COMMUNICATIONS IN
ANALYSIS AND GEOMETRY
Volume 13, Number 5, 999-1037, 2005

Constructing Associative 3-folds by Evolution Equations

JASON LOTAY

1. Introduction.

This paper gives two methods for constructing associative 3-folds in $\mathbb{R}^7$, based around the fundamental idea of evolution equations, and uses them to produce examples. It is a generalisation of the work by Joyce in [6, 7, 8, 9] on special Lagrangian (SL) 3-folds in $\mathbb{C}^3$. The methods described involve the use of an affine evolution equation with affine evolution data and the area of ruled submanifolds.

We begin in Section 2 by introducing the exceptional Lie group G_2 and its relationship with the geometry of associative 3-folds in $\mathbb{R}^7$. In Section 3, we review the work by Joyce in [6, 7, 8] on evolution equation constructions for SL m-folds in $\mathbb{C}^m$. We follow this in Section 4 with a derivation of an evolution equation for associative 3-folds.

In Section 5, we derive an affine evolution equation using affine evolution data. This is used on an example of such data to construct a 14-dimensional family of associative 3-folds. One of the main results of the paper is an explicit solution of the system of differential equations generated in a particular case to give a 12-dimensional family of associative 3-folds. Moreover, we find a straightforward condition which ensures that the associative 3-folds constructed are closed and diffeomorphic to $\mathcal{S}^1 \times \mathbb{R}^2$, rather than $\mathbb{R}^3$.

In the final section, Section 6, we define ruled associative 3-folds and derive an evolution equation for them. This allows us to characterise a family of ruled associative 3-folds using a pair of real analytic maps satisfying two partial differential equations. We finish by giving a means of constructing ruled associative 3-folds M from r-oriented two-sided associative cones M_0 such that M is asymptotically conical to M_0 with order $O(r^{-1})$.

2. Introduction to G_2 and Associative 3-folds.

We give two equivalent definitions of G_2, which relate to the geometry of $\mathbb{R}^7$ and the octonions respectively. The first follows [5, page 242].

Definition 2.1. Let $(x_1, \ldots, x_7)$ be coordinates on $\mathbb{R}^7$. We shall write $d\mathbf{x}_{ij\ldots k}$ for the form $dx_i \wedge dx_j \wedge \ldots \wedge dx_k$ on $\mathbb{R}^7$. Define a 3-form φ on $\mathbb{R}^7$ by

$$\varphi = d\mathbf{x}_{123} + d\mathbf{x}_{145} + d\mathbf{x}_{167} + d\mathbf{x}_{246} - d\mathbf{x}_{257} - d\mathbf{x}_{347} - d\mathbf{x}_{356}. \tag{2.1}$$

Then, $G_2 = \{\gamma \in GL(7, \mathbb{R}) \,:\, \gamma^*\varphi = \varphi\}$.

We note that G_2 is a compact, connected, simply connected, simple, 14-dimensional Lie group, which preserves the Euclidean metric and the orientation on $\mathbb{R}^7$. It also preserves the 4-form $*\varphi$ given by

$$*\varphi = d\mathbf{x}_{4567} + d\mathbf{x}_{2367} + d\mathbf{x}_{2345} + d\mathbf{x}_{1357} - d\mathbf{x}_{1346} - d\mathbf{x}_{1256} - d\mathbf{x}_{1247}, \tag{2.2}$$

where φ and $*\varphi$ are related by the Hodge star.

The second definition, taken from [3], comes from considering the algebra of the octonions, or Cayley numbers, $\mathbb{O}$.

Definition 2.2. The group of automorphisms of $\mathbb{O}$ is G_2.

Suppose we take the latter definition of G_2 and note that $x \in \operatorname{Im}\mathbb{O}$ if and only if x^2 is real, but x is not. Therefore, for all $\gamma \in G_2$ and for $x \in \mathbb{O}$, $\gamma(x) \in \operatorname{Im}\mathbb{O} \Leftrightarrow \gamma(x)^2 = \gamma(x^2) \in \mathbb{R}$, $\gamma(x) \notin \mathbb{R} \Leftrightarrow x^2 \in \mathbb{R}$, $x \notin \mathbb{R} \Leftrightarrow x \in \operatorname{Im}\mathbb{O}$. Hence, G_2 is the subgroup of the group of automorphisms of $\operatorname{Im}\mathbb{O} \cong \mathbb{R}^7$ preserving the octonionic multiplication on $\operatorname{Im}\mathbb{O}$. This multiplication defines a cross product $\times : \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}^7$ by

$$x \times y = \frac{1}{2}(xy - yx), \tag{2.3}$$

where the right-hand side is defined by considering x and y as imaginary octonions. Note that we can recover the octonionic multiplication from the cross product and also that the cross product can be written as follows:

$$(x \times y)^d = \varphi_{abc} x^a y^b g^{cd} \tag{2.4}$$

using index notation for tensors on $\mathbb{R}^7$, where g^{cd} is the inverse of the Euclidean metric on $\mathbb{R}^7$. This can be verified using (2.1), (2.3) and a Cayley multiplication table for the octonions. We deduce from (2.4) that

$$\varphi(x,y,z) = g(x \times y, z) \tag{2.5}$$

for $x, y, z \in \mathbb{R}^7$, where g is the Euclidean metric on $\mathbb{R}^7$.

For this article, we take manifolds to be smooth and non-singular almost everywhere and submanifolds to be immersed, unless otherwise stated. We define *calibrations* and *calibrated submanifolds* following the approach in [3].

Definition 2.3. Let (M, g) be a Riemannian manifold. An *oriented tangent k-plane V* on M is an oriented k-dimensional vector subspace V of T_xM, for some x in M. Given an oriented tangent k-plane V on M, $g|_V$ is a Euclidean metric on V and hence, using $g|_V$ and the orientation on V, we have a natural volume form, vol_V, which is a k-form on V.
Let η be a closed k-form on M. Then η is a *calibration* on M if $\eta|_V \le \mathrm{vol}_V$ for all oriented tangent k-planes V on M, where $\eta|_V = \alpha \cdot \mathrm{vol}_V$ for some $\alpha \in \mathbb{R}$, and so $\eta|_V \le \mathrm{vol}_V$ if $\alpha \le 1$.
Let N be an oriented k-dimensional submanifold of M. Then N is a *calibrated submanifold* or *η−submanifold* if $\eta|_{T_xN} = \mathrm{vol}_{T_xN}$ for all $x \in N$.

Calibrated submanifolds are *minimal* submanifolds [3, Theorem II.4.2]. We now define *associative 3-folds*.

Definition 2.4. Let N be a 3-dimensional submanifold of $\mathbb{R}^7$. Note that, by [3, Theorem IV.1.4], φ as given by (2.1) is a calibration on $\mathbb{R}^7$. An oriented 3-plane V in $\mathbb{R}^7$ is *associative* if $\varphi|_V = \mathrm{vol}_V$. N is an *associative 3-fold* if T_xN is associative for all $x \in N$, i.e. if N is a φ−submanifold.

An alternative description of associative 3-planes is given in [3] which requires the definition of the *associator* of three octonions.

Definition 2.5. The *associator* $[x, y, z]$ of $x, y, z \in \mathbb{O}$ is given by

$$[x, y, z] = (xy)z - x(yz). \tag{2.6}$$

Whereas the commutator measures the extent to which commutativity fails, the associator gives the degree to which associativity fails in $\mathbb{O}$. Note that we can write an alternative formula, in index notation, for the associator of three vectors $x, y, z \in \mathbb{R}^7$ using $*\varphi$ and the inverse of the Euclidean metric g on $\mathbb{R}^7$ as follows:

$$\frac{1}{2}[x, y, z]^e = (*\varphi)_{abcd} x^a y^b z^c g^{de}. \tag{2.7}$$

This can be verified using (2.2), (2.6) and a Cayley multiplication table for $\mathbb{O}$. We then have the following result [3, Corollary IV.1.7].

Proposition 2.6. *Let V be a 3-plane in $\operatorname{Im}\mathbb{O} \cong \mathbb{R}^7$ with basis (x, y, z). Then V, with an appropriate orientation, is associative if and only if $[x, y, z] = 0$.*

In Section 5, we require some properties of the associator which we state as a proposition taken from [3, Proposition IV.B.16].

Proposition 2.7. *The associator $[x, y, z]$ of $x, y, z \in \mathbb{O}$ is:*

(i) *alternating,*
(ii) *imaginary valued,*
(iii) *orthogonal to x, y, z and to $[a, b] = ab - ba$ for any subset $\{a, b\}$ of $\{x, y, z\}$.*

3. Special Lagrangian m-folds in $\mathbb{C}^m$ [6, 7, 8].

We review the work in Joyce's papers [6, 7, 8] on the construction of special Lagrangian (SL) m-folds in $\mathbb{C}^m$ using evolution equations, upon which this paper is based. We begin by defining the SL calibration form on $\mathbb{C}^m$ and hence SL m-folds.

Definition 3.1. Let $(z_1, \ldots, z_m)$ be complex coordinates on $\mathbb{C}^m$ with complex structure I. Define a metric g, a real 2-form ω and a complex m-form Ω on $\mathbb{C}^m$ by

$$\begin{aligned} g &= |dz_1|^2 + \ldots + |dz_m|^2, \\ \omega &= \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + \ldots + dz_m \wedge d\bar{z}_m), \\ \Omega &= dz_1 \wedge \ldots \wedge dz_m. \end{aligned}$$

Let L be a real oriented m-dimensional submanifold of $\mathbb{C}^m$. Then L is a *special Lagrangian* (SL) m-fold in $\mathbb{C}^m$ with *phase* $e^{i\theta}$ if L is calibrated with respect to the real m-form $\cos\theta \operatorname{Re}\Omega + \sin\theta \operatorname{Im}\Omega$. If the phase of L is unspecified, it is taken to be one so that L is calibrated with respect to $\operatorname{Re}\Omega$.

Harvey and Lawson [3, Corollary III.1.11] give the following alternative characterisation of SL m-folds.

Proposition 3.2. *Let L be a real m-dimensional submanifold of $\mathbb{C}^m$. Then L admits an orientation making it into an SL m-fold in $\mathbb{C}^m$ with phase $e^{i\theta}$ if and only if $\omega|_L \equiv 0$ and $(\sin\theta \operatorname{Re}\Omega - \cos\theta \operatorname{Im}\Omega)|_L \equiv 0$.*

Joyce, in [6], derives an evolution equation for SL m-folds, the proof of which requires the following result [3, Theorem III.5.5].

Theorem 3.3. *Let P be a real analytic $(m-1)$-dimensional submanifold of $\mathbb{C}^m$ with $\omega|_P \equiv 0$. Then there exists a unique SL m-fold in $\mathbb{C}^m$ containing P.*

The requirement that P be real analytic is due to the fact that the proof uses the *Cartan–Kähler Theorem*, which is only applicable in the real analytic category. We now give the main result [6, Theorem 3.3].

Theorem 3.4. *Let P be a compact, orientable, $(m-1)$-dimensional, real analytic manifold, let χ be a real analytic nowhere vanishing section of $\Lambda^{m-1}TP$ and let $\psi : P \to \mathbb{C}^m$ be a real analytic embedding (immersion) such that $\psi^*(\omega) \equiv 0$ on P. Then there exist $\epsilon > 0$ and a unique family $\{\psi_t : t \in (-\epsilon, \epsilon)\}$ of real analytic maps $\psi_t : P \to \mathbb{C}^m$ with $\psi_0 = \psi$ satisfying*

$$\left(\frac{d\psi_t}{dt}\right)^b = (\psi_t)_*(\chi)^{a_1 \dots a_{m-1}} (\operatorname{Re}\Omega)_{a_1 \dots a_{m-1} a_m} g^{a_m b}$$

using index notation for tensors on $\mathbb{C}^m$. Define $\Psi : (-\epsilon, \epsilon) \times P \to \mathbb{C}^m$ by $\Psi(t,p) = \psi_t(p)$. Then $M = \operatorname{Image}\Psi$ is a non-singular embedded (immersed) SL m-fold in $\mathbb{C}^m$.

In [7, Section 3], Joyce introduces the idea of affine evolution data with which he is able to derive an affine evolution equation, and therefore reduces the infinite-dimensional problem of Theorem 3.4 to a finite-dimensional one.

Definition 3.5. Let $2 \le m \le n$ be integers. A set of *affine evolution data* is a pair (P, χ), where P is an $(m-1)$-dimensional submanifold of $\mathbb{R}^n$ and $\chi : \mathbb{R}^n \to \Lambda^{m-1}\mathbb{R}^n$ is an affine map, such that $\chi(p)$ is a non-zero element of $\Lambda^{m-1}TP$ in $\Lambda^{m-1}\mathbb{R}^n$ for each non-singular $p \in P$. We suppose also that P is not contained in any proper affine subspace $\mathbb{R}^k$ of $\mathbb{R}^n$.
Let $\operatorname{Aff}(\mathbb{R}^n, \mathbb{C}^m)$ be the affine space of affine maps $\psi : \mathbb{R}^n \to \mathbb{C}^m$ and define $\mathcal{C}_P$ to be the set of $\psi \in \operatorname{Aff}(\mathbb{R}^n, \mathbb{C}^m)$ satisfying:

(i) $\psi^*(\omega)|_P \equiv 0$,

(ii) $\psi|_{T_pP} : T_pP \to \mathbb{C}^m$ is injective for all p in a dense open subset of P.

Then (i) is a quadratic condition on ψ and (ii) is an open condition on ψ, so $\mathcal{C}_P$ is a non-empty open set in the intersection of a finite number of quadrics in $\mathrm{Aff}(\mathbb{R}^n, \mathbb{C}^m)$.

The conditions upon χ in Definition 3.5 are strong. The result is that there are few known examples of affine evolution data. The evolution equation derived in [7] is given below [7, Theorem 3.5].

Theorem 3.6. *Let (P, χ) be a set of affine evolution data and let $\psi \in \mathcal{C}_P$, where $\mathcal{C}_P$ is defined in Definition 3.5. Then there exist $\epsilon > 0$ and a unique real analytic family $\{\psi_t : t \in (-\epsilon, \epsilon)\}$ in $\mathcal{C}_P$ with $\psi_0 = \psi$, satisfying*

$$\left(\frac{d\psi_t}{dt}(x)\right)^b = (\psi_t)_*(\chi(x))^{a_1\dots a_{m-1}}(\mathrm{Re}\,\Omega)_{a_1\dots a_{m-1}a_m} g^{a_m b}$$

for all $x \in \mathbb{R}^n$, using index notation for tensors in $\mathbb{C}^m$. Furthermore, $M = \{\psi_t(p) : t \in (-\epsilon, \epsilon),\ p \in P\}$ is an SL m-fold in $\mathbb{C}^m$ wherever it is non-singular.

We conclude this section by discussing the material in [8], which is particularly pertinent to Section 5, where Joyce, for the majority of the paper, focuses on constructing SL 3-folds in $\mathbb{C}^3$ using the set of affine evolution data given below [8, p. 352].

Example 3.7. Let $\phi : \mathbb{R}^2 \to \mathbb{R}^5$ be the embedding of $\mathbb{R}^2$ in $\mathbb{R}^5$ given by

$$\phi(y_1, y_2) = \left(\frac{1}{2}(y_1^2 + y_2^2), \frac{1}{2}(y_1^2 - y_2^2), y_1y_2, y_1, y_2\right). \tag{3.1}$$

Then $P = \mathrm{Image}\,\phi$ can be written as

$$P = \left\{(x_1, \dots, x_5) \in \mathbb{R}^5 : x_1 = \frac{1}{2}(x_4^2 + x_5^2),\ x_2 = \frac{1}{2}(x_4^2 - x_5^2),\ x_3 = x_4x_5\right\},$$

which is diffeomorphic to $\mathbb{R}^2$. From (3.1), we calculate, writing $e_j = \frac{\partial}{\partial x_j}$:

$$\phi_*\left(\frac{\partial}{\partial y_1}\right) = y_1e_1 + y_1e_2 + y_2e_3 + e_4,$$
$$\phi_*\left(\frac{\partial}{\partial y_2}\right) = y_2e_1 - y_2e_2 + y_1e_3 + e_5,$$

and thus

$$\phi_* \left(\frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2} \right) = (y_1^2 + y_2^2) e_2 \wedge e_3 + (y_1^2 - y_2^2) e_1 \wedge e_3 - 2y_1 y_2 e_1 \wedge e_2 \\ + y_1 (e_1 \wedge e_5 + e_2 \wedge e_5 - e_3 \wedge e_4) + e_4 \wedge e_5 \\ + y_2 (-e_1 \wedge e_4 + e_2 \wedge e_4 + e_3 \wedge e_5).$$

Hence, if we define an affine map $\chi : \mathbb{R}^5 \to \Lambda^2 \mathbb{R}^5$ by

(3.2) $$\chi(x_1, \ldots, x_5) = 2x_1 e_2 \wedge e_3 + 2x_2 e_1 \wedge e_3 - 2x_3 e_1 \wedge e_2 + e_4 \wedge e_5 \\ + x_4 (e_1 \wedge e_5 + e_2 \wedge e_5 - e_3 \wedge e_4) \\ + x_5 (-e_1 \wedge e_4 + e_2 \wedge e_4 + e_3 \wedge e_5),$$

then $\chi = \phi_* \left(\frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2} \right)$ on P. Therefore, (P, χ) is a set of affine evolution data with $m = 3$ and $n = 5$.

The main result [8, Theorem 5.1] requires the definition of a cross product $\times : \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C}^3$, given in index notation by

(3.3) $$(\mathbf{u} \times \mathbf{v})^d = (\mathrm{Re}\, \Omega)_{abc} \mathbf{u}^a \mathbf{v}^b g^{cd}$$

for $\mathbf{u}, \mathbf{v} \in \mathbb{C}^3$, regarding $\mathbb{C}^3$ as a real vector space.

Theorem 3.8. *Suppose that $\mathbf{z}_1, \ldots, \mathbf{z}_6 : \mathbb{R} \to \mathbb{C}^3$ are differentiable functions satisfying:*

(3.4) $$\omega(\mathbf{z}_2, \mathbf{z}_3) = \omega(\mathbf{z}_1, \mathbf{z}_3) = \omega(\mathbf{z}_1, \mathbf{z}_2) = 0,$$

(3.5) $$\omega(\mathbf{z}_1, \mathbf{z}_5) + \omega(\mathbf{z}_2, \mathbf{z}_5) - \omega(\mathbf{z}_3, \mathbf{z}_4) = 0,$$

(3.6) $$-\omega(\mathbf{z}_1, \mathbf{z}_4) + \omega(\mathbf{z}_2, \mathbf{z}_4) + \omega(\mathbf{z}_3, \mathbf{z}_5) = 0,$$

(3.7) $$\omega(\mathbf{z}_4, \mathbf{z}_5) = 0,$$

at $t = 0$, and the equations:

(3.8) $$\frac{d\mathbf{z}_1}{dt} = 2\mathbf{z}_2 \times \mathbf{z}_3,$$

(3.9) $$\frac{d\mathbf{z}_2}{dt} = 2\mathbf{z}_1 \times \mathbf{z}_3,$$

(3.10) $$\frac{d\mathbf{z}_3}{dt} = -2\mathbf{z}_1 \times \mathbf{z}_2,$$

(3.11) $$\frac{d\mathbf{z}_4}{dt} = \mathbf{z}_1 \times \mathbf{z}_5 + \mathbf{z}_2 \times \mathbf{z}_5 - \mathbf{z}_3 \times \mathbf{z}_4,$$

(3.12) $$\frac{d\mathbf{z}_5}{dt} = -\mathbf{z}_1 \times \mathbf{z}_4 + \mathbf{z}_2 \times \mathbf{z}_4 + \mathbf{z}_3 \times \mathbf{z}_5,$$

(3.13) $$\frac{d\mathbf{z}_6}{dt} = \mathbf{z}_4 \times \mathbf{z}_5,$$

for all $t \in \mathbb{R}$*, where* $\times$ *is defined by* (3.3)*. Let* $M \subseteq \mathbb{C}^3$ *be defined by:*

$$M = \Big\{ \frac{1}{2}(y_1^2 + y_2^2)\mathbf{z}_1(t) + \frac{1}{2}(y_1^2 - y_2^2)\mathbf{z}_2(t) + y_1 y_2 \mathbf{z}_3(t) + y_1 \mathbf{z}_4(t) + y_2 \mathbf{z}_5(t) + \mathbf{z}_6(t) : y_1, y_2 \in \mathbb{R},\ t \in (-\epsilon, \epsilon) \Big\}.$$

Then M *is a special Lagrangian 3-fold in* $\mathbb{C}^3$ *wherever it is non-singular.*

Joyce [8] solves (3.8)–(3.13) subject to the conditions (3.4)–(3.7), dividing the solutions into cases based on the dimension of $\langle \mathbf{z}_1(t), \mathbf{z}_2(t), \mathbf{z}_3(t)\rangle_{\mathbb{R}}$ for generic $t \in \mathbb{R}$. We shall be concerned with the case where $\dim\langle \mathbf{z}_1(t), \mathbf{z}_2(t), \mathbf{z}_3(t)\rangle_{\mathbb{R}} = 3$, which forms the bulk of the results of [8]. The solutions in this case involve the *Jacobi elliptic functions*, which we now give a brief description of, following the material in [2, Chapter VII].

For $k \in [0,1]$, the Jacobi elliptic functions, $\operatorname{sn}(u,k)$, $\operatorname{cn}(u,k)$, $\operatorname{dn}(u,k)$, with modulus k are the unique solutions to the equations:

$$\begin{aligned}
\left(\frac{d}{du}\operatorname{sn}(u,k)\right)^2 &= (1 - \operatorname{sn}^2(u,k))(1 - k^2 \operatorname{sn}^2(u,k)),\\
\left(\frac{d}{du}\operatorname{cn}(u,k)\right)^2 &= (1 - \operatorname{cn}^2(u,k))(1 - k^2 + k^2 \operatorname{cn}^2(u,k)),\\
\left(\frac{d}{du}\operatorname{dn}(u,k)\right)^2 &= -(1 - \operatorname{dn}^2(u,k))(1 - k^2 - \operatorname{dn}^2(u,k)),
\end{aligned}$$

with the initial conditions

$$\begin{aligned}
\operatorname{sn}(0,k) &= 0, & \operatorname{cn}(0,k) &= 1, & \operatorname{dn}(0,k) &= 1,\\
\tfrac{d}{du}\operatorname{sn}(0,k) &= 1, & \tfrac{d}{du}\operatorname{cn}(0,k) &= 0, & \tfrac{d}{du}\operatorname{dn}(0,k) &= 0.
\end{aligned}$$

They also satisfy the following identities and differential equations:

$$\begin{aligned}
\operatorname{sn}^2(u,k) + \operatorname{cn}^2(u,k) &= 1,\\
k^2 \operatorname{sn}^2(u,k) + \operatorname{dn}^2(u,k) &= 1,\\
\frac{d}{du}\operatorname{sn}(u,k) &= \operatorname{cn}(u,k)\operatorname{dn}(u,k),\\
\frac{d}{du}\operatorname{cn}(u,k) &= -\operatorname{sn}(u,k)\operatorname{dn}(u,k),\\
\frac{d}{du}\operatorname{dn}(u,k) &= -k^2 \operatorname{sn}(u,k)\operatorname{cn}(u,k).
\end{aligned}$$

For $k = 0, 1$, they reduce to familiar functions:

$$\begin{array}{lll} \mathrm{sn}(u,0) = \sin u, & \mathrm{cn}(u,0) = \cos u, & \mathrm{dn}(u,0) = 1, \\ \mathrm{sn}(u,1) = \tanh u, & \mathrm{cn}(u,1) = \mathrm{sech}\, u, & \mathrm{dn}(u,1) = \mathrm{sech}\, u. \end{array}$$

For each $k \in [0, 1)$, they are periodic functions.

The embedding given in Example 3.7 was constructed by considering the action of $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ on $\mathbb{R}^2$. Hence, Joyce [8, Proposition 9.1] shows that solutions of (3.8)–(3.10), satisfying the condition (3.4), are equivalent under the natural actions of $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(3)$ to a solution of the form $\mathbf{z}_1 = (z_1, 0, 0)$, $\mathbf{z}_2 = (0, z_2, 0)$, $\mathbf{z}_3 = (0, 0, z_3)$, for differentiable functions $z_1, z_2, z_3 : \mathbb{R} \to \mathbb{C}$. Therefore, we assume that the solution is of this form. Equations (3.8)–(3.10) become:

$$\frac{dz_1}{dt} = 2\,\overline{z_2 z_3}, \quad \frac{dz_2}{dt} = -2\,\overline{z_3 z_1}, \quad \frac{dz_3}{dt} = -2\,\overline{z_1 z_2}. \tag{3.14}$$

The next result is taken from [8, Proposition 9.2].

Proposition 3.9. *Given any initial data $z_1(0), z_2(0), z_3(0)$, solutions to (3.14) exist for all $t \in \mathbb{R}$. Wherever the $z_j(t)$ are non-zero, they may be written as:*

$$2z_1 = e^{i\theta_1}\sqrt{\alpha_1^2 + v}, \quad 2z_2 = e^{i\theta_2}\sqrt{\alpha_2^2 - v}, \quad 2z_3 = e^{i\theta_3}\sqrt{\alpha_3^2 - v},$$

where $\alpha_j \in \mathbb{R}$ for all j and $v, \theta_1, \theta_2, \theta_3 : \mathbb{R} \to \mathbb{R}$ are differentiable functions. Let $\theta = \theta_1 + \theta_2 + \theta_3$ and let $Q(v) = (\alpha_1^2 + v)(\alpha_2^2 - v)(\alpha_3^2 - v)$. Then there exists $A \in \mathbb{R}$ such that $Q(v)^{\frac{1}{2}} \sin\theta = A$.

We state the main theorem that we shall require in Section 5, [8, Theorem 9.3].

Theorem 3.10. *Using the notation of Proposition 3.9, let $\alpha_j > 0$ for all j and $\alpha_1^{-2} = \alpha_2^{-2} + \alpha_3^{-2}$. Suppose that v has a minimum at $t = 0$, that $\theta_2(0) = \theta_3(0) = 0$, $A \geq 0$ and that $\alpha_2 \leq \alpha_3$. Then exactly one of the following four cases holds:*

(i) *$A = 0$ and $\alpha_2 = \alpha_3$, and z_1, z_2, z_3 are given by:*

$$2z_1(t) = \sqrt{3}\alpha_1 \tanh\left(\sqrt{3}\alpha_1 t\right),$$
$$2z_2(t) = 2z_3(t) = \sqrt{3}\alpha_1\, \mathrm{sech}\left(\sqrt{3}\alpha_1 t\right);$$

(ii) $A = 0$ *and* $\alpha_2 < \alpha_3$*, and* z_1, z_2, z_3 *are given by:*

$$2z_1(t) = \sqrt{\alpha_1^2 + \alpha_2^2}\,\mathrm{sn}(\sigma t, \tau),$$
$$2z_2(t) = \sqrt{\alpha_1^2 + \alpha_2^2}\,\mathrm{cn}(\sigma t, \tau),$$
$$2z_3(t) = \sqrt{\alpha_1^2 + \alpha_3^2}\,\mathrm{dn}(\sigma t, \tau),$$

where $\sigma = \sqrt{\alpha_1^2 + \alpha_3^2}$ *and* $\tau = \sqrt{\frac{\alpha_1^2+\alpha_2^2}{\alpha_1^2+\alpha_3^2}}$;

(iii) $0 < A < \alpha_1\alpha_2\alpha_3$*. Let the roots of* $Q(v) - A^2$ *be* $\gamma_1, \gamma_2, \gamma_3$*, ordered such that* $\gamma_1 \leq 0 \leq \gamma_2 \leq \gamma_3$*. Then* $v, \theta_1, \theta_2, \theta_3$ *are given by:*

$$v(t) = \gamma_1 + (\gamma_2 - \gamma_1)\,\mathrm{sn}^2(\sigma t, \tau),$$
$$\theta_1(t) = \theta_1(0) - A\int_0^t \frac{ds}{\alpha_1^2 + \gamma_1 + (\gamma_2 - \gamma_1)\,\mathrm{sn}^2(\sigma s, \tau)},$$
$$\theta_2(t) = A\int_0^t \frac{ds}{\alpha_2^2 - \gamma_1 - (\gamma_2 - \gamma_1)\,\mathrm{sn}^2(\sigma s, \tau)},$$
$$\theta_3(t) = A\int_0^t \frac{ds}{\alpha_3^2 - \gamma_1 - (\gamma_2 - \gamma_1)\,\mathrm{sn}^2(\sigma s, \tau)},$$

where $\sigma = \sqrt{\gamma_3 - \gamma_1}$ *and* $\tau = \sqrt{\frac{\gamma_2-\gamma_1}{\gamma_3-\gamma_1}}$;

(iv) $A = \alpha_1\alpha_2\alpha_3$*. Define* $a_1, a_2, a_3 \in \mathbb{R}$ *by:*

$$a_1 = -\frac{\alpha_2\alpha_3}{\alpha_1}, \quad a_2 = \frac{\alpha_3\alpha_1}{\alpha_2}, \quad a_3 = \frac{\alpha_1\alpha_2}{\alpha_3},$$

then $a_1 + a_2 + a_3 = 0$ *since* $\alpha_1^{-2} = \alpha_2^{-2} + \alpha_3^{-2}$ *and* z_1, z_2, z_3 *are given by:*

$$2z_1(t) = i\alpha_1 e^{ia_1 t}, \quad 2z_2(t) = \alpha_2 e^{ia_2 t}, \quad 2z_3(t) = \alpha_3 e^{ia_3 t}.$$

4. The First Evolution Equation.

To derive our evolution equation, we shall require two results related to *real analyticity*. The first follows from the minimality of associative 3-folds, as discussed in [3].

Theorem 4.1. *Let* N *be an associative 3-fold in* $\mathbb{R}^7$*. Then* N *is real analytic wherever it is non-singular.*

The proof of the next result [3, Theorem IV.4.1] relies on the *Cartan–Kähler Theorem*, which is only applicable in the real analytic category.

Theorem 4.2. *Let P be a 2-dimensional real analytic submanifold of* $\mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$*. Then there exists a unique real analytic associative 3-fold N in $\mathbb{R}^7$ which contains P.*

We now formulate an evolution equation for associative 3-folds, given a 2-dimensional real analytic submanifold of $\mathbb{R}^7$, following Theorem 3.4.

Theorem 4.3. *Let P be a compact, orientable, 2-dimensional, real analytic manifold, let χ be a real analytic nowhere vanishing section of $\Lambda^2 TP$, and let $\psi : P \to \mathbb{R}^7$ be a real analytic embedding (immersion). Then there exist $\epsilon > 0$ and a unique family $\{\psi_t : t \in (-\epsilon, \epsilon)\}$ of real analytic maps $\psi_t : P \to \mathbb{R}^7$ with $\psi_0 = \psi$ satisfying*

$$\left(\frac{d\psi_t}{dt}\right)^d = (\psi_t)_*(\chi)^{ab}\varphi_{abc}g^{cd}, \tag{4.1}$$

where g^{cd} is the inverse of the Euclidean metric on $\mathbb{R}^7$, using index notation for tensors on $\mathbb{R}^7$. Define $\Psi : (-\epsilon, \epsilon) \times P \to \mathbb{R}^7$ by $\Psi(t, p) = \psi_t(p)$. Then $M = \mathrm{Image}\,\Psi$ is a non-singular embedded (immersed) associative 3-fold in $\mathbb{R}^7$.

Note that we are realising M as the total space of a one parameter family of two-dimensional manifolds $\{P_t : t \in (-\epsilon, \epsilon)\}$, where each P_t is diffeomorphic to P, satisfying a first-order ordinary differential equation in t with initial condition $P_0 = P$.

Proof. Equation (4.1) is an evolution equation for maps $\psi_t : P \to \mathbb{R}^7$ with the initial condition $\psi_0 = \psi$. Since P is compact and P, χ, ψ are real analytic, the *Cauchy–Kowalevsky Theorem* [12, p. 234] from the theory of partial differential equations gives $\epsilon > 0$ such that a unique solution to the evolution equation exists for $t \in (-\epsilon, \epsilon)$.

By Theorem 4.2, there exists a unique real analytic associative 3-fold $N \subseteq \mathbb{R}^7$ such that $\psi(P) \subseteq N$. Consider a family $\{\tilde{\psi}_t : t \in (-\tilde{\epsilon}, \tilde{\epsilon})\}$, for some $\tilde{\epsilon} > 0$, of real analytic maps $\tilde{\psi}_t : P \to N$, with $\tilde{\psi}_0 = \psi$, satisfying

$$\left(\frac{d\tilde{\psi}_t}{dt}\right)^d = (\tilde{\psi}_t)_*(\chi)^{ab}(\varphi|_N)_{abc}(g|_N)^{cd}, \tag{4.2}$$

using index notation for tensors on N. By the same argument as above, a unique solution exists to (4.2) for some $\tilde{\epsilon} > 0$.

Let $p \in P$, $t \in (-\tilde{\epsilon}, \tilde{\epsilon})$ and set $x = \tilde{\psi}_t(p) \in N$. Let $V = (T_xN)^\perp$ in $\mathbb{R}^7$, so $\mathbb{R}^7 = T_xN \oplus V$ and $(\mathbb{R}^7)^* = T_x^*N \oplus V^*$. This induces a splitting:

$$\Lambda^3(\mathbb{R}^7)^* = \sum_{k=0}^{3} \Lambda^k T_x^*N \otimes \Lambda^{3-k}V^*.$$

Note that $\varphi \in \Lambda^3(\mathbb{R}^7)^*$ and that N is calibrated with respect to φ as N is an associative 3-fold. Therefore, the component of φ in $\Lambda^2 T_x^*N \otimes V^*$ is zero since this measures the change in $\varphi|_{T_xN}$ under small variations of T_xN, but $\varphi|_{T_xN}$ is maximum and therefore stationary. Since $(\tilde{\psi}_t)_*(\chi)|_p$ lies in $\Lambda^2 T_xN$, $(\tilde{\psi}_t)_*(\chi)^{ab}|_p\varphi_{abc}$ lies in T_x^*N, because the component in V^* comes from the component of φ in $\Lambda^2 T_x^*N \otimes V^*$, which is zero by above. Therefore,

$$(\tilde{\psi}_t)_*(\chi)^{ab}|_p\varphi_{abc} = (\tilde{\psi}_t)_*(\chi)^{ab}|_p(\varphi|_{T_xN})_{abc}.$$

As $(\mathbb{R}^7)^* = T_x^*N \oplus V^*$ is an orthogonal decomposition, $g^{cd} = (g|_{T_xN})^{cd} + h^{cd}$ for some $h \in S^2V$. Then, $(\tilde{\psi}_t)_*(\chi)^{ab}|_p(\varphi|_{T_xN})_{abc}h^{cd}$ is zero because $(\tilde{\psi}_t)_*(\chi)^{ab}|_p(\varphi|_{T_xN})_{abc} \in T_x^*N$ and $h \in S^2V$, so their contraction is zero. Hence,

$$(\tilde{\psi}_t)_*(\chi)^{ab}\varphi_{abc}g^{cd} = (\tilde{\psi}_t)_*(\chi)^{ab}(\varphi|_N)_{abc}(g|_N)^{cd}$$

for all $p \in P$ and $t \in (-\tilde{\epsilon}, \tilde{\epsilon})$. Thus the family $\{\tilde{\psi}_t : t \in (-\tilde{\epsilon}, \tilde{\epsilon})\}$ satisfies (4.1) and $\tilde{\psi}_0 = \psi$, which implies that $\tilde{\psi}_t = \psi_t$ by uniqueness.

Hence, ψ_t maps P to N and Ψ maps $(-\epsilon, \epsilon) \times P$ to N for ϵ sufficiently small. Suppose ψ is an embedding. Then $\psi_t : P \to N$ is an embedding for small t. Moreover, $\frac{d\psi_t}{dt}$ is a normal vector field to $\psi_t(P)$ in N with length $|(\psi_t)_*(\chi)|$, so, since χ is nowhere vanishing, this vector field is non-zero. We deduce that Ψ is an embedding for small ϵ, with Image $\Psi = M$ an open subset of N, and conclude that M is an associative 3-fold. Similarly if ψ is an immersion. □

5. The Second Evolution Equation.

In general, it is difficult to use Theorem 4.3 as stated to construct associative 3-folds, since it is an *infinite-dimensional* evolution problem. We follow the material in [7, Section 3] to reduce the theorem to a *finite-dimensional* problem.

Definition 5.1. Let $n \geq 3$ be an integer. A set of *affine evolution data* is a pair (P, χ), where P is a 2-dimensional submanifold of $\mathbb{R}^n$ and $\chi : \mathbb{R}^n \to \Lambda^2\mathbb{R}^n$ is an affine map, such that $\chi(p)$ is a non-zero element of $\Lambda^2 TP$ in $\Lambda^2\mathbb{R}^n$ for each non-singular point $p \in P$. Further, suppose that P is not contained in any proper affine subspace $\mathbb{R}^k$ of $\mathbb{R}^n$.

Let $\text{Aff}(\mathbb{R}^n, \mathbb{R}^7)$ be the affine space of affine maps $\psi : \mathbb{R}^n \to \mathbb{R}^7$. Define $\mathcal{C}_P$ as the set of $\psi \in \text{Aff}(\mathbb{R}^n, \mathbb{R}^7)$, such that $\psi|_{T_pP} : T_pP \to \mathbb{R}^7$ is injective for all p in a dense open subset of P. Let M be an associative 3-plane in $\mathbb{R}^7$. Then, generic linear maps $\psi : \mathbb{R}^n \to M$ will satisfy the condition to be members of $\mathcal{C}_P$. Hence $\mathcal{C}_P$ is non-empty.

We formulate our second evolution equation following Theorem 3.6.

Theorem 5.2. *Let (P, χ) be a set of affine evolution data and n, $\text{Aff}(\mathbb{R}^n, \mathbb{R}^7)$ and $\mathcal{C}_P$ be as in Definition 5.1. Suppose $\psi \in \mathcal{C}_P$. Then there exist $\epsilon > 0$ and a unique one parameter family $\{\psi_t : t \in (-\epsilon, \epsilon)\} \subseteq \mathcal{C}_P$ of real analytic maps with $\psi_0 = \psi$ satisfying*

$$\left(\frac{d\psi_t}{dt}(x)\right)^d = (\psi_t)_*(\chi(x))^{ab}\varphi_{abc}g^{cd} \tag{5.1}$$

for all $x \in \mathbb{R}^n$, using index notation for tensors on $\mathbb{R}^7$, where g^{cd} is the inverse of the Euclidean metric on $\mathbb{R}^7$. Define $\Psi : (-\epsilon, \epsilon) \times P \to \mathbb{R}^7$ by $\Psi(t, p) = \psi_t(p)$. Then $M = \text{Image}\ \Psi$ is an associative 3-fold wherever it is non-singular.

Proof. It is sufficient to restrict to the case of linear maps $\psi : \mathbb{R}^n \to \mathbb{R}^7$ since $\mathbb{R}^n$ can be regarded as $\mathbb{R}^n \times \{1\} \subseteq \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, and therefore any affine map $\psi : \mathbb{R}^n \to \mathbb{R}^7$ can be uniquely extended to a linear map $\tilde{\psi} : \mathbb{R}^{n+1} \to \mathbb{R}^7$. We denote the space of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^7$ by $\text{Hom}(\mathbb{R}^n, \mathbb{R}^7)$. Therefore, (5.1) is a well-defined first-order ordinary differential equation upon the maps $\psi_t \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^7)$ of the form $\frac{d\psi_t}{dt} = Q(\psi_t)$, where Q is a quadratic. Hence, by the theory of ordinary differential equations, there exist $\epsilon > 0$ and a unique real analytic family $\{\psi_t : t \in (-\epsilon, \epsilon)\} \subseteq \text{Hom}(\mathbb{R}^n, \mathbb{R}^7)$, with $\psi_0 = \psi$, satisfying equation (5.1).

Having established existence and uniqueness, we can then follow the proof of Theorem 4.3, noticing that we may drop the assumption made there of the compactness of P, since it was only used to establish the existence of the required family of maps. Note that (4.1) is precisely the restriction of (5.1) to $x \in P$, so we deduce that M is an associative 3-fold wherever it is non-singular.

We need only show now that the family constructed lies in $\mathcal{C}_P$. Note that the requirement that $\psi_t|_{T_pP} : T_pP \to \mathbb{R}^7$ is injective for all p in an open dense subset of P is clearly an open condition, and that it holds at $\psi_0 = \psi$ since $\psi \in \mathcal{C}_P$. Thus, by selecting a sufficiently small value of ϵ, we see that $\psi_t \in \mathcal{C}_P$ for all $t \in (-\epsilon, \epsilon)$ and the proof is complete. ☐

Before we construct associative 3-folds using this result, it is worth noting that using quadrics to provide affine evolution data as in [7] would not be a worthwhile enterprise. Suppose $Q \subseteq \mathbb{R}^3$ is a quadric and that $L : \mathbb{R}^3 \to \mathbb{R}^7$ is a linear map. Then we can transform $\mathbb{R}^7$ using G_2 such that, if we write $\mathbb{R}^7 = \mathbb{R}\oplus\mathbb{C}^3$, then $L(\mathbb{R}^3) \subseteq \mathbb{C}^3$ is a Lagrangian plane. Therefore, evolving Q using (5.1) will only produce SL 3-folds, which have already been studied in [7].

Let us now return to the affine evolution data given in Example 3.7 and use Theorem 5.2 to construct associative 3-folds. Let (P, χ) be as in Example 3.7 and define affine maps $\psi_t : \mathbb{R}^5 \to \mathbb{R}^7$ by:

$$\psi_t(x_1, \dots, x_5) = \mathbf{w}_1(t)x_1 + \dots + \mathbf{w}_5(t)x_5 + \mathbf{w}_6(t), \tag{5.2}$$

where $\mathbf{w}_j : \mathbb{R} \to \mathbb{R}^7$ are smooth functions for all j. Using the notation of Example 3.7, we see that $(\psi_t)_*(e_j) = \mathbf{w}_j$ for $j = 1, \dots, 5$. Hence, by equation (3.2) for χ, equation (2.4) for the cross product on $\mathbb{R}^7$ and (5.1) we have that

(5.3)
$$\begin{aligned}\frac{d\psi_t}{dt}(x_1, \dots, x_5) = {} & 2x_1\mathbf{w}_2 \times \mathbf{w}_3 + 2x_2\mathbf{w}_1 \times \mathbf{w}_3 - 2x_3\mathbf{w}_1 \times \mathbf{w}_2 \\ & + x_4(\mathbf{w}_1 \times \mathbf{w}_5 + \mathbf{w}_2 \times \mathbf{w}_5 - \mathbf{w}_3 \times \mathbf{w}_4) \\ & + x_5(-\mathbf{w}_1 \times \mathbf{w}_4 + \mathbf{w}_2 \times \mathbf{w}_4 + \mathbf{w}_3 \times \mathbf{w}_5) + \mathbf{w}_4 \times \mathbf{w}_5\end{aligned}$$

for all $(x_1, \dots, x_5) \in \mathbb{R}^5$. Therefore, from (5.2) and (5.3), we get the following result.

Theorem 5.3. *Let $\mathbf{w}_1, \dots, \mathbf{w}_6 : \mathbb{R} \to \mathbb{R}^7$ be differentiable functions satisfying*

$$\frac{d\mathbf{w}_1}{dt} = 2\mathbf{w}_2 \times \mathbf{w}_3, \tag{5.4}$$

$$\frac{d\mathbf{w}_2}{dt} = 2\mathbf{w}_1 \times \mathbf{w}_3, \tag{5.5}$$

$$\frac{d\mathbf{w}_3}{dt} = -2\mathbf{w}_1 \times \mathbf{w}_2, \tag{5.6}$$

$$\frac{d\mathbf{w}_4}{dt} = \mathbf{w}_1 \times \mathbf{w}_5 + \mathbf{w}_2 \times \mathbf{w}_5 - \mathbf{w}_3 \times \mathbf{w}_4, \tag{5.7}$$

$$\frac{d\mathbf{w}_5}{dt} = -\mathbf{w}_1 \times \mathbf{w}_4 + \mathbf{w}_2 \times \mathbf{w}_4 + \mathbf{w}_3 \times \mathbf{w}_5, \tag{5.8}$$

$$\frac{d\mathbf{w}_6}{dt} = \mathbf{w}_4 \times \mathbf{w}_5. \tag{5.9}$$

Then M, given by:

$$M = \left\{ \frac{1}{2}(y_1^2 + y_2^2)\mathbf{w}_1(t) + \frac{1}{2}(y_1^2 - y_2^2)\mathbf{w}_2(t) + y_1 y_2 \mathbf{w}_3(t) \right.$$
$$\left. + y_1 \mathbf{w}_4(t) + y_2 \mathbf{w}_5(t) + \mathbf{w}_6(t) : y_1, y_2, t \in \mathbb{R} \right\},$$

is an associative 3-fold in $\mathbb{R}^7$ wherever it is non-singular.

Theorem 5.2 only gives us that the associative 3-fold M is defined for t in some small open neighbourhood of zero, but work later in this section shows that M is indeed defined for all t as stated in the above theorem.

The equations we have just obtained fall naturally into three parts: (5.4)–(5.6) show that $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ evolve amongst themselves; (5.7)–(5.8) are *linear* equations for $\mathbf{w}_4$ and $\mathbf{w}_5$, once $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are known; and (5.9) defines $\mathbf{w}_6$ once the functions $\mathbf{w}_4$ and $\mathbf{w}_5$ are known. Moreover, these equations are very similar to (3.8)–(3.13), given in Theorem 3.8, the only difference being that here our functions and cross products are defined on $\mathbb{R}^7$ rather than $\mathbb{C}^3$. If we could show that any solutions $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are equivalent to functions $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$, lying in $\mathbb{C}^3$, satisfying (3.8)–(3.10) and (3.4), then we would be able to use results from [8] to hopefully construct associative 3-folds which are not SL 3-folds. It is to this end that we now proceed.

Suppose that $\mathbf{w}_1(t), \mathbf{w}_2(t), \mathbf{w}_3(t)$ are solutions to (5.4)–(5.6). Let $w_j = \mathbf{w}_j(0)$ for all j and let $v = [w_1, w_2, w_3]$, as defined by (2.7).

If $v = 0$, then, by Proposition 2.6, $\langle w_1, w_2, w_3 \rangle_{\mathbb{R}}$ lies in an associative 3-plane which we can map to $\mathbb{R}^3 \subseteq \mathbb{C}^3 \subseteq \mathbb{R}^7 = \mathbb{R} \oplus \mathbb{C}^3$, since G_2 acts transitively on associative 3-planes [3, Theorem IV.1.8]. Let z_1, z_2, z_3 be the images of w_1, w_2, w_3 under this transformation and let ω be the standard symplectic form on $\mathbb{C}^3$, which in terms of coordinates $(x_1, \ldots, x_7)$ on $\mathbb{R}^7$ is given by:

$$\omega = dx_2 \wedge dx_3 + dx_4 \wedge dx_5 + dx_6 \wedge dx_7.$$

Then, z_1, z_2, z_3 lie in $\mathbb{R}^3 \subseteq \mathbb{C}^3$ and so $\omega(z_j, z_k) = 0$ for $j \neq k$.

If $v \neq 0$, then v is orthogonal to w_j for all j by Proposition 2.7, so we can split $\mathbb{R}^7 = \mathbb{R} \oplus \mathbb{C}^3$ where $\mathbb{R} = \langle v \rangle$ and $\mathbb{C}^3 = \langle v \rangle^{\perp}$. Hence, w_j lies in $\mathbb{C}^3$

for all j with respect to this splitting. By Proposition 2.7, v is orthogonal to $[w_j, w_k] = w_j w_k - w_k w_j = 2w_j \times w_k$ and therefore, from (2.5),

$$\varphi_{abc} v^a {w_j}^b {w_k}^c = 0$$

using index notation for tensors on $\mathbb{R}^7$. Note that we can write:

$$\varphi = dx_1 \wedge \omega + \operatorname{Re} \Omega, \tag{5.10}$$

where Ω is the holomorphic volume form on $\mathbb{C}^3$. Therefore, $\varphi_{abc} v^a = |v| \omega_{bc}$ and hence, since $|v| \neq 0$, $\omega(w_j, w_k) = 0$.

From equations (2.4) and (3.3) defining the cross products on $\mathbb{R}^7$ and $\mathbb{C}^3$ respectively and (5.10) above, we see that, for vectors $\mathbf{x}, \mathbf{y} \in \mathbb{C}^3 \subseteq \mathbb{R}^7$,

$$\mathbf{x} \times \mathbf{y} = \mathbf{x} \times' \mathbf{y} + \omega(\mathbf{x}, \mathbf{y}) \mathbf{e}_1, \tag{5.11}$$

where $\times'$ is the cross product on $\mathbb{C}^3$ and $\mathbf{e}_1 = (1, \mathbf{0}) \in \mathbb{R} \oplus \mathbb{C}^3 = \mathbb{R}^7$. We have shown that, using a G_2 transformation, we can map the solutions $\mathbf{w}_1(t), \mathbf{w}_2(t), \mathbf{w}_3(t)$ to solutions $\mathbf{z}_1(t), \mathbf{z}_2(t), \mathbf{z}_3(t)$ such that $\mathbf{z}_j(0) \in \mathbb{C}^3 \subseteq \mathbb{R}^7$ and $\omega(\mathbf{z}_j(0), \mathbf{z}_k(0)) = 0$. Our remarks above about (5.4)–(5.6), and the relationship (5.11) between the cross products on $\mathbb{C}^3$ and $\mathbb{R}^7$, show that $\mathbf{z}_1(t), \mathbf{z}_2(t), \mathbf{z}_3(t)$ must remain in $\mathbb{C}^3$ and satisfy (3.8)–(3.10) along with condition (3.4). Hence, any solution of (5.4)–(5.6) is equivalent up to a G_2 transformation to a solution to the corresponding equations in Theorem 3.8.

We now perform a parameter count in order to calculate the dimension of the family of associative 3-folds constructed by Theorem 5.3. The initial data $\mathbf{w}_1(0), \ldots, \mathbf{w}_6(0)$ has 42 real parameters, which implies that $\dim \mathcal{C}_P = 42$ (using the notation of Definition 5.1), and so the family of curves in $\mathcal{C}_P$ has dimension 41, which corresponds to factoring out translation in t. It is shown in [8] that $\mathrm{GL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$ acts on this family of curves and, because of the internal symmetry of the evolution data, any two curves related by this group action give the same 3-fold. Therefore, we have to reduce the dimension of distinct associative 3-folds up to this group action by 6 to 35. We can also identify any two associative 3-folds which are isomorphic under automorphisms of $\mathbb{R}^7$, i.e. up to the action of $\mathrm{G}_2 \ltimes \mathbb{R}^7$, and so we reduce the dimension by 21 to 14.

In conclusion, the family of associative 3-folds constructed in this section has dimension 14, whereas the dimension of the family of SL 3-folds constructed in Theorem 3.8 has dimension 9, so not only do we know that we have constructed new geometric objects, but also how many more interesting parameters we expect to find.

5.1. Singularities of these associative 3-folds.

We study the singularities of the 3-folds constructed by Theorem 5.3, by introducing the function $F : \mathbb{R}^3 \to \mathbb{R}^7$ defined by:

$$F(y_1, y_2, t) = \tfrac{1}{2}(y_1^2 + y_2^2)\mathbf{w}_1(t) + \tfrac{1}{2}(y_1^2 - y_2^2)\mathbf{w}_2(t) + y_1y_2\mathbf{w}_3(t) + y_1\mathbf{w}_4(t) + y_2\mathbf{w}_5(t) + \mathbf{w}_6(t). \tag{5.12}$$

Clearly, F is smooth and, if $dF|_{(y_1,y_2,t)} : \mathbb{R}^3 \to \mathbb{R}^7$ is injective for all $(y_1, y_2, t) \in \mathbb{R}$, then F is an immersion and $M = \operatorname{Image} F$ is non-singular. Therefore, the possible singularities of M correspond to points where dF is not injective. Since we have from (5.4)–(5.9) that

$$\frac{\partial F}{\partial y_1} \times \frac{\partial F}{\partial y_2} = \frac{\partial F}{\partial t},$$

$\frac{\partial F}{\partial t}$ is perpendicular to the other two partial derivatives, and it is zero if and only if the y_1 and y_2 partial derivatives are linearly dependent. We deduce that F is an immersion if and only if $\frac{\partial F}{\partial y_1}$ and $\frac{\partial F}{\partial y_2}$ are linearly independent, since dF is injective if and only if the three partial derivatives of F are linearly independent. The condition for F to be an immersion at $(0, 0, 0)$ is that $\mathbf{w}_4(0)$ and $\mathbf{w}_5(0)$ are linearly independent.

We perform a parameter count for the family of singular associative 3-folds constructed by Theorem 5.3. The set of initial data $\mathbf{w}_1(0), \ldots, \mathbf{w}_6(0)$, with $\mathbf{w}_4(0)$ and $\mathbf{w}_5(0)$ linearly dependent, has dimension $28 + 8 = 36$, since the set of linearly dependent pairs in $\mathbb{R}^7$ has dimension 8. We saw in the earlier parameter count above that the set of initial data without any restrictions had dimension 42. Hence, the condition that F is not an immersion at $(0, 0, 0)$ is of real codimension 6, but this is clearly true for any point in $\mathbb{R}^3$ and therefore, it is expected that the family of singular associative 3-folds will be of codimension $6 - 3 = 3$ in the family of all associative 3-folds constructed by Theorem 5.3. Therefore, the family of distinct singular associative 3-folds up to automorphisms of $\mathbb{R}^7$ should have dimension $14 - 3 = 11$. Thus, generic associative 3-folds constructed by Theorem 5.3 will be non-singular. Moreover, the dimension of the family of singular associative 3-folds is greater than the dimension of the family of singular SL 3-folds constructed from the same evolution data (which has dimension 8).

We now model $M = \text{Image } F$ near a singular point, which we take to be the origin without loss of generality. Therefore, we expand $\mathbf{w}_1(t), \ldots, \mathbf{w}_6(t)$ about $t = 0$ to study the singularity. Since dF is not injective at the origin, $\mathbf{w}_4(0)$ and $\mathbf{w}_5(0)$ are linearly dependent. As mentioned above, Joyce [8, Section 5.1] describes how internal symmetry of the evolution data gives rise to an action of $\mathrm{GL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ upon $\mathbf{w}_1(t), \dots, \mathbf{w}_6(t)$, under which the associative 3-fold constructed is invariant. A rotation of $\mathbb{R}^2$ by an angle θ transforms $\mathbf{w}_4(0)$ and $\mathbf{w}_5(0)$ to

$$\begin{aligned}\tilde{\mathbf{w}}_4(0) &= \cos\theta \mathbf{w}_4(0) - \sin\theta \mathbf{w}_5(0),\\ \tilde{\mathbf{w}}_5(0) &= \sin\theta \mathbf{w}_5(0) + \cos\theta \mathbf{w}_5(0).\end{aligned}$$

Since $\mathbf{w}_4(0)$ and $\mathbf{w}_5(0)$ are linearly dependent, θ may be chosen so that $\tilde{\mathbf{w}}_5(0) = 0$. We may therefore suppose that $\mathbf{w}_5(0) = 0$ and take our initial data to be:

$$\begin{aligned}&\mathbf{w}_1(0) = \mathbf{v} + \mathbf{w}, &&\mathbf{w}_2(0) = \mathbf{v} - \mathbf{w},\\ &\mathbf{w}_3(0) = \mathbf{x}, &&\mathbf{w}_4(0) = \mathbf{u},\\ &\mathbf{w}_5(0) = \mathbf{w}_6(0) = 0,\end{aligned}$$

for vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x} \in \mathbb{R}^7$. Expanding our solutions to (5.4)–(5.9) to low order in t:

$$\begin{aligned}\mathbf{w}_1(t) &= \mathbf{v} + \mathbf{w} + 2t(\mathbf{v} - \mathbf{w}) \times \mathbf{x} + O(t^2),\\ \mathbf{w}_2(t) &= \mathbf{v} - \mathbf{w} + 2t(\mathbf{v} + \mathbf{w}) \times \mathbf{x} + O(t^2),\\ \mathbf{w}_3(t) &= \mathbf{x} + 4t\mathbf{v} \times \mathbf{w} + O(t^2),\\ \mathbf{w}_4(t) &= \mathbf{u} + t\mathbf{u} \times \mathbf{x} + O(t^2),\\ \mathbf{w}_5(t) &= 2t\mathbf{u} \times \mathbf{w} + 8t^2\mathbf{x} \times (\mathbf{u} \times \mathbf{w}) + O(t^3),\\ \mathbf{w}_6(0) &= 10t^3\mathbf{u} \times (\mathbf{x} \times (\mathbf{u} \times \mathbf{w})) + O(t^4).\end{aligned}$$

Calculating $F(y_1, y_2, t)$ near the origin, we see that the dominant terms in the expansion are dependent upon $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$, which we have shown to be equivalent under G_2 to solutions as given in Theorem 3.8. Following Joyce [8, p. 363–364], we consider $F(\epsilon^2 y_1, \epsilon y_2, \epsilon t)$ for small ϵ, which is given by:

$$\begin{aligned}F(\epsilon^2 y_1, \epsilon y_2, \epsilon t) = \epsilon^2&[(y_1 + \frac{1}{4}g(\mathbf{u},\mathbf{w})t^2)\mathbf{u} + (y_2^2 - \frac{1}{4}|\mathbf{u}|^2t^2)\mathbf{w} + 2y_2t\mathbf{u} \times \mathbf{w}]\\ &+ \epsilon^3[4y_2^2t\mathbf{x} \times \mathbf{w} + y_1y_2\mathbf{x} + y_1t\mathbf{u} \times \mathbf{x} + 8y_2t^2\mathbf{x} \times (\mathbf{u} \times \mathbf{w})\\ &\quad + 10t^3\mathbf{u} \times (\mathbf{x} \times (\mathbf{u} \times \mathbf{w}))] + O(\epsilon^4).\end{aligned} \tag{5.13}$$

Here, we have assumed that $\omega(\mathbf{u}, \mathbf{w}) = 0$ in order to simplify the coefficient of $\mathbf{u}$. The ϵ^2 terms in (5.13) give us the lowest order description of the singularity. If we suppose that $\mathbf{u}$ and $\mathbf{w}$ are linearly independent, which will be true in the generic case, then $\mathbf{u}$, $\mathbf{w}$ and $\mathbf{u}\times\mathbf{w}$ are linearly independent and therefore generate an SL $\mathbb{R}^3$. Hence, near the origin to lowest order, M is the image of the map from $\mathbb{R}^3$ to $\mathbb{R}^3$ given by

$$(y_1, y_2, t) \mapsto (y_1 + \frac{1}{4}g(\mathbf{u},\mathbf{w})t^2, y_2^2 - \frac{1}{4}|\mathbf{u}|^2t^2, 2y_2t). \tag{5.14}$$

Note that the first coordinate axis is fixed under (5.14) and, moreover, y_2 and t are allowed to take either sign. Therefore, (5.14) is a *double cover* of an SL $\mathbb{R}^3$ which is branched over the first coordinate axis. This is the same behaviour as occurs in the SL case [8, p. 364].

In order to study the singularity further, we consider the ϵ^3 terms in (5.13). It is generally not possible to simplify the final cross product in the ϵ^3 terms to give a neat expression using only four vectors. However, if we choose $\{\mathbf{v},\mathbf{w},\mathbf{x}\}$ to be the usual basis for the standard $\mathbb{R}^3$ in $\mathbb{C}^3\subseteq\mathbb{R}^7$, we have that $\mathbf{v}=\mathbf{w}\times\mathbf{x}$ and the t^3 term vanishes. Thus, using (5.13) and (5.14), the next order of the singularity is the image of the following map from $\mathbb{R}^3$ to $\mathbb{R}^7$:

$$(y_1, y_2, t) \mapsto \left(y_1 + \frac{1}{4}g(\mathbf{u},\mathbf{w})t^2, y_2^2 - \frac{1}{4}|\mathbf{u}|^2t^2, 2y_2t, -4\epsilon y_2^2t, \epsilon y_1y_2, \epsilon y_1t, 8\epsilon y_2t^2\right).$$

Note that the singularity does not lie within $\mathbb{C}^3\subseteq\mathbb{R}^7$ and so we have a model for a singularity which is different from the SL case.

5.2. Solving the equations.

From the work above, any solution $\mathbf{w}_1(t),\mathbf{w}_2(t),\mathbf{w}_3(t)$ in $\mathbb{R}^7$ to (5.4)–(5.6) is equivalent under a G_2 transformation to a solution $\mathbf{z}_1(t),\mathbf{z}_2(t),\mathbf{z}_3(t)$ in $\mathbb{C}^3$ to (3.8)–(3.10) satisfying (3.4). We can thus use results from [8] to produce some associative 3-folds. However, we must exercise some caution: we require that $\langle\mathbf{z}_1(t),\mathbf{z}_2(t),\mathbf{z}_3(t) : t\in\mathbb{R}\rangle_{\mathbb{R}}=\mathbb{C}^3$. If this does not occur, there may be a further G_2 transformation that preserves the subspace spanned by the $\mathbf{z}_j(t)$, but transforms $\mathbb{C}^3$ so that $\mathbf{w}_4$ and $\mathbf{w}_5$ are mapped into $\mathbb{C}^3$, and thus the submanifold constructed will be an SL 3-fold embedded in $\mathbb{R}^7$.

When $\dim\langle\mathbf{z}_1(t),\mathbf{z}_2(t),\mathbf{z}_3(t)\rangle_{\mathbb{R}}<3$, for generic $t\in\mathbb{R}$, the $\mathbf{z}_j(t)$ define a subspace of an SL $\mathbb{R}^3$ in $\mathbb{C}^3$, which corresponds to an associative $\mathbb{R}^3$ in $\mathbb{R}^7$. The subgroup of G_2 preserving an associative $\mathbb{R}^3$ is SO(4) [3, Theorem IV.1.8], and the subgroup of SU(3), which is the automorphism group of $\mathbb{C}^3$, preserving the standard $\mathbb{R}^3$ is SO(3). Hence, the family of different ways of identifying $\mathbb{R}^7\cong\mathbb{R}\oplus\mathbb{C}^3$ such that $\langle\mathbf{z}_1(t),\mathbf{z}_2(t),\mathbf{z}_3(t)\rangle_{\mathbb{R}}$ is mapped into the standard $\mathbb{R}^3$ in $\mathbb{C}^3$ contains $\mathrm{SO}(4)/\mathrm{SO}(3)\cong\mathcal{S}^3$. We therefore have sufficient freedom left in using the G_2 symmetry, after mapping $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ into $\mathbb{C}^3$, to map $\mathbf{w}_4$ and $\mathbf{w}_5$ into $\mathbb{C}^3$ as well. This means that these cases will only produce SL 3-folds.

It is also true in (i) and (ii) of Theorem 3.10 that the solutions $\mathbf{z}_j(t)$ define a subspace of an SL $\mathbb{R}^3$ in $\mathbb{C}^3$ and so these cases will not provide any new associative 3-folds either. Therefore, we need only consider (iii) and (iv) in Theorem 3.10.

Suppose we are in the situation of Theorem 3.10 so that, if we write $\mathbb{R}^7 = \mathbb{R} \oplus \mathbb{C}^3$, $\mathbf{w}_1 = (0, w_1, 0, 0)$, $\mathbf{w}_2 = (0, 0, w_2, 0)$, $\mathbf{w}_3 = (0, 0, 0, w_3)$ for differentiable functions $w_1, w_2, w_3 : \mathbb{R} \to \mathbb{C}$. Let $\mathbf{w}_4 = (y, p_1, p_2, q_3)$ and $\mathbf{w}_5 = (-x, q_1, -q_2, p_3)$, where all the various functions defined here are differentiable. Equations (5.7)–(5.8) become

$$\frac{dx}{dt} = \operatorname{Im}(\bar{w}_1 p_1 - \bar{w}_2 p_2 - \bar{w}_3 p_3), \tag{5.15}$$

$$\frac{dp_1}{dt} = ixw_1 + \overline{w_2 p_3} + \overline{w_3 p_2}, \tag{5.16}$$

$$\frac{dp_2}{dt} = ixw_2 - \overline{w_3 p_1} - \overline{w_1 p_3}, \tag{5.17}$$

$$\frac{dp_3}{dt} = ixw_3 - \overline{w_1 p_2} - \overline{w_2 p_1}; \tag{5.18}$$

$$\frac{dy}{dt} = \operatorname{Im}(\bar{w}_1 q_1 - \bar{w}_2 q_2 - \bar{w}_3 q_3), \tag{5.19}$$

$$\frac{dq_1}{dt} = iyw_1 + \overline{w_2 q_3} + \overline{w_3 q_2}, \tag{5.20}$$

$$\frac{dq_2}{dt} = iyw_2 - \overline{w_3 q_1} - \overline{w_1 q_3}, \tag{5.21}$$

$$\frac{dq_3}{dt} = iyw_3 - \overline{w_1 q_2} - \overline{w_2 q_1}. \tag{5.22}$$

Note that the equations on (x, p_1, p_2, p_3) are the same as on (y, q_1, q_2, q_3). Moreover, $(x, p_1, p_2, p_3) = (0, w_1, w_2, w_3)$ gives an automatic solution to (5.15)–(5.18) and $(y, q_1, q_2, q_3) = (0, w_1, w_2, w_3)$ solves (5.19)–(5.22).

If we write $\mathbf{w}_6 = (z, r_1, r_2, r_3)$, where $z : \mathbb{R} \to \mathbb{R}$ and $r_1, r_2, r_3 : \mathbb{R} \to \mathbb{C}$ are differentiable functions, (5.9) becomes

$$\frac{dz}{dt} = \operatorname{Im}(\bar{p}_1 q_1 - \bar{p}_2 q_2 - \bar{p}_3 q_3), \tag{5.23}$$

$$\frac{dr_1}{dt} = ixp_1 + iyq_1 + \overline{p_2 p_3} + \overline{q_2 q_3}, \tag{5.24}$$

$$\frac{dr_2}{dt} = ixp_2 - iyq_2 - \overline{p_3 p_1} + \overline{q_3 q_1}, \tag{5.25}$$

$$\frac{dr_3}{dt} = ixq_3 + iyp_3 - \overline{p_1 q_2} - \overline{p_2 q_1}. \tag{5.26}$$

Note that the conditions that x, y, z are constant correspond to (3.6), (3.5) and (3.7) in Theorem 3.8 respectively. Calculation using (5.15)–(5.18) gives

$$\frac{d^2 x}{dt^2} = x(|w_1|^2 - |w_2|^2 - |w_3|^2).$$

Suppose that x is a non-zero constant. Then $|w_1|^2 - |w_2|^2 - |w_3|^2 \equiv 0$. Using (2.5), (5.4)–(5.6) and the alternating properties of φ:

$$\begin{aligned}
&\frac{d}{dt}\left(|w_1|^2 - |w_2|^2 - |w_3|^2\right) \\
&\quad = 2g\left(\frac{d\mathbf{w}_1}{dt}, \mathbf{w}_1\right) - 2g\left(\frac{d\mathbf{w}_2}{dt}, \mathbf{w}_2\right) - 2g\left(\frac{d\mathbf{w}_3}{dt}, \mathbf{w}_3\right) \\
&\quad = 4(g(\mathbf{w}_2 \times \mathbf{w}_3, \mathbf{w}_1) - g(\mathbf{w}_1 \times \mathbf{w}_3, \mathbf{w}_2) + g(\mathbf{w}_1 \times \mathbf{w}_2, \mathbf{w}_3)) \\
&\quad = 4(\varphi(\mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_1) - \varphi(\mathbf{w}_1, \mathbf{w}_3, \mathbf{w}_2) + \varphi(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)) \\
&\quad = 12\varphi(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3).
\end{aligned}$$

Therefore, $\varphi(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) = \operatorname{Re}(w_1 w_2 w_3) \equiv 0$, which occurs if and only if (iv) of Theorem 3.10 holds. However, in case (iv), $|w_1|^2 - |w_2|^2 - |w_3|^2 = \alpha_1^2 - \alpha_2^2 - \alpha_3^2$, which, together with the condition $\alpha_1^{-2} = \alpha_2^{-2} + \alpha_3^{-2}$, forces $\alpha_j = 0$ for all j which is a contradiction. Hence, if x is constant, then x has to be zero, and we have a similar result for y. Therefore (3.5)–(3.7) correspond to $x = y = 0$ and z constant. This is unsurprising since having $x = y = 0$ and z constant corresponds to $\mathbf{w}_4, \mathbf{w}_5, \mathbf{w}_6$ remaining in $\mathbb{C}^3$ and thus the associative 3-fold M constructed will be SL and hence satisfy $\omega|_M \equiv 0$.

Following the discussion earlier in this subsection, we consider (iii) and (iv) of Theorem 3.10. However, no solutions are known in case (iii), so we focus on case (iv). We therefore let $\alpha_1, \alpha_2, \alpha_3$ be positive real numbers satisfying $\alpha_1^{-2} = \alpha_2^{-2} + \alpha_3^{-2}$ and define a_1, a_2, a_3 by:

$$a_1 = -\frac{\alpha_2\alpha_3}{\alpha_1}, \quad a_2 = \frac{\alpha_3\alpha_1}{\alpha_2}, \quad a_3 = \frac{\alpha_1\alpha_2}{\alpha_3}. \tag{5.27}$$

By Theorem 3.10, we have that

$$2w_1(t) = i\alpha_1 e^{ia_1 t}, \quad 2w_2(t) = \alpha_2 e^{ia_2 t}, \quad 2w_3(t) = \alpha_3 e^{ia_3 t}.$$

Hence, if we let $\beta_1, \beta_1, \beta_3 : \mathbb{R} \to \mathbb{C}$ be differentiable functions such that

$$p_1(t) = ie^{ia_1 t}\beta_1(t), \quad p_2(t) = e^{ia_2 t}\beta_2(t), \quad p_3(t) = e^{ia_3 t}\beta_3(t),$$

we have the following result.

Proposition 5.4. *Using the notation above,* (5.15)–(5.18) *can be written as the following matrix equation for the functions* $x, \beta_1, \beta_2, \beta_3$*:*

$$\frac{d}{dt}\begin{pmatrix} x \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \bar{\beta}_1 \\ \bar{\beta}_2 \\ \bar{\beta}_3 \end{pmatrix} = \frac{i}{2}\begin{pmatrix} 0 & -\frac{\alpha_1}{2} & \frac{\alpha_2}{2} & \frac{\alpha_3}{2} & \frac{\alpha_1}{2} & -\frac{\alpha_2}{2} & -\frac{\alpha_3}{2} \\ \alpha_1 & -2a_1 & 0 & 0 & 0 & -\alpha_3 & -\alpha_2 \\ \alpha_2 & 0 & -2a_2 & 0 & \alpha_3 & 0 & \alpha_1 \\ \alpha_3 & 0 & 0 & -2a_3 & \alpha_2 & \alpha_1 & 0 \\ -\alpha_1 & 0 & \alpha_3 & \alpha_2 & 2a_1 & 0 & 0 \\ -\alpha_2 & -\alpha_3 & 0 & -\alpha_1 & 0 & 2a_2 & 0 \\ -\alpha_3 & -\alpha_2 & -\alpha_1 & 0 & 0 & 0 & 2a_3 \end{pmatrix}\begin{pmatrix} x \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \bar{\beta}_1 \\ \bar{\beta}_2 \\ \bar{\beta}_3 \end{pmatrix}.$$

Proof. Using (5.15),

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{2}\mathrm{Im}(\alpha_1\beta_1 - \alpha_2\beta_2 - \alpha_3\beta_3) \\ &= \frac{i}{4}\left(\alpha_1(\bar{\beta}_1 - \beta_1) + \alpha_2(\beta_2 - \bar{\beta}_2) + \alpha_3(\beta_3 - \bar{\beta}_3)\right),\end{aligned}$$

which gives the first row in the matrix equation above. Since $a_1+a_2+a_3 = 0$, equation (5.16) for p_1 shows that

$$i\frac{d\beta_1}{dt} - a_1\beta_1 = \frac{1}{2}(-\alpha_1 x + \alpha_2\bar{\beta}_3 + \alpha_3\bar{\beta}_2),$$

which, upon rearrangement, gives the second row in the matrix equation above. The calculation of the rest of the rows follows in a similar fashion. □

In order to solve the matrix equation given in Proposition 5.4, we find the eigenvalues and corresponding eigenvectors of the matrix.

Proposition 5.5. *Let* T *denote the* 7×7 *real matrix given in Proposition* 5.4 *and let* $\mathbf{a} = (0, \alpha_1, \alpha_2, \alpha_3, \alpha_1, \alpha_2, \alpha_3)^{\mathrm{T}}$*, where* $^{\mathrm{T}}$ *denotes transpose. Then there exist non-zero vectors* $\mathbf{b}_\pm$*,* $\mathbf{c}_\pm$*,* $\mathbf{d}_\pm \in \mathbb{R}^7$ *such that*

$$T\mathbf{a} = 0, \quad T\mathbf{b}_\pm = \pm\lambda\mathbf{b}_\pm, \quad T\mathbf{c}_\pm = \pm\lambda\mathbf{c}_\pm, \quad T\mathbf{d}_\pm = \pm 3\lambda\mathbf{d}_\pm,$$

where $\lambda > 0$ *is such that* $\lambda^2 = a_2^2 - a_1a_3$ *and*

$$\begin{aligned} &\mathbf{b}_+ = (b_1, b_2, 0, b_3, b_4, 0, b_5)^{\mathrm{T}}, && \mathbf{b}_- = (b_1, b_4, 0, b_5, b_2, 0, b_3)^{\mathrm{T}}, \\ &\mathbf{c}_+ = (c_1, 0, c_2, c_3, 0, c_4, c_5)^{\mathrm{T}}, && \mathbf{c}_- = (c_1, 0, c_4, c_5, 0, c_2, c_3)^{\mathrm{T}}, \\ &\mathbf{d}_+ = (0, d_1, d_2, d_3, d_4, d_5, d_6)^{\mathrm{T}}, && \mathbf{d}_- = (0, d_4, d_5, d_6, d_1, d_2, d_3)^{\mathrm{T}}, \end{aligned} \tag{5.28}$$

for constants $b_1, \ldots, b_5, c_1, \ldots, c_5, d_1, \ldots, d_6 \in \mathbb{R}$*. In particular, the pairs* $\{\mathbf{b}_\pm, \mathbf{c}_\pm\}$ *are linearly independent.*

Proof. Most of the results in this proposition are found by direct calculation using Maple. The only point to note is that if $\mathbf{w}$ is a μ-eigenvector of T, for some $\mu \in \mathbb{R}$, and we write $\mathbf{w} = (\ x \quad \mathbf{y} \quad \mathbf{z}\ )^{\mathrm{T}}$, where $x \in \mathbb{R}$ and $\mathbf{y}, \mathbf{z} \in \mathbb{R}^3$, then $\tilde{\mathbf{w}} = (\ x \quad \mathbf{z} \quad \mathbf{y}\ )^{\mathrm{T}}$ is a $-\mu$-eigenvector of T, and hence we can cast the eigenvectors of T into the form as given in (5.28). □

From this result, we can write down the general solution to the matrix equation given in Proposition 5.4:

(5.29)

$$\begin{pmatrix} x \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \bar{\beta}_1 \\ \bar{\beta}_2 \\ \bar{\beta}_3 \end{pmatrix} = A \begin{pmatrix} 0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} + B_+ e^{\frac{i}{2}\lambda t} \begin{pmatrix} b_1 \\ b_2 \\ 0 \\ b_3 \\ b_4 \\ 0 \\ b_5 \end{pmatrix} + B_- e^{-\frac{i}{2}\lambda t} \begin{pmatrix} b_1 \\ b_4 \\ 0 \\ b_5 \\ b_2 \\ 0 \\ b_3 \end{pmatrix} + C_+ e^{\frac{i}{2}\lambda t} \begin{pmatrix} c_1 \\ 0 \\ c_2 \\ c_3 \\ 0 \\ c_4 \\ c_5 \end{pmatrix}$$
$$+ C_- e^{-\frac{i}{2}\lambda t} \begin{pmatrix} c_1 \\ 0 \\ c_4 \\ c_5 \\ 0 \\ c_2 \\ c_3 \end{pmatrix} + D_+ e^{\frac{3i}{2}\lambda t} \begin{pmatrix} 0 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{pmatrix} + D_- e^{-\frac{3i}{2}\lambda t} \begin{pmatrix} 0 \\ d_4 \\ d_5 \\ d_6 \\ d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

for constants $A, B_\pm, C_\pm, D_\pm \in \mathbb{C}$. However, the last three rows in this equation are equal to the complex conjugate of the three rows above them, which implies that $B_- = \bar{B}_+$, $C_- = \bar{C}_+$, and $D_- = \bar{D}_+$. Moreover, if we translate $\mathbb{R}^2$, as given in the evolution data, from (y_1, y_2) to $(y_1 - A, y_2)$, then $\mathbf{w}_j$ is unaltered for $j = 1, 2, 3$, but $\mathbf{w}_4$ is mapped to $\mathbf{w}_4 - A\mathbf{w}_1$. Therefore, we can set $A = 0$.

From the discussion above, we may now write down the general solution to (5.15)–(5.18) and (5.19)–(5.22) and then simply integrate equations (5.23)–(5.26) to give an explicit description of some associative 3-folds constructed using our second evolution equation. This result is given below.

Theorem 5.6. *Define functions $x, y, z : \mathbb{R} \to \mathbb{R}$ and $w_j, p_j, q_j, r_j : \mathbb{R} \to \mathbb{C}$ for $j = 1, 2, 3$ by:*

$$2w_1(t) = i\alpha_1 e^{ia_1 t}, \quad 2w_2(t) = \alpha_2 e^{ia_2 t}, \quad 2w_3(t) = \alpha_3 e^{ia_3 t},$$

where $\alpha_1, \alpha_2, \alpha_3$ *are positive constants such that* $\alpha_1^{-2} = \alpha_2^{-2} + \alpha_3^{-2}$ *and* a_1, a_2, a_3 *are given in* (5.27)*;*

$$
\begin{aligned}
x(t) &= 2\,\mathrm{Re}\left(Bb_1 e^{\frac{i}{2}\lambda t} + Cc_1 e^{\frac{i}{2}\lambda t}\right),\\
p_1(t) &= ie^{ia_1 t}\left(Bb_2 e^{\frac{i}{2}\lambda t} + \bar{B}b_4 e^{-\frac{i}{2}\lambda t} + Dd_1 e^{\frac{3i}{2}\lambda t} + \bar{D}d_4 e^{-\frac{3i}{2}\lambda t}\right),\\
p_2(t) &= e^{ia_2 t}\left(Cc_2 e^{\frac{i}{2}\lambda t} + \bar{C}c_4 e^{-\frac{i}{2}\lambda t} + Dd_2 e^{\frac{3i}{2}\lambda t} + \bar{D}d_5 e^{-\frac{3i}{2}\lambda t}\right),\\
p_3(t) &= e^{ia_3 t}\left((Bb_3 + Cc_3)e^{\frac{i}{2}\lambda t} + (\bar{B}b_5 + \bar{C}c_5)e^{-\frac{i}{2}\lambda t} + Dd_3 e^{\frac{3i}{2}\lambda t} + \bar{D}d_6 e^{-\frac{3i}{2}\lambda t}\right),\\
y(t) &= 2\,\mathrm{Re}\left(B'b_1 e^{\frac{i}{2}\lambda t} + C'c_1 e^{\frac{i}{2}\lambda t}\right),\\
q_1(t) &= ie^{ia_1 t}\left(B'b_2 e^{\frac{i}{2}\lambda t} + \bar{B}'b_4 e^{-\frac{i}{2}\lambda t} + D'd_1 e^{\frac{3i}{2}\lambda t} + \bar{D}'d_4 e^{-\frac{3i}{2}\lambda t}\right),\\
q_2(t) &= e^{ia_2 t}\left(C'c_2 e^{\frac{i}{2}\lambda t} + \bar{C}'c_4 e^{-\frac{i}{2}\lambda t} + D'd_2 e^{\frac{3i}{2}\lambda t} + \bar{D}'d_5 e^{-\frac{3i}{2}\lambda t}\right),\\
q_3(t) &= e^{ia_3 t}\left((B'b_3 + C'c_3)e^{\frac{i}{2}\lambda t} + (\bar{B}'b_5 + \bar{C}'c_5)e^{-\frac{i}{2}\lambda t} + D'd_3 e^{\frac{3i}{2}\lambda t} + \bar{D}'d_6 e^{-\frac{3i}{2}\lambda t}\right),\\
\frac{dz}{dt} &= \mathrm{Im}(\bar{p}_1 q_1 - \bar{p}_2 q_2 - \bar{p}_3 q_3),\\
\frac{dr_1}{dt} &= ixp_1 + iyq_1 + \overline{p_2 p_3} + \overline{q_2 q_3},\\
\frac{dr_2}{dt} &= ixp_2 - iyq_2 - \overline{p_3 p_1} + \overline{q_3 q_1},\\
\frac{dr_3}{dt} &= ixq_3 + iyp_3 - \overline{p_1 q_2} - \overline{p_2 q_1},
\end{aligned}
$$

where the real constants λ *and* b_j, c_j, d_j *are as defined in Proposition* 5.5 *and* $B, B', C, C', D, D' \in \mathbb{C}$ *are arbitrary constants.*

Define a subset M *of* $\mathbb{R}^7 = \mathbb{R} \oplus \mathbb{C}^3$ *by:*

$$
\begin{aligned}
M = \Big\{ \Big(& y_1 y(t) - y_2 x(t) + z(t),\ \frac{1}{2}(y_1^2 + y_2^2)w_1(t) + y_1 p_1(t) + y_2 q_1(t) + r_1(t),\\
& \frac{1}{2}(y_1^2 - y_2^2)w_2(t) + y_1 p_2(t) - y_2 q_2(t) + r_2(t),\\
& y_1 y_2 w_3(t) + y_1 q_3(t) + y_2 p_3(t) + r_3(t) \Big) : y_1, y_2, t \in \mathbb{R} \Big\}.
\end{aligned} \tag{5.30}
$$

Then, M *is an associative 3-fold in* $\mathbb{R}^7$*.*

We now count parameters for the associative 3-folds constructed by Theorem 5.6. There are four real parameters ($\alpha_1, \alpha_2, \alpha_3$ and the constant of integration for $z(t)$) and nine complex parameters (B, B', C, C', D, D' and the three constants of integration for $r_1(t), r_2(t), r_3(t)$), which makes a total of 22 real parameters. The relationship $\alpha_1^{-2} = \alpha_2^{-2} + \alpha_3^{-2}$ then reduces the number of parameters by one to 21. Recall that we have the symmetry groups $\mathrm{GL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ and $\mathrm{G}_2 \ltimes \mathbb{R}^7$ for these associative 3-folds. By the arguments proceeding Theorem 5.3 and the proof of [8, Proposition 9.1], we have used the freedom in G_2 transformations and rotations in $\mathrm{GL}(2,\mathbb{R})$ to transform our solutions $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ of (5.4)–(5.6) to solutions of (3.8)–(3.10), satisfying (3.4), of the form $\mathbf{w}_1 = (0, w_1, 0, 0)$, $\mathbf{w}_2 = (0, 0, w_2, 0)$, $\mathbf{w}_3 = (0, 0, 0, w_3)$. We have also used translations in $\mathbb{R}^2$ to set the constant A in (5.29) and the corresponding constant A' in the general solution to (5.19)–(5.22) both to zero. Therefore, the remaining symmetries available are dilations in $\mathrm{GL}(2,\mathbb{R})$ and translations in $\mathbb{R}^7$, which reduce the number of parameters by eight to 13. Translation in time, say $t \mapsto t + t_0$, corresponds to multiplying B, B', C, C' by $e^{\frac{i}{2}\lambda t_0}$ and D, D' by $e^{\frac{3i}{2}\lambda t_0}$, which thus lowers the parameter count by one. We conclude that the dimension of the family of associative 3-folds generated by Theorem 5.6 is 12, whereas the dimension of the whole family generated by Theorem 5.3 is 14.

5.3. Periodicity.

Note that the solutions to Theorem 5.6 are all linear combinations of terms of the form $e^{i(a_j + m\lambda)t}$ for $j = 1, 2, 3$ and $m = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2, \pm 3$, since $a_j \pm n\lambda \neq 0$ for $n = 0, 1, 2, 3$, which ensures that r_1, r_2, r_3 do not have any linear terms in t. It is therefore reasonable to search for associative 3-folds M as in (5.30) that are *periodic* in t. Define a map $F : \mathbb{R}^3 \to \mathbb{R}^7$ by (5.12), so that $M = \mathrm{Image}\, F$. Then M is periodic if and only if there exists some constant $T > 0$ such that $F(y_1, y_2, t + T) = F(y_1, y_2, t)$ for all $y_1, y_2, t \in \mathbb{R}$.

From above, the periods of the exponentials in the functions defined in Theorem 5.6 are proportional to $(a_j + m\lambda)^{-1}$ for $j = 1, 2, 3$ and the values of m given above. In general, F will be periodic if and only if these periods have a common multiple. By the definition of the constants a_j, we can write $a_2 = -xa_1$ and $a_3 = (x - 1)a_1$ for some $x \in (0, 1)$. Then $\lambda^2 = a_2^2 - a_1 a_3 = a_1^2(x^2 - x + 1)$ and, if we let $y = \sqrt{x^2 - x + 1}$, we deduce that $\lambda = -ya_1$ since $a_1 < 0$ and $\lambda, y > 0$. The periods thus have a common multiple if and only if x and y are rational. We have therefore reduced the problem to finding rational points on the conic $y^2 = x^2 - x + 1$. This is a standard problem in number theory and is identical to the one solved by Joyce [8, Section 11.2], so we are able to prove the following result.

Theorem 5.7. *Given* $s \in (0, \frac{1}{2}) \cap \mathbb{Q}$*, Theorem* 5.6 *gives a family of closed associative 3-folds in* $\mathbb{R}^7$ *whose generic members are non-singular immersed 3-folds diffeomorphic to* $\mathcal{S}^1 \times \mathbb{R}^2$*.*

Proof. Let $s \in (0, \frac{1}{2}) \cap \mathbb{Q}$ and write $s = \frac{p}{q}$ where p, q are coprime positive integers. Then, by the work in [8, p. 390], we define a_1, a_2, a_3, λ either by

$$a_1 = p^2 - q^2, \quad a_2 = q^2 - 2pq, \quad a_3 = 2pq - p^2, \quad \lambda = p^2 - pq + q^2;$$

or, if $p + q$ is divisible by 3, by

$$3a_1 = p^2 - q^2, \quad 3a_2 = q^2 - 2pq, \quad 3a_3 = 2pq - p^2, \quad 3\lambda = p^2 - pq + q^2.$$

In both cases, $\mathrm{hcf}\,(a_1, a_2, a_3) = \mathrm{hcf}\,(a_1, a_2, a_3, \lambda) = 1$. We also note that λ is odd since at least one of p, q is odd. Thus, $a_j + m\lambda$ is an integer for integer values of m and half an integer, but not an integer, for non-integer values of m. Hence, by the form of the functions given in Theorem 5.6 and equation (5.12) for F, $F(y_1, y_2, t + 2\pi) = F(-y_1, -y_2, t)$ for all y_1, y_2, t. We deduce that F has period 4π, using the condition that $\mathrm{hcf}(a_1, a_2, a_3) = 1$.

If we define an action of $\mathbb{Z}$ on $\mathbb{R}^3$ by requiring, for $n \in \mathbb{Z}$, that (y_1, y_2, t) maps to $((-1)^n y_1, (-1)^n y_2, t + 2n\pi)$, then we can consider F as a map from the quotient of $\mathbb{R}^3$ by $\mathbb{Z}$ under this action. Since this quotient is diffeomorphic to $\mathcal{S}^1 \times \mathbb{R}^2$ and generically F is an immersion, $M = \mathrm{Image}\, F$ is generically an immersed 3-fold diffeomorphic to $\mathcal{S}^1 \times \mathbb{R}^2$. □

Joyce [8] has considered the asymptotic behaviour of the SL 3-folds constructed by Theorem 3.10(iv) at infinity, which is dependent on the quadratic terms in F. However, since solutions $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ in Theorem 5.3 are essentially equivalent to solutions $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ in Theorem 3.10, the asymptotic behaviour of the 3-folds given by Theorem 5.7 must be identical to that found by Joyce [8, p. 391]. We first make a definition and then state our result.

Definition 5.8. Let M, M_0 be closed m-dimensional submanifolds of $\mathbb{R}^n$ and let $k < 1$. We say that M is *asymptotic with order* $O(r^k)$ *at infinity in* $\mathbb{R}^n$ *to* M_0 if there exist $R > 0$, some compact subset K of M and a diffeomorphism $\Phi : M_0 \setminus \bar{B}_R \to M \setminus K$ such that

$$|\Phi(\mathbf{x}) - \mathbf{x}| = O(r^k) \quad \text{as } r \to \infty,$$

where r is the radius function on $\mathbb{R}^n$ and $\bar{B}_R$ is the closed ball of radius R.

Theorem 5.9. *Every closed associative 3-fold defined by* $s \in (0, \frac{1}{2}) \cap \mathbb{Q}$*, as given in Theorem* 5.7*, is asymptotic with order* $O(r^{\frac{1}{2}})$ *at infinity in* $\mathbb{R}^7$ *to a double cover of the SL* T^2 *cone defined by:*

$$\left\{(0, ie^{ia_1 t}x_1, e^{ia_2 t}x_2, e^{ia_3 t}x_3) : x_1, x_2, x_3, t \in \mathbb{R},\ x_1 \geq 0,\ \sum_{i=1}^{3} a_i x_i^2 = 0\right\}$$

where the constants a_1, a_2, a_3 *are defined by* s *as in the proof of Theorem* 5.7*.*

The associative 3-folds in Theorem 5.7 actually diverge away from the SL cone given above, but Theorem 5.9 gives a measure of the rate of divergence.

We now show that if an associative 3-fold M were to converge to an SL 3-fold at infinity, then M would in fact be SL, which we know is not the case for generic members of the family given by Theorem 5.7.

Theorem 5.10. *Suppose* M *is an associative 3-fold in* $\mathbb{R}^7 = \mathbb{R} \oplus \mathbb{C}^3$ *and that* L *is an SL 3-fold in* $\mathbb{C}^3$*. Suppose further that* M *is asymptotic with order* $O(r^k)$ *at infinity in* $\mathbb{R}^7$ *to* L*, where* $k < 0$*. Then* M *is an SL 3-fold in* $\mathbb{C}^3$ *embedded in* $\mathbb{R}^7$*.*

To prove Theorem 5.10, we need two results. The first is a *maximum principle* for harmonic functions due to Hopf [10, p. 12].

Theorem 5.11. *Let* f *be a smooth function on a Riemannian manifold* M*. Suppose* f *is harmonic, i.e.* $d^*df = 0$*, where* d^* *is the formal adjoint of* d*. If* f *assumes a local maximum (or minimum) at a point in* $M \setminus \partial M$*, then* f *is constant.*

The second is an elementary result from the theory of minimal submanifolds [10, Corollary 9].

Theorem 5.12. *Let* M *be a submanifold of* $\mathbb{R}^n$*, for some* n*, with immersion* ι*. Then* M *is a minimal submanifold if and only if* ι *is harmonic.*

Here, the function $\iota : M \to \mathbb{R}^n$ is harmonic if and only if each of the components of ι mapping to $\mathbb{R}$ is harmonic. We now prove Theorem 5.10.

Proof of Theorem 5.10. Since M is an associative 3-fold in $\mathbb{R}^7$, M is a minimal submanifold of $\mathbb{R}^7$ [3, Theorem II.4.2]. Therefore, the embedding of M in $\mathbb{R}^7$ is harmonic by Theorem 5.12. In particular, if we write coordinates on M as $(x_1, \ldots, x_7)$, x_1 is harmonic. We may assume, without loss of generality, that the SL 3-fold L to which M converges lies in $\{0\} \times \mathbb{C}^3 \subseteq \mathbb{R}^7$. Since M is asymptotic to L at infinity with order $O(r^k)$, where $k < 0$, $x_1 \to 0$ as $r \to \infty$. Suppose x_1 is not identically zero. Then x_1 assumes a strict maximum or minimum at some point in the interior of M. By Theorem 5.11, x_1 is therefore constant, which contradicts the assumption that x_1 was not identically zero. Hence, $x_1 \equiv 0$ and M is an SL 3-fold in $\mathbb{C}^3$. □

6. Ruled Associative 3-folds.

In this final section we focus on *ruled* 3-folds and apply our ideas of evolution equations to give methods for constructing associative examples. This is a generalisation of the work in Joyce's paper [9] on ruled SL 3-folds in $\mathbb{C}^3$ and it is from this source that we take the definitions below. By a *cone* in $\mathbb{R}^7$, we shall mean a submanifold of $\mathbb{R}^7$ which is invariant under dilations and is non-singular except possibly at 0. A cone C is said to be *two-sided* if $C = -C$.

Definition 6.1. Let M be a 3-dimensional submanifold of $\mathbb{R}^7$. A *ruling* of M is a pair (Σ, π), where Σ is a 2-dimensional manifold and $\pi : M \to \Sigma$ is a smooth map, such that for all $\sigma \in \Sigma$, there exist $\mathbf{v}_\sigma \in \mathcal{S}^6$, $\mathbf{w}_\sigma \in \mathbb{R}^7$ such that $\pi^{-1}(\sigma) = \{r\mathbf{v}_\sigma + \mathbf{w}_\sigma : r \in \mathbb{R}\}$. Then the triple (M, Σ, π) is a *ruled submanifold* of $\mathbb{R}^7$.

An *r-orientation* for a ruling (Σ, π) of M is a choice of orientation for the affine straight line $\pi^{-1}(\sigma)$ in $\mathbb{R}^7$, for each $\sigma \in \Sigma$, which varies smoothly with σ. A ruled submanifold with an r-orientation for the ruling is called an *r-oriented ruled submanifold*.

Let (M, Σ, π) be an r-oriented ruled submanifold. For each $\sigma \in \Sigma$, let $\phi(\sigma)$ be the unique unit vector in $\mathbb{R}^7$ parallel to $\pi^{-1}(\sigma)$ and in the positive direction with respect to the orientation on $\pi^{-1}(\sigma)$, given by the r-orientation. Then $\phi : \Sigma \to \mathcal{S}^6$ is a smooth map. Define $\psi : \Sigma \to \mathbb{R}^7$ such that, for all $\sigma \in \Sigma$, $\psi(\sigma)$ is the unique vector in $\pi^{-1}(\sigma)$ orthogonal to $\phi(\sigma)$. Then ψ is a smooth map and we may write:

$$M = \{r\phi(\sigma) + \psi(\sigma) : \sigma \in \Sigma,\ r \in \mathbb{R}\}. \tag{6.1}$$

Define the *asymptotic cone* M_0 of a ruled submanifold M by:

$$M_0 = \{\mathbf{v} \in \mathbb{R}^7 : \mathbf{v} \text{ is parallel to } \pi^{-1}(\sigma) \text{ for some } \sigma \in \Sigma\}.$$

If M is also r-oriented then

$$M_0 = \{r\phi(\sigma) : \sigma \in \Sigma,\ r \in \mathbb{R}\} \tag{6.2}$$

and is usually a 3-dimensional two-sided cone; that is, whenever ϕ is an immersion.

Note that we can consider any r-oriented ruled submanifold as being defined by two maps ϕ, ψ as given in Definition 6.1. Hence, r-oriented ruled associative 3-folds may be constructed by evolution equations for ϕ, ψ.

Suppose we have a 3-dimensional two-sided cone M_0 in $\mathbb{R}^7$. The *link* of M_0, $M_0 \cap \mathcal{S}^6$, is a non-singular 2-dimensional submanifold of $\mathcal{S}^6$ closed under the action of $-1 : \mathcal{S}^6 \to \mathcal{S}^6$. Let Σ be the quotient of the link by the ± 1 maps on $\mathcal{S}^6$. Clearly, Σ is a non-singular 2-dimensional manifold. Define $\tilde{M_0} \subseteq \Sigma \times \mathbb{R}^7$ by:

$$\tilde{M_0} = \{(\{\pm\sigma\}, r\sigma) : \sigma \in M_0 \cap \mathcal{S}^6,\ r \in \mathbb{R}\}.$$

Then $\tilde{M_0}$ is a non-singular 3-fold. Define $\pi : \tilde{M_0} \to \Sigma$ by $\pi(\{\pm\sigma\}, r\sigma) = \{\pm\sigma\}$ and $\iota : \tilde{M_0} \to \mathbb{R}^7$ by $\iota(\{\pm\sigma\}, r\sigma) = r\sigma$. Note that $\iota(\tilde{M_0}) = M_0$ and that ι is an immersion except on $\iota^{-1}(0) \cong \Sigma$, so we may consider $\tilde{M_0}$ as a singular immersed submanifold of $\mathbb{R}^7$. Hence $(\tilde{M_0}, \Sigma, \pi)$ is a ruled submanifold of $\mathbb{R}^7$. Therefore, we can regard M_0 as a ruled submanifold and dispense with $\tilde{M_0}$. Suppose further that M_0 is an r-oriented two-sided cone. We can thus write M_0 in the form (6.1) for maps ϕ, ψ, as given in Definition 6.1, and see that ψ must be identically zero. It is also clear that any ruled submanifold defined by ϕ, ψ with $\psi \equiv 0$ is an r-oriented two-sided cone.

We now justify the terminology of asymptotic cone as given in Definition 6.1. For this, we need to define the term *asymptotically conical with order* $O(r^\alpha)$, where r is the radius function on $\mathbb{R}^7$.

Definition 6.2. Let M_0 be a closed cone in $\mathbb{R}^7$ and let M be a closed non-singular submanifold in $\mathbb{R}^7$. We say that M is *asymptotically conical to* M_0 *with order* $O(r^\alpha)$, for some $\alpha < 1$, if there exist some constant $R > 0$, a compact subset K of M and a diffeomorphism $\Phi : M_0 \setminus \bar{B}_R \to M \setminus K$ such that

$$|\nabla^k(\Phi(\mathbf{x}) - I(\mathbf{x}))| = O(r^{\alpha-k}) \quad \text{for } k = 0, 1, 2, \dots \text{ as } r \to \infty, \tag{6.3}$$

where $\bar{B}_R$ is the closed ball of radius R in $\mathbb{R}^7$ and $I : M_0 \to \mathbb{R}^7$ is the inclusion map. Here $|\,.\,|$ is calculated using the cone metric on $M_0 \setminus \bar{B}_R$, and ∇ is a combination of the Levi–Civita connection derived from the cone metric and the flat connection on $\mathbb{R}^n$, which acts as partial differentiation.

Suppose that M is an r-oriented ruled submanifold and let M_0 be its asymptotic cone. Writing M in the form (6.1) and M_0 in the form (6.2) for maps ϕ, ψ, define a diffeomorphism $\Phi : M_0 \setminus \bar{B}_1 \to M \setminus K$, where K is some compact subset of M, by $\Phi(r\phi(\sigma)) = r\phi(\sigma) + \psi(\sigma)$ for all $\sigma \in \Sigma$ and $|r| > 1$. If Σ is compact, so that ψ is bounded, then Φ satisfies (6.3) as given in Definition 6.2 for $\alpha = 0$, which shows that M is asymptotically conical to M_0 with order $O(1)$.

6.1. The associative condition.

Let Σ be a 2-dimensional, connected, real analytic manifold, let $\phi : \Sigma \to \mathcal{S}^6$ be a real analytic immersion and let $\psi : \Sigma \to \mathbb{R}^7$ be a real analytic map. Define M by (6.1), so that M is the image of the map $\iota : \mathbb{R} \times \Sigma \to \mathbb{R}^7$ given by $\iota(r, \sigma) = r\phi(\sigma) + \psi(\sigma)$. Clearly, $\mathbb{R} \times \Sigma$ is an r-oriented ruled submanifold with ruling (Σ, π), where π is given by $\pi(r, \sigma) = \sigma$. Since ϕ is an immersion, ι is an immersion almost everywhere in $\mathbb{R} \times \Sigma$ and thus M is an r-oriented ruled submanifold.

We now suppose that M is associative in order to discover the conditions that this imposes upon ϕ, ψ. Note that the asymptotic cone M_0 of M, given by (6.2), is the image of $\mathbb{R} \times \Sigma$ under the map ι_0, defined by $\iota_0(r, \sigma) = r\phi(\sigma)$. Since ϕ is an immersion, ι_0 is an immersion except at $r = 0$, so M_0 is a 3-dimensional cone which is non-singular except at 0.

Let $p \in M$. There exist $r \in \mathbb{R}$, $\sigma \in \Sigma$ such that $p = r\phi(\sigma) + \psi(\sigma)$. Choose local coordinates (s, t) near σ in Σ. Then $T_pM = \langle x, y, z\rangle_{\mathbb{R}}$, where $x = \phi(\sigma)$, $y = r\frac{\partial\phi}{\partial s}(\sigma) + \frac{\partial\psi}{\partial s}(\sigma)$ and $z = r\frac{\partial\phi}{\partial t}(\sigma) + \frac{\partial\psi}{\partial t}(\sigma)$. Since M is associative, T_pM is an associative 3-plane, which by Proposition 2.6 occurs if and only if $[x, y, z] = 0$. This condition forces a quadratic in r to vanish, and thus the coefficient of each power of r must be zero as this condition should hold for all $r \in \mathbb{R}$. The following set of equations must therefore hold in Σ:

$$\left[\phi, \frac{\partial\phi}{\partial s}, \frac{\partial\phi}{\partial t}\right] = 0, \tag{6.4}$$

$$\left[\phi, \frac{\partial\phi}{\partial s}, \frac{\partial\psi}{\partial t}\right] + \left[\phi, \frac{\partial\psi}{\partial s}, \frac{\partial\phi}{\partial t}\right] = 0, \tag{6.5}$$

$$\left[\phi, \frac{\partial\psi}{\partial s}, \frac{\partial\psi}{\partial t}\right] = 0. \tag{6.6}$$

Note firstly that, if we do not suppose M to be associative, but that (6.4)–(6.6) hold locally in Σ, then following the argument above, we see that each tangent space to M must be associative and hence that M is associative.

Moreover, (6.4) is equivalent to having that tangent spaces to points of the form $r\phi(\sigma)$, for $r \in \mathbb{R}, \sigma \in \Sigma$, are associative, which is precisely the condition for the asymptotic cone M_0 to be associative. We may therefore deduce the following result.

Proposition 6.3. *Let M be an r-oriented ruled associative 3-fold in $\mathbb{R}^7$ and let M_0 be the asymptotic cone of M. Then M_0 is an associative cone in $\mathbb{R}^7$ provided it is 3-dimensional.*

Since M_0 is associative, φ is a non-vanishing 3-form on M_0 that defines the orientation on M_0. This forces Σ to be oriented, for if (s,t) are some local coordinates on Σ, then we can define them to be oriented by imposing the condition that

$$\varphi\left(\phi, \frac{\partial\phi}{\partial s}, \frac{\partial\phi}{\partial t}\right) > 0.$$

In addition, if g is the natural metric on $\mathcal{S}^6$, then the pullback $\phi^*(g)$ is a metric on Σ making it a *Riemannian* 2-fold, since $\phi : \Sigma \to \mathcal{S}^6$ is an immersion. Therefore we can consider Σ as an oriented Riemannian 2-fold and hence it has a natural *complex structure*, which we denote as J. Locally, in Σ, we can choose a holomorphic coordinate $u = s + it$, and so the corresponding real coordinates (s,t) satisfy the condition $J(\frac{\partial}{\partial s}) = \frac{\partial}{\partial t}$. Following Joyce [9, p. 241], we say that local real coordinates (s,t) on Σ that have this property are *oriented conformal coordinates*.

We now use oriented conformal coordinates in the proof of the next result, which gives neater equations for maps ϕ, ψ defining an r-oriented ruled associative 3-fold.

Theorem 6.4. *Let Σ be a connected real analytic 2-fold, let $\phi : \Sigma \to \mathcal{S}^6$ be a real analytic immersion and let $\psi : \Sigma \to \mathbb{R}^7$ be a real analytic map. Let M be defined by* (6.1)*. Then M is associative if and only if*

$$\frac{\partial\phi}{\partial t} = \phi \times \frac{\partial\phi}{\partial s} \tag{6.7}$$

and ψ satisfies

(i) $\frac{\partial\psi}{\partial t} = \phi \times \frac{\partial\psi}{\partial s} + f\phi$ *for some real analytic function $f : \Sigma \to \mathbb{R}$,*
or
(ii) $\frac{\partial\psi}{\partial s}(\sigma), \frac{\partial\psi}{\partial t}(\sigma) \in \langle\phi(\sigma), \frac{\partial\phi}{\partial s}(\sigma), \frac{\partial\phi}{\partial t}(\sigma)\rangle_{\mathbb{R}}$ *for all $\sigma \in \Sigma$,*

where $\times$ is defined by (2.4) *and (s,t) are oriented conformal coordinates on Σ.*

Proof. Above, we noted that (6.4)–(6.6) were equivalent to the condition that M is associative, so we show that (6.7) is equivalent to (6.4) and that (i) and (ii) are equivalent to (6.5) and (6.6).

Let $\sigma \in \Sigma$, $C = |\frac{\partial\phi}{\partial s}(\sigma)| > 0$. Since ϕ maps to the unit sphere in $\mathbb{R}^7$, $\phi(\sigma)$ is orthogonal to $\frac{\partial\phi}{\partial s}(\sigma)$ and $\frac{\partial\phi}{\partial t}(\sigma)$. As (s,t) are oriented conformal coordinates, we also see that $\frac{\partial\phi}{\partial s}(\sigma)$ and $\frac{\partial\phi}{\partial t}(\sigma)$ are orthogonal and that $|\frac{\partial\phi}{\partial t}(\sigma)| = C$. We conclude that the triple $(\phi(\sigma), C^{-1}\frac{\partial\phi}{\partial s}(\sigma), C^{-1}\frac{\partial\phi}{\partial t}(\sigma))$ is an oriented orthonormal triad in $\mathbb{R}^7$, and it is the basis for an associative 3-plane in $\mathbb{R}^7$ if and only if (6.4) holds at σ. Since G_2 acts transitively on the set of associative 3-planes [3, Theorem IV.1.8], if (6.4) holds at σ, then we can transform coordinates on $\mathbb{R}^7$ using G_2 so that

$$\phi(\sigma) = e_1, \qquad \frac{\partial\phi}{\partial s}(\sigma) = Ce_2, \qquad \frac{\partial\phi}{\partial t}(\sigma) = Ce_3,$$

where $\{e_1, \ldots, e_7\}$ is a basis for $\mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$. We note here that (6.7) holds at σ, since the cross product is invariant under G_2. It is clear that, if (6.7) holds at σ, then the 3-plane generated by $\{\phi(\sigma), \frac{\partial\phi}{\partial s}(\sigma), \frac{\partial\phi}{\partial t}(\sigma)\}$ is associative, simply by the definition of the cross product.

Under the change of coordinates of $\mathbb{R}^7$ above, we can write $\frac{\partial\psi}{\partial s}(\sigma) = a_1e_1 + \ldots + a_7e_7$ and $\frac{\partial\psi}{\partial t}(\sigma) = b_1e_1 + \ldots + b_7e_7$ for real constants a_j, b_j for $j = 1, \ldots, 7$. Calculations show that (6.5) holds at σ if and only if

$$b_4 = -a_5, \quad b_5 = a_4, \quad b_6 = -a_7, \quad b_7 = a_6, \tag{6.8}$$

and (6.6) holds at σ if and only if

$$-a_4b_7 - a_5b_6 + a_6b_5 + b_4a_7 = 0, \tag{6.9}$$
$$-a_4b_6 + a_5b_7 + a_6b_4 - a_7b_5 = 0, \tag{6.10}$$
$$a_2b_7 + a_3b_6 - a_6b_3 - a_7b_2 = 0, \tag{6.11}$$
$$a_2b_6 - a_3b_7 - a_6b_2 + a_7b_3 = 0, \tag{6.12}$$
$$-a_2b_5 - a_3b_4 + a_4b_3 + a_5b_2 = 0, \tag{6.13}$$
$$-a_2b_4 + a_3b_5 + a_4b_2 - a_5b_3 = 0. \tag{6.14}$$

Substituting condition (6.8) into the above equations, (6.9) and (6.10) are satisfied immediately and (6.11)–(6.14) become:

$$a_6(a_2 - b_3) - a_7(a_3 + b_2) = 0,$$
$$-a_6(a_3 + b_2) - a_7(a_2 - b_3) = 0,$$
$$-a_4(a_2 - b_3) + a_5(a_3 + b_2) = 0,$$

$$a_4(a_3+b_2)+a_5(a_2-b_3)=0.$$

These equations can then be written in matrix form:

$$\begin{pmatrix} -a_6 & a_7 \\ a_7 & a_6 \end{pmatrix}\begin{pmatrix} a_2-b_3 \\ a_3+b_2 \end{pmatrix}=0, \tag{6.15}$$

$$\begin{pmatrix} -a_4 & a_5 \\ a_5 & a_4 \end{pmatrix}\begin{pmatrix} a_2-b_3 \\ a_3+b_2 \end{pmatrix}=0. \tag{6.16}$$

We see that equations (6.15) and (6.16) hold if and only if the vector appearing in both equations is zero or the determinants of the matrices are zero. We thus have two conditions which we shall show correspond to (i) and (ii):

$$a_2=b_3, \qquad -a_3=b_2; \tag{6.17}$$

$$a_4=a_5=0=a_6=a_7. \tag{6.18}$$

Using the fact that $\phi(\sigma)=e_1$, (6.17) holds if and only if

$$\begin{aligned}\frac{\partial\psi}{\partial t}(\sigma)&=b_1e_1-a_3e_2+a_2e_3-a_5e_4+a_4e_5-a_7e_6+a_6e_7\\&=\phi(\sigma)\times\frac{\partial\psi}{\partial s}(\sigma)+f(\sigma)\phi(\sigma),\end{aligned}$$

where $f(\sigma)=b_1$. Therefore, (6.17) corresponds to condition (i) holding at σ by virtue of the invariance of the cross product under G_2. The fact that f is real analytic is immediate from the hypotheses that ϕ,ψ are real analytic and that ϕ is non-zero, since ϕ maps to $\mathcal{S}^6$.

Similarly, (6.18) holds if and only if

$$\frac{\partial\psi}{\partial s}(\sigma)=a_1e_1+a_2e_2+a_3e_3 \quad\text{and}\quad \frac{\partial\psi}{\partial t}(\sigma)=b_1e_1+b_2e_2+b_3e_3,$$

which is equivalent to condition (ii) holding at σ, since we may note here that $\langle e_1,e_2,e_3\rangle_{\mathbb{R}}=\langle\phi(\sigma),\frac{\partial\phi}{\partial s}(\sigma),\frac{\partial\phi}{\partial t}(\sigma)\rangle_{\mathbb{R}}$.

In conclusion, at each point $\sigma\in\Sigma$, condition (i) or (ii) holds. Let $\Sigma_1=\{\sigma\in\Sigma : \text{(i) holds at } \sigma\}$ and let $\Sigma_2=\{\sigma\in\Sigma : \text{(ii) holds at } \sigma\}$. Note that (i) and (ii) are closed conditions on the real analytic maps ϕ,ψ. Therefore, Σ_1 and Σ_2 are closed real analytic subsets of Σ. Since Σ is real analytic and connected, Σ_j must either coincide with Σ or else be of zero measure in Σ for $j=1,2$. However, not both Σ_1 and Σ_2 can be of zero measure in Σ since $\Sigma_1\cup\Sigma_2=\Sigma$. Hence, $\Sigma_1=\Sigma$ or $\Sigma_2=\Sigma$, which completes the proof. □

It is worth making some remarks about Theorem 6.4. Note that (i) and (ii) are *linear* conditions on ψ and, by the remarks made above, (6.7) is the condition which makes the asymptotic cone M_0 associative. So, if we start with an associative two-sided cone M_0 defined by a map ϕ, then ϕ and a function ψ satisfying (i) or (ii) will define an r-oriented ruled associative 3-fold M with asymptotic cone M_0. We also note that conditions (i) and (ii) are unchanged if ϕ is fixed and satisfies (6.7), but ψ is replaced by $\psi + \tilde{f}\phi$ where $\tilde{f}$ is a real analytic function. We can thus always locally transform ψ such that f in condition (i) is zero.

6.2. Evolution equations for ruled associative 3-folds.

Our first result follows [9, Proposition 5.2]. Here we make the definition that a function is real analytic on a compact interval I in $\mathbb{R}$ if it extends to a real analytic function on an open set containing I.

Theorem 6.5. *Let I be a compact interval in $\mathbb{R}$, let s be a coordinate on I, and let $\phi_0 : I \to \mathcal{S}^6$ and $\psi_0 : I \to \mathbb{R}^7$ be real analytic maps. Then there exist $\epsilon > 0$ and unique real analytic maps $\phi : I \times (-\epsilon, \epsilon) \to \mathcal{S}^6$ and $\psi : I \times (-\epsilon, \epsilon) \to \mathbb{R}^7$ satisfying $\phi(s, 0) = \phi_0(s)$, $\psi(s, 0) = \psi_0(s)$ for all $s \in I$ and*

$$\frac{\partial \phi}{\partial t} = \phi \times \frac{\partial \phi}{\partial s}, \quad \frac{\partial \psi}{\partial t} = \phi \times \frac{\partial \psi}{\partial s}, \tag{6.19}$$

where t is a coordinate on $(-\epsilon, \epsilon)$. Let M be defined by:

$$M = \{r\phi(s,t) + \psi(s,t) : r \in \mathbb{R},\ s \in I, t \in (-\epsilon, \epsilon)\}.$$

Then M is an r-oriented ruled associative 3-fold in $\mathbb{R}^7$.

Proof. Since I is compact and ϕ_0, ψ_0 are real analytic, we may use the *Cauchy–Kowalevsky Theorem* [12, p. 234] to give us functions $\phi : I \times (-\epsilon, \epsilon) \to \mathbb{R}^7$ and $\psi : I \times (-\epsilon, \epsilon) \to \mathbb{R}^7$ satisfying the initial conditions and (6.19). It is clear that $\frac{\partial}{\partial t} g(\phi, \phi) = 2g(\phi, \frac{\partial \phi}{\partial t}) = 0$, since $\frac{\partial \phi}{\partial t}$ is defined by a cross product involving ϕ and hence is orthogonal to ϕ. We may deduce that $|\phi|$ is independent of t and is therefore one, so that ϕ maps to $\mathcal{S}^6$. We conclude that M is an r-oriented ruled associative 3-fold using (i) of Theorem 6.4. □

Theorem 6.5 shows that (6.19) can be considered as evolution equations for maps ϕ, ψ satisfying (i) of Theorem 6.4. We now show that condition (ii) of Theorem 6.4 does not produce any interesting ruled associative 3-folds. We make the definition that two rulings (Σ,π) and $(\tilde{\Sigma},\tilde{\pi})$ are *distinct* if the families of affine straight lines $\mathcal{F}_\Sigma=\{\pi^{-1}(\sigma):\sigma\in\Sigma\}$ and $\mathcal{F}_{\tilde{\Sigma}}=\{\tilde{\pi}^{-1}(\tilde{\sigma}):\tilde{\sigma}\in\tilde{\Sigma}\}$ are different.

Proposition 6.6. *Any r-oriented ruled associative 3-fold (M,Σ,π) satisfying condition (ii) but not (i) of Theorem* 6.4 *is locally isomorphic to an affine associative 3-plane in* $\mathbb{R}^7$.

Proof. By Theorem 4.1, M is real analytic wherever it is non-singular and so we can take (Σ,π) to be locally real analytic. Let $I=[0,1]$, let $\gamma:I\to\Sigma$ be a real analytic curve in Σ and let ϕ,ψ be the functions defining M. Then we can use Theorem 6.5 with initial conditions $\phi_0=\phi(\gamma(s))$ and $\psi_0=\psi(\gamma(s))$ to give us functions $\tilde{\phi},\tilde{\psi}$, which define an r-oriented ruled associative 3-fold $\tilde{M}$ satisfying (i) of Theorem 6.4. However, M and $\tilde{M}$ coincide in the real analytic 2-fold $\pi^{-1}(\gamma(I))$, and hence, by Theorem 4.2, they must be locally equal. We conclude that M locally admits a ruling $(\tilde{\Sigma},\tilde{\pi})$ satisfying (i) of Theorem 6.4, which must therefore be distinct from the ruling (Σ,π).

The families of affine straight lines $\mathcal{F}_\Sigma$ and $\mathcal{F}_{\tilde{\Sigma}}$, using the notation above, coincide in the family of affine straight lines defined by points on γ, denoted $\mathcal{F}_\gamma$. Using local real analyticity of the families, either $\mathcal{F}_\Sigma$ is equal to $\mathcal{F}_{\tilde{\Sigma}}$ locally or they only meet in $\mathcal{F}_\gamma$ locally. The former possibility is excluded because the rulings (Σ,π) and $(\tilde{\Sigma},\tilde{\pi})$ are distinct and thus the latter is true.

Let γ_1 and γ_2 be distinct real analytic curves near γ in Σ defining rulings (Σ_1,π_1) and (Σ_2,π_2), respectively, as above. Then $\mathcal{F}_\Sigma\cap\mathcal{F}_{\Sigma_j}$ is locally equal to $\mathcal{F}_{\gamma_j}$ for $j=1,2$. Hence, (Σ_1,π_1) and (Σ_2,π_2) are not distinct (that is, $\mathcal{F}_{\Sigma_1}=\mathcal{F}_{\Sigma_2}$) if and only if $\mathcal{F}_{\gamma_1}=\mathcal{F}_{\gamma_2}$, which implies that $\gamma_1=\gamma_2$. Therefore, distinct curves near γ in Σ produce different rulings of M and thus M has infinitely many rulings.

Suppose that $\{\gamma_t:t\in\mathbb{R}\}$ is a one parameter family of distinct curves near γ in Σ, with $\gamma_0=\gamma$. Each curve in the family defines a distinct ruling (Σ_t,π_t), and hence there exists $p\in M$ with M non-singular at p such that $L_t=\pi_t^{-1}(\pi_t(p))$ is not constant as a line in $\mathbb{R}^7$. We therefore get a one parameter family of lines L_t in M through p with $\frac{dL_t}{dt}\neq 0$ at some point, i.e. such that L_t changes non-trivially. We have thus constructed a real analytic one-dimensional family of lines $\{L_t:t\in\mathbb{R}\}$ whose total space is a real analytic 2-fold N contained in M. Moreover, every line in M through p is a line in the affine associative 3-plane $p+T_pM$, and so N is contained in $p+T_pM$. Then, since N has non-singular points in the intersection between

M and $p + T_pM$, Theorem 4.2 shows that M and $p + T_pM$ coincide on a connected component of M. Hence, M is planar, i.e. M is locally isomorphic to an affine associative 3-plane in $\mathbb{R}^7$. □

We now state our main result on ruled associative 3-folds, which follows from the results in this section.

Theorem 6.7. *Let (M, Σ, π) be a non-planar, r-oriented, ruled associative 3-fold in $\mathbb{R}^7$. Then there exist real analytic maps $\phi : \Sigma \to \mathcal{S}^6$ and $\psi : \Sigma \to \mathbb{R}^7$ such that:*

$$M = \{r\phi(\sigma) + \psi(\sigma) : r \in \mathbb{R},\, \sigma \in \Sigma\},$$

$$\frac{\partial \phi}{\partial t} = \phi \times \frac{\partial \phi}{\partial s}, \tag{6.20}$$

$$\frac{\partial \psi}{\partial t} = \phi \times \frac{\partial \psi}{\partial s} + f\phi, \tag{6.21}$$

where (s, t) are oriented conformal coordinates on Σ and $f : \Sigma \to \mathbb{R}$ is some real analytic function.

Conversely, suppose $\phi : \Sigma \to \mathcal{S}^6$ and $\psi : \Sigma \to \mathbb{R}^7$ are real analytic maps satisfying (6.20) and (6.21) on a connected real analytic 2-fold Σ. If M is defined as above, then M is an r-oriented ruled associative 3-fold wherever it is non-singular.

6.3. Holomorphic vector fields.

We now follow [9, Section 6] and use a *holomorphic vector field* on a Riemann surface Σ to construct ruled associative 3-folds.

Proposition 6.8. *Let M_0 be an r-oriented two-sided associative cone in $\mathbb{R}^7$. We can then write M_0 in the form (6.2) for a real analytic map $\phi : \Sigma \to \mathcal{S}^6$, where Σ is a Riemann surface. Let w be a holomorphic vector field on Σ and define a map $\psi : \Sigma \to \mathbb{R}^7$ by $\psi = \mathcal{L}_w\phi$, where $\mathcal{L}_w$ is the Lie derivative with respect to w. If we define M by equation (6.1), then M is an r-oriented ruled associative 3-fold in $\mathbb{R}^7$ with asymptotic cone M_0.*

Proof. We need only consider the case where w is not identically zero since this is trivial. Then w has only isolated zeros and, since the fact that M is associative is a closed condition on the non-singular part of M, it is sufficient to prove that (6.21) holds at any point $\sigma \in \Sigma$ such that $w(\sigma) \neq 0$. Suppose

σ is such a point. Then, since w is a holomorphic vector field, there exists an open set in Σ containing σ on which oriented conformal coordinates (s,t) may be chosen such that $w = \frac{\partial}{\partial s}$. Hence, $\psi = \frac{\partial \phi}{\partial s}$ in a neighbourhood of σ, and differentiating (6.20) gives:

$$\frac{\partial^2 \phi}{\partial s \partial t} = \frac{\partial \phi}{\partial s} \times \frac{\partial \phi}{\partial s} + \phi \times \frac{\partial^2 \phi}{\partial s^2}.$$

Interchanging the order of the partial derivatives on the left-hand side and noting that the cross product is alternating, we have that

$$\frac{\partial \psi}{\partial t} = \frac{\partial^2 \phi}{\partial s \partial t} = \phi \times \frac{\partial \psi}{\partial s}.$$

The result follows from Theorem 6.7. □

Having proved a result which enables us to construct ruled associative 3-folds given an associative cone on a Riemann surface Σ, we consider which choices for Σ will produce interesting examples. The only non-trivial vector spaces for holomorphic vector fields on a compact connected Riemann surface occur for genus zero or one. We therefore focus our attention upon the cases where we take Σ to be $\mathcal{S}^2$ or T^2. The space of holomorphic vector fields on $\mathcal{S}^2$ is 6-dimensional, and on T^2 it is 2-dimensional. In the SL case, any SL cone on $\mathcal{S}^2$ has to be an SL 3-plane [4, Theorem B]; Bryant [1, Section 4] shows that this is not true in the associative case and that, in fact, there are many non-trivial associative cones on $\mathcal{S}^2$.

Theorem 6.9. *Let M_0 be an r-oriented, two-sided, associative cone on a Riemann surface $\Sigma \cong \mathcal{S}^2$ (or T^2) with associated real analytic map $\phi : \Sigma \to \mathcal{S}^6$ as in* (6.2). *Then there exists a 6-dimensional (or 2-dimensional) family of distinct r-oriented ruled associative 3-folds with asymptotic cone M_0, which are asymptotically conical to M_0 with order $O(r^{-1})$.*

Proof. If (s,t) are oriented conformal coordinates on Σ, we may write holomorphic vector fields on Σ in the form:

$$w = u(s,t)\frac{\partial}{\partial s} + v(s,t)\frac{\partial}{\partial t}\,, \tag{6.22}$$

where $u, v : \mathbb{R}^2 \to \mathbb{R}$ satisfy the Cauchy–Riemann equations. For each holomorphic vector field w, as written in (6.22), define a 3-fold M_w by:

$$M_w = \left\{ r\phi(s,t) + u(s,t)\frac{\partial \phi}{\partial s}(s,t) + v(s,t)\frac{\partial \phi}{\partial t}(s,t) : r \in \mathbb{R}, (s,t) \in \Sigma \right\}.$$

By Proposition 6.8, M_w is an r-oriented ruled associative 3-fold with asymptotic cone M_0, and it is clear that each holomorphic vector field w will give a distinct 3-fold.

We now construct a diffeomorphism Φ as in Definition 6.2 satisfying (6.3) for $\alpha = -1$. Let $R > 0$, w be a holomorphic vector field as in (6.22), and let $\bar{B}_R$ denote the closed ball of radius R in $\mathbb{R}^7$. Define $\Phi : M_0 \setminus \bar{B}_R \to M_w$ by:

$$\Phi(r\phi(s,t)) = r\phi\left(s - \frac{u}{r}, t - \frac{v}{r}\right) + u\frac{\partial\phi}{\partial s}\left(s - \frac{u}{r}, t - \frac{v}{r}\right) + v\frac{\partial\phi}{\partial t}\left(s - \frac{u}{r}, t - \frac{v}{r}\right),$$

where $|r| > R$. Clearly, Φ is a well-defined map with image in $M_w \setminus K$ for some compact subset K of M_w. Note that, by choosing R sufficiently large, we can expand the various terms defining Φ in powers of r^{-1} as follows:

$$\begin{aligned}
\phi\left(s - \frac{u(s,t)}{r}, t - \frac{v(s,t)}{r}\right) &= \phi(s,t) - \frac{u(s,t)}{r}\frac{\partial\phi}{\partial s}(s,t) - \frac{v(s,t)}{r}\frac{\partial\phi}{\partial t} + O(r^{-2}),\\
\frac{\partial\phi}{\partial s}\left(s - \frac{u(s,t)}{r}, t - \frac{v(s,t)}{r}\right) &= \frac{\partial\phi}{\partial s}(s,t) + O(r^{-1}),\\
\frac{\partial\phi}{\partial t}\left(s - \frac{u(s,t)}{r}, t - \frac{v(s,t)}{r}\right) &= \frac{\partial\phi}{\partial t}(s,t) + O(r^{-1}).
\end{aligned}$$

We deduce that

$$|\Phi(r\phi(s,t)) - r\phi(s,t)| = O(r^{-1}),$$

and the other conditions in (6.3) can be derived similarly. We conclude that M_w is asymptotically conical to M_0 with order $O(r^{-1})$. □

There are many examples of associative cones over T^2 given by the SL tori constructed by Haskins [4], Joyce [6] and McIntosh [11] and others. However, by Theorem 5.10, applying Theorem 6.9 to them will only produce ruled SL 3-folds and the result reduces to [9, Theorem 6.3].

7. Acknowledgements.

I owe a great deal of gratitude to my supervisor Dominic Joyce for his help, guidance and foundational work. Thanks are also due to the referees for their work in providing helpful suggestions and corrections.

References.

[1] R. L. Bryant, *Submanifolds and Special Structures on the Octonions*, J. Diff. Geom. **17** (1982), 185-232.

[2] K. Chandrasekharan, *Elliptic Functions*, Grundlehren der mathematische Wissenschaften **281**, Springer-Verlag, Berlin, 1985.

[3] R. Harvey and H. B. Lawson, *Calibrated Geometries*, Acta Mathematica **148** (1982), 47-152.

[4] M. Haskins, *Special Lagrangian Cones*, Amer. J. Math. **126** (2004), 845-871.

[5] D. D. Joyce, *Compact Manifolds with Special Holonomy*, OUP, 2000.

[6] D. D. Joyce, *Special Lagrangian m-folds in* $\mathbb{C}^m$ *with Symmetries*, Duke Math. J. **115** (2002), 1-51.

[7] D. D. Joyce, *Constructing Special Lagrangian m-folds in* $\mathbb{C}^m$ *by Evolving Quadrics*, Math. Ann. **320** (2001), 757-797.

[8] D. D. Joyce, *Evolution Equations for Special Lagrangian 3-Folds in* $\mathbb{C}^3$, Ann. Global Anal. Geom. **20** (2001), 345-403.

[9] D. D. Joyce, *Ruled Special Lagrangian 3-folds in* $\mathbb{C}^3$, Proc. London Math Soc. **85** (2002), 233-256.

[10] H. B. Lawson, *Lectures on Minimal Submanifolds Volume 1*, Publish or Perish, Inc., 1980.

[11] I. McIntosh, *Special Lagrangian Cones in* $\mathbb{C}^3$ *and Primitive Harmonic Maps*, J. London Math. Soc. **67** (2003), 769-789.

[12] R. Racke, *Lectures on Nonlinear Evolution Equations*, Aspects of Mathematics **E19**, Max-Planck Institute, Bonn, 1992.

Christ Church
Oxford

Received January 14, 2004.