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MONOTONICITY FORMULAS FOR PARABOLIC FLOWS ON MANIFOLDS

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Recently Michael Struwe [S] and Gerhard Huisken [Hu2] have independently derived monotonicity formulas for the Harmonic Map heat flow on a Euclidean domain and for the Mean Curvature flow of a hypersurface in Euclidean space. In this paper we show how to generalize these results to the case of flows on a general compact manifold, and we also give the analogous monotonicity formula for the Yang-Mills heat flow. The key ingredient is a matrix Harnack estimate for positive solutions to the scalar heat equation given in [H]. In [GrH] the authors show how to use the monotonicity formula to prove that rapidly forming singularities in the Harmonic Map heat flow are asymptotic to homothetically shrinking solitons; similar results may be expected in other cases, as Huisken does in [Hu2] for the Mean Curvature flow in Euclidean space.

We only obtain strict monotonicity for a special class of metrics, but in general there is an error term which is small enough to give the same effect. (Chen Yummei and Michael Struwe [CS] give a different approach to the error on manifolds.) The special class of metrics are those which are Ricci parallel (so that $D_i R_{jk} = 0$) and have weakly positive sectional curvature (so that $R_{ijk\ell} V_i W_j V_k W_\ell \geq 0$ for all vectors V and W). This holds for example if M is flat or a sphere or a complex projective space, or a product of such, or a quotient of a product by a finite free group of isometries.

In each case we consider a solution to our parabolic equation on a compact manifold M for some finite time interval $0 \leq t < T$, and we let k be any

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positive solution to the backward heat equation

$$\frac{\partial k}{\partial t} = -\Delta k$$

on the same interval. Many such solutions exist, as we can specify k to be any positive function at the end time $t = T$ and solve backwards. Usually we take k to evolve backwards from a delta function at a point or along a set where the equation is singular.

The Harmonic Map heat flow evolves a map $F : M \to N$ of one compact Riemannian manifold to another by the equation

$$\frac{\partial F}{\partial t} = \Delta F$$

introduced by Eells and Sampson [ES]. It is the gradient flow of the energy

$$E = \frac{1}{2} \int_M |DF|^2 .$$

Theorem A. *If F solves the Harmonic Map heat flow on $0 \leq t < T$ and if k is any positive backward solution to the scalar heat equation on M, with $\int_M k = 1$, then the quantity*

$$Z(t) = (T - t) \int_M |DF|^2 k$$

is monotone decreasing in t when M is Ricci parallel with weakly positive sectional curvatures; while on a general M we have

$$Z(t) \leq CZ(\tau) + C(t - \tau)E_0$$

whenever $T - 1 \leq \tau \leq t \leq T$, where E_0 is the initial energy at time 0 and C is a constant depending only on the geometry of M (the diameter, the volume, the Riemannian curvature, and the covariant derivative of the Ricci curvature). Moreover if

$$W(t) = (T - t) \int_M \left| \frac{\partial F}{\partial t} + \frac{Dk}{k} \cdot DF \right|^2 k$$

then also

$$\int_\tau^T W(t) dt \leq CZ(\tau) + CE_0 .$$

The Mean Curvature Flow for a compact submanifold S of dimension s in a compact Riemannian manifold M of dimension m evolves the surface by the equation

$$\frac{\partial P}{\partial t} = H$$

where P is the position of a point and H is the mean curvature vector. It is the gradient flow for the area function

$$A = \int_S 1$$

and was first studied by Gage and Hamilton [GaH] for plane curves, by Huisken [Hu1] for hypersurfaces, and by Altschuler [A] for space curves.

Theorem B. *If S solves the Mean Curvature flow on $0 \leq t < T$ and if k is any positive backward solution to the scalar heat equation on M with $\int_M k = 1$, then the quantity*

$$Z(t) = (T-t)^{(m-s)/2} \int_S k$$

is monotone decreasing in t when M is Ricci parallel with non-negative sectional curvatures; while on a general M we have

$$Z(t) \leq CZ(\tau) + C(t-\tau)A_0$$

whenever $T-1 \leq \tau \leq t \leq T$, where A_0 is the initial area at time 0 and C is a constant depending only on the geometry of M as before. Moreover if

$$W(t) = (T-t)^{(m-s)/2} \int_S \left| H - \frac{Dk}{k} \right|_\perp^2 k$$

where $| \ |_\perp$ denotes the norm of the component of a vector perpendicular to S then

$$\int_\tau^T W(t)dt \leq CZ(\tau) + CA_0 .$$

The Yang-Mills heat flow evolves a connection A on a bundle over a compact manifold M by the divergence of its curvature F_A under the formula

$$\frac{\partial}{\partial t} A = \operatorname{div}\ F_A .$$

This is the gradient flow for the energy

$$E = \frac{1}{2} \int_M |F|^2 .$$

Recently Johan Råde [R] has proved that if M has dimension 2 or 3, the solution exists for all time and converges (without modification by the gauge group) to a Yang-Mills connection, one with div $F_A = 0$. The most interesting case is where M has dimension 4. Here one expects that there will be a weak solution with only slowly-forming isolated singularities in space-time. These must occur, but the author does not know if they can occur in finite time. For higher dimensions, one expects rapidly forming singularities modeled on homothetically shrinking solitons.

Theorem C. *If A solves the Yang-Mills heat flow on $0 \le t < T$ and if k is any positive backward solution to the scalar heat equation on M with $\int_M k = 1$, then the quantity*

$$Z(t) = (T-t)\int_M |F|^2 k$$

is monotone decreasing in t when M is Ricci parallel with weakly positive sectional curvatures; while on a general M we have

$$Z(t) \le CZ(\tau) + C(t-\tau)E_0$$

whenever $T-1 \le \tau \le t \le T$, where E_0 is the initial energy at time 0 and C is a constant depending only on the geometry of M as before. Moreover if

$$W(t) = (T-t)\int_M \left| \text{div } F_A + \frac{Dk}{k}\cdot F_A \right|^2 k$$

then

$$\int_\tau^T W(t)dt \le CZ(\tau) + CE_0$$

1. The Monotonicity Formula for the Harmonic Map Heat Flow

Let M^m and N^n be compact Riemannian manifolds with matrices g_{ij} on M and $h_{\alpha\beta}$ on N. We define the energy of a map $F\colon M \to N$ by

$$E = \frac{1}{2}\int_M |DF|^2 = \frac{1}{2}\int_M g^{ij}h_{\alpha\beta}D_iF^\alpha D_jF^\beta .$$

The associated gradient flow is the harmonic map heat equation

$$\frac{\partial F}{\partial t} = \triangle F$$

where the intrinsic Laplacian is given in local coordinates by

$$\Delta F^\alpha = g^{ij}\left\{\frac{\partial^2 F^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial F^\alpha}{\partial x^k} + \Delta_{\beta\gamma}^\alpha(F)\frac{\partial F^\beta}{\partial x^i}\cdot\frac{\partial F^\gamma}{\partial x^j}\right\}$$

in terms of the Christoffel symbols Γ_{ij}^k on M and $\Delta_{\beta\gamma}^\alpha$ on N.

It is useful to consider how the energy density accumulates at a point or along some singular set. We let $k = k(x,t)$ be a positive scalar solution of the backwards heat equation

$$\frac{\partial k}{\partial t} + \Delta k = 0$$

and consider the energy concentrated along k. This leads us to the following monotonicity formula, which occurs in Struwe [S] for the case $M = R^m$.

Theorem 1.1. *For any solution F of the harmonic map heat flow and any positive solution k of the backward heat equation on $0 \le t \le T$ we have the formula*

$$\frac{d}{dt}(T-t)\int_M |DF|^2 k + 2(T-t)\int_M \left|\Delta F + \frac{Dk}{k}\cdot DF\right|^2 k$$
$$+ 2(T-t)\int_M \left[D_i D_j k - \frac{D_i k D_j k}{k} + \frac{1}{2(T-t)} k g_{ij}\right] D_i F^\alpha D_j F^\alpha = 0\,.$$

Proof. This is a straightforward integration by parts, when we are careful never to interchange D_i and D_j so we can avoid curvature terms. □

This formula is very useful because all the terms are positive, or nearly so. In Euclidean space $M = R^m$ the term

$$D_i D_j k - \frac{D_i k D_j k}{k} + \frac{1}{2(T-t)} k g_{ij}$$

vanishes identically when k is a fundamental solution converging to a δ-function at some point X as $t \to T$. In [H] we show that if M is compact with non-negative sectional curvature and is Ricci parallel, then for any positive solution of the scalar heat equation this expression is non-negative. In this case we see that

$$Z(t) = (T-t)\int_M |DF|^2 k$$

is monotone decreasing; hence it is called a monotonicity formula. Also we have

$$2\int_{T-1}^{T}(T-t)\int_M\left|\frac{\partial F}{\partial t}+\frac{Dk}{k}\cdot DF\right|^2 k\le Z(T-1)$$

which is also useful. In general, without any curvature assumptions, we show in [H] that there exist constants B and C depending only on M such that if $\int k=1$ then for $T-1\le t\le T$ we have

$$D_iD_jk-\frac{D_ikD_jk}{k}+\frac{1}{2(T-t)}kg_{ij}\ge -Ck\left[1+\log(B/(T-t)^{m/2}k)\right]g_{ij}$$

and this is almost as good.

Lemma 1.2. *For all $x>0$ and $y>0$ we have*

$$x[1+\log(y/x)]\le 1+x\log y\,.$$

Proof. We know

$$\log x\le x-1$$

and replacing x by $1/x$

$$\log(1/x)\le(1/x)-1\,.$$

Multiply by x and add x to both sides and get

$$x[1+\log(1/x)]\le 1$$

and add $x\log y$ to each side and get

$$x[1+\log(y/x)]\le 1+x\log y\,.\quad\square$$

Corollary 1.3. *We have*

$$k\left[1+\log\left(B/(T-t)^{m/2}k\right)\right]\le 1+k\log\left(B/(T-t)^{m/2}\right)\,.$$

Applying this estimate to our bound on the last term in the monotonicity formula gives

$$\frac{d}{dt}(T-t)\int_M|DF|^2k+2(T-t)\int_M\left|\Delta F+\frac{Dk}{k}\cdot DF\right|^2k$$
$$\le 2C(T-t)\int_M|DF|^2+2C(T-t)\log\left(B/(T-t)^{m/2}\right)\int_M|DF|^2k\,.$$

Now the energy is decreasing since

$$\frac{d}{dt}\frac{1}{2}\int |DF|^2 + \int \left|\frac{\partial F}{\partial t}\right|^2 = 0$$

so if E_0 is the initial energy then

$$\frac{1}{2}\int_M |DF|^2 \leq E_0 .$$

Let us put

$$Z = (T-t)\int_M |DF|^2 k$$

$$W = (T-t)\int_M \left|\triangle F + \frac{Dk}{k}\cdot DF\right|^2 k$$

$$\varphi = (T-t)\left[\frac{m}{2} + \log(B/(T-t)^{m/2})\right] .$$

Then we compute

$$\frac{d\varphi}{dt} = -\log\left(B/(T-t)^{m/2}\right)$$

and hence for $T-1 \leq t \leq T$

$$\frac{d}{dt}e^{2C\varphi}Z + 2e^{2C\varphi}W \leq 4CE_0 \cdot e^{2C\varphi} .$$

Now we can find a constant so that

$$0 \leq \varphi \leq C \quad \text{for} \quad T-1 \leq t \leq T$$

and integrating gives us the result in Theorem A.

2. The Monotonicity Formula for the Mean Curvature Flow

We let the submanifold S^s of the Riemannian manifold M^m evolve by its mean curvature vector H. We use subscripts $i, j, k, \ldots$ for a basis tangent to S and $\alpha, \beta, \gamma, \ldots$ for a basis normal to S. Then $H = \{H^\alpha\}$ is the trace of the second fundamental form H_{ij}^α. We let g_{ij} denote the induced metric on S and $g_{\alpha\beta}$ the induced metric on the normal bundle. Then

$$H^\alpha = g^{ij}H_{ij}^\alpha .$$

The metric on S evolves by the formula

$$\frac{\partial}{\partial t}g_{ij} = -2g_{\alpha\beta}H^\alpha H_{ij}^\beta$$

and hence the induced area da on S evolves by

$$\frac{\partial}{\partial t} da = -|H|^2 da.$$

The Laplacian on M is related to the Laplacian on S by

$$\triangle_M k = \triangle_S k + g^{\alpha\beta} D_\alpha D_\beta k - H^\alpha D_\alpha k$$

as can be readily seen by choosing coordinates first on S, and then extending by normal geodesics, and writing the Laplacian as the divergence with respect to the volume form of the gradient of f, and noting that the normal variation of the volume form is given by the mean curvature vector. The evolution of k is

$$\frac{\partial k}{\partial t} = -\triangle_M k$$

and considering also the motion of S and the change in the area element

$$\frac{d}{dt}\int_S k = \int_S \frac{\partial k}{\partial t} + H^\alpha D_\alpha k - |H|^2 k.$$

It is now a straightforward calculation that

$$\frac{d}{dt}(T-t)^{(m-s)/2}\int_S k + (T-t)^{(m-s)/2}\int_S \left|H - \frac{Dk}{k}\right|_\perp^2 k$$
$$+ (T-t)^{(m-s)/2}\int_S g^{\alpha\beta}\left[D_\alpha D_\beta k - \frac{D_\alpha k D_\beta k}{k} + \frac{1}{2(T-t)} k g_{\alpha\beta}\right] = 0.$$

Now if M is Ricci parallel with weakly positive sectional curvatures, the Harnack estimate implies that the matrix in the last integral is weakly positive, and we are done. Otherwise it is entirely analogous to the previous case to estimate the error, and we leave it to the reader.

3. The Monotonicity Principle for the Yang-Mills Heat Flow

. Let M be a compact manifold with a Riemannian metric g_{ij}, and consider a vector bundle over M with a metric $h_{\alpha\beta}$ on the fibres (where we let $i, j, k, \ldots$ denote manifold indices and $\alpha, \beta, \gamma, \ldots$ denote bundle indices). A connection A on the vector bundle is given locally by $A^\alpha_{i\beta}$, where the covariant derivative of a section V^α is given by

$$D_i V^\alpha = \frac{\partial}{\partial x^i} V^\alpha + A^\alpha_{i\beta} V^\beta.$$

The curvature F_A of the connection is given locally by $F^\alpha_{ij\beta}$ where

$$F^\alpha_{ij\beta} = \frac{\partial}{\partial x^i} A^\alpha_{j\beta} - \frac{\partial}{\partial x^j} A^\alpha_{i\beta} + A^\alpha_{i\gamma} A^\gamma_{j\beta} - A^\alpha_{j\gamma} A^\gamma_{i\beta} \,.$$

The connection A is compatible with the metric h if $D_A k = 0$, where

$$D_A h = \{D_i h_{\alpha\beta}\}$$

is the covariant derivative of k with respect to A. In this case the curvature is symmetric in the bundle indices, so

$$h_{\alpha\gamma} F^\gamma_{ij\beta} = h_{\beta\gamma} F^\gamma_{ij\alpha} \,.$$

The divergence of the curvature is given by

$$\text{div } F_A = \{g^{ij} D_i F^\alpha_{jp\beta}\}$$

and the Yang-Mills heat flow is

$$\frac{\partial}{\partial t} A^\alpha_{p\beta} = g^{ij} D_i F^\alpha_{jp\beta} \,.$$

$$\frac{\partial}{\partial t} F^{\alpha}_{ij\beta} = D_i \left(\frac{\partial}{\partial t} A^{\alpha}_{j\beta} \right) - D_j \left(\frac{\partial}{\partial t} A^{\alpha}_{i\beta} \right)$$

and from this one can compute an evolution equation

$$\frac{\partial}{\partial t} F = \Delta F + F * F + Rm * F$$

where Rm is the curvature of M and $*$ denotes some tensor product; but here it is best not to do this, but to integrate by parts first. Then one finds readily enough that if k is a backwards solution to the scalar heat equation on M one has

$$\frac{d}{dt}(T-t) \int |F^{\alpha}_{ij\beta}|^2 k + 4(T-t) \int \left| g^{ij} \left(D_i F^{\alpha}_{jp\beta} + \frac{D_i k}{k} F^{\alpha}_{jp\beta} \right) \right|^2 k$$
$$+4(T-t) \int g^{ip} g^{jq} g^{rs} h_{\alpha\beta} h^{\gamma\delta} F^{\alpha}_{pr\gamma} F^{\beta}_{qs\delta} \left[D_i D_j k - \frac{D_i k D_j k}{k} + \frac{1}{2(T-t)} k g_{ij} \right] = 0.$$

From this we immediately see that if M is Ricci parallel with weakly positive sectional curvatures the quantity

$$Z(t) = (T-t) \int |F|^2 k$$

is monotone decreasing. For a general M we make the same estimates as before.

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