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Desingularization of Lie groupoids and pseudodifferential operators on singular spaces

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We study the integral kernels of certain natural operators on desingularization (or blown-up) spaces. A useful desingularization $\Sigma(X)$ of a singular space X is obtained by successively blowing up the lowest dimensional singular strata of X. To study integral kernel operators on $\Sigma(X)$, we introduce and study the "desingularization" $[[\mathcal{G} : L]] \rightrightarrows [M : L]$ of a Lie groupoid $\mathcal{G} \rightrightarrows M$ along an "$A(\mathcal{G})$-tame" submanifold L of its space of units M, where $A(\mathcal{G})$ denotes the Lie algebroid of $\mathcal{G}$. An $A(\mathcal{G})$*-tame* submanifold $L \subset M$ is one that has, by definition, a tubular neighborhood on which $A(\mathcal{G})$ becomes the thick pull-back Lie algebroid of an algebroid on L. Here $[M : L]$ denotes the usual (real) blow-up of M with respect to L and M is obtained from X by a sequence of blow-ups. (In particular, M is an intermediate desingularization step between X and $\Sigma(X)$.) The construction of the desingularization $[[\mathcal{G} : L]]$ of $\mathcal{G}$ along L is based on a canonical fibered pull-back groupoid structure result for $\mathcal{G}$ in a neighborhood of the tame $A(\mathcal{G})$-submanifold $L \subset M$ (Theorem 3.3). Technically, this local structure result is obtained by using an integration result of Moerdijk and Mrčun (Amer. J. Math. 2002). Locally, the desingularization $[[\mathcal{G} : L]]$ is defined using the gauge adiabatic groupoid of Debord and Skandalis (Advances in Math., 2014). The space of units of the desingularization $[[\mathcal{G} : L]]$ is $[M : L]$, the blow-up of M along L. The desingularization groupoid $[[\mathcal{G} : L]]$ is constructed using a gluing construction of Gualtieri and Li (IMRN 2014). The glueing construction is applied to a groupoid that is Morita equivalent to the gauge-adiabatic groupoid and to $\mathcal{G}_{M \smallsetminus L}^{M \smallsetminus L}$, the reduction of the given groupoid to the complement of L. We provide an explicit description of the structure of the desingularized groupoid $[[\mathcal{G} : L]]$ and we identify its Lie algebroid, which is significant in analysis applications. We also discuss a variant of our construction that is useful for analysis on asymptotically hyperbolic manifolds. We conclude with an example discussing the groupoid associated to one of the simplest singularities, namely an edge-type singularity. The resulting groupoid is related to the so called "edge pseudodifferential calculus," which is quite important in applications. The paper also provides an introduction to Lie groupoids for applications to analysis on singular spaces.

Introduction

A typical approach to analysis on a singular space X is to successively blow-up its lowest dimensional singular strata. Recall that the blow-up of a smooth, compact manifold M with respect to a closed submanifold $L \subset M$ replaces L with the unit sphere bundle SNL of the normal bundle $NL \to L$ of L in M. In this paper, we make an important step towards understanding the integral kernels on the resulting final blown-up space $\Sigma(X)$. Namely, one has a good understanding of many natural integral kernel operators on $\Sigma(X)$ once one knows that they are obtained from a Lie groupoid [20, 35, 37, 53, 64], and in this paper we provide the essential step in the construction of the natural groupoid on $\Sigma(X)$.

More precisely, the resulting iterated blown-up space $\Sigma(X)$ is a manifold with corners that is endowed with a natural Lie algebroid $A(X) \to \Sigma(X)$. The main result of this paper is to provide the essential step in the construction of a canonical groupoid $\mathcal{G}_X$ with Lie algebroid $A(\mathcal{G}_X) \simeq A(X)$. We thus give a direct solution to the problem of integrating the natural Lie algebroid $A(X)$ on the iterated blown-up space $\Sigma(X)$. This integration problem has been treated in a related, but more general context in [18, 19, 51]. However, the constructions of those papers are usually not explicit enough and, moreover, may yield a different groupoid than the one that is needed in analysis applications. The manifold M is an intermediate step between X and $\Sigma(X)$, and is thus obtained from X by a sequence of blow-ups. To obtain the final construction of the groupoid on $\Sigma(X)$ one has to perform several times the desingularization procedure. Thus, from now on we shall essentially forget the initial singular space X, and rather concentrate of a single step (or desingularization) in the iterated blow-up procedure leading from X to $\Sigma(X)$ and its natural Lie groupoid $\mathcal{G}_X$.

On a technical level, the main thrust of this paper is to introduce and study the *desingularization of a Lie groupoid* $\mathcal{G} \rightrightarrows M$ *with respect to an* $A(\mathcal{G})$*-tame submanifold* L *of its set of units* M. The resulting groupoid has as units $[M : L]$, the blow-up of M with respect to L, and it has as space of sections of its Lie algebroid the set $r_L C^\infty([M : L])\Gamma(A(\mathcal{G}))$, where r_L is the distance to L, suitably smoothed outside L. Let us denote for any Lie algebroid $\mathcal{G}$ by $\mathrm{Lie}(\mathcal{G}) := \Gamma(A(\mathcal{G}))$, the spaces of sections of $A(\mathcal{G})$. The main properties of the desingularization groupoid $[[\mathcal{G} : L]] \rightrightarrows [M : L]$ are therefore:

$$
(1) \qquad \begin{gathered} \mathrm{Lie}([[\mathcal{G} : L]]) = r_L C^\infty([M : L])\,\mathrm{Lie}(\mathcal{G}) =: \Gamma([[A(\mathcal{G}) : L]]) \\ \text{and} \quad [[\mathcal{G} : L]]_{M \smallsetminus L}^{M \smallsetminus L} = \mathcal{G}_{M \smallsetminus L}^{M \smallsetminus L}. \end{gathered}
$$

The desingularization groupoid $[[\mathcal{G} : L]] \rightrightarrows [M : L]$ thus solves a *constrained integration problem.* The integration problem is expressed in the condition that $\mathrm{Lie}([[\mathcal{G} : L]]) = r_L C^\infty([M : L])\,\mathrm{Lie}(\mathcal{G})$, that is, the first condition of Equation (1). The constrain is provided by the second condition of that equation. Using the notation introduced already, if $X = M$ has only one singular stratum $L \subset M$, then $\Sigma(X) = [M : L]$ and $\Gamma(A(X)) = r_L C^\infty([M : L])\,\mathrm{Lie}(\mathcal{G})$.

The constrains in this integration problems provides, in fact, more flexibility in the construction of our Lie groupoid than other constructions. Indeed, typically, the known integration constructions provide either the smallest or the largest integrating groupoid. However, the "right groupoid" for analysis in a specific situation may be neither the smallest nor the largest integrating groupoid, but the one satisfying the constraint condition. In practical situations, both the blow-up of the base space and the desingularization of the corresponding groupoid have to be performed several times.

Let us try to give here a quick idea of the details of our desingularization procedure. To this end, we need to first introduce the concept of an "A-tame submanifold." Let $A \to M$ be a Lie algebroid over a manifold with corners M and let $L \subset M$ be a submanifold. Recall that L is called A*-tame* if it has a tubular neighborhood $\pi : U \to L$ in M such that the restriction $A|_U$ is isomorphic to the thick pull-back Lie algebroid $\pi^{\downarrow\downarrow}(B)$, for some Lie algebroid $B \to L$. A *tubular neighborhood* of L in M is an open subset U, $L \subset U \subset M$, together with a smooth vector bundle structure $\pi : U \to L$, with π the identity on L. Let $\mathcal{G}$ be a Lie groupoid with units M and Lie algebroid $A(\mathcal{G})$. Let $L \subset M$ be an $A(\mathcal{G})$-tame submanifold. The blow-up $[M : L]$ is then defined, since L is tame. Let us also assume that the fibration $\pi : U \to L$ is a ball bundle over L. The reduction groupoid $\mathcal{G}_U^U$ will then have a fibered pull-back groupoid structure (Theorem 3.3), and hence it can be replaced with a modification of the "gauge-adiabatic groupoid" [21] to define the *desingularization* $[[\mathcal{G} : L]]$ of $\mathcal{G}$ along L. To this end, we use also a gluing construction for Lie groupoids [25].

As it is hopefully apparent by now, our definition of the desingularization of a Lie groupoid with respect to a tame submanifold is motivated by the method of successively blowing-up the lowest dimensional strata of a singular space, which was successfully used in the analysis on singular spaces. The successive blow-up of the lowest dimensional singular strata of a (suitable) singular space leads to the eventual removal of all singularities. This approach was used in [8] to obtain a well-posedness result for the Poisson problem in weighted Sobolev spaces on n-dimensional polyhedral domains using *energy methods* (the Lax-Milgram lemma). To use the *method of layer potentials* one would need also an understanding of the resulting integral kernel operators. This is our main motivation for this paper.

In fact, our definition of the desingularization groupoid $[[\mathcal{G} : L]]$ provides the necessary results for the construction of many integral kernel operators on the resulting blown-up spaces as functions (or distributions) on $[[\mathcal{G} : L]]$. It turns out that quite general operators can be obtained using integral kernel and pseudodifferential operators on the desingularization groupoid [1, 4–6, 47, 53]. For example, by combining our desingularization construction with the construction of psedodifferential operators on groupoids, one can essentially recover the pseudodifferential calculi of Grushin [24], Mazzeo [44], and Schulze [61, 62].

The blow-up procedure leads naturally to manifolds with corners, as follows: the blow-up of a smooth manifold with respect to a submanifold is a manifold with boundary, but the blow-up of a manifold with boundary along a tame submanifold is a manifold with corners of codimension two. In general, the blow-up of a manifold with corners with respect to a tame submanifold is a manifold with corners of higher maximum codimension (i.e. rank). Thus, even if one is interested in analysis on smooth manifolds, sometimes one is lead to consider also manifolds with corners. See, for example, [8, 17, 32, 48] for further motivation and references. This paper will thus provide the background for the construction of the integral kernel (or pseudodifferential) operators on the resulting blown-up spaces. The results of this paper may also turn out to be useful in index theory, although this is not our main motivation. See however [20], where a related, but different construction was used for some index problems. The construction in [20] models a *different type* of singular spaces.

The paper is organized as follows. The first section is devoted mostly to background material. We thus review manifolds with corners and tame submersions and establish a canonical (i.e. fibration) local form for a tame submersion that generalizes to manifolds with corners the corresponding classical result in the smooth case. We then recall the definitions of a Lie groupoid, of a Lie algebroid, and of the Lie algebroid associated to a Lie groupoid. We do that in the framework that we need, that is, that of manifolds with corners. Almost all basic constructions and results on Lie groupoids and Lie algebroids extend to the setting of manifolds with corners without any significant changes. One must be careful, however, to use *tame submersions* instead of (plain) submersions. One of the main results of this paper is the construction of the desingularization of a Lie groupoid $\mathcal{G}$ along an $A(\mathcal{G})$-tame submanifold. This requires several other, intermediate constructions, such as that of the adiabatic (deformation) groupoid and of the thick pull-back Lie algebroid. In the second section, we thus review and extend all these examples as well as others, more basic ones that are needed in the construction of the desingularization groupoid. In particular, we introduce the so called "edge modification" of a groupoid using the gauge-adiabatic groupoid of Debord and Skandalis [21]. We combine this with a gluing construction due to Gualtieri and Li [25], which we also review and extend to our setting. The third section contains most of our main results. We first prove a local structure theorem for a Lie groupoid $\mathcal{G}$ with units M in a tubular neighborhood $\pi : U \to L$ of an $A(\mathcal{G})$-tame submanifold $L \subset M$ using results on the integration of Lie algebroid morphisms due to Moerdijk and Mrčun [45]. More precisely, we prove that the reduction of $\mathcal{G}$ to U is isomorphic to $\pi^{\downarrow\downarrow}(\mathcal{G}_L^L)$, the fibered pull-back groupoid to U of the reduction of $\mathcal{G}$ to L. This allows us to define the desingularization for this type of fibered pull-back groupoids, in which case we obtain a groupoid that is Morita equivalent to the gauge-adiabatic groupoid. The general case is obtained using the gluing procedure mentioned above. We identify the Lie algebroid of the desingularization as the desingularization of its Lie algebroid (the desingularization of a Lie algebroid was introduced in [2]). We conclude with an example related to the 'edge'-calculus (see [33] and the references therein).

The paper is written such that it provides also an introduction to Lie groupoids for students and researchers interested in applications to analysis on singular spaces. This is the reason for which the first two sections contain additional material that will explain the role of Lie groupoids. For instance, we discuss the convolution algebras of some classes of Lie groupoids. We also provide most of the needed definitions to make the paper as self-contained as possible. We also study in detail the needed classes of Lie algebroids and Lie groupoids.

A note on notation and terminology

We shall use manifolds with corners extensively. They are defined in Subsection 1.1. We shall use the term *smooth manifold* to mean a $\mathcal{C}^\infty$–manifold without corners. We take the point of view that all maps, submanifolds, and so on, will be defined in the same way in the corner case as in the smooth case, except that all our submanifolds will be assumed to be *closed*. Sometimes, we need maps and submanifolds with special properties; they will usually be termed “tame”, for instance, a tame submersion (of manifolds with corners) will be a submersion of manifolds with corners that maps inward pointing vectors to inward pointing vectors, and hence it has the property that all its fibers are smooth manifolds. This property is not shared by general submersions, however. Also, we use only *real* vector bundles and functions, to avoid confusion and simplify notation. The results extend without any difficulty to the complex case, when it makes sense. Moreover, all our manifolds will be paracompact, but we do not require them to be Hausdorff in general. However, all the spaces of units of groupoids and the bases of Lie groupoids will be Hausdorff. We also note that a “submanifold of a manifold with corners” in the sense of this paper is not the same thing as the more restrictive concept of a “submanifold with corners of a manifold with corners” used in [4].

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1. Preliminaries on Lie algebroids

We now recall the needed definitions and properties of Lie groupoids and of Lie algebroids. We shall work with manifolds with corners, so we also recall some basic definitions and results on manifolds with corners. Few results in this section are new, although the presentation probably is. We refer to

Mackenzie's books [38, 39] for a nice introduction to the subject, as well as to further references and historical comments on Lie groupoids and on Lie algebroids. See also [11, 42, 46, 57] for some of the more specialized results used in this paper.

1.1. Manifolds with corners and notation

In the following, by a *manifold*, we shall mean a $\mathcal{C}^\infty$-manifold, possibly with corners. By a *smooth manifold* we shall mean a $\mathcal{C}^\infty$-manifold *without corners.* All our manifolds will be assumed to be paracompact. Recall [30, 32, 43] (and the references therein) that M is a *manifold with corners* of dimension n if it is locally diffeomorphic to an open subset of $[-1,1]^n$ with smooth changes of coordinates. A smooth map $h : [0,a)^k \times (-a,a)^{n-k} \to [0,b)^d \times (-b,b)^{m-d}$ is simply the restriction of a smooth map $\tilde{h} : (-a,a)^n \to (-b,b)^m$ such that $\tilde{h}([0,a)^k \times (-a,a)^{n-k}) \subset [0,b)^d \times (-b,b)^{m-d}$. A point $p \in M$ is called of *depth k* if it has a neighborhood V_p diffeomorphic to $[0,a)^k \times (-a,a)^{n-k}$, $a > 0$, by a diffeomorphism (i.e. a coordinate chart) $\phi_p : V_p \to [0,a)^k \times (-a,a)^{n-k}$ mapping p to the origin: $\phi_p(p) = 0$. Such a neighborhood will be called *standard.* A function $f : M \to M_1$ between two manifolds with corners will be called *smooth* if its components are smooth in all coordinate charts.

A connected component F of the set of points of depth k will be called an *open face (of codimension k)* of M. The maximum depths of a point in M will be called the *rank* of M. Thus the smooth manifolds are exactly the manifolds of rank zero. The closure in M of an open face F of M will be called a *closed* face of M. The closed faces of M may not be manifolds with corners on their own.

We define the tangent space to a manifold with corners TM as usual, that is, as follows: the vector space T_pM is the set of derivations $D_p : \mathcal{C}^\infty(M) \to \mathbb{R}$ satisfying $D_p(fg) = f(p)D_p(g) + D_p(f)g(p)$ and TM is the disjoint union of the vector spaces T_pM, with $p \in M$. Let $v \in T_pM$ be a tangent vector to M at $p \in M$. We say that v is *inward pointing* if, by definition, there exists a smooth curve $\gamma : [0,1] \to M$ such that $\gamma'(0) = v$ (so, in particular, $\gamma(0) = p$). The set of inward pointing vectors in $v \in T_x(M)$ will form a closed cone denoted $T_x^+(M)$. If, close to x, our manifold with corners is given by the conditions $\{f_i(y) \geq 0\}$ with df_i linearly independent at x, then the cone $T_x^+(M)$ is given by

$$T_x^+(M) = \{v \in T_xM,\, df_i(v) \geq 0\}. \tag{2}$$

Let M and M_1 be manifolds with corners and $f : M_1 \to M$ be a smooth map. Then f induces a vector bundle map $df : TM_1 \to TM$, as in the smooth case, satisfying also $df(T_z^+(M_1)) \subset T_{f(z)}^+ M$. If the smooth map $f : M_1 \to M$ is injective, has injective differential df, and has closed range, then we say that $f(M_1)$ is a (closed) *submanifold* of M. All our submanifolds will be closed, so we shall simply say "manifold" instead of "closed manifold." Except for the condition that our submanifolds be locally closed, we are thus imposing the least restrictions on smooth maps and submanifolds, unlike [30], for example. This is, of course, just a matter of taste, choice, and terminology, and has no mathematical content, but allows us to navigate easier through the jungle of the terminology for manifolds with corners. For instance, a smooth map f between manifolds with corners is a *submersion* if, by definition, the differential $df = f_*$ is surjective (as in the case of smooth manifolds). However, we will typically need a special class of submersions with additional properties, the *tame submersions*. More precisely, we have the following definition.

Definition 1.1. A *tame submersion* $h : M_1 \to M$ is a smooth map $h : M_1 \to M$ such that its differential dh is surjective everywhere (i.e. h is submersion in the usual sense) *and*

$$(dh_x)^{-1}(T_{h(x)}^+ M) = T_x^+ M_1.$$

(That is, $dh(v)$ is an inward pointing vector of M if, and only if, v is an inward pointing vector of M_1.)

We do not require our tame submersions to be surjective (although, as we will see soon below, they are open, as in the smooth case). We shall need the following lemma.

Lemma 1.2. *Let $h : M_1 \to M$ be a tame submersion of manifolds with corners. Then x and $h(x)$ have the same depth.*

Proof. This is because the depth of x in M is the same as the depth of the origin 0 in $T_x^+ M_1$, which, in turn, is the same as the depth of the origin 0 in $T_{h(x)}^+ M$, since dh_x is surjective and $(dh_x)^{-1}(T_{h(x)}^+ M) = T_x^+ M_1$. □

The following lemma is probably known, but we could not find a suitable reference, so we include a proof.

Lemma 1.3. *Let $h : M_1 \to M$ be a tame submersion of manifolds with corners.*

(i) The rank of M_1 is $\leq$ the rank of M.

(ii) For any $m_1 \in M_1$, there exists an open neighborhood U_1 of m_1 in M_1 such that $U := h(U_1)$ is open and the restriction of h to U_1 is a fibration $U_1 \to U$ with fibers smooth manifolds (i.e. without corners).

(iii) Let $L \subset M$ be a submanifold, then $L_1 := h^{-1}(L)$ is a submanifold of M_1 of rank $\leq$ the rank of L.

Proof. We have that (i) is a consequence of Lemma 1.2 and (iii) is a consequence of (ii), so let us concentrate on proving (ii).

Let $m_1 \in M_1$ be of depth k. We can choose a standard neighborhood W_1 of m_1 in M_1 and a standard neighborhood W of $h(m_1)$ in M such that $h(W_1) \subset W$. Since our problem is local, we may assume that $M_1 = W_1 = [0,a)^k \times (-a,a)^{n_1-k}$ and that $M = W = [0,b)^k \times (-b,b)^{n-k}$, $a, b > 0$, with m_1 and $h(m_1)$ being the corresponding origins. Note that both M and M_1 will then be manifolds with corners of the same rank k, which is possible since h preserves the depth (see Lemma 1.2). We can then extend h to a map $h_0 : Y_1 := (-a,a)^{n_1} \to \mathbb{R}^n$ that is a (usual) submersion at $0 = m_1$ (not necessarily tame). By decreasing a, if necessary, we may assume that h_0 is a (usual) submersion everywhere and hence that $h_0(Y_1)$ is open in $\mathbb{R}^n$. By standard differential geometry results, we can then choose an open neighborhood V of $0 = h_0(m_1)$ in $\mathbb{R}^n$ and an open neighborhood V_1 of $0 = m_1$ in $Y_1 := (-a,a)^{n_1}$ such that the restriction h_1 of h_0 to V_1 is a fibration $h_1 : V_1 \to V$ with fibers diffeomorphic to $(-1,1)^{n_1-n}$. By further decreasing V and V_1, we may assume that V is an open ball centered at 0.

Next, we notice that our reductions mean that $M \cap V$ consists of the vectors in V that have the first k components ≥ 0. By construction, we therefore have that

$$h_1(M_1 \cap V_1) = h_0(M_1 \cap V_1) \subset M \cap V = \Big([0,b)^k \times (-b,b)^{n-k}\Big) \cap V.$$

Let $U_1 := M_1 \cap V_1$. We will show that we have in fact more, namely, that we have

$$U_1 = h_1^{-1}(M \cap V) \quad \text{and} \quad h_1(U_1) = M \cap V, \tag{3}$$

which will prove (ii) for $U_1 := M_1 \cap V_1$, since $h_1 : V_1 \to V$ is a fibration with fibers diffeomorphic to $(-1,1)^{n_1-n}$ and $h(U_1) = h_1(U_1) = M \cap V$ is open in M.

Indeed, in order to prove the relations in Equation (3), let us notice that, since h_1 is surjective, it is enough to prove that $U_1 = h_1^{-1}(M \cap V)$, since that will then give right away that $h_1(U_1) = M \cap V$. The relations in Equation (3) will be enough to complete the proof of (ii). Let us assume then, by contradiction, that it is *not* true that $U_1 = h_1^{-1}(M \cap V)$. This means that there exists $p = (p_i) \in V_1 \smallsetminus M_1$ such that $h_1(p) = h_0(p) \in M \cap V = \big([0,b)^k \times (-b,b)^{n-k}\big) \cap V$. Let us choose $q = (q_i)$ in $M_1 \cap V_1$ of depth zero. That means that q is an interior point of $M_1 \cap V_1$. Then the two points $h_1(p) = h_0(p)$ and $h_1(q) = h_0(q) = h(q)$ both belong to M, more precisely,

$$h_1(p), h_1(q) \in M \cap V = \big([0,b)^k \times (-b,b)^{n-k}\big) \cap V,$$

which is the first octant in a ball. Therefore $h_1(p)$ and $h_1(q)$ can be joined by a path $\gamma = (\gamma_i) : [0,1] \to M \cap V$, with p corresponding to 1 and q corresponding to 0 (that is, $\gamma(1) = h_1(p)$ and $\gamma(0) = h_1(q)$). All paths are assumed to be continuous, by definition. Since h preserves the depth, $h_1(q) = h_0(q) = h(q)$ is moreover an interior point of $M \cap V$. Therefore we may assume that the path $\gamma(t)$ consists completely of interior points of M for $t < 1$.

We can lift the path γ to a path $\tilde\gamma : [0,1] \to V_1$ with $\tilde\gamma(0) = q$, $\tilde\gamma(1) = p$, $\gamma = h_1 \circ \tilde\gamma$, since

$$h_1 := h_0|_{V_1} : V_1 \to V$$

is a fibration. We have $\tilde\gamma_i(0) = q_i > 0$ for $i = 1, \ldots, k$, since $q = (q_i)$ is an interior point of $V_1 \cap M_1$. On the other hand, since $p \notin M_1$, there exists at least one i, $1 \le i \le k$, such that $\tilde\gamma_i(1) = p_i < 0$. Since $\tilde\gamma_i(0) = q_i > 0$ and the functions $\tilde\gamma_j$ are continuous, we obtain that the set

$$Z := \cup_{j=1}^{n} \tilde\gamma_j^{-1}(0) = \{t \in [0,1], \text{ there exists } 1 \le j \le k \text{ such that } \tilde\gamma_j(t) = 0\}$$

is closed and non-empty. Let $t_* = \inf Z$. Then $t_* \in Z$, since Z is closed. Moreover, $t_* > 0$, since $q = (q_i) = (\tilde\gamma_i(0))$ is of depth zero, meaning that $\tilde\gamma_j(0) > 0$ for $1 \le j \le k$, and hence that $0 \notin Z$. Using again $\tilde\gamma_j(0) > 0$, we obtain $\tilde\gamma_i(s) > 0$ for all $0 \le s < t_*$, by the minimality of t_*, since the functions $\tilde\gamma_j$ are continuous. Hence $\tilde\gamma(s) \in M_1 \subset Y_1$ for $s < t_*$. (Recall that $h_0 : Y_1 := (-a,a)^{n_1} \to \mathbb{R}^n$ and that we are assuming $M_1 = [0,1)^k \times (-1,1)^{n-k}$.) We obtain that $\tilde\gamma(t_*) \in M_1 \cap V_1$, because M_1 is closed in Y_1. Therefore $t_* < 1$, because $p = \tilde\gamma(1) \notin M_1$. Since $\tilde\gamma_j(t_*) = 0$ for some j, we have that $\tilde\gamma(t_*)$ is a boundary point of M_1, and hence it has depth > 0. Hence the depth of $\gamma(t_*) = h_0(\tilde\gamma(t_*)) = h(\tilde\gamma(t_*))$ is also > 0 since h preserves the depth. But this is a contradiction since $\gamma(t)$ was constructed to consist entirely of interior points for $t < 1$. This proves (ii). $\square$

We shall use the above result in the following way:

Corollary 1.4. *Let $h : M_1 \to M$ be a tame submersion of manifolds with corners.*

(i) h is an open map.

(ii) The fibers $h^{-1}(m)$, $m \in M$, are smooth manifolds (that is, they have no corners).

(iii) Let us denote by $\Delta \subset M \times M$ the diagonal and by $h \times h : M_1 \times M_1 \to M \times M$ the product map $h \times h(m, m') = (h(m), h(m'))$. Then $(h \times h)^{-1}(\Delta)$ is a submanifold of $M_1 \times M_1$ of the same rank as M_1.

Proof. The first part follows from Lemma 1.3(ii). The second and third parts follow from Lemma 1.3(iii), by taking $L = \{m\}$ for (ii) and $L = \Delta$ for (iii). $\square$

We shall use the following conventions and notations.

Notations 1.5. If $E \to X$ is a smooth vector bundle, we denote by $\Gamma(X; E)$ (respectively, by $\Gamma_c(X; E)$) the space of smooth (respectively, smooth, compactly supported) sections of E. Sometimes, when no confusion can arise, we simply write $\Gamma(E)$, or, respectively, $\Gamma_c(E)$ instead of $\Gamma(X; E)$, respectively $\Gamma_c(X; E)$. If M is a manifold with corners, we shall denote by

$$\mathcal{V}_b(M) := \{X \in \Gamma(M; TM), X \text{ tangent to all faces of } M\}$$

the set of vector fields on M that are tangent to all faces of M [32].

For further reference, let us recall a classical result of Serre and Swan [31], which we formulate in the way that we will use.

Theorem 1.6 (Serre-Swan, [31]). *Let M be a compact Hausdorff manifold with corners and $\mathcal{V}$ be a finitely generated, projective $\mathcal{C}^\infty(M)$-module. Then there exists a real vector bundle $E_\mathcal{V} \to M$, uniquely determined up to an isomorphism, such that $\mathcal{V} \simeq \Gamma(M; E_\mathcal{V})$ as $\mathcal{C}^\infty(M)$-module. We can choose $E_\mathcal{V}$ to depend functorially on $\mathcal{V}$, in particular, any $\mathcal{C}^\infty(M)$-module morphism $f : \mathcal{V} \to \mathcal{W} \simeq \Gamma(M; E_\mathcal{W})$ induces a unique smooth vector bundle morphism $\tilde{f} : E_\mathcal{V} \to E_\mathcal{W}$ compatible with the isomorphisms $\mathcal{V} \simeq \Gamma(M; E_\mathcal{V})$ and $\mathcal{W} \simeq \Gamma(M; E_\mathcal{W})$.*

For instance, on can take $E_\mathcal{V}$ to be the disjoint union of the sets $\mathcal{V}/I_m\mathcal{V}$, $m \in M$, where $I_m := \{f \in \mathcal{C}^\infty(M) \mid f(m) = 0\}$, endowed with a suitable topology and smooth structure. In particular, there exists a (unique up to an isomorphism) vector bundle T^bM such that $\Gamma(T^bM) \simeq \mathcal{V}_b(M)$ as $\mathcal{C}^\infty(M)$-modules [32], where $\mathcal{V}_b$ is as introduced in 1.5, and one can take $T^bM := \cup_{m\in M}\mathcal{V}_b/I_m\mathcal{V}_b$. By localization, we can use the Serre-Swan Theorem also on non-compact manifolds.

1.2. Definition of Lie groupoids and Lie algebroids

Recall that a category $\mathcal{C}$ is given by a class of objects $\mathcal{C}^{(0)}$, and, for any two objects A and B in $\mathcal{C}^{(0)}$, by a set of morphisms $\mathrm{Hom}_{\mathcal{C}}(A,B)$ between them, together with a composition

$$\mathrm{Hom}_{\mathcal{C}}(A,B) \times \mathrm{Hom}_{\mathcal{C}}(B,C) \ni (\phi,\psi) \to \psi \circ \phi \in \mathrm{Hom}_{\mathcal{C}}(A,C)$$

satisfying the usual axioms (such as "associativity" and existence of identity morphism $id_A \in \mathrm{Hom}_{\mathcal{C}}(A,A)$, for all objects A of $\mathcal{C}$). A morphism $\phi \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ is said to be *invertible* if there exists $\psi \in \mathrm{Hom}_{\mathcal{C}}(B,A)$ such that $\phi \circ \psi = id_B$ and $\psi \circ \phi = id_A$. A standard example is given by the class of all sets, with morphisms given by functions. A category is *small* if its class of objects is a set.

The simplest version of the definition of *a groupoid* $\mathcal{G}$ is that it is a small category in which every morphism is invertible. The class of objects of $\mathcal{G}$, denoted $\mathcal{G}^{(0)}$, is thus a set. The set of morphisms $\mathcal{G} := \mathcal{G}^{(1)}$ is thus also a set. For convenience, we shall denote $M := \mathcal{G}^{(0)}$.

One typically thinks of a groupoid in terms of its structural morphisms. First of all, the domain and range of a morphism give rise to maps $d, r : \mathcal{G} \to M$. We shall therefore write $d, r : \mathcal{G} \rightrightarrows M$ (or, simply, $\mathcal{G} \rightrightarrows M$) for a groupoid with units M and domain and range maps d and r. Two morphisms $g, h \in \mathcal{G} := \mathcal{G}^{(1)}$ are composable if, and only if, $d(g) = r(h)$, and we shall denote by $\mu(g,h) = gh$ their composition. It is a map

$$\mu : \mathcal{G}^{(2)} := \{(g,h) \in \mathcal{G} \times \mathcal{G} |\ d(g) = r(h)\} \to \mathcal{G}. \tag{4}$$

The objects of $\mathcal{G}$ will also be called *units* and the morphisms of $\mathcal{G}$ will also be called *arrows*. To the groupoid $\mathcal{G}$ there are also associated the inverse map $i(g) = g^{-1}$ and the embedding $u : M \to \mathcal{G}$, which associates to each object of $\mathcal{G}$ its identity morphism. If M and $\mathcal{G}$ are manifolds with corners, if i is smooth, and if d is a tame submersion of manifolds with corners, then r is also a tame submersion of manifolds with corners. The structural morphisms d, r, μ, i, u will then satisfy the following conditions [11, 39, 45, 57]:

1) $d(gh) = d(h)$ and $r(gh) = r(g)$, $g, h \in \mathcal{G}$.

2) $g_1(g_2g_3) = (g_1g_2)g_3$ for all $g_i \in \mathcal{G}$ such that $d(g_i) = r(g_{i+1})$.

3) $d(u(x)) = x = r(u(x))$, for all $x \in M$.

4) $gu(d(g)) = g$ and $u(r(g))g = g$ for all $g \in \mathcal{G}$.

5) $gi(g) = u(r(g))$ and $i(g)g = u(d(g))$ for all $g \in \mathcal{G}$.

Let us assume that M and $\mathcal{G}$ are manifolds with corners and that d and r are tame submersions (of manifolds with corners). We then notice that, by Corollary 1.4, the set $\mathcal{G}^{(2)}$ of Equation (4) is a manifold with corners as well. Also, if d is a tame submersion and i is a diffeomorphism, then $r = d \circ i$ is also a tame submersion. Recall then the following fundamental definition

Definition 1.7. A *Lie groupoid* is a groupoid $\mathcal{G} \rightrightarrows M$ such that:

1) M and $\mathcal{G}$ are manifolds (possibly with corners) and M is Hausdorff.

2) The structural morphisms d, r, i, u are smooth.

3) d is a tame submersion and $\mu : \mathcal{G}^{(2)} \to \mathcal{G}$ is smooth.

Lie groupoids were introduced by Ehresmann. See [39] for a comprehensive introduction to the subject as well as for more references. Note that $\mathcal{G}$ is *not* required to be Hausdorff, as this will needlessly remove a large class of important examples, such as the ones arising in the study of foliations [13]. However, all groupoids used in this paper will be either assumed or proved to be Hausdorff. We shall use the following standard notation.

Notations 1.8. *Let $d, r : \mathcal{G} \rightrightarrows M$ be a groupoid and $K, L \subset M$, then we denote $\mathcal{G}_K := d^{-1}(K)$, $\mathcal{G}^K := r^{-1}(K)$, and $\mathcal{G}_K^L := r^{-1}(L) \cap d^{-1}(K)$. We shall also write $\mathcal{G}_x := d^{-1}(x)$.*

In particular, $\mathcal{G}_K^K$ is a groupoid with units K, called the *reduction of $\mathcal{G}$ to K*. If $K \subset M$ is open, then $\mathcal{G}_K^K$ will be a Lie groupoid if $\mathcal{G}$ is a Lie groupoid. In general, it will not be a Lie groupoid even if $\mathcal{G}$ is a Lie groupoid. If $K \subset M$ is $\mathcal{G}$-invariant, meaning that $\mathcal{G}_K^K = \mathcal{G}_K = \mathcal{G}^K$, then $\mathcal{G}_K$ will be a groupoid, called the *restriction* of $\mathcal{G}$ to (the invariant subset) K.

We are interested in Lie groupoids since many operators of interest have distribution kernels that are naturally defined on a Lie groupoid. This is convenient since it yields a quick proof of the composition formula for these natural operators. Let us introduce now the composition in the case of regularizing operators. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and let us choose a metric on $A(\mathcal{G})$. Let $d, r : \mathcal{G} \rightrightarrows M$ be a Lie groupoid and denote

$$A(\mathcal{G}) := \ker(d_* : T\mathcal{G} \to TM)|_M = \cup_{x \in M} T_x \mathcal{G}_x. \tag{5}$$

That is $A(\mathcal{G})$ is the restriction to the units of the kernel of the differential of the domain map d. It is a vector bundle on M. We can use this metric and the projections $r : \mathcal{G}_x \to M$ that satisfy $T\mathcal{G}_x \simeq r^*(A(\mathcal{G}))$ to obtain a family of metrics g_x on $\mathcal{G}_x$. By constructions, these metrics will be right invariant. The associated volume forms $d\,\mathrm{vol}_x$ on $\mathcal{G}_x$ will hence also be right invariant for the action of $\mathcal{G}$. Let us assume, for simplicity that $\mathcal{G}$ is Hausdorff (this will be the case throughout the paper). We then define a *convolution product* on $\mathcal{C}_c^\infty(\mathcal{G})$ by the formula

$$\phi * \psi(g) := \int_{\mathcal{G}_{d(g)}} \phi(gh^{-1})\psi(h)\, d\,\mathrm{vol}_{d(g)}(h). \tag{6}$$

A *subgroupoid* of a groupoid $\mathcal{G}$ is a subset $\mathcal{H}$ such that the structural morphisms of $\mathcal{G}$ induce a groupoid structure on $\mathcal{H}$. We shall need the notion of a *Lie subgroupoid* of a Lie groupoid, which is closely modeled on the definition in [39]. Recall that if M is a manifold with corners and $L \subset M$ is a subset, we say that L is a submanifold of M if it is a *closed* subset, if it is a manifold with corners in its own for the topology induced from M, and if the inclusion $L \to M$ is smooth and has injective differential.

Definition 1.9. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. A Lie groupoid $\mathcal{H} \rightrightarrows L$ is a *Lie subgroupoid* of $\mathcal{G}$ if L is a submanifold of M and $\mathcal{H}$ is a submanifold of $\mathcal{G}$ with the groupoid structural maps induced from $\mathcal{G}$. (So L and $\mathcal{H}$ are closed subsets, according to our conventions.)

Lie groupoids generalize Lie groups. By analogy, a Lie groupoid $\mathcal{G}$ will have an associated infinitesimal object $A(\mathcal{G})$, the "Lie algebroid associated to $\mathcal{G}$." To recall its definition, let us first recall the definition of a Lie algebroid. See Pradines' paper [56] for the original definition and Mackenzie's books [39] for a comprehensive introduction to their general theory.

Definition 1.10. A *Lie algebroid* $A \to M$ is a real vector bundle over a Hausdorff manifold with corners M together with a *Lie algebra* structure on $\Gamma(M; A)$ (with bracket $[\ ,\ ]$) and a vector bundle map $\varrho : A \to TM$, called *anchor*, such that the induced map $\varrho_* : \Gamma(M; A) \to \Gamma(M; TM)$ satisfies the following two conditions:

(i) $\varrho_*([X, Y]) = [\varrho_*(X), \varrho_*(Y)]$ and

(ii) $[X, fY] = f[X,Y] + (\varrho_*(X)f)Y$, for all $X, Y \in \Gamma(M; A)$ and $f \in \mathcal{C}^\infty(M)$.

We shall write Xf instead of $\varrho(X)f$ in what follows.

Let $A(\mathcal{G}) = \cup_{x\in M} T_x\mathcal{G}_x$, as in Equation (5). The groupoid $\mathcal{G}$ acts by right translations on $\mathcal{G}$ (or, more precisely, on the fibers of d) in the sense that if $\gamma \in \mathcal{G}$ has $r(\gamma) = x$ and $d(\gamma) = y$, then the map $\mathcal{G}_x \ni h \to h\gamma \in \mathcal{G}_y$ is a diffeomorphism. The sections of $A(\mathcal{G})$ identify with the space of d–vertical, right invariant vector fields on $\mathcal{G}$ (that is, the vector fields on $\mathcal{G}$ that are tangent to the submanifolds $\mathcal{G}_x := d^{-1}(x)$ and are invariant with respect to the natural action of $\mathcal{G}$ by right translations on the fibers of d). In particular, the space of sections of $A(\mathcal{G}) \to M$ has a natural Lie bracket that makes it into a Lie algebroid, since the space of d–vertical vector fields on $\mathcal{G}$ is closed under the Lie bracket and the Lie bracket is invariant for right translations. This definition is due to Pradines [56].

Definition 1.11 (Pradines). Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid, then the Lie algebroid $A(\mathcal{G})$ is called the *Lie algebroid associated to* $\mathcal{G}$*.* The anchor is the differential of r restricted at the units.

Recall the following definition (see [39, 58]).

Definition 1.12. Let R be a commutative, associative, unital, real algebra and let $\mathfrak{g}$ be a Lie algebra and an R-module such that $\mathfrak{g}$ acts by derivations on R and the Lie bracket satisfies the compatibility relation

$$[X, rY] = r[X,Y] + X(r)Y, \quad \text{for all } \ r \in R \ \text{ and } \ X, Y \in \mathfrak{g}.$$

Then we say that $\mathfrak{g}$ is an *R-Lie-Rinehart algebra.*

Let M be a compact manifold with corners. We thus see that the category of Lie algebroids with base M is equivalent to the category of finitely-generated, projective $\mathcal{C}^\infty(M)$-Lie-Rinehart algebras, by the Serre-Swan Theorem, Theorem 1.6. It is useful in Analysis to think of Lie algebroids as coming from suitable Lie-Rinehart algebras.

Morphisms of Lie algebroids are tricky to define in general (see for instance 4.3.1 of [39]), but we will need only special cases. For instance, the isomorphisms of Lie algebroids are easy to define. Indeed, two algebroids $A_i \to M_i$ are *isomorphic* if there exists a vector bundle isomorphism $\phi : A_1 \to A_2$ that preserves the corresponding Lie brackets. The other case of morphisms of Lie algebroids that we will consider will be that of a *Lie algebroid morphisms over* M. They are obtained when $M_1 = M_2 = M$.

Definition 1.13. Let $A_i \to M$ be two Lie algebroids with anchor maps $\rho_i : A_i \to TM$. A *morphism over* M of A_1 to A_2 is a vector bundle morphism $\phi : A_1 \to A_2$ that induces the identity over M and is compatible with the anchor maps and the Lie brackets. More precisely, $\varrho_2(\phi(X)) = \varrho_1(X)$ and $\phi([X, Y]_1) = [\phi(X), \phi(Y)]_2$ for all sections X and Y of A_1.

See 3.3.1 of [39] for more details. This definition is easily dualized to $\mathcal{C}^\infty(M)$-Lie-Rinehart algebras by requiring that ϕ induces a $\mathcal{C}^\infty(M)$-linear Lie-algebra morphism $\mathfrak{g}_1 \to \mathfrak{g}_2$. Unless explicitly stated otherwise, we will consider in this paper only morphisms of Lie algebroids *over* M (thus morphisms that induce the identity on the base), with the exception when the morphism is an isomorphism that comes from the action of a Lie group. The same convention applies to the isomorphisms of Lie groupoids.

The following simple remark will be useful in the proof of Theorem 3.17.

Lemma 1.14. *Let $A \to M$ be a Lie algebroid and $f \in \mathcal{C}^\infty(M)$ be such that $\{f = 0\}$ has an empty interior. Then $f\Gamma(M; A) \subset \Gamma(M; A)$ is a Lie subalgebra and there exists a Lie algebroid, denoted fA, such that $\Gamma(fA) := \Gamma(M; fA) \simeq f\Gamma(A)$, as $\mathcal{C}^\infty(M)$-Lie-Rinehart algebras.*

Proof. The proof of the Lemma relies on a simple calculation, which nevertheless will be useful in what follows. Let $X, Y \in \Gamma(A) := \Gamma(M; A)$. We have

$$[fX, fY] = fX(f)Y - fY(f)X + f^2[X, Y] \in \Gamma(fA). \tag{7}$$

The assumption that the interior of $\{f = 0\}$ be empty guarantees that the multiplication by f is an isomorphism $\Gamma(A) \to f\Gamma(A)$. In particular, we can choose, $fA = A$ as a vector bundle, but with a different bracket on fA: $[X, Y]_{fA} = X(f)Y - Y(f)X + f[X, Y]$. □

For further reference, let us recall also the *isotropy* of a Lie algebroid.

Definition 1.15. Let $\varrho : A \to TM$ be a Lie algebroid on M with anchor ϱ. Then the kernel $\ker(\varrho_x : A_x \to T_xM)$ of the anchor is the *isotropy* of A at $x \in M$.

It is known that the isotropy at any point is a Lie algebra.

1.3. Direct products and pull-backs of Lie algebroids

For the purpose of proving Theorems 3.3 and 3.22 below, we need a good understanding of thick pull-back Lie algebroids and of their relation to vector bundle pull-backs. We thus recall the definition of the thick pull-back of a Lie algebroid and of the direct product of two Lie algebroids. We use a simplified approach that is enough for our purposes, however, more details can be found in [39]. We therefore adapt accordingly our notation and terminology. For instance, we shall use the term "thick pull-back of Lie algebroids" (as in [4]) in order to avoid confusion with the ordinary (i.e. vector bundle) pull-back, which will also play a role in what follows. For example, vector bundle pull-backs appear in the next lemma, Lemma 1.16, which states that a constant family of Lie algebroids defines a new Lie algebroid. We first make the following observations.

Lemma 1.16. *Let $A_2 \to M_2$ be a vector bundle and M_1 be another manifold. Let $A := p_2^*(A_2)$ be the vector bundle pull-back of A_2 to the product $M_1 \times M_2$ via projection $p_2 : M_1 \times M_2 \to M_2$. If $A_2 \to M_2$ is a Lie algebroid, then $A \to M_1 \times M_2$ is also Lie algebroid with $[f \otimes X, g \otimes Y] = fg \otimes [X, Y]$ for all $f, g \in \mathcal{C}^\infty(M_1)$ and $X, Y \in \Gamma(A_2)$, where we regard $\mathcal{C}^\infty(M_1) \otimes \Gamma(M_2, A_2) \subset \Gamma(M_1 \times M_2, A)$ in the obvious natural way.*

Proof. This follows from definitions. □

Remark 1.17. A slight generalization of Lemma 1.16 would be that if $\mathfrak{g}$ is an R-Lie-Rinehart algebra and R_1 is another ring, then $R_1 \otimes \mathfrak{g}$ (tensor product over the real numbers) is an $R_1 \otimes R$-Lie-Rinehart algebra, except that, in our case, we are really considering also completions (of $R_1 \otimes R$ and of $R_1 \otimes \mathfrak{g}$) with respect to the natural topologies.

We now make the Lie algebroid structure in Lemma 1.16 more explicit.

Remark 1.18. The isomorphism $\Gamma(M_1 \times M_2; A) \simeq \mathcal{C}^\infty(M_1; \Gamma(M_2; A_2))$ identifies the Lie bracket on the space of sections of the vector bundle $A \to M_1 \times M_2$ of Lemma 1.16 with

$$[X, Y](m) := [X(m), Y(m)],$$

where $m \in M$ and $X, Y \in \Gamma(M_1 \times M_2; A) \simeq \mathcal{C}^\infty(M_1; \Gamma(M_2; A_2))$, so that the evaluations $X(m), Y(m) \in \Gamma(M_2; A_2)$ are defined. The anchor identifies with the composition

$$A := p_2^*(A_2) \to p_2^*(TM_2) = M_1 \times TM_2 \subset T(M_1 \times M_2),$$

where the first map is induced by the anchor of A_2.

We now introduce products of Lie algebroids [39] (our notation is slightly different from the one in that book).

Corollary 1.19. *Let $A_i \to M_i$, $i = 1, 2$, be Lie algebroids and let $p_1^*(A_1)$ and $p_2^*(A_2)$ be their vector bundle pull-backs to $M_1 \times M_2$ with their natural Lie algebroid structures (introduced in Lemma 1.16). Then*

$$A_1 \boxtimes A_2 := p_1^*(A_1) \oplus p_2^*(A_2) \simeq A_1 \times A_2 \to M_1 \times M_2$$

has a natural Lie algebroid structure $A_1 \boxtimes A_2 \to M_1 \times M_2$ such that $\Gamma(M_1; A_1)$ and $\Gamma(M_2; A_2)$ commute in $\Gamma(M_1 \times M_2; A_1 \boxtimes A_2)$. We notice that $\Gamma(M_1 \times M_2; p_i^*(A_i))$ is thus a sub Lie algebra of $\Gamma(M_1 \times M_2; A_1 \boxtimes A_2)$, $i = 1, 2$.

The Lie algebroid $A_1 \boxtimes A_2$ just defined is called the *direct product Lie algebroid* (see, for instance, [39]) and is thus isomorphic, as a vector bundle, to the product $A_1 \times A_2 \to M_1 \times M_2$. We shall need the following important related construction. The following definition is from [27], pages 202–203. See [39] for more details.

Definition 1.20 (Higgins-Mackenzie). Let $A \to L$ be a Lie algebroid over L with anchor $\varrho : A \to TL$. Let $f : M \to L$ be a smooth map and define

$$A \oplus_{TL} TM := \{(\xi, X) \in A \times TM,\ \varrho(\xi) = f_*(X) \in TL\}.$$

Assume $A \oplus_{TL} TM$ defines a smooth vector bundle over M. Then we define the *thick pull-back* Lie algebroid of A by f by $f^{\downarrow\downarrow}(A) := A \oplus_{TL} TM$, with the obvious anchor and bracket.

As we will see shortly, it is easy to verify that if f is a tame submersion of manifolds with corners, then $f^{\downarrow\downarrow}(A)$ is defined and is a Lie algebroid. The anchor and bracket will be made fully explicit in that case. We shall use Lemma 1.3(ii) to reduce the proof of this fact to the case of products, which we treat first.

Lemma 1.21. *Let $A \to L$ be a Lie algebroid over a manifold with corners L and let Y be a smooth manifold. If f denotes the projection $L \times Y \to L$, then*

$$f^{\downarrow\downarrow}(A) \simeq A \boxtimes TY \simeq f^*(A) \oplus (L \times TY),$$

the first isomorphism being an isomorphism of Lie algebroids and the second isomorphism being simply an isomorphism of vector bundles. The bundle $L \times TY \to L \times Y$ is the pull-back vector bundle of $TY \to Y$ to $L \times Y$ via the projection $L \times Y \to Y$.

Proof. The result follows from Corollary 1.19 and Definition 1.20. □

Thus, in general, the Lie algebroid pull-back (or thick pull-back) $f^{\downarrow\downarrow}(A)$ will not be isomorphic to the vector bundle pull-back $f^*(A)$. The following was stated in the smooth case in [39].

Proposition 1.22. *Let $f : M \to L$ be a surjective tame submersion of manifolds with corners and $A \to L$ be a Lie algebroid. Then the thick pull-back $f^{\downarrow\downarrow}(A)$ is defined (that is, it is a Lie algebroid). Let $T_{vert}(f) := \ker(f_* : TM \to TL)$, then $T_{vert}(f) \subset f^{\downarrow\downarrow}(A)$ is an inclusion of Lie algebroids and $f^{\downarrow\downarrow}(A)/T_{vert}(f) \simeq f^*(A)$ as vector bundles.*

Proof. This is a local result, so it follows from Lemmas 1.3 and 1.21. □

2. Constructions with Lie groupoids

We now introduce some basic constructions using Lie groupoids.

2.1. Basic examples of groupoids

We continue with various examples of constructions of Lie groupoids and Lie algebroids that will be needed in what follows.

We begin with three basic examples. Most of these examples are extensions to the category of manifolds with corners of some examples from the category of locally compact spaces. The category of locally compact spaces will not be considered separately, however. Recall the definition of the convolution product on $\mathcal{C}_c^\infty(\mathcal{G})$ from Equation (6).

Example 2.1. Any *Lie group* G is a Lie groupoid with associated Lie algebroid $A(G) = Lie(G)$, the Lie algebra of G. Let us assume G unimodular, for simplicity, then the product on $\mathcal{C}_c^\infty(\mathcal{G}) = \mathcal{C}_c^\infty(G)$ is simply the convolution product with respect to a Haar measure.

At the other end of the spectrum, we have the following example.

Example 2.2. Let M be a manifold with corners and let $\mathcal{G}^{(1)} = \mathcal{G}^{(0)} = M$, so the groupoid of this example contains only units. We shall call a groupoid with these properties a *space*. We have $A(M) = M \times \{0\}$, the zero vector bundle over M. The product on $\mathcal{C}_c^\infty(\mathcal{G}) = \mathcal{C}_c^\infty(M)$ is nothing but the pointwise product of two functions.

We thus see that the category of Lie groupoids contains the subcategory of Lie groups and the subcategory of manifolds (possibly with corners). The last basic example is that of a product.

Example 2.3. Let $\mathcal{G}_i \rightrightarrows M_i$, $i = 1, 2$, be two Lie groupoids. Then $\mathcal{G}_1 \times \mathcal{G}_2$ is a Lie groupoid with units $M_1 \times M_2$. We have $A(\mathcal{G}_1 \times \mathcal{G}_2) \simeq A(\mathcal{G}_1) \boxtimes A(\mathcal{G}_2)$, by Proposition 4.3.10 in [39].

We shall need some more specific classes of Lie groupoids. The goal is to successively build more and more general examples that will lead us to our desired desingularization procedure. We proceed by small steps, mainly due to the complicated nature of this construction, but also because particular or intermediate cases of this construction are needed on their own. The following example is crucial in what follows, since it will be used in the definition of the desingularization groupoid.

Example 2.4. Let G be a Lie group with automorphism group $\mathrm{Aut}(G)$ and let $P \to M$ be a principal $\mathrm{Aut}(G)$-bundle. Then the associated fiber bundle $\mathcal{G} := P \times_{\mathrm{Aut}(G)} G$ (with fiber G) is a Lie groupoid called a *Lie group bundle* or a *bundle of Lie groups*. We have $d = r : \mathcal{G} \to M$ and $A(\mathcal{G}) \simeq P \times_{\mathrm{Aut}(G)} Lie(G)$ in this example. We shall be concerned with this example especially in the following two particular situations. Let $\pi : E \to M$ be a smooth *real vector bundle* over a manifold with corners. Then each fiber $E_m := \pi^{-1}(m)$ is a commutative Lie group, and hence E is a Lie groupoid with the corresponding Lie group bundle structure. A related frequently used example is obtained as follows. Let $\mathbb{R}_+^* = (0, \infty)$ act on the fibers of the vector bundle $\pi : E \to M$ by dilation. This yields, for each $m \in M$, the semi-direct product $G_m := E_m \rtimes \mathbb{R}_+^*$. Then $\mathcal{G} := \cup G_m$ is a Lie group bundle, and hence has a natural Lie groupoid structure. Typically, we will use this construction for $E = A(\mathcal{H})$, the Lie algebroid of some Lie groupoid $\mathcal{H}$, in which case these constructions appear in the definitions of the adiabatic groupoid and of the edge modification, and hence in the definition of the desingularization of a Lie groupoid. Equation (6) becomes the fiberwise group convolution product.

Example 2.5. Let M be a smooth manifold (thus M *does not have corners*). Then we define the *pair groupoid* of M as $\mathcal{G} := M \times M$, a groupoid with units M and with d the second projection, r the first projection, and $(m_1, m_2)(m_2, m_3) = (m_1, m_3)$. We have $A(M \times M) = TM$, with anchor map the identity map. A related example is that of $\mathcal{P}M$, the *path groupoid* of M, defined as the set of fixed end point homotopy classes of paths in M. It has the same Lie algebroid as the pair groupoid: $A(\mathcal{P}M) = TM$, but it leads to differential operators with completely different properties (and hence to a different type of Analysis). See [25] for a description of all groupoids integrating TM.

Remark 2.6. Let $\mathcal{G} := M \times M$ be the pair groupoid for a *smooth manifold* M. The product on $\mathcal{C}_c^\infty(\mathcal{G}) = \mathcal{C}_c^\infty(M \times M)$ is then simply the product of integral kernels. Indeed, let us fix a metric on $A(M \times M) = TM$ and hence a volume form (i.e. measure) $d\,\mathrm{vol}$ on M. Then Equation (6) becomes

$$\phi * \psi(x, z) = \int_M \phi(x, y)\psi(y, z)\, d\,\mathrm{vol}(y). \tag{8}$$

This is the reason why the pair groupoids are so basic in our considerations. (In fact, any Radon measure on M with full support could be considered.)

We need to recall the concept of a *morphism* of two groupoids, because we want equivariance properties of our constructions.

Definition 2.7. Let $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows L$ be two groupoids. A *morphism* $\phi : \mathcal{H} \to \mathcal{G}$ is a functor of the corresponding categories.

More concretely, a morphism $\phi : \mathcal{H} \to \mathcal{G}$ is required to satisfy $\phi(gh) = \phi(g)\phi(h)$. Then there will also exists a map $\phi_0 : L \to M$ such that $d(\phi(g)) = \phi_0(d(g))$, $r(\phi(g)) = \phi_0(r(g))$, and $\phi(u(x)) = u(\phi_0(x))$. If $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows L$ are Lie groupoids and the groupoid morphism $\phi : \mathcal{H} \to \mathcal{G}$ is smooth, we shall say that ϕ is a *Lie groupoid morphism.*

If Γ is a Lie group and $\mathcal{G} \rightrightarrows M$ is a Lie groupoid, we shall say that Γ *acts* on $\mathcal{G}$ if there exists a smooth map $\alpha : \Gamma \times \mathcal{G} \to \mathcal{G}$ such that, for each $\gamma \in \Gamma$, the induced map $\alpha_\gamma : \mathcal{G} \ni g \to \alpha(\gamma, g) \in \mathcal{G}$ is a Lie groupoid morphism and $\alpha_\gamma \alpha_\delta = \alpha_{\gamma\delta}$.

We now recall the important construction of *fibered pull-back groupoids* [27, 28].

Example 2.8. Let $f : M \to L$ be a function and $d, r : \mathcal{H} \rightrightarrows L$ be a groupoid (so L is the set of units of $\mathcal{H}$), the *fibered pull-back groupoid* $f^{\downarrow\downarrow}(\mathcal{H})$ is then

$$f^{\downarrow\downarrow}(\mathcal{H}) := \{(m, g, m') \in M \times \mathcal{H} \times M | f(m) = r(g),\, d(g) = f(m')\}.$$

It is a groupoid with units M and with $d(m, g, m') = m'$, $r(m, g, m') = m$, and product $(m, g, m')(m', g', m'') = (m, gg', m'')$. We shall also sometimes write $M \times_f \mathcal{H} \times_f M = f^{\downarrow\downarrow}(\mathcal{H})$ for the fibered pull-back groupoid. We shall use this construction in the case when f is a tame submersion of manifolds with corners and $\mathcal{H}$ is a Lie groupoid. Then $f^{\downarrow\downarrow}(\mathcal{H})$ is a Lie groupoid (the fibered pull-back Lie groupoid). Indeed, to see that d is a tame submersion, it is enough to write that f is locally a product, see Lemma 1.3(ii). The groupoid $f^{\downarrow\downarrow}(\mathcal{H})$ is a subgroupoid of the product $M \times M \times \mathcal{H}$ of the pair groupoid $M \times M$ and $\mathcal{H}$. Also by Proposition 4.3.11 in [39], we have

$$A\big(f^{\downarrow\downarrow}(\mathcal{H})\big) \simeq f^{\downarrow\downarrow}\big(A(\mathcal{H})\big) \tag{9}$$

(see Definition 1.20). Thus the Lie algebroid of the fibered pull-back groupoid $f^{\downarrow\downarrow}(\mathcal{H})$ is the thick pull-back Lie algebroid $f^{\downarrow\downarrow}\big(A(\mathcal{H})\big)$ and hence it contains as a Lie algebroid the space $\ker(f_*)$ of f-vertical tangent vector fields on M. We note that if a Lie group Γ acts (smoothly by groupoid automorphisms) on $\mathcal{H} \rightrightarrows L$ and if the map $f : M \to L$ is Γ-equivariant, then Γ will act on $f^{\downarrow\downarrow}(\mathcal{H})$.

2.2. Adiabatic groupoids and the edge-modification

Our desingularization uses in an essential way *adiabatic groupoids.* In this subsection, we shall thus recall in detail the construction of the adiabatic groupoid $\mathcal{G}_{ad}$ associated to a Lie groupoid $\mathcal{G}$, as well as some related constructions [13, 21, 29, 53]. For the purpose of further applications, we stress the smooth action of a Lie group Γ (by Lie groupoid automorphisms) and thus the functoriality of our constructions.

Let $\mathcal{G}$ be a Lie groupoid with units M and Lie algebroid $A := A(\mathcal{G}) \to M$. The *adiabatic groupoid* $\mathcal{G}_{ad}$ *associated to* $\mathcal{G}$ will have units $M \times [0, \infty)$. We shall define $\mathcal{G}_{ad}$ in several steps: first we define its Lie algebroid, then we define it as a set, then we recall the unique smooth structure that yields the desired Lie algebroid, and, finally, we show that this construction is functorial and thus preserves group actions.

2.2.1. The Lie algebroid of the adiabatic groupoid. We first define a Lie algebroid $A_{ad} \to M \times [0, \infty)$ that will turn out to be isomorphic to

$A(\mathcal{G}_{ad})$, as in [53]. As vector bundles, we have

$$A_{ad} := A \times [0, \infty) \to M \times [0, \infty).$$

That is, A_{ad} is the vector bundle pull-back of $A \to M$ to $M \times [0, \infty)$ via the canonical projection $\pi : M \times [0, \infty) \to M$. To define the Lie algebra structure on the space of sections of A_{ad}, let $X(t)$ and $Y(t)$ be sections of A_{ad}, regarded as smooth functions $[0, \infty) \to \Gamma(M; A(\mathcal{G}))$. Then

$$[X, Y](t) := t[X(t), Y(t)]. \tag{10}$$

Let us denote by $\pi^*(A)$ the Lie algebroid defined by the vector bundle pull-back, as in Lemma 1.16. Thus we see that $A_{ad} \simeq \pi^*(A)$ as *vector bundles* but **not** as Lie algebroids. Nevertheless, we do have a natural Lie algebroid morphism (over $M \times [0, \infty)$, not injective!)

$$A_{ad} \simeq t\pi^*(A) \to \pi^*(A), \tag{11}$$

where the second Lie algebroid is defined by Lemma 1.14 and the isomorphism is by Equation (10). The induced map identifies $\Gamma(A_{ad})$ with $t\Gamma(\pi^*(A))$, however.

2.2.2. The underlying groupoid of $\mathcal{G}_{ad}$. We shall define the adiabatic groupoid $\mathcal{G}_{ad}$ as the *disjoint* union of two Lie groupoids, denoted $\mathcal{G}_1$ and $\mathcal{G}_2$, which we define first. This will also define the groupoid structure on $\mathcal{G}_{ad}$ (but not the smooth structure yet!). We let $\mathcal{G}_1 := A(\mathcal{G}) \times \{0\}$ with the Lie groupoid structure of a bundle of commutative Lie groups $A(\mathcal{G}) \times \{0\} \to M$. (That is $\mathcal{G}_1$ is simply a vector bundle, regarded as a Lie groupoid as in Example 2.4.) The groupoid $\mathcal{G}_2$ is given by $\mathcal{G}_2 := \mathcal{G} \times (0, \infty)$, with the product Lie groupoid structure, where $(0, \infty)$ is regarded as a space (as in Example 2.2). As a set, we then define the adiabatic groupoid $\mathcal{G}_{ad}$ associated to $\mathcal{G}$ as the *disjoint* union

$$\mathcal{G}_{ad} := \mathcal{G}_1 \sqcup \mathcal{G}_2 := \big(A(\mathcal{G}) \times \{0\}\big) \sqcup \big(\mathcal{G} \times (0, \infty)\big). \tag{12}$$

We endow $\mathcal{G}_{ad}$ with the natural groupoid structure $d, r : \mathcal{G}_{ad} \to M \times [0, \infty)$, where d and r restrict on each of $\mathcal{G}_1$ and $\mathcal{G}_2$ to the corresponding domain and range maps, respectively.

2.2.3. The Lie groupoid structure on $\mathcal{G}_{ad}$. We endow $\mathcal{G}_{ad} := \mathcal{G}_1 \cup \mathcal{G}_2$ with the unique smooth structure that makes it a Lie groupoid with Lie algebroid A_{ad}, as in [51]. We proceed as in [13, 21, 29] using a (real version of) the "deformation to the normal cone" considered in those papers. Let us make that construction explicit in our case. We thus choose connections $\nabla : \Gamma(T\mathcal{G}_x) \to \Gamma(T\mathcal{G}_x \otimes T^*\mathcal{G}_x)$ on all the manifolds $\mathcal{G}_x := d^{-1}(x)$, $x \in M$. As in [53], we can choose these connections such that the resulting family of connections is invariant with respect to right multiplication by elements in $\mathcal{G}$. (One way to achieve this is to consider an embedding of $A(\mathcal{G})$ into a trivial bundle. This gives an equivariant family of embeddings of each $T\mathcal{G}_x$ into a trivial bundle, and then we can choose the corresponding orthogonal connections.) This gives rise to a smooth map $\exp_\nabla : A = A(\mathcal{G}) \to \mathcal{G}$ that maps the zero section of $A(\mathcal{G})$ to the set of units of $\mathcal{G}$. There exists a neighborhood U of the zero section of $A(\mathcal{G})$ on which $\exp_\nabla$ is a diffeomorphism onto its image. Let us define then $W = W_U \subset A \times [0, \infty) = A_{ad}$ to be the set of pairs $(X, t) \in A \times [0, \infty)$ such that $tX \in U$ and define $\Phi : W \to \mathcal{G}_{ad}$ by the formula

$$\Phi(X,t) := \begin{cases} (\exp_\nabla(tX), t) \in \mathcal{G} \times (0, \infty) & \text{if } t > 0 \\ (X, 0) \in A(\mathcal{G}) \times \{0\} & \text{if } t = 0. \end{cases} \tag{13}$$

We define the smooth structure on $\mathcal{G}_{ad}$ such that both the image of Φ and the set $\mathcal{G} \times (0, \infty)$ are open subsets of $\mathcal{G}_{ad}$, with the induced smooth structures on W_U and $\mathcal{G} \times (0, \infty)$ coinciding with the original ones. We obtain a manifold structure on $\mathcal{G}_{ad}$ since transition functions are smooth. The fact that the resulting smooth structure makes $\mathcal{G}_{ad}$ a Lie groupoid follows from the differentiability with respect to parameters (including initial data) of solutions of ordinary differential equations. This smooth structure does not depend on the choice of the connection ∇, since the choice of a different connection would just amount to the conjugation with a local diffeomorphism ψ of $\mathcal{G}$ in a neighborhood of the units. By construction, the space of sections of $A(\mathcal{G}_{ad})$ identifies with $t\Gamma(\pi^*(A))$, and hence $A(\mathcal{G}_{ad}) \simeq A_{ad}$, as desired. (Note that by [51, 53], it is known that there exists a unique Lie groupoid structure on $\mathcal{G}_{ad}$ such that the associated Lie algebroid is A_{ad} and in this remark we have done nothing but to make more explicit the construction in [51].)

2.2.4. Actions of Lie groups. The following lemma states that the adiabatic construction is compatible with Lie group actions.

Lemma 2.9. *Let Γ be a Lie group and assume that Γ acts on $\mathcal{G} \rightrightarrows M$, then Γ acts on $\mathcal{G}_{ad}$ as well.*

Proof. We use the notation in 2.2.2. We obtain immediately an action of Γ on each of $\mathcal{G}_1$ and $\mathcal{G}_2$. To see that this extends to an action on the adiabatic groupoid, we need to check the compatibility with the coordinate map Φ. Let V be a compact neighborhood of the identity in Γ. We can choose an open neighborhood $U_1 \subset U$ of the set of units of $\mathcal{G}$ such that the action of Γ on M maps $V \times U_1$ to U. Then $V \times W_{U_1}$ maps to W_U and the resulting map is smooth by the invariance of the smooth structure on $\mathcal{G}_{ad}$ with respect to the choice of connection. More precisely, denoting all the actions induced by $\gamma \in \Gamma$ by α_γ, we obtain

$$\alpha_\gamma\big(\exp_\nabla(t)\big)\big) = \exp_{\alpha_\gamma(\nabla)}\big(t(\alpha_\gamma)_*(X)\big) = \exp_{\alpha_\gamma(\nabla)}\big((\alpha_\gamma)_*(tX)\big).$$

□

2.2.5. Extensions of the adiabatic groupoid construction. We shall need two slight generalizations of the adiabatic groupoid construction. We shall use the reduction of a groupoid $\mathcal{G}$ to a subset K, which, we recall, is denoted $\mathcal{G}_K^K := r^{-1}(K) \cap d^{-1}(K)$.

Example 2.10. Let again M and L be manifolds with corners and $f : M \to L$ be a tame submersion of manifolds with corners. Let $\mathcal{H} \rightrightarrows L$ be a Lie groupoid with adiabatic groupoid $\mathcal{H}_{ad} \rightrightarrows L \times [0, \infty)$. Let $\mathcal{G} := f^{\downarrow\downarrow}(\mathcal{H}) = M \times_f \mathcal{H} \times_f M$ be the fibered pull-back groupoid. Then the *adiabatic groupoid of $\mathcal{G}$ with respect to f* has units $M \times [0, \infty)$ and is defined by

$$\mathcal{G}_{ad,f} := f_1^{\downarrow\downarrow}(\mathcal{H}_{ad}),$$

where $f_1 := (f, id) : M \times [0, \infty) \to L \times [0, \infty)$. Unlike $\mathcal{G}_{ad}$, the groupoid $\mathcal{G}_{ad,f}$ will not be a bundle of Lie groups at time 0, but will be the fibered pull-back of the Lie groupoid $A(\mathcal{H}) \to L$, regarded as a bundle of Lie groups, by the map $f : M \to L$. More precisely, let $X := M \times \{0\}$, which is an invariant subset of the set of units of $M \times [0, \infty)$. Then the restriction of $\mathcal{G}_{ad,f}$ to X satisfies

$$(\mathcal{G}_{ad,f})_X \simeq M \times_f A(\mathcal{H}) \times_f M =: f^{\downarrow\downarrow}\big(A(\mathcal{H})\big). \tag{14}$$

Remark 2.11. We use the notation in Example 2.10. If $\mathcal{H} = L \times L$, then $\mathcal{G} = M \times M$ (so both $\mathcal{H}$ and $\mathcal{G}$ are pair groupoids in this particular case) and $\mathcal{G}_{ad,f}$ at time 0 will be the fibered pull-back to M of the Lie groupoid $A(\mathcal{H}) = TL \to L$. In this particular case, the associated differential operators on $\mathcal{G}_{ad,f}$ model adiabatic limits, hence the name of these groupoids (this explains the choice of the name "adiabatic groupoid" in [53]).

For the next example, we shall need to introduce an action of $\mathbb{R}_+^*$ on the groupoid in the last example, as in [21].

Remark 2.12. We use the same setting and notation as in Example 2.10 above and let $\mathbb{R}_+^* = (0,\infty)$ act by dilations on the time variable $[0,\infty)$. This action induces a family of automorphisms of $\mathcal{H}_{ad}$, as in [21] if we let $s \in \mathbb{R}_+^* = (0,\infty)$ act by $s \cdot (g,t) = (g, s^{-1}t)$ on $(g,t) \in \mathcal{H} \times (0,\infty) \subset \mathcal{H}_{ad}$. Referring to Equation (13) that defines a parametrization of a neighborhood of $A(\mathcal{H}) \times \{0\} \subset \mathcal{H}_{ad}$, we obtain

$$\begin{aligned} s \cdot \Phi(X,t) := s \cdot (\exp_\nabla(tX), t) &:= (\exp_\nabla(tX), s^{-1}t) \\ &= (\exp_\nabla(s^{-1}tsX), s^{-1}t) =: \Phi(sX, s^{-1}t). \end{aligned}$$

By setting $t = 0$ in this equation, we obtain by continuity that the action of s on $(X,0)$ is $s(X,0) = (sX,0)$.

We shall use this remark to obtain a (slight extension of a) construction in [21]. Recall that if a Lie group Γ acts on a Lie groupoid $\mathcal{G} \rightrightarrows M$, then the *semi-direct product* [39, 45] $\mathcal{G} \rtimes \Gamma$ is defined by

(i) $\mathcal{G} \rtimes \Gamma = \mathcal{G} \times \Gamma$, as manifolds.

(ii) $\mathcal{G} \rtimes \Gamma$ has units M (same as $\mathcal{G}$).

(iii) $(g_1,\gamma_1)(g_2,\gamma_2) := (g_1\gamma_1(g_2), \gamma_1\gamma_2)$, when $g_1\gamma_1(g_2)$ is defined in $\mathcal{G}$.

Example 2.13. We use the notation in Example 2.10 and in Remark 2.12. In particular, we denote $f_1 := (f, id) : M \times [0,\infty) \to L \times [0,\infty)$. The action of $\mathbb{R}_+^*$ commutes with f_1 and induces an action on $\mathcal{G}_{ad,f} := f_1^{\downarrow\downarrow}(\mathcal{H}_{ad})$ and we let

$$\mathcal{E}(M,f,\mathcal{H}) := \mathcal{G}_{ad,f} \rtimes \mathbb{R}_+^* := f_1^{\downarrow\downarrow}(\mathcal{H}_{ad}) \rtimes \mathbb{R}_+^* = f_1^{\downarrow\downarrow}(\mathcal{H}_{ad} \rtimes \mathbb{R}_+^*),$$

be the associated semi-direct product groupoid. The space of units of $\mathcal{E}(M,f,\mathcal{H})$ is $M \times [0,\infty)$. The groupoid $\mathcal{H}_{ad} \rtimes \mathbb{R}_+^*$ was introduced and studied in [21] under the name *gauge adiabatic groupoid.*

Let us spell out in detail the structure of the groupoid $\mathcal{E}(M,f,\mathcal{H})$.

Remark 2.14. To describe $\mathcal{E}(M,f,\mathcal{H})$ as a set, we shall describe its reductions to $M \times \{0\}$ and to $M \times (0,\infty)$ (that is, we shall describe its reductions at time $t = 0$ and at time $t > 0$). Let us endow $A(\mathcal{H})$ with the Lie groupoid structure of a (commutative) bundle of Lie groups with units $L \times \{0\}$. Then, at time $t = 0$, $\mathcal{E}(M, f, \mathcal{H})$ is the semi-direct product $f^{\downarrow\downarrow}(A(\mathcal{H})) \rtimes \mathbb{R}_+^* \simeq f^{\downarrow\downarrow}(A(\mathcal{H}) \rtimes \mathbb{R}_+^*)$, with $\mathbb{R}_+^*$ acting by dilations on the fibers of $A(\mathcal{H})$. That is

$$\begin{aligned} \mathcal{E}(M, f, \mathcal{H})_{\{0\}\times M} &\simeq (M \times_f A(\mathcal{H}) \times_f M) \rtimes \mathbb{R}_+^* \\ &= M \times_f (A(\mathcal{H}) \rtimes \mathbb{R}_+^*) \times_f M \end{aligned} \tag{15}$$

Thus $\mathcal{E}(M, f, \mathcal{H})_{M\times\{0\}}$ is the fibered pull-back to $M \times \{0\}$ via f of a bundle of solvable Lie groups on L. On the other hand, the complement, that is, the reduction of $\mathcal{E}(M, f, \mathcal{H})$ to $M \times (0, \infty)$ is isomorphic to the product groupoid $f^{\downarrow\downarrow}(\mathcal{H}) \times (0, \infty)^2$, where the first factor in the product is the fibered pull-back of $\mathcal{H}$ to M and the second factor is the pair groupoid of $(0, \infty)$.

For the pair groupoid $\mathcal{G} = M \times M$ with M smooth, compact, the example of the adiabatic groupoid is due to Connes [13] and was studied in connection with the index theorem for smooth, compact manifolds. See [13, 21, 53] for more details.

The construction of the edge modifications is equivariant.

Lemma 2.15. *Let us assume with the same notation that a Lie group Γ acts on $\mathcal{H} \rightrightarrows L$ and that the tame submersion $f : M \to L$ is Γ invariant. Then Γ acts on $\mathcal{E}(M, f, \mathcal{H})$ in a way that is compatible with the structure provided by Remark 2.14.*

Proof. The group Γ acts on $\mathcal{H}_{ad}$ by Lemma 2.9. This action commutes with the action of $\mathbb{R}_+^*$ by naturality. Hence we obtain an action of Γ on $\mathcal{H}_{ad} \rtimes \mathbb{R}_+^*$. The result follows since $f : M \to L$ is Γ invariant. $\square$

2.3. Glueing Lie groupoids

We shall need to "glue" two Lie groupoids along an open subset of the set of units above which they are isomorphic. This can be done under certain conditions, and we review now this construction following Theorem 3.4 in [25].

Let $\mathcal{G}_i \rightrightarrows M_i$, $i = 1, 2$, be two Lie groupoids. (Thus, the sets of units, M_i, are Hausdorff manifolds, possibly with corners.) Let us assume that we are given open subsets $U_i \subset M_i$ such that the reductions $(\mathcal{G}_i)_{U_i}^{U_i}$, $i = 1, 2$, are isomorphic via an isomorphism $\phi : (\mathcal{G}_1)_{U_1}^{U_1} \to (\mathcal{G}_2)_{U_2}^{U_2}$ that covers a diffeomorphism $U_1 \to U_2$, also denoted by ϕ. We define $M := M_1 \cup_\phi M_2$ as follows.

Let us consider on the disjoint union $M_1 \sqcup M_2$ the equivalence relation $\sim_\phi$ generated by $x \sim_\phi \phi(x)$ if $x \in U_1$. Then $M_1 \cup_\phi M_2 := M_1 \sqcup M_2/\sim_\phi$. We define similarly

$$\mathcal{H} := \mathcal{G}_1 \cup_\phi \mathcal{G}_2 := (\mathcal{G}_1 \sqcup \mathcal{G}_2)/\sim_\phi . \tag{16}$$

We shall denote by $U_1^c := M_1 \smallsetminus U_1$ the complement of U_1 in M_1 and by

$$U_1 \cap \mathcal{G}_1 U_1^c \mathcal{G}_1 := \{x \in U_1 | (\exists) g \in \mathcal{G}_1,\, d(g) = x,\, r(g) \notin U_1\},$$

the $\mathcal{G}_1$-orbit (or saturation) of U_1^c in M_1. We shall use a similar notation for $\mathcal{G}_2$.

Proposition 2.16. *Let us assume that the set $\phi(U_1 \cap \mathcal{G}_1 U_1^c \mathcal{G}_1)$ does not intersect $U_2 \cap \mathcal{G}_2 U_2^c \mathcal{G}_2$ and that $M := M_1 \cup_\phi M_2$ is a Hausdorff manifold (possibly with corners). Then the set $\mathcal{H}$ of Equation* (16) *has a natural Lie groupoid structure with units M. We have $\mathcal{G}_i \simeq (\mathcal{H})_{M_i}^{M_i}$.*

Proof. This is basically a consequence of the definitions. We define the domain map $d : \mathcal{H} \to M$ by restriction to each of the groupoids $\mathcal{G}_i$, which is possible since $h \sim h'$ implies $d(h) \sim d(h')$. We proceed similarly to define the range map r.

Let us identify $\mathcal{G}_i$ with subsets of $\mathcal{H}$, for simplicity. Hence now $U_1 = U_2$ and ϕ is the identity. To define the product of $g_j \in \mathcal{H}$, $j = 1, 2$, just note that the assumptions ensure that, if $g_j \in \mathcal{G}_j$, for $j = 1, 2$, with $d(g_1) = r(g_2) \in M$, then, first of all, $x := d(g_1) = r(g_2) \in M_1 \cap M_2 = U_1 = U_2$. Next, either $r(g_1) \in U_1$ or $d(g_2) \in U_2 = \phi(U_1) = U_1$, because otherwise

$$d(g_1) = r(g_2) \in U_1 \cap \mathcal{G}_1 U_1^c \mathcal{G}_1 \cap \mathcal{G}_2 U_2^c \mathcal{G}_2,$$

which is in direct contradiction with the hypothesis. This means that, in fact, $g_j \in \mathcal{G}_i$, for the same i, and we can define the multiplication using the multiplication in $\mathcal{G}_i$. □

One of the differences between our result, Proposition 2.16, and Theorem 3.4 in [25] is that we are not starting with a Lie algebroid that needs to be integrated, and hence we do not have to consider orbits. The paper [25] also contains a discussion of the gluing procedure in the framework of manifolds and many other useful results.

3. Desingularization groupoids

We now introduce our desingularization of a Lie groupoid along a tame submanifold of its unit space. We also identify its Lie algebroid and prove some other properties of the desingularization.

3.1. A structure theorem near tame submanifolds

A *tubular neighborhood* of L in M is an open subset U, $L \subset U \subset M$, together with a smooth vector bundle structure $\pi : U \to L$, with π the identity on L. In the framework of manifolds with corners, a tubular neighborhood is thus one that is locally of the form

$$L_0 \simeq L_0 \times \{0\} \subset L_0 \times \mathbb{R}^n \to L_0,$$

as in the smooth case. For example, if we let $\Delta_M := \{(m, m) | m \in M\} \subset M \times M$, where M is a manifold with (non-empty) boundary, then Δ_M is a submanifold of $M \times M$ in the sense of this paper, but is not a submanifold with corners of $M \times M$ in the sense of [4]. Moreover, Δ_M *does not* have a tubular neighborhood in $M \times M$ (recall that we are working in the smooth category).

We can now introduce the following definition that is central for what follows.

Definition 3.1. Let $A \to M$ be a Lie algebroid over a manifold M. Let $L \subset M$ be a submanifold of M such that there exists a tubular neighborhood U of L in M with projection map $\pi : U \to L$. We shall say that L is an A-*tame submanifold* of M if there exists a Lie algebroid $A_1 \to L$ such that the restriction of A to U is isomorphic to the thick pull-back Lie algebroid of A_1 to U via π, that is,

$$A|_U \simeq \pi^{\downarrow\downarrow}(A_1), \tag{17}$$

via an isomorphism that is the identity on U. Both M and L may have corners.

Remark 3.2. We note that, by Proposition 1.22, we have that the Lie algebroid A_1 of Definition 3.1 satisfies $A_1 \simeq (A/\ker(\pi_* \circ \varrho))|_L$, and hence A_1 is determined up to an isomorphism by A. We also have that the joint map $(d, r) : \mathcal{G} \to M \times M$ is transverse to $L \times L$. This transversality property follows from the A-tameness of L, which in turn implies that, for every $g \in U$, we have that

$$(d_*, r_*)(T_g\mathcal{G}) \supset T_{d(g)}U \times T_{vert,r(g)}\pi.$$

Note however, that the condition that (d, r) be transverse to $L \times L$ does not involve in an obvious way the Lie algebroid structure of $A(\mathcal{G})$, so it is not clear whether this condition is sufficient for L to be $A(\mathcal{G})$-tame.

Recall that if $\mathcal{G} \rightrightarrows M$ is a groupoid and $K \subset M$, then $\mathcal{G}_K^K := r^{-1}(K) \cap d^{-1}(K)$ is the *reduction of* $\mathcal{G}$ *to* K. *We shall repeatedly use the fact that, if* $L \subset K$, *then* $(\mathcal{G}_K^K)_L^L = \mathcal{G}_L^L$. Also, recall that a topological space is called *simply-connected* if it is path connected and its first homotopy group $\pi_1(X)$ is trivial. A groupoid $\mathcal{G}$ is called *d-simply connected* if the fibers $\mathcal{G}_x := d^{-1}(x)$ of the domain map are simply-connected.

Here is one of our main technical results that provides a canonical form for a Lie groupoid in the neighborhood of a tame submanifold. All the isomorphisms of Lie groupoids are smooth morphisms.

Theorem 3.3. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and let $L \subset M$ be an $A(\mathcal{G})$-tame submanifold of M. Let $U \subset M$ be a tubular neighborhood of L as in Definition 3.1, with $\pi : U \to L \subset U$ the associated structural projection. Then the reduction groupoids $\mathcal{G}_L^L$ and $\mathcal{G}_U^U$ are Lie groupoids. Assume, furthermore, that the fibers of $\pi : U \to L$ are simply-connected. Then there exists an isomorphism*

$$\mathcal{G}_U^U \simeq \pi^{\downarrow\downarrow}(\mathcal{G}_L^L) := U \times_\pi \mathcal{G}_L^L \times_\pi U$$

of Lie groupoids that is the identity on the set of units U.

Proof. First of all, we have that $\mathcal{G}_L^L$ is a Lie groupoid by [39, Proposition 1.5.16] since the joint map $(d, r) : \mathcal{G} \to M \times M$ is transverse to $L \times L$ by Remark 3.2. Then, the *fibered pair* groupoid $\mathcal{H} := U \times_\pi U = \{(u_1, u_2) | \pi(u_1) = \pi(u_2)\}$ is a Lie groupoid with Lie algebroid $T_{vert}\pi = \ker(\pi_*)$. The assumption that the fibers of $\pi : U \to L$ are simply-connected shows that $\mathcal{H}$ is d-simply connected. Since $T_{vert}\pi$ is contained in $A(\mathcal{G})|_U$ as a Lie subalgebroid, by the definition of a $A(\mathcal{G})$-tame submanifold, Proposition 3.4 of [45] (which extends right away to manifolds with corners see also [51]) gives that there exists a morphism of Lie groupoids over U

$$\Phi : \mathcal{H} := U \times_\pi U \to \mathcal{G}_U^U. \tag{18}$$

Recall that "over U" means that Φ preserves the units, in the sense that $d(\Phi(\gamma)) = d(\gamma)$ and $r(\Phi(\gamma)) = r(\gamma)$. In particular, Φ is injective.

Let $(u_1, u_2) \in U \times U$ with $\pi(u_1) = \pi(u_2)$. Then $(u_1, u_2) \in \mathcal{H} := U \times_\pi U$. In particular, $(\pi(u), u) \in L \times_\pi U \in \mathcal{H}$ for any $u \in U$. Let $g(u) := \Phi(\pi(u), u) \in \Phi(\mathcal{H}) \subset \mathcal{G}$, so that $g : U \to \mathcal{G}$ is a smooth map, since Φ is smooth. Then, for any $\gamma \in r^{-1}(U) \cap d^{-1}(U) =: \mathcal{G}_U^U$ and any $u \in U$, we have

$$\begin{gathered} d(g(u)) = d(\pi(u), u) = u, \quad r(g(u)) = r(\pi(u), u) = \pi(u) \in L, \\ h_\gamma := g(r(\gamma))\gamma g(d(\gamma))^{-1} \in \mathcal{G}_L^L, \quad \text{and} \\ \Psi(\gamma) := (r(\gamma), h_\gamma, d(\gamma)) \in U \times_\pi \mathcal{G}_L^L \times_\pi U. \end{gathered}$$

The map Ψ is the desired isomorphism. $\square$

3.2. The edge modification

We shall now use the structure theorem, Theorem 3.3, to introduce a desingularization of a Lie groupoid $\mathcal{G} \rightrightarrows M$ in the neighborhood of a tame submanifold L of its set of units M. We need, however, to first discuss the (real) blow-up of a tame submanifold. We use the standard approach, see for example [2, 32].

Notations 3.4. In what follows, L will be a *tame* submanifold of a manifold with corners (to be specified each time), that is, a submanifold with the property that it has a tubular neighborhood U with structural projection $\pi : U \to L$. We let $S := \partial U$. We shall denote by NL the normal bundle of L in M. We assume that U identifies with the set of vectors of length < 1 in NL, for some arbitrary metric. In particular, $S \simeq SNL$, the set of unit vectors in NL, and $U \smallsetminus L \simeq S \times (0, 1)$.

We now recall the definition of the real blow-up of a manifold with respect to a tame submanifold. We use the notation introduced in 3.4. Let us assume that L is a tame submanifold of a manifold with corners M. Informally, the *real blow-up* or, simply, the *blow-up* of M along L is the manifold with corners obtained by removing L from M and by gluing back $S \simeq SNL$ in a compatible way. The following definition formalizes this idea.

Definition 3.5. Let L be a tame submanifold of a manifold with corners M. We use the notation introduced in 3.4 and we let ϕ be the diffeomorphism $U \smallsetminus L \simeq S \times (0, 1)$. Then the *real blow-up* of M along L, denoted $[M : L]$ is defined by glueing $M \smallsetminus L$ and $S \times [0, 1)$ using ϕ, that is,

$$[M : L] := (M \smallsetminus L) \cup_\phi (S \times [0, 1)) = \big((M \smallsetminus L) \sqcup S \times [0, 1)\big) / \sim_\phi, \tag{19}$$

where $\sim_\phi$ is the equivalence relation generated by $\phi(x) \sim_\phi x$, as in Subsection 2.3.

Remark 3.6. By construction, there exists an associated natural smooth map

$$\kappa : [M : L] \to M,$$

the *blow-down map*, which is uniquely determined by the condition that it be continuous and it be the identity on $M \setminus L$. For example,

$$[\mathbb{R}^{n+k} : \{0\} \times \mathbb{R}^k] \simeq S^{n-1} \times [0, \infty) \times \mathbb{R}^k, \tag{20}$$

with $r \in [0, \infty)$ representing the distance to the submanifold $L = \{0\} \times \mathbb{R}^k$ and S^p denoting the sphere of dimension p (the unit sphere in $\mathbb{R}^{p+1}$). Locally, all blow-ups that we consider are of this form. In Equation (19), the blow-down map is simply $\kappa(x', r, x'') = (rx', x'') \in \mathbb{R}^n \times \mathbb{R}^k$.

The definition of the blow-up in this paper is the one common in Analysis [2, 8, 24, 32, 44]; it is *different*, however, from the one in [7, 25, 55], where one replaces L with $PN = SNL/\mathbb{Z}_2$ instead of $S := SNL$. We are ready now to introduce the desingularization of a Lie groupoid with respect to a tame submanifold in the particular case of a suitable pull-back.

Definition 3.7. Let $\pi : E \to L$ be a euclidean vector bundle. We choose $U \subset E$ to be the set of vectors of length < 1 and $S := \partial U \subset E$, as before. The various restrictions of π will also be denoted by π. Let $\mathcal{H} \rightrightarrows L$ be a Lie groupoid and $\mathcal{G} := \pi^{\downarrow\downarrow}(\mathcal{H}) = U \times_\pi \mathcal{H} \times_\pi U$. Then *the edge modification* of $\mathcal{G}$ is the fibered pull-back groupoid

$$\mathcal{E}(S, \pi, \mathcal{H}) := (S \times_\pi \mathcal{H}_{ad} \times_\pi S) \rtimes \mathbb{R}_+^* \simeq S \times_\pi (\mathcal{H}_{ad} \rtimes \mathbb{R}_+^*) \times_\pi S.$$

Remark 3.8. The edge modification is thus a particular case for $f = \pi : M = S \to L$ of the example 2.13. It is a Lie groupoid with units $S \times [0, \infty)$. We extend in an obvious way the definition of the edge modification to groupoids isomorphic to groupoids of the form $\mathcal{G} = \pi^{\downarrow\downarrow}(\mathcal{H}) = U \times_\pi \mathcal{H} \times_\pi U$.

It will be convenient to fix the following further notation.

Notations 3.9. In what follows, $\mathcal{G} \rightrightarrows M$ will denote a Lie groupoid and $L \subset M$ will be an $A(\mathcal{G})$-tame submanifold. The sets U and $S := \partial U$ have the same meaning as in 3.4. In particular, $\pi : U \to L$ is a tubular neighborhood of L that is chosen as in Definition 3.1, and hence has simply connected fibers. Using Theorem 3.3, we obtain that the reduction $\mathcal{G}_U^U$ is, up to an isomorphism, of the form $\pi^{\downarrow\downarrow}(\mathcal{H}) := U \times_\pi \mathcal{H} \times_\pi U$, and hence its edge-modification $\mathcal{E}(S, \pi, \mathcal{H})$ is defined. (Note that $\mathcal{H}$ is determined by $\mathcal{G}$: $\mathcal{H} = \mathcal{G}_L^L$.) Let $M_1 = S \times [0, 1)$, which is an open subset of the set $S \times [0, \infty)$ of units of $\mathcal{E}(S, \pi, \mathcal{H})$. We let $\mathcal{G}_1 := \mathcal{E}(S, \pi, \mathcal{H})_{M_1}^{M_1}$, the reduction of $\mathcal{E}(S, \pi, \mathcal{H})$ to M_1, and $U_1 := U \smallsetminus L = S \times (0, 1) \subset M_1$. Similarly, $\mathcal{G}_2 := \mathcal{G}_{M \smallsetminus L}^{M \smallsetminus L}$ denotes the reduction of the groupoid $\mathcal{G}$ to $M \smallsetminus L$.

Remark 3.10. Using the notation and assumptions of Definition 3.7 and the notation introduced in 3.4 and 3.9, we have that the reduction of $\mathcal{G}_1$ to U_1 (which, by the definition of $\mathcal{G}_1$ is the reduction of $\mathcal{E}(S, \pi, \mathcal{H})$ to $U_1 := U \smallsetminus L = S \times (0, 1) \subset M_1$) is isomorphic to

$$(\mathcal{G}_1)_{U_1}^{U_1} \simeq (\mathcal{E}(S, \pi, \mathcal{H}))_{U_1}^{U_1} \simeq (S \times_\pi \mathcal{H} \times_\pi S) \times (0, 1)^2 \simeq U_1 \times_\pi \mathcal{H} \times_\pi U_1, \tag{21}$$

where $(0, 1)^2$ is the pair groupoid. Since the reduction of $\mathcal{G}$ to U is isomorphic to $U \times_\pi \mathcal{H} \times_\pi U$, by Theorem 3.3, it follows that the reduction of $\mathcal{G}$ to U_1 is isomorphic to $U_1 \times_\pi \mathcal{H} \times_\pi U_1$. Hence the reduction of $\mathcal{G}_2$ to U_1 is also isomorphic to $U_1 \times_\pi \mathcal{H} \times_\pi U_1$. We thus obtain an isomorphism of Lie groupoids

$$\phi : (\mathcal{G}_1)_{U_1}^{U_1} \to (\mathcal{G}_2)_{U_1}^{U_1} \simeq U_1 \times_\pi \mathcal{H} \times_\pi U_1. \tag{22}$$

We are thus in position to glue the groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ along their isomorphic reductions to U_1 using Proposition 2.16 (for $U_2 = U_1$).

We can now define the desingularization of a groupoid with respect to a tame submanifold.

Definition 3.11. Let $L \subset M$ be an $A(\mathcal{G})$-tame manifold. Let us use the notation we have just defined in Remark 3.10. The *desingularization of $\mathcal{G}$ along L* is the groupoid obtained by glueing the groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ along their isomorphic reductions to $U_1 = S \times (0, 1)$, using Proposition 2.16. We shall denote this desingularization groupoid by $[[\mathcal{G} : L]]$.

Remark 3.12. We note that the hypothesis of Proposition 2.16 are satisfied because U_1^c is an invariant subset of M_1.

Remark 3.13. To summarize the construction of the desingularization, let us denote by ϕ the natural isomorphism of the following two groupoids:

$\mathcal{E}(S, \pi, \mathcal{H})_{U_1}^{U_1}$ (reduction to $U_1 \simeq S \times (0,1)$) and $\mathcal{G}_{U_1}^{U_1} = (\mathcal{G}_2)_{U_1}^{U_1}$. Then

$$[[\mathcal{G} : L]] := \mathcal{E}(S, \pi, \mathcal{H})_{U_1}^{U_1} \cup_\phi \mathcal{G}_{M \smallsetminus L}^{M \smallsetminus L} =: \mathcal{G}_1 \cup_\phi \mathcal{G}_2 = (\mathcal{G}_1 \sqcup \mathcal{G}_2)/ \sim_\phi . \tag{23}$$

One should not confuse $[[\mathcal{G} : L]]$ with $[\mathcal{G} : L]$, the blow-up of the manifold $\mathcal{G}$ with respect to the submanifold L.

Recall that $\kappa : [M : L] \to M$ denotes the blow-down map, see Remark 3.6. The following result is crucial in studying the desingularization $[[\mathcal{G} : L]]$. Recall that we endow $A(\mathcal{H}) \to L$ with the Lie groupoid structure of a bundle of Lie groups, that $S = \partial U = \kappa^{-1}(L) = [M : L] \smallsetminus (M \smallsetminus L)$ is a closed subset of $[M : L]$. As agreed, here U is the given tubular neighborhood of L and $\pi : S \to L$ denotes the natural (fiber bundle) projection, as in 3.4 and 3.9.

Proposition 3.14. *The space of units of $[[\mathcal{G} : L]]$ is $[M : L]$ and $S := \kappa^{-1}(L)$ is a $[[\mathcal{G} : L]]$-invariant subset of $[M : L]$ with complement $[M : L] \smallsetminus S = M \smallsetminus L$. Also,*

$$[[\mathcal{G} : L]]_S \simeq \pi^{\downarrow\downarrow}\big(A(\mathcal{H}) \rtimes \mathbb{R}_+^*\big) \quad \textit{and} \quad [[\mathcal{G} : L]]_{M \smallsetminus L} = \mathcal{G}_{M \smallsetminus L}^{M \smallsetminus L}.$$

Proof. When we glue groupoids, we also glue their units, which gives that the set of units of $[[\mathcal{G} : L]]$ is indeed $M_1 \cup_\phi M_2 =: [M : L]$. We have that $S \simeq S \times \{0\}$ is a closed, invariant subset of the set $S \times [0, 1)$ of units of $\mathcal{G}_1 := \mathcal{E}(S, \pi, \mathcal{H})_{M_1}^{M_1}$, the reduction of $\mathcal{E}(S, \pi, \mathcal{H})$ to M_1. Moreover, S is an invariant subset of the desingularization of $\mathcal{G}_1$ with respect to L, and hence also an invariant subset for $[[\mathcal{G} : L]]$. (See also Remark 3.12.) Since $(\mathcal{G}_1)_S = \mathcal{G}_1 \smallsetminus \mathcal{G}_2$, we have

$$[[\mathcal{G} : L]]_S = (\mathcal{G}_1)_S = \mathcal{E}(S, \pi, \mathcal{H})_S.$$

The rest follows from the construction of $[[\mathcal{G} : L]]$ and the discussion in Example 2.13, Remark 2.14, and, especially, Equation (15). □

Similar structures arise in other situations; see, for instance, [16, 23, 33, 40, 41, 49, 52, 60, 63]. See also the discussion at the end of Example 2.4. Proposition 3.14 is important in Index theory and Spectral theory because it gives rise to exact sequences of algebras [15, 52].

The local structure of the desingularization construction is discussed in Subsection 4.2. We now show that the desingularization is compatible with Lie group actions.

Proposition 3.15. *Let us assume that a Lie group Γ acts on M such that it leaves invariant the tame submanifold with corners $L \subset M$. Then Γ acts* *on $[M : L]$ as well. If, moreover, L is $A(\mathcal{G})$-tame for some groupoid $\mathcal{G} \rightrightarrows M$ on which Γ acts, then we obtain that Γ acts on $[[\mathcal{G} : L]]$ also.*

Proof. The action on $[M : L]$ is obtained by the same argument as in the proof of Lemma 2.9 by considering a compact neighborhood of the identity in Γ. We now show that Γ acts on $[[\mathcal{G} : L]]$. Since $M \smallsetminus L$ is Γ-invariant, Γ will act on $\mathcal{G}_2 := \mathcal{G}_{M \smallsetminus L}^{M \smallsetminus L}$. By Lemma 2.15, Γ acts on $\mathcal{E}(S, \pi, \mathcal{H})$. These actions coincide on the common domain, and hence Γ acts on $[[\mathcal{G} : L]]$. $\square$

3.3. The Lie algebroid of the desingularization

We can now describe the Lie algebroid of the desingularization $[[\mathcal{G} : L]]$ of a Lie groupoid $\mathcal{G}$ with respect to an $A(\mathcal{G})$-tame submanifold $L \subset M$.

Notations 3.16. *In the following, $A \to M$ will be a Lie algebroid and $L \subset M$ will be an A-tame submanifold of M. Also, $r_L : M \to [0, \infty)$ will be a function that is smooth and > 0 on $M \smallsetminus L$ and coincides with the distance to L close to L. Also, $[M : L]$ will continue to denote the blow-up of M along L.*

We notice that the function r_L lifts to a *smooth* function on $[M : L]$ (not just continuous, as on M), which is the main reason for introducing the blow-up $[M : L]$. Recall the definition of R-Lie-Rinehart algebras 1.12. We have the following extension of [2, Theorem 3.10] that was proved originally for Lie manifolds.

Theorem 3.17. *Let $\mathcal{W} := \mathcal{C}^\infty([M : L]) \otimes_{\mathcal{C}^\infty(M)} r_L \Gamma(M; A)$, where we use the notation 3.16. Then $\mathcal{W}$ is a finitely generated, projective $\mathcal{C}^\infty([M : L])$-module with the property that the given Lie bracket on $\mathcal{C}_c^\infty(M \smallsetminus L; A) \subset \mathcal{W}$ extends to $\mathcal{W}$. Hence, there exists a Lie algebroid $B \to [M : L]$ such that $\Gamma([M : L]; B) \simeq \mathcal{W}$.*

We shall denote $[[A : L]] := B$ the Lie algebroid introduced in Theorem 3.17. The isomorphism $\Gamma([M : L]; B) \simeq \mathcal{W}$ is an isomorphism of vector bundles inducing the identity over $[M : L]$ and an isomorphism of Lie algebras, hence it is an isomorphism of $\mathcal{C}^\infty([M : L])$-Lie-Rinehart algebras.

Proof. The proof follows the lines of the proof of Theorem 3.10 in [2], using the A-tameness of L in order to construct the Lie algebra structure on $\Gamma(M; A)$. We include the details for the benefit of the reader, taking also advantage of the results in Subsection 1.2. In particular, we shall use the local product structure of the thick pull-back of Lie algebroids, Corollary 1.19 and Lemma 1.21.

We have that $\Gamma(M;A)$ is a finitely generated, projective $\mathcal{C}^\infty(M)$-module, hence $r_L\Gamma(M;A)$ is a finitely generated, projective $\mathcal{C}^\infty(M)$-module, and hence $\mathcal{W} := \mathcal{C}^\infty([M:L]) \otimes_{\mathcal{C}^\infty(M)} r_L\Gamma(M;A)$ is a finitely generated, projective $\mathcal{C}^\infty([M:L])$-module. It remains to define the Lie bracket on $\mathcal{W}$. We shall prove, in fact, more than that, namely, we shall obtain in Equation (28) a local structure result for $\mathcal{W}$, which will be formalized in a few corollaries that will follow the proof.

We shall use the notation introduced in 3.4. In particular, $\pi : U \to L$, $L \subset U$ is the tubular neighborhood used to define the thick pull-back algebroid $\pi^{\downarrow\downarrow}(A_1) \simeq A|_U$, as in the definition of a tame submanifold, Definition 3.1. The problem is local, so we may assume that $U = L \times \mathbb{R}^n$ and that π is the first projection. Since A is the thick pull-back of the Lie algebroid $A_1 \to L$ to U, we have by Lemma 1.21 that

$$A|_U \simeq \pi^{\downarrow\downarrow}(A_1) \simeq A_1 \boxtimes T\mathbb{R}^n = \pi^*(A_1) \oplus (L \times T\mathbb{R}^n). \tag{24}$$

We want to lift the sections of A on U to the blow-up $[U:L]$. This is, of course, possible for the sections of $\pi^*(A_1) \to U$, but not for the sections of $L \times T\mathbb{R}^n \to L$. This is why we need to multiply with the factor r_L.

Let us consider first the simplified case when L is reduced to a point. We then use a lifting result for vector fields from $\mathbb{R}^n$ to

$$R_0 := [\mathbb{R}^n : 0] = S^{n-1} \times [0,\infty).$$

Let $r(x) := |x|$ denote the distance to the origin in $\mathbb{R}^n$. We recall [2] that a vector field $X \in r\Gamma(\mathbb{R}^n, T\mathbb{R}^n)$ lifts to the blow-up R_0 and the resulting lift is tangent to the boundary of the blow-up (which, we recall, is S^{n-1}). Thus

$$\begin{aligned} \mathcal{C}^\infty(R_0) \otimes_{\mathcal{C}^\infty(\mathbb{R}^n)} r\Gamma(\mathbb{R}^n, T\mathbb{R}^n) &\simeq \mathcal{V}_b(R_0) \\ &\simeq \Gamma(R_0; TS^{n-1} \boxtimes T^b[0,\infty)) \\ &\simeq \Gamma(R_0; TS^{n-1}) \oplus r\Gamma(R_0; T[0,\infty)), \end{aligned} \tag{25}$$

where $\mathcal{V}_b(R_0)$ is as defined in 1.5.

Let us come back now to the case of a general L. Again since the problem is local, we may also assume that $r_L : M = U = L \times \mathbb{R}^n \to [0,\infty)$ is given by $r_L(x,y) = r(y)$. Hence

$$M_1 := [M : L] = L \times [\mathbb{R}^n : 0] = L \times S^{n-1} \times [0,\infty).$$

We now identify the spaces of sections of the vector bundles of interest using Equation (24), the isomorphisms below being isomorphisms of $\mathcal{C}^\infty(M_1)$-modules

$$
\begin{aligned}
(26) \qquad \mathcal{W} &:= \mathcal{C}^\infty(M_1) \otimes_{\mathcal{C}^\infty(M)} r_L\Gamma(M; A)\\
&\simeq \mathcal{C}^\infty(M_1) \otimes_{\mathcal{C}^\infty(M)} r_L\Gamma(M; A_1 \boxtimes T\mathbb{R}^n)\\
&\simeq \mathcal{C}^\infty(M_1) \otimes_{\mathcal{C}^\infty(M)} \Big(r_L\Gamma(M; p_1^*(A_1)) \oplus r_L\Gamma(M; p_2^*(T\mathbb{R}^n))\Big)\\
&\simeq \mathcal{C}^\infty(M_1) \otimes_{\mathcal{C}^\infty(L)} r_L\Gamma(L; A_1) \oplus \mathcal{C}^\infty(M_1) \otimes_{\mathcal{C}^\infty(M)} r_L\Gamma(M; p_2^*(T\mathbb{R}^n)).
\end{aligned}
$$

Next, Equation (25) gives

$$
\begin{aligned}
&\mathcal{C}^\infty(M_1) \otimes_{\mathcal{C}^\infty(M)} r_L\Gamma(M; p_2^*(T\mathbb{R}^n))\\
&\simeq \mathcal{C}^\infty(M_1) \otimes_{\mathcal{C}^\infty(\mathbb{R}^n)} r_L\Gamma(\mathbb{R}^n; T\mathbb{R}^n)\\
&\simeq \mathcal{C}^\infty(M_1) \otimes_{\mathcal{C}^\infty(R_0)} \mathcal{C}^\infty(R_0) \otimes_{\mathcal{C}^\infty(\mathbb{R}^n)} r_L\Gamma(\mathbb{R}^n; T\mathbb{R}^n)\\
&\simeq \mathcal{C}^\infty(M_1) \otimes_{\mathcal{C}^\infty(R_0)} \mathcal{V}_b(R_0)\\
&\simeq \mathcal{C}^\infty(M_1) \otimes_{\mathcal{C}^\infty(R_0)} \Big(r\Gamma(R_0; T[0,\infty)) \oplus \Gamma(R_0; TS^{n-1})\Big).
\end{aligned}
$$

Let p_i be the three projections of $M_1 := L \times S^{n-1} \times [0,\infty)$ onto its three components and let $A_1 \to L$, $A_2 := T[0,\infty) \to [0,\infty)$, and $A_3 := TS^{n-1} \to S^{n-1}$, be the corresponding three Lie algebroids (with the last two being simply the tangent bundles of the corresponding spaces). Since $\mathcal{C}^\infty(M_1) \otimes_{\mathcal{C}^\infty(L)} r_L\Gamma(L; A_1) \simeq r_L\Gamma(M_1; p_1^*(A_1))$, the above calculations then identify $\mathcal{W}$ with the submodule

$$
\begin{aligned}
(27) \qquad \mathcal{W} &\simeq r_L\Gamma(M_1; p_1^*(A_1)) \oplus r_L\Gamma(M_1; p_2^*(A_2)) \oplus \Gamma(M_1; p_3^*(A_3))\\
&\subset \Gamma(M_1; p_1^*(A_1)) \oplus \Gamma(M_1; p_2^*(A_2)) \oplus \Gamma(M_1; p_3^*(A_3))\\
&\simeq \Gamma(M_1; A_1 \boxtimes A_2 \boxtimes A_3).
\end{aligned}
$$

More precisely, let us denote by $p := (\pi, r_L) : M_1 := [M : L] \to L \times [0,\infty)$ the natural fibration, where r_L is the distance to L, as before. Let $r_L(A_1 \boxtimes A_2)$ be as in Lemma 1.14. Then

$$
(28) \qquad [[A : L]] \simeq p^{\downarrow\downarrow}\big(r_L(A_1 \boxtimes A_2)\big).
$$

This equation is the local structure result we had anticipated. It just remains to show that $\mathcal{W}$ is closed under the Lie bracket defined on the dense, open subset $M \smallsetminus L \subset [M : L]$. Indeed, this follows from Equation (28) and Lemma 1.14. □

Definition 3.18. Let us use the notation introduced in 3.16 and in Theorem 3.17. Then the Lie algebroid $[[A : L]] = B$ defined in that theorem will be called the *desingularization* Lie algebroid of A with respect to L.

Remark 3.19. In [25], Gualtieri and Li introduced the "lower elementary modification" $[A : B]_{lower}$ of a Lie algebroid $A \to M$ with respect to a Lie subalgebroid $B \to L$, with L a submanifold of M and $B \subset A|_L$. It is defined by

$$\Gamma([A : B]_{lower}) := \{X \in \Gamma(A) | X|_L \in \Gamma(B)\}.$$

One can see right away that their modification is different from ours. In fact, if $B \neq A|_L$, one can see that the right hand side of the equation is a projective $\mathcal{C}^\infty(M)$-module if, and only if, L is of codimension one in M. In that case (codimension one) one obtains a vector bundle over the same base M, and not over the blow-up manifold $[M : L]$.

We have to following consequence of the proof of Theorem 3.17.

Corollary 3.20. *Let $\pi : M \to L$ be a vector bundle, let $A_1 \to L$ be a Lie algebroid, and let $A = \pi^{\downarrow\downarrow}(A_1)$. Let $A_2 := T[0, \infty)$, let $r_L : [M : L] \to [0, \infty)$ be as in 3.16, and let $p := (\pi, r_L) : [M : L] \to L \times [0, \infty)$ be the natural fibration. Let $r_L(A_1 \boxtimes A_2)$ be as in Lemma 1.14. Then*

$$[[A : L]] \simeq p^{\downarrow\downarrow}\big(r_L(A_1 \boxtimes A_2)\big).$$

Proof. Locally, this reduces to Equation (27) (but see also Equation (28)). $\square$

A more general form of Corollary 3.20 is the following corollary, which is a direct consequence of the proof of Theorem 3.17 (see Equation (27)).

Corollary 3.21. *Using the notation of Theorem 3.17 and of its proof (summarized to a large extend in Corollary 3.20), we have that $\mathcal{W}$ be the set of sections ξ of A over $M \smallsetminus L$ such that, in the neighborhood of every point of $M_1 := [M : L]$, ξ is the restriction of a section of*

$$r_L\Gamma(M_1; p_1^*(A_1)) \oplus r_L\Gamma(M_1; p_2^*(A_2)) \oplus \Gamma(M_1; p_3^*(A_3)). \tag{29}$$

Here is now another of our main results. Recall the definition of the desingularization $[[\mathcal{G} : L]]$ of a Lie groupoid $\mathcal{G}$ along an $A(\mathcal{G})$-tame submanifold $L \subset M$, Definition 3.11.

Theorem 3.22. *Let $\mathcal{G}$ be a Lie groupoid with units M and $L \subset M$ be an $A(\mathcal{G})$-tame submanifold $L \subset M$. Then the Lie algebroid of $[[\mathcal{G} : L]]$ is canonically isomorphic to $[[A(\mathcal{G}) : L]]$ by an isomorphism that induces the identity on $[M : L]$.*

Proof. Recall the notation introduced in 3.9. In particular, $\mathcal{G}_2 \subset [[\mathcal{G} : L]]$ denotes the reduction of $\mathcal{G}$ to $U_2 := M \smallsetminus L$. We have that $\mathcal{G}_2 = [[\mathcal{G} : L]]|_{U_2}$ as well, and hence,

$$A([[\mathcal{G} : L]])|_{U_2} = A(\mathcal{G}_2) = A(\mathcal{G})|_{U_2} = [[A(\mathcal{G}) : L]]|_{U_2}.$$

(This simply means that, up to an isomorphism, nothing changes outside L.) Recall that U is the distinguished tubular neighborhood of L used to define the desingularization groupoid $[[\mathcal{G} : L]]$. Also, $\mathcal{G}_1$ is the edge modification of $\mathcal{G}$ and hence $\mathcal{G}_1$ is the reduction of $[[\mathcal{G} : L]]$ to $U_1 := [U : L]$. See 3.9 and Proposition 2.16. It suffices then to show that $A([[\mathcal{G}_1 : L]])|_{U_1} = [[A(\mathcal{G}_1) : L]]|_{U_1}$, because then

$$A([[\mathcal{G} : L]])|_{U_1} = A([[\mathcal{G}_1 : L]])|_{U_1} = [[A(\mathcal{G}_1) : L]]|_{U_1} = [[A(\mathcal{G}) : L]]|_{U_1}. \tag{30}$$

Let $\pi : U \to L$ denote the projection, as before. Without loss of generality, we may assume that $M = U$, that $\pi : M = U \to L$ is a vector bundle, and hence that $\mathcal{G} = \pi^{\downarrow\downarrow}(\mathcal{H})$. It follows that $A(\mathcal{G}) \simeq \pi^{\downarrow\downarrow}A(\mathcal{H})$, by Proposition 4.3.11 in [39] (used already in Example 2.8).

We use the notation of Corollary 3.20. Let p_i be the two projections of $L \times [0, \infty)$ onto its components. Let $A_2 = T[0, \infty)$. Then we have that

$$A(\mathcal{H}_{ad}) \simeq rp_1^*(A(\mathcal{H})) \subset A(\mathcal{H}) \boxtimes A_2, \tag{31}$$

by Equation (11) (see also Equation (10)). Next, the Lie algebroid of the semi-direct product $\mathcal{H}_{ad} \rtimes \mathbb{R}_+^*$ is

$$A(\mathcal{H}_{ad} \rtimes \mathbb{R}_+^*) \simeq rp_1^*(A(\mathcal{H})) \oplus rp_2^*(A_2) \simeq r(A(\mathcal{H}) \boxtimes A_2), \tag{32}$$

by Equation (31) and since the action of $\mathbb{R}_+^*$ on $[0, \infty)$ has infinitesimal generator $r\partial_r$, $r \in [0, \infty)$. Finally, the fibered pull-back $p^{\downarrow\downarrow}(\mathcal{H}_{ad} \rtimes \mathbb{R}_+^*)$ of $\mathcal{H}_{ad} \rtimes \mathbb{R}_+^*$ to $[M : L]$ via the projection $p := (\pi, r) : [M : L] \to L \times [0, \infty)$ is isomorphic to $[[\mathcal{G} : L]]$, since $U = M$. It has Lie algebroid

$$p^{\downarrow\downarrow}\big(r(A(\mathcal{H}) \boxtimes A_2)\big).$$

That is,

$$\begin{aligned} A([[\mathcal{G} : L]]) &\simeq A(p^{\downarrow\downarrow}(\mathcal{H}_{ad} \rtimes \mathbb{R}_+^*)) \\ &\simeq p^{\downarrow\downarrow} A(\mathcal{H}_{ad} \rtimes \mathbb{R}_+^*) \simeq p^{\downarrow\downarrow}\big(r(A(\mathcal{H}) \boxtimes A_2)\big) \simeq [[A(\mathcal{G}) : L]], \end{aligned}$$

where the last isomorphism is by Corollary 3.20, since $A(\mathcal{G}) \simeq \pi^{\downarrow\downarrow} A(\mathcal{H})$. $\square$

Remark 3.23. The above theorem, Theorem 3.22, is the *raison d'être* for our definition of a desingularization of a Lie groupoid. Indeed, there are good reasons in Analysis and Poisson Geometry for considering generalized polar coordinates in the form of coordinates on the blow-up space $[M : L]$ (think of cylindrical coordinates, which amount to the blow-up of a line in the three dimensional Euclidean space). This is especially convenient when studying the conformal change of metrics that replaces the original metric g with $r_L^{-2} g$. Some of the vector fields on the base manifold become singular in the new coordinates (in our language, they do not *lift* to the blow-up). Multiplying them with the distance function r_L eliminates this singularity and does not affect too much the resulting differential operators. At the level of metrics, this corresponds to the conformal change of metric $g \to r_L^{-2} g$ mentioned above. We are thus lead to study vector fields of the form $r_L \mathcal{V}$, where $\mathcal{V}$ is a given Lie algebra of vector fields (a finitely generated, projective module in all our examples). This motivates our definition of the desingularization of Lie algebroids. In Analysis, one may want then to integrate the resulting desingularized Lie algebroid. Relevant result in this sense were obtained in [18, 51]. However, what our results show is that, if one is given a natural groupoid integrating the *original* (non-desingularized) Lie algebroid (with sections $\mathcal{V}$), then one can construct starting from the initial groupoid a new groupoid that will integrate the desingularized Lie groupoid and at the same time preserve the basic properties of the original groupoid (such as the structure of the orbits).

Related to the above remark, let us mention that it would be interesting to see, given a Poisson groupoid structure on $\mathcal{G}$, whether this structure lifts to a Poisson groupoid structure on $[[\mathcal{G} : L]]$ (probably not). Some possibly relevant results in this direction can be found in [25, 34, 46, 55].

4. Extensions and examples

This final section contains an extension of the results of the last subsection to asymptotically hyperbolic spaces and an example related to the so called "edge calculus."

4.1. The asymptotically hyperbolic modification

One can consider also the case when $L \subset M$ has a tubular neighborhood that is not a ball bundle, but something similar. Let us assume then that $L \subset M$ is a face of codimension $n = 1$ that is a manifold with corners in its own. We assume that the neighborhood U of L is such that $U \simeq L \times [0, 1)$, with $\pi : U \simeq L \times [0, 1) \to L$ being the projection onto the first component. Then our methods extend without change in this case, the result being quite similar. Theorem 3.3 and its proof extend without change to this setting, and so does the definition of $[[\mathcal{G} : L]]$ as well as the result on its structure and Lie algebroid. One just has to consider $S := L$. This example is related to the study of asymptotically hyperbolic spaces, see, for instance [3, 10, 22, 26, 37] and the references therein.

4.2. The local structure of the desingularization for pair groupoids

Let us see what these constructions become in the particular, but important case when we apply these constructions to the pair groupoid. For the purpose of further reference, let us introduce the groupoid $\mathcal{H}_k$ defined as the semi-direct product with $\mathbb{R}_+^*$ of the adiabatic groupoid $(\mathbb{R}^k)^2_{ad}$ of the pair groupoid $(\mathbb{R}^k)^2$, that is,

$$\mathcal{H}_k := (\mathbb{R}^k)^2_{ad} \rtimes \mathbb{R}_+^* = \mathbb{R}^k \times G \sqcup \left(\mathbb{R}^k \times (0, \infty)\right)^2,$$

where G is the semi-direct product $\mathbb{R}^k \rtimes \mathbb{R}_+^*$ and $\sqcup$ denotes again the disjoint union.

Example 4.1. Let us assume that $\mathcal{G} := \mathbb{R}^{n+k} \times \mathbb{R}^{n+k}$ is the pair groupoid and that $L = \mathbb{R}^k \times \{0\} \subset \mathbb{R}^{n+k} =: M$. This gives $\mathcal{H} = L \times L$. We have $A(\mathcal{G}) = T\mathbb{R}^{n+k}$, and hence L is an $A(\mathcal{G})$-tame submanifold. We are, in fact, in the setting of Definition 3.7, with $E = M$ and $\pi : E \to L$ the natural projection. We have already seen that $[M : L] \simeq S^{n-1} \times [0, \infty) \times \mathbb{R}^k$. By definition $[[\mathcal{G} : L]] := \pi^{\downarrow\downarrow}(\mathcal{H}_k)$. Thus

$$\begin{aligned}[][\mathcal{G} : L]] = \left(S^{n-1}\right)^2 \times \mathcal{H}_k &\simeq \left(S^{n-1}\right)^2 \times \left[\mathbb{R}^k \times G \sqcup \left(\mathbb{R}^k \times (0, \infty)\right)^2\right] \\ &\simeq \left(S^{n-1}\right)^2 \times \mathbb{R}^k \times G \sqcup \left(S^{n-1} \times \mathbb{R}^k \times (0, \infty)\right)^2,\end{aligned}$$

where the first set in the disjoint union corresponds to the restriction to S, all sets of the form X^2 represent pair groupoids, and $G = \mathbb{R}^k \rtimes \mathbb{R}_+^*$, as before.

The case of an asymptotically hyperbolic modification is completely similar.

Example 4.2. The simplest case is the one that models a true hyperbolic space, that is, $L = \mathbb{R}^k$ and $M = L \times [0, \infty)$. Then we have $[[\mathcal{G} : L]] = \mathcal{H}_k$.

4.3. An example: the 'edge calculus' groupoid

Let us conclude with one of the simplest non-trivial examples, in which we consider the desingularization of a groupoid with a smooth set of units over a smooth manifold M with respect to a (closed) submanifold $L \subset M$. Thus neither the large manifold nor its submanifold have corners. This example is the one needed to recover the pseudodifferential calculi of Grushin [24], Mazzeo [44], and Schulze [61].

Remark 4.3. Let M be a smooth, compact, connected manifold (so M has no corners). Recall the path groupoid of M, consisting of homotopy classes of end-point preserving paths $[0,1] \to M$. It is a d-simply-connected Lie groupoid integrating TM (that is, its Lie algebroid is isomorphic to TM), so it is the maximal d-connected Lie groupoid with this property. On the other hand, the minimal groupoid integrating TM is $\mathcal{G} = M \times M$. In general, a d-connected groupoid $\mathcal{G}$ integrating TM will be a quotient of $\mathcal{P}(M)$, explicitly described in [25] (see also [45]). For analysis questions, it is typically more natural to choose for $\mathcal{G}$ the minimal integrating groupoid $M \times M$. We notice that in analysis one has to use sometimes groupoids that are not d-connected [12].

We shall fix in what follows a smooth, compact, connected manifold M (so M has no corners) and a d-connected Lie groupoid $\mathcal{G}$ integrating the Lie algebroid $TM \to M$. The following example is related to the so-called "edge calculus"of [24, 44, 61].

Example 4.4. Let $L \subset M$ be an embedded smooth submanifold with tubular neighborhood U that we identify with the set of vectors of length < 1 in NL, the normal bundle to L in M, as in 3.4. We denote by $\pi : S := \partial U \to L$ the natural projection. Then recall that the blow-up $[M : L]$ of M with respect to L is the disjoint union

$$[M : L] := (M \smallsetminus L) \sqcup S,$$

with the topology of a manifold with boundary S. We have that L is automatically $A(\mathcal{G}) = TM$-tame, so we can define $[[\mathcal{G} : L]]$ (Definition 3.11), which is a Lie groupoid with base $[M : L]$. Pseudodifferential operators on the resulting groupoid [1, 6, 47, 53] can be used to recover the pseudodifferential calculi of Grushin [24], Mazzeo [44], Schulze [61], and others. See also Coriasco-Schulze [14], Guillarmou-Moroianu-Park [26], Lauter-Moroianu [36], Lauter-Nistor [37], and many others.

Let us spell out the structure of $[[\mathcal{G} : L]]$ in the simple case of the edge calculus, in order to better understand the desingularization construction.

Remark 4.5. We continue to use the notation introduced in Example 4.4. By the definition of the groupoid $[[\mathcal{G} : L]]$, the open set $U_0 := M \smallsetminus L = [M : L] \smallsetminus S$ is a $[[\mathcal{G} : L]]$-invariant subset and the restriction $[[\mathcal{G} : L]]_{U_0}$ coincides with the reduction $\mathcal{G}_{U_0}^{U_0}$. In particular, if $\mathcal{G} = M \times M$, then $[[\mathcal{G} : L]]_{U_0} = \mathcal{G}_{U_0}^{U_0} = U_0 \times U_0$, the pair groupoid. On the other hand, the restriction of $[[\mathcal{G} : L]]$ to $S := [M : L] \smallsetminus U_0$ is a fibered pull-back groupoid defined as follows. We consider first $TL \to L$, regarded as a bundle of (commutative) Lie groups. We let $\mathbb{R}_+^*$ act on the fibers of $TL \to L$ by dilation and define the bundle of Lie groups $G_S \to L$ by $G_S := TL \rtimes \mathbb{R}_+^* \to L$, that is, the group bundle over L obtained by taking the semi-direct product of TL, by the action of $\mathbb{R}_+^*$ by dilations. (See also Example 2.4.) Then $[[\mathcal{G} : L]]_S := \pi^{\downarrow\downarrow}(G)$. In particular, $[[\mathcal{G} : L]]_S$ does not depend on the choice of integrating groupoid $\mathcal{G}$ of Remark 4.3.

Remark 4.6. Let us choose $\mathcal{G} := M \times M$. As mentioned above, if a Lie group acts on M leaving L invariant, then it will act on $\mathcal{G}$, and hence also on $[\mathcal{G} : L]$, by Proposition 3.15. This yields hence also an action of Γ on the edge calculus [24, 44, 59, 61]. See also [33, 54, 63, 65].

Remark 4.7. By choosing $\mathcal{G} := \mathcal{P}(M)$, one obtains a "covering edge calculus," that is, a calculus that is on the universal covering manifold $\widetilde{M} \to M$, is invariant with respect to the group of deck transformations, and respects the edge structure along the lift of L to $\widetilde{M}$. In particular, we see that if $\mathcal{G}$ is the path groupoid and the fundamental group is not trivial, the groupoid $[[\mathcal{G} : L]]$ *is not* Morita equivalent to the gauge adiabatic groupoid of [21]. In general, if U does not intersect all orbits of $\mathcal{G}$ on M, then $[[\mathcal{G} : L]]$ will not be

Morita equivalent to the gauge-adiabatic groupoid. See [50] for applications of the covering calculus.

By iterating this construction as in [2], one obtains integral kernel operators on polyhedral domains. It would be interesting to extend this example to the pseudodifferential calculus on manifolds with boundary [9].

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