communications in
analysis and geometry
Volume 13, Number 5, 939-961, 2005

Small knots and large handle additions

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We construct a hyperbolic 3-manifold M (with ∂M totally geodesic) which contains no essential closed surfaces, but for any even integer $g > 0$, there are infinitely many separating slopes r on ∂M so that $M[r]$, the 3-manifold obtained by attaching 2-handle to M along r, contains an essential separating closed surface of genus g and is still hyperbolic. The result contrasts sharply with those known finiteness results for the cases $g = 0, 1$. Our 3-manifold M is the complement of a simple small knot in a handlebody.

1. Introduction.

All manifolds in this paper are orientable. All submanifolds are embedded and proper ($F \subset M$ is *proper* if $F \cap \partial M = \partial F$), unless otherwise specified. A connected 1-manifold (an arc or a circle) on a surface F is *non-trivial* if it does not separate a disc from F.

Let M be a compact 3-manifold with the boundary $\partial M \neq \emptyset$, F be a surface in M which is not the 2-sphere S^2. Say F is *incompressible* if a circle $c \subset F$ bounds a disk in M implies that c bounds a disc in F. Say a surface in M is *essential* if either it is incompressible and is not parallel to a sub-surface of ∂M, or it is a 2-sphere which does not bound a 3-ball in M. Say a 3-manifold M is *irreducible* if each 2-sphere in M bounds a 3-ball. Say M is *∂-irreducible* if ∂M is incompressible. Say M is atoroidal if it contains no essential tori; Say M is *anannular* if it contains no essential annuli.

Say a 3-manifold M is *simple* if M is irreducible, ∂-irreducible, anannular and atoroidal. Suppose M is a simple 3-manifold with $\partial M \neq \emptyset$. By Thurston's theorem, M admits a complete finite volume hyperbolic structure with totally geodesic boundary (with torus components in ∂M removed) [8]. A knot K in M is *simple* if M_K, the complement of K in M, is simple. A 3-manifold M is *small* if M contains no essential closed surface. A knot K in M is *small* if M_K is small.

[1]Both authors are supported by NSFC, the second author is also supported by MOSTC.

A (*separating*) *slope* r in ∂M is the isotopy class of a non-oriented non-trivial (separating) circle in ∂M. We denote by $M[r]$ the manifold obtained by adding a 2-handle to M along a regular neighborhood of r in ∂M and then capping off spherical components with 3-balls. Specially, if r lies in a torus component of ∂M, this operation is known as Dehn filling.

Essential surface is a basic tool to study 3-manifolds and handle addition is a basic method to construct 3-manifolds. A central topic connecting those two aspects in 3-manifold topology is the following:

Question 1.1. *Let M be a simple 3-manifold with $\partial M \neq \emptyset$ which contains no essential closed surface of genus g. How many slopes $r \subset \partial M$ are there so that $M[r]$ contains an essential closed surface of genus g?*

Remark on Question 1.1. The mapping class group of a simple 3-manifold M with $\partial M \neq \emptyset$ is finite. The question is asked only for simple 3-manifolds to avoid possibly infinitely many slopes produced from Dehn twists along essential discs or annuli. The main result in this paper is the following:

Theorem 1.2. *There is a simple small knot K in the handlebody H of genus 3 such that for any even integer $g > 0$, there are infinitely many separating slopes r in ∂H so that $H_K[r]$ contains an essential separating closed surface of genus g. Moreover, those $H_K[r]$ are still simple.*

Remarks on Theorem 1.2

(1) Suppose M is a simple 3-manifolds with $\partial M \neq \emptyset$.

(i) ∂M is a torus. Thurston's pioneering result claims that there are at most finitely many slopes r on ∂M so that $M[r]$ is not hyperbolic [8], hence the number of slopes are finite in Question 1.1 when $g = 0, 1$. The sharp upper bound of such slopes are given by Gordon and Luecke and by Gordon when $g = 0, 1$, see [2] for a survey. Hatcher proved the number of slopes in Question 1.1 is finitely many for all g [3].

(ii) ∂M has genus > 1. Scharlemann and Y-Q Wu [7] have shown that if $g = 0, 1$, then there are only finitely many separating slopes r so that $M[r]$ contains an essential closed surface of genus g. Very recently, Lackenby [5] generalized Thurston's finiteness result to handlebody attaching, that is to adding 2-handles simultaneously. He proved that for a hyperbolic 3-manifold M, there is a finite set $\mathcal{C}$ of exceptional circles on ∂M so that attaching a handlebody to M is still hyperbolic if none of those circles is attached to a meridian disc of the handlebody.

Theorem 1.2 and those finiteness results of [8], [3], [7] and [5] give a global view about the answer of Question 1.1. In particular, those finiteness results of [8], [3] and [7] do not hold in general. We think the example in Theorem 1.2 also indicates that the finiteness result of [5] does not hold in general (a working project of the authors).

(2) It is unusual to the authors that a given manifold M provides non-finiteness answer to Question 1.1 for all even genus $g \geq 2$. From an aesthetic point of view, one may wonder if there is a manifold that provides non-finiteness answer to Question 1.1 for all genus $g \geq 2$. We think that the answer is positive. In this case, the knot K is complicated and then the proof of that H_K is small will be much more difficult (a working project of the authors).

(3) Without handle addition, the 3-manifold M itself in Theorem 1.2 is interesting independently. First, the construction of the small knot in Theorem 1.2 can be modified to provided infinitely many small knots in handlebodies of any genus $g > 1$ (a working project of the authors). To our knowledge, no examples of simple small knots in the handlebody of genus > 1 were explicitly presented before. Secondly, M provides a hyperbolic 3-manifold with totally geodesic boundary which splits over essential surfaces of genus g in infinitely many different ways for each even $g > 0$.

Remarks on the Proof of Theorem 1.2 **and the organization of the paper.** In Section 2, we construct a knot K and infinitely many separating surfaces $S_{g,l}$ of genus g for each even $g > 0$ in the handlebody H of genus 3, such that all those surfaces $S_{g,l}$ are disjoint from K and have connected boundaries. Those $\partial S_{g,l}$ will serve as the slopes r in Theorem 1.2. Some elementary properties of $S_{g,l}$ and of K are also described in Section 2. Let $\hat{S}_{g,l} \subset H_K[\partial S_{g,l}]$ be the closed surface obtained by capping off $\partial S_{g,l}$ with a disk. In Section 3, we will prove that $\hat{S}_{g,l}$ is incompressible in $H_K[\partial S_{g,l}]$ as well as that $\partial S_{g,l}$ and $\partial S_{g,l'}$ are not isotopic in ∂H when $l \neq l'$. Sections 4 and 5 are devoted to proving that the knot K is simple and small.

A result in [4] is quoted in Section 3, which is a crucial step for the proof of Proposition 3.2, and a result in [1] is quoted in Section 4, which is used to shorten the argument of Case 2 in the proof of Lemma 4.4. Up to those two results and the knowledge in the beginning of standard textbooks of elementary algebraic topology, combinatorial groups and 3-manifolds, the paper is self-contained. Even so, the argument of Case 1 (2) in the proof of Lemma 4.4 is initiated by Gordon–Litherland in the mid 1980's.

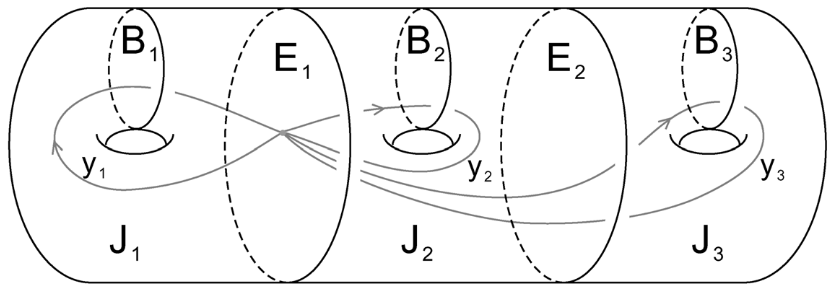


Figure 1.

2. Construction of the surfaces $S_{g,l}$ and the knot K in H.

Suppose X_1 and X_2 are connected proper sub-manifolds of M with complementary dimensions and meeting tranversely. Let $||X_1, X_2||$ be the absolute value of their algebraic intersection number. Since all manifolds are orientable, $||X_1, X_2||$ is well defined. For a compact manifold X, $|X|$ denotes the number of components of X. If X is an arc or an annulus, we often use $\partial_1 X$ to denote one component of ∂X and $\partial_2 X$ to denote another.

Let H be the handlebody of genus 3. Let $\{B_1, B_2, B_3\}$ be a set of basis disks of H, and $\{E_1, E_2\}$ be two separating disks of H which separate H into three solid tori J_1, J_2 and J_3. See Figure 1.

The orientable surface S_g of even genus $g > 0$ with $|\partial S_g| = 1$ can be presented as in Figure 2 (where $g = 4$). Each surface $S_{g,l}$ we are going to construct in H can be viewed as a properly embedded image of S_g, where the disk in Figure 2 is sent to E_1 (approximately) and the 1-handle ended at v_i and u_i is sent to the 1-handle $N(\alpha_i)$ attached to E_1, which will be shown in Figures 3 and 4

Remark on Figure 2 In Figure 2, if we attach g 1-handles on each side of the disc for odd g in the same way, we get a surface of genus $g - 1$ with three boundary components rather than a surface of genus g with one boundary component.

Let C be a closed curve in ∂H (with one self-intersection) as in Figure 3. Then $\partial E_1 \cup \partial E_2$ separates C into eight embedded arcs $c_1, \ldots, c_8$, where $c_3, c_7 \subset J_1$ with $||\partial B_1, c_7|| = 3$, $||\partial B_1, c_3|| = 1$; $c_2, c_4, c_6, c_8 \subset J_2$ with $||\partial B_2, c_4|| = 1$, $||\partial B_2, c_6|| = 3$, $||\partial B_2, c_2|| = ||\partial B_2, c_8|| = 0$; $c_1, c_5 \subset J_3$ with $||\partial B_3, c_1|| = 3$, $||\partial B_3, c_5|| = 1$.

Let $u_1, \ldots, u_{2g}, v_1, \ldots, v_{2g}$ be $4g$ points located on ∂E_1 in the cyclic order

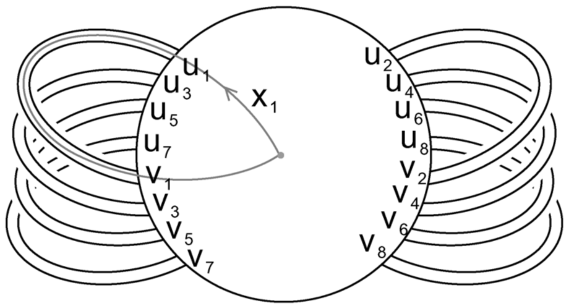


Figure 2.

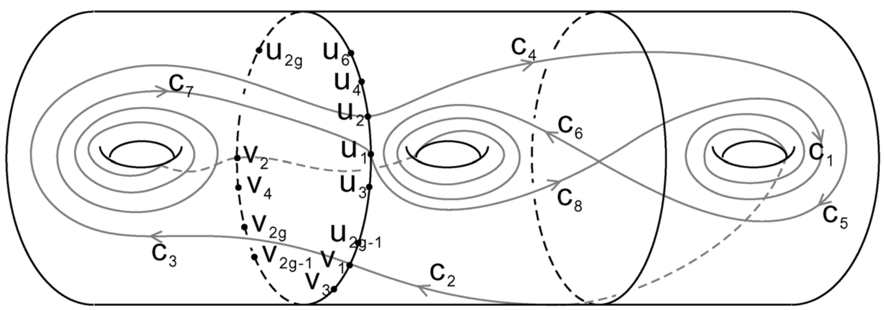


Figure 3.

u_1, u_3, $\ldots$, u_{2g-3}, u_{2g-1}, v_1, v_3, $\ldots$, v_{2g-3}, v_{2g-1}, v_{2g}, v_{2g-2},$\ldots$,v_4,v_2, u_{2g}, u_{2g-2}, $\ldots$,u_4, u_2 as in Figure 3 (see also Figure 2).

By the order of those points, we can assume that the isotopy has been made so that $\partial(c_8 \cup c_1 \cup c_2) = \{u_1, v_1\}$, $\partial c_3 = \{v_1, u_2\}$, $\partial c_7 = \{v_2, u_1\}$. Then pick a proper arc c_*, (resp. $c_\#$) in $\partial H \cap (J_2 \cup J_3)$ connecting v_3 and v_2 (resp. u_2 and u_3) as in Figure 4, where $||c_*, \partial B_2|| = l$, $l \geq 3$. By construction, $intc_\# \cap C = \emptyset$ and $c_* \cap C \neq \emptyset$.

Now, we define oriented arcs on ∂H to connect some pairs in $\{u_i, v_j; i, j = 1, \ldots, 2g\}$ as follows: First let $\overline{u_1v_1} = c_8 \cup c_1 \cup c_2$, $\overline{v_1u_2} = c_3$, $\overline{v_2u_1} = c_7$, $\overline{u_2u_3} = c_\#$, $\overline{v_3v_2} = c_*$. Then, let $\overline{v_{2i}u_{2i-1}}$ and $\overline{v_{2i-1}u_{2i}}$ be proper arcs on $\partial H \cap J_1$ parallel to c_7 and c_3 respectively, $i = 2, \ldots, g$ and $\overline{u_{2i}u_{2i+1}}$ and $\overline{v_{2i+1}v_{2i}}$ be a proper arcs on $\partial H \cap (J_2 \cup J_3)$ parallel to $c_\#$ and c_* respectively, $i = 2, \ldots, g-1$. See $\overline{u_3v_4}$ and $\overline{u_4u_5}$ in Figure 4. Now define

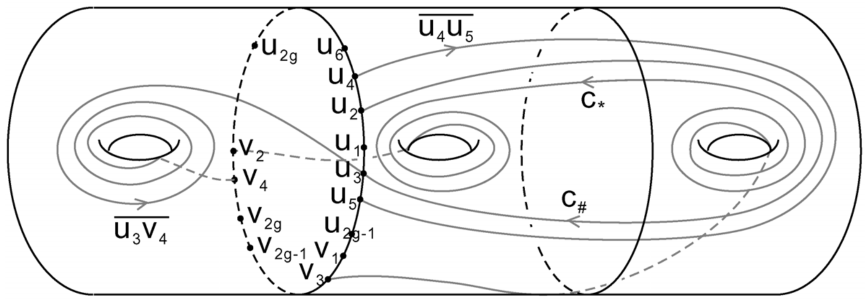


Figure 4.

$$\alpha_1 = \overline{u_1 v_1}, \tag{2.0}$$

and for $1 < 4k + j \leq 2g, j = 1, 2, 3, 4,$

$$\alpha_{4k+1} = \overline{u_{4k+1}u_{4k}} \cup \alpha_{4k} \cup \overline{v_{4k}v_{4k+1}}, \tag{2.1}$$

$$\alpha_{4k+2} = \overline{v_{4k+2}u_{4k+1}} \cup \alpha_{4k+1} \cup \overline{v_{4k+1}u_{4k+2}}, \tag{2.2}$$

$$\alpha_{4k+3} = \overline{v_{4k+3}v_{4k+2}} \cup \alpha_{4k+2} \cup \overline{u_{4k+2}u_{4k+3}}, \tag{2.3}$$

$$\alpha_{4k+4} = \overline{u_{4k+4}v_{4k+3}} \cup \alpha_{4k+3} \cup \overline{u_{4k+3}v_{4k+4}}. \tag{2.4}$$

Hence, $\alpha_{k-1} \subset \alpha_k$ is an increasing sequence of embedded arcs on ∂H.

Let $\alpha \subset \partial H$ be an arc which meets ∂S exactly in its two ends for a proper separating surfaces $S \subset H$. The resulting proper surface by tubing S along α in H, denoted by $S(\alpha)$, is obtained by first attaching 2-dimensional 1-handle $N(\alpha) \subset \partial H$ to S, then making the surface $S \cup N(\alpha)$ to be proper, that is, pushing the interior of $S \cup N(\alpha)$ into the interior of H. The image of $N(\alpha)$ after the pushing is still denoted by $N(\alpha)$. Since S is orientable and separating, it is a direct observation that $S(\alpha)$ is still orientable and separating.

Since α_1 meets E_1 exactly in its two ends, we do tubing of E_1 along α_1 to get $E_1(\alpha_1)$. Now, α_2 meets $E_1(\alpha_1)$ exactly in its two ends, we do tubing of $E_1(\alpha_1)$ along α_2 to get $E_1(\alpha_1, \alpha_2) = E_1(\alpha_1)(\alpha_2)$, where the tube $N(\alpha_2)$ is thinner and closer to ∂H so that it goes over the tube $N(\alpha_1)$. Hence, $E_1(\alpha_1, \alpha_2)$ is a proper embedded surface. Repeating this process by tubing along $\alpha_3, \ldots, \alpha_{2g}$ in order, we get a surface $E_1(\alpha_1, \ldots, \alpha_{2g})$, denoted by $S_{g,l}$, in H. Clearly, $S_{g,l}$ is a proper embedding of S_g into H for each even $g > 0$. We survey this fact as

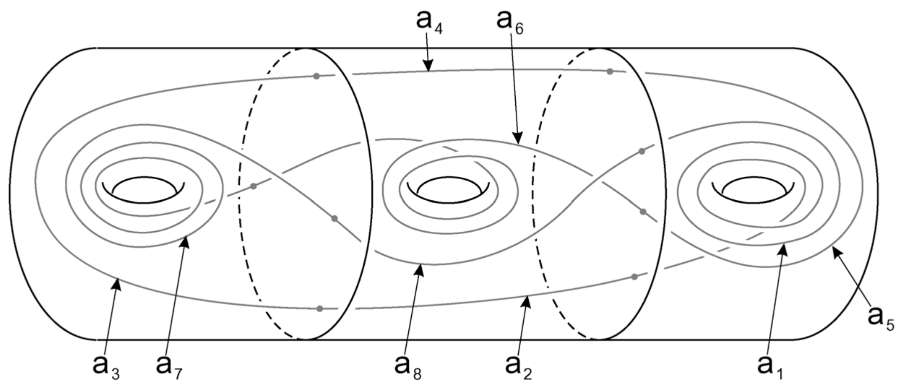


Figure 5.

Lemma 2.1. *$S_{g,l}$ is an orientable separating surface in H. Moreover, $S_{g,l}$ is of genus g with $|\partial S_{g,l}| = 1$ for even $g > 0$ (and of genus $g-1$ with $|\partial S_{g,l}| = 3$ for odd g).*

In the construction of $S_{g,l}$ for all g, l, we may assume that (i) the positions of the arcs α_1, α_2 are fixed; (ii) each tube $N(\alpha_i)$ has distance δ/i from α_i for some $\delta > 0$. By (i) and (ii), we have (iii) $N(\alpha_1)$, $N(\alpha_2)$ and the part of $N(\alpha_3)$ that goes over $N(\alpha_2)$ are fixed for all g, l.

Now, our knot K is obtained by pushing C into the interior of H along the inward normal direction of ∂H in the following way: (iv) first push the arc $c_7 \cup c_8 \cup c_1 \cup c_2 \cup c_3$ to stay between $N(\alpha_2)$ and $N(\alpha_3)$, (v) then push the arc $c_4 \cup c_5 \cup c_6$ so that it has distance larger than $\delta/3$ from ∂H and is disjoint from $N(\alpha_1)$. Below, we use a_i to denote the image of c_i after pushing. Then $E_1 \cup E_2$ separates K into 8 arcs $a_1, \ldots, a_8$. See Figure 5 for $K, a_i \subset H$, where a_6 is crossing under a_8, and $||a_i, B_k||(\text{in } H) = ||c_i, \partial B_k||(\text{in } \partial H)$, $i = 1, \ldots, 8$ and $k = 1, 2, 3$.

Lemma 2.2. *$K \cap S_{g,l} = \emptyset$ for all g, l.*

Proof. By (iii) and (iv), the part $a_7 \cup a_8 \cup a_1 \cup a_2 \cup a_3$ of K is disjoint from $S_{g,l}$. By (ii), (iii) and (v), the part $a_4 \cup a_5 \cup a_6 \subset J_2 \cup J_3$ of K is also disjoint from $S_{g,l}$. Hence $K \cap S_{g,l} = \emptyset$ for all g, l. □

Let $N(K) = K \times D$ be the regular neighborhood of K in H such that (i) $S_{g,l} \subset H_K = H - intN(K)$ for all g, l, (ii) the product structure has been

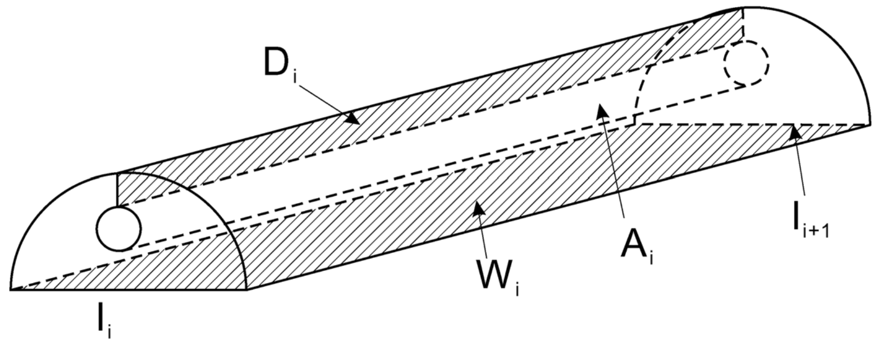


Figure 6.

adjusted so that $\cup_{i=1}^{8}\partial a_i \times D \subset E_1 \cup E_2$. Let $F_j = E_j - intN(K)$, $j = 1, 2$; $M_k = H_K \cap J_k$, $k = 1, 2, 3$; and $T = \partial(K \times D)$. Then $F_1 \cup F_2$ separates T into eight annuli $A_1, \ldots, A_8$, where $A_i = a_i \times \partial D$. Moreover, K and C bound a non-embedded annulus in H (the trace of pushing C to K) which is cut by $E_1 \cup E_2$ into eight disk D_{i*}, with $a_i \subset \partial D_{i*}$, $i = 1, \ldots, 8$. Suppose $a_i \subset M_k$, then $D_i = D_{i*} \cap M_k$ is a proper disc in M_k. Let $W_i = \overline{\partial N(D_i \cup A_i) - \partial M_k}$, where $N(D_i \cup A_i)$ is a regular neighborhood of $D_i \cup A_i$ in M_k. Then W_i is a proper separating disk in M_k. Each W_i intersects $F_1 \cup F_2$ in two arcs l_i and l_{i+1}. Note $W = \cup_{i \neq 6} W_i$ is still a (non-proper) disc. We use μ to denote the meridian slope on T. See Figure 6 for $A_i, D_i, W_i, l_i, l_{i+1} \subset M_k$.

The following facts about K and a_i, which are based on Figure 5 and whose proofs involve only elementary algebraic topology rather than 3-manifold topology, will be used in Sections 4 and 5.

Lemma 2.3. *(1) K is not contractible in H.*

(2) Suppose $a_i, a_j \subset J_k$. There is no relative homotopy on $(J_k, E_1 \cup E_2)$ which either sends a_i to $E_1 \cup E_2$; or sends a_i to a_j unless (i, j) is $(2, 8)$.

(3) Suppose $a_i, a_j \subset J_k$. The meridians of A_i and A_j are not homotopic in M_k.

(4) Suppose B is a proper disc of J_k with $|B \cap E_j| \leq 1$, $j = 1, 2$. If $|B \cap (\cup_{a_i \subset J_k} a_i)| < 3 - |B \cap (E_1 \cup E_2)|$, then B separates a 3-ball from J_k.

(5) There is no annulus $A \subset H$ such that (i) $\partial_1 A = K$ and $\partial_2 A \subset \partial H$, (ii) each component of $A \cap (E_1 \cup E_2)$ is non-trivial in A.

Proof. The proofs of (1), (2), and (3) are direct.

(4) If B is a separating disk in J_k, then B separates a 3-ball from J_k, since J_k is a solid torus. So we need only to show that each non-separating disk in J_k does not meet the inequality in (4).

Note $B \cap E_j$ is either an arc or the empty-set. We suppose B is a non-separating disk in J_2 which meets each E_j in an arc d_j, $j = 1, 2$ (the remaining cases are more direct). Let b_j be an arc in E_j connecting the two endpoints of a_6 and a_8. Then $c = b_1 \cup a_6 \cup b_2 \cup a_8$ is a circle which goes around J_3 three times. Hence

$$\begin{aligned} 3 = ||B, c|| &\leq ||B, a_6 \cup a_8|| + ||\partial B, b_1|| + ||\partial B, b_2|| \\ &= ||B, a_6 \cup a_8|| + ||d_1, b_1|| + ||d_2, b_2||. \end{aligned}$$

By the Jordan Curve Theorem, $||d_j, b_j|| \leq 1 = |B \cap E_j|$. Hence,

$$|B \cap (\cup_{a_i \subset J_k} a_i)| \geq ||B, b_1 \cup a_6 \cup b_2 \cup a_8|| \geq 3 - |B \cap (E_1 \cup E_2)|.$$

(5) Otherwise, there is an annulus A that meets (i) and (ii) in (5). Then by (1), A is cut by $E_1 \cup E_2$ into eight rectangles R_i, $i = 1, \ldots, 8$, each R_i has two opposite sides in $E_1 \cup E_2$ and remaining two sides a_i in K and $a_i^* \subset \partial H$. Let b_i be an arc in $E_1 \cup E_2$ connecting ∂a_i^*, and denote the circle $b_i \cup a_i^* \subset J_k$ by e_i, $i = 1, 3, 7, 5$. In a basis of $H_1(\partial J_3, Z)$, e_1 and e_5 have coordinates $(3, p)$ and $(1, q)$ respectively, and hence $||b_1, b_5|| = ||e_1, e_5|| = |3q - p| \neq 0$, since p and 3 are co-prime. It follows that $\partial b_1 = \partial a_1^*$ and $\partial b_5 = \partial a_5^*$ are alternating on ∂E_2. By the same reason, ∂a_3^* and ∂a_7^* are alternating on ∂E_1.

Now, back to J_2, ∂a_i^*'s have the cyclic order $4, 8, 2, 6$ in ∂E_1, and the cyclic order $4, 8, 6, 2$ in ∂E_2. Hence, there are four disjoint arcs on $E_1 \cup E_2$ such that the two with $a_4^* \cup a_8^*$ form a circle $e_{4,8}$ on ∂J_2, and the other two with $a_2^* \cup a_6^*$ form a circle $e_{2,6}$ on ∂J_2, moreover $e_{2,6}$ and $e_{4,8}$ are disjoint, therefore they are parallel on ∂J_2. But in a basis of $H_1(\partial J_2, Z)$, those two circles have coordinates $(3, p)$ and $(1, q)$, and $||e_{2,6}, e_{4,8}|| = |3q - p| \neq 0$, since 3 and p are coprime. A contradiction. □

3. Proof of Theorem 1.2 by assuming that K is simple and small.

In this section, $g > 0$ will be even integer. By Lemma 2.1, let $\hat{S}_{g,l} \subset H_K[\partial S_{g,l}] \subset H(\partial S_{g,l})$ the surface obtained by capping off the boundary of $S_{g,l}$ with a disk. Then $\hat{S}_{g,l}$ is a closed surface of genus g.

Now, Theorem 1.2 follows from the following two propositions (the "Moreover" part of Theorem 1.2 follows directly from [7]).

Proposition 3.1. *$K \subset H$ is a simple and small knot.*

Proposition 3.2. (*1*) *$\hat{S}_{g,l}$ is incompressible in $H_K[\partial S_{g,l}]$.*

(*2*) *for given g, $\partial S_{g,l}$ and $\partial S_{g,l'}$ are not the same slope in ∂H_K when $l \neq l'$.*

We choose the center of E_1 as the common base point for the fundamental groups of H and of all surfaces $S_{g,l}$. Now, $\pi_1(H)$ is the free group of rank three generated by y_1, y_2, y_3 indicated in Figure 1. and $\pi_1(S_{g,l})$ is the free group of rank $2g$ generated by $x_1, \dots, x_{2g}$, where x_i is the generator given by α_i and two arcs in E_1 (see Figure 2). Let $\phi : S_{g,l} \to H$ be the inclusion (precisely ϕ should be $\phi_{g,l}$, we omit sub-index without making confusion), $\phi_* : \pi_1(S_{g,l}) \to \pi_1(H)$ be the induced homomorphism. It is easy to see from Figures 1, 3 and 4,

$$(*) \qquad \tilde{c}_3 = y_1,\ \tilde{c}_7 = y_1^3,\ \tilde{c}_\# = y_2 y_3,\ \tilde{c}_* = y_3^{-3} y_2^{-l}$$

where $\tilde{c}_3 \in \pi_1(H)$ is given by c_3 and two arcs in E_1 and so on.

Recall that $\overline{v_{2i}u_{2i-1}}$, $\overline{v_{2i-1}u_{2i}}$ $\overline{u_{2i}u_{2i+1}}$ and $\overline{v_{2i+1}v_{2i}}$ are parallel copies of c_7, c_3, $c_\#$ and c_* respectively. One can read $\phi_*(x_i)$ directly as words in y_1, y_2, y_3 by (2.0)–(2.4) and (*). They are:

$$(3.0) \qquad \phi_*(x_1) = y_3^3,$$

and for $1 < 4i + j \leq 2g$, $j = 1, 2, 3, 4$,

$$(3.1) \quad \phi_*(x_{4i+1}) = y_3^{-1} y_2^{-1} \phi_*(x_{4i}) y_2^l y_3^3 = (w_1^{-1} w_2)^i y_3^3 (w_1 w_2^{-1})^i,$$

$$(3.2) \quad \phi_*(x_{4i+2}) = y_1^3 \phi_*(x_{4i+1}) y_1 = y_1^3 (w_1^{-1} w_2)^i y_3^3 (w_1 w_2^{-1})^i y_1,$$

$$(3.3) \quad \phi_*(x_{4i+3}) = y_3^{-3} y_2^{-l} \phi_*(x_{4i+2}) y_2 y_3 = w_2 (w_1^{-1} w_2)^i y_3^3 (w_1 w_2^{-1})^i w_1,$$

$$(3.4) \quad \phi_*(x_{4i+4}) = y_1^{-1} \phi_*(x_{4i+3}) y_1^{-3} = y_1^{-1} w_2 (w_1^{-1} w_2)^i y_3^3 (w_1 w_2^{-1})^i w_1 y_1^{-3},$$

$$(3.5) \quad \text{where } w_1 = y_1 y_2 y_3 \text{ and } w_2 = y_3^{-3} y_2^{-l} y_1^3.$$

Obviously

$$(3.6) \qquad (w_1 w_2^{-1})^j w_1 (w_1^{-1} w_2)^i = w_2 (w_1^{-1} w_2)^{i-j-1} \text{ if } i > j \text{ and}$$
$$(w_1 w_2^{-1})^j w_1 (w_1^{-1} w_2)^i = (w_1 w_2^{-1})^{j-i} w_1 \text{ if } i \leq j.$$

$$(3.7) \qquad (w_1 w_2^{-1})^j w_2 (w_1^{-1} w_2)^i = w_2 (w_1^{-1} w_2)^{i-j} \text{ if } i \geq j \text{ and}$$
$$(w_1 w_2^{-1})^j w_2 (w_1^{-1} w_2)^i = (w_1 w_2^{-1})^{j-i-1} w_1 \text{ if } i < j.$$

Lemma 3.3. (*1*) *$S_{g,l}$ is incompressible in H.*

(*2*) *for given g, $\partial S_{g,l}$ and $\partial S_{g,l'}$ are not in the same slope in ∂H if $l \neq l'$.*

Proof. By (3.5), the right sides of (3.0)–(3.4) are reduced words in $< y_1, y_2, y_3 >$. Now, we present $\pi_1(S_{l,g})$ as the free product $G_1 * G_2$, where $G_1 =< x_1, x_3, \ldots, x_{2g-1} >$ and $G_2 =< x_2, x_4, \ldots, x_{2g} >$.

(1) We need only to show that $\phi_* : \pi_1(S_{g,l}) \to \pi_1(H)$ is injective.

For each $w_2 \in G_2$, we may suppose that w_2 is a reduced form in $< x_2, \ldots, x_{2g} >$. Now, we can present $\phi_*(w_2)$ as a word in $< y_1, y_2, y_3 >$ by first replacing each $x_{2i}^{\pm l} \in w_2$ by $\phi_*(x_{2i})^{\pm l}$ and then applying (3.2) and (3.4). By (3.5), (3.6), (3.7) and obvious cancellations, one can get a reduced form of $\phi_*(w_2)$ in $< y_1, y_2, y_3 >$. Indeed by an induction on the length of the reduced form w_2, it is easy to see that if $w_2 \neq 1$, then $\phi_*(w_2) \neq 1$ and $\phi_*(w_2)$ has the reduced form started from and ended by the non-zero powers of y_1. Similarly, one can argue that for $1 \neq w_1 \in G_1$, $\phi_*(w_1) \neq 1$ and $\phi_*(w_1)$ has the reduced form started from and ended by the non-zero powers of y_3 and y_2.

Now, present each $1 \neq w \in G_1 * G_2$ in a reduced form $g_1, g_2, \ldots, g_n$ of $G_1 * G_2$, and each g_i in a reduced form in G_1 or G_2. It is clear that $\phi_*(w) \neq 1$.

(2) For given g, l, the conjugacy class corresponding to $\partial S_{g,l}$ in $\pi_1(S_{g,l})$ can be presented by a reduced word below (see Figure 2):

(**)

$$x_1 x_3^{-1} \ldots x_{2g-3} x_{2g-1}^{-1} x_1^{-1} x_3 \ldots x_{2g-3}^{-1} x_{2g-1} x_{2g}^{-1} x_{2g-2} \ldots x_4^{-1} x_2 x_{2g} x_{2g-2}^{-1} \ldots x_4 x_2^{-1}$$

Now, we can present $\phi_*([\partial S_{g,l}])$ in $\pi_1(H)$ as a word of $< y_1, y_2, y_3 >$ by (**) and (3.0)–(3.4). Then doing cancellations to get the reduced form of $\phi_*([\partial S_{g,l}])$ is very direct and all powers of y_2 are untouched in this process. It follows that $\phi_*([\partial S_{g,l}])$ and $\phi_*([\partial S_{g,l'}])$ do not have the same cyclic reduced form when $l \neq l'$. Hence, if $l \neq l'$, $S_{g,l}$ and $S_{g,l'}$ are not homotopic in H, and therefore, they are not isotopic in ∂H. □

Now, $S_{g,l}$ separates H into two components P_1 and P_2, with $\partial P_1 = T_1 \cup S_{g,l}$ and $\partial P_2 = T_2 \cup S_{g,l}$, where $T_1 \cup T_2 = \partial H$ and $\partial T_1 = \partial T_2 = \partial S_{g,l}$.

Lemma 3.4. *T_i is incompressible in H.*

Proof. Let $\phi_\# : H_1(S_{g,l}, Z) \to H_1(H, Z)$ be the induced homomorphism on the first homology groups. Note that $H_1(H, Z) = Z + Z + Z$ generate $\bar{y}_1$, $\bar{y}_2$ and $\bar{y}_3$, where $\bar{y}_i = \pi(y_i)$, and $\pi : \pi_1(H, Z) \to H_1(H, Z)$ is the abelization. By (3.0)–(3.4), it is easy to see that $i_\#(H_1(S_{g,l}, Z))$ is a subgroup of $H_1(H, Z)$ generated by $4\bar{y}_1$, $(l+1)\bar{y}_2$, $\bar{y}_3$. Thus, $H_1(H, Z)/\phi_\#(H_1(S_{g,l}, Z))$ is a finite group (of order $4l + 4$).

If T_i, $i = 1$ or 2, is compressible, then there is a compressing disk B_1' in H for T_i. Since $\partial B_1' \cap \partial S_{g,l} = \emptyset$ and $S_{g,l}$ is incompressible in H, by standard argument in 3-manifold topology, we may assume that $B_1' \cap S_{g,l} = \emptyset$. Furthermore, since H is a handlebody, we may also assume that B_1' is non-separating in H. Thus, there are two properly embedded disks B_2' and B_3' in H such that, (B_1', B_2', B_3') is a set of basis disks of H. Let z_1, z_2 and z_3 be generators of $\pi_1(H)$ corresponding to B_1', B_2' and B_3'. Since $S_{g,l}$ misses B_1', $\phi_*(\pi_1(S_{g,l})) \subset G \subset \pi_1(H)$, where G is generated by z_2 and z_3. Then, clearly, $H_1(H, Z)/\phi_\#(H_1(S_{g,l}, Z))$ is infinite group, a contradiction. □

Jaco's Lemma [4]. *Let M be a compact 3-manifold with compressible ∂M and r be a circle in ∂M. If $\partial M - r$ is incompressible in M, then either $M[r]$ is a 3-ball or $\partial M[r]$ is incompressible.*

Proof of Proposition 3.2**.** Since $S_{g,l}$ is incompressible in H by Lemma 3.3, and H contains no closed incompressible surface, ∂P_i is compressible in $P_i, i = 1, 2$;

Since T_1, T_2 and $S_{g,l}$ are incompressible in H by Lemma 3.4 and Lemma 3.3, T_i and $S_{g,l}$ are incompressible in P_i. Hence $\partial P_i - \partial S_{g,l}$ is incompressible in P_i, $i = 1, 2$. Since, clearly, $P_i[\partial S_{g,l}]$ is not a 3-ball, $\partial P_i[\partial S_{g,l}]$ is incompressible by Jaco's Lemma. It follows that $\hat{S}_{g,l}$, which is parallel to a component of $\partial P_i[\partial S_{g,l}]$, is incompressible in $P_i[\partial S_{g,l}]$. Since $H[\partial S_{g,l}]$ is a union of $P_1[\partial S_{g,l}]$ and $P_2[\partial S_{g,l}]$ along $\hat{S}_{g,l}$, $\hat{S}_{g,l}$ is incompressible in $H[\partial S_{g,l}]$. Therefore, $\hat{S}_{g,l}$ is incompressible in $H_K[\partial S_{g,l}]$. We proved Proposition 3.2 (1).

Proposition 3.2 (2) follows Lemma 3.3. (2)

4. H_k is irreducible, ∂-irreducible, anannular.

Recall E_j, F_j, J_k, M_k, B_k, a_i, A_i, D_i, T, μ defined in Section 2.

Lemma 4.1. *$F_1 \cup F_2$ is incompressible and ∂-incompressible in H_K.*

Proof. Suppose first $F_1 \cup F_2$ is compressible in H_K. Then there is a disk $B \subset M_k$ such that $B \cap (F_1 \cup F_2) = \partial B$ and ∂B is a non-trivial circle on $F_1 \cup F_2$. Denote by B' the disk bounded by ∂B in $E_1 \cup E_2$. Then, $B \cup B'$ is a 2-sphere S^2 in the solid torus J_k, so $B \cup B'$ bounds a 3-ball B^3 in J_k. Since ∂B is non-trivial in $F_1 \cup F_2$, B' contains $\partial_1 a_i$ for some $a_i \subset J_k$. Since S^2 is separating and a_i is connected, we must have $(a_i, \partial a_i) \subset (B^3, B')$, which provides a relative homotopy on $(J_k, E_1 \cup E_2)$ sending a_i to $E_1 \cup E_2$, see Figure 7 (a), which contradicts Lemma 2.3 (2).

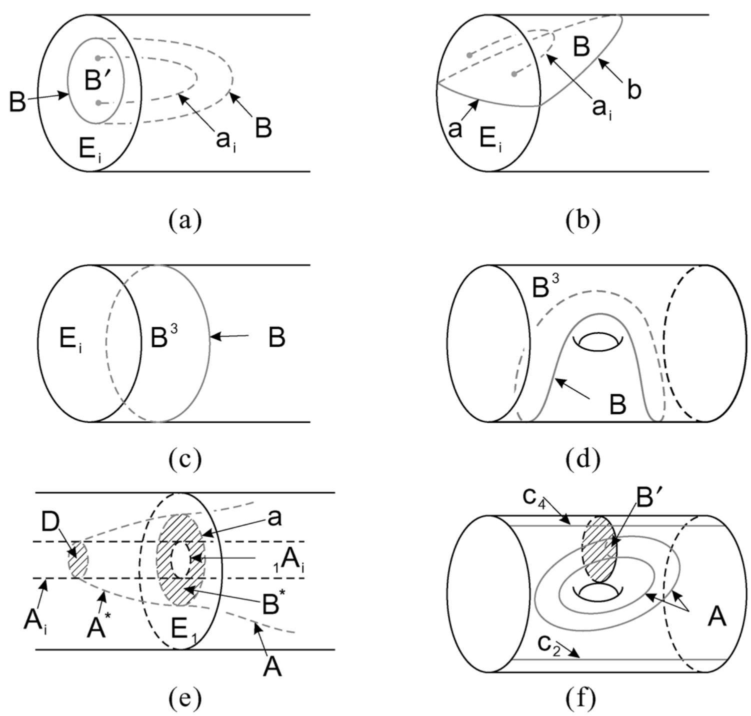


Figure 7.

Suppose then $F_1 \cup F_2$ is ∂-compressible in H_K. Then there is a non-trivial arc a in $F_1 \cup F_2$ and an arc b in ∂H_K bound a proper disk B in M_k. There are two cases:

(1) $b \subset T$. Then b is a proper arc in A_i, $i = 1, 5, 3, 7$. Now, either b is a trivial arc in A_i, then there is an arc b' in ∂A_i such that the circle $a \cup b'$ is non-trivial in $F_1 \cup F_2$ but bounds a disc in M_k, which contradicts the incompressibility of $F_1 \cup F_2$ we just proved; or b is a non-trivial arc in A_i, then the disc B provides a relative homotopy on $(J_k, E_1 \cup E_2)$ sending a_i to $E_1 \cup E_2$, which contradicts Lemma 2.3 (2).

(2) $b \subset \partial H$. Since $|B \cap (E_1 \cup E_2)| = 1$, B separates a 3-ball B^3 from J_k by Lemma 2.3 (4). Since a is non-trivial in $F_1 \cup F_2$, by the same reason as the end of the first paragraph, one of a_i lies in B^3 with ∂a_i lies in a disc in $\partial B^3 \cap E_1$, see Figure 7 (b), which contradicts Lemma 2.3 (2). □

Lemma 4.2. *H_K is irreducible.*

Proof. Otherwise, there is an essential 2-sphere S^2 in H_K. Since H is irreducible, S^2 bounds a 3-ball B^3 in H with $K \subset B^3$, which contradicts Lemma 2.3 (1). □

Lemma 4.3. *H_K is ∂-irreducible.*

Proof. Suppose H_K is ∂-reducible. Let B be a compressing disk of ∂H_K. If $\partial B \subset T$, then H_K contains an essential 2-sphere, which contradicts Lemma 4.2. Below, we assume that $\partial B \subset \partial H$. Furthermore, we assume that

(*) $|B \cap (F_1 \cup F_2)|$ is minimal among all compressing disks B of ∂H_k.

Suppose first $B \subset M_k$. Since $B \cap (E_1 \cup E_2) = \emptyset$, B separates a 3-ball B^3 from J_k by Lemma 2.3 (4). Since ∂B is non-trivial in ∂H_K, then either B^3 contains only one of E_1 and E_2, see Figure 7 (c), and then $||a_i, B|| \neq \emptyset$ for all $a_i \subset J_k$, a contradiction; or B^3 contains both E_1 and E_2 and $k = 2$ in this case, see Figure 7 (d), a_4 and a_8 are properly homotopic in $(B^3, E_1 \cup E_2) \subset (J_k, E_1 \cup E_2)$, which contradicts Lemma 2.3 (2).

Suppose then $B \cap (F_1 \cup F_2) \neq \emptyset$. By Lemma 4.1 and the assumption (*), $B \cap (F_1 \cup F_2)$ consists of arcs. Then an outmost arc a of $B \cap (F_1 \cup F_2) \subset B$ separates a disk B_0 from B with $B_0 \subset M_k$ for some k. By (*) a must be non-trivial in $F_1 \cup F_2$, (otherwise $|B \cap (F_1 \cup F_2)|$ can be reduced by pushing B_0 to a suitable side). Since $|B_0 \cap (E_1 \cup E_2)| = 1$, B_0 separates a 3-ball from J_k by Lemma 2.3 (4). Then we reach a contradiction by the same reason in the end of the proof of Lemma 4.1 □

Lemma 4.4. *M is anannular.*

Proof. Suppose H_K contains an essential annulus A. Assume that

(**) $|A \cap (F_1 \cup F_2)|$ is minimal among all essential annuli in H_K.

By Lemma 4.1 and (**), each component of $A \cap (F_1 \cup F_2)$ is non-trivial in both A and $(F_1 \cup F_2)$. There are three cases:

Case 1. $\partial A \subset T$. There are two sub-cases:

(1) $||\partial_1 A, \mu|| = 0$. Let us assume $\partial A \cap (F_1 \cup F_2) = \emptyset$.

Suppose first $A \cap (F_1 \cup F_2) = \emptyset$. Let us assume that A is contained in M_2 (the remaining cases are more direct). The whole ∂A must lie in the same $A_i \subset M_2$ by Lemma 2.3 (3). Since A is essential, $A \cap D_i = \emptyset$ for $i = 2, 4, 8$. Then $\partial A \subset A_6$. Let M' be obtained by cutting M_2 along D_2, D_4, D_8. Then A is still an incompressible annulus in $M^{'}$, and $D_6 \subset M_2$ become a properly embedding disk $D_6' \subset M^{'}$ with $\partial D_6' \cap A_6 = \partial D_6 \cap A_6$, a non-trivial arc of A_6. Since $\partial A \subset A_6$, $A \cap D_6'$ is an arc in both A and $D_6^{'}$. Hence, there is a

∂-compressing disk of A in M' which is also a ∂-compressing disk of A in M_2, which contradicts that A is essential in M_2.

Suppose then $A \cap (F_1 \cup F_2) \neq \emptyset$, which must be a union of circles. An outmost circle a of $A \cap (F_1 \cup F_2) \subset A$ and $\partial_1 A$ bound an annulus $A^* \subset A$. May assume that $a \subset F_1$, and $\partial_1 A \subset A_i \subset M_k$. Let B^* be the disk bounded by a on E_1 and D be the meridian disk of $N(K)$ bounded by $\partial_1 A$. $B^* \cup A^* \cup D$ is a separating 2-sphere S^2 which bounds a 3-ball $B^3 \subset J_k$, see Figure 7 (e). Since $|(A^* \cup D) \cap (\cup_{a_j \subset J_k} a_j)| = |(A^* \cup D) \cap a_i)| = 1$, by applying Lemma 2.3 (2) as before, we have $|B^* \cap (\cup_{a_j \subset J_k} a_j)| = |B^* \cap a_i| = 1$. Hence, a and $\partial_1 A_i$ bound an annulus A' in F_1. Now, by pushing the annulus $\overline{A - A^*} \cup A'$ to a suitable side of F_1, $|A \cap (F_1 \cup F_2)|$ is reduced, which contradicts (**).

(2) $||\partial_1 A, \mu|| \geq 1$. Now, $A \cap (F_1 \cup F_2)$ consists of non-trivial arcs in A, which cut A into $8||\partial_1 A, \mu||$ rectangles and each rectangle has two opposite edges on $F_1 \cup F_2$ and two opposite edges on A_i and $A_{\pi(i)}$, where $\pi(i) = i + l \mod 8$. If $l = 0$, then the two ends of each arc of $A \cap (F_1 \cup F_2)$ lie in a same component of $\partial(F_1 \cup F_2)$, and an inner most arc is trivial in $F_1 \cup F_2$, a contradiction [2]. If $l \neq 0 \bmod 8$, then a_6 and a_{6+l} are properly isotopy in M_2, which contradicts to Lemma 2.3 (2).

Case 2. $\partial_1 A \subset T$ and $\partial_2 A \subset \partial H$.

By Lemma 4.3, both ∂H and T are incompressible in H_K. Clearly, H_K is not homeomorphic to $T \times I$. Since both Dehn fillings along μ and ∂A_1 compress ∂H_K, by [1, 2.4.3], $\Delta(\partial_1 A, \mu) \leq 1$. There are two sub-cases.

(1) $||\partial_1 A, \mu|| = 0$. Since $\partial_1 A$ is disjoint from $F_1 \cup F_2$, $\partial_2 A$ is disjoint from $F_1 \cup F_2$ (otherwise, there is an arc in $A \cap (F_1 \cup F_2)$ with two ends in $\partial_2 A$ which is trivial in A). Then it follows that A is disjoint from $F_1 \cup F_2$ by the proof of Case 1. Suppose $\partial_1 A \subset A_i \subset M_k$ for some i, k. Let D be the meridian disk of $N(K)$ bounded by $\partial_1 A$ and $B = A \cup_{\partial_1 A} D$. Then B is a proper disc in J_k, ∂B is non-trivial in $\partial H \cap J_k$, and $|B, \cup_{a_j \subset M_k} a_j| = |B, a_i| = 1$. Since $B \cap (E_1 \cup E_2) = \emptyset$, B separates a 3-ball B^3 from J_2 by Lemma 2.3 (4). If B^3 contains only one of E_1 and E_2, then B meets all a_j in J_k. If B^3 contains both E_1 and E_2, then B meets a_i in non-zero even number. In each case, we reach a contradiction.

(2) $||\partial_1 A, \mu|| = 1$. It is easy to see that this case is ruled out by Lemma 2.3 (5).

Case 3. $\partial A \subset \partial H$.

Suppose first $A \cap (F_1 \cup F_2) = \emptyset$. Since A is essential, A is disjoint from D_i for $i \neq 6$. Let us assume that $A \subset M_2$ (the remaining cases are the same). Since $\partial A \subset \partial H \cap J_2$ and A is disjoint from c_4, c_2, A separates J_2 into two solid torus J^* and J' such that $(E_1 \cup E_2) \cap J' = \emptyset$ and separates a disc

$B' \subset J'$ from B_2, see Figure 7 (f). Since J' is disjoint from all $A_i \subset M_2$, A is ∂-compressible in M_2, which contradicts that A is essential in M_2.

Suppose then $A \cap (F_1 \cup F_2) \neq \emptyset$. There are two sub-cases:

(1) $A \cap (F_1 \cup F_2)$ consists of circles. Then an outmost circle a of $A \cap (F_1 \cup F_2) \subset A$ and $\partial_1 A$ bound an annulus $A^* \subset A$ such that $A^* \subset M_k$. Let B^* be the disk bounded by a on $E_1 \cup E_2$ and $D^* = A^* \cup B^*$. By slightly pushing, we have $D^* \subset J_k$, moreover, (i) $D^* \cap a_i \neq \emptyset$ for some $a_i \subset J_k$, (ii) for each $a_j \subset J_k$, $|D^* \cap a_j| \leq 2$, and ≤ 1 if $k = 2$. Since $\partial D^* \subset J_k \cap \partial H$, D^* separates a 3-ball B^3 from J_k by Lemma 2.3 (4). Now either B^3 contains both E_1 and E_2, $k = 2$ in this case, and D^* meets each $a_j \subset J_2$ in even number of points, which contradicts (i) and (ii) above; or B^3 contains only one E_i, say E_1, then $\partial_1 A = \partial D^*$ is parallel to ∂E_1. Since A^* is disjoint from K, $|B^* \cap (\cup_{a_j \subset J_k} a_j)| = |D^* \cap (\cup_{a_j \subset J_k} a_j)| = |E_1 \cap (\cup_{a_j \subset J_k} a_j)| = 4$. Hence, a and ∂E_1 bound an annulus A' in F_1 by applying Jordan Curve Theorem. Now, we reach a contradiction by the same argument at the end of Case 1 (1).

(2) $A \cap (F_1 \cup F_2)$ consists of arcs. Then $F_1 \cup F_2$ cut A into rectangles R_i, and each R_i has two opposite sides in $F_1 \cup F_2$ and remaining two sides in ∂H. Then R_i separates a 3-ball B_i^3 from J_k by Lemma 2.3 (4). Let D_i^1 and D_i^2 be two disks of $B_i^3 \cap (E_1 \cup E_2)$. By Lemma 2.3 (2), we have (i) $\partial_1 a_j \subset D_i^1$ if and only if $\partial_2 a_j \subset D_i^2$. (ii) a_j and a_l are contained in the same B_i^3 implies that $(j, l) = (2, 8)$.

If a_2 and a_8 belong to the same B_i^3, then so do a_3 and a_7, which contradicts (ii). Hence, there is only one a_j in each B_i^3. Hence, A separates from H a solid torus with K as centerline (up to isotopy). Then K and a component of $\partial A \subset \partial H$ bound an annulus, which contradicts Lemma 2.3 (5). □

5. H_K contains no closed essential surfaces.

Recall W, W_i, l_i defined in Section 2.

Suppose H_K contains closed essential surfaces F. We define the complexity of F by an ordered pair

$$C(F) = (|F \cap W|, |F \cap (F_1 \cup F_2)|).$$

Suppose F realizes the minimality of $C(F)$. By the minimality of $C(F)$, Lemma 4.1 and the standard argument in 3-manifold topology, we have

Lemma 5.1. (*1*) *Each component of* $F \cap (F_1 \cup F_2)$ *is a non-trivial circle in both* F *and* $F_1 \cup F_2$*,*

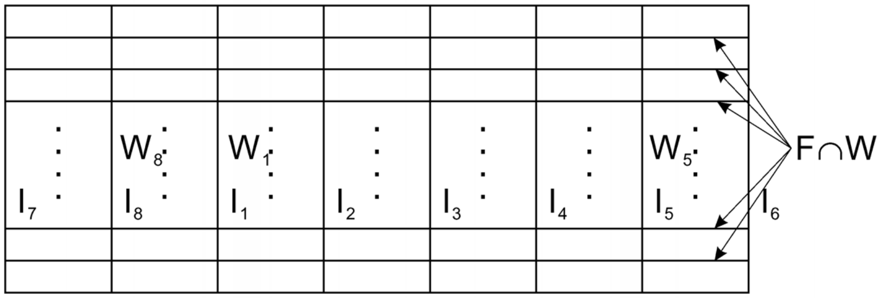
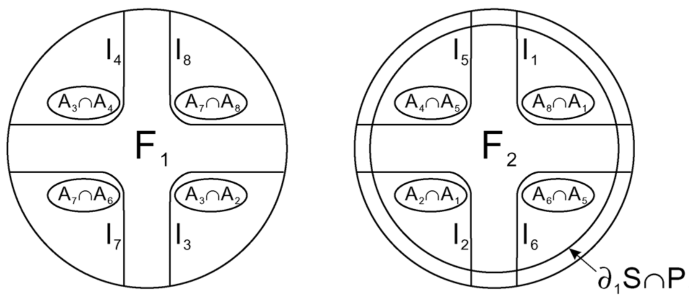


Figure 8.



(a) (b)

Figure 9.

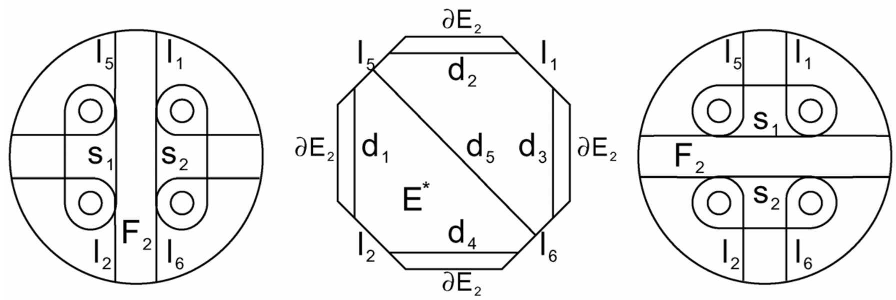
(2) $F \cap W \subset W$ is a union of arcs as in Figure 8. *Hence, $|F \cap l_i| = |F \cap l_j|$ for all i, j.*

The positions of ∂A_i and l_i in $F_1 \cup F_2$ are indicated as in Figure 9.

Below, we will use $\tilde{s}$ to denote a given family of parallel disjoint proper 1-manifolds on some surface, and use s to denote a representive (a component) of $\tilde{s}$.

Lemma 5.2. *Each component of $F \cap M_k$ is isotopic to either $M_k \cap \partial H$ or some $A_i \subset M_k$, where $k = 1, 3$.*

Proof. The proofs for $k = 1$ and 3 are the same. Assume $k = 3$. First, we need



(a) (b) (c)

Figure 10.

Lemma 5.3. *Components of $F \cap F_2$ in F_2 which are not parallel to a component of ∂F_2 are divided into two families of circles $\tilde{s}_1$ and $\tilde{s}_2$ in Figure* 10 (*a*) *and* (*c*). *Moreover, in each case,* $|\tilde{s}_1| = |\tilde{s}_2|$.

Proof. Since each component of $F \cap F_2$ isotopic to a component of ∂F_2 contributes the same to $|F \cap l_k|$ for all $l_k \subset F_2$, we may assume that $F \cap F_2$ contains no such components when we apply Lemma 5.1 (2) to prove Lemma 5.3.

Note that l_1, l_2, l_5, l_6 separate F_2 into four annuli and one disc which is presented as an octagon E^* in Figure 10 (b), where $F \cap E^*$ are presented as five families of proper disjoint arcs $\tilde{d}_1, \dots, \tilde{d}_4, \tilde{d}_5$ with ∂d_i in different l_j and l_k for each i. By Lemma 5.1 (2), we have

$$|\tilde{d}_4| + |\tilde{d}_1| = |\tilde{d}_2| + |\tilde{d}_1| + |\tilde{d}_5| = |\tilde{d}_2| + |\tilde{d}_3| = |\tilde{d}_4| + |\tilde{d}_3| + |\tilde{d}_5|.$$

It follows $|\tilde{d}_5| = 0$, $|\tilde{d}_1| = |\tilde{d}_3|$ and $|\tilde{d}_2| = |\tilde{d}_4|$. Back to Figure 9 (b), since no component is isotopic to ∂F_2, it follows that either $|\tilde{d}_2| = 0$, which is Figure 10 (a), or $|\tilde{d}_1| = 0$, which is Figure 10 (c). □

Let us return to the proof of Lemma 5.2. Let S be a component of $F \cap M_3$. Each W_i separates a solid tori P_i from M_3, $i = 1, 5$. Let $M_3' = M_3 - (P_1 \cup P_5)$, which is a solid torus. There are three cases to discuss.

Case 1. If a component of ∂S is isotopic to a component of ∂A_i, $i = 1, 5$. By the minimality of the complexity $C(F)$, S is disjoint from W_i, and therefore $S \subset P_i$, which is an annulus isotopy to A_i, $i = 1$ or 5.

Case 2. If a component of ∂S is isotopic to ∂E_2, let $\partial_1 S$ be the outmost component of $\partial S \subset F_2$ which is isotopic to ∂E_2. Now, $\partial_1 S$ intersects P_i

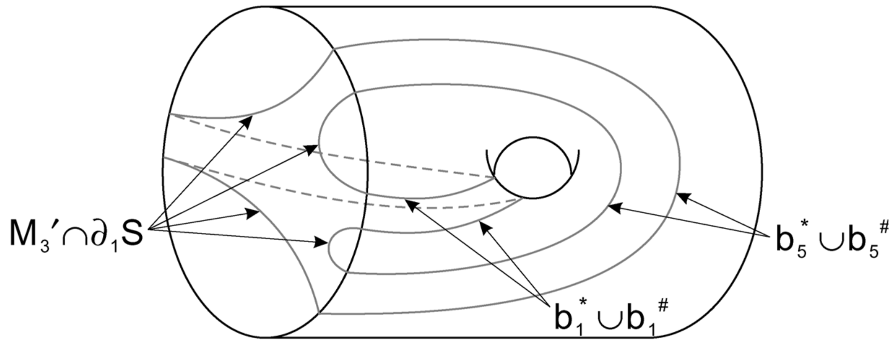


Figure 11.

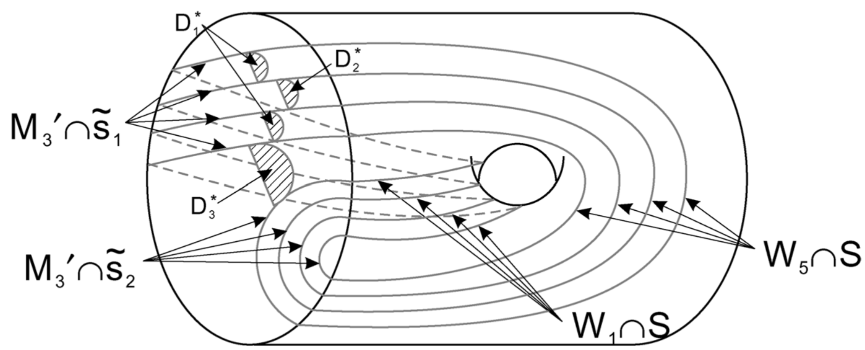


Figure 12.

as in Figure 9 (b) and $W_i \cap S$ contains two arcs b_i^* and $b_i^{\#}$ with ends in $\partial_1 S$. Let S_i be the component of $S \cap P_i$ which meets $\partial_1 S$, $i = 1, 5$. Then S_i is incompressible in P_i. Since $\partial S_i = (\partial_1 S \cap P_i) \cup (b_i^* \cup b_i^{\#})$ bounds a disk in P_i parallel to ∂M_3, S_i itself is such a disc, $i = 1, 5$. Let S_3 be a component of $S \cap M_3'$ which meets $\partial_1 S$, then S_3 is incompressible in the solid torus M_3' and ∂S_3 has a component $(\partial_1 S \cap M_3') \cup (b_1^* \cup b_1^{\#}) \cup (b_5^* \cup b_5^{\#})$ which bounds a disk in $\partial M_3'$ as in Figure 11. Hence, S_3 itself is such a disk. Thus $S = S_1 \cup_{b_1^* \cup b_1^{\#}} S_3 \cup_{b_5^* \cup b_5^{\#}} S_5$ is isotopic to $M_3 \cap \partial H$.

By Lemma 5.3, to finish the proof, we need only to rule out Case 3 below.

Case 3. $|\tilde{s}_1| = |\tilde{s}_2| \neq 0$ in Figure 10 (a) or (b). Since the discussion for (a) and (c) in Figure 10 are the same, we just discuss the former case.

We may assume that no component of S is isotopic to a component of ∂F_2 by Case 1 and Case 2 we just discussed. Let $S_3' = S \cap M_3'$. Then $\partial S_3'$

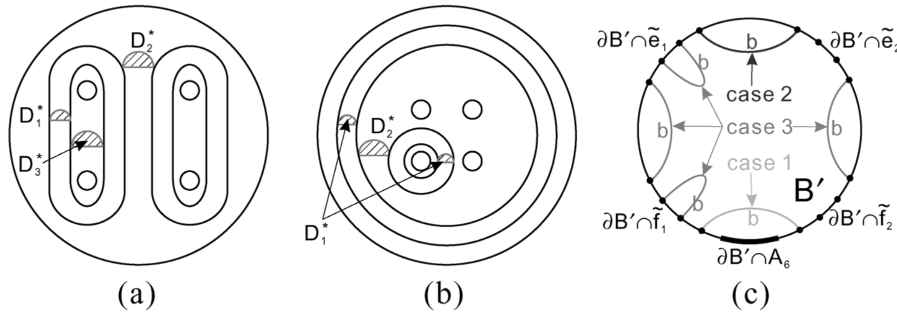


Figure 13.

contains $2|\tilde{s}_1|$ circles which are produced from the arcs $(\tilde{s}_1 \cup \tilde{s}_2) \cap M'_3$ and the arcs $(W_1 \cup W_5) \cap S$, as in Figure 12, where $|\tilde{s}_1| = 2$. Note each circle in $\partial S'_3$ is non-trivial in $\partial M'_3$. Since S'_3 is incompressible in the solid torus M'_3, each component of S'_3 is an annulus which is ∂-compressible. Now, $B'_3 \cap S'_3$ is a union of arcs, where $B'_3 = B_3 \cap M'_3$. An outmost arc b of $B'_3 \cap S'_3 \subset B'_3$ separates a disc D^* from B'_3. As a ∂-compressing disc of S'_3, D^* can be moved into the positions of D^*_1, D^*_2, D^*_3, indicated as in Figure 12. Now, back to M_3, those D^*_i's in Figure 12 are corresponding to those D^*_i's in Figure 13 (a), $i = 1, 2, 3$. In the cases of D^*_1 and D^*_3 in Figure 13 (a), one can push F along the disc to reduce $|F \cap W|$; in the case of D^*_2 in Figure 13 (a), one can push F along the disc to reduce $|F \cap (F_1 \cup F_2)|$, but not to increase $|F \cap W|$. In each case, it contradicts the minimality of $C(F)$. □

Remark on Figures 11, 12 and 14. In Figure 11, to simplify the picture, W_1 does not meet B_3 in three arcs as it should be. But one verifies easily that this simplification does not affect the proof. The same remark is needed for Figures 12 and 14. Moreover in Figure 14, we only draw a representative e_i for a families $\tilde{e}_i$ and so on.

Proposition 5.4. *H_K contains no closed essential surface.*

Proof. Now, we consider $F \cap M_2$. Each component of $F \cap (F_1 \cup F_2)$ is isotopic to a component of $\partial F_2 \cup \partial F_1$ by Lemma 5.2. Applying Lemma 5.1 (2) again, we have $|\widetilde{\partial E_2}| = |\widetilde{\partial E_1}|$, where $\widetilde{\partial E_i} \subset F_i$ are components of $F \cap F_i$ isotopic to ∂E_i, $i = 1, 2$. Each W_i separates a solid tori P_i from M_2, $i = 2, 4, 8$. Let $M'_2 = \overline{M_2 - (P_2 \cup P_4 \cup P_8)}$, which is a handlebody of genus 2.

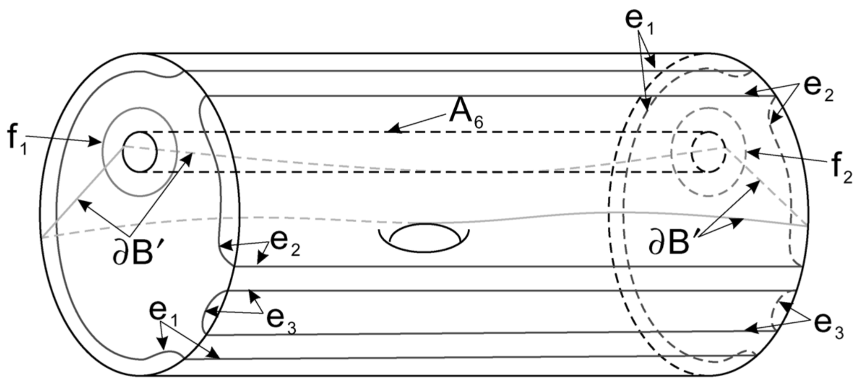


Figure 14.

Note if $F \cap M_2$ has a component S such that a component of ∂S is isotopic to ∂A_i, $i = 2, 4, 8$, by the minimality of $C(F)$, $S \subset P_i$ and hence S is isotopic to A_i, $i = 2, 4, 8$.

Let $S_2' = F \cap M_2'$. Then $\partial S_2'$ consists of possibly five families of circles $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$, $\tilde{f}_1$ and $\tilde{f}_2$, where $\tilde{e}_1$, $\tilde{e}_2$ and $\tilde{e}_3$ with $|\tilde{e}_i| = |\widetilde{\partial E_1}|$ are produced from the arcs $(\widetilde{\partial E_1} \cup \widetilde{\partial E_2}) \cap M_2'$ and the arcs of $(W_2 \cup W_4 \cup W_8) \cap F$ with end points lying $\widetilde{\partial E_1} \cup \widetilde{\partial E_2}$, $\tilde{f}_1 \subset F_1$ and $\tilde{f}_2 \subset F_2$ are parallel copies of the two components of ∂A_6 respectively. All those are indicated in Figure 14 (see Remark on Figures 11., 12, and 14). Moreover,

(i) each component of $\tilde{e}_3$ bounds a disk in $\partial M_2'$, hence bounds also a disk in F.

(ii) any two components in $\tilde{e}_1 \cup \tilde{e}_2$ bound an annulus in $\partial M_2'$ disjoint from A_6.

There is a proper disc B' in M_2' with $\partial B'$ shown in Figure 14 such that

(iii) $\partial B'$ meets those four families in the cyclic order $\tilde{e}_1$, $\tilde{e}_2$, $\tilde{f}_2$, $\tilde{f}_1$,

(iv) $\partial B'$ meets each component of $\tilde{e}_1 \cup \tilde{e}_2 \cup \tilde{f}_1 \cup \tilde{f}_2$ in one point and $\partial B' \cap A_6$ is a non-trivial arc in A_6,

Let S' be a component of S_2'. Since S' is incompressible in M_2', $S' \cap B'$ consists of arcs. By (iv), there is an outmost arc b of $S' \cap B' \subset B'$ which separates a disk D^* from B' so that

(v) ∂D^* disjoint from A_6 and D^* is a ∂-compressing disk of S'.

We divide the remaining discussion into three cases by (iii). (Figure 13 (c) is helpful to understand (iii), (iv) and (v) above and each case below.)

Case 1. One end of b is in $f_1 \in \tilde{f}_1$ and the other is in $f_2 \in \tilde{f}_2$. In this case, $|\widetilde{\partial E_1}| = 0$ and all $\tilde{e}_i$'s do not exist by (iii) and (v). One can show that S' is isotopic to A_6 by cutting and pasting argument in 3-manifold topology, the detail is contained in what we will do in Case 2.

Case 2. One end of b is in $e_1 \in \tilde{e}_1$ and the other is in $e_2 \in \tilde{e}_2$. By (ii), e_1 and e_2 bound an annulus A in $\partial M_2'$ disjoint from A_6. Let M_2'', S'', A', A_6', e_1', e_2' be the images of M_2', S', A, A_6, e_1, e_2 respectively after cutting M_2' along B'. By (iv), e_i' is an arc, $i = 1, 2$. Let b_i, D_i^*, $i = 1, 2$, be the two copies of b and D^* after cutting M_2' along B'. By (v), the circle $c' \subset \partial S''$ formed by four arcs e_1', e_2', b_1' and b_2' bound a disc $D_1^* \cup A' \cup D_2^*$ in $\partial M_2''$ which is disjoint from A_6'. Since S'' is incompressible in M_2'', S'' is such a disc up to isotopy. Back to M_2', S' is isotopic to the annulus $A \subset M_2'$. Back to M_2, by (i) and similar argument in Case 2 in the proof of Lemma 5.2, S is isotopic to $M_2 \cap \partial H$.

Case 3. If either ∂b lie in one of the four families $\tilde{e}_1$, $\tilde{e}_2$, $\tilde{f}_1$ and $\tilde{f}_2$, or one end of b is in $\tilde{e}_i$ and the other in $\tilde{f}_i$, $i = 1$ or 2, then D^* can be moved in M_2' keeping to be a ∂-compressing disk of S' so that when we go back to M_2, it is a ∂-compressing disk of $F \cap M_2$ in the position of either D_1^* or D_2^* in Figure 13 (b). One can push F along either D_1^* or D_2^* to reduce $C(F)$, which contradicts the minimality of $C(F)$. (Refer the end of the argument in Case 3 of the Proof of Lemma 5.2).

So, each component S of $F \cap M_2$ is isotopic to either $M_2 \cap \partial H$ or A_i, $i = 2, 4, 6, 8$. In the former case, ∂S is ∂E_1 and ∂E_2 which bound (up to isotopy) $\partial H \cap M_1$ and $\partial H \cap M_3$ respectively by Lemma 5.2, and then F is isotopic to ∂H. In the later case, by Lemma 5.2, each component of $F \cap (M_1 \cup M_3)$ is an annulus isotopic to one of A_1, A_3, A_5, A_7. Since F is closed, it follows that F is a torus isotopic to T. □

Proposition 3.1 follows from Lemmas 4.2, 4.3, 4.4 and Proposition 5.4. Hence, Theorem 1.2 is proved.

6. Acknowledgement.

We would like to thank Doctor Zheng Hao for all the figures in this paper. We would like to express our thanks to the referee for his comments on this paper.

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Received February 12, 2004.