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Intrinsic flat Arzela-Ascoli theorems

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One of the most powerful theorems in metric geometry is the Arzela-Ascoli Theorem which provides a continuous limit for sequences of equicontinuous functions between two compact spaces. This theorem has been extended by Gromov and Grove-Petersen to sequences of functions with varying domains and ranges where the domains and the ranges respectively converge in the Gromov-Hausdorff sense to compact limit spaces. However such a powerful theorem does not hold when the domains and ranges only converge in the intrinsic flat sense due to the possible disappearance of points in the limit.

In this paper two Arzela-Ascoli Theorems are proven for intrinsic flat converging sequences of manifolds: one for uniformly Lipschitz functions with fixed range whose domains are converging in the intrinsic flat sense, and one for sequences of uniformly local isometries between spaces which are converging in the intrinsic flat sense. A basic Bolzano-Weierstrass Theorem is proven for sequences of points in such sequences of spaces. In addition it is proven that when a sequence of manifolds has a precompact intrinsic flat limit then the metric completion of the limit is the Gromov-Hausdorff limit of regions within those manifolds. Applications and suggested applications of these results are described in the final section of this paper.

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1. Introduction

When studying sequences of Riemannian manifolds, one may use a variety of notions of convergence from $C^{k,\alpha}$ smooth convergence to Gromov-Hausdorff convergence as metric spaces. One needs to understand whether points and balls in the sequences converge to points and balls in limit spaces. So one proves Bolzano-Weierstrass theorems to produce converging subsequences of points. One needs to understand the limits of functions on these spaces and local isometries between these spaces. So one proves Arzela-Ascoli theorems for sequences of uniformly Lipschitz functions between converging spaces. Such theorems have been proven for Gromov-Hausdorff convergence by Gromov and by Grove-Petersen [13] [12] [15]. They have been applied in these works as well as that of Cheeger-Colding, Cheeger-Naber, the author, Wei, and numerous other papers including Perelman's solution of the Poincare Conjecture (c.f. [6] [8] [34] [35] and [27]).

There are many questions concerning Riemannian manifolds which cannot be addressed using these relatively strong notions of convergence. The intrinsic flat convergence is a more flexible notion allowing a larger class of sequences of manifolds to converge. Gromov has proposed that this notion would be natural to study questions arising in [14]. Lakzian has applied intrinsic flat convergence to prove continuity of Ricci flow through a singularity [18]. Dan Lee and the author have shown intrinsic flat convergence is well adapted to questions arising in General Relativity [20]. Additional applications of intrinsic flat convergence are described in the final section of this paper relating to work of Basilio, Burago, Ivanov, Ding, Fukaya, Gromov, Huang, Lakzian, LeFloch, Lee, Li, Matveev, Munn, Perales, Portegies, Sinaei and Wei [3] [5] [9] [11] [14] [16] [21] [24] [23] [25][22][26][22][30] [32] [35].

The flexibility of intrinsic flat convergence is that it allows points to disappear in the limit. As a consequence, many of the techniques used to study Gromov-Hausdorff limits including the Arzela-Ascoli Theorem fail to hold when the domains and ranges of the functions only converge in the intrinsic flat sense (c.f. Remark 6.3) . In this paper additional hypothesis are provided to produce two Arzela-Ascoli Theorems [Theorems 6.2 and 8.1] as well as a basic Bolzano-Weierstrass Theorem [Theorem 7.1]. A new relationship between Gromov-Hausdorff and intrinsic flat convergence is also proven [Theorem 5.1]. Direct applications of these theorems are described in the final section of this paper.

Intrinsic flat convergence was introduced by Wenger and the author in [39] building upon work of Ambrosio-Kirchheim in [2]. It is defined for oriented Riemannian manifolds, M_j^m with boundary such that

$$\mathrm{Vol}(M_j) \le V_j \quad \text{and} \quad \mathrm{Vol}(\partial M_j) \le A_j. \tag{1}$$

The limit spaces obtained under this convergence are called integral current spaces. They are either countably $\mathcal{H}^m$ rectifiable metric spaces of the same dimension as the sequence of manifolds or possibly the $\mathbf{0}$ space. When there is a Gromov-Hausdorff limit, $M_j^m \xrightarrow{\text{GH}} Y$, and one has uniform bounds on volume and boundary volume,

$$\mathrm{Vol}(M_j) \le V_0 \quad \text{and} \quad \mathrm{Vol}(\partial M_j) \le A_0, \tag{2}$$

then a subsequence has an intrinsic flat limit, $M_{j_i} \xrightarrow{\mathcal{F}} X$ where $X \subset Y$ with the restricted distance, $d_X = d_Y$ [39]. It is possible that X is the $\mathbf{0}$ space or a strict subset of Y either because the sequence is collapsing or due to cancellation (see examples in [39]). When the sequence is collapsing to a GH limit Y with a Hausdorff dimension that is strictly less than m, then X must be the $\mathbf{0}$ space. Intrinsic flat limits may exist for sequences of manifolds with no Gromov-Hausdorff limit [39]. In fact Wenger's Compactness Theorem implies that any sequence M_j satisfying (2) and a uniform bound on diameter has a subsequence converging in the intrinsic flat sense possibly to the $\mathbf{0}$ space [41]. See Section 2 for a review.

This paper focuses on sequences of oriented Riemannian manifolds, M_j^m satisfying (1), or more generally integral current spaces satisfying a similar condition, which converge in the intrinsic flat sense. The paper begins with the definition of converging and disappearing sequences of points [Definitions 3.1 and 3.2] and a proof that diameter is lower semicontinuous [Theorem 3.6]. Viewing balls within integral current spaces as integral current spaces themselves [Lemma 2.34] it is proven that, for almost every radius, balls around converging points have subsequences which converge to balls about their limit points [Lemma 4.1]. The necessity of taking a subsequence is shown in Example 4.3. If a sequence of points disappears, the balls of small radius about those points converge to the **0** space [Lemma 4.1]. Lemma 4.4 examines how the intrinsic flat distance may be estimated when the spaces are rescaled. Although technical, these lemmas are key steps in the subsequent theorems.

It is shown in Theorem 5.1 that if Riemannian manifolds M_i converge in the intrinsic flat sense to a nonzero precompact limit space, M, then there are open submanifolds $N_i \subset M_i$ such that $N_i \xrightarrow{\text{GH}} M$. This theorem and Remark 5.2 also describe the volumes of these submanifolds as well as what happens when M_i are integral current spaces. Section 5 also contains a few related open questions within remarks concerning possible extensions and applications of this theorem.

Theorem 6.1 is the simplest Intrinsic Flat Arzela-Arzela Theorem. It states that if a sequence of functions, $F_i : M_i \to W$ where $M_i \xrightarrow{\mathcal{F}} M_\infty$ and W is compact and $\mathrm{Lip}(F_i) \le K$, then there is a converging subsequence $F_i \to F_\infty$ where $F_\infty : M_\infty \to W$ also has $\mathrm{Lip}(F_\infty) \le K$. A precise description as to exactly how $F_i \to F_\infty$ is given. More general is Theorem 6.2 which allows the target spaces to converge in the GH sense. Remark 6.3 explains the impossibility of extending this theorem to allow the target spaces to converge in the intrinsic flat sense.

Theorem 7.1 is an Intrinsic Flat Bolzano-Weierstrass theorem for points $p_i \in M_i$ such that $M_i \xrightarrow{\mathcal{F}} M_\infty$. Since it is known that points may disappear in the limit [Remark 3.3], it is necessary to add a condition to obtain a subsequence with a limit point p_∞. In Theorem 7.1, the extra condition is that for almost every sufficiently small radius there is a uniform bound on the intrinsic flat distance between the balls about p_i and **0**. Remark 2.38 discusses how one can obtain such a bound when needed.

Theorem 8.1 is the second Intrinsic Flat Arzela-Ascoli Theorem proven here. In this theorem the domains and ranges of the functions converge in the intrinsic flat sense and have uniform upper bounds as in (2). The functions are assumed to be local isometries which are isometries on balls of fixed radius. It is shown that a subsequence of the functions converges to a limit function which is also a local isometry. If the functions are surjective, then so is the limit. The case where the limit spaces are possibly the **0** space is also considered. Remark 8.4 discusses a possible extension of this theorem to uniformly locally bi-Lipschitz functions or more simply uniformly bi-Lipschitz functions. Remark 8.2 discusses the necessity of various conditions in Theorem 8.1.

In Section 9 an example is presented showing how these theorems can be applied to prove certain sequences of Riemannian manifolds have no intrinsic flat limit. Additional applications to construct examples which do have specific limits will appear in joint work with Basilio [3].

Section 10 includes remarks describing the possible additional applications of the various theorems in this paper. In particular one may be able to apply Theorem 8.1 to answer a question posed by Gromov in [14] concerning the intrinsic flat limits of tori whose universal covers have almost maximal volume growth in the sense described by Burago-Ivanov in [5]. See Remark 10.1. Additional possible applications of Theorem 8.1 to extend work of the author with Wei are described in Remarks 10.2 and 10.3. It may also be possible to apply Theorem 6.2 to study the limits of harmonic functions, eigenfunctions and heat kernels. See Remark 10.4. In Remark 10.5, it is described how one may be able to apply Theorem 7.1 to prove that the intrinsic flat and Gromov-Hausdorff limits of Riemannian manifolds with uniform lower Ricci curvature bounds agree extending a theorem of the author with Wenger in [38]. Finally there are three remarks discussing how various theorems in the paper may be applied to a variety of questions and conjectures related to questions in General Relativity.

The author would like to thank Blaine Lawson (SUNYSB) for suggesting that the basic properties of intrinsic flat convergence should appear in their own paper separate from the more technical theorems involving the Gromov Filling Volume which appear in [29]. That lead to the creation of this paper. The author is also indebted to doctoral students, Jacobus Portegies and Raquel Perales, and the referee for their careful reading of this paper and their extensive feedback.

2. Background

Here the key definitions and theorems applied in this paper are reviewed. Please keep in mind that this is by no means a complete introduction to Gromov-Hausdorff convergence and Intrinsic Flat convergence. Only the notions that are applied in this paper are reviewed. In fact, the primary reason for combining Theorems 5.1, 6.2, 7.1, and 8.1 together into this paper is because these four theorems can be proven using the same background material. Other related theorems appearing in [29] all require additional results of Gromov and Ambrosio-Kirchheim.

Those who have already studied the notion of Intrinsic Flat convergence in the initial paper by the author with Wenger [39], should still review Subsections 2.1, 2.3 and 2.7 which cover material not presented there. Those who have never studied Gromov-Hausdorff or Intrinsic Flat convergence will find the entire background section useful as a very brief but self contained introduction to the subjects. As the author sees no reason to restate theorems, definitions and remarks, some of these statements have been repeated exactly as stated in prior background sections written by the author elsewhere.

2.1. A review of Gromov-Hausdorff convergence

Throughout this paper, Gromov's definition of an isometric embedding will be used:

Definition 2.1. A map $\varphi : X \to Y$ between metric spaces, (X, d_X) and (Y, d_Y), is an isometric embedding iff it is distance preserving:

$$d_Y(\varphi(x_1), \varphi(x_2)) = d_X(x_1, x_2) \qquad \forall x_1, x_2 \in X. \tag{3}$$

Observe that this does not agree with the Riemannian notion of an isometric embedding.

The following is one of the more beautiful definitions of the Gromov-Hausdorff distance:

Definition 2.2 (Gromov). The Gromov-Hausdorff distance between two compact metric spaces (X_1, d_{X_1}) and (X_2, d_{X_2}) is defined as

$$d_{GH}\left(X_1, X_2\right) := \inf\left\{d_H^Z\left(\varphi_1\left(X_1\right), \varphi_2\left(X_2\right)\right) \mid \varphi_i : X_i \to Z\right\} \tag{4}$$

where the infimum is taken over compact metric spaces, Z, and isometric embeddings, $\varphi_i : X_i \to Z$, and where the Hausdorff distance in Z is defined as

$$d_H^Z\left(A, B\right) = \inf\{\epsilon > 0 \mid A \subset T_\epsilon\left(B\right) \text{ and } B \subset T_\epsilon\left(A\right)\} \tag{5}$$

where $T_\epsilon(A) = \{x \in Z : \exists\, a \in A \; s.t. \; d_Z(x, a) < \epsilon\}$.

Gromov proved that this is indeed a distance on compact metric spaces in the sense that $d_{GH}(X,Y)=0$ iff there is an isometry between X and Y in [13]. Gromov proved the following embedding theorem in [12]:

Theorem 2.3 (Gromov). *If a sequence of compact metric spaces, X_j, converges in the Gromov-Hausdorff sense to a compact metric space X_∞,*

$$X_j \xrightarrow{GH} X_\infty \tag{6}$$

then in fact there is a compact metric space, Z, and isometric embeddings $\varphi_j : X_j \to Z$ for $j \in \{1,2,\ldots,\infty\}$ such that

$$d_H^Z\left(\varphi_j(X_j), \varphi_\infty(X_\infty)\right) \to 0. \tag{7}$$

This theorem allows one to define converging sequences of points:

Definition 2.4. One says that $x_j \in X_j$ converges to $x_\infty \in X_\infty$, if there is a common space Z as in Theorem 2.3 such that $\varphi_j(x_j) \to \varphi_\infty(x)$ as points in Z. If one discusses the limits of multiple sequences of points then one uses a common Z and the same collection of φ_j to determine the convergence. This avoids difficulties arising from isometries in the limit space. Then one immediately has

$$\lim_{j\to\infty} d_{X_j}(x_j, x_j') = d_{X_\infty}(x_\infty, x_\infty') \tag{8}$$

whenever $x_j \to x_\infty$ and $x_j' \to x_\infty'$ via a common Z.

One can apply Theorem 2.3 to see that for any $x_\infty \in X_\infty$ there exists $x_j \in X_j$ converging to x_∞ in this sense. Also observe that whenever x_j converges to x_∞ in this sense,

$$\begin{aligned} &d_{GH}\left(B(x_j,r), B(x_\infty,r)\right) \\ &\le d_H^Z\left(B(\varphi_j(x_j),r), B(\varphi_\infty(x_\infty),r)\right) \to 0 \qquad \forall r>0 \end{aligned} \tag{9}$$

if one views the balls $B(x_j,r) \subset X_j$ as metric spaces endowed with the restricted metric, d_{X_j}, from X_j. See the appendix of joint work of the author with Wei [37] for a theorem concerning the induced length metrics. Theorem 2.3 also implies the following basic Bolzano-Weierstrass Theorem:

Theorem 2.5 (Gromov). *Given compact metric spaces, $X_j \xrightarrow{GH} X_\infty$, and $x_j \in X_j$ then a subsequence also denoted x_j converges to a point $x_\infty \in X_\infty$ in the sense described above.*

In particular, one sees that

$$X_j \to X_\infty \implies \lim_{j\to\infty} \mathrm{Diam}(X_j) = \mathrm{Diam}(X_\infty). \tag{10}$$

Gromov's embedding theorem can also be applied in combination with other extension theorems to obtain the following Gromov-Hausdorff Arzela-Ascoli Theorem.

Theorem 2.6 (Gromov). *Given compact metric spaces, $X_j \xrightarrow{GH} X_\infty$ and $Y_j \to Y_\infty$, and equicontinuous functions, $f_j : X_j \to Y_j$, in the sense that*

$$\forall \epsilon > 0\ \exists \delta_\epsilon > 0 \text{ such that } d_{X_j}(x,x') < \delta_\epsilon \implies d_{Y_j}(f_j(x), f_j(x')) \le \epsilon. \tag{11}$$

Then there exists a subsequence, also denoted $f_j : X_j \to Y_j$, which converges to a continuous function, $f_\infty : X_\infty \to Y_\infty$, in the sense that there exists common compact metric spaces, Z, W, and isometric embeddings, $\varphi_j : X_j \to Z$, $\psi_j : Y_j \to W$, such that

$$\lim_{j\to\infty} \psi_j(f_j(x_j)) = \psi_\infty(f_\infty(x_\infty)) \text{ whenever } \lim_{j\to\infty} \varphi_j(x_j) = \varphi_\infty(x_\infty). \tag{12}$$

Furthermore, if $\mathrm{Lip}(f_j) \le K$ *then* $\mathrm{Lip}(f_\infty) \le K$.

Gromov used this idea to prove that geodesic metric spaces converge to geodesic metric spaces but did not include a general proof in [13]. For completeness of exposition, we include a proof here. One may also find a discussion of the proof in a paper of Grove-Petersen [15] and a more general statement in Theorem 2.3 of [34] by the author.

Proof. By Gromov's Embedding Theorem, one has isometric embeddings $\varphi_j : X_j \to Z$ and $\psi_j : Y_j \to W$ such that

$$d_H^Z(\varphi_j(X_j), \varphi_\infty(X_\infty)) \to 0 \text{ and } d_H^W(\psi_j(Y_j), \psi_\infty(Y_\infty)) \to 0.$$

Let $X_0 \subset X_\infty$ be a countable dense subset. For each $p_\infty \in X_0$, there exists $p_j \in X_j$ such that $\varphi_j(p_j) \to \varphi_\infty(p_\infty)$. Since W is compact, a subsequence of $\psi_j(f_j(p_j)) \in W$ converges to some point $w_\infty \in W$.

We claim $w_\infty \in \psi_\infty(Y_\infty)$. If not, then there exists $r > 0$ such that

$$B(w_\infty, r) \cap \psi_\infty(Y_\infty) = \emptyset.$$

Then for j sufficiently large

$$B(\psi_j(f_j(p_j)), r/2) \cap \psi_\infty(Y_\infty) = \emptyset.$$

This implies that

$$d_H(\psi_j(Y_j), \psi_\infty(Y_\infty)) \geq r/2,$$

which is a contradiction.

Thus we have a point we call $f_\infty(p_\infty) \in Y_\infty$ such that $\psi_\infty(f_\infty(p_\infty)) = w_\infty$. Applying a diagonalization process to choose a subsequence, we have thus defined $f_\infty : X_0 \subset X_\infty \to Y_\infty$ satisfying (11). Extending this function continuously to $f_\infty : X_\infty \to Y_\infty$, it still satisfies (11).

To see that (12) holds consider $x_j \in X_j$ such that

$$\lim_{j\to\infty} \varphi_j(x_j) = \varphi_\infty(x_\infty).$$

Taking $p_\infty = x_\infty$ as in the top of the proof, there exists $p_j \in X_j$ such that $\varphi_j(p_j) \to \varphi_\infty(p_\infty)$ and $\psi_j(f_j(p_j)) \to \psi_\infty(f_\infty(p_\infty))$. Observe that

$$\begin{aligned} d_{X_j}(x_j, p_j) &= d_Z(\varphi_j(x_j), \varphi_j(p_j)) \\ &\leq d_Z(\varphi_j(x_j), \varphi_\infty(x_\infty)) + d_Z(\varphi_\infty(x_\infty), \varphi_j(p_j)) \to 0. \end{aligned}$$

For any $\epsilon > 0$ take j sufficiently large that $d_X(x_j, p_j) < \delta_\epsilon$ of (11), then

$$\begin{aligned} &\lim_{j\to\infty} d_W(\psi_j(f_j(x_j)), \psi_\infty(f_\infty(x_\infty))) \\ &\leq \lim_{j\to\infty} d_W(\psi_j(f_j(x_j)), \psi_j(f_j(p_j))) + d_W(\psi_j(f_j(p_j)), \psi_\infty(f_\infty(x_\infty))) \\ &= \lim_{j\to\infty} d_{Y_j}(f_j(x_j), f_j(p_j)) + d_W(\psi_j(f_j(p_j)), \psi_\infty(f_\infty(p_\infty))) \\ &< \epsilon + 0 \qquad \forall \epsilon > 0. \end{aligned}$$

Thus $\psi_j(f_j(x_j)) \to \psi_\infty(f_\infty(x_\infty))$. □

All these theorems are key ingredients in the many important works applying Gromov-Hausdorff convergence to better understand Riemannian Geometry. See the classic textbook of Burago-Burago-Ivanov [4], the work of Cheeger-Colding [6] and the work of the author with Wei [35].

In this paper these theorems are extended, as far as possible, in the setting where one only has intrinsic flat convergence. Of course it is known that these theorems do not hold in their full strength in the setting where sequences of Riemannian manifolds are converging in the intrinsic flat sense. Examples in joint work of the author with Wenger in [39] demonstrate that (10) fails in general and that geodesics need not converge to geodesics. Nevertheless there are versions of these theorems which do hold.

2.2. Review of Ambrosio-Kirchheim currents on metric spaces

In order to rigorously review the definition of the intrinsic flat distance, one needs a few key results of Ambrosio-Kirchheim. These results will also be applied later to prove the main theorems of the paper.

In [2], Ambrosio-Kirchheim extend Federer-Fleming's notion of integral currents on Euclidean space to an arbitrary complete metric space, Z. In Federer-Fleming, currents were defined as linear functionals on differential forms [10]. This approach extends naturally to smooth manifolds but not to complete metric spaces which do not have differential forms. In the place of differential forms, Ambrosio-Kirchheim use DiGeorgi's $m+1$ tuples, $\omega \in \mathcal{D}^m(Z)$,

$$\omega = f\pi = (f, \pi_1, \ldots, \pi_m) \in \mathcal{D}^m(Z) \tag{13}$$

where $f : X \to \mathbb{R}$ is a bounded Lipschitz function and $\pi_i : X \to \mathbb{R}$ are Lipschitz.

In [2] Definitions 2.1, 2.2, 2.6 and 3.1, an m dimensional current $T \in \mathbf{M}_m(Z)$ is defined. Here these are combined into a single definition:

Definition 2.7. On a complete metric space, Z, an m dimensional ***current***, denoted $T \in \mathbf{M}_m(Z)$, is a real valued *multilinear functional* on $\mathcal{D}^m(Z)$, with the following three required properties:

i) *Locality*:

$$T(f, \pi_1, \ldots, \pi_m) = 0$$
$$\text{if } \exists i \in \{1, \ldots, m\} \text{ s.t. } \pi_i \text{ is constant on a nbd of } \{f \neq 0\}.$$

ii) *Continuity*: Continuity of T with respect to the ptwise convergence of π_i such that $\mathrm{Lip}(\pi_i) \leq 1$.

iii) *Finite mass*:

$\exists$ finite Borel μ

$$s.t.\ \ |T(f, \pi_1, \ldots, \pi_m)| \leq \prod_{i=1}^{m} \mathrm{Lip}(\pi_i) \int_Z |f|\, d\mu \ \ \forall (f, \pi_1, \ldots, \pi_m) \in \mathcal{D}^m(Z).$$

In [2] Definition 2.6 Ambrosio-Kirchheim introduce their mass measure:

Definition 2.8. The mass measure $\|T\|$ of a current $T \in \mathbf{M}_m(Z)$, is the smallest Borel measure μ such that

$$\left|T(f,\pi)\right| \le \int_X |f| d\mu \qquad \forall\ (f,\pi) \text{ where } \operatorname{Lip}(\pi_i) \le 1. \tag{14}$$

The mass of T is defined

$$M(T) = ||T||(Z) = \int_Z d\|T\|. \tag{15}$$

In particular

$$\left|T(f,\pi_1,\ldots,\pi_m)\right| \le \mathbf{M}(T)|f|_\infty \operatorname{Lip}(\pi_1)\cdots\operatorname{Lip}(\pi_m). \tag{16}$$

Ambrosio-Kirchheim then define restrictions and push forwards:

Definition 2.9. [2][Defn 2.5] The *restriction* $T \llcorner \omega \in \mathbf{M}_m(Z)$ of a current $T \in M_{m+k}(Z)$ by a $k+1$ tuple $\omega = (g, \tau_1, \ldots, \tau_k) \in \mathcal{D}^k(Z)$:

$$(T \llcorner \omega)(f, \pi_1, \ldots, \pi_m) := T(f \cdot g, \tau_1, \ldots, \tau_k, \pi_1, \ldots, \pi_m). \tag{17}$$

Given a Borel set, A,

$$T \llcorner A := T \llcorner \omega \tag{18}$$

where $\omega = 1_A \in \mathcal{D}^0(Z)$ is the indicator function of the set. In this case,

$$\mathbf{M}(T \llcorner \omega) = ||T||(A). \tag{19}$$

Definition 2.10. Given a Lipschitz map $\varphi : Z \to Z'$, the *push forward* of a current $T \in \mathbf{M}_m(Z)$ to a current $\varphi_\# T \in \mathbf{M}_m(Z')$ is given in [2][Defn 2.4] by

$$\varphi_\# T(f, \pi_1, \ldots, \pi_m) := T(f \circ \varphi, \pi_1 \circ \varphi, \ldots, \pi_m \circ \varphi). \tag{20}$$

Remark 2.11. Observe that

$$(\varphi_\# T) \llcorner (f, \pi_1, \ldots, \pi_k)) = \varphi_\#(T \llcorner (f \circ \varphi, \pi_1 \circ \varphi, \ldots, \pi_k \circ \varphi)) \tag{21}$$

and

$$(\varphi_\# T) \llcorner A = (\varphi_\# T) \llcorner (1_A) = \varphi_\#(T \llcorner (1_A \circ \varphi)) = \varphi_\#(T \llcorner \varphi^{-1}(A)). \tag{22}$$

In (2.4) [2], Ambrosio-Kirchheim show that

$$||\varphi_\# T|| \le [\mathrm{Lip}(\varphi)]^m \varphi_\# ||T||, \tag{23}$$

so that when φ is an isometric embedding

$$||\varphi_\# T|| = \varphi_\# ||T|| \text{ and } \mathbf{M}(T) = \mathbf{M}(\varphi_\# T). \tag{24}$$

The simplest example of a current is:

Example 2.12. Given a Lebesgue function $h \in L^1(A, \mathbb{Z})$ where $A \subset \mathbb{R}^m$ is Borel, then we can define an m dimensional current in $\mathbb{R}^m$, $[\![h]\!] \in \mathbf{M}_m(\mathbb{R}^m)$, as follows

$$[\![h]\!](f, \pi_1, \ldots, \pi_m) = \int_{A \subset \mathbb{R}^m} h \cdot f \, d\pi_1 \wedge \cdots \wedge d\pi_m. \tag{25}$$

Here the mass measure and mass are

$$||[\![h]\!]|| = |h| \, d\mathcal{L}_m \qquad \mathbf{M}([\![h]\!]) = \int_A |h| \, d\mathcal{L}_m \tag{26}$$

respectively. If one has a bi-Lipschitz map, $\varphi : \mathbb{R}^m \to Z$, then we can define an m dimensional current in Z using the pushforward map

$$\varphi_\# [\![h]\!](f, \pi_1, \ldots, \pi_m) = \int_{A \subset \mathbb{R}^m} (h \circ \varphi)(f \circ \varphi) d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi) \tag{27}$$

where $d(\pi_i \circ \varphi)$ is well defined almost everywhere by Rademacher's Theorem.

In [2][Theorem 4.6] Ambrosio-Kirchheim define the following set associated with any integer rectifiable current:

Definition 2.13. The (canonical) set of a current, T, is the collection of points in Z with positive lower density:

$$\mathrm{set}\,(T) = \{p \in Z : \Theta_{*m}\,(\|T\|, p) > 0\}, \tag{28}$$

where the definition of lower density is

$$\Theta_{*m}\,(\mu, p) = \liminf_{r \to 0} \frac{\mu(B_p(r))}{\omega_m r^m}. \tag{29}$$

In [2] Definition 4.2 and Theorems 4.5-4.6, an integer rectifiable current is defined using the Hausdorff measure, $\mathcal{H}^m$:

Definition 2.14. Let $m \geq 1$. A current, $T \in \mathcal{D}_m(Z)$, is rectifiable if $\mathrm{set}(T)$ is countably $\mathcal{H}^m$ rectifiable and if $||T||(A) = 0$ for any set $A \subset Z$ whose Hausdorff measure is zero, $\mathcal{H}^k(A) = 0$. One writes $T \in \mathcal{R}_m(Z)$.

One says $T \in \mathcal{R}_m(Z)$ is integer rectifiable, denoted $T \in \mathcal{I}_m(Z)$, if for any $\varphi \in \mathrm{Lip}(Z, \mathbb{R}^m)$ and any open set $A \subset Z$, then

$$\exists\, \theta \in \mathcal{L}^1(\mathbb{R}^k, Z) \;\; s.t. \;\; \varphi_\#(T \llcorner A) = [\![\theta]\!] \text{ as in (25).} \tag{30}$$

In fact, $T \in \mathbf{I}_m(Z)$ iff it has a parametrization. A parametrization $(\{\varphi_i\}, \{\theta_i\})$ of an integer rectifiable current $T \in \mathcal{I}^m(Z)$ is a collection of bi-Lipschitz maps $\varphi_i : A_i \to Z$ with $A_i \subset \mathbb{R}^m$ precompact Borel measurable and with pairwise disjoint images and weight functions $\theta_i \in L^1(A_i, \mathbb{N})$ such that

$$T = \sum_{i=1}^{\infty} \varphi_{i\#}[\![\theta_i]\!] \quad \text{and} \quad \mathbf{M}(T) = \sum_{i=1}^{\infty} \mathbf{M}\left(\varphi_{i\#}[\![\theta_i]\!]\right). \tag{31}$$

A 0 dimensional rectifiable current is defined by the existence of countably many distinct points, $\{x_i\} \in Z$, weights $\theta_i \in \mathbb{R}^+$ and orientation, $\sigma_i \in \{-1, +1\}$ such that

$$T(f) = \sum_h \sigma_i \theta_i f(x_i) \qquad \forall f \in \mathcal{B}^\infty(Z). \tag{32}$$

where $\mathcal{B}^\infty(Z)$ is the class of bounded Borel functions on Z and where

$$\mathbf{M}(T) = \sum_h \theta_i < \infty \tag{33}$$

If T is integer rectifiable $\theta_i \in \mathbb{Z}^+$, so the sum must be finite.

In particular, the mass measure of $T \in \mathbf{I}_m(Z)$ satisfies

$$||T|| = \sum_{i=1}^{\infty} ||\varphi_{i\#}[\![\theta_i]\!]||. \tag{34}$$

Theorems 4.3 and 8.8 of [2] provide necessary and sufficient criteria for determining when a current is integer rectifiable.

Note that the current in Example 2.12 is an integer rectifiable current.

Example 2.15. *If one has a Riemannian manifold, M^m, and a bi-Lipschitz map $\varphi : M^m \to Z$, then $T = \varphi_\#[\![1_M]\!]$ is an integer rectifiable current of dimension m in Z. If φ is an isometric embedding, and $Z = M$ then* $\mathbf{M}(T) = \mathrm{Vol}(M^m)$. *Note further that* $\mathrm{set}(T) = \varphi(M)$.

Definition 2.16. [2][Defn 2.3] The *boundary* of $T \in \mathbf{M}_m(Z)$ is defined

$$\partial T(f, \pi_1, \ldots, \pi_{m-1}) := T(1, f, \pi_1, \ldots, \pi_{m-1}) \in M_{m-1}(Z) \tag{35}$$

When $m = 0$, set $\partial T = 0$.

Note that $\varphi_\#(\partial T) = \partial(\varphi_\# T)$.

Definition 2.17. [2][Defn 3.4 and 4.2] An integer rectifiable current $T \in \mathcal{I}_m(Z)$ is called an integral current, denoted $T \in \mathbf{I}_m(Z)$, if ∂T defined as

$$\partial T\,(f, \pi_1, \ldots, \pi_{m-1}) := T\,(1, f, \pi_1, \ldots, \pi_{m-1}) \tag{36}$$

has finite mass. The total mass of an integral current is

$$\mathbf{N}(T) = \mathbf{M}(T) + \mathbf{M}(\partial T). \tag{37}$$

Observe that $\partial\partial T = 0$. In [2] Theorem 8.6, Ambrosio-Kirchheim prove that

$$\partial : \mathbf{I}_m(Z) \to \mathbf{I}_{m-1}(Z) \tag{38}$$

whenever $m \geq 1$.

Recall Definition 2.10 of the push forward of a current. By (23) one can see that if $\varphi : Z_1 \to Z_2$ is Lipschitz, then

$$\varphi_\# : \mathbf{I}_m(Z_1) \to \mathbf{I}_m(Z_2). \tag{39}$$

Recall Definition 2.9 of the restriction of a current. The restriction of an integral current need not be an integral current except in special circumstances. For example, T might be integration over $[0,1]^2$ with the Euclidean metric and $A \subset [0,1]^2$ could have an infinitely long boundary, so that $T \llcorner A \notin \mathbf{I}_2([0,1]^2)$ because $\partial(T \llcorner A)$ has infinite mass. The Ambrosio-Kirchheim Slicing Theorem, presented next, allows one to prove $T \llcorner A$ is an integral current for a large collection of open sets defined using Lipschitz functions. See in particular (44) below.

2.3. Ambrosio-Kirchheim slicing theorem

As in the work of Federer-Fleming, Ambrosio-Kirchheim consider the slices of currents:

Theorem 2.18. **[Ambrosio-Kirchheim]** *[2][Theorems 5.6-5.7] Let Z be a complete metric space, $T \in \mathbf{I}_m Z$ and $f : Z \to \mathbb{R}$ a Lipschitz function. For almost every $s \in \mathbb{R}$ one can define an integral current*

$$< T, f, s > := -\partial \left(T \llcorner f^{-1}(s, \infty) \right) + (\partial T) \llcorner f^{-1}(s, \infty), \tag{40}$$

so that

$$\partial < T, f, s > = < -\partial T, f, s > \tag{41}$$

and $< T_1 + T_2, f, s > = < T_1, f, s > + < T_2, f, s >$. In addition, one can integrate the masses to obtain:

$$\int_{s \in \mathbb{R}} \mathbf{M}(< T, f, s >)\, ds = \mathbf{M}(T \llcorner df) \leq \mathrm{Lip}(f)\, \mathbf{M}(T) \tag{42}$$

where

$$(T \llcorner df)(h, \pi_1, \ldots, \pi_{m-1}) = T(h, f, \pi_1, \ldots, \pi_{m-1}). \tag{43}$$

In particular, for almost every $s > 0$ one has

$$T \llcorner f^{-1}(s, \infty) \in \mathbf{I}_{m-1}(Z)\,. \tag{44}$$

Remark 2.19. Observe that for any $T \in \mathbf{I}_m(Z')$, and any Lipschitz functions, $\varphi : Z \to Z'$ and $f : Z' \to \mathbb{R}$ and any $s > 0$, one has

$$< \varphi_\# T, f, s > = \varphi_\# < T, (f \circ \varphi), s > . \tag{45}$$

2.4. Review of convergence of currents

Ambrosio Kirchheim's Compactness Theorem, which extends Federer-Fleming's Flat Norm Compactness Theorem, is stated in terms of weak convergence of currents. Definition 3.6 of [2] extends Federer-Fleming's notion of weak convergence (except that they do not require compact support):

Definition 2.20. A sequence of integral currents $T_j \in \mathbf{I}_m(Z)$ is said to converge weakly to a current T iff the pointwise limits satisfy

$$\lim_{j \to \infty} T_j (f, \pi_1, \ldots, \pi_m) = T (f, \pi_1, \ldots, \pi_m) \tag{46}$$

for all bounded Lipschitz $f : Z \to \mathbb{R}$ and Lipschitz $\pi_i : Z \to \mathbb{R}$. One writes

$$T_j \to T \tag{47}$$

One sees immediately that $T_j \to T$ implies

$$\partial T_j \to \partial T, \tag{48}$$

and

$$\varphi_\# T_j \to \varphi_\# T. \tag{49}$$

However $T_j \llcorner A$ need not converge weakly to $T \llcorner A$ as seen in the following example:

Example 2.21. Let $Z = \mathbb{R}^2$ with the Euclidean metric. Let $\varphi_j : [0,1] \to Z$ be $\varphi_j(t) = (1/j, t)$ and $\varphi_\infty(t) = (0,t)$. Let $S \in \mathbf{I}_1([0,1])$ be

$$S(f, \pi_1) = \int_0^1 f \, d\pi_1. \tag{50}$$

Let $T_j \in \mathbf{I}_1(Z)$ be defined $T_j = \varphi_{j\#}(S)$. Then $T_j \to T_\infty$. Taking $A = [0,1] \times (0,1)$, one has $T_j \llcorner A = T_j$ but $T_\infty \llcorner A = 0$.

Immediately below the definition of weak convergence [2] Definition 3.6, Ambrosio-Kirchheim prove the lower semicontinuity of mass: *If T_j converges weakly to T, then*

$$\liminf_{j \to \infty} \mathbf{M}(T_j) \geq \mathbf{M}(T). \tag{51}$$

and for any open set, $A \subset Z$,

$$\liminf_{j \to \infty} ||T_j||(A) \geq ||T||(A). \tag{52}$$

Theorem 2.22 (Ambrosio-Kirchheim Compactness). *Given any complete metric space Z, a compact set $K \subset Z$ and $A_0, V_0 > 0$. Given any sequence of integral currents $T_j \in \mathbf{I}_m(Z)$ satisfying*

$$\mathbf{M}(T_j) \leq V_0,\ \mathbf{M}(\partial T_j) \leq A_0 \text{ and } \operatorname{set}(T_j) \subset K, \tag{53}$$

there exists a subsequence, T_{j_i}, and a limit current $T \in \mathbf{I}_m(Z)$ such that T_{j_i} converges weakly to T.

2.5. Review of integral current spaces

The notion of an integral current space was introduced by the author and Stefan Wenger in [39]:

Definition 2.23. An m dimensional metric space (X, d, T) is called an integral current space if it has a integral current structure $T \in \mathbf{I}_m\left(\bar{X}\right)$ where $\bar{X}$ is the metric completion of X and $\mathrm{set}(T) = X$. Also included in the m dimensional integral current spaces is the $\mathbf{0}$ space, denoted $\mathbf{0}$. The integral current structure of the $\mathbf{0}$ space is $T = 0$ and it has an empty metric space.

Note that $\mathrm{set}\,(\partial T) \subset \bar{X}$. The boundary of (X, d, T) is then the integral current space:

$$\partial\,(X, d_X, T) := (\mathrm{set}\,(\partial T)\,, d_{\bar{X}}, \partial T)\,. \tag{54}$$

If $\partial T = 0$ then one says (X, d, T) is an integral current without boundary. The $\mathbf{0}$ space has no boundary.

Definition 2.24. The space of $m \geq 0$ dimensional integral current spaces, $\mathcal{M}^m$, consists of all metric spaces which are integral current spaces with currents of dimension m as in Definition 2.23 as well as the $\mathbf{0}$ spaces. Then $\partial : \mathcal{M}^{m+1} \to \mathcal{M}^m$.

Remark 2.25. Any m dimensional integral current space is countably $\mathcal{H}^m$ rectifiable with oriented charts, φ_i and weights θ_i provided as in (31).

Example 2.26. A compact oriented Riemannian manifold with boundary, M^m, is an integral current space, where $X = M^m$, d is the standard metric on M and T is integration over M. In this case $\mathbf{M}(M) = \mathrm{Vol}(M)$ and ∂M is the boundary manifold. When M has no boundary, $\partial M = 0$.

2.6. Review of the intrinsic flat convergence

Recall that the flat distance between m dimensional integral currents $S, T \in \mathbf{I}_m\,(Z)$ is given by

$$d_F^Z\,(S,T) := \inf\{\mathbf{M}\,(U) + \mathbf{M}\,(V) : S - T = U + \partial V\} \tag{55}$$

where $U \in \mathbf{I}_m(Z)$ and $V \in \mathbf{I}_{m+1}(Z)$. This notion of a flat distance was first introduced by Whitney in [42] for chains and later adapted to rectifiable currents by Federer-Fleming [10]. The flat distance between Ambrosio-Kirchheim's integral currents was studied by Wenger in [40]. In particular, Wenger proved that if $T_j \in \mathbf{I}_m(Z)$ has $\mathbf{M}(T_j) \le V_0$ and $\mathbf{M}(\partial T_j) \le A_0$ then T_j converges weakly to T as currents iff $d_F^Z(T_j, T) \to 0$. A similar result was proven by Federer-Fleming for currents in Euclidean space in [10].

The intrinsic flat distance between integral current spaces was first defined in [39][Defn 1.1]:

Definition 2.27. For $M_1 = (X_1, d_1, T_1)$ and $M_2 = (X_2, d_2, T_2) \in \mathcal{M}^m$ let the intrinsic flat distance be defined:

$$d_{\mathcal{F}}(M_1, M_2) := \inf d_F^Z(\varphi_{1\#}T_1, \varphi_{2\#}T_2), \tag{56}$$

where the infimum is taken over all complete metric spaces (Z, d) and isometric embeddings $\varphi_1 : (\bar{X}_1, d_1) \to (Z, d)$ and $\varphi_2 : (\bar{X}_2, d_2) \to (Z, d)$ and the flat norm d_F^Z is taken in Z. Here $\bar{X}_i$ denotes the metric completion of X_i and d_i is the extension of d_i on $\bar{X}_i$ and $\phi_\# T$ denotes the push forward of T by the map ϕ.

In [39], it is observed that

$$d_{\mathcal{F}}(M_1, M_2) \le d_{\mathcal{F}}(M_1, 0) + d_{\mathcal{F}}(0, M_2) \le \mathbf{M}(M_1) + \mathbf{M}(M_2). \tag{57}$$

There it is also proven that $d_{\mathcal{F}}$ satisfies the triangle inequality [39][Thm 3.2] and is a distance [39][Thm3.27] on the class of precompact integral current spaces up to current preserving isometries:

$$F : X_1 \to X_2 \text{ s.t. } F_\# T_1 = T_2 \text{ and } d_2(F(p), F(q)) = d_1(p, q)\ \forall p, q \in X_1.$$

In particular it is a distance on the class of oriented compact manifolds with boundary of a given dimension.

In [39] Theorem 3.23 it is also proven that

Theorem 2.28. *[39][Thm 3.23] Given a pair of precompact integral current spaces, $M_1^m = (X_1, d_1, T_1)$ and $M_2^m = (X_2, d_2, T_2)$, there exists a compact metric space, (Z, d_Z), integral currents $U \in \mathbf{I}_m(Z)$ and $V \in \mathbf{I}_{m+1}(Z)$, and* *isometric embeddings* $\varphi_1 : \bar{X}_1 \to Z$ *and* $\varphi_2 : \bar{X}_2 \to Z$ *with*

$$\varphi_{1\#}T_1 - \varphi_{2\#}T_2 = U + \partial V \tag{58}$$

such that

$$d_{\mathcal{F}}(M_1, M_2) = \mathbf{M}(U) + \mathbf{M}(V). \tag{59}$$

Remark 2.29. The metric space Z in Theorem 2.28 has

$$\mathrm{Diam}(Z) \le 3\,\mathrm{Diam}(X_1) + 3\,\mathrm{Diam}(X_2). \tag{60}$$

This is seen by consulting the proof of Theorem 3.23 in [39], where Z is constructed as the injective envelope of the Gromov-Hausdorff limit of a sequence of spaces Z_n with this same diameter bound.

The following theorem in [39] is an immediate consequence of Gromov and Ambrosio-Kirchheim's Compactness Theorems:

Theorem 2.30. *Given a sequence of precompact* m *dimensional integral current spaces* $M_j = (X_j, d_j, T_j)$ *such that*

$$\left(\bar{X}_j, d_j\right) \xrightarrow{GH} (Y, d_Y), \ \ \mathbf{M}(M_j) \le V_0 \quad \textit{and} \quad \mathbf{M}(\partial M_j) \le A_0 \tag{61}$$

then a subsequence converges in the intrinsic flat sense

$$(X_{j_i}, d_{j_i}, T_{j_i}) \xrightarrow{\mathcal{F}} (X, d_X, T) \tag{62}$$

where either (X, d_X, T) *is the* $\mathbf{0}$ *current space or* (X, d_X, T) *is an* m *dimensional integral current space with* $X \subset Y$ *with the restricted metric* $d_X = d_Y$*.*

Immediately one notes that if Y has Hausdorff dimension less than m, then $(X, d, T) = \mathbf{0}$. There are many examples of sequences of Riemannian manifolds which have no Gromov-Hausdorff limit but have an intrinsic flat limit. The first is Ilmanen's Example of an increasingly hairy three sphere with positive scalar curvature described in [39] Example A.7.

The following three theorems are proven in work of the author with Wenger [39]. These theorems with the work of Ambrosio-Kirchheim reviewed are key ingredients in the proofs of the theorems in this paper.

Theorem 2.31. *[39][Thm 4.2] If a sequence of integral current spaces has*

$$M_j = (X_j, d_j, T_j) \xrightarrow{\mathcal{F}} M_0 = (X_0, d_0, T_0)\,, \tag{63}$$

then there is a separable complete metric space, Z, and isometric embeddings $\varphi_j : X_j \to Z$ such that

$$d_F^Z(\varphi_{j\#}T_j, \varphi_{0\#}T_0) \to 0 \tag{64}$$

and thus $\varphi_{j\#}T_j$ converges weakly to $\varphi_{0\#}T_0$ as well.

Theorem 2.32. *[39][Thm 4.3] If a sequence of integral current spaces has*

$$M_j = (X_j, d_j, T_j) \xrightarrow{\mathcal{F}} \mathbf{0} \tag{65}$$

then one may choose points $x_j \in X_j$ and a separable complete metric space, Z, and isometric embeddings $\varphi_j : X_j \to Z$ such that $\varphi_j(x_j) = z_0 \in Z$ and

$$d_F^Z(\varphi_{j\#}T_j, 0) \to 0 \tag{66}$$

and thus $\varphi_{j\#}T_j$ converges weakly to 0 in Z as well.

Theorems 2.31 and 2.32 combined with Ambrosio-Kirchheim's lower semicontinuity of mass [c.f. Remark 2.33] imply the following:

Theorem 2.33. *If a sequence of integral current spaces M_j converges in the intrinsic flat sense to an integral current space, M_∞, then*

$$\liminf_{i\to\infty} \mathbf{M}(M_i) \geq \mathbf{M}(M_\infty) \tag{67}$$

Note that Theorems 2.31, 2.32 and 2.33 do not require uniform bounds on the masses or volumes of the M_j and ∂M_j.

2.7. Balls in integral current spaces

Many theorems in Riemannian geometry involve open and closed balls,

$$B(p,r) = \{x \in X : d_X(x,p) < r\} \quad \bar{B}(p,r) = \{x \in X : d_X(x,p) \leq r\}. \tag{68}$$

Here a few basic lemmas are proven about balls in integral current spaces. These lemmas are new but so basic that they are best placed in this background section.

Lemma 2.34. *A ball in an integral current space, $M = (X, d, T)$, with the current restricted from the current structure of M is an integral current space itself,*

$$S(p,r) := (\,\mathrm{set}(T \llcorner B(p,r)), d, T \llcorner B\,(p,r)\,) \tag{69}$$

for almost every $r > 0$. Furthermore,

$$B(p,r) \subset \mathrm{set}(S(p,r)) \subset \bar{B}(p,r) \subset X. \tag{70}$$

Proof. First one shows that $S(p,r) = T \llcorner B(p,r)$ is an integer rectifiable current. Let $\rho_p : \bar{X} \to \mathbb{R}$ be the distance function from p. Then by Ambrosio-Kirchheim's Slicing Theorem, applied to $f(x) = -\rho_p(x)$, one has

$$\partial(T \llcorner B(p,r)) = \partial(T \llcorner \rho_p^{-1}(-\infty, r)) \tag{71}$$
$$= < T, -\rho_p, -r > + (\partial T) \llcorner \rho_p^{-1}(-\infty, r) \tag{72}$$
$$= < T, -\rho_p, -r > + (\partial T) \llcorner B(p,r) \tag{73}$$

where the mass of the slice $< T, \rho_p, r >$ is bounded for almost every r. Thus

$$\mathbf{M}(\partial(T \llcorner B(p,r))) \le \mathbf{M}(< T, -\rho_p, -r >) + \mathbf{M}((\partial T) \llcorner B(p,r)) \tag{74}$$
$$\le \mathbf{M}(< T, -\rho_p, -r >) + \mathbf{M}(\partial T) < \infty. \tag{75}$$

So $T \llcorner B(p,r)$ is an integral current in $\bar{X}$ for almost every r.

Next one proves (70). Recall that $x \in \mathrm{set}(S(p,r)) \subset \bar{X}$ iff

$$0 < \liminf_{s \to 0} \frac{||S(p,r)||(B(x,s))}{\omega_m s^m} \tag{76}$$
$$= \liminf_{s \to 0} \frac{||T||(B(p,r) \cap B(x,s))}{\omega_m s^m} \tag{77}$$

If $x \in B(p,r) \subset X$, then eventually $B(x,s) \subset B(p,r)$ and the liminf is just the lower density of T at x. Since $x \in X = \mathrm{set}(T)$, this lower density is positive. If $x \in \bar{X} \setminus X$, then the liminf is 0 because it is smaller than the density of T at x, which is 0. If $x \notin \bar{B}(p,r)$, then the liminf is 0 because eventually the balls do not intersect. □

One may imagine that it is possible that a ball is cusp shaped and that some points in the closure of the ball that lie in X do not lie in the set of $S(p,r)$. In a manifold, the set of $S(p,r)$ is a closed ball:

Lemma 2.35. *When M is a Riemannian manifold with boundary we have*

$$S\left(p,r\right)=\left(\bar{B}\left(p,r\right),d,T\llcorner B\left(p,r\right)\right) \tag{78}$$

is an integral current space for all $r>0$.

Proof. In this setting,

$$\begin{aligned}
\partial(T\llcorner B(p,r))(f,\pi_1,\dots,\pi_m) &= (T\llcorner B(p,r))(1,f,\pi_1,\dots,\pi_m) &(79)\\
&= T\left(\chi_{B(p,r)},f,\pi_1,\dots,\pi_m\right) &(80)\\
&= \int_M \chi_{B(p,r)} df\wedge d\pi_1\wedge\cdots\wedge d\pi_m &(81)\\
&= \int_{B(p,r)} df\wedge d\pi_1\wedge\cdots\wedge d\pi_m &(82)\\
&= \int_{\partial B(p,r)} f\, d\pi_1\wedge\cdots\wedge d\pi_m &(83)
\end{aligned}$$

Thus for every $r>0$ we have,

$$\mathbf{M}\Big(\partial(T\llcorner B(p,r))\Big)=\mathrm{Vol}_{m-1}\left(\partial B_p(r)\right)<\infty. \tag{84}$$

By (70) we need only show $\partial B(p,r)\subset \mathrm{set}(S(p,r))$. If $d(x,p)=r$, then let $\gamma:[0,r]\to M$ be a curve parametrized by arclength running minimally from x to p. Then

$$B(\gamma(s/2),s/2)\subset B(x,s)\cap B(p,r). \tag{85}$$

and

$$\begin{aligned}
\liminf_{s\to 0}\frac{||S(p,r)||(B(x,s))}{\omega_m s^m} &= \liminf_{s\to 0}\frac{||T||(B(p,r)\cap B(x,s))}{\omega_m s^m} &(86)\\
&\ge \liminf_{s\to 0}\frac{||T||(B(\gamma(s/2),s/2)}{\omega_m s^m} &(87)\\
&\ge \liminf_{s\to 0}\frac{\mathrm{Vol}(B(\gamma(s/2),s/2)}{2^m\omega_m(s/2)^m}\ge\frac{C_{\gamma(s/2)}}{2^m} &(88)
\end{aligned}$$

where $C_q=1/2$ when $q\in\partial M$ and $C_q=1$ when $q\in M\setminus\partial M$. In either case we have a positive liminf and thus $x\in\mathrm{set}(S(p,r))$. □

Example 2.36. There exist integral current spaces with balls that are not integral current spaces.

Proof. Suppose one defines an integral current space, (X, d, T) where $X = S^2$ with the following generalized metric

$$g = dr^2 + \left(\cos(r)/\sqrt{r}\right)^2 d\theta^2 \qquad r \in [-\pi/2, \pi/2]. \tag{89}$$

The metric is defined as

$$d(p_1, p_2) = \inf\left\{L_g(\gamma) : \ \gamma(0) = p_1,\ \gamma(1) = p_2\right\} \tag{90}$$

where

$$L_g(\gamma) = \int_0^1 g(\gamma'(t), \gamma'(t))^{1/2}\, dt \tag{91}$$

as in a Riemannian manifold. In fact this metric space consists of two open isometric Riemannian manifolds diffeomorphic to disks whose metric completions are glued together along corresponding points. The current structure T is defined by

$$T(f, \pi_1, \ldots, \pi_m) = \int_{-\pi/2}^{0} \int_{S^1} f\, d\pi_1 \wedge d\pi_2 + \int_0^{\pi/2} \int_{S^1} f\, d\pi_1 \wedge d\pi_2 \tag{92}$$

so that $\partial T = 0$ and

$$\mathbf{M}(T) = \mathrm{Vol}_m\left(r^{-1}[-\pi/2, 0)\right) + \mathrm{Vol}_m\left(r^{-1}(0, \pi/2]\right) \tag{93}$$

$$= 2\int_0^{\pi/2} 2\pi\left(\cos(r) r^{-1/2}\right) dr \tag{94}$$

$$\leq 4\pi \int_0^{\pi/2} r^{-1/2}\, dr = 8\pi(\pi/2)^{1/2} < \infty. \tag{95}$$

Taking p such that $r(p) = -\pi/2$, then $S(p, \pi/2)$ is a rectifiable current but its boundary does not have finite mass. This can be seen by taking q such that $(r(q), \theta(q)) = (0, 0)$, setting $\pi_1 = \rho_q$ and $f = \rho_p = r + \pi/2$ and observing that

$$\left|\partial(S(p, \pi/2))(f, \pi_1)\right| = \left|S(p, \pi/2)(1, f, \pi_1)\right| \tag{96}$$

$$= \left|\int_{B(p,\pi/2)} df \wedge d\pi_1\right| \tag{97}$$

$$\geq \left|\int_{B(p,\pi/2-\delta)} df \wedge d\pi_1\right| \tag{98}$$

$$= \left| \int_{\partial B(p,\pi/2-\delta)} f \, d\pi_1 \right| \tag{99}$$

$$= \left| \int_{\theta=-\pi}^{\pi} (\pi/2-\delta) \, \frac{d\pi_1}{d\theta} \, d\theta \right| \tag{100}$$

$$= \left| \int_{\theta=-\pi}^{\pi} (\pi/2-\delta) \, \frac{\cos(r)}{r^{1/2}} \, d\theta \right| \tag{101}$$

$$\geq 2\pi(\pi/2-\delta) \, \cos(-\delta) \, \delta^{-1/2} \tag{102}$$

which is unbounded as δ decreases to 0. □

Remark 2.37. Note that the outside of the ball, $(M \setminus B(p,r), d, T - S(p,r))$, is also an integral current space for almost every $r > 0$.

Remark 2.38. In some of the theorems in this paper, it will be important to estimate $d_{\mathcal{F}}(S(p,r), \mathbf{0})$. There are various ways to estimate this value. First observe that

$$d_{\mathcal{F}}\left(S(p,r), \mathbf{0}\right) \leq \min\left\{ \mathbf{M}(S(p,r)), \ \mathbf{M}(\partial(S(p,r))\right\}. \tag{103}$$

In addition, if one finds a comparison integral current space, N, such that

$$d_{\mathcal{F}}(S(p,r), N) < d_{\mathcal{F}}(N, \mathbf{0})/2 \tag{104}$$

then by the triangle inequality

$$d_{\mathcal{F}}(S(p,r), \mathbf{0}) > d_{\mathcal{F}}(N, \mathbf{0})/2. \tag{105}$$

Recall that in joint work with Wenger [39], in joint work with Lakzian [19], and in joint work with Lee [20] various means of estimating the intrinsic flat distance are provided.

3. Converging points and diameters

In this section the limits of points in sequences of integral current spaces that converge in the intrinsic flat sense are examined. See Definitions 3.1 and 3.2 and Lemma 3.4. The diameter is then proven to be lower semicontinuous. See Definition 3.5 and Theorem 3.6. The depth is proven to be semicontinuous. See Definition 3.7 and Theorem 3.8. The section ends with two open questions.

First we define what it means for a sequence of points $x_i \in M_i$ to converge to a point $x_\infty \in M_\infty$. We imitate the definition used for points in sequences of metric spaces with a Gromov-Hausdorff limit. That definition was made rigorous using the Gromov Embedding Theorem which embeds the M_i into a common metric space (cf. Theorem 2.3). Our definition is made rigorous using the embedding theorem for $\mathcal{F}$-converging integral current spaces (cf. Theorem 2.31).

Definition 3.1. If $M_i = (X_i, d_i, T_i) \xrightarrow{\mathcal{F}} M_\infty = (X_\infty, d_\infty, T_\infty)$, then one says $x_i \in X_i$ are a converging sequence that converges to $x_\infty \in \bar{X}_\infty$ if

$$\exists\ Z \text{ and } \exists\ \varphi_i : X_i \to Z \text{ such that } d_F^Z\left(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty\right) \to 0 \tag{106}$$

where Z is a complete metric space and φ_i are isometric embeddings, and

$$\varphi_i(x_i) \to \varphi_\infty(x_\infty). \tag{107}$$

A collection of points, $\{p_{1,i}, p_{2,i}, \ldots, p_{k,i}\}$, converges to a corresponding collection of points, $\{p_{1,\infty}, p_{2,\infty}, \ldots, p_{k,\infty}\}$, if $\varphi_i(p_{j,i}) \to \varphi_\infty(p_{j,\infty})$ for $j \in \{1, \ldots, k\}$.

Note that as in Gromov-Hausdorff convergence (c.f. Definition 2.4), there is the possibility that a constant sequence of points, $x_i = x$, in a constant sequence of spaces, $X_i = X$, can converge to any point $x_\infty \in X$ such that there exists an (orientation preserving) isometry $\Phi : X \to X$ such that $\Phi(x) = x_\infty$. This is a natural consequence of the fact that both the Gromov-Hausdorff distance and the Intrinsic Flat distance are only defined up to (orientation preserving) isometries on Riemannian manifolds. See Remark 3.9.

Unlike in Gromov-Hausdorff convergence, there is a possibility of disappearing sequences of points:

Definition 3.2. If $M_i = (X_i, d_i, T_i) \xrightarrow{\mathcal{F}} M_\infty = (X_\infty, d_\infty, T_\infty)$, then one says $x_i \in X_i$ are Cauchy if one has (106) where

$$\varphi_i(x_i) \to z_\infty \in Z. \tag{108}$$

One says the sequence is disappearing if one has (106) and (108) where $z_\infty \notin \varphi_\infty(X_\infty)$. One says the sequence has no limit in $\bar{X}_\infty$ if one has (106) and (108) where $z_\infty \notin \varphi_\infty(\bar{X}_\infty)$.

Note that in this definition we only require the existence of a sequence of φ_i. A constant sequence of points, $x_i = x$, in a constant sequence of spaces,

$X_i = X$, is always Cauchy in this sense as can be seen by taking $X = Z$ and φ_i to be the identity map. If the space X has a current preserving isometry, $\Psi : X \to X$, then the sequence of points $\varphi_i(\Psi^i(x)) \subset Z$ alternates between x and $\Psi(x)$. So one cannot expect Cauchy sequences to always be embedded as Cauchy sequences in Z for any choice of converging embeddings. This is a natural consequence of the fact that intrinsic flat convergence is only defined up to current preserving isometries. See Remark 3.10.

Remark 3.3. Examples with disappearing splines from [39] demonstrate that there exist Cauchy sequences of points which disappear. In fact z_∞ may not even lie in the metric completion of the limit space, $\varphi_\infty(\bar{X}_\infty)$.

Lemma 3.4. *If a sequence of integral current spaces, $M_i = (X_i, d_i, T_i)$, converges to an integral current space, $M_\infty = (X_\infty, d_\infty, T_\infty)$, in the intrinsic flat sense, then every point x in the limit space X_∞ is the limit of points $x_i \in M_i$. In fact given any sequence of embeddings $\varphi_i : X_i \to Z$ satisfying (106), we can find $x_i \in M_i$ satisfying (107).*

Furthermore, for any such sequence of embeddings, φ_i, there exists a sequence of maps $H_i : X \to X_i$ such that $x_i = H_i(x)$ converges to x in the sense that

$$\lim_{i\to\infty} d_i(H_i(x), H_i(y)) = d_\infty(x, y) \;\; \forall x, y \in X$$

and

$$H_i(x) \in \operatorname{set}(\partial T_i) \;\; \textit{whenever } x \in \operatorname{set}(\partial T).$$

This sequence of maps H_i are not uniquely defined and are not even unique up to isometry.

Proof. By Theorem 2.31 there exists a common metric space Z and isometric embeddings $\varphi_i : X_i \to Z$ such that

$$\varphi_{\infty\#}T_\infty - \varphi_{i\#}T_i = U_i + \partial V_i$$

where $m_i = \mathbf{M}(U_i) + \mathbf{M}(V_i) \to 0$. So $\varphi_{i\#}T_i$ converges in the flat and the weak sense to $\varphi_{\infty\#}T_\infty$. Furthermore $\varphi_{i\#}\partial T_i$ converges in the flat and the weak sense to $\varphi_{\infty\#}\partial T_\infty$

Let ρ_x be the distance function from $\varphi_\infty(x)$. Since $x \in \operatorname{spt}(T_\infty)$, for any $\varepsilon > 0$,

$$||\varphi_{\infty\#}T_\infty||(\rho_x^{-1}[0,\varepsilon)) > 0.$$

By the lower semicontinuity of mass,

$$\liminf_{i\to\infty} ||\varphi_{i\#}T_i|| \left(\rho_x^{-1}[0,\varepsilon)\right) \geq ||\varphi_{\infty\#}T_\infty|| \left(\rho_x^{-1}[0,\varepsilon)\right) > 0. \tag{109}$$

In particular,

$$\exists N'_{\epsilon,x} \in \mathbb{N} \ s.t.\ \varphi_{i\#}T_i \llcorner \left(\rho_x^{-1}[0,\varepsilon)\right) \neq 0 \qquad \forall i \geq N'_{\epsilon,x}. \tag{110}$$

So for all $x \in X$ and any $j \in \mathbb{N}$

$$\exists N_{j,x}\ s.t.\ \exists s_{i,j,x} \in \operatorname{set}(\varphi_{i\#}T_i) \cap B\left(x, 1/j\right) \qquad \forall i \geq N_{j,x}. \tag{111}$$

Without loss of generality, assume $N_{j,x}$ is increasing in j. For $i \in \{1,\ldots,N_{1,x}\}$ take $j_i = 1$. Then for $i \in \{N_{j-1,x}+1,\ldots,N_{j,x}\}$ let $j_i = j$. Thus $i \geq N_{j_i,x}$. Let

$$x_i = \varphi_i^{-1}(s_{i,j_i,x}). \tag{112}$$

Then $\varphi_i(x_i) \in B(x, 1/j_i)$ and $\varphi_i(x_i) \to \varphi(x)$. Note that if $x \in \operatorname{set}(\partial T_\infty)$ then these x_i can be chosen in $\operatorname{set}(\partial T_i)$ using the exact same argument.

Since this process can be completed for any $x \in X_\infty$, one has defined maps $H_i : X_\infty \to X_i$ such that

$$\varphi_i(H_i(x)) \to \varphi_\infty(x). \tag{113}$$

Finally, for all $x, y \in X_\infty$,

$$\begin{aligned} d_i(H_i(x), H_i(y)) &= d_Z(\varphi_i(H_i(x)), \varphi_i(H_i(y))) \\ &\to d_Z(\varphi_\infty(x), \varphi_\infty(y)) = d(x,y). \end{aligned}$$

Now for $x \in \operatorname{set}(\partial T_\infty)$ we may use the fact that

$$d_F^Z(\varphi_{i\#}\partial T_i, \varphi_{\infty\#}\partial T_\infty) \to 0$$

and repeat the proof above to select $H_i(x) \in \operatorname{set}(\partial T_i)$ satisfying (113). □

Definition 3.5. Like any metric space, one can define the diameter of an integral current space, $M = (X, d, T)$, to be

$$\mathrm{Diam}(M) = \sup\{d_X(x,y) :\ x, y \in X\} \in [0,\infty]\}\,.$$

One explicitly defines the diameter of the **0** integral current space to be 0. A space is said to be bounded if the diameter is finite.

Theorem 3.6. *Suppose $M_i \xrightarrow{\mathcal{F}} M$ are integral current spaces then*

$$\mathrm{Diam}(M) \le \liminf_{i\to\infty} \mathrm{Diam}(M_i) \subset [0,\infty] \tag{114}$$

Proof. Note that by the definition, $\mathrm{Diam}(M_i) \ge 0$, so the liminf is always ≥ 0. Thus the inequality is trivial when M is the **0** space. Assuming M is not the **0** space, for any $\epsilon > 0$, there exists $x, y \in X$ such that

$$\mathrm{Diam}(M) \le d(x,y) + \epsilon. \tag{115}$$

By Lemma 3.4, there exists $x_i, y_i \in X_i$ converging to $x, y \in X$ so that

$$\mathrm{Diam}(M) \le \lim_{i\to\infty} d_i(x_i, y_i) + \epsilon \le \liminf_{i\to\infty} \mathrm{Diam}(X_i) + \epsilon. \tag{116}$$

□

The following notion of depth was introduced in joint work of the author with LeFloch [21] as part of an intrinsic flat compactness theorem. In that paper M always has a boundary. Here we extend the definition to include M without boundary as suggested by the referee of this paper.

Definition 3.7. Like any metric space, one can define the depth of an integral current space, $M = (X, d, T)$, to be

$$\mathit{Depth}(M) = \sup\{d_X(x,y) :\ x \in X,\ y \in \mathrm{set}(\partial T)\,\} \in [0,\infty].$$

One explicitly defines the depth of any space with $\partial T = 0$ to be 0.

Theorem 3.8. *Suppose $M_i \xrightarrow{\mathcal{F}} M$ are integral current spaces then*

$$\mathrm{Depth}(M) \le \liminf_{i\to\infty} \mathrm{Depth}(M_i) \subset [0,\infty]$$

Proof. Note that by the definition, $\mathrm{Depth}(M_i) \ge 0$, so the liminf is always ≥ 0. Thus the inequality is trivial when ∂M is the **0** space. Assuming ∂M

is not the **0** space, for any $\epsilon > 0$, there exists $x \in X$ and $y \in \mathrm{set}(\partial T)$ such that

$$\mathrm{Depth}(M) \le d(x,y) + \epsilon. \tag{117}$$

By Lemma 3.4, there exists $x_i = H_i(x) \in X_i$ and $y_i = H_i(y) \in \mathrm{set}(\partial T_i)$ converging to x and y respectively so that

$$\mathrm{Depth}(M) \le \lim_{i\to\infty} d_i(x_i,y_i) + \epsilon \le \liminf_{i\to\infty} \mathrm{Depth}(M_i) + \epsilon. \tag{118}$$

□

Remark 3.9. Suppose one has $M_i = (X_i, d_i, T_i) \xrightarrow{\mathcal{F}} M_\infty = (X_\infty, d_\infty, T_\infty)$, with isometric embeddings $\varphi_i : X_i \to Z$ such that

$$d_F^Z(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty) \to 0 \text{ and } \varphi_i(x_i) \to \varphi_\infty(x_\infty)$$

and also suppose there are isometric embeddings $\varphi_i' : X_i \to Z$ such that

$$d_F^Z(\varphi'_{i\#}T_i, \varphi_{\infty\#}T_\infty) \to 0 \text{ and } \varphi_i'(x_i) \to \varphi'_\infty(x'_\infty).$$

Is there a current preserving isometry $F : X_\infty \to X_\infty$ such that $F(x_\infty) = x'_\infty$? It is possible that one of the Arzela-Ascoli or Bolzano-Weierstrass Theorems proven below or in work of the author with Portegies [29] may be useful towards proving this.

Remark 3.10. Suppose $M_i = (X_i, d_i, T_i) \xrightarrow{\mathcal{F}} M_\infty = (X_\infty, d_\infty, T_\infty)$, and $x_i \in X_i$ is a Cauchy sequence, can one prove that for any sequence of isometric embeddings $\varphi_i' : M_i \to Z$ such that

$$d_F^Z(\varphi'_{i\#}T_i, \varphi'_{\infty\#}T_\infty) \to 0$$

we have a sequence of isometries $F_i : X_i \to X_i$ such that $\varphi_i'(F_i(x_i))$ converges to some point $z \in Z_\infty$? Can one prove that if the sequence is disappearing, then $z \notin \varphi'_\infty(X_\infty)$ and if the sequence has no limit in $\bar{X}_\infty$ then $z \notin \varphi'_\infty(\bar{X}_\infty)$ regardless of the original choice of φ_i'? It is possible that Lemma 4.1 below could be applied to prove this.

4. Convergence of balls and spheres

In this section the following key lemma concerning the convergence of balls and spheres is proven. It is an essential ingredient when trying to prove intrinsic flat limits are not the zero space or that points do not disappear. See Remark 4.2. It will be applied to prove Theorem 7.1, Theorem 8.1, and Example 9.1.

Lemma 4.1. *If $M_j = (X_j, d_j, T_j) \xrightarrow{\mathcal{F}} M_\infty = (X_\infty, d_\infty, T_\infty)$ and $p_j \to p_\infty \in \bar{X}_\infty$, then there exists a subsequence of M_j also denoted M_j such that for almost every $r > 0$,*

$$S(p_j, r) = (\mathrm{set}(T_j \llcorner B(p_j, r)),\, d_j,\, T_j \llcorner B(p_j, r)) \tag{119}$$

are integral current spaces for $j \in \{1, 2, \ldots, \infty\}$ and

$$S(p_j, r) \xrightarrow{\mathcal{F}} S(p_\infty, r). \tag{120}$$

If p_j are Cauchy with no limit in $\bar{X}_\infty$ then there exists $\delta > 0$ such that for almost every $r \in (0, \delta)$ such that $S(p_j, r)$ are integral current spaces for $j \in \{1, 2, \ldots\}$ and

$$S(p_j, r) \xrightarrow{\mathcal{F}} 0. \tag{121}$$

If $M_j \xrightarrow{\mathcal{F}} \mathbf{0}$ then for almost every r and for all sequences p_j one has (121).

Example 4.3 demonstrates why it is necessary to choose a subsequence. Observe that this lemma does not require a uniform upper bound on volume and boundary volume.

Remark 4.2. Some parts of this lemma appeared in a paper by the author and Stefan Wenger in [39]. However a few mathematicians voiced concern that in [39] we did not adequately address the changing basepoints $p_j \neq p_\infty$. Here all details are provided.

Lemma 4.1 is now proven:

Proof. By Theorem 2.31 and 2.32 there exists a common complete metric space, Z, and isometric embeddings, $\varphi_j : X_j \to Z$ and $\varphi_\infty : X_\infty \to Z$, such that

$$\varphi_{j\#} T_j - T = \partial B_j + A_j \tag{122}$$

where $A_j \in \mathbf{I}_m(Z)$ and $B_j \in \mathbf{I}_{m+1}(Z)$ with

$$\mathbf{M}(A_j) + \mathbf{M}(B_j) \to 0 \tag{123}$$

and where

$$T = \varphi_{\infty\#} T_\infty \in \mathbf{I}_m(Z) \text{ when } M_\infty \neq \mathbf{0} \text{ and } T = 0 \text{ when } M_\infty = \mathbf{0}. \tag{124}$$

Since p_j are Cauchy,

$$z_j = \varphi_j(p_j) \to z_\infty \in Z. \tag{125}$$

When $p_j \to p_\infty$ then $z_\infty = \varphi_\infty(p_\infty)$. Then for almost every r

$$(\varphi_{j\#} T_j) \llcorner B(z_j, r) = \varphi_{j\#} T_j \llcorner B(p_j, r). \tag{126}$$

and

$$T \llcorner B(z_\infty, r) = \varphi_{\infty\#} T_\infty \llcorner B(p_\infty, r). \tag{127}$$

If p_j has no limit in $\bar{X}_\infty$, then $z_\infty \notin \varphi_\infty(\bar{X}_\infty)$ and so there exists $\delta > 0$ such that for all $r < \delta$,

$$B(z_\infty, r) \cap \varphi_\infty(\bar{X}_\infty) = 0. \tag{128}$$

So

$$T \llcorner B(z_\infty, r) = 0. \tag{129}$$

If $M_j \xrightarrow{\mathcal{F}} 0$, then one has this as well without requiring $r < \delta$.

So to prove the lemma in all cases one need only show that one can find a subsequence of the M_j also denoted M_j such that for almost every r, the $S(p_j, r)$ are integral current spaces and

$$d_F^Z \left((\varphi_{j\#} T_j) \llcorner \rho_j^{-1}(-\infty, r), T \llcorner \rho_\infty^{-1}(-\infty, r) \right) \to 0 \tag{130}$$

where $\rho_j(z) = d_Z(z_j, z)$.

By Lemma 2.34 for almost every r these are integral current spaces.

Observe that by (122), for almost every r:

$$(\varphi_{j\#} T_j) \llcorner \rho_j^{-1}(-\infty, r) - T \llcorner \rho_j^{-1}(-\infty, r) \tag{131}$$
$$= (\partial B_j) \llcorner \rho_j^{-1}(-\infty, r) + A_j \llcorner \rho_j^{-1}(-\infty, r) \tag{132}$$
$$= < B_j, -\rho_j, -r > + \partial \left(B_j \llcorner \rho_j^{-1}(-\infty, r) \right) \tag{133}$$
$$+ A_j \llcorner \rho_j^{-1}(-\infty, r). \tag{134}$$

Thus $d_F^Z\left(\varphi_{j\#}T_j \llcorner \rho_j^{-1}(-\infty,r), T\llcorner \rho_j^{-1}(-\infty,r)\right) \le$

$$\le f_j(r) + \mathbf{M}(B_j \llcorner \rho_j^{-1}(-\infty,r)) + \mathbf{M}(A_j \llcorner \rho_j^{-1}(-\infty,r)) \tag{135}$$
$$\le f_j(r) + \mathbf{M}(B_j) + \mathbf{M}(A_j) \tag{136}$$

where

$$f_j(r) = \mathbf{M}(< B_j, -\rho_j, -r >). \tag{137}$$

By the Ambrosio-Kirchheim Slicing Theorem

$$\int_{-\infty}^{\infty} f_j(r)\,dr = \int_{-\infty}^{\infty} \mathbf{M}(< B_j, \rho_j, r >)\,dr \tag{138}$$
$$= \mathbf{M}(B_j \llcorner d\rho_j) \le \operatorname{Lip}(\rho_j)\mathbf{M}(B_j) \le \mathbf{M}(B_j) \to 0. \tag{139}$$

Since f_j converge in L^1 to 0, there exists a subsequence, also denoted f_j, such that for almost every $r > 0$, $f_j(r)$ converge to 0 pointwise (c.f. [31] Theorem 3.12).

Thus there is a subsequence such that for almost every $r > 0$

$$\lim_{j\to\infty} d_F^Z\left(\varphi_{j\#}T_j \llcorner \rho_j^{-1}(-\infty,r), T\llcorner \rho_j^{-1}(-\infty,r)\right) = 0. \tag{140}$$

Next observe that the set

$$K = \left(\rho_j^{-1}(-\infty,r) \setminus \rho_\infty^{-1}(-\infty,r)\right) \cup \left(\rho_\infty^{-1}(-\infty,r) \setminus \rho_j^{-1}(-\infty,r)\right) \tag{141}$$

satisfies

$$K \subset \rho_\infty^{-1}(r-\delta_j, r+\delta_j) \tag{142}$$

where

$$\delta_j = d_Z(z_j, z_\infty). \tag{143}$$

Then

$$\begin{aligned}
&d_F^Z\left(T\llcorner \rho_j^{-1}(-\infty,r), T\llcorner \rho_\infty^{-1}(-\infty,r)\right)\\
&\le \mathbf{M}\left(T\llcorner \rho_j^{-1}(-\infty,r) \,-\, T\llcorner \rho_\infty^{-1}(-\infty,r)\right)\\
&\le \mathbf{M}(T\llcorner K)\\
&\le ||T||\left(\rho_\infty^{-1}(r-\delta_j, r+\delta_j)\right)
\end{aligned}$$

Since $\lim_{j\to\infty} \delta_j = 0$, one has

$$\lim_{j\to\infty} ||T|| \left(\rho_\infty^{-1}(r-\delta_j, r+\delta_j)\right) = \lim_{j\to\infty} ||\rho_{\infty\#}T|| \left(\rho_\infty^{-1}(r-\delta_j, r+\delta_j)\right) \tag{144}$$

$$= ||\rho_{\infty\#}T||\{r\} \tag{145}$$

Since $||\rho_{\infty\#}T||$ is a finite measure on $\mathbb{R}$, $||\rho_{\infty\#}T||\{r\} = 0$ except on a countable set of values of r. Thus, for almost every r,

$$\lim_{j\to\infty} d_F^Z(T \llcorner \rho_j^{-1}(-\infty, r), T \llcorner \rho_\infty^{-1}(-\infty, r)) = 0. \tag{146}$$

Combining this with (140) one has (130) and the proof is complete. $\square$

Example 4.3. There exists a sequence of Riemannian manifolds M_j diffeomorphic to a sphere with $vol(M_j) \le V_0$ such that $M_j \xrightarrow{\mathcal{F}} 0$ but there exists a Cauchy sequence $p_j \in M_j$ such that $S(p_j, r)$ does not have an intrinsic flat limit for any $r \in (0, \pi)$.

Proof. Take the metric

$$g_j = dr^2 + f_j^2(r) d\theta^2 \qquad r \in [0, \pi] \tag{147}$$

with $f_j(0) = 0,\ f_j(\pi) = 0,\ f_j'(0) = 1,\ f_j'(\pi) = -1$ so that M_j is a smooth Riemannian manifold. Choose $f_j > 0$ smooth on $(0, \pi)$ such that

$$\int_0^\pi f_j^2(r)\, dr \to 0 \tag{148}$$

and such that

$$f_j(r) > 1 \text{ for } r \in [j \bmod \pi, j + 1/j \bmod \pi] \cap (1/j^2, \pi - 1/j^2) \tag{149}$$

and

$$f_j(r) < 1/j \text{ for } r \in [j + 2/j \bmod \pi, j + 3/j \bmod \pi] \cap (1/j^2, \pi - 1/j^2) \tag{150}$$

and f_j smoothly decreasing in between. Since

$$\mathrm{Vol}(M_j) = 4\pi \int_0^{2\pi} f_j^2(r)\, dr \to 0 \tag{151}$$

one has $M_j \xrightarrow{\mathcal{F}} \mathbf{0}$. Take p_j to be the point where $r = 0$. Suppose one has r' such that the balls converge to the zero integral current space, $S(p_j, r') \xrightarrow{\mathcal{F}} \mathbf{0}$, then the spheres also converge to the zero space, $\partial S(p_j, r') \xrightarrow{\mathcal{F}} \mathbf{0}$.

However there exists a subsequence $j' \to \infty$ such that $r \in [j' \mod \pi, j' + 1/j' \mod \pi]$. On this set $\partial S(p_{j'}, r)$ is bi-Lipschitz close to a circle S^1 endowed with the restricted metric from the disk. So

$$\partial S(p_{j'}, r) \xrightarrow{\mathcal{F}} \left(S^1, d_{D^2}, \int_{S^1}\right). \tag{152}$$

□

Also useful for some applications is the following lemma:

Lemma 4.4. *Let $M_j = (X_j, d_j, T_j)$ be precompact and let $R > 0$. Then one has rescaled integral current spaces, $M_j' = (X_j, d_j/R, T_j)$, one of which may possibly be $\mathbf{0}$, and*

$$d_{\mathcal{F}}(M_1, M_2) \le d_{\mathcal{F}}(M_1', M_2')R^m(1 + R). \tag{153}$$

In particular taking almost any $r = R \in (0, \delta)$ and $p_j \in X_j$ one can rescale

$$S(p_j, r) = (\operatorname{set}(T_j \llcorner B(p_j, r)), d_j, T_j \llcorner B\,(p_j, r)) \tag{154}$$

by r to obtain

$$S'(p_j, 1) = (\operatorname{set}(T_j \llcorner B(p_j, 1)), d_j/R, T_j \llcorner B\,(p_j, r)) \tag{155}$$

and

$$d_{\mathcal{F}}(S(p_1, r), S(p_2, r)) \le d_{\mathcal{F}}(S'(p_1, 1), S'(p_2, 1))r^m(1 + \delta). \tag{156}$$

Proof. By the Theorem 2.28, there exists isometric embeddings $\varphi_j : X_j \to Z$

$$d_Z(\varphi_j(x), \varphi_j(y))/R = d_j(x, y)/R \qquad \forall x, y \in X_j \tag{157}$$

and $A \in \mathbf{I}_m(Z)$, $B \in \mathbf{I}_{m+1}(Z)$ such that

$$\varphi_{1\#}T_1 - \varphi_{2\#}T_2 = A + \partial B \tag{158}$$

and

$$d_{\mathcal{F}}(M_1', M_2') = \mathbf{M}(A) + \mathbf{M}(B) \tag{159}$$

where these masses are defined using d_Z/R. Then $\varphi_j : X_j \to Z$

$$d_Z(\varphi_j(x), \varphi_j(y)) = d_j(x,y) \qquad \forall x, y \in X_j \tag{160}$$

and so by definition of intrinsic flat distance

$$d_{\mathcal{F}}(M_1, M_2) \le \mathbf{M}'(A) + \mathbf{M}'(B) \tag{161}$$

where these masses are defined using d_Z. Thus

$$d_{\mathcal{F}}(M_1, M_2) \le \mathbf{M}(A)R^m + \mathbf{M}(B)R^{m+1} \tag{162}$$
$$\le (\mathbf{M}(A) + \mathbf{M}(B))R^m(1+R) \tag{163}$$
$$\le d_{\mathcal{F}}(M_1', M_2')R^m(1+R). \tag{164}$$

It is easy to see this argument also works when $M_2 = \mathbf{0}$ taking $\varphi_{2\#}T_2 = 0$. □

5. Flat convergence to Gromov-Hausdorff convergence

In this subsection, Theorem 5.1 is proven:

Theorem 5.1. *If a sequence of precompact integral current spaces, $M_i = (X_i, d_i, T_i)$, converges to a nonzero precompact integral current space, $M = (X, d, T)$, in the intrinsic flat sense, then there exists $S_i \in \mathbf{I}_m\left(\bar{X}_i\right)$ such that $N_i = \left(\operatorname{set}\left(S_i\right), d_i\right)$ converges to $\left(\bar{X}, d\right)$ in the Gromov-Hausdorff sense*

$$d_{GH}(N_i, M) \to 0 \tag{165}$$

and

$$\liminf_{i\to\infty} \mathbf{M}(S_i) \ge \mathbf{M}(M). \tag{166}$$

When the M_i are Riemannian manifolds, the N_i can be taken to be settled completions of open submanifolds of M_i.

Remark 5.2. If in addition it is assumed that $\lim_{i\to\infty}\mathbf{M}(M_i) = \mathbf{M}(M)$, then by (166),

$$\lim_{i\to\infty} \mathbf{M}(\operatorname{set}(T_i - S_i), d_i, T_i - S_i) = 0. \tag{167}$$

In the Riemannian setting,

$$\lim_{i\to\infty} \operatorname{Vol}(M_i \setminus N_i) = 0. \tag{168}$$

Remark 5.3. In Ilmanen's example [39] of a sphere with increasingly many splines, the S_i may be chosen to be integration over the spherical part of M_i with balls around the tips removed. Then set(S_i) are manifolds with boundary converging to the sphere in the Gromov-Hausdorff and intrinsic flat sense.

Remark 5.4. The precompactness of the limit integral current spaces is necessary in this theorem because a noncompact limit space can never be the Gromov-Hausdorff limit of precompact spaces. In fact there are sequences of compact Riemannian manifolds, M_j, whose intrinsic flat limit is an unbounded complete Riemannian manifold of finite volume [39][Ex A.10] and another example of such spaces whose Intrinsic Flat limit is a bounded noncompact integral current space [39][Ex A.11].

Remark 5.5. Gromov's Compactness Theorem combined with Theorem 5.1 implies that that any sequence of $x_i \in N_i \subset M_i$ has a subsequence converging to a point x in the metric completion of M. Other points need not have limit points, as can be seen when the tips of thin splines disappear in the examples from [39]. A more general Bolzano-Weierstrass Theorem precisely identifying those points which do not disappear is proven later in Section 7 and in joint work with Portegies appearing in [29].

Theorem 5.1 is now proven:

Proof. By Theorem 2.31 there exists a common metric space Z and isometric embeddings $\varphi_i : X_i \to Z$ and $\varphi : X \to Z$ such that

$$\varphi_{\#}T - \varphi_{i\#}T_i = U_i + \partial V_i \tag{169}$$

where $m_i = \mathbf{M}(U_i) + \mathbf{M}(V_i) \to 0$. So $\varphi_{i\#}T_i$ converges in the flat and thus the weak sense to $\varphi_{\#}T$.

Since M is precompact, $\varphi(X)$ is precompact. Let $\rho : Z \to \mathbb{R}$ be the distance function from $\varphi(X)$.

By the Ambrosio-Kirchheim Slicing Theorem [Theorem 2.18] applied to $f(s) = -\rho(s)$, one has

$$S_{i,\epsilon} := \varphi_{i\#}T_i \llcorner \rho^{-1}([0,\epsilon)) \in \mathbf{I}_m(Z) \tag{170}$$

for almost every $\epsilon > 0$. Fix any such ϵ.

Before choosing the S_i mentioned in the statement of the theorem, one may examine the mass of $S_{i,\epsilon}$ and the Hausdorff distance between set$(S_{i\epsilon})$

and $\varphi(X)$. Note that $\varphi_{\#}T = \varphi_{\#}T \llcorner \rho^{-1}[0,\epsilon)$. So

$$||T||(\rho^{-1}[0,\epsilon)) = \mathbf{M}(T). \tag{171}$$

By lower semicontinuity of mass one has

$$\liminf_{i\to\infty} ||\varphi_{i\#}T_i||(\rho^{-1}[0,\epsilon)) \geq ||\varphi_{\#}T||(\rho^{-1}[0,\epsilon)). \tag{172}$$

Combining this with (170) and (171) and the definition of liminf one has:

$$\text{for a.e. } \epsilon > 0\ \exists N'_\epsilon \in \mathbb{N} \text{ such that } \mathbf{M}(S_{i,\epsilon}) \geq \mathbf{M}(T) - \epsilon\ \ \forall i \geq N'_\epsilon. \tag{173}$$

To see that the Hausdorff distance between $S_{i,\epsilon}$ and $\varphi(X)$ is small,

$$d_H^Z(S_{i,\epsilon}, \varphi(X)) < 2\epsilon,$$

first immediately observe that

$$\mathrm{set}(S_{i,\epsilon}) \subset \bar{T}_\epsilon(\varphi(X)) \subset T_{2\epsilon}(\varphi(X)). \tag{174}$$

One needs only show

$$\varphi\left(X\right) \subset T_{2\epsilon}\left(\mathrm{set}\left(S_{i,\epsilon}\right)\right) \qquad \forall i \geq N_\epsilon. \tag{175}$$

To prove (175), first note that for any $x \in X$, one can let ρ_x be the distance function from $\varphi\left(x\right)$. By the lower semicontinuity of mass of open sets one has,

$$\liminf_{i\to\infty} ||\varphi_{i\#}T_i||(\rho_x^{-1}[0,\epsilon)) \geq ||\varphi_{\#}T||(\rho_x^{-1}[0,\epsilon)) > 0\ \ \forall \epsilon > 0. \tag{176}$$

Thus one has

$$\text{for a.e. } \epsilon > 0\ \exists N_{\epsilon,x} \geq N'_\epsilon\ s.t.\ \varphi_{i\#}T_i \llcorner \rho_x^{-1}[0,\epsilon) \neq 0\ \ \forall i \geq N_{\epsilon,x}. \tag{177}$$

Recall N'_ϵ was defined in (173). Combining this with (170), and the fact that

$$\rho_x^{-1}[0,\epsilon) = B(x,\epsilon) \subset \rho^{-1}[0,\epsilon) = T_\epsilon(\varphi(X)) \tag{178}$$

one has

$$\forall x \in X \quad \text{for a.e. } \epsilon > 0\ \exists N_{\epsilon,x} \geq N'_\epsilon \text{ and } s_{i,\epsilon,x} \in \mathrm{set}(S_i) \cap B(\varphi(x),\epsilon). \tag{179}$$

By the precompactness of X, there is a finite ϵ net, $X_\epsilon = \{x_1, \dots, x_N\}$ on $\varphi(X)$ (i.e. the union of $B(x_i, \epsilon)$ contains X_ϵ). Define

$$N_\epsilon = \max\left\{N_{\epsilon,x_j} : x_j \in X_\epsilon\right\} \geq N'_\epsilon \tag{180}$$

then taking $s_x := s_{i,\epsilon,x_j} \in \operatorname{set}(S_{i,\epsilon})$ we have

$$\forall x \in X\, \exists x_j \in X_\epsilon \; s.t.\; \forall i \geq N_\epsilon \; \exists s_x \in \operatorname{set}(S_{i,\epsilon}) \text{ s.t. } d_Z(s_x, \varphi(x)) < 2\epsilon. \tag{181}$$

So (175) has been proven.

Combining (175) with (174), the Hausdorff distance satisfies

$$d_H^Z\left(\operatorname{set}(S_{i,\epsilon}), \varphi(X)\right) \leq 2\epsilon \qquad \forall i \geq N_\epsilon. \tag{182}$$

Recall the definition of S_i as in the statement of the theorem. One must prove (165) and (166).

Let $\epsilon_k \to 0$ be a decreasing sequence of ϵ for which all these currents are defined. Let $N_k := N_{\epsilon_k}$. Let

$$S_i = T_i \in \mathbf{I}_m(X_i) \text{ for } i = 1 \text{ to } N_1 \tag{183}$$

$$S_i = \varphi_{i\#}^{-1} S_{i,\epsilon_1} \in \mathbf{I}_m(X_i) \text{ for } i = N_1 + 1 \text{ to } N_2 \tag{184}$$

and so on:

$$S_i = \varphi_{i\#}^{-1} S_{i,\epsilon_j} \in \mathbf{I}_m(X_i) \text{ for } i = N_j + 1 \text{ to } N_{j+1} \tag{185}$$

Then by (182),

$$d_H^Z\left(\varphi_i(\operatorname{set}(S_i)), \varphi(X)\right) \leq 2\epsilon_i. \tag{186}$$

This implies (165).

By (173) and $N_k = N_{\epsilon_k} \geq N'_{\epsilon_k}$ one has, one has

$$\mathbf{M}(S_i) \geq \mathbf{M}(T) - \epsilon_i \tag{187}$$

which gives us (166) and completes the proof of the theorem. □

Remark 5.6. One could construct a common metric space Z for Examples A.10 and A.11 of [39] and find $S_{i,\epsilon}$ as in the above proof satisfying (174). However, in that example, (175) will fail to hold. This is where the precompactness of the limit space is essential in the proof.

Remark 5.7. Examples in [39] demonstrate that the metric space of a current space need not be a length space. In general, when a sequence of Riemannian manifolds converges in the intrinsic flat sense to an integral current space it need not be a geodesic length space. If the set(S_i) are length spaces or approximate length spaces, then the limit current space is in fact a length space. This occurs for example in Ilmanen's example of [39]. It also occurs whenever the Gromov-Hausdorff limits and flat limits of length spaces agree. It might be interesting to develop a notion of an approximate length space that suffices to give a geodesic limit space. What properties must hold on M_i to guarantee that their limit is a geodesic length space?

Remark 5.8. It is not immediately clear whether the integral current spaces, N_i, constructed in the proof of Theorem 5.1 actually converge in the intrinsic flat sense to M. One expects an extra assumption on total mass would be needed to interchange between flat and weak convergence, but even so it is not completely clear. One would need to uniformly control the masses of ∂N_i using a common upper bound on $\mathbf{M}(N)$ which can be done using theorems in Section 5 of [2], but is highly technical. It is only worth investigating if one has an application in mind.

6. Arzela-Ascoli theorem for equicontinuous functions

In this section we prove Theorems 6.1 and 6.2. See also Remark 6.3.

Theorem 6.1. *Fix $K > 0$. Suppose $M_i = (X_i, d_i, T_i)$ are integral current spaces for $i \in \{1, 2, \dots, \infty\}$ and $M_i \xrightarrow{\mathcal{F}} M_\infty$ and $F_i : X_i \to W$ are Lipschitz maps into a compact metric space W with*

$$\mathrm{Lip}(F_i) \le K, \tag{188}$$

then a subsequence converges to a Lipschitz map $F_\infty : X_\infty \to W$ with

$$\mathrm{Lip}(F_\infty) \le K. \tag{189}$$

More specifically, there exists isometric embeddings of the subsequence, $\varphi_i : X_i \to Z$, such that $d_F^Z(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty) \to 0$ and for any sequence $x_i \in X_i$ converging to $x \in X_\infty$ as in

$$d_Z(\varphi_i(x_i), \varphi_\infty(x)) \to 0, \tag{190}$$

one has converging images,

$$d_W(F_i(x_i), F_\infty(x)) \to 0. \tag{191}$$

This theorem is an immediate consequence of the following more general theorem which is proven using only Theorem 2.31 and Lemma 3.4.

Theorem 6.2. *Suppose $M_i = (X_i, d_i, T_i)$ are integral current spaces with $M_i \xrightarrow{\mathcal{F}} M_\infty$, suppose Y_i are compact metric spaces where $Y_i \xrightarrow{GH} Y_\infty$, and suppose $f_i : X_i \to Y_i$ are equicontinuous maps satisfying,*

$$d_{X_j}(x, x') < \delta \implies d_{Y_j}(f_j(x), f_j(x')) \le \epsilon(\delta) \tag{192}$$

for some function such that $\lim_{\delta\to 0} \epsilon(\delta) = 0$. Then a subsequence converges to a map $f_\infty : X_\infty \to Y_\infty$ satisfying (192) with the same function $\epsilon(\delta)$. More specifically, there exists isometric embeddings of the subsequence, $\varphi_i : X_i \to Z$, and $\psi_i : Y_i \to W$ such that

$$d_F^Z\Big(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty\Big) \to 0 \tag{193}$$

and

$$d_H^W\Big(\psi_i(Y_i), \psi_\infty(Y_\infty)\Big) \to 0 \tag{194}$$

such that

$$d_Z\Big(\varphi_i(x_i), \varphi_\infty(x)\Big) \to 0, \tag{195}$$

implies

$$d_W\Big(\psi_i(f_i(x_i)), \psi_\infty(f_\infty(x))\Big) \to 0. \tag{196}$$

Proof. By Theorem 2.31, $\exists\, \varphi_i : X_i \to Z$ satisfying (193). By Gromov's Embedding Theorem (cf. Theorem 2.3), $\exists\, \psi_i : Y_i \to W$ with W compact such that (194) holds. Take any $p_\infty \in X_\infty$. By Lemma 3.4, there exists $p_i \in X_i$ such that $\lim_{i\to\infty} \varphi_i(p_i) = \varphi_\infty(p_\infty)$. Since $\psi_i(f_i(p_i)) \in W$ and W is compact, there is a subsequence which converges to some $w \in W$.

We claim $w_\infty \in \psi_\infty(Y_\infty)$. If not, then there exists $r > 0$ such that

$$B(w_\infty, r) \cap \psi_\infty(Y_\infty) = \emptyset. \tag{197}$$

Then for j large

$$B(\psi_j(f_j(p_j)), r/2) \cap \psi_\infty(Y_\infty) = \emptyset \tag{198}$$

which implies that

$$d_H(\psi_j(Y_j), \psi_\infty(Y_\infty)) \geq r/2, \tag{199}$$

which is a contradiction.

Thus we have a point we call $f_\infty(p_\infty) \in Y_\infty$ such that $\psi_\infty(f_\infty(p_\infty)) = w_\infty$. Applying a diagonalization process to choose a subsequence, we have thus defined $f_\infty : X_0 \subset X_\infty \to Y_\infty$ satisfying (192). Extending this function continuously to $f_\infty : X_\infty \to Y_\infty$, it still satisfies (192).

To see that (195) implies (196), consider $x_j \in X_j$ satisfying (195). Taking $p_\infty = x_\infty$ as in the top of the proof, there exists $p_j \in X_j$ such that $\varphi_j(p_j) \to \varphi_\infty(p_\infty)$ and $\psi_j(f_j(p_j)) \to \psi_\infty(f_\infty(p_\infty))$. Observe that

$$\begin{aligned} d_{X_j}(x_j, p_j) &= d_Z(\varphi_j(x_j), \varphi_j(p_j)) && (200)\\ &\leq d_Z(\varphi_j(x_j), \varphi_\infty(x_\infty)) + d_Z(\varphi_\infty(x_\infty), \varphi_j(p_j)) \to 0. && (201) \end{aligned}$$

For any $\varepsilon > 0$ take δ_ε sufficiently small that $\epsilon(\delta_\varepsilon) < \varepsilon$ and j sufficiently large that $d_X(x_j, p_j) < \delta_\varepsilon$ of (192), then

$$\begin{aligned} D &= \lim_{j\to\infty} d_W\Big(\psi_j(f_j(x_j)), \psi_\infty(f_\infty(x_\infty))\Big) && (202)\\ &\leq \lim_{j\to\infty} d_W\Big(\psi_j(f_j(x_j)), \psi_\infty(f_j(p_j))\Big) && (203)\\ &\quad + d_W\Big(\psi_j(f_j(p_j)), \psi_\infty(f_\infty(x_\infty))\Big) \\ &= \lim_{j\to\infty} d_{Y_j}\Big(f_j(x_j), f_j(p_j)\Big) + d_W\Big(\psi_j(f_j(p_j)), \psi_\infty(f_\infty(p_\infty))\Big) && (204)\\ &< \varepsilon + 0 \qquad \forall \varepsilon > 0. && (205) \end{aligned}$$

Thus $\psi_j(f_j(x_j)) \to \psi_\infty(f_\infty(x_\infty))$. □

Remark 6.3. If we allow both $X_i \xrightarrow{\mathcal{F}} X$ and $Y_i \xrightarrow{\mathcal{F}} Y$ in the above theorem statements, then they are false. For example, one may have a sequence of compact connected manifolds, Y_i, which converge in the intrinsic flat sense to a compact metric space, Y, that is not connected [39]. In that setting one has a sequence of Lipschitz maps which are unit speed geodesics, $F_i : [0,1] \to Y_i$ where $Y_i \xrightarrow{\mathcal{F}} Y$ with no limiting function $F : [0,1] \to Y$. One key step in the proof above where we used the Gromov-Hausdorff convergence of the target spaces was in (199). It is also crucial that the Gromov Embedding Theorem produces a common compact metric space W. The Ilmanen Example in [39] shows that $\mathcal{F}$ converging sequences may be impossible to embed in a common compact metric space.

7. Basic Bolzano-Weierstrass theorem

In this section, Theorem 7.1 is proven. Recall Lemma 2.34 states that for almost every radius $S(p,r)$ of (69) is an integral current space. Recall also that, like any integral current space, $d_{\mathcal{F}}(S(p,r),\mathbf{0}) = 0$ iff $S(p,r) = \mathbf{0}$ [38]. If one considers a sequence of integral current spaces, M_i with points p_i, then for almost every r, $S(p_i,r)$ is an integral current space for all i in the sequence. In this basic Bolzano-Weierstrass Theorem one assumes these $S(p_i,r)$ are kept a definite distance away from $\mathbf{0}$ where this distance depends upon on the radius. A different Bolzano-Weierstrass Theorem which involves the Gromov Filling Volume appears in [29].

Theorem 7.1. *Suppose $M_i^m = (X_i, d_i, T_i)$ are integral current spaces which converge in the intrinsic flat sense to a nonzero integral current space $M_\infty^m = (X_\infty, d_\infty, T_\infty)$. Suppose there exists $r_0 > 0$, a positive function $h : (0,r_0) \to (0,r_0)$, and a sequence $p_i \in M_i$ such that for almost every $r \in (0,r_0)$*

$$\liminf_{i\to\infty} d_{\mathcal{F}}(S(p_i,r),0) \geq h(r) > 0. \tag{206}$$

Then there exists a subsequence, also denoted M_i, such that p_i converges to $p_\infty \in \bar{X}_\infty$.

Remark 7.2. Note that M_i and M_∞ are not required to be precompact. The M_i are not required to have uniformly bounded mass or volume. The key hypothesis is that the $M_i \xrightarrow{\mathcal{F}} M_\infty$ and that M_∞ has finite mass. For this reason there is not enough room to fit too many balls of mass $h(r)$ in M_∞. This allows us to produce a converging subsequence in the style of a classical Bolzano-Weierstrass Theorem.

Remark 7.3. It is possible that $p_\infty \notin X_\infty$ as can be seen by taking all the $M_i = M_\infty$ a manifold M with a cusp singularity at p_∞ so that $M_\infty = M \setminus p_\infty$ and p_i a sequence of points approaching p_∞.

Proof. By Theorem 2.31 there exists a common metric space Z and isometric embeddings $\varphi_j : X_j \to Z$ and $\varphi_\infty : X_\infty \to Z$ such that

$$\varphi_{j\#}T_j - \varphi_{\infty\#}T = \partial B_j + A_j \tag{207}$$

where $A_j \in \mathbf{I}_m(Z)$ and $B_j \in \mathbf{I}_{m+1}(Z)$ with

$$\mathbf{M}(A_j) + \mathbf{M}(B_j) \to 0 \tag{208}$$

and where

$$T = \varphi_{\infty\#}T_\infty \in \mathbf{I}_m(Z). \tag{209}$$

One needs only show that a subsequence of $\varphi_i(p_i)$ is a Cauchy sequence. Once this is done, one can apply Lemma 4.1 to the subsequence. In that lemma, it is shown that a Cauchy sequence, p_i, converges to $p_\infty \in \bar{X}_\infty$ unless there is a radius r sufficiently small that $S(p_i, r) \xrightarrow{\mathcal{F}} \mathbf{0}$. Since this is not allowed by the hypothesis of the theorem being proven, one sees that the subsequence converges to $p_\infty \in \bar{X}_\infty$ as desired.

So one needs only prove that a subsequence $\varphi_i(p_i)$ converges in Z. This is not immediate because Z is only complete and need not be compact.

Assume on the contrary that

$$\exists \delta > 0 \;\; s.t. \;\; d_Z(\varphi_i(p_i), \varphi_j(p_j)) \geq \delta \;\; \forall i, j \in \mathbb{N}. \tag{210}$$

Let $\rho_i(x) = d_Z(\varphi_i(p_i), x)$, then for almost every $r \in (0, r_0) \cap (0, \delta/2)$,

$$\rho_i^{-1}(-\infty, r) \cap \rho_j^{-1}(-\infty, r) = \emptyset \qquad \forall i, j \in \mathbb{N}. \tag{211}$$

Now

$$(\varphi_{i\#}T_i) \llcorner \rho_j^{-1}(-\infty, r) - \varphi_{\infty\#}T_\infty \llcorner \rho_j^{-1}(-\infty, r) \tag{212}$$
$$= (\partial B_i) \llcorner \rho_j^{-1}(-\infty, r) + A_i \llcorner \rho_j^{-1}(-\infty, r) \tag{213}$$
$$= < B_i, \rho_j, r > + \partial \left(B_i \llcorner \rho_j^{-1}(-\infty, r) \right) \tag{214}$$
$$+ A_i \llcorner \rho_j^{-1}(-\infty, r). \tag{215}$$

Thus $d_F^Z \left(\varphi_{i\#}T_i \llcorner \rho_j^{-1}(-\infty, r), \varphi_{\infty\#}T_\infty \llcorner \rho_j^{-1}(-\infty, r) \right) \leq$

$$\leq f_{ij}(r) + \mathbf{M}(B_i \llcorner \rho_j^{-1}(-\infty, r)) + \mathbf{M}(A_i \llcorner \rho_j^{-1}(-\infty, r)) \tag{216}$$
$$\leq f_{ij}(r) + \mathbf{M}(B_i) + \mathbf{M}(A_i) \tag{217}$$

where

$$f_{ij}(r) = \mathbf{M}(< B_i, \rho_j, r >). \tag{218}$$

By the Ambrosio-Kirchheim Slicing Theorem, for fixed $j \in \mathbb{N}$,

$$\int_{-\infty}^{\infty} f_{ij}(r)\, dr = \int_{-\infty}^{\infty} \mathbf{M}(< B_i, \rho_j, r >)\, dr \tag{219}$$

$$= \mathbf{M}(B_i \llcorner d\rho_j) \le Lip(\rho_j)\mathbf{M}(B_i) \le \mathbf{M}(B_i) \tag{220}$$

which converges to 0 as $i \to \infty$. Thus for fixed j and almost every r there is a subsequence $i' \to \infty$ such that $\lim_{i' \to \infty} f_{i'j}(r) = 0$ pointwise. Diagonalizing, there is a subsequence i" such that for all j, $\lim_{i' \to \infty} f_{i''j}(r) = 0$ pointwise.

Thus for almost every $r \in (0, r_0) \cap (0, \delta/2)$, there is a subsequence i'' such that for all $j \in \mathbb{N}$,

$$d_F^Z\left(\varphi_{i''\#}T_{i''} \llcorner \rho_j^{-1}(-\infty, r), \varphi_{\infty\#}T_\infty \llcorner \rho_j^{-1}(-\infty, r)\right) \to 0 \tag{221}$$

Since the balls are disjoint,

$$\mathbf{M}(T_\infty) \ge \sum_{j=1}^{N} \mathbf{M}\left(\varphi_{\infty\#}T_\infty \llcorner \rho_j^{-1}(-\infty, r)\right). \tag{222}$$

Thus

$$\limsup_{j \to \infty} \mathbf{M}\left(\varphi_{\infty\#}T_\infty \llcorner \rho_j^{-1}(-\infty, r)\right) = 0. \tag{223}$$

So

$$\limsup_{j \to \infty} d_F^Z\left(\varphi_{\infty\#}T_\infty \llcorner \rho_j^{-1}(-\infty, r), \mathbf{0}\right) = 0. \tag{224}$$

In particular, for j sufficiently large

$$d_F^Z\left(\varphi_{\infty\#}T_\infty \llcorner \rho_j^{-1}(-\infty, r), \mathbf{0}\right) < h(r)/2. \tag{225}$$

Combining this with (221), for i" sufficiently large

$$d_{\mathcal{F}}\left(S(p_{i"}, r), \mathbf{0}\right) \le d_F^Z\left(\varphi_{i"\#}T_{i"} \llcorner \rho_j^{-1}(-\infty, r), \mathbf{0}\right) < h(r)/2 \tag{226}$$

which contradicts the hypothesis. Thus there is a subsequence $\varphi_i(p_i)$ which converges to some point $z_\infty \in Z$ exactly as needed. $\square$

8. Limits of uniformly local isometries

In this section we prove an Arzela-Ascoli Theorem which allows both the domain and the target spaces to converge in the intrinsic flat sense. This theorem applies to sequences of oriented Riemannian manifolds M_i with

$$\mathrm{Vol}(M_i) \le V_i \quad \text{and} \quad \mathrm{Vol}(\partial M_i) \le A_i \tag{227}$$

and functions $F_i : M_i \to M_i'$ which are orientation preserving local isometries that are isometries on balls of a fixed radius, $\delta > 0$ which is uniform for the sequence.

Theorem 8.1. *Let $M_i = (X_i, d_i, T_i)$ and $M_i' = (X_i', d_i', T_i')$ be integral current spaces such that*

$$M_i \xrightarrow{\mathcal{F}} M_\infty \quad \textit{and} \quad M_i' \xrightarrow{\mathcal{F}} M_\infty'. \tag{228}$$

Fix $\delta > 0$. Let $F_i : M_i \to M_i'$ be continuous maps which are current preserving isometries on balls of radius δ in the sense that:

$$\forall x \in X_i,\ F_i : \bar{B}(x,\delta) \to \bar{B}(F_i(x),\delta) \textit{ is an isometry} \tag{229}$$

and

$$F_{i\#}(T_i \llcorner B(x,r)) = T_i' \llcorner B(F(x),r) \textit{ for almost every } r \in (0,\delta). \tag{230}$$

Then, when $M_\infty \neq \mathbf{0}$, one has $M_\infty' \neq \mathbf{0}$ and there is a subsequence, also denoted F_i, which converges to a (surjective) local isometry

$$F_\infty : \bar{X}_\infty \to \bar{X}_\infty'. \tag{231}$$

To be more precise, there exists isometric embeddings of the subsequence $\varphi_i : X_i \to Z$ and $\varphi_i' : X_i' \to Z'$, such that

$$d_F^Z(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty) \to 0 \quad \textit{and} \quad d_F^{Z'}(\varphi_{i\#}'T_i', \varphi_{\infty\#}'T_\infty') \to 0 \tag{232}$$

such that for any sequence $x_i \in X_i$ converging to $x \in X_\infty$ as in

$$\lim_{i\to\infty} \varphi_i(x_i) = \varphi_\infty(x) \in Z, \tag{233}$$

one has

$$\lim_{i\to\infty} \varphi_i'(F_i(x_i)) = \varphi_\infty'(F_\infty(x_\infty)) \in Z'. \tag{234}$$

When $M_\infty = \mathbf{0}$ and F_i are surjective, one has $M'_\infty = \mathbf{0}$.

Remark 8.2. Example 8.5 describes the necessity of the uniformity condition (229) in Theorem 8.1.

Remark 8.3. It may be possible to prove that the limit map here is also current preserving on balls of radius less than δ. This is technical and not needed for present applications but might be an interesting investigation in the future.

Remark 8.4. It may be possible to prove a similar theorem replacing the surjective uniformly local isometries with surjective uniformly local uniformly bi-Lipschitz maps but the proof would be fairly technical and there is no immediate application for this at this time.

Theorem 8.1 is now proven:

Proof. By Theorem 2.31 there exists $\varphi_i : M_i \to Z$ such that

$$d_F^Z(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty) \to 0 \tag{235}$$

and $\varphi'_i : M'_i \to Z'$ such that

$$d_F^{Z'}(\varphi'_{i\#}T'_i, \varphi'_{\infty\#}T'_\infty) \to 0. \tag{236}$$

Assuming $M'_\infty \neq \mathbf{0}$, one must first find a subsequence and construct the limit function $F_\infty : P \to X'_\infty$ satisfying (234) for all $p \in P$ where P is a countably dense collection of points in X_∞.

Take any $p \in P$. Recall $S(p,r) = (\mathrm{set}(T_\infty \llcorner B(p,r)), d_\infty, T_\infty \llcorner B(p,r))$ is defined for almost every r. Since $p \in X_\infty$, and $X_\infty = \mathrm{set}(T_\infty)$,

$$\liminf_{r\to 0} \mathbf{M}(S(p,r))/r^m = \liminf_{r\to 0} ||T_\infty||(B(p,r))/r^m > 0. \tag{237}$$

In particular

$$S(p,r) \neq \mathbf{0}. \tag{238}$$

By Lemma 3.4 there exists $p_i \in X_i$ such that

$$\lim_{i\to\infty} \varphi_i(p_i) = \varphi_\infty(p). \tag{239}$$

By Lemma 4.1, for almost every $r_\infty > 0$, there is a subsequence (also denoted i) such that

$$d_{\mathcal{F}}\left(S(p_i, r_\infty), S(p, r_\infty)\right) \to 0. \tag{240}$$

Taking $r_\infty < \delta$, applying (230) one has

$$F_{i\#}S(p_i, r_\infty) = S(p_i', r_\infty) \text{ where } p_i' = F_i(p_i) \tag{241}$$

so

$$d_{\mathcal{F}}\big(S(p_i', r_\infty), S(p, r_\infty)\big) \to 0. \tag{242}$$

Combining via the triangle inequality with (238),

$$\liminf_{i\to\infty} d_{\mathcal{F}}\big(S(p_i', r_\infty), \mathbf{0}\big) > 0. \tag{243}$$

Thus applying the basic Bolzano-Weierstrass Theorem [Theorem 7.1] to $S(p_i', r_\infty)$, one sees that there is a $p_\infty \in \bar{X}'_\infty$ and a further subsequence (also denoted i) such that $p_i' \to p'_\infty$ in the sense that

$$\varphi_i'(p_i') \to \varphi'_\infty(p'_\infty) \in Z'. \tag{244}$$

Define $F_\infty(p) = p_\infty$.

Repeat this process to choose subsequences and p_∞ for each p in the countable collection $P \subset X_\infty$. Diagonalize to obtain the subsequence in that statement of the theorem (also denoted M_i). Thus $F : P \to \bar{X}'_\infty$ is defined such that

$$\varphi_\infty(F_\infty(p)) = \lim_{i\to\infty} \varphi_i'(F_i(p_i)) \in Z'. \tag{245}$$

To see that F is distance preserving for any p, q in a ball of radius δ in X_∞:

$$\begin{aligned}
d_{\bar{X}_\infty}\left(F_\infty(p), F_\infty(q)\right) &= d_{Z'}\left(\varphi_\infty(F_\infty(p)), \varphi_\infty(F_\infty(q))\right) && (246)\\
&= \lim_{i\to\infty} d_{Z'}\big(\varphi_i'(F_i(p_i)), \varphi_i'(F_i(q_i))\big) && (247)\\
&= \lim_{i\to\infty} d_Z\left(\varphi_i(p_i), \varphi_i(q_i)\right) && (248)\\
&= d_Z(\varphi_\infty(p), \varphi_\infty(q)) = d_{X_\infty}(p, q). && (249)
\end{aligned}$$

In particular $F : P \to \bar{X}'_\infty$ is continuous and can be extended to the metric completion, $F_\infty : \bar{X}_\infty \to \bar{X}'_\infty$ which is an isometry on balls of radius δ.

To see that (233) implies (234), consider $x_i \in X_i$ satisfying (233). Taking $p_\infty = x_\infty$ as in the top of the proof where we define F_∞, there exists $p_i \in X_i$ such that $\varphi_i(p_i) \to \varphi_\infty(p_\infty)$ with $\varphi'_i(F_i(p_i)) \to \varphi'_\infty(F_\infty(p_\infty))$. Let

$$D_i = d_{Z'}(\, \varphi'_i(F_i(x_i)),\, \varphi'_\infty(F_\infty(x_\infty))\,) \tag{250}$$

then we have (233) as follows:

$$\lim_{i\to\infty} D_i \le \lim_{i\to\infty} d_{Z'}(\, \varphi'_i(F_i(x_i)),\, \varphi'_i(F_i(p_i))\,) \tag{251}$$
$$= \lim_{i\to\infty} d_{X'_i}(F_i(x_i), F_i(p_i)) \tag{252}$$
$$= \lim_{i\to\infty} d_{X_i}(x_i, p_i) \tag{253}$$
$$= \lim_{i\to\infty} d_Z(\varphi_i(x_i), \varphi_i(p_i)) \tag{254}$$
$$\le \lim_{i\to\infty} d_Z(\varphi_i(x_i), \varphi_\infty(x_\infty)) + d_Z(\varphi_\infty(x_\infty), \varphi_i(p_i)) = 0. \tag{255}$$

To see that F_∞ is surjective when F_i are surjective, take any $x \in X'_\infty$. so

$$\liminf_{r\to 0} \mathbf{M}(S(x,r))/r^m > 0. \tag{256}$$

In particular

$$\exists r_x > 0 \ s.t.\ \ S(x,r) \neq \mathbf{0} \qquad a.e.\ r < r_x. \tag{257}$$

By Lemma 3.4 there exists $x_i \in X'_i$ such that

$$\lim_{i\to\infty} \varphi'_i(x_i) = \varphi'_\infty(x)$$

and by Lemma 4.1, for almost every $r > 0$ there is a subsequence (also denoted i) such that

$$d_{\mathcal{F}}(S(x_i, r), S(x,r)) \to 0. \tag{258}$$

Since F_i are surjective, there exists $p_i \in X_i$ such that $F_i(p_i) = x_i$. However, for almost every $r < \delta$,

$$F_{i\#}S(p_i, r) = S(x_i, r) \tag{259}$$

so

$$d_{\mathcal{F}}(S(p_i, r), S(x,r)) \to 0. \tag{260}$$

and

$$\liminf_{i\to\infty} d_{\mathcal{F}}(S(p_i,r),\mathbf{0}) = h(r) > 0. \tag{261}$$

Thus applying the basic Bolzano-Weierstrass Theorem [Theorem 7.1], there is a further subsequence of the p_i which converges to a $p_\infty \in X_\infty$. To see that $F_\infty(p_\infty) = x$ observe that

$$\varphi_\infty(F_\infty(p_\infty)) = \lim_{i\to\infty} \varphi_i(F_i(p_i)) \tag{262}$$

$$= \lim_{i\to\infty} \varphi_i(x_i) = \varphi_\infty(x_\infty). \tag{263}$$

Now suppose $M_\infty = \mathbf{0}$. One needs only show that $M'_\infty = \mathbf{0}$. If not there exists $x \in X'_\infty$ such that (256)-(261) hold. However by Lemma 4.1

$$\lim_{i\to\infty} d_{\mathcal{F}}(S(p_i,r),\mathbf{0}) = 0 \tag{264}$$

which contradicts (261). □

Example 8.5. The hypothesis that a uniform $\delta > 0$ exists such that (229) holds is necessary. This can be seen by taking M_i to be standard flat 1×1 tori and M'_i to be flat $1\times(1/i)$ tori. Let $F_i : M_i \to M'_i$ be the i fold covering maps which are surjective local isometries on balls of radius $\delta_i = 1/(2i)$. Then M_i converges in the intrinsic flat sense to a standard flat torus while M'_i converges in the intrinsic flat sense to the $\mathbf{0}$ integral current space. Thus there cannot be any limit map F_∞.

9. Example with no intrinsic flat limit

The theorems in this paper may be applied to prove certain sequences of Riemannian manifolds do not converge or converge to specific Riemannian manifolds. One such example is provided here. Further examples will appear in joint work with Basilio [3].

Example 9.1. There exists a sequence of smooth Riemannian manifolds with boundary with constant sectional curvature such that $\mathrm{Vol}_{m-1}(\partial M_j) \le A_0$, $\mathrm{Diam}(M_j) \le D_0$ such that no subsequence converges in the intrinsic flat or Gromov-Hausdorff sense not even to $\mathbf{0}$.

Proof. Let M_j be the j fold covering space of

$$N_j = \mathbb{S}^2 \setminus \left(B_{p_+}(1/j) \cup B_{p_-}(1/j)\right) \tag{265}$$

where $\mathbb{S}^2$ is endowed with the standard metric tensor $g_{\mathbb{S}^2}$ which is lifted to M_j and p_+ and p_- are opposite poles. Let d_j be the length metric on M_j defined by this metric tensor.

Then

$$\mathrm{Diam}(M_j) \le \pi + j2\pi(1/j) + \pi = 4\pi \tag{266}$$

and

$$\mathrm{Vol}_{m-1}(\partial M_j) \le j\,\mathrm{Vol}_{m-1}(\partial N_j) \le j2(2\pi/j) = 4\pi \tag{267}$$

but

$$\lim_{j\to\infty} \frac{\mathrm{Vol}_m(M_j)}{j} = \lim_{j\to\infty} \mathrm{Vol}(N_j) = \mathrm{Vol}(\mathbb{S}^2) = 4\pi. \tag{268}$$

Suppose on the contrary that a subsequence converges $M_{j'} \xrightarrow{\mathcal{F}} M_\infty$.

Case I: $M_\infty = \mathbf{0}$. If so, then by Lemma 4.1, any sequence $q_j \in M_j$ and almost every $r > 0$, there is a subsequence $S(q_{j''}, r) \xrightarrow{\mathcal{F}} \mathbf{0}$. Take q_j lying on the equator and choose an $r < 1/2$. Then by the convexity of balls one has

$$S(q_j, r) = \left(B(p_0, r), d_{\mathbb{S}^2}, \int_{B(p_0,r)}\right) \tag{269}$$

are all isometric to one another. Thus they do not converge to $\mathbf{0}$ and there is a contradiction.

Case II: $M_\infty \ne \mathbf{0}$. Let $x_{j,1}, x_{j,2}, \ldots, x_{j,j}$ lie on the equator of X_j so that

$$d_{X_j}(x_{j,i}, x_{j,k}) \ge \pi \qquad \forall i, k \in \{1, 2, \ldots, j\}. \tag{270}$$

Observe also that $B(x_{j,k}, \pi/4)$ are disjoint and are all isometric to a ball $B(x, \pi/4)$ in a standard sphere. Thus

$$d_{\mathcal{F}}(S(x_{j,k}, \pi/4), S(x, \pi/4)) = 0 \qquad \forall k \in \{1, 2, \ldots, j\}. \tag{271}$$

and

$$d_{\mathcal{F}}(S(x_{j,k}, \pi/4), \mathbf{0}) = h_0 = d_{\mathcal{F}}(S(x, \pi/4), \mathbf{0}) > 0 \qquad \forall k \in \{1, 2, \ldots, j\}. \tag{272}$$

Applying Theorem 7.1, there is a subsequence of each $x_{j,k}$ must converge to some $x_k \in \bar{X}_\infty$. Diagonalizing, there is a subsequence (also denoted M_j) such that $x_{j,k} \to x_k$ for all k: so that

$$d_{X_\infty}(x_k, x_{k'}) \geq \pi \tag{273}$$

so that $B(x_{j,k}, \pi/4)$ are disjoint. Applying Lemma 4.1,

$$\lim_{j\to\infty} d_{\mathcal{F}}(S(x_{j,k}, \pi/4), S(x_k, \pi/4)) = 0. \tag{274}$$

and so

$$d_{\mathcal{F}}(S(x_k, \pi/4), S(x, \pi/4)) = 0. \tag{275}$$

Thus M_∞ contains infinitely many balls of the same mass, which contradicts the fact that $\mathbf{M}(T_\infty)$ is finite. □

10. Applications

In this section, we describe some existing and potential applications for the results in this paper.

Remark 10.1. In [5], Burago and Ivanov prove that the volume growth of the universal cover of a Riemannian manifold homeomorphic to a torus is at least that of Euclidean space. If it is exactly equal, then they have a rigidity theorem stating that the Riemannian manifold is flat. Theorem 8.1 may be useful in the study of questions arising in Gromov's work [14] analyzing the almost rigidity of Burago-Ivanov's Theorem (where the volume growth is close to that of Euclidean space). Examples related to this question applying Theorem 8.1 will appear in upcoming work of the author with her doctoral student, Jorge Basilio [3].

Remark 10.2. Theorem 8.1 should be useful when wishing to study limits of covering maps and analyzing the existence of a universal cover of an intrinsic flat limit. Recall that in joint work with Guofang Wei, the author has conducted such an analysis of Gromov-Hausdorff limits [35]. Zahra Sinaei and the author have completed applications in this direction in [33]. More work may be done in this direction.

Remark 10.3. Theorem 8.1 should also be useful when studying how covering spectra behave under intrinsic flat convergence. See joint work of the author with Guofang Wei in which it was shown that covering spectra behave continuously under Gromov-Hausdorff convergence [36]. Zahra Sinaei and the author have completed applications in this direction in [33], but more work may be done in this direction as well.

Remark 10.4. Theorem 6.2 may possibly be applied to study the limits of harmonic functions, eigenfunctions and heat kernels. Recall that Cheeger-Colding proved the convergence of eigenfunctions and eigenvalues when the Riemannian manifolds are converging in the measured Gromov-Hausdorff sense with a uniform lower bound on Ricci curvature [7]. Ding has proved the convergence of heat kernels in this setting [9]. Building on work of Fukaya [11], Sinaei has proven the convergence of harmonic maps in this setting with additional conditions [32]. Portegies has shown that eigenvalues need not converge when one only has intrinsic flat convergence without a volume bound, but building on work of Fukaya [11] has shown the eigenvalues semiconverge as long as the volume converges [30]. It would be interesting to examine what happens to the eigenfunctions and heat kernels in this setting.

Remark 10.5. Under certain conditions one may prove intrinsic flat and Gromov-Hausdorff limits of noncollapsing sequences of manifolds agree by demonstrating that no points disappear in the limit. For example in [38], the author and Wenger proved that these limits agree when the sequence of manifolds has nonnegative Ricci curvature and no boundary. In that paper, Gromov's filling volumes and a contractibility theorem of Perelman was required to complete the argument. The theorems in this paper enable mathematicians to control the disappearance of points without using such powerful theorems. These theorems are applied by Munn to prove $\mathcal{F}$ and GH limits agree for noncollapsing sequences with two sided Ricci curvature bounds in [24] and by Matveev-Portegies to prove these results for sequences with uniform lower Ricci curvature bounds in [23]. Perales has applied these theorems to study noncollapsing sequences of manifolds with boundary and various curvature bounds in [25]. In joint work with Li, Perales has applied these theorems to Alexandrov spaces in [22], It would be interesting to study the limits of sequences of spaces with RCD bounds as in Ambrosio-Gigli-Savare [1] or integral Ricci curvature bounds as in Petersen-Wei [28].

Remark 10.6. In joint work with Dan Lee [20], it has been conjectured that sequences of manifolds with nonnegative scalar curvature and no interior minimal surfaces whose ADM mass converges to 0 must converge in the pointed intrinsic flat sense to Euclidean space. The conjecture is proven in that paper in the rotationally symmetric case. In [16], Huang, Lee and the author have applied the Arzela-Ascoli theorems from this paper to prove this conjecture when the manifolds are graphs satisfying various technical conditions. One should be able to apply these theorems towards proving this conjecture in far more general settings.

Remark 10.7. It can be very difficult to prove a sequence of manifolds converges in the intrinsic flat sense to a particular limit. In the original paper introducing intrinsic flat convergence [39], the author and Stefan Wenger had to construct sequences of filling manifolds explicitly to prove these examples converged. In joint work of the author with Sajjad Lakzian, theorems were proven to allow one to construct intrinsic flat limits as long as the manifolds were converging smoothly on sufficiently nice subregions [19]. Additional such theorems were proven by Lakzian in [17] and applied to Ricci flow through singularities by Lakzian in [18]. Theorem 7.1 may now be applied to prove sequences converge in the intrinsic flat sense to limits even when there is no smooth convergence anywhere. In joint work of the author with Jorge Basilio [3], Theorem 7.1 is applied to prove a collection of examples of sequences of manifolds with nonnegative scalar curvature that have surprising limits.

Remark 10.8. In joint work of the author with LeFloch [21] it is proven that sequences of rotationally symmetric regions with nonnegative scalar curvature, no interior minimal surfaces and uniformly bounded Hawking mass have subsequences which converge in the Intrinsic Flat sense. The proof consists of first proving a Sobolev limit of the metric tensors exist for a well chosen gauge and then showing the sequence converges in the intrinsic flat sense to the Sobolev limit. In order to extend this relationship between Sobolev limits and intrinsic flat limits to the nonrotationally symmetric setting, one may try to apply theorems from this paper in the same way that they are being applied as described in Remark 10.7.

Remark 10.9. In early work, the author studied the stability of the spacelike Friedmann model of cosmology using the Gromov-Hausdorff distance [34]. The Arzela-Ascoli Theorem for Gromov-Hausdorff convergence was a key ingredient in this work. In order to apply Gromov-Hausdorff convergence, one could not allow the universes under consideration to develop thin deep wells. However in work with Dan Lee [20], it is seen that thin deep gravity wells naturally occur even in regions of small mass. In order to study the stability of the spacelike Friedmann model of cosmology in a way which permits thin deep gravity wells, one needs to use the intrinsic flat distance (otherwise there are counterexamples). The new Arzela-Ascoli Theorems provided in this paper should now allow one to extend the techniques in [34] to prove a new intrinsic flat stability theorem for the spacelike Friedmann model which allows for gravity wells.

If a reader is interested in studying any of these questions, please contact the author. More details can be provided and the author can coordinate the research of those working on these problems. Funding to visit the author may be available.

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