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The width of ellipsoids

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We compute the k-width of a round 2-sphere for $k = 1, \dots, 8$ and we use this result to show that unstable embedded closed geodesics can arise with multiplicity as a min-max critical varifold.

1. Introduction

The aim of this work is to compute some of the k-width of the 2-sphere and to provide a concrete counterexample to the Multiplicity One Conjecture in the case of closed geodesics on a surface. The conjecture appears on [13], by F. C. Marques and A. Neves, and it states that the two-sided unstable components of a closed minimal hypersurface obtained by a min-max method should have multiplicity one when the ambient dimension is $3 \leq n+1 \leq 7$. This was recently proved by X. Zhou in [21]. It is related to variational

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approaches to show the existence of infinitely many closed minimal hypersurfaces on closed manifolds, originally conjectured by S.-T. Yau in the case of immersed surfaces in dimension 3.

Even in the simple case of the round 2-sphere the full width spectrum is not known. One of the motivations to compute it was to prove a Weyl type law for the width as it was proposed in [9], where the author suggests that the width should be considered as a non-linear spectrum analogue to the spectrum of the laplacian. Recently the Weyl law for codimension one cycles on manifolds as well as arbitrary codimension on Lipschitz domains was proved by Y. Liokumovich, Marques and Neves on [11].

In a closed Riemannian manifold M of dimension n the Weyl law says that $\frac{\lambda_p}{p^{\frac{2}{n}}} \to c_n \mathrm{vol}(M)^{-\frac{2}{n}}$ for a known constant c_n, where λ_p denotes the p^{th} eigenvalue of the laplacian. In the case of curves in a 2-dimensional manifold M we have that

$$\frac{\omega_p}{p^{\frac{1}{2}}} \to C_2 \mathrm{vol}(M)^{-\frac{1}{2}},$$

where $C_2 > 0$ is a constant. Our computation suggests what should be the optimal constant in the case of the round 2-sphere.

By making a contrast with classical Morse theory one could ask the following two naive questions about the index and nullity of a varifold that achieves the width:

Question 1. Let (M, g) be a Riemannian manifold and $V \in \mathrm{I}\mathcal{V}_l(M)$ be a critical varifold for the k-width $\omega_k(M, g)$. Then

$$k \leq \mathrm{index}(V) + \mathrm{null}(V).$$

Question 2. Let (M, g) be a Riemannian manifold and $V \in \mathrm{I}\mathcal{V}_l(M)$ be a critical varifold for the k-width $\omega_k(M, g)$. Then

$$\mathrm{index}(V) \leq k.$$

Where $\mathrm{index}(V)$ and $\mathrm{null}(V)$ are the index and nullity of the second variation $\delta^{(2)}V$ on the space of vectorfields in M. By a critical varifold we mean that V is obtained as the accumulation point of a min-max sequence.

To illustrate these questions let us present a situation in which **Question 1** holds and compare to our context. Say we are studying closed geodesics by analyzing the energy functional E in the free loop space $\Lambda = W^{1,2}(S^1, M)$, in which case we can apply infinite dimensional Morse theory. Take $a < b$ regular values and suppose we can find a non-trivial homology class $\alpha \in H_k(\Lambda^b, \Lambda^b)$ ($\Lambda^a = \{E \le a\}$) then we can find a closed geodesic γ satisfying

$$E(\gamma) = \inf_{A \in \alpha} \sup_{x \in \operatorname{supp} A} E(x).$$

In this case it is known that $\operatorname{index}(\gamma) \le k \le \operatorname{index}(\gamma) + \operatorname{null}(\gamma)$ (this is encoded in [8, §1 Lemma 2], alternatively see [5, Chapter 2 Corollary 1.3]). Compared to our case γ would correspond to V, a non-trivial k-dimensional homology class corresponds to a k-sweepout and the min-max quantity is analogue to the k-width.

In the Almgren-Pitts min-max set up we work with varifolds instead of parametrized curves, which allow degenerations. On the other hand we compute the index and nullity in the same way, by using vectorfield variations.

As an example, consider the union of two great circles in the 2-sphere. It divides the sphere into four discs and for each of them we take a 1-parameter contraction to a point. If we follow the boundary of the contractions of two opposite disks simultaneously we would have a 1-parameter family of cycles that decreases length. However, this is not generated by an ambient vectorfield, so it does not contribute to the index of the stationary varifold.

Furthermore, classical Morse theory will always produce a closed geodesic as a critical point of energy, but Almgren-Pitts min-max only produces a geodesic network. In this paper we address this issue and show the following result (see Theorems 4.4 and 4.13 for precise statements).

Theorem. *A stationary integral* 1*-varifold obtained by min-max method on cycles with coefficients* $\mathbb{Z}_2$ *must have integer density everywhere.*

This allow us to exclude networks with triple junctions as critical varifolds because the density at the singularity is $\frac{3}{2}$. However, this is not sufficient to prove that they correspond to closed geodesics. For example, we are not able to exclude a union of two triple junctions with a small angle between them. In this case the density at the singular point is 3 but it is not the union of 3 closed geodesics.

As a perturbation of our results we will show that **Question 1** is false for 1-varifolds on a surface. Regarding **Question 2**, it was recently shown by Marques-Neves in [13] that $\operatorname{index}(\operatorname{supp}(V)) \le k$ in the case of codimension one and $3 \le dim(M) \le 7$. The authors also prove the Multiplicity One Conjecture for min-max with one parameter. In the hypersurface case Pitts' min-max theorem gives us an embedded minimal hypersurface, whereas the dimension 1 case allows self-intersections. That is why they do not expected the conjecture to hold for curves. This work provides a concrete example of how it fails to be true in the dimension 1 case.

Theorem (see Corollary 5.8). *For ellipsoids sufficiently close to the round sphere it is possible to produce a multiplicity* 2 *unstable closed geodesic using min-max methods.*

The idea of the proof is the following. Firstly observe that the width of such ellipsoid has to be close to the width of S^2. We use this estimate to show that a critical varifold produced with at most 8 parameters on an ellipsoid cannot have length greater than the union of two principle geodesics.

Secondly we use the fact that it cannot have triple junctions. Together with a simple density estimate we conclude that it must be a union of closed geodesics. If the ellipsoid is sufficiently close to the round sphere it is well known that the prime closed geodesics other than the equators must have arbitrarily large length. We conclude that the critical varifolds must be the union of at most two principle geodesics.

It follows from a Lusternik-Schnirelmann type result that critical varifolds obtained with different number of parameters must be distinct. The ones produced with 1, 2 and 3 parameters will correspond exactly to the three principle geodesics with multiplicity 1. Because all the possible combinations must be exhausted, we will have that either the min-max varifold produced with 4 or 5 parameters must have multiplicity 2, that is, it is given by the union of two if the same geodesic. Since the index and nullity of the principle geodesics are well known, it also serves and an negative answer to **Question 1**.

This article is divided as follows. In section 2 we briefly overview properties of sweepouts, currents and varifolds. In section 3 we define geodesic networks and we prove a structure result for 1-dimensional stationary integral varifolds. In section 4 we define almost minimising varifolds and characterize its singularities. In section 5 we compute the k-width of S^2 for $k = 1, \dots, 8$. Then we use the regularity results to find the critical varifolds for a generic ellipsoid.

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2. Preliminaries

In this section definitions and notations are established. Throughout this section M denotes a closed Riemannian manifold of dimension m isometrically embedded in $\mathbb{R}^n$ for some $n > 0$.

Let us denote by $\mathcal{Z}_k(M)$ the space of **flat k-cycles** in M with coefficients in $\mathbb{Z}_2$ endowed with the flat topology. We write $\mathcal{F}$ for the **flat norm** and $\underline{\underline{\mathbf{M}}}$ for the **mass** of a cycle.

We adopt the definition of varifolds in [15]. We denote the spaces of k-**varifolds**, **rectifiable varifolds** and **integral varifolds** by $\mathcal{V}_k(M)$, $\mathrm{R}\mathcal{V}_k(M)$ and $\mathrm{I}\mathcal{V}_k(M)$, respectively. These spaces are endowed with the weak topology induced by the metric $\underline{\underline{\mathbf{F}}}$.

Given a rectifiable varifold $V \in \mathrm{R}\mathcal{V}_k(M)$ we write $\mathscr{C}_p V$ for the tangent cone of V at the point $p \in \operatorname{supp}\|V\|$. We also denote by $G(k,n)$ the space of k-planes in $\mathbb{R}^n$ and $G_k(M) = \{(x,P) \in \mathbb{R}^n \times G(k,n) : x \in M, P \subset T_x M\}$ the k-Grassmanian bundle over M. For a rectifiable set $S \subset \mathbb{R}^n$ and θ and integrable function in $G_k(\mathbb{R}^n)$ we write $\underline{\underline{v}}(S,\theta)$ the varifold associated to S with density θ.

Now we establish a relation between currents and varifolds. Given a k-current T (not necessarily closed) we denoted by $|T| \in \mathcal{V}_k(M)$ the varifold induced by the support of T and its coefficients. Reversely, given a k-varifold V we denote by $[V]$ the unique k-current such that $\Theta^k(|[V]|,x) = \Theta^k(V,x)\, mod\, 2$ for all $x \in \operatorname{supp}\|V\|$ (see [18]).

2.1. Sweepouts and the width

In [3] Almgren proved, in particular, that $\pi_i(\mathcal{Z}_k(M)) = H_{i+k}(M;\mathbb{Z}_2)$ for all $i > 0$. We call it the **Almgren isomorphism** and denote it by F_A. It follows from the Universal Coefficient Theorem that $H^{n-k}(\mathcal{Z}_k(M);\mathbb{Z}_2) = \mathbb{Z}_2$, denote its generator by $\bar{\lambda}$ and $\bar{\lambda}^p$ the cup product with itself p times. For the next definition we follow [10] and [12].

Definition 2.1. Let $X \subset [0,1]^N$ be a cubical subcomplex for some $N > 0$ and $f : X \to \mathcal{Z}_k(M)$ a flat continuous map. We say that f is a p-**sweepout** if

$$f^*(\bar{\lambda}^p) \neq 0 \in H^{p(n-k)}(X;\mathbb{Z}_2).$$

Denote the set of p-sweepouts with *no concentration of mass* (see definition 4.3) in M by $\mathcal{P}_p(M)$.

We define the p-**width** of (M, g) as

$$\omega_p(M, g) = \inf_{f \in \mathcal{P}_p} \sup_{x \in \mathrm{dmn}(f)} \underline{\underline{\mathbf{M}}}(f(x)),$$

where $\mathrm{dmn}(f)$ denotes the domain of f.

Note that $\omega_p \leq \omega_{p+1}$ since every $(p+1)$-sweepout is also a p-sweepout.

2.2. Varifolds in S^n

Let (S^n, g_{S^n}) denote the round sphere of radius 1 in $\mathbb{R}^{n+1}$. Given a varifold $V \in \mathcal{V}_k(S^n)$ we can define the cone generated by V in $\mathbb{R}^{n+1}$. It is sufficient to define a positive functional in the space $C_c(G_{k+1}(\mathbb{R}^{n+1}))$ (see for example [1, §5.2(3)]).

Definition 2.2. Given $V \in \mathcal{V}_k(S^n)$ define $C(V) \in \mathcal{V}_{k+1}(\mathbb{R}^{n+1})$ to be the measure corresponding to the functional

$$C(V)(f) = \int_0^\infty \tau^k V(f_\tau) d\tau$$

where $f \in C_c(G_{k+1}(\mathbb{R}^{n+1}))$ and $f_\tau \in C_c(G_k(\mathbb{R}^{n+1}))$ is given by

$$f_\tau(x, P) = \begin{cases} f\left(\tau x, P \oplus \mathbb{R}\langle x \rangle\right), & \text{if } x \in S^n \text{ and } P \subset T_x S^n; \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 2.3. *The cone map $C : \mathcal{V}_k(S^n) \to \mathcal{V}_{k+1}(\mathbb{R}^{k+1})$ satisfy the following properties:*

(i) $C(V)$ is a cone varifold;

(ii) If $a, b \in \mathbb{R}_{\geq 0}$ and $V, W \in \mathcal{V}_k(S^n)$ then $C(aV + bW) = aC(V) + bC(W)$;

(iii) If $V \in \mathrm{R}\mathcal{V}_k(S^n)$ then $C(V)$ is given by

$$C(V)(f) = \int_0^\infty \tau^k \int_{S^n} f\left(\tau x, T_V(x) \oplus \mathbb{R}\langle x \rangle\right) \Theta^k(V, x) d\mathcal{H}^k_x d\tau,$$

where $f \in C_c(G_{k+1}(\mathbb{R}^{n+1}))$ and $T_V(x) \subset T_x S^n$ is the tangent space of V defined $\|V\|$-almost everywhere in S^n;

Proof. **(i):** We must show that ${\eta_{0,\lambda}}_{\#}C(V) = C(V)$ for all $\lambda > 0$, where $\eta_{0,\lambda}(x) = \lambda x$ for $x \in \mathbb{R}^{n+1}$. Take any $f \in C_c(G_{k+1}(\mathbb{R}^{n+1}))$ and compute

$$\begin{aligned}
{\eta_{0,\lambda}}_{\#}C(V)(f) &= \int f(x,\tilde{P})d({\eta_{0,\lambda}}_{\#}C(V))_{x,\tilde{P}} \\
&= \int [J^{k+1}\eta_{0,\lambda}](x) f(\eta_{0,\lambda}(x), D\eta_{0,\lambda}\big|_x \cdot \tilde{P}) dC(V)_{x,\tilde{P}} \\
&= \frac{1}{\lambda^{k+1}} \int f\left(\frac{x}{\lambda}, \tilde{P}\right) dC(V)_{x,\tilde{P}} \\
&= \frac{1}{\lambda^{k+1}} \int_0^\infty \tau^k V(f_{\frac{\tau}{\lambda}}) d\tau \\
&= \frac{1}{\lambda^{k+1}} \int_0^\infty (t\lambda)^k V(f_t) \lambda dt \\
&= C(V)(f).
\end{aligned}$$

Here it was used the definition of pushforward and change of variables $t = \frac{\tau}{\lambda}$ in the second last line.
(ii): This is straightforward from the definition.
(iii): To prove this formula simply use that

$$V(f_\tau) = \int_{S^n} f_\tau\left(x, T_V(x)\right)) d\|V\|_x$$

holds for rectifiable varifolds and $d\|V\|_x = \Theta^k(V,x) d\mathcal{H}^k_x$. □

We now want to prove that this cone map is continuous with respect to the weak convergence.

Lemma 2.4. *Let $\{V_n\} \subset \mathcal{V}_k(S^n)$ be a sequence of varifolds converging to $V \in \mathcal{V}_k(S^n)$ in the $\underline{\underline{\boldsymbol{F}}}$-metric. Then $C(V_n) \to C(V)$ with respect to $\underline{\underline{\boldsymbol{F}}}$.*

Proof. It is enough to prove that $C(V_n)(f) \to C(V)(f)$ for any compactly supported function in $G_{k+1}(\mathbb{R}^{n+1})$.

There exist $R_0 > 0$ such that $\operatorname{supp}(f) \subset B(0, R_0) \times G(k+1, n+1)$. For $\tau > R_0$ we have

$$f_\tau(\tau x, P) = 0,$$

for all $x \in S^n$ and $P \in G(k, n+1)$. Thus, whenever $\tau > R_0$,

$$V_n(f_\tau) = \int f(\tau x, P) d\left(V_n\right)_{x,P} = 0$$

for all $n>0$, and $V(f_\tau)=0$. This implies that the sequence $h_n(\tau)=\tau^k V_n(f_\tau)$ is uniformly bounded. By the Dominated Convergence theorem we obtain

$$\begin{aligned}\lim_{n\to\infty} C(V_n)(f) &= \lim_{n\to\infty}\int_0^{R_0} h_n(\tau)d\tau = \int_0^{R_0} \lim_{n\to\infty} h_n(\tau)d\tau\\ &= \int_0^{R_0} \tau^k V(f_\tau)d\tau\\ &= C(V)(f). \qquad \square\end{aligned}$$

Next we show that the cone of a varifold associated to a rectifiable set in S^n is defined by the cone of the set, as one would expect.

Lemma 2.5. *Let $R\subset S^n$ be a k-rectifiable set, $\theta : G_k(S^n)\to\mathbb{R}_{\geq 0}$ a locally integrable function and $\underline{\underline{v}}(R,\theta)\in \mathrm{R}\mathcal{V}_k(S^n)$. Then*

$$C\left(\underline{\underline{v}}(R,\theta)\right) = \underline{\underline{v}}\left(\tilde{R},\tilde{\theta}\right),$$

where $\tilde{R}=\{\lambda x\in\mathbb{R}^{n+1}|\lambda\geq 0,\ x\in R\}$ and $\tilde{\theta} : G_{k+1}(\mathbb{R}^{n+1})\to\mathbb{R}_{\geq 0}$ is a given by

$$\tilde{\theta}(x,\tilde{P}) = \begin{cases}\theta\left(\frac{x}{|x|},P\right), & \text{if } x\neq 0 \text{ and } \tilde{P}=P\oplus\langle x\rangle;\\ 0, & \text{otherwise.}\end{cases}$$

Proof. It is easy to see that $\tilde{\theta}$ is locally integrable in $G_{k+1}(\mathbb{R}^{n+1})$, $\tilde{R}$ is $(k+1)$-rectifiable and its tangent space is given by $T_{\tilde{R}}(x)=T_R(\frac{x}{|x|})\oplus\langle\frac{x}{|x|}\rangle$ for x $(\mathcal{H}^{k+1}\llcorner\tilde{R})$-almost-everywhere. For $f\in C_c(G_{k+1}(\mathbb{R}^{n+1}))$ compute

$$\begin{aligned}\underline{\underline{v}}\left(\tilde{R},\tilde{\theta}\right)(f) &= \int_{\mathbb{R}^{n+1}} f\left(x,T_{\tilde{R}}(x)\right)\tilde{\theta}(x,T_{\tilde{R}}(x))d(\mathcal{H}^{k+1}\llcorner\tilde{R})_x\\ (*)\qquad &= \int_{\mathbb{R}^{n+1}\setminus\{0\}} f\left(x,T_R\left(\frac{x}{|x|}\right)\oplus\langle\frac{x}{|x|}\rangle\right)\\ &\qquad\times\theta\left(\frac{x}{|x|},T_R\left(\frac{x}{|x|}\right)\right)d(\mathcal{H}^{k+1}\llcorner\tilde{R})_x.\end{aligned}$$

We want to use the Co-area formula (see [7, §3.2.22]), we clarify notation and make some remarks.

Define the warped product metric on $(0,+\infty)\times S^n$ as $g_{(\tau,x)}=d\tau^2+\tau^2(g_{S^n})_x$, where g_{S^n} is the round Riemannian metric on S^n. Let d_g, d_{S^n} and d_0 be the metrics induced by g on $(0,+\infty)\times S^n$, g_{S^n} on S^n and the Euclidian metric g_0 on $\mathbb{R}^{n+1}$ respectively. Given any metric d we denote by $\mathcal{H}^k(d)$ the k-dimensional Hausdorff measure associated to d.

`Claim` **1.** *The metrics g, g_{S^n} and g_0 satisfy:*

(a) $F : (\tau, z) \in ((0, +\infty) \times S^n, g) \mapsto \tau z \in (\mathbb{R}^{n+1} \setminus \{0\}, g_0)$ *is an isometry;*

(b) $d_g\left((\tau, z), (\tau, y)\right) = \tau d_{S^n}(z, y)$*;*

(c) $(\iota_\tau)_* \mathcal{H}^k(d_{S^n}) = \tau^{-k}\mathcal{H}^k(d_g)$, *where* $\iota_\tau : S^n \to (0, +\infty) \times S^n$ *is the inclusion in the slice* $\{\tau\} \times S^n$.

Firstly, (a) is a well known fact and (b) follows easily from the definition. Lastly, (b) implies that ι_τ is τ^{-1}-Lipschitz so (c) follows from basic properties of $\mathcal{H}^k$.

For simplicity denote $h(x)$ the integrand in (*). Applying a change of variables and the Co-area formula for the projection $(\tau, z) \mapsto \tau$ we obtain

$$
\begin{aligned}
\underline{\underline{v}}\left(\tilde{R}, \tilde{\theta}\right)(f) &= \int_{\mathbb{R}^{n+1}\setminus\{0\}} h(x) dF_* F_*^{-1}(\mathcal{H}^{k+1}(d_0) \llcorner \tilde{R})_x \\
&= \int_{(0,+\infty)\times S^n} h \circ F(\lambda, z) d\left(\mathcal{H}^{k+1}(d_g) \llcorner (0, +\infty) \times R\right)_{(\lambda,z)} \\
&= \int_{(0,+\infty)\times R} h \circ F(\lambda, z) d\mathcal{H}^{k+1}(d_g)_{(\lambda,z)} \\
&= \int_0^\infty \left(\int_{\{\tau\}\times R} h \circ F(\lambda, z) d\mathcal{H}^k(d_g)_{(\lambda,z)}\right) d\mathcal{H}^1(d_0)_\tau .
\end{aligned}
$$

Changing variables again and using (c) we conclude

$$
\begin{aligned}
\underline{\underline{v}}\left(\tilde{R}, \tilde{\theta}\right)(f) &= \int_0^\infty \left(\int_{\{\tau\}\times R} h \circ F(\lambda, z) d\left(\tau^k \iota_{\tau *} \mathcal{H}^k(d_{S^n})\right)_{(\lambda,z)}\right) d\mathcal{H}^1(d_0)_\tau \\
&= \int_0^\infty \left(\int_R h \circ F \circ \iota_\tau(z) \tau^k d\mathcal{H}^k(d_{S^n})_z\right) d\tau \\
&= \int_0^\infty \tau^k \int_{S^n} h(\tau z) d(\mathcal{H}^k \llcorner R)_z d\tau .
\end{aligned}
$$

The proof is finished by replacing h in the formula and 2.3(iii). □

Finally, we can prove the main properties of the cone $C(V)$.

Proposition 2.6. *Let* $V \in \mathrm{R}\mathcal{V}_k(S^n)$ *and* $C(V) \in \mathcal{V}_{k+1}(\mathbb{R}^{n+1})$. *Then the following is true:*

(i) $C(V)$ *is rectifiable;*

(ii) if V is integral then so is $C(V)$;

(iii) $\operatorname{supp}\|C(V)\| = \{\lambda x \in \mathbb{R}^{n+1} | x \in \operatorname{supp}\|V\|$ *and* $\lambda \geq 0\}$*;*

(iv) if $y \neq 0$ then $\Theta^{k+1}(C(V), y) = \Theta^k(V, \frac{y}{|y|})$;

(v) if $y \neq 0$ then

$$\|V\|(S^n) = (k+1) \lim_{r \to \infty} \frac{\|C(V)\|B(y,r)}{r^{k+1}};$$

(vi) if V is stationary and $k \geq 1$ then so is $C(V)$.

Proof. **(i):** Let $V = \lim_{n\to\infty} \sum_{i=1}^n \underline{\underline{v}}(R_i, \theta_i)$, where $R_i \subset S^n$ is k-rectifiable and $\theta_i : G_k(S^n) \to \mathbb{R}_{\geq 0}$ is locally integrable for all $i > 0$. The result follows directly from 2.3(ii), Lemma 2.4 and Lemma 2.5.

(ii): Just note in the proof of Lemma 2.5, if θ is integer-valued then so is $\tilde{\theta}$.

(iii): First show that $\operatorname{supp}\|C(V)\| \supset \{\lambda x \in \mathbb{R}^{n+1} | x \in \operatorname{supp}\|V\|$ *and* $\lambda \geq 0\}$. Take $y \notin \operatorname{supp}\|C(V)\|$ and a positive continuous function $\tilde{f} : \mathbb{R}^{n+1} \to \mathbb{R}_{\geq 0}$ supported in $B(y,r)$, for some $r > 0$. Define $f(x) = \tilde{f}(ax)$, where $a = \min\{1, |y|\}$. So f is supported in $B(\frac{y}{|y|}, r)$. If we assume $\|C(V)\|(\tilde{f}) = 0$ then it is easy to check that $\|V\|(f) = 0$, so $\frac{y}{|y|} \notin \operatorname{supp}\|V\|$. The other inclusion is similar.

(iv): for simplicity put $C = C(V)$. It is enough to show that

$$\int_{\mathbb{R}^{n+1}\setminus\{0\}} g(y) d\|C\|_y = \int_{\mathbb{R}^{n+1}\setminus\{0\}} g(y)\Theta^k\left(V, \frac{y}{|y|}\right) d\mathcal{H}^{k+1}_y$$

for every continuous function g compactly supported in $\mathbb{R}^{n+1} \setminus \{0\}$.

If $f \in C_c(G_{k+1}(\mathbb{R}^{n+1}))$ satifies $f(0, \tilde{P}) = 0$ for all $\tilde{P} \in G(k+1, n+1)$ then, from rectifiability (property (i)), it follows that

$$\begin{aligned} C(f) &= \int_{G_{k+1}(\mathbb{R}^{n+1})} f(y, \tilde{P}) dC_{(y,\tilde{P})} \\ &= \int_{\mathbb{R}^{n+1}\setminus\{0\}} f(y, T_C(y)) d\|C\|_y. \end{aligned}$$

On the other hand, by property 2.3(iii), we have

$$C(f) = \int_0^\infty \tau^k \int_{S^n} f(\tau x, T_V(x) \oplus \mathbb{R}\langle x \rangle)\Theta^k(V, x) d\mathcal{H}^k_x d\tau.$$

Following a computation similar to the proof of Lemma 2.5 we conclude

$$C(f) = \int_{\mathbb{R}^{n+1}\setminus\{0\}} f\left(y, T_V\left(\frac{y}{|y|}\right)\right)\Theta^k\left(V, \frac{y}{|y|}\right) d\mathcal{H}^{k+1}_y$$

Take a continuous function $g : \mathbb{R}^{n+1} \to \mathbb{R}$ compactly supported in $\mathbb{R}^{n+1} \setminus \{0\}$ and define $f(y, \tilde{P}) = g(y)$ for all $y \in \mathbb{R}^{n+1}$ and $\tilde{P} \in G(k+1, n+1)$. The result follows by replacing such f in the previous formulas.

(v): Fix $y \neq 0$, $r > 0$ and let $F : \mathbb{R}^{n+1} \setminus \{0\} \to (0, +\infty) \times S^n$ be the isometry $F(z) = (|z|, \frac{z}{|z|})$. Denote $A(r) = F(B(y,r))$, $\mathrm{pr}_1(\tau, x) = \tau$.

If $r > |y|$ then $0 \in B(y,r)$ and

$$\begin{aligned}
\|C\|B(y,r) &= \int_{B(y,r)} d\|C\|_z \\
&= \int_{B(y,r)\setminus\{0\}} \Theta^{k+1}(C, z) d\mathcal{H}^{k+1}(d_0)_z \\
&= \int_{B(y,r)\setminus\{0\}} \Theta^k\left(V, \frac{z}{|z|}\right) d\mathcal{H}^{k+1}(d_0)_z \\
&= \int_{A(r)} \Theta^k(V, x) d\mathcal{H}^{k+1}(d_g)_{(\tau,x)}.
\end{aligned}$$

Furthermore, $\mathrm{pr}_1(A(r)) = (a(r), b(r))$, with $a(r) = \inf_{z\in B(y,r)} |z| = 0$ and $b(r) = \sup_{z\in B(y,r)} |z| = |y| + r$. Note also that $\mathrm{pr}_1^{-1}(\tau) = \{\tau\} \times S^n$ for $\tau < r - |y|$. Applying the Co-area formula with respect to pr_1we get

$$\begin{aligned}
\|C\|B(y,r) &= \int_0^{|y|+r} \int_{\mathrm{pr}_1^{-1}(\tau)} \Theta^k(V, x) d\mathcal{H}^k(d_g)_{(\lambda,x)} d\tau \\
&= \int_0^{r-|y|} \int_{\{\tau\}\times S^n} \Theta^k(V, x) d\mathcal{H}^k(d_g)_{(\lambda,x)} d\tau \\
&\quad + \int_{r-|y|}^{r+|y|} \int_{\mathrm{pr}_1^{-1}(\tau)} \Theta^k(V, x) d\mathcal{H}^k(d_g)_{(\lambda,x)} d\tau \\
&= \int_0^{r-|y|} \tau^k \int_{S^n} \Theta^k(V, x) d\mathcal{H}^k(d_{S^n})_{(x)} d\tau \\
&\quad + \int_{r-|y|}^{r+|y|} \int_{\mathrm{pr}_1^{-1}(\tau)} \Theta^k(V, x) d\mathcal{H}^k(d_g)_{(\lambda,x)} d\tau.
\end{aligned}$$

The first term in the sum is given by

$$\int_0^{r-|y|} \tau^k \int_{S^n} \Theta^k(V, x) d\mathcal{H}^k(d_{S^n})_{(x)} d\tau = \frac{(r - |y|)^{k+1}}{k+1} \|V\|(S^n).$$

Since $\mathrm{pr}_1^{-1}(\tau) \subset \{\tau\} \times S^n$, the second term is bounded by

$$\begin{aligned}
&\int_{r-|y|}^{r+|y|} \int_{\mathrm{pr}_1^{-1}(\tau)} \Theta^k(V,x) d\mathcal{H}^k(d_g)_{(\lambda,x)} d\tau \\
&\leq \frac{(r+|y|)^{k+1} - (r-|y|)^{k+1}}{k+1} \|V\|(S^n).
\end{aligned}$$

When we divide by r^{k+1} and take the limit $r \to \infty$, the first term converges to $\frac{\|V\|(S^n)}{k+1}$ and the second term tends to zero.

(vi): Since we assume $k \geq 1$ it is enough to prove that $C(V)$ is stationary outside the origin.

Fix a vector field Y with compact support $\mathrm{supp}(Y) \subset \mathbb{R}^{n+1} \setminus \{0\}$. We can write $Y(y) = h(y)y + X(\frac{y}{|y|})$ where X is a compactly supported vector field in S^n and h is a compactly supported function. The first variation is given by $\delta C(Y) = \delta C(h(y)y) + \delta C(X(\frac{y}{|y|}))$. Let us compute the first term:

$$\begin{aligned}
\delta C(h(y)y) &= \int_{G_{k+1}(\mathbb{R}^{n+1})} \mathrm{div}_{\tilde{P}}(h(y)y) dC_{(y,\tilde{P})} \\
&= \int_0^\infty \tau^k \int_{S^n} \mathrm{div}_{T_V(x)\oplus\mathbb{R}\langle x\rangle}(h(\tau x)\tau x) d\|V\|_x d\tau \\
&= \int_0^\infty \tau^k \int_{S^n} Dh\big|_{\tau x} \cdot \tau x + h(\tau x)\mathrm{div}_{T_V(x)\oplus\mathbb{R}\langle x\rangle}(\tau x) d\|V\|_x d\tau \\
&= \int_0^\infty \int_{S^n} \tau^k \left(\tau Dh\big|_{\tau x} \cdot x + h(\tau x)(k+1)\right) d\|V\|_x d\tau \\
&= \int_{S^n} \left[\int_0^\infty \left(\frac{d}{dt}\Big|_{t=\tau} t^k h(tx)\right) d\tau\right] d\|V\|_x \\
&= 0
\end{aligned}$$

In the last line we used that h has compact support away from 0.

Using that X doesn't depend on the radial direction, that is, $\mathrm{div}_{\langle x\rangle}(X) = 0$, we compute the second term

$$\begin{aligned}
\delta C(X(\frac{y}{|y|})) &= \int_{G_{k+1}(\mathbb{R}^{n+1})} \mathrm{div}_{\tilde{P}}(X(\frac{y}{|y|})) dC_{(y,\tilde{P})} \\
&= \int_0^\infty \tau^k \int_{S^n} \mathrm{div}_{T_V(x)\oplus\mathbb{R}\langle x\rangle}(X(x)) d\|V\|_x d\tau \\
&= \int_0^\infty \tau^k \int_{S^n} \mathrm{div}_{T_V(x)}(X) + \mathrm{div}_{\mathbb{R}\langle x\rangle}(X) d\|V\|_x d\tau \\
&= \int_0^\infty \tau^k \delta V(X) d\tau = 0.
\end{aligned}$$

Thus finishing the proof of the proposition. □

3. Geodesic networks

In this section we are concerned with 1-dimensional varifolds whose support is represented by geodesic segments. Our aim is to prove that any stationary integral 1-varifold has this structure.

Definition 3.1. Let $U \subset M$ be an open subset. A varifold $V \in I\mathcal{V}_1(M)$ is called a **geodesic network** in U if there exist geodesic segments $\{\alpha_1, \ldots, \alpha_l\}$ in M and $\{\theta_1, \ldots, \theta_l\} \subset \mathbb{Z}_{>0}$ such that

(a)

$$V \llcorner G_k(U) = \sum_{j=1}^{l} \underline{\underline{v}}(\alpha_j \cap U, \theta_j).$$

(b) Let $\Sigma_V = \cup_{j=1}^{l}(\partial\alpha_j) \cap U$, we require each $p \in \Sigma_V$ to belong to exactly $m = m(p) \geq 3$ geodesic segments $\{\alpha_{j_1}, \ldots \alpha_{j_m}\}$ and

$$\sum_{k=1}^{m} \theta_{j_k} \dot{\alpha}_{j_k}(0) = 0.$$

Here we are taking the arc-length parametrization with start point at p.

We call a point in Σ_V a **junction**. We say that a junction is **singular** if there exist at least 1 geodesic segment α_{j_k} with $\theta_{j_k} \dot{\alpha_{j_k}}(0) \neq -\theta_{j_{k'}} \dot{\alpha_{j_{k'}}}(0)$ for every other $j_k \neq j_{k'}$ and **regular** otherwise. A **triple junction** is a point $p \in \Sigma_V$ such that p is the boundary of only 3 geodesic segments with multiplicity 1 each.

The following properties can be derived straightforwardly from the definition.

Proposition 3.2. *Let V be a geodesic network in $U \subset M$. The following holds:*

(i) V is stationary in U;

(ii) if $p \in \Sigma_V$ and $\{(\alpha_{j_1}, \theta_{j_1}), \dots, (\alpha_{j_m}, \theta_{j_m})\}$ define this junction then the tangent cone at p is given by

$$\mathscr{C}_p V = \sum_{k=1}^{m} \underline{\underline{v}}(\mathrm{cone}(\dot{\alpha}_{j_k}(0)), \theta_{j_k})$$

where $\mathrm{cone}(\dot{\alpha}_{j_k}(0)) = \{\lambda \dot{\alpha}_{j_k}(0) \in T_pM | \lambda \geq 0\}$ *and* $\alpha_{j_k}(0) = p$.

Corollary 3.3. *Let $U \subset M$ be an open set. If V is a geodesic network in U and $\Theta^1(V, x) < 2$ for all $x \in \mathrm{supp}\|V\|$, then every $p \in \Sigma_V$ is a triple junction.*

Proof. First note that the condition $\Theta^1(V, x) < 2$ at regular points imply that $\theta_j = 1$ for all j. By proposition 3.2(ii) the density is given by

$$\Theta^1(V, p) = \Theta^1(\mathscr{C}_p V, 0) = \sum_{k=1}^{m} \frac{\theta_{j_k}}{2}.$$

Since $\theta_{j_k} = 1$ we must have $m < 4$ thus $m = 3$. □

In the two dimensional case we can infer further on the regularity of junctions.

Corollary 3.4. *Let M be a surface and $V \in \mathrm{I}\mathcal{V}_1(M)$ be a geodesic network with density $\Theta^1(V, p) \leq 2$ for all $p \in \mathrm{supp}\|V\|$. Then **either***

*(i) Σ_V contains at least one triple junctions **or***

(ii) Σ_V has no triple junctions, all junctions are regular and V is given by

$$V = \sum_{i=1}^{l} \underline{\underline{v}}(\gamma_i, 1)$$

where γ_i are closed geodesics (possibly repeated) and $\gamma_{i_1} \cap \gamma_{i_2} \cap \gamma_{i_3} = \emptyset$ for i_1, i_2, i_3 all distinct.

Proof. In view of Corollary 3.3 all of the junctions with multiplicity less than 2 are triple junctions. Let us assume that (i) is false and we will show that V must satisfy (ii), that is, V has no triple junctions so all of the singular points have multiplicity 2. If there is a geodesic segment of multiplicity 2, then it cannot intersect any junction, because of the multiplicity bound.

The only possible junction is one formed by 4 distinct geodesic segments of multiplicity one each. We want to show that in this case it must be regular. That is, at least two of the segments must have opposite directions at the singular point, which implies that so do the other two.

Denote by v_1, v_2, v_3, v_4 the unitary tangent direction of each geodesic segment at the singularity. Let us suppose that at least 2 of these are distinct and not opposite to each other. Without loss of generality we may assume it is v_1 and v_2. Since we are in dimension 2 we can use them as a basis and write v_3 and v_4 in terms of v_1 and v_2. If one solves the system

$$\begin{cases} v_1 + v_2 + v_3 + v_4 = 0 \,(\text{stationary condition}) \\ \|v_i\| = 1 \,(\text{multiplicity one}) \end{cases}$$

then it is easy to see that, for example, v_3 must be opposite to either v_1 or v_2.

For the second part of (ii), take $C \subset \operatorname{supp}\|V\|$ a connected component. If $V \llcorner C$ is given by a closed geodesic with multiplicity 2 then the density condition implies that it cannot have junctions and the statement is true. Otherwise, by what we showed above, each geodesic segment can be extended through the singular points. Again, because of the density hypothesis we cannot have 3 geodesics intersecting at the same point. □

The main result is a structure theorem for 1-varifolds proved in [2]. Here we state a particular case and refer to the original article for a proof.

Theorem 3.5. *Let M be a closed manifold and $U \subset M$ an open set. If $V \in \mathrm{I}\mathcal{V}_1(M)$ is stationary in U then V is a geodesic network in U.*

Proof. Simply note that the definition of interval in [2, §1] is equivalent to being the image of a geodesic segment. The hypothesis for the theorem in [2, §3] are true because V is integral. Finally, note that the set S_V is the same as our set of junctions Σ_V. □

Now we prove the property that we are mainly interested for geodesic networks in S^n

Proposition 3.6. *Let (S^n, g_0) be the round sphere of radius 1, $U \subset S^n$ and $V \in \mathrm{I}\mathcal{V}_1(S^n, U)$ be stationary in U with total mass $\|V\|(S^n) < 2\pi d$ for some positive integer d. Then V is a geodesic network satisfying $\Theta^1(V, x) < d$ for all $x \in S^n$.*

Proof. We know by Theorem 3.5 that V is a geodesic network. Let us prove that $\|V\|(S^n) < 2\pi d$ implies $\Theta^1(V,x) < d$ for every $x \in \operatorname{supp}\|V\|$.

Using proposition 2.6(iv) and (v), the Monotonicity formula for stationary varifolds (see [16, §17.8]) and $\alpha_2 = \pi$ we compute

$$\begin{aligned}\Theta^1(V,x) = \Theta^2(C(V),x) &= \lim_{r\to 0}\frac{\|C\|B(x,r)}{\alpha_2 r^2}\\ &\le \lim_{r\to\infty}\frac{\|C\|B(x,r)}{\alpha_2 r^2}\\ &= \frac{\|V\|(S^n)}{2\alpha_2}\\ &< d.\end{aligned}$$

□

We can also prove a weaker version of this theorem for metrics that are sufficiently close to the round metric in S^n.

Theorem 3.7. *Let g be a Riemannian metric in S^n. If g is sufficiently C^∞-close to the round metric then any varifold $W \in \mathcal{IV}_1(S^n)$ stationary with respect to the metric g satisfying $\|W\|(S^n) < 2\pi(d+\frac{1}{3})$ is a geodesic network such that $\Theta^1(W,x) \le d$ for all $x \in \operatorname{supp}\|W\|$.*

Proof. By Theorem 3.5 we know that W is a geodesic network with respect to the metric g. It remains to prove the second statement.

Assume false, that is, there exist a sequence of metrics g_i converging to g_0 and a sequence of integral varifolds W_i stationary with respect to g_i satisfying $\|W_i\|(S^n) < 2\pi(d+\frac{1}{3})$ and $\Theta^1(W_i,p_i) > d$ for some $p_i \in \operatorname{supp}\|W_i\|$. In fact we must have $\Theta^1(W_i,p_i) \ge (d+\frac{1}{2})$ because W_i is a geodesic network.

Since the first variation is continuous with respect to the metric, we may assume that each W_i has bounded first variation in the metric g_0. By the Compactness Theorem we may suppose that W_i converges to an integral varifold V stationary in the round metric and p_i converges to $p \in \operatorname{supp}\|V\|$.

Furthermore, we have $\|V\|(S^n) \le \liminf_{i\to\infty}\|W_i\|(S^n) < 2\pi(d+\frac{1}{3})$ because the mass is lower semicontinuous with respect to varifold convergence. Following a computation similar to Proposition 3.6 we obtain that $\Theta^1(V,x) < (d+\frac{1}{3})$ for all $x \in \operatorname{supp}\|V\|$. Since V is a geodesic network we must have $\Theta^1(V,x) \le d$. On the other hand, the density is upper semicontinuous with respect to weak convergence of varifolds. Hence, $(d+\frac{1}{2}) \le \limsup_{i\to\infty}\Theta^1(W_i,p_i) \le \Theta^1(V,p) \le d$, which is a contradiction. □

4. Almost minimising varifolds

In this section we define $\mathbb{Z}_2$-almost minimising varifolds and show that such 1-dimensional varifolds cannot admit triple junctions.

Definition 4.1. Let $U \subset M$ be an open set, $\varepsilon > 0$ and $\delta > 0$. We define

$$\mathfrak{A}_k(U;\varepsilon,\delta) \subset \mathcal{Z}_k(M)$$

as the set $T \in \mathcal{Z}_k(M)$ such that any finite sequence $\{T_1,\ldots,T_m\} \subset \mathcal{Z}_k(M)$ satisfying

(a) $\operatorname{supp}(T - T_i) \subset U$ for all $i = 1,2,\ldots,m$;

(b) $\mathcal{F}(T_i, T_{i-1}) \leq \delta$ for all $i = 1,2,\ldots,m$ and

(c) $\underline{\underline{\mathbf{M}}}(T_i) \leq \underline{\underline{\mathbf{M}}}(T) + \delta$

must also satisfy

$$\underline{\underline{\mathbf{M}}}(T_m) \geq \underline{\underline{\mathbf{M}}}(T) - \varepsilon.$$

Roughly speaking, if T belongs to $\mathfrak{A}_k(U;\varepsilon,\delta)$ then any deformation of T supported in U that does not increase mass at least ε must be δ-far from T in the $\mathcal{F}$ metric. Note that we define the elements of $\mathfrak{A}_k$ as closed cycles in M instead of relative cycles as defined in [15].

Definition 4.2. We say that a varifold $V \in \mathcal{V}_k(M)$ is $\mathbb{Z}_2$-**almost minimising** in U if for every $\varepsilon > 0$ there exists $\delta > 0$ and $T \in \mathfrak{A}_k(U;\varepsilon,\delta)$ such that

$$\underline{\underline{\mathbf{F}}}(V, |T|) < \varepsilon.$$

A varifold $V \in \mathcal{V}_k(M)$ is said to be $\mathbb{Z}_2$-**almost minimising in annuli** if for every $p \in \operatorname{supp}\|V\|$ there exists $r > 0$ such that V is $\mathbb{Z}_2$-almost minimising in the annulus $A = A(p;s,r)$ for all $0 < s < r$.

Definition 4.3. For a cubical subcomplex $X \subset I^N$, we say that a flat continuous map $f : X \to \mathcal{Z}_k(M)$ has *no concentration of mass* if

$$\lim_{r\to 0} \sup\{\|f(x)\|(B(q,r)) : x \in X \text{ and } q \in M\} = 0.$$

The next theorem shows the existence of $\mathbb{Z}_2$-almost minimising varifolds.

Theorem 4.4. *Let $X \subset I^N$ be a cubical subcomplex and $f : X \to \mathcal{Z}_k(M)$ be a p-sweepout with no concentration of mass. Denote Π_f the class of all flat continuous maps $g : X \to \mathcal{Z}_k(M)$ with no concentration of mass that are flat homotopic to f and write*

$$L[\Pi_f] = \inf_{g\in\Pi_f} \sup_{x\in X} \underline{\underline{\boldsymbol{M}}}(g(x)).$$

If $L[\Pi_f] > 0$ then there exists $V \in \mathrm{I}\mathcal{V}_k(M)$ such that

(i) $\|V\|(M) = L[\Pi_f]$;

(ii) V *is stationary in* M;

(iii) V *is* $\mathbb{Z}_2$*-almost minimising in annuli.*

This was first proven by Pitts (see [15, §4.10]), for another proof (when $k = dim(M) - 1$) we refer to [12].

Note that this is a weaker statement than in [12], but it remains true for all dimensions and codimensions. This is because for every flat continuous homotopy class we can construct a discrete homotopy class just as in [12, Theorem 3.9] with the same width. The final statement then follows from [15, §4.10].

Definition 4.5. Let $T \in \mathcal{Z}_k(M)$ and $W \subset M$ be an open set. We say that T is locally mass minimising in W if for every $p \in \operatorname{supp}(T) \bigcap W$ there exists $r_p > 0$ such that $B(p, r_p) \subset W$ and for all $S \in \mathcal{Z}_k(M)$ satisfying $\operatorname{supp}(T - S) \subset B(p, r_p)$ we have

$$\underline{\underline{\mathbf{M}}}(S) \geq \underline{\underline{\mathbf{M}}}(T).$$

In the one dimensional case we have the following characterization:

Proposition 4.6. *Let $W \subset M$ be an open set, $Z \subset W$ compact and $T \in \mathcal{Z}_1(M)$ be locally mass minimising in W. Then each connected component of $\operatorname{supp}(T) \bigcap Z$ is the restriction of a geodesic segment with endpoints in $W \setminus Z$.*

Proof. Let $A \subset \operatorname{supp}(T) \bigcap Z$ be a connected component. Cover A by finitely many balls $B_i = B(p_i, r)$, $i = 1, \ldots, m$ such that each ball is contained in a convex neighborhood and $r < r_{p_i}$ for all i. Denote $C = \operatorname{supp}(T) \bigcap (B_1 \cup \cdots \cup B_m)$, then each component $C \bigcap B_i$ is the unique minimising geodesic connecting the two points in $C \bigcap \partial B_i$. In particular the endpoints $A \bigcap \partial Z$ belong to the interior of a geodesic segment with endpoints in $\operatorname{int}(Z)$ and

$W \setminus Z$. We conclude that A is given by the image of a broken geodesic with singular points in the interior of Z.

Now, for each singular point $q \in A$ there exist r_q such that $T \llcorner B(q, r_q)$ is mass minimising relative to its boundary. Thus it must be a geodesic segment, that is, q is a smooth point in A. This implies that C is the image of a geodesic segment with endpoints in $W \setminus Z$. The proof finishes by simply noting that $A = \overline{C \bigcap Z}$. $\square$

Corollary 4.7. *Let $W \subset M$ be an open set, $Z \subset W$ a compact set and $T \in \mathcal{Z}_1(M)$ be locally mass minimising in W. Then, viewing T as an integer coefficient current,*

$$T \llcorner Z = \sum_{i=1}^{k} \underline{\underline{\mathbf{t}}}(\beta_i, [1], \dot{\beta}_i),$$

where $\beta_i : [0,1] \to Z$ are geodesic segments for each $i = 1, \dots, k$ with endpoints in ∂Z.

In particular, the associated varifold $|T| \in \mathbf{I}\mathcal{V}_1(M)$ is stationary in W.

Proof. We simply need to apply the Constancy Theorem (see [16, §41]) to each connected component. Since we are working with $\mathbb{Z}_2$ coefficients the density in each segment must be constant 1. $\square$

The replacement theorem for almost minimising varifolds can be stated as follows:

Theorem 4.8. *Let $U \subset M$ be an open set, $K \subset U$ compact and $V \in \mathcal{V}_k(M)$ $\mathbb{Z}_2$-almost minimising in U. There exists a non-empty set $\mathcal{R}(V; U, K) \subset \mathcal{V}_k(M)$ such that every $V^* \in \mathcal{R}(V; U, K)$ satisfy:*

(i) $V^* \llcorner G_k(M \setminus K) = V \llcorner G_k(M \setminus K)$;

(ii) $\|V^*\|(M) = \|V\|(M)$;

(iii) V^* *is $\mathbb{Z}_2$-almost minimising in U;*

(iv) $V^* \llcorner G_k(\mathrm{int}\,(K)) \in \mathbf{I}\mathcal{V}_k(M)$ *and*

(v) for each $\varepsilon > 0$ there exists $T \in \mathcal{Z}_k(M)$ locally mass minimising in $\mathrm{int}\,(K)$ *such that* $\underline{\underline{\boldsymbol{F}}}(V^*, |T|) < \varepsilon$.

Proof. The proof of (i)-(iv) is exactly as in [15, §3.11]. To show (v) one need to modify the construction in [15, §3.10] using our definition of almost minimising. $\square$

Remark. Note that if V is stationary on all of M then so is V^*.

In fact, V^* is almost-minimising in U (property 4.8(iii)) so it is also stationary in U. Since V^* coincides with V on $M \setminus K$ then it is also stationary in $M \setminus K$. That is, V^* is stationary in U, $M \setminus K$ and $U \cap (M \setminus K)$. Hence V^* is stationary in M.

4.1. Almost minimising geodesics networks

Here we will treat the particular case when V is a geodesic network. Our main goal is to prove that the almost minimising property excludes the existence of triple junctions.

The rough idea is to use the replacement theorem and approximate V^* by closed currents with coefficients in $\mathbb{Z}_2$. We will show that V^* can be described as a non-zero $\mathbb{Z}_2$-cycle but triple junctions always have boundary in $\mathbb{Z}_2$. From now on, given a varifold V we will denote by V^* a replacement given by Theorem 4.8 whenever V satisfies the conditions of the theorem.

To prove the next technical lemma we will need the following theorem proven in [19] by B. White and is used to prove a maximum principle for varifolds.

Theorem 4.9. *Let N be a n-dimensional Riemannian manifold with boundary and $p \in \partial N$ such that $\kappa_1(p) + \cdots + \kappa_m(p) > \eta$, where $\kappa_1 \leq \cdots \leq \kappa_{n-1}$ are the principal curvatures of ∂N with respect to the inward normal vectorfield ν_N. Then, given $\varepsilon > 0$ there exists a supported vectorfield X on N such that $X(p) \neq 0$ is normal to ∂N and*

$$\langle X, \nu_N \rangle \geq 0 \text{ in } \partial N$$

and

$$\delta V(X) \leq -\eta \int |X| d\|V\|$$

for every $V \in \mathcal{V}_m(N)$.

We remark that the same theorem is true with all its inequalities reversed, the proof is exactly the same (see [19]).

Corollary 4.10. *Let M be a closed Riemannian manifold and N an open set with strictly convex boundary with respect to the inward normal vectorfield ($\kappa_1 \geq \eta > 0$).*

If $V \in \mathcal{V}_1(M)$ is stationary, $p \in \operatorname{supp}\|V\| \cap \partial N$ and $\operatorname{supp}\|V\| \cap B(p,\varepsilon) \cap N \neq \emptyset$ then $\operatorname{supp}\|V\| \cap B(p,\varepsilon) \cap (M \setminus \overline{N}) \neq \emptyset$.

Proof. First suppose there exists $\varepsilon > 0$ such that $\operatorname{supp}\|V\| \cap B(p,\varepsilon) \subset \overline{N}$, that is, $W = V \llcorner G_1(B(p,\varepsilon)) \in \mathcal{V}_1(\overline{N})$. Since ∂N is strictly convex, we can choose $\eta > 0$ in Theorem 4.9 and obtain a vectorfield X in $\overline{N}$ such that $\operatorname{supp}(X) \subset \overline{N} \cap B(p,\varepsilon)$ and

$$\delta W(X) + \frac{\eta}{2}\int |X| < 0.$$

This is not a contradiction yet because X is not a smooth vectorfield in M. However, we can construct a extension $\tilde{X}$ such that $\operatorname{supp}(\tilde{X}) \subset B(p,\varepsilon)$, $\tilde{X}$ is C^1-close to X and

$$\delta W(\tilde{X}) + \frac{\eta}{2}\int |\tilde{X}| < 0.$$

By construction $\operatorname{supp}(\tilde{X}) \subset B(p,\varepsilon)$ hence $\delta V(\tilde{X}) = \delta W(\tilde{X}) < 0$. This is a contradiction because V is stationary, thus $\operatorname{supp}\|V\| \cap B(p,\varepsilon) \cap (M \setminus \overline{N}) \neq \emptyset$. □

We now show that an almost-minimising geodesic network is its own replacement. To simplify notation, from now on we write $V \llcorner U = V \llcorner G_1(U)$ whenever $V \in \mathrm{I}\mathcal{V}_1(M)$ and $U \subset M$ is an open set.

Lemma 4.11. *Let M be a closed Riemannian manifold and $V \in \mathrm{I}\mathcal{V}_1(M)$ be a geodesic network and $p \in \Sigma_V$ be a junction point. If V is almost minimising in annuli at p then there exists $r > 0$ and a compact set $K \subset A(p;r,3r)$ such that*

(i) V is almost minimising in $A(p;r,3r)$ and

(ii) $\mathcal{R}(V;A,K) = \{V\}$.

Proof. Since V is a geodesic network, then its singularities are isolated. That is, there exists $r_p > 0$ such that p is the only singularity in $B(p,r_p)$.

Firstly choose $r > 0$ such that $4r < r_p$, $B = B(p,4r)$ is a convex ball and V is almost minimising in $A = A(p;r,3r)$. It follows from the structure of a geodesic network that

$$V \llcorner B = \sum_{j=1}^{m} \underline{\underline{v}}(\alpha_j,\theta_j)$$

where $\alpha_j : [0,4r] \to \overline{B}$ is a minimising geodesic parametrized by arc-length for each $j = 1,\ldots,m$. By abuse of notation we identify the curves α_j with its image.

Secondly, we can choose $\delta < r$ sufficiently small such that the balls $K_j = \bar{B}(\alpha_j(2r), \delta)$ have strictly convex boundary with respect to the inward normal vector and are pairwise disjoint. Define $a_j = \alpha_j(2r - \delta)$, $b_j = \alpha_j(2r + \delta)$ and $K = K_1 \cup \cdots \cup K_m \subset A$.

Finally we take $V^* \in \mathcal{R}(V; A, K)$ a replacement for V and define $V_j^* = V^* \llcorner \operatorname{int}(K_j)$ and $V_j = V \llcorner \operatorname{int}(K_j)$. By property 4.8(i) it is sufficient to show that $V_j^* = V_j$ for each j.

Claim **1.** $\sum_{j=1}^m \|V_j^*\|(M) = \sum_{j=1}^m \|V_j\|(M)$

This follows directly from properties 4.8(i) and (ii).

Claim **2.** *For each $j = 1, \ldots, m$ either $V_j^* = 0$ or $\operatorname{supp}\|V_j^*\|$ contains a rectifiable curve connecting a_j to b_j.*

Note that $\operatorname{supp}\|V_j^*\|$ only intersects $\operatorname{bdry}(K_j)$ at the points a_j and b_j. In fact, suppose there is another point of intersection. Then, by the maximum principle (Corollary 4.10) it follows that $\operatorname{supp}\|V^*\| \setminus \operatorname{int}(K_j) = \operatorname{supp}\|V^* \llcorner M \setminus K\|$ also contains that point, but this contradicts property 4.8(i).

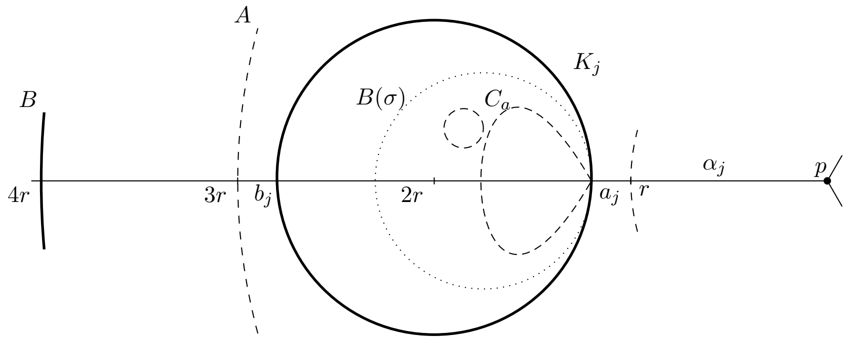
Now, suppose $\operatorname{supp}\|V_j^*\|$ contains no curve joining a_j and b_j. In that case, we can write $\operatorname{supp}\|V_j^*\| = C_a \cup C_b$ where C_a and C_b are closed disjoint sets containing a_j and b_j respectively (these are not unique and not necessarily connected). Take U_a and U_b open and disjoint neighbourhoods of C_a and C_b in the interior of K_j respectively. We will show that $V_j^* \llcorner U_a = V_j^* \llcorner U_b = 0$.

Take for example $V_j^* \llcorner U_a$, which is stationary (see remark after Theorem 4.8). Now, consider $B(\sigma) = B(\alpha(2r\sigma), \sigma\delta)$ then $V_j^* \llcorner U_a$ is entirely contained in $B(1) = \operatorname{int}(K_j)$ and it only intersects the boundary at the point a_j. Since $\partial B(\sigma)$ is strictly convex for all σ, a maximum principle argument shows that $V_j^* \llcorner U_a$ is contained in $B(\sigma)$ for all $\sigma < 1$ thus proving that $V_j^* \llcorner U_a = 0$. The same argument shows that $V_j^* \llcorner U_b = 0$ and we prove the claim.

Claim **3.** $V_j^* \neq 0$ *for all* $j = 1, \ldots, m$

Consider $B_j' = B(\alpha_j(2r), \delta')$ with $\delta < \delta' < r$ such that $K_j \subset B_j' \subset A$ are still pairwise disjoint. Then property 4.8(i) implies that

$$V^* \llcorner B_j' \setminus K_j = \underline{\underline{v}}(\alpha_j \cap (B_j' \setminus K_j), \theta_j).$$



If V_j^* was zero, then in particular $V_j^* = \underline{\underline{v}}(\alpha_j, 0)$. But

$$V^* \llcorner B_j' = V^* \llcorner B_j' \setminus K_j + V_j^*$$

is stationary and its support is contained in α_j. From the Constancy Theorem we conclude that $\theta_j = 0$ which is a contradiction, thus $V_j^* \neq 0$.

This means that $\operatorname{supp}\|V_j^*\|$ contains a rectifiable curve C_j connecting a_j to b_j for all $j = 1, \ldots, m$. In particular this implies that $l(C_j) \geq d(a_j, b_j)$. Since V_j^* is integral (see property 4.8(iv)) it follows that $\|V_j^*\|(M) \geq d(a_j, b_j)$. However, $\alpha_j \cap K_j$ is a minimising geodesic connecting a_j to b_j, so $\|V_j\|(M) = d(a_j, b_j)$. We conclude that $\|V_j^*\|(M) \geq \|V_j\|(M)$ for all $j = 1, \ldots, m$. Claim 1 implies that we have in fact

$$\|V_j^*\|(M) = \|V_j\|(M) \text{ for all } j = 1, \ldots, m.$$

On the other hand, we have $d(a_j, b_j) = \|V_j^*\|(M) \geq l(C_j) \geq d(a_j, b_j)$, that is, C_j is a minimising curve and it must be a geodesic. Since $\alpha_j \cap K_j$ is the unique geodesic connecting a_j to b_j we conclude that $C_j = \alpha_j \cap K_j$. Finally, this implies that $\operatorname{supp}\|V_j^*\| = \operatorname{supp}\|V_j\|$ because otherwise there would be more contribution of mass. Applying the Constancy Theorem again we show that $V_j^* = V_j$ and this finishes the proof. $\square$

The last result we need relates flat convergence of $\mathbb{Z}_2$-currents and the weak convergence of the associated varifold. This was proven in [18] by B. White.

Theorem 4.12. *Let M be a Riemannian manifold, $\{W_i\} \subset \mathbf{I}\mathcal{V}_k(M)$ be a sequence converging to an integral varifold W. Suppose that:*

(I) Each W_i has locally bounded first variation;

(II) $\partial[W_i]$ converges in the flat topology.

Then $[W_i]$ converges to $[W]$ in the flat topology.

Finally we prove our main result of this section.

Theorem 4.13. *Let M be a Riemannian manifold, $V \in \mathbf{I}\mathcal{V}_1(M)$ a geodesic network and $p \in \Sigma_V$ a junction point. If V is $\mathbb{Z}_2$-almost minimising in annuli at p, then*

$$\Theta^1(V, p) \in \mathbb{N}.$$

In particular p is not a triple junction.

Proof. Let $r > 0$, $B = B(p, 4r)$, $A = A(p; r, 3r)$ and $K \subset A$ as in Lemma 4.11. Applying property 4.8(v), Corollary 4.7 and the Compactness theorem for $\mathbb{Z}_2$-chains (see [18, Theorem 5.1]) we may assume there exists a convergent sequence $\{T_i\}_{i\in\mathbb{N}} \subset \mathcal{Z}_1(M)$ and $T \in \mathcal{Z}_1(M)$ such that

(a) $T_i \to T$ in the $\mathcal{F}$-norm;

(b) $V_i = |T_i|$ is stationary in $\mathrm{int}\,(K)$ and

(c) $V_i \to V$ in the $\underline{\underline{\mathbf{F}}}$-metric.

Even though convergence of chains in the flat norm do not correspond to weak convergence for varifolds, in the stationary case, with convergent boundary, it does.

We want to apply Theorem 4.12 for the sequence $\{V_i \llcorner \mathrm{int}\,(K)\}_{i\geq 1}$. We know that $\partial[V_i \llcorner \mathrm{int}\,(K)] \to \partial T \llcorner \mathrm{int}\,(K)$ by the definition of V_i. Together with property (b) it means that the sequence satisfies the hypothesis of the theorem. We conclude that

$$T \llcorner \mathrm{int}\,(K) = [V \llcorner \mathrm{int}\,(K)].$$

Since $V \llcorner B = \sum_{j=1}^m \underline{\underline{v}}(\alpha_j, \theta_j)$ for some geodesic segments α_j and $\theta_j \in \mathbb{Z}_{>0}$, we have

$$[V \llcorner \mathrm{int}\,(K)] = \sum_{j=1}^m \underline{\underline{v}}(\alpha_j \cap \mathrm{int}\,(K), [\theta_j])$$

and $[\theta_j]$ is non-zero only when θ_j is odd.

If θ_j is even for all j then the density at p must be an integer and we finish our proof because geodesic segments with even multiplicity contribute to the density at p with an integer number.

In case some θ_j is odd we have that $T \neq 0$ and $\operatorname{supp}(T) \subset \operatorname{supp}\|V\|$. We can view $T \llcorner B$ as an integer chain and apply the Constancy theorem for integral currents (see [16, §26.27]) and the fact that T and $[V]$ coincide in $\operatorname{int}(K)$ to conclude that

$$T \llcorner B = \sum_{j=1}^{m} \underline{\underline{\mathbf{t}}}(\alpha_j, [\theta_j], \dot{\alpha_j}).$$

Now we simply note that p is a boundary point for T unless the number of θ_j such that $[\theta_j] \neq 0$ is even. That is, there is an even number of geodesic segments α_j with odd multiplicity and in particular its density contribution is an integer number. This finishes the proof because T is a closed chain. $\square$

5. The width of an ellipsoid

Here we will apply the previous results to estimate some of the k-widths of ellipsoids sufficiently close to the round sphere.

5.1. Sweepouts of S^2

Let (S^2, g_0) denote the round 2-dimensional sphere with radius 1 in $\mathbb{R}^3$. We will construct k-sweepouts of S^2 as families of algebraic sets in $\mathbb{R}^3$. This is similar to how it is done in [10] for the unit ball.

Denote by x_1, x_2, x_3 the coordinates in $\mathbb{R}^3$ with respect to the standard basis. Let $p_i : \mathbb{R}^3 \to \mathbb{R}$ denote the following polynomials for $i = 1, \dots, 8$:

$$\begin{aligned}
&p_j(x) = x_j \text{ for } j = 1, 2, 3;\\
&p_4(x) = {x_1}^2;\\
&p_5(x) = x_1 x_2;\\
&p_6(x) = x_1 x_3;\\
&p_7(x) = x_2 x_3;\\
&p_8(x) = {x_3}^2.
\end{aligned}$$

Note that we skipped the polynomial ${x_2}^2$. The reason for this is because we are only interested in the zero set restricted to the sphere, which is given by the equation ${x_1}^2 + {x_2}^2 + {x_3}^2 - 1 = 0$. That is, p_4, p_8 and $x \mapsto {x_2}^2$ are linearly dependent so their linear combinations will define the same algebraic sets.

Now, put $A_k = \mathrm{span}_{\mathbb{R}}\left(1 \cup_{j=1}^{k} p_j\right) \setminus \{0\}$ and define the relation $q \sim \lambda q$, for $\lambda > 0$ and $q \in A_k$. Note that the zero set is invariant under this relation, that is, $\{\lambda q = 0\} = \{q = 0\}$ so it makes sense to define the map $F_k : (A_k/\sim) \to \mathcal{Z}_1(S^2)$ as

$$F_k([q]) = \partial[\{q < 0\} \cap S^2],$$

where $[R]$ denotes the mod 2 current associated with $R \subset S^2$. It is clear that F_k is well defined and it takes values in $\mathcal{Z}_1(S^2)$. Also, we can identify $(A_k/\sim)$ with $\mathbb{R}P^k$.

Lemma 5.1. *F_k is continuous with respect to the flat topology.*

Proof. Let p and q be two polynomials in A_k.

Observe that $\partial[\{p < 0\}\Delta\{q < 0\} \cap S^2] = \partial[\{p < 0\} \cap S^2] - \partial[\{q < 0\} \cap S^2]$ and $\{p < 0\}\Delta\{q < 0\} = \{pq \le 0\}$, where Δ denotes the symmetric difference of sets. Now, note that $\underline{\underline{\mathbf{M}}}([\{pq \le 0\} \cap S^2]) = \underline{\underline{\mathbf{M}}}([\{pq < 0\} \cap S^2])$ unless $\{pq = 0\} \cap S^2$ is an open set of S^2 with positive measure. In this case we must have $\{pq = 0\} \cap S^2 = S^2$ which implies that $\partial[\{p < 0\} \cap S^2] - \partial[\{q < 0\} \cap S^2] = 0$, that is, $F_k([p]) = F_k([q])$.

Finally, let $[q_i]$ converge to $[p]$ in $A_k/\sim$. Without loss of generality we can suppose q_i converges to λp, for some $\lambda > 0$. We conclude that

$$\mathcal{F}(\partial[\{p < 0\} \cap S^2] - \partial[\{q_i < 0\} \cap S^2]) \le \underline{\underline{\mathbf{M}}}([\{pq_i < 0\} \cap S^2]),$$

unless $F_k([p]) = F_k([q_i])$. We can suppose that $F_k([p]) \ne F_k([q_i])$ for all i sufficiently large. If q_i tends to p then $\underline{\underline{\mathbf{M}}}([\{pq_i < 0\} \cap S^2])$ converges to $\underline{\underline{\mathbf{M}}}([\{\lambda p^2 < 0\} \cap S^2]) = 0$. This proves that the flat norm converges to zero, that is, F_k is flat continuous. $\square$

Lemma 5.2. *Let $F_k : \mathbb{R}P^k \to \mathcal{Z}_1(S^2)$, k=1,…,8, be the family of cycles defined above. Then F_k has no concentration of mass.*

Proof. Take $p \in S^2$ and $0 < r < \pi$ and denote by α_p the equator given by $p^\perp \cap S^2$, where $p^\perp$ is the plane normal to p in $\mathbb{R}^3$. Consider the ball $B(p, r) \subset S^2$. We can parametrize the space of geodesics that go through $B(p, r)$ as $G(r) = \{q^\perp \cap S^2 : d(q, \alpha_p) < r\}$. The set $G(r)$ defines a spherical segment whose area is $area(G(r)) = 4\pi \sin(r)$.

If $x \in \mathbb{R}P^k$ is such that $F_k(x) \cap B(p,r) \neq \emptyset$ then it follows from the Crofton formula that

$$\underline{\underline{\mathbf{M}}}(F_k(x) \llcorner B(p,r)) = \frac{1}{4} \int_{\Gamma \in G(r)} \#(\Gamma \cap F_k(x)).$$

Since $\Gamma \cap F_k(x)$ is the intersection of a plane with S^2 and $F_k(x)$ then it is the solution of a system of 3 polynomials of degree 1, 2 and at most 2 (1 if $k = 1, 2, 3$ or 2 if $k = 4, \ldots, 8$), respectively. It follows that the intersection is generically $\#(\Gamma \cap F_k(x)) \leq 4$. Hence,

$$\underline{\underline{\mathbf{M}}}(F_k(x) \llcorner B(p,r)) \leq 4\pi \sin(r).$$

If we take $r \to 0$ we conclude that F_k has no concentration of mass at p. Since p was arbitrary we conclude the proof. □

Remark. Note that the same proof is valid for any family of algebraic curves in S^2 with bounded degree.

Theorem 5.3. *If S^2 is the round 2-sphere of radius 1, then*

(i) $\omega_1(S^2) = \omega_2(S^2) = \omega_3(S^2) = 2\pi$;

(ii) $\omega_4(S^2) = \omega_5(S^2) = \omega_6(S^2) = \omega_7(S^2) = \omega_8(S^2) = 4\pi$.

Proof. **(i):** By the Crofton formula we have that, $\underline{\underline{\mathbf{M}}}(F_k(q)) \leq 2\pi$ for all $q \in \mathbb{R}P^k$ and $k = 1, 2, 3$. In fact, it is not hard to see that $\sup \underline{\underline{\mathbf{M}}}(F_k(q)) = 2\pi$. That is, $\omega_k \leq 2\pi$.

Suppose $\omega_k < 2\pi$, then there exists another k-sweepout with no concentration of mass $\tilde{F}$ such that $L[\Pi_{\tilde{F}}] < 2\pi$. Hence, Theorem 4.4 would give us a stationary $\mathbb{Z}_2$-almost minimising integral varifold with $\|V\|(S^2) < 2\pi$. This is a contradiction because Proposition 3.6 tells us that the density would be lower than 1 everywhere. So F_k is optimal and $\omega_k = 2\pi$ for $k = 1, 2, 3$.

For the next item we need a lemma whose proof we give in the Appendix.

Lemma 5.4. *Let S^2 be the round 2-sphere of radius 1, then $\omega_4 > 2\pi$.*

(ii): When $k = 4, 5, 6, 7, 8$ the degree of the polynomials are less than or equal to 2, thus, using the Crofton formula again, $\underline{\underline{\mathbf{M}}}(F_k(q)) \leq 4\pi$ for all $q \in \mathbb{R}P^k$. As before, it is trivial to check that $\sup \underline{\underline{\mathbf{M}}}(F_k(q)) = 4\pi$ from which we get $\omega_k \leq 4\pi$.

By Lemma 5.4 and the previous item we already know that $\omega_k \geq \omega_4 > 2\pi$. Suppose $\omega_k < 4\pi$ then, as before, we have a k-sweepout $\tilde{F}$ with no concentration of mass such that $\omega_k \leq L[\Pi_{\tilde{F}}] < 4\pi$. From Theorem 4.4 we produce $V \in \mathbf{I}\mathcal{V}_1(S^2)$ stationary and $\mathbb{Z}_2$-almost minimising. It follows from Theorems 3.5 and 4.13 that V has constant density equal to 1. Hence V corresponds to a closed regular geodesic, that is, $\|V\|(S^2) = 2\pi$, which is a contradiction. $\square$

5.2. Geodesics on ellipsoids

Our goal here is to find the varifold that realizes the k-width of an ellipsoid sufficiently close to the round sphere.

Let $E^2 = E^2(a_1, a_2, a_3)$ be an ellipsoid defined by the equation $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 - 1 = 0$ in $\mathbb{R}^3$. If the parameters a_1, a_2, a_3 are all sufficiently close to 1 then it is clear that the induced metric in E^2 is C^∞-close to the round metric in S^2. We can assume other properties that we summarize here.

Proposition 5.5. *Let $\gamma_i = \{x_i = 0\} \cap E^2$ for $i = 1, 2, 3$ be the three principal geodesics in E^2, $\gamma_i^{(r)}$ be the r-covering of γ_i for $r \in \mathbb{N}$ and $\omega_k(E^2)$ denote the k-width for $k \in \mathbb{N}$. If we choose $a_1 < a_2 < a_3$ sufficiently close to 1 then the following is true:*

(i) $2\pi(1 - \frac{1}{4}) < L(\gamma_1) < L(\gamma_2) < L(\gamma_3) < 2\pi(1 + \frac{1}{4})$;

(ii) $\mathrm{index}(\gamma_i^{(r)}) = i + 2(r-1)$ *and* $\mathrm{null}(\gamma_i^{(r)}) = 0$ *for* $i = 1, 2, 3$ *and* $r < 100$;

(iii) if α is a smooth closed geodesic with $L(\alpha) < 100\pi$ then $\alpha = \gamma_i^{(r)}$ for some $i = 1, 2$ or 3 and $r > 0$;

(iv) $|\omega_k(E^2) - \omega_k(S^2)| < \frac{1}{4}$ *for all* $k < 100$;

By $\mathrm{index}(\gamma)$ *and* $\mathrm{null}(\gamma)$ *we mean the Morse index and nullity as smooth closed geodesics, that is, critical points of the energy functional.*

Proof. **(i)**: Note, for example, that γ_1 is a planar ellipsis with axes $\frac{1}{a_2}$ and $\frac{1}{a_3}$, similarly for the other two. So, as long as a_i are close to 1 each ellipsis is close to a circle of length 2π.

(ii): See [14, XI,Theorem 3.3].

(iii): See [14, XI,Theorem 4.1].

(iv): Note that every sweepout of E^2 is also a sweepout for S^2, simply by the fact they are both diffeomorphic and the definition of sweepout does not depend on the metric. Since the metric in E^2 can be chosen sufficently close to the round metric we can prove that each k-width is continuous by simply using the same approximating sweepouts. The uniform convergence follows directly because we are considering only finitely many k-widths. $\square$

Given these three main ellipses we are able to define the varifolds that will be candidates to realize the first 8 widths of E^2. Define

$$\begin{aligned}
W_j &= \underline{\underline{v}}(\gamma_j, 1), j = 1, 2, 3;\\
W_4 &= \underline{\underline{v}}(\gamma_1, 2);\\
W_5 &= \underline{\underline{v}}(\gamma_1, 1) + \underline{\underline{v}}(\gamma_2, 1);\\
W_6 &= \underline{\underline{v}}(\gamma_2, 2);\\
W_7 &= \underline{\underline{v}}(\gamma_1, 1) + \underline{\underline{v}}(\gamma_3, 1);\\
W_8 &= \underline{\underline{v}}(\gamma_2, 1) + \underline{\underline{v}}(\gamma_3, 1);\\
W_9 &= \underline{\underline{v}}(\gamma_3, 2).
\end{aligned}$$

Remark. Suppose E^2 is sufficiently close to the round sphere of radius 1. Since these are all possible combinations of the three principal geodesic with density less than or equal to 2, Theorems 3.7, 4.13 and Corollary 3.4 imply that these are also the only almost minimising geodesic networks with mass less than $2\pi(2 + \frac{1}{3})$.

They also correspond to the zero set (counted with multiplicity) of the polynomials p_j, defined in the previous section, intersected with E^2 (except for W_6).

Before proceeding to the main theorem we need a technical lemma that was proved in [12, §6] under a different context. We explain how to obtain our result from their proof in the Appendix.

Lemma 5.6. *Let E^2 be an ellipsoid as in Proposition 5.5. Then $\omega_i < \omega_{i+1}$ for $i = 1, \dots, 7$.*

Theorem 5.7. *Let E^2 be an ellipsoid as in Proposition 5.5. The following holds:*

(i) if $i = 1, 2$ or 3 then $\omega_i(E^2) = \|W_i\|(E^2)$;

(ii) if $i = 4, \dots 8$ then $\omega_i(E^2) = \|W_l\|(E^2)$ for some $l = 4, \dots, 9$ without repetition.

Proof. Firstly, it follows from Proposition 5.5(iv) and Theorem 5.3(i) that

$$\omega_i(E^2) < 2\pi\left(1 + \frac{1}{4}\right), \quad \text{for } i = 1, 2, 3 \text{ and}$$
$$\omega_i(E^2) < 2\pi\left(2 + \frac{1}{4}\right), \quad \text{for } i = 4, \ldots, 8.$$

In either case we claim that there exists an optimal sweepout for ω_i. Indeed, if no such map existed for some i we would have a sequence of sweepouts $\{F_k\}$ satisfying $\omega_i < L[F_{k+1}] < L[F_k] < 2\pi(2 + \frac{1}{4})$. Each F_k provides us a distinct almost-minimising geodesic network with mass less than $2\pi(2 + \frac{1}{4})$ by Theorem 4.4, and the characterization of stationary integral varifolds (Theorem 3.5). However, as we have already remarked, there only finitely many such varifolds (that is to say, the previously defined W_j) so we have a contradiction.

Secondly, Lemma 5.6 tells us that $\omega_1 < \cdots < \omega_8$. Hence, each optimal sweepout gives us an almost-minimising geodesic network V_i satisfying

$$\|V_i\|(E^2) = \omega_i(E^2) \quad \text{and} \quad \|V_i\|(E^2) < \|V_{i+1}\|(E^2) \quad \text{for} \quad i = 1, \ldots, 8.$$

(i): For $i = 1, 2, 3$ we have $\|V_i\|(E^2) < 2\pi(1 + \frac{1}{4})$ so each one of these must correspond to one W_j, $j = 1, 2, 3$. Since their masses are ordered as

$$\|W_1\|(E^2) < \|W_2\|(E^2) < \|W_3\|(E^2)$$

we must have $V_i = W_i$, $i = 1, 2, 3$.

(ii): For $j = 4, \ldots, 8$ the W_j's are not necessarily ordered by their mass. To be specific, we cannot guarantee for a general ellipsoid that $\|W_6\|(E^2) < \|W_7\|(E^2)$ or vice-versa. However, we know that each V_i corresponds to one of the W_j's and this correspondence must be one to one, which finishes the proof. □

At last we construct an unstable min-max varifold with multiplicity 2 and give a counterexample to Question 1.

Corollary 5.8. *Let E^2 be as in Proposition 5.5, then **Question 1** is false for E^2. Furthermore, out of W_4, W_6 and W_9 at least two must be a min-max varifold.*

Proof. First of all we observe that if the support of V is given by a smooth closed geodesic γ then index(V) and null(V) as a varifold are the same as index(γ) and null(γ) as a critical point for the energy functional.

Now, Theorem 5.7(ii) tells us that $\omega_4(E^2) = \|W_j\|(E^2)$ for some $j = 4, \ldots, 9$. Where W_j, $j = 1, \ldots, 8$ are as before. Since there are 6 varifolds to choose for 5 widths we know that one, and only one, will not correspond to a width. This proves the second statement.

The first 3 varifolds are ordered as $\|W_4\|(E^2) < \|W_5\|(E^2) < \|W_6\|(E^2)$ which implies that ω_4 must correspond to either W_4 or W_5. If $\omega_4(E^2) = \|W_4\|(E^2)$ then the number of parameters is 4 but $\mathrm{index}(V) + \mathrm{null}(V) = 3 + 0 < 4$, as given by property 5.5(ii).

If this is not the case, then W_4 is the only varifold that does not correspond to any width and all the other ones must correspond to one, and only one, width. As we have already pointed out, the comparison between $\|W_6\|(E^2)$ and $\|W_7\|(E^2)$ is not known in general. In any case, W_6 must correspond to either ω_5 (if $\|W_6\|(E^2) < \|W_7\|(E^2)$) or ω_6 (if $\|W_7\|(E^2) < \|W_6\|(E^2)$). On the other hand, $\mathrm{index}(W_6) + \mathrm{null}(W_6) = 4 + 0 < 5$, which disproves the conjecture in either case. □

Remark. If one picks the ellipsoid $E_2(1-\varepsilon, 1, 1+\varepsilon)$ we have $\|W_6\|(E^2) = \|W_7\|(E^2)$ thus forcing one of them to not be a critical varifold. In this case W_4 must correspond to $\omega_4(E^2)$.

As a final remark we observe that W_4, W_6 and W_9 correspond to an unstable embedded closed geodesic with multiplicity 2. This provides a concrete counterexample to the Multiplicity One Conjecture in the case of 1-cycles on surfaces. The main difference to higher dimensions is that in the hypersurface case one could be able to de-singularize two minimal surfaces (for example two great spheres in S^3 approaching a sphere with multiplicity 2) along their intersection and obtain an embedded "competitor" with less area, but with different topology. This cannot be done for curves.

6. Further problems

We would like to propose a general formula for the width of the round sphere S^2. First let us give our conjecture and then explain the motivation. We expect that

$$\omega_j = 2\pi k, \text{ if } j \in \{k^2, \ldots, (k+1)^2 - 1\}.$$

A simple computation shows that this would imply the Weyl law for S^2. Of course, to prove the Weyl law it is not necessary to compute the width spectrum, one is only interested in its asymptotic behavior. This is a much stronger conjecture.

Denote by $P[\mathbb{R}^3, d]$ the space of real polynomials of degree less than or equal to d in 3 variables. For each $p \in P$ we can define $\{p = 0\} \cap S^2$ as we have already done. However, any polynomial that contains the fact $({x_1}^2 + {x_2}^2 + {x_3}^2 - 1)$ do not define a 1-cycle in S^2 so we have to quotient these out. That is, we are interested in the space $A_d = P[\mathbb{R}^3, d]/\langle {x_1}^2 + {x_2}^2 + {x_3}^2 - 1\rangle_d$, where $\langle {x_1}^2 + {x_2}^2 + {x_3}^2 - 1\rangle_d$ denotes the ideal generated by $({x_1}^2 + {x_2}^2 + {x_3}^2 - 1)$ intersected with $P[\mathbb{R}^3, d]$.

Now, note that we can write $A_k = A_{k-1} \oplus H_k$ where H_k is the space of homogeneous polynomials in 3 variables of degree k. The space H_k is isomorphic to the eigenspace of the k^{th} eigenvalue of the Laplacian in S^2 and its dimension is $2k + 1 = (k + 1)^2 - k^2$. Using the polynomials in A_{k-1} and a basis for H_k we can construct j-sweepouts for $j = k^2, \ldots, (k + 1)^2 - 1$ whose minmax values are $2\pi k$ as given by the Crofton formula. We expect these sweepouts to be optimal for the round sphere.

This is motivated by Lusternik-Schnirelmann theory on manifolds. We believe that the width will be realised by a combination of great circles with possible multiplicities. Lusternik-Schnirelmann theory indicates that if $\omega_k = \omega_{k+N}$ then there exists a N-parameter family of varifolds with constant mass ω_k. More generally, one would expect the space of critical varifolds with mass ω_k to have Lusternik-Schnirelmann category greater or equal to N In the case of S^2 the space of k-combinations of great circles is simply the space of unordered k-tuples of great circles, that is, it is given by $SP^k(\mathbb{R}P^2)$. We denote by $SP^k(X)$ the quotient of X^k by the action of the k-symmetry group S_k. It is known that $SP^k(\mathbb{R}P^2) = \mathbb{R}P^{2k}$ (see [4]), whose Lusternik-Schnirelmann category is $2k + 1$. Finally our conjecture implies that the equality gaps in the width spectrum are given by $\omega_{k^2} = \omega_{(k+1)^2-1}$, which is consistent with the Lusternik-Schnirelmann motivation. As a brief remark we would like to point out the for higher dimensions the same ideas would violate the category of the critical set.

Unfortunately none of this has been proved. Neither the category ideas or the optimality of the polynomial sweepouts are known. The Lusternik-Schnirelmann theory for smooth functions on manifolds (see [6]) does not carry over to our case directly.

Appendix A.

First let us extract a weaker version of the results in [12]. From the proof of [12, Theorem 6.1] we can obtain the following general, but weaker, proposition.

Proposition A.1. *Let M be a closed Riemannian manifold and $\{\omega_k(M)\}_{k\in\mathbb{N}}$ be the width spectrum corresponding to 1-cycles in M. If $\omega_k(M) = \omega_{k+1}(M)$ for some k, then there exist infinitely many geodesic networks with mass $\omega_k(M)$ and that are almost-minimising in annuli at every point.*

The proof is similar to [12, Theorem 6.1], however we cannot use Schoen-Simon's Regularity Theorem or the Constancy Theorem (as in [12, Claim 6.2]). To overcome this one notes that if a sequence of varifolds converges to a geodesic network then the sequence of associated currents converge to a subnetwork of the limit.

More precisely, let $\{T_i\}_{i\in\mathbb{N}} \subset \mathcal{Z}_1(M)$ be a sequence of flat cycles such that $|T_i| \to V$ and $T_i \to T$. If V is a geodesic network defined by geodesic segments $\{\gamma_1, \ldots, \gamma_m\}$ and its respective multiplicities, then T is a cycle (not necessarily stationary) defined by a subset of geodesics $\Omega \subset \{\gamma_1, \ldots, \gamma_m\}$ with multiplicity one each.

This is true because the support of the limit is contained in the varifold geodesic network, then we can apply the Constancy Theorem to each geodesic segment whose intersection is non-empty. If we assume that the set of geodesic networks is finite, then so is the set of all possible subnetworks (not necessarily stationary) and the rest of the proof is the same as in [12].

With this proposition we can prove Lemmas 5.4 and 5.6.

Proof of Lemma 5.4. Suppose $\omega_4(S^2) < 2\pi(1 + \frac{1}{6})$ and choose an ellipsoid E^2 sufficiently close to S^2 so that Proposition 5.5 holds. In particular there are only 3 almost minimising geodesic networks with length less than $2\pi(1 + \frac{1}{4})$ in E^2 (namely, the three principal geodesics). Indeed, any such geodesic network must have density less than 2 by Theorem 3.7 and the almost minimising condition excludes triple junctions. Note also that $\omega_4(E^2) < 2\pi(1 + \frac{1}{3})$.

We claim that there exist an optimal sweepout for $\omega_4(E^2)$. If that is not the case we would be able to produce a sequence of sweepouts F_i with no concentration of mass such that $L[\Pi_{F_{i+1}}] < L[\Pi_{F_i}] < 2\pi(1 + \frac{1}{3})$. Thus, each F_i would give us a distinct almost-minimising geodesic network with length less than $2\pi(1 + \frac{1}{3})$ (Theorem 4.4), which is a contradiction.

It follows that there exists an almost-minimising geodesic network V such that $\|V\|(E^2) = \omega_4(E^2)$. Thus, V must be one of the three principal geodesics which implies that $\omega_k(E^2) = \omega_{k+1}(E^2)$ for some $k = 1, 2, 3$. This is a contradiction because Proposition A.1 would imply the existence of infinitely many almost-minimising geodesic networks with length $\omega_k(E^2)$ and we already know that this is not possible.

We conclude that our initial assumption is false, thus $\omega_4(S^2) > 2\pi$. $\square$

The next proof is very similar to the previous one.

Proof of Lemma 5.6. If the ellipsoid is sufficiently close to S^2 then we can assume that $\omega_i(E^2) < 2\pi(2 + \frac{1}{4})$, by Theorem 5.3. As we have already remarked, there are only 9 almost-minimising geodesic networks with mass less than $2\pi(2 + \frac{1}{4})$ (namely the W_j previously described). If we had equality $\omega_i = \omega_{i+1}$ for any $i = 1, \ldots, 7$ then Proposition A.1 would give us infinitely many almost-minimising geodesic networks with mass ω_i, which is a contradiction. $\square$

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