \leq N$ then*

$$\mathcal{S}^\delta_\hbar(|M\rangle) = \mathcal{S}^\delta_\hbar(|N\rangle) = \Big\{ \sum_{M \leq n \leq N} v_n \left|n\right\rangle : v_n \in \mathbb{C} \Big\} .$$

Proof. Let us provide a proof of (3), since the other two cases are proved analogously. The assumptions immediately gives that

$$A_+ \left|N\right\rangle = f\big((N + \tfrac{1}{2} + \delta)\hbar\big) \left|N + 1\right\rangle = 0$$

$$A_- |M\rangle = f\big((M - \tfrac{1}{2} + \delta)\hbar\big) |M-1\rangle = 0 .$$

Moreover,

$$\begin{aligned} A_+ |n\rangle &= f\big((n + \tfrac{1}{2} + \delta)\hbar\big) |n+1\rangle \neq 0 \quad \text{for } M \leq n < N \\ A_- |n\rangle &= f\big((n - \tfrac{1}{2} + \delta)\hbar\big) |n-1\rangle \\ &= f\big((n - 1 + \tfrac{1}{2} + \delta)\hbar\big) |n-1\rangle \neq 0 \quad \text{for } M < n \leq N \end{aligned}$$

since $f\big((n + \frac{1}{2} + \delta)\hbar\big) \neq 0$ for $M \leq n \leq N$, by assumption. It follows that $\mathcal{S}_\hbar^\delta(|M\rangle) = \mathcal{S}_\hbar(|N\rangle) = \Big\{ \sum_{M \leq n \leq N} v_n |n\rangle : v_n \in \mathbb{C} \Big\}$. □

Note that Proposition 4.2 implies that the modules in the above result are simple. In the following we shall mostly be interested in the type of finite dimensional representations described by (3) in Proposition 4.3.

Example 4.4. *Let $\mathcal{F}$ be the polynomial functions on $\mathbb{R}$ and let $f(u) = \frac{9}{4} - u^2$. Let us now construct a three dimensional representation of $\mathcal{A}_\hbar(\mathcal{F}, f)$ with $\hbar = 1$. Since*

$$f\Big(\hbar(1 + \frac{1}{2})\Big) = f\Big(\hbar(-1 - \frac{1}{2})\Big) = 0$$

we choose $\delta = 0$, $M = -1$ and $N = 1$ and $\mathcal{S}_1^0(|1\rangle)$ is spanned by $|-1\rangle, |0\rangle, |1\rangle$ according to Proposition 4.3. One finds that

$$\begin{aligned} A_+ |-1\rangle &= f(-1 + \tfrac{1}{2}) |0\rangle = 2 |0\rangle \quad & A_+ |0\rangle &= f(0 + \tfrac{1}{2}) |1\rangle = 2 |1\rangle \\ A_- |1\rangle &= f(1 - \tfrac{1}{2}) |0\rangle = 2 |0\rangle \quad & A_- |0\rangle &= f(0 - \tfrac{1}{2}) |-1\rangle = 2 |-1\rangle \end{aligned}$$

Identifying

$$|-1\rangle \sim \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |0\rangle \sim \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |1\rangle \sim \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

gives the matrix representation

$$A_+ = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \qquad A_- = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Moreover, $g \in \mathcal{F}$ is represented as the diagonal matrix $\hat{g}$, given by

$$\hat{g} = \begin{pmatrix} g(-1) & 0 & 0 \\ 0 & g(0) & 0 \\ 0 & 0 & g(1) \end{pmatrix}.$$

Let us now describe how one can construct a sequence of algebras

$$\mathcal{A}_{\hbar_1}(\mathcal{F}, f), \mathcal{A}_{\hbar_2}(\mathcal{F}, f), \mathcal{A}_{\hbar_3}(\mathcal{F}, f) \dots$$

together with simple finite dimensional modules of increasing dimension, providing a "fuzzy" analogue of the surface defined by (4). To this end, assume that $f \in \mathcal{F}$ such that $f(u_1) = f(u_2) = 0$, where $u_1 < u_2$, and $f(u) \neq 0$ for $u_1 < u < u_2$. Furthermore, let $N_1, N_2, \dots$ be an increasing sequence of positive integers and set

$$\hbar_k = \frac{u_2 - u_1}{N_k}. \tag{30}$$

Choosing $M_k \in \mathbb{Z}$ and $\delta_k \in [0, 1)$ such that

$$u_1 = \big(M_k - \tfrac{1}{2} + \delta_k\big)\hbar$$

one checks that

$$\begin{aligned} f\big((M_k - \tfrac{1}{2} + \delta_k)\hbar\big) &= f(u_1) = 0 \\ f\big((M_k + N_k - 1 + \tfrac{1}{2} + \delta_k)\hbar\big) &= f(u_2) = 0 \end{aligned}$$

and Proposition 4.3 implies that $\mathcal{S}_{\hbar_k}^{\delta_k}(|M_k\rangle)$ is a simple N_k-dimensional $\mathcal{A}_{\hbar_k}(\mathcal{F}, f)$-module. Clearly, because of (30), $\hbar_k$ decreases as N_k increases.

Example 4.5. *As an illustration of the procedure given above, let us construct a sequence of odd-dimensional modules related to the algebra given in Example 4.4. In the notation above, let $u_1 = -3/2$, $u_2 = 3/2$ giving $u_2 - u_1 = 3$. For a sequence of positive integers $N_1 < N_2 < \cdots$ we set $\hbar_k = 3/N_k$, and find $M_k \in \mathbb{Z}$ and $\delta_k \in [0, 1)$ such that*

$$u_1 = \big(M - \tfrac{1}{2} + \delta_k\big)\hbar \quad \Leftrightarrow \quad -\frac{3}{2} = \big(M_k - \tfrac{1}{2} + \delta_k\big)\frac{3}{N_k}$$

which results in

$$\begin{cases} M_k = \frac{1-N_k}{2},\ \delta_k = 0, & \textit{for } N_k \textit{ odd} \\ M_k = -\frac{N_k}{2},\ \delta_k = \frac{1}{2}, & \textit{for } N_k \textit{ even}. \end{cases}$$

With these values, $\mathcal{S}_{\hbar_k}^{\delta_k}(|M_k\rangle)$ is a simple N_k-dimensional $\mathcal{A}_{3/N_k}(\mathcal{F}, f)$-module for $k \geq 1$. The matrix elements of the operators A_+, A_- are easily deduced from

$$\begin{aligned} A_+ |n\rangle &= f\big((n + \tfrac{1}{2} + \delta_k)\hbar_k\big) |n+1\rangle \\ &= \big(\tfrac{9}{4} - (n + \tfrac{1}{2} + \delta_k)^2 \tfrac{9}{N_k^2})\big) |n+1\rangle \\ A_- |n\rangle &= f\big((n - \tfrac{1}{2} + \delta_k)\hbar_k\big) |n-1\rangle \\ &= \big(\tfrac{9}{4} - (n - \tfrac{1}{2} + \delta_k)^2 \tfrac{9}{N_k^2})\big) |n-1\rangle \ . \end{aligned}$$

It is interesting that we see a "spin" in δ_k, related to even and odd dimension of the module.

In the classical limit, the algebra elements A_+ and A_- are identified with the complexified coordinate $x + iy$ and its conjugate. The coordinates x and y then fulfill

$$x^2 + y^2 = \left(\frac{9}{4} - u^2\right)^2$$

where u is restricted to the interval $[u_1 = -\frac{3}{2}, u_2 = \frac{3}{2}]$. The resulting surface is spindle shaped with peaks at u_1 and u_2 (cf. Figure 1).

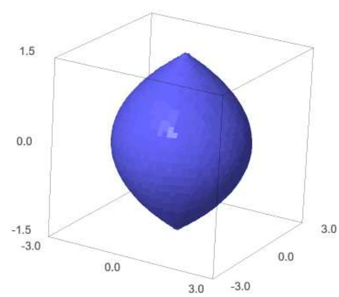

Figure 1: The level set related to the classical limit in Example 4.5.

In the following, we will provide several examples, in which the generators have the form $A_i = \sqrt{f_n} U^n$ (no summation over n), where $f_n \in \mathcal{F}$ is a *real-valued* function. However, f_n is not assumed to be strictly positive and we shall use the convention

$$\sqrt{x} = i\sqrt{|x|}\,, \text{ for } x < 0\,.$$

Example 4.6 (Fuzzy sphere). *We start with*

$$A_+ = \sqrt{R^2 - u(u-\hbar)}U^{-1}, \quad A_- = \sqrt{R^2 - u(u+\hbar)}U$$

where R is a real positive number. With this definition

$$[A_+, A_-] = 2\hbar u, \quad \frac{1}{2}(A_+A_- + A_-A_+) + 2u^2 = 2R^2\,,$$

in agreement with the defining relations of the fuzzy sphere.

On the representation $\mathcal{S}_\hbar^\delta(|k\rangle)$, this becomes

$$\begin{aligned} A_+\,|n\rangle &= \sqrt{R^2 - \hbar^2(n+\delta+1)(n+\delta)}\,|n+1\rangle \\ A_- &= \sqrt{R^2 - \hbar^2(n+\delta)(n+\delta-1)}\,|n-1\rangle\ . \end{aligned}$$

The parameters R, δ and $\hbar$ need not be interrelated for infinite dimensional representations. However, if we demand that there exists subrepresentations of $\mathcal{S}_\hbar^\delta(|k\rangle)$ according to Proposition 4.3, then this puts restrictions on the parameters.

In particular, in the notation of Proposition 4.3

$$f(u) = \sqrt{R_k^2 - \big(u + \frac{\hbar}{2}\big)\big(u - \frac{\hbar}{2}\big)}$$

where k is a natural number labeling the representation. Requiring $A_+\,|N_k\rangle = \mathcal{A}_-\,|M_k\rangle = 0$ gives

$$N_k + \frac{1}{2} + \delta_k = \pm\sqrt{\frac{R_k^2}{\hbar_k^2} - \frac{1}{4}}, \quad M_k - \frac{1}{2} + \delta_k = \pm\sqrt{\frac{R_k^2}{\hbar^2} - \frac{1}{4}}$$

For the choice of different signs of N_k and M_k, one can add the two equations resulting in

$$N_k + M_k + 2\delta_k = 0\,.$$

Since N_k and M_k are integers, δ_k is either 0 or $\frac{1}{2}$. Furthermore, we can set $N_k = k$ and $M_k = -k - 2\delta_k$. If we set

$$S_k = \begin{cases} \frac{N_k}{2}\,, & \textit{for } N_k \textit{ even} \\ \frac{N_k+1}{2}\,, & \textit{for } N_k \textit{ odd} \end{cases}$$

we arrive at the spin-S_k-representations of $su(2)$.

Example 4.7 (Fuzzy catenoid). *Analogously to the previous example, we start with*

$$A_+ = \sqrt{R^2 + u(u-\hbar)}U^{-1}\,, \quad A_- = \sqrt{R^2 + (u+\hbar)u}U$$

where $R > \hbar$ is again a real positive number. Due to the sign change compared to the previous example, the subalgebra generated by these two elements fulfills

$$[A_+, A_-] = -2\hbar u\,, \quad \frac{1}{2}(A_+A_- + A_-A_+) - u^2 = R^2\,.$$

In the classical limit one can identify A_+ and A_- with $x + iy$ and $x - iy$ and the second equation becomes $x^2 + y^2 - u^2 = R^2$. We can think of the subalgebra as a fuzzy version of a catenoid. The representations are infinite dimensional, and one can choose arbitrary values for R, $\hbar$ and δ.

Example 4.8 (Fuzzy cone and fuzzy plane). *In the previous examples, one finds quadratic polynomials in the square root. Here, we investigate a linear polynomial*

$$A_+ = \sqrt{u + \hbar c}\,U^{-1}\,, \quad A_- = \sqrt{u + \hbar(c+1)}\,U$$

where $c \in \mathbb{R}$ is a constant. This results in a constant commutator

$$[A_+, A_-] = -\hbar\,, \quad \frac{1}{2}(A_+A_- + A_-A_+) = u + \hbar(c + \frac{1}{2})\,.$$

In the classical limit the second equation becomes $x^2 + y^2 = u$. This is only solvable for $u \geq 0$. In the representation $\mathcal{S}_\hbar^\delta(|k\rangle)$, the action of the generators is

$$A_+\,|n\rangle = \sqrt{\hbar(n+\delta+c+1)}\,|n+1\rangle\;,$$
$$A_- = \sqrt{\hbar(n+\delta+c)}\,|n-1\rangle\;.$$

When restricting the representation $\mathcal{S}_\hbar^\delta(|k\rangle)$ to vectors with $n \geq N$, where N is an integer, it is necessary that $\hbar(\delta + c) = N$. The non-integer part of the constant c has to compensate δ. Let us assume that $N = 0$, since any other N can be reached by redefining $n \to n - N$. In the case $\hbar(\delta + c) = 0$, the representation is restricted to $n \geq 0$, where $\hbar$ can be arbitrary, and the action of the generators becomes

$$A_+\,|n\rangle = \sqrt{\hbar(n+1)}\,|n+1\rangle\;, \qquad A_- = \sqrt{\hbar n}\,|n-1\rangle\;.$$

The representation restricted to $n \geq 0$ can be seen as a fuzzy version of a cone. Note that a single cone with an opening angle of 180° is a plane. Since

$$\rho = \frac{1}{2}(A_+A_- + A_-A_+)\,|n\rangle = \hbar(n+\frac{1}{2})\,|n\rangle$$

we can identify ρ with r^2, where r is the distance to the origin. In this way, the representation can be viewed as a fuzzy plane.

4.2. Shift subalgebras for $D = 2$

In the case when $D = 2$, the algebra $\mathcal{A}^2_{\hbar,q}(\mathcal{F})$ is generated by U_1, U_2 together with functions in an appropriate subalgebra $\mathcal{F} \subseteq F(\mathbb{R}^2, \mathbb{C})$. We write $u = u_1$ and $v = u_2$ and consider the shift subalgebra generated by

$$U_+ = f\left(u - \frac{\hbar}{2}, v\right) U_1^{-1}\,, \quad U_- = f\left(u + \frac{\hbar}{2}, v\right) U_1\,, \tag{31}$$

$$V_+ = g\left(u, v - \frac{\hbar}{2}\right) U_2^{-1}\,, \quad V_- = g\left(u, v + \frac{\hbar}{2}\right) U_2\,, \tag{32}$$

for some $f, g \in \mathcal{F}$. In analogy with the one dimensional case, we denote the shift subalgebra of $\mathcal{A}^2_{\hbar,q}(\mathcal{F})$ generated by these generators and by $f \in \mathcal{F}$ by $\mathcal{A}^2_{\hbar,q}(\mathcal{F}; f, g)$.

The generators of $\mathcal{A}^2_{\hbar,q}(\mathcal{F}; f, g)$ fulfill

$$\begin{aligned} U_+U_- + U_-U_+ &= f^2(u - \frac{\hbar}{2}, v) + f^2(u + \frac{\hbar}{2}, v) \\ V_+V_- + V_-V_+ &= g^2(u, v - \frac{\hbar}{2}) + g^2(u, v + \frac{\hbar}{2})\,, \end{aligned}$$

which can be seen as defining relations of a fuzzy space. Namely, in the classical limit, when one identifies $U \sim x + iy$, $V \sim a + ib$, these relations become

$$x^2 + y^2 = f^2(u, v)\,, \quad a^2 + b^2 = g^2(u, v)\,. \tag{33}$$

Thus, the algebra $\mathcal{A}^2_{\hbar,q}(\mathcal{F}; f, g)$ can be seen as a fuzzy version of a (generically four dimensional) level set immersed in $\mathbb{R}^6$, defined by (33).

The representation $\mathcal{S}^\delta_\hbar(|k\rangle)$ of $\mathcal{A}^2_{\hbar,q}(\mathcal{F})$ has basis vectors $|n, m\rangle$, where $n, m \in \mathbb{Z}$. As in the one dimensional case, it is possible to construct irreducible representations of the subalgebra $\mathcal{A}^2_{\hbar,q}(\mathcal{F}; f, g)$ by choosing f and g such that their zero loci "cut out" a part of the $\mathbb{Z}^2$-lattice. We will show some examples of this in the following.

Example 4.9. *In this example, we shall construct a subalgebra of* $\mathcal{A}^2_{\hbar,q}(\mathcal{F})$ *with finite dimensional representations. In particular, let* N *be a natural number and let the generators of the subalgebra be defined by*

$$\begin{aligned}
U_+ &= \sqrt{(u-\hbar\delta_1)\,(\hbar(\delta_1+\delta_2+N+1)-u-v)}\tilde{f}\,(u,v)\,U_1^{-1}\\
U_- &= \sqrt{(u-\hbar(\delta_1+1))\,(\hbar(\delta_1+\delta_2+N)-u-v)}\tilde{f}\,(u,v)\,U_1\\
V_+ &= \sqrt{(v-\hbar\delta_2)\,(\hbar(\delta_1+\delta_2+N+1)-u-v)}\tilde{g}\,(u,v)\,U_2^{-1}\\
V_- &= \sqrt{(v-\hbar\delta_2)\,(\hbar(\delta_1+\delta_2+N)-u-v)}\tilde{g}\,(u,v)\,U_2\,,
\end{aligned}$$

where $\tilde{f},\tilde{g}\in\mathcal{F}$ *denote arbitrary positive functions. The action of the above generators is given by*

$$\begin{aligned}
U_+\left|n,m\right\rangle &= \hbar\sqrt{(n+1)(N-n-m)}\\
&\quad\times\tilde{f}\big(\hbar(n+1+\delta_1),\hbar(m+\delta_2)\big)\left|n+1,m\right\rangle\\
U_-\left|n,m\right\rangle &= \hbar\sqrt{n(N-1-n-m)}\\
&\quad\times\tilde{f}\big(\hbar(n+\delta_1),\hbar(m+\delta_2)\big)\left|n-1,m\right\rangle\\
V_+\left|n,m\right\rangle &= \hbar\sqrt{(m+1)(N-n+m)}\\
&\quad\times\tilde{g}\big(\hbar(n+\delta_1),\hbar(m+1+\delta_2)\big)\left|n,m+1\right\rangle\\
V_-\left|n,m\right\rangle &= \hbar\sqrt{m(N-1-n-m)}\\
&\quad\times\tilde{g}\big((\hbar(n+\delta_1),\hbar(m+\delta_2)\big)\left|n,m-1\right\rangle\,.
\end{aligned}$$

The important observations are now that $U_-\left|0,m\right\rangle=0$, $V_-\left|n,0\right\rangle=0$ *and that* $U_+\left|\hat{n},\hat{m}\right\rangle=V_+\left|\hat{n},\hat{m}\right\rangle=0$ *for* $\hat{n}+\hat{m}=N$. *One sees that when one starts with the generating vector* $\left|0,0\right\rangle$ *and applies a monomial in* U_+, V_+, U_- *and* V_-, *it is not possible to leave the part of the lattice defined by* $n\geq 0$, $m\geq 0$ *and* $n+m\leq N$. *Thus, the irreducible representation generated from* $\left|0,0\right\rangle$ *is finite dimensional and has dimension* $\frac{1}{2}(N+1)(N+2)$. *For the classical limit, where* $N\to\infty$, *one assumes that* $\hbar N=C$ *for some* $C\in\mathbb{R}$, *i.e. that when* N *increases then* $\hbar$ *decreases. This is analogous to the fuzzy sphere, where the assumption that the radius should be constant also interrelates* $\hbar$ *with the size of the representation* N.

In the classical limit, the algebra relations give two equations, which define a four dimensional level set in $\mathbb{R}^6$ *as follows*

$$x^2+y^2=u(C-u-v)\tilde{f}^2,\qquad a^2+b^2=v(C-u-v)\tilde{g}^2\,.$$

For instance, when $\tilde{f}=1$, *a section through the level set at constant* v *is a sphere with radius* $(C-v)/2$.

Example 4.10. *Let us now construct a subalgebra with infinite dimensional representations. The generators of the subalgebra of* $\mathcal{A}^2_{\hbar,q}(\mathcal{F})$ *are*

$$\begin{aligned} U_+ &= \sqrt{(u-\hbar\delta_1)\,(v-\hbar\delta_2)}\tilde{f}\,(u,v)\,U_1^{-1} \\ U_- &= \sqrt{(u-\hbar(1+\delta_1))\,(v-\hbar\delta_2)}\tilde{f}\,(u,v)\,U_1 \\ V_+ &= \sqrt{v-\hbar\delta_2}\tilde{g}\,(u,v)\,U_2^{-1} \\ V_- &= \sqrt{(v-\hbar(\delta_2+1)}\tilde{g}\,(u,v)\,U_2\,. \end{aligned}$$

Again, $\tilde{f},\tilde{g}\in\mathcal{F}$ *denote arbitrary positive functions. We will now show that the irreducible representation generated by* $|0,0\rangle$ *is restricted to* $\{|n,m\rangle : n\geq 0,\, m\geq 0\}$*. To this end, let us calculate the action of the generators*

$$\begin{aligned} U_+\,|n,m\rangle &= \hbar\sqrt{(n+1)m}\;\tilde{f}\big(\hbar(n+1+\delta_1),\hbar(m+\delta_2)\big)\,|n+1,m\rangle \\ U_-\,|n,m\rangle &= \hbar\sqrt{nm}\;\tilde{f}\big(\hbar(n+\delta_1),\hbar(m+\delta_2)\big)\,|n-1,m\rangle \\ V_+\,|n,m\rangle &= \sqrt{\hbar(m+1)}\;\tilde{g}\big(\hbar(n+\delta_1),\hbar(m+1+\delta_2)\big)\,|n,m+1\rangle \\ V_-\,|n,m\rangle &= \sqrt{\hbar m}\;\tilde{g}\big((\hbar(n+\delta_1),\hbar(m+\delta_2)\big)\,|n,m-1\rangle\;. \end{aligned}$$

It follows that $U_-\,|0,m\rangle=0$ *and that for* $n>0$, $U_-\,|n,m\rangle\neq 0$ *. Vice versa, for* $n\geq 0$, $U_+\,|n,m\rangle\neq 0$*. In the same way,* $V_-\,|n,0\rangle=0$ *and for* $m>0$ $V_-\,|n,m\rangle\neq 0$*. Vice versa, for* $m\geq 0$ $V_+\,|n,m\rangle\neq 0$*. Starting from* $|0,0\rangle$*, every vector* $|n,m\rangle$ *for* $n\geq 0$ *and* $m\geq 0$ *is reachable by applying* U_+ *and* V_+*. On the other hand, the vectors* $|n,m\rangle$ *for* $n<0$ *and* $m<0$ *are not reachable from* $|0,0\rangle$*, since it is not possible to move outside the lines* $n=0$ *and* $m=0$ *in* $\mathbb{Z}^2$*. Therefore, the representation* $\mathcal{S}^\delta_\hbar(|k\rangle)$ *is infinite dimensional and has basis vectors* $|n,m\rangle$ *with* $n,m\geq 0$*.*

In the classical limit the algebra relations become

$$x^2+y^2=uv\tilde{f}(u,v)\,,\quad a^2+b^2=v\tilde{g}(u,v)\,,$$

defining a non-compact level set in $\mathbb{R}^6$*. Note that the restriction to* $n\geq 0$ *and* $m\geq 0$ *in the representation restricts the coordinates in the classical limit to* $u\geq 0$ *and* $v\geq 0$*. This example is particularly interesting, since if one sets* $u=r+z$*,* $v=r-z$ *and* $\tilde{f}=1$*, then, in the classical limit, the algebra relations become*

$$x^2+y^2+z^2=r^2\,,\quad a^2+b^2=(r-z)\tilde{g}(r,z)\,.$$

Since $r=\frac{1}{2}(u+v)$*, one may interpret* r *as a radius since it is non-negative because of* $u\geq 0$ *and* $v\geq 0$*. Additionally, we can parametrize the coordinates* *a and b with a compact parameter* τ*:*

$$a = \sqrt{(r-z)\tilde{g}(r,z)}\cos\tau\,, \quad b = \sqrt{(r-z)\tilde{g}(r,z)}\sin\tau\,.$$

Thus, the level set can be parametrized by $\mathbb{R}^3 \times S^1$*, where* S^1 *is a circle with radius* $\sqrt{(r-z)\tilde{g}(r,z)}$*, vanishing when* $r = z$*.*

5. Representations of Lie algebras

In this section we show that, for any finite dimensional Lie algebra $\mathfrak{g}$, one can define a shift subalgebra where the generators satisfy the commutation relations of $\mathfrak{g}$. Consequently, one may construct irreducible representations $\mathcal{S}^\delta_\hbar(|k\rangle)$ of $\mathfrak{g}$ by using Proposition 4.2. Moreover, the Lie algebra generators in the shift subalgebra provide a natural set of inner derivations, suggesting the interpretation of the shift subalgebra as a "noncommutative homogeneous space". A lot of literature exists, in which such fuzzy homogeneous spaces are constructed. For instance, it is possible to use coherent states [14], highest weight representations of Lie algebras [2], [7] or subspaces of Fock space representations [13], [15]. In this section, we illustrate another way of arriving at such fuzzy homogeneous spaces.

We start with a shift algebra $\mathcal{A}^D_{\hbar,q}(\mathcal{F})$, and assume in this section that $q = 1$. For notational convenience, we write $S_i = S_{\hbar,i}$ (cf. Section 2) such that $(S_i f)(u_1,\dots,u_D) = f(u_1,\dots,u_i+\hbar,\dots,u_D)$.

Lemma 5.1. *Let* $\mathcal{A}^D_{\hbar,1}$ *be a shift algebra and let* $f_i \in \mathcal{F}$*, for* $i = 1,\dots,D$*, such that* $S_k f_i = f_i$ *for* $k \neq i$*. Then the shift subalgebra generated by*

$$E_{ij} = f_i U_i^{-1} U_j f_j \qquad (i,j = 1,\dots,D)$$

(no summation over i *and* j*) fulfills the commutator relations*

$$[E_{ij}, E_{kl}] = \delta_{jk}\Delta_j E_{il} - \delta_{il}\Delta_i E_{kj} \tag{34}$$

where $\Delta_i = S_i(f_i^2) - f_i^2$*.*

Proof. By direct calculation one shows that for $i \neq j \neq k \neq l$

$$\begin{aligned}
&E_{ij}E_{kl} = E_{kl}E_{ij} \quad \Rightarrow \quad [E_{ij}, E_{kl}] = 0\\
&E_{ii}E_{jj} = E_{jj}E_{ii} \quad \Rightarrow \quad [E_{ii}, E_{jj}] = 0\\
&E_{ij}E_{kj} = E_{kj}E_{ij} \quad \Rightarrow \quad [E_{ij}, E_{kj}] = 0\\
&E_{ij}E_{ik} = E_{ik}E_{ij} \quad \Rightarrow \quad [E_{ij}, E_{ik}] = 0\\
&E_{ij}E_{jk} = S_j(f_j^2)E_{ik}\,,
\end{aligned}$$

$$\begin{aligned}
&E_{jk}E_{ij} = f_j^2 E_{ik} \quad \Rightarrow \quad [E_{ij}, E_{jk}] = \left(S_j(f_j^2) - f_j^2\right) E_{ik} \\
&E_{ij}E_{jj} = S_j(f_j^2)E_{ij}\,, \\
&E_{jj}E_{ij} = f_j^2 E_{ij} \quad \Rightarrow \quad [E_{ij}, E_{jj}] = \left(S_j(f_j^2) - f_j^2\right) E_{ij} \\
&E_{ij}E_{ji} = f_i^2\, S_j(f_j^2) = S_j(f_j^2)E_{ii} \\
&\quad \Rightarrow \quad [E_{ij}, E_{ji}] = \left(S_j(f_j^2) - f_j^2\right) E_{ii} - \left(S_i(f_i^2) - f_i^2\right) E_{jj}\,.
\end{aligned}$$

These relations can be summarized into (34). □

If Δ_i is constant and independent of i, then in Lemma 5.1, the commutator relations (34) reduce to the Lie algebra relations of $u(D)$. In this case, the shift subalgebra generated by the E_{ij} is an enveloping algebra of $u(D)$ (but not the universal one). Setting $\Delta_i = \hbar$ can be achieved by choosing $f_i = \sqrt{u_i + c}$, for arbitrary $c \in \mathbb{R}$. In the following we show that this can be generalized to any finite dimensional Lie algebra $\mathfrak{g}$.

Proposition 5.2. *Let $\mathcal{A}^D_{\hbar,1}(\mathcal{F})$ be a shift algebra such that $\sqrt{u_i + c_i} \in \mathcal{F}$ for $c_i \in \mathbb{R}$ and $i = 1, \ldots, D$, and let $\mathfrak{g}$ denote a finite dimensional Lie algebra with basis $\{X^a\}_{a=1}^n$ such that $[X^a, X^b] = f^{ab}{}_c X^c$. Furthermore, let (X^a_{ij}) denote the matrices of a N-dimensional representation of $\mathfrak{g}$. Then the elements*

$$\hat{X}^a = \sum_{i,j=1}^N X^a_{ij}\sqrt{u_i + c_i}U_i^{-1}U_j\sqrt{u_j + c_j}$$

satisfy $[\hat{X}^a, \hat{X}^b] = \hbar f^{ab}{}_c \hat{X}^c$ for $a, b = 1, \ldots, n$.

Proof. In the notation of Lemma 5.1, one finds that $\Delta_i = u_i + \hbar + c_i - u_i - c_i = \hbar$. Defining

$$E_{ij} = \sqrt{u_i + c_i}U_i^{-1}U_j\sqrt{u_j + c_j} \tag{35}$$

and setting

$$\hat{X}^a = \sum_{i,j=1}^N X^a_{ij}E_{ij}\,,$$

it follows immediately from (34) that

$$[\hat{X}^a, \hat{X}^b] = \hbar f^{ab}{}_c \hat{X}^c\,.$$

□

Thus, Proposition 5.2 implies that the shift subalgebra generated by $\{\hat{X}^a\}_{a=1}^n$ is an enveloping algebra of the Lie algebra $\mathfrak{g}$.

Let us now present the main result of this section, showing that one can construct irreducible representations of an arbitrary finite dimensional Lie algebra by considering representations of shift subalgebras.

Proposition 5.3. *Let $\mathfrak{g}$ be a n-dimensional Lie algebra, and let $\mathcal{A}$ be the shift subalgebra generated by $\{\hat{X}^a\}_{a=1}^n$, as defined in Proposition 5.2. For $n = (n_1, \dots, n_D) \in \mathbb{Z}^D$, the simple $\mathcal{A}$-module $\mathcal{S}_\hbar^\delta(|n\rangle)$ (cf. Proposition 4.2) is an irreducible representation of $\mathfrak{g}$. Moreover,*

$$\mathcal{S}_\hbar^\delta(|n_1, \dots, n_D\rangle) \subseteq \operatorname{span}_{\mathbb{C}}\{|k_1, \dots, k_D\rangle : k_1 + \cdots + k_D = n_1 + \cdots + n_D\}.$$

If $\hbar\delta_i + c_i = 0$ and $n_i \geq 0$ for $i = 1, \dots, D$ then $\mathcal{S}_\hbar^\delta(|n_1, \dots, n_D\rangle)$ is finite dimensional.

Proof. We know from Proposition 4.2 that $\mathcal{S}_\hbar^\delta(|n_1, \dots, n_D\rangle)$ is a simple $\mathcal{A}$-module. Moreover, there it is shown that every vector is cyclic, which is also the case for the $\hat{X}^a$, which represent the Lie algebra basis elements. Therefore, $\mathcal{S}_\hbar^\delta(|n_1, \dots, n_D\rangle)$ is also simple (or irreducible) as a representation of the Lie algebra $\mathfrak{g}$.

Furthermore, the sum $k_1 + k_2 + \cdots + k_D$ is constant under the action of the E_{ij} of (35). Namely, for $i \neq j$ it follows that

$$\begin{aligned} E_{ij}\,|k_1, \dots, k_D\rangle &= \sqrt{\hbar(k_i + \delta_i + 1) + c_i)(\hbar(k_j + \delta_j) + c_j)} \\ &\qquad \times |k_1, \dots, k_i + 1, \dots, k_j - 1, \dots, k_D\rangle \\ E_{ii}\,|k_1, \dots, k_D\rangle &= \big(\hbar(k_i + \delta_i) + c_i\big)\,|k_1, \dots, k_D\rangle\ . \end{aligned}$$

Consequently, the sum $k_1 + \cdots + k_D$ is also constant under the action of an arbitrary element in the shift subalgebra, since the shift subalgebra is generated by the $\hat{X}^a$, which are linear combinations of the E_{ij}.

Assuming that $\hbar\delta_i + c_i = 0$, the action of the E_{ij} becomes

$$\begin{aligned} E_{ij}\,|k_1, \dots, k_D\rangle &= \hbar\sqrt{\big(k_i + 1\big)k_j} \\ &\qquad \times |k_1, \dots, k_i + 1, \dots, k_j - 1, \dots, k_D\rangle \\ E_{ii}\,|k_1, \dots, k_D\rangle &= \hbar k_i\,|k_1, \dots, k_i, \dots, k_D\rangle\ . \end{aligned}$$

From this follows that if $n_i \geq 0$, the module $\mathcal{S}_\hbar^\delta(|n_1, \dots, n_D\rangle)$ is restricted to a subset of the basis elements $|k_1, \dots, k_D\rangle$ with $k_i \geq 0$, since the factor

$\sqrt{\big(k_i+1\big)k_j}$ for the operators E_{ij} with $i \neq j$ becomes 0, when applied to a vector with $k_j = 0$. In the case $\hbar\delta_i + c_i = 0$ and $n_i \geq 0$ this means that $\mathcal{S}^{\delta}_{\hbar}(|n_1, \ldots, n_D\rangle)$ is a subspace of the vector space spanned by $|k_1, \ldots, k_D\rangle$ where $k_1 + \cdots + k_D = n_1 + \cdots + n_D$ and $k_i \geq 0$. These $\{k_1, \ldots, k_D\}$ form a finite subset of $\mathbb{Z}^D$.

The shift subalgebra $\mathcal{A}$ generated by the $\hat{X}^a$ is a subalgebra of the shift subalgebra generated by the E_{ij} , since the $\hat{X}^a$ are linear combinations of the E_{ij}. Thus, because the module $\mathcal{S}^{\delta}_{\hbar}(|n_1, \ldots, n_D\rangle)$ is finite dimensional for the shift subalgebra generated by the E_{ij}, this is also the case for $\mathcal{A}$. □

It is an interesting observation that the hyperplane in $\mathbb{Z}^D$, which is defined by $k_1 + \cdots + k_D = N$ with N some integer forms a triangular lattice. The condition $k_i \geq 0$ cuts out a regular D-simplex from this hyperplane.

Let us end with two examples as an illustration of the above results.

Example 5.4 ($su(2) \simeq so(3)$)**.** *We explore the simplest non-trivial example, the Lie algebra $su(2)$ and its fundamental representation, which is provided by the Pauli matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This is a two-dimensional representation, implying that the subalgebra defined in Proposition 5.2 is a shift subalgebra of $\mathcal{A}^2_{\hbar,1}(\mathcal{F})$. The generators are given by

$$\begin{aligned}
X_1 &= \sqrt{u_1+c_1}U_1^{-1}U_2\sqrt{u_2+c_2} + \sqrt{u_2+c_2}U_2^{-1}U_1\sqrt{u_1+c_1} \\
X_2 &= -i\sqrt{u_1+c_1}U_1^{-1}U_2\sqrt{u_2+c_2} + i\sqrt{u_2+c_2}U_2^{-1}U_1\sqrt{u_1+c_1} \\
X_3 &= u_1 + c_1 - u_2 - c_2 \,.
\end{aligned}$$

We know from Proposition 5.3 that the representation $\mathcal{S}^{\delta}_{\hbar}(|n_1, n_2\rangle)$ is restricted at least to the vectors $|k, N-k\rangle$ with $N = n_1 + n_2$. The operators become

$$\begin{aligned}
A_+ |k, N-k\rangle &= \sqrt{\big(\hbar(k+1+\delta_1)+c_1\big)\hbar(N-k+\delta_2)+c_2} \\
&\quad \times |k+1, N-k-1\rangle \\
A_- |k, N-k\rangle &= \sqrt{\big(\hbar(N-k+1+\delta_2)+c_2\big)\big(\hbar(k+\delta_1)+c_1\big)} \\
&\quad \times |k-1, N-k+1\rangle \\
Z |k, N-k\rangle &= \big(\hbar(k+\delta_1)+c_1-\hbar(N-k+\delta_2)-c_2\big)
\end{aligned}$$

$$\times |k, N-k\rangle \ ,$$

where we have introduced the ladder operators

$$A_+ = \frac{1}{2}(X + iY), \quad A_- = \frac{1}{2i}(X - iY), \quad Z = X_3 \,.$$

Depending on k*, the expressions below the square roots can be positive and negative. As shown in Proposition 5.3 we see that a restricted representation is only possible, when the expression below the square root can become* 0 *for some* k*, which is only possible, when the* c_i *compensate the* δ_i*. Let us therefore assume that* $\hbar\delta_i + c_i = 0$*. We then arrive at*

$$\begin{aligned}
A_+ |k, N-k\rangle &= \hbar\sqrt{(k+1)(N-k)}\, |k+1, N-k-1\rangle \\
A_- |k, N-k\rangle &= \hbar\sqrt{(N-k+1)k}\, |k-1, N-k+1\rangle \\
Z\, |k, N-k\rangle &= \hbar(2k-N)\, |k, N-k\rangle \ .
\end{aligned}$$

The expressions below the square roots are positive for $0 < k < N$ *otherwise they are negative. Furthermore,* $A_+ |N, 0\rangle = 0$ *and* $A_- |0, N\rangle = 0$*. The irreducible representations are the representations* $\mathcal{S}^{\delta}_{\hbar}(|N, 0\rangle)$ *and are up to a shift the representations of the fuzzy sphere.*

Example 5.5 ($su(D)$)**.** *To generalize the previous example, for* $su(D)$ *we can directly use the algebra elements* E_{ij} *from Proposition 5.3. From this proposition we know that the representation* $\mathcal{S}^{\delta}_{\hbar}(|n_1, \ldots, n_D\rangle)$ *can be restricted to basis vectors* $|k_1, \ldots, k_D\rangle$ *with* $k_1 + \cdots + k_D = n_1 + \cdots + n_D = N$*. Since for* $k_i > 0$*, the factors under the square root of the* E_{ij} *are all greater than* 0*, the representation sweeps out the complete set* $k_1 + \cdots + k_D = N$ *restricted to* $k_i > 0$*. For the same* N *the representations are equivalent. For every* N*, one obtains an irreducible representation of* $su(D)$*.*

The operators E_{ii} *in the representation are linearly dependent. They correspond to the diagonal generator* Z *of the previous example. Since the factors under the square root of the operators* E_{ij} *with* $i \neq j$ *are all nonnegative for all* $k_1, \ldots, k_D$ *with* $k_i \geq 0$ *it follows that the operators* E_{ij} *are conjugate to the operators* E_{ji}*. These operators correspond to the conjugate generators* A_+ *and* A_- *of the previous example.*

The representations generated in this way are usually denoted by

$$[N, 0, \ldots, 0],$$

where each of the numbers in the square brackets indicates a highest weight vector of $su(D)$. It is known that these representations are fuzzy complex projective spaces; see e.g. [2, 7].

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