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The Ricci flow on surfaces with boundary

Jean C. Cortissoz and Alexander Murcia

We study a boundary value problem for the Ricci flow on a surface with boundary, where the (constant in space) geodesic curvature of the boundary is prescribed.

1. Introduction

The Ricci flow on surfaces, compact and noncompact, has been an intense subject of study since the appearance of Hamilton's seminal work [19], where the asymptotic behavior of the flow is studied on closed surfaces, and it is used as a tool towards giving a proof of the Uniformization Theorem via parabolic methods. In addition to its obvious geometric appeal, it is of note that the study of the Ricci flow on surfaces is related to the study of the logarithmic diffusion equation, and hence, the interest on this problem goes beyond its geometric applications (see [22]).

However, not much is known about the behavior of the Ricci flow on manifolds with boundary. One of the main difficulties in studying this problem arises from the fact that even trying to impose meaningful boundary conditions for the Ricci flow, for which existence and uniqueness results can be proved so interesting geometric applications can be hoped for, seems to be a challenging task. For the reader to get an idea of the difficulty of the problem, we recommend the interesting works of Y. Shen [34], S. Brendle [4], A. Pulemotov [32, 33], and P. Gianniotis [17, 18]. In the case of the boundary conditions imposed by Shen [34], satisfactory convergence results have been given for manifolds of positive Ricci curvature and totally geodesic boundary, and also when the boundary is convex and the metric is rotationally symmetric. In the case of surfaces, the Ricci flow is parabolic, and imposing natural geometric boundary conditions is not difficult: one can for instance control the geodesic curvature of the boundary. In this case, Brendle [4] has shown that when the boundary is totally geodesic, then the behavior is completely analogous to the behavior of the Ricci flow in closed surfaces ([8, 19]). In this case, also for non totally geodesic boundary, the first author has proved, under the hypothesis of rotational symmetry of the metrics involved, results on the asymptotic behavior of the Ricci flow in the case of positive curvature and convex boundary, and for certain families of metrics with non convex boundary ([12]).

The purpose of this paper is to contribute towards the understanding of the behavior of the Ricci flow on surfaces with boundary. To be more precise, let M be a compact surface with boundary ($\partial M \neq \emptyset$), endowed with a smooth metric g_0; we will study the equation

$$(1) \qquad \begin{cases} \frac{\partial g}{\partial t} = -R_g g & \text{in} \quad M \times (0,T) \\ k_g(\cdot,t) = \psi(\cdot,t) & \text{on} \quad \partial M \times (0,T) \\ g(\cdot,0) = g_0(\cdot) & \text{in} \quad M, \end{cases}$$

where R_g represents the scalar curvature of M and k_g the geodesic curvature of ∂M, both with respect to the time evolving metric g, and ψ is a smooth real valued function, which is constant in space, defined on $\partial M \times [0,\infty)$, and which satisfies the compatibility condition $\psi(\cdot,0) = k_{g_0}$.

The short-time existence theory of equation (1) is well understood. Indeed, since the deformation given by (1) is conformal, if we write $g(p,t) = e^{u(p,t)} g_0$, problem (1) is equivalent to a nonlinear parabolic equation with

Robin boundary conditions, and initial data $u(p,0)=1$. Hence, via the Inverse Function Theorem and standard methods from the theory of parabolic equations [26], it can be shown that (1) has a unique solution for a short time, and that this solution is in the parabolic Hölder space $H^{2+\alpha,1+\frac{\alpha}{2}}$, $0<\alpha<1$, on $\overline{M}\times[0,T)$, and smooth away from the corner.

Before we state the results we intend to prove in this paper, we must introduce a normalization of (1). As it is well known, the solution to (1) can be normalized to keep the area of the surface constant. This is done as follows: Let us assume without loss of generality that the area of M with respect to g_0 is 2π, and choose $\phi(t)$ such that $\phi(t)\,A_g(t)=2\pi$, where $A_g(t)$ is the area of the surface at time t with respect to the metric g. Then define

$$\tilde{t}(t)=\int_0^t \phi(\tau)\,d\tau \quad \text{and} \quad \tilde{g}=\phi g. \tag{2}$$

If the family of metrics $g(t)$ satisfies (1), then the family of metrics $\tilde{g}\left(\tilde{t}\right)$ satisfies the evolution equation

$$\begin{cases} \frac{\partial \tilde{g}}{\partial \tilde{t}}=\left(\tilde{r}_{\tilde{g}}-\tilde{R}_{\tilde{g}}\right)\tilde{g} & \text{in} \quad M\times\left(0,\tilde{T}\right)\\ k_{\tilde{g}}(\cdot,t)=\tilde{\psi}\left(\cdot,\tilde{t}\right) & \text{on} \quad \partial M\times\left(0,\tilde{T}\right)\\ \tilde{g}(\cdot,0)=g_0(\cdot) & \text{on} \quad M, \end{cases} \tag{3}$$

where $\tilde{\psi}$ is the normalization of the function ψ, $\tilde{R}_{\tilde{g}}$ is the scalar curvature of the metric $\tilde{g}$, and

$$\tilde{r}_{\tilde{g}}=\frac{\int_M \tilde{R}_{\tilde{g}}\,dA_{\tilde{g}}}{\int_M dA_{\tilde{g}}}=\frac{1}{2\pi}\int_M \tilde{R}_{\tilde{g}}\,d\tilde{A}_g.$$

Here $dA_{\tilde{g}}$ denotes the area element of M with respect to the metric $\tilde{g}$. We refer to (3) as the *normalized Ricci flow.*

We can now state our first result.

Theorem 1.1. *Let $\left(M^2,g_0\right)$ be a compact surface with boundary with positive scalar curvature ($R_{g_0}>0$), and such that the geodesic curvature of its boundary is a nonnegative constant ($k_{g_0}\geq 0$), and assume that ψ, as defined above, is nonnegative, constant in space with $\psi(0)=k_{g_0}$, and also satisfies that $\frac{d}{dt}\psi\leq 0$. Let $g(t)$ be the solution to (1) with initial condition g_0. Then the corresponding solution to the normalized flow, $\tilde{g}\left(\tilde{t}\right)$, exists for all time, and for any sequence $\tilde{t}_n\to\infty$, there is a subsequence $\tilde{t}_{n_k}\to\infty$ such that the metrics $\tilde{g}\left(\tilde{t}_{n_k}\right)$ converge smoothly to a metric of constant curvature and totally geodesic boundary.*

Theorem 1.1 partially extends results on the asymptotic behavior of solutions to the Ricci flow known for $\mathbb{S}^2$ (Hamilton [20] and Chow [8]), and for the case of surfaces with totally geodesic boundary (Brendle [4]) and with rotational symmetry ([12]).

Before going any further, let us give an outline of the proof of Theorem 1.1. First of all, given an initial metric g_0 of positive scalar curvature and convex boundary, it can be shown that the curvature of $g(t)$ blows up in finite time, say $T < \infty$; the idea then is to take a blow-up limit of the solution $(M, g(t))$ as $t \to T$, and to show that the only possibility for this blow-up limit is to be a round hemisphere: this would, essentially, give a proof of Theorem 1.1. It remains then to remove two technical difficulties: we must be able to produce this blow-up limit, and hence, on the one hand, we will have to show how to bound derivatives of the curvature in terms of bounds on the curvature; and, on the other hand, we must show how to estimate the injectivity radius of the surface, and, because it has boundary, we are required to show that there are no geodesics hitting the boundary orthogonally that are too short with respect to the inverse of the square root of the maximum of the curvature. The first technical difficulty is dealt with using the recent work of Gianniotis [18] (here is where the hypothesis of the constancy in space of ψ is needed; we thank the referee who referred us to Gianniotis' paper). To deal with the second technical difficulty, we have introduced an extension procedure for surfaces with boundary that allows to have some control over the maximum curvature and the size of the extension, and which reduces the estimation of the injectivity radius of the original surface to the same problem but in a closed surface where it is embedded (this part of the proof does not depend on the hypothesis of constancy in space of the geodesic curvature of the boundary).

At this point it might be convenient to compare Theorem 1.1 with the main result (Theorem 3) in [13]. As a matter of fact, Theorem 1.1 is the two-dimensional version of the main theorem of [13], and the general arguments employed to prove both results are essentially the same. However, in contrast to the three-dimensional case (which is the one dealt with in [13]), in the two-dimensional case we have been able to avoid assuming constant geodesic curvature of the boundary (which corresponds to the hypothesis of weak umbilicity of the boundary in [13]) to obtain the injectivity radius estimate needed in the blow-up arguments. Also, whereas in the three-dimensional case a pinching estimate is needed to rule out unwanted possible blow-up limits in the blow-up argument, in the case studied in this paper it is Perelman's formula that is used to rule out the unwanted possible blow-up limits in the blow-up argument: this shows that the proof of Theorem 1.1 presented here does not follow as an immediate application of the techniques employed in [13].

Our second result is concerned with the behavior of the Ricci flow when the geodesic curvature of the boundary is nonpositive. Again, using blow up analysis techniques, we prove the following theorem, which generalizes similar results from [12] (notice that we make no requirements on the sign of R).

Theorem 1.2. *Let g_0 be a rotationally symmetric metric on the two-ball. Assume that $k_{g_0} \leq 0$, and that the boundary data is given by $\psi = k_{g_0}$. Then the normalized flow corresponding to the solution to (1) with initial data g_0 and boundary data ψ exists for all time.*

The layout of this paper is as follows. In Section 2 we prove the basic evolution equation for the scalar curvature when the metric evolves under (1), and show that under certain conditions the curvature R blows up in finite time; in Section 3 we prove a monotonicity formula for Perelman's functionals on surfaces with boundary; in Section 4 it is shown that it is possible to take blow up limits for solutions to (1), by proving that we can control the injectivity radius of the surface in terms of the scalar curvature and the geodesic curvature of the boundary, and then using a compactness theorem for sequences of pointed Ricci flows on manifolds with boundary due to Gianniotis [18]; in Section 5 we use the results from the previous sections to give a proof of Theorem 1.1. Finally, in Section 6 we give a proof of Theorem 1.2. This paper is complemented by an appendix where among other things we discuss a procedure to obtain bounds on the derivatives of the conformal factor of solutions to (1) -and hence to (3)- in terms of bounds on the curvature and the boundary data (and its derivatives), and a doubling procedure for the Ricci flow on surfaces with boundary.

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2. Evolution equations

In the following proposition, which is stated in [12] without proof, we compute the evolution of the curvature of a metric g when it is evolving under (1).

Proposition 2.1. *Let $(M, g(t))$ be a solution to (1). The scalar curvature satisfies the evolution equation*

$$\begin{cases} \frac{\partial R_g}{\partial t} = \Delta_g R_g + R_g^2 & \text{in} \quad M \times (0,T) \\ \frac{\partial R}{\partial \eta_g} = k_g R_g - 2k_g' = \psi R_g - 2\psi' & \text{on} \quad \partial M \times (0,T) \end{cases}$$

where η_g is the outward pointing unit normal with respect to the metric g, and the prime ($'$) represents differentiation with respect to time.

Proof. Since the evolution equation satisfied by R in the interior of M is known ([20]), we will just compute its normal derivative, with respect to the outward normal, at the boundary. To do so, we choose local coordinates $\left(x^1, x^2\right)$ at $p \in \partial M$ such that $x^2 = 0$ is a defining function for ∂M, so that the corresponding coordinate frame $\{\partial_1, \partial_2\}$ is orthonormal at $p \in \partial M$ and time $t = t_0$ (i.e., the point and instant where and when we want to compute the normal derivative), and so that ∂_2 coincides with the outward unit normal to the boundary in the whole coordinate patch (this also at time $t = t_0 > 0$). Since the deformation is conformal, ∂_2 remains normal to the boundary. Therefore the geodesic curvature is given (as long as the flow is defined for $t \geq t_0$) by the formula

$$k_g g_{11} = -\frac{\Gamma_{11}^2}{\left(g^{22}\right)^{\frac{1}{2}}} = -\left(g_{22}\right)^{\frac{1}{2}} \Gamma_{11}^2.$$

Computing the time derivative, the previous identity yields

$$\begin{aligned} \left(k_g g_{11}\right)' &= -\frac{1}{2\left(g_{22}\right)^{\frac{1}{2}}} \left(g_{22}\right)' \Gamma_{11}^2 - \left(g_{22}\right)^{\frac{1}{2}} \left(\Gamma_{11}^2\right)' \\ &= \frac{1}{2} R_g \left(g_{22}\right)^{\frac{1}{2}} \Gamma_{11}^2 - \left(g_{22}\right)^{\frac{1}{2}} \left(\Gamma_{11}^2\right)'. \end{aligned}$$

Let us calculate $\left(\Gamma_{11}^2\right)'$ (as is customary ∇_j denotes covariant differentiation with respect to ∂_j, and recall that $g_{12} = 0$ and $g_{ii} = 1$)

$$\begin{aligned}\left(\Gamma_{11}^{2}\right)' &= \frac{1}{2}g^{2j}\left(\nabla_1 g'_{1j} + \nabla_1 g'_{1j} - \nabla_j g'_{11}\right)\\ &= \frac{1}{2}g^{22}\left(-2\nabla_1\left(R_g g_{12}\right) + \nabla_2\left(R_g g_{11}\right)\right)\\ &= \frac{1}{2}g^{22}\left(\partial_2 R_g\right) g_{11} = \frac{1}{2}g^{22}\left(\partial_2 R_g\right).\end{aligned}$$

Therefore

$$k'_g g_{11} - k_g R g_{11} = -\frac{1}{2}k_g R - \frac{1}{2\left(g_{22}\right)^{\frac{1}{2}}}\partial_2 R_g = -\frac{1}{2}k_g R_g - \frac{1}{2}\frac{\partial R_g}{\partial \eta_g},$$

and the result follows. □

As a consequence from Hopf Maximum Principle, since $k'_g = \psi' \leq 0$ in the case we are considering, we obtain the following result.

Proposition 2.2. *Let $(M, g(t))$, M compact, be a solution to (1). Assume that ψ, the boundary data, satisfies $\psi' \leq 0$. Then, if $R_g \geq 0$ at time $t = 0$, it remains so as long as the solution exists. Furthermore, if the initial data has positive scalar curvature, then R_g remains strictly positive. Also, if $R_g > 0$ at $t = 0$ and $\psi \geq 0$ then R_g blows up in finite time.*

Proof. We leave the proof that R remains strictly positive to the reader, and show that the solution of (1) must blow-up in finite time. By Hopf Maximum Principle, from the hypotheses at the boundary we have

$$\frac{\partial R}{\partial \eta} \geq 0,$$

so the minimum $R_{\min}(t)$ of R at time t occurs in the interior of M. Hence, $R_{\min}$ satisfies a differential inequality

$$\frac{d}{dt}R_{\min} \geq R_{\min}^2.$$

Therefore, comparing with the solution of the ODE

$$\frac{du}{dt} = u^2, \quad u(0) = R_{\min}(0),$$

we have that $R_{\min} \geq u$, and since $u > 0$, u blows up in finite time, and so must $R_{\min}$. □

We must point out that if $R \geq 0$ at time $t = 0$, and it is strictly positive at a point, under the assumption $\psi \geq 0$, $\psi' \leq 0$, it becomes strictly positive instantaneously, so the hypotheses in the previous proposition may be relaxed a bit.

2.1.

In view of Proposition 2.2, this seems a good place to discuss the following fact. Let $(0, T)$ be the maximal interval of existence of a solution to (1), with $0 < T < \infty$, then

$$\limsup_{t \to T} \left(\sup_{p \in M} R_g (p, t) \right) = \infty.$$

First of all if g_0 is the initial metric, then as the Ricci flow preserves the conformal structure, we have that the evolving metric can be represented as $g = e^u g_0$. Hence, if R_{g_0} is the scalar curvature of the initial metric, at a fixed (but arbitrary) time, we have that u satisfies the elliptic boundary value problem

$$\begin{cases} \Delta_{g_0} u + R_{g_0} = R_g e^u & \text{in} \quad M, \\ \frac{\partial}{\partial \eta_{g_0}} u + 2k_{g_0} = 2k_{g_0} e^{\frac{u}{2}} & \text{on} \quad \partial M. \end{cases} \tag{4}$$

To reach a contradiction assume that R_g remains uniformly bounded on $(0, T)$. A consequence of this assumption is that e^u remains bounded away from 0 and uniformly bounded above on $(0, T)$. Now, since from bounds on the curvature and also on the geodesic curvature of the boundary (i.e., on ψ) and its derivatives, we can obtain bounds on the derivatives of u (see Theorems A.1 and A.2 in the Appendix), u and its derivatives (including those with respect to t) are uniformly bounded on $(0, T)$, and consequently $u(\cdot, t)$ converges as $t \to T$ to a smooth function, say $\hat{u}$. If we start the Ricci flow at $t = T$ with initial data $e^{\hat{u}} g_0$ and the same boundary data, then we would be able to continue the original solution past T, which contradicts the hypothesis. Therefore, if the Ricci flow (1) cannot be extended past $T < \infty$, the curvature blows up. We invite the reader to consult the recent work of Gianniotis [17], where this property of the Ricci flow on manifolds with boundary is discussed in a more general context.

2.2.

There are other interesting cases when solutions to (1) blow up. We have for instance the following proposition.

Proposition 2.3. *Let (M, g_0) be a compact surface with boundary, and assume that $\int_M R_{g_0}\, dA_{g_0} + \int_{\partial M} 2k_{g_0}\, ds_{g_0} > 0$ with $k_{g_0} \leq 0$. Let $\psi \leq 0$ be such that $\psi = k_{g_0}$ at $t = 0$. Then the solution to (1), with initial condition g_0 and boundary data ψ, blows up in finite time.*

Proof. Let $g\,(t)$ be the solution to (1) with initial data g_0 and boundary data ψ. If $A\,(t)$ represents the area of M with respect to $g\,(t)$, we can calculate

$$\frac{dA}{dt} = -\int_M R_g\, dA_g = -4\pi\chi\,(M) + 2\int_{\partial M} k_g\, ds_g \leq -4\pi\chi\,(M) < 0.$$

Therefore, the area cannot remain positive for all time, hence a singularity must occur in finite time. □

3. Monotonicity of Perelman's Functionals on surfaces with boundary

The purpose of this section is to show a monotonicity formula for Perelman's celebrated $\mathcal{F}$ and $\mathcal{W}$ functionals (see [29]) in the case of surfaces with boundary. The results in this section are stated, although with no carefully crafted proofs, more or less in the same way as it is done here, in [12] (the reader is also advised to consult the interesting work [33], which was pointed out to us by a referee, and where a monotonicity formula for Perelman's functionals is proved for certain manifolds with boundary). As usual, all curvature quantities, scalar products and operators depend on the time-varying metric g (some of them will not bear a subindex to show this dependence). We will use the Einstein summation convention freely, and the raising and lowering of indices is done, by means of the metric g, in the usual way.

In order to proceed, recall the definition of Perelman's $\mathcal{F}$-functional:

$$\mathcal{F}\,(g_{ij}, f) = \int_M \left(R_g + |\nabla f|^2\right) \exp\,(-f)\; dV_g,$$

where dV_g represents the volume (in the case of a surface, area) element of the manifold M with respect to the metric g. Let us compute the first variation of this functional on a manifold with boundary.

Proposition 3.1. *Let* $\delta g_{ij} = v_{ij}, \delta f = h, g^{ij} v_{ij} = v$. *Then we have,*

$$\begin{aligned}\delta\mathcal{F} = &\int_M \exp(-f)\left[-v^{ij}\left(R_{ij}+\nabla_i\nabla_j f\right)+\left(\frac{v}{2}-h\right)\left(2\Delta_g f-|\nabla f|^2+R_g\right)\right] dV_g \\ &-\int_{\partial M}\left[\frac{\partial v}{\partial\eta_g}+(v-2h)\frac{\partial f}{\partial\eta_g}\right]\exp(-f)\,d\sigma_g \\ &+\int_{\partial M}\exp(-f)\nabla_i v^{ij}\eta_j\,d\sigma_g-\int_{\partial M}\nabla_j\exp(-f)\,v^{ij}\eta_i\,d\sigma_g.\end{aligned}$$

Here, R_{ij} *represents the Ricci tensor of the metric* g, $\dfrac{\partial}{\partial\eta_g}$ $(=\eta^i\partial_i$ *in local coordinates) is the outward unit normal to* ∂M *with respect to* g, ∇ *represents covariant differentiation with respect to the metric* g, *and* $d\sigma_g$ *represents the volume element of* ∂M.

Proof. As in [23], we have that

$$\begin{aligned}\delta\mathcal{F}(v_{ij},h) = \int_M e^{-f}\Big[&-\Delta_g v+\nabla_i\nabla_j v^{ij}-R_{ij}v^{ij} \\ &-v^{ij}\nabla_i f\nabla_j f+2g(\nabla f,\nabla h)+\left(R_g+|\nabla f|^2\right)\left(\frac{v}{2}-h\right)\Big]dV_g.\end{aligned}$$

We must compute the integrals on the righthand side of the previous identity, using as our main tool integration by parts. We start by calculating

$$\begin{aligned}&\int_M e^{-f}(-\Delta_g v)\,dV_g \\ &=-\int_M \Delta_g e^{-f}v\,dV_g+\int_{\partial M} v\frac{\partial e^{-f}}{\partial\eta_g}\,d\sigma_g-\int_{\partial M}e^{-f}\frac{\partial v}{\partial\eta_g}\,d\sigma_g \\ &=-\int_M \Delta_g e^{-f}v\,dV_g-\int_{\partial M}\left(\frac{\partial v}{\partial\eta_g}+v\frac{\partial f}{\partial\eta_g}\right)\exp(-f)\,d\sigma_g.\end{aligned}$$

Now we compute

$$\begin{aligned}\int_M e^{-f}\nabla_i\nabla_j v^{ij}\,dV_g &= -\int_M \nabla_i e^{-f}\nabla_j v^{ij}\,dV_g+\int_{\partial M}e^{-f}\nabla_j v^{ij}\eta_i\,d\sigma_g \\ &=\int_M \nabla_i\nabla_j e^{-f}v^{ij}\,dV_g-\int_{\partial M}\nabla_i e^{-f}v^{ij}\eta_j\,d\sigma_g \\ &\quad+\int_{\partial M}e^{-f}\nabla_j v^{ij}\eta_i\,d\sigma_g.\end{aligned}$$

Finally,

$$
\begin{aligned}
2\int_M e^{-f} g\left(\nabla f, \nabla h\right) dV_g &= -2\int_M g\left(\nabla e^{-f}, \nabla h\right) dV_g \\
&= 2\int_M \left(\Delta_g e^{-f}\right) h\, dV_g - \int_{\partial M} h \frac{\partial e^{-f}}{\partial \eta_g}\, d\sigma_g .
\end{aligned}
$$

Putting all these calculations together proves the result. □

Consider the evolution equations on a surface with boundary given by

$$
\begin{cases}
\dfrac{\partial}{\partial t} g_{ij} = -R_g g_{ij} = -2R_{ij} & \text{in} \quad M \times (0,T)\,, \\
k_g\left(\cdot, t\right) = \psi\left(\cdot\right) & \text{on} \quad \partial M \times (0,T)\,, \\
\dfrac{\partial f}{\partial t} = -\Delta_g f + |\nabla f|^2 - R_g & \text{in} \quad M \times (0,T)\,, \\
\dfrac{\partial}{\partial \eta_g} f = 0 & \text{on} \quad \partial M \times (0,T)\,.
\end{cases} \tag{5}
$$

A formula for $\dfrac{d}{dt}\mathcal{F}$ is given by the following result.

Theorem 3.1. *Under (5) the functional $\mathcal{F}$ satisfies*

$$
\begin{aligned}
\frac{d}{dt}\mathcal{F} = 2\int_M &\left|R_{ij} + \nabla_i \nabla_j f\right|^2 \exp\left(-f\right) dA_g \\
&+ \int_{\partial M} \left(k_g R_g - 2k_g'\right) \exp\left(-f\right) ds_g \\
&+ 2\int_{\partial M} k_g \left|\nabla^\top f\right|^2 \exp\left(-f\right) ds_g ,
\end{aligned}
$$

and here $\nabla^\top f$ represents the component of ∇f tangent to ∂M, dA_g the area element of the surface, and ds_g the length element of the boundary.

Proof. Let us first introduce some notation and conventions. Since parts of these computations apply to manifolds of higher dimensions, in this proof we will fix coordinates $x^1, x^2, \ldots, x^{n-1}, x^n$ at a boundary point and at fixed (but arbitrary) time t, so that $x^n = 0$ is a defining function for ∂M. We will assume that on ∂M, $\dfrac{\partial}{\partial x^n} = \dfrac{\partial}{\partial \eta_g}$ represents the outward unit normal, and hence we will denote by a subscript or superscript n quantities that are evaluated, at a boundary point, with respect to the outward unit normal. By a greek letter we will represent indices running from $1, 2, 3, \ldots, n-1$, and therefore at a boundary point the vector fields $\dfrac{\partial}{\partial x^{\alpha}}$ are tangent to the boundary. Let us transform the evolution equations given by (5) using the one-parameter family of diffeomorphisms φ_t generated by $-\nabla f$; notice that the boundary is sent to itself via this family of diffeomorphisms due to the fact that $\dfrac{\partial f}{\partial \eta_g} = 0$. Now, by defining $f(\cdot, t) = f(\varphi_t(\cdot), t)$ and $g = (\varphi_t)_* g$ (forgive the abuse of notation), instead of (5) we must take the variations given by

$$v_{ij} = \delta g_{ij} = -2\left(R_{ij} + \nabla_i \nabla_j f\right), \quad h = \delta f = -\Delta_g f - R.$$

For a moment let us denote with a subindex $(\varphi_t)_* g$ the quantities that depend on the pullback metric. Observe that then we have

$$\frac{\partial}{\partial \eta_{(\varphi_t)_* g}} R_{(\varphi_t)_* g}(\cdot, t) = \left(k_g R_g - 2k_g'\right)(\varphi_t(\cdot), t),$$

so keeping on with the abuse of notation, we will write, for the metric $g = (\varphi_t)_* g$,

$$\frac{\partial R}{\partial \eta_g} = k_g R_g - 2k_g',$$

where the prime ($'$) now means that we differentiate k_g (or ψ) with respect to its second variable (t) and then it is evaluated at $(\varphi_t(\cdot), t)$. Notice that for the pullback metric we still have $\dfrac{\partial f}{\partial \eta_g} = 0$.

We will now compute each of the boundary integrals in the first variation of Perelman's functional given by Proposition 3.1, which will prove the theorem, since the computations for the integrals over M are known from the work of Perelman. We change the notation from the previous proposition as follows: $dV_g = dA_g$ and $d\sigma_g = ds_g$. We start with

$$\int_{\partial M} \left[\frac{\partial v}{\partial \eta_g} + (v - 2h)\frac{\partial f}{\partial \eta_g}\right] \exp(-f)\, ds_g, \quad \text{and} \quad \int_{\partial M} \exp(-f)\, \nabla_i v^{ij} \eta_j\, ds_g.$$

To compute these integrals, let us first calculate $\nabla_i v^{ij} \eta_j$. We have

$$\begin{aligned} \nabla_i v^i{}_n &= -2\nabla_i R^i{}_n - 2\nabla_i \nabla^i \nabla_n f \\ &= -\nabla_n R_g - 2\Delta_g \nabla_n f \quad \text{(by the contracted Bianchi identity).} \end{aligned}$$

By the Ricci identity, using the fact that at the boundary $\nabla_n f = 0$ and also $R_{\alpha n} = 0$, we obtain

$$\Delta_g \nabla_n f = \nabla_n \Delta_g f + R_n{}^k \nabla_k f = \nabla_n \Delta_g f,$$

and therefore,

$$\nabla_i v^i{}_n = -\nabla_n R_g - 2\nabla_n \Delta_g f.$$

Using the evolution equation $f_t = -\Delta_g f - R$, we get, at the boundary,

$$\nabla_n \Delta_g f = -\nabla_n R_g.$$

This last identity has two consequences. On the one hand, it implies that

$$\nabla_i v^i{}_n = -\nabla_n R_g + 2\nabla_n R = \nabla_n R_g = k_g R_g - 2k_g',$$

which shows that

$$\int_{\partial M} \exp(-f)\, \nabla_i v^{ij} \eta_j \, ds_g = \int_{\partial M} \left(k_g R_g - 2k_g'\right) \exp(-f)\; ds_g.$$

On the other hand, it implies that $\dfrac{\partial v}{\partial \eta_g} = 0$, so we obtain

$$\int_{\partial M} \left[\frac{\partial v}{\partial \eta_g} + (v - 2h)\,\frac{\partial f}{\partial \eta_g}\right] \exp(-f)\; ds_g = 0.$$

Let us now compute the integral

$$II = -\int_{\partial M} \nabla_i \exp(-f)\, v^{ij} \eta_j \, ds_g = -\int_{\partial M} \nabla_i \exp(-f)\, v^i{}_n \, ds_g.$$

Under the previous conventions,

$$II = \int_{\partial M} \nabla_\alpha f \exp(-f)\, v^\alpha_n \, ds_g + \int_{\partial M} \nabla_n f \exp(-f)\, v^n_n \, ds_g.$$

Using the fact that $\nabla_n f = \partial_n f = 0$ on ∂M, we can compute

$$\nabla^\alpha \nabla_n f \nabla_\alpha f = -\mathcal{A}\left(\nabla^\top f, \nabla^\top f\right),$$

where $\mathcal{A}$ denotes the second fundamental form of the boundary. Hence, using the definition of $v_{\alpha n}$, we get

$$II = \int_{\partial M} 2\mathcal{A}\left(\nabla^\top f, \nabla^\top f\right) \exp(-f)\; ds_g = \int_{\partial M} 2k_g \left|\nabla^\top f\right|^2 \exp(-f)\; ds_g,$$

and the formula is proved. □

Next, we consider Perelman's $\mathcal{W}$-functional, namely,

$$\mathcal{W}(g,f,\tau)=\int_M\left[\tau\left(|\nabla f|^2+R_g\right)+f-2\right](4\pi\tau)^{-1}\exp(-f)\,dA_g.$$

Under the unnormalized Ricci flow (1), and the evolution equations

$$\begin{cases}\dfrac{\partial f}{\partial t}=-\Delta_g f+|\nabla f|^2-R_g+\dfrac{1}{\tau} & \text{in} \quad M\times(0,T),\\ \dfrac{d\tau}{dt}=-1 & \text{in} \quad (0,T),\\ \dfrac{\partial f}{\partial \eta_g}=0 & \text{on} \quad \partial M\times(0,T),\end{cases} \tag{6}$$

we have the following formula, which shows a monotonicity property as long as $k_g\geq 0$ and $k_g'=\psi'\leq 0$, for the functional $\mathcal{W}$.

Theorem 3.2.

$$\begin{aligned}\frac{d}{dt}\mathcal{W}=&\int_M 2\tau\left|R_{ij}+\nabla_i\nabla_j f-\frac{1}{2\tau}g_{ij}\right|^2(4\pi\tau)^{-1}\exp(-f)\,dA_g\\ &+\frac{1}{4\pi}\left(\int_{\partial M}\left(k_gR_g-2k_g'+2k_g\left|\nabla^{\top}f\right|^2\right)\exp(-f)\,ds_g\right).\end{aligned}$$

Proof. Using the fact that

$$0=\int_{\partial M}\frac{\partial e^{-f}}{\partial\eta_g}\,ds_g=\int_M\Delta_g e^{-f}\,dA_g=\int_M\left(|\nabla f|^2-\Delta_g f\right)e^{-f}\,dA_g,$$

and recalling that under (6), $\delta\left(\frac{1}{4\pi\tau}e^{-f}dA_g\right)=0$ ([23, Eq. 12.3]), we can compute the contribution to the formula due to the variation of the term

$$\frac{1}{4\pi\tau}\int_M(f-2)\exp(-f)\,dA_g.$$

From this, and the computations in the proof of Theorem 3.1, the theorem easily follows. □

4. Controlling the injectivity radius of a surface with boundary

4.1. An extension procedure

Here we show an extension procedure for surfaces with boundary of positive scalar curvature and convex boundary that allows us to control the maximum of the curvature of the extension (compare with the results in [25]).

Theorem 4.1. *Let (M, g) be a compact surface with boundary, and assume that its Gaussian curvature and the geodesic curvature of its boundary are strictly positive. Let $z_0 > 0$ be arbitrary. Then there exists a closed surface $\left(\hat{M}, \hat{g}\right)$, $\hat{g}$ a C^2 metric, such that M is isometrically embedded in $\hat{M}$, the Gaussian curvature $\hat{K}$ of $\hat{M}$ is strictly positive and satisfies*

$$0 < \hat{K} \leq K_+ + \frac{2\alpha_+}{z_0},$$

where K_+ is the maximum of the Gaussian curvature of M, and α_+ is the maximum of the geodesic curvature of ∂M.

Proof. Let $\theta \in \partial M$. Given $K(\theta) > 0$ the (Gaussian) curvature function of M restricted to ∂M, define the following family of functions. First for $y < 0$:

$$K_y(\theta, \zeta) = \begin{cases} K(\theta) + \dfrac{yK(\theta)}{z_0}\zeta & \text{if} \quad 0 \leq \zeta < \dfrac{z_0}{1-y}, \\ \dfrac{K(\theta)}{1-y} & \text{if} \quad \dfrac{z_0}{1-y} \leq \zeta \leq z_0, \end{cases}$$

and for $y \geq 0$:

$$K_y(\theta, \zeta) = K(\theta) + y\zeta, \quad 0 \leq \zeta \leq z_0.$$

Observe that for a given $\alpha > 0$ there exists exactly one member of the previously defined family, say $K_{y(\alpha)}$, such that

$$\alpha(\theta) = \int_0^{z_0} K_{y(\alpha)}(\theta, \zeta)\, d\zeta. \tag{7}$$

Indeed, notice that for fixed θ we have that

$$K_{y_1}(\theta, \zeta) < K_{y_2}(\theta, \zeta), \quad \zeta > 0, \quad \text{whenever} \quad y_1 < y_2,$$

$$\int_0^{z_0} K_y(\theta,\zeta)\,d\zeta \to 0 \quad \text{as} \quad y \to -\infty,$$

$$\int_0^{z_0} K_y(\theta,\zeta)\,d\zeta \to \infty \quad \text{as} \quad y \to \infty,$$

and also, for θ fixed, if $\alpha' \leq \alpha$, the corresponding functions $K_{y(\alpha')}$ and $K_{y(\alpha)}$ (which satisfy (7) for α' and α respectively) satisfy

$$K_{y(\alpha')}(\theta,\zeta) \leq K_{y(\alpha)}(\theta,\zeta).$$

We are ready to extend the metric from a convex surface with boundary to a compact closed surface, keeping control over the maximum of the curvature. Define the warping function

$$f(\theta,z) = 1 + \alpha(\theta)\,z - \int_0^z \int_0^\zeta K_{y(\alpha(\theta))}(\theta,\xi)\,d\xi\,d\zeta,$$

where $\alpha(\theta)$ is the geodesic curvature of ∂M at the point $\theta \in \partial M$. Notice that $z_0 > 0$ can be chosen arbitrarily, and also that $\dfrac{\partial f}{\partial z} \geq 0$ on $0 \leq z \leq z_0$ and hence $f \geq 1$ on the same interval.

If $g_{\partial M}$ is the metric of M restricted to its boundary, we define a metric $\hat{g}$ on $N = \partial M \times [0, z_0]$ by

$$\hat{g} = dz^2 + f^2 g_{\partial M}.$$

This metric defines an extension of the metric on the surface M to the surface $\hat{M}_0 = M \cup N$ where $\partial M \subset M$ is identified with $\partial M \times \{0\} \subset N$. It is clear that this metric is C^2, that $\partial \hat{M}_0 = M \times \{z_0\}$, and that it is totally geodesic.

Let us now estimate the maximum of the curvature in our extension. Define

$$K_+(\zeta) = K_+ + \frac{2\alpha_+}{z_0^2}\zeta,$$

i.e., take from the family of functions defined above, in the case when $y \geq 0$, $K(\theta) = K_+$ and $y = \dfrac{2\alpha_+}{z_0^2}$. Observe that

$$\int_0^{z_0} K_+(\zeta)\,d\zeta = K_+ z_0 + \alpha_+ > \alpha_+.$$

Therefore, by the properties discussed above for the family of functions K_y, we have

$$K_{y(\alpha(\theta))}(\theta, \zeta) \leq K_+ + \frac{2\alpha_+}{z_0^2}\zeta.$$

Notice now that $K_{y(\alpha(\theta))} = -\frac{\partial^2}{\partial z^2} f(z)$ and hence, by taking $\zeta = z_0$, we obtain

$$-\frac{\partial^2}{\partial z^2} f(z) \leq K_+ + \frac{2\alpha_+}{z_0}.$$

Since $f \geq 1$, it follows that the Gaussian curvature of $\hat{M}_0$, which is equal to $-f_{zz}/f$, is at most $K_+ + \frac{2\alpha_+}{z_0}$. Given the fact that the produced extension $\hat{M}_0$ is a surface with a C^2 metric and with a totally geodesic boundary, we can double it to obtain a closed surface endowed with a C^2 metric of positive Gaussian curvature which is bounded above by $K_+ + \frac{2\alpha_+}{z_0}$. □

From the proof of the previous theorem we can extract the following useful corollary.

Corollary 4.1. *If there is a geodesic in M of length l that hits the boundary orthogonally at both its endpoints, then there is a closed geodesic (which is C^3) in the extension $\hat{M}$ (as constructed in the proof of Theorem 4.1) of length $2l + 2z_0$.*

4.2.

Let (M, g) be a compact surface with boundary. We will assume that its scalar curvature is positive as well as the geodesic curvature of its boundary. We will assume that the bounds $0 < R \leq 2K_+$ and $0 \leq k_g \leq \alpha_+$ hold, and also, without loss of generality as this is all that is needed in the applications, that $K_+ \geq 1$.

Before we state the main result of this section, let us review the concept of injectivity radius of a surface with boundary. We shall need the following definitions:

Definition 4.1. Let M be a manifold with boundary and p a point in its interior ($p \in M \setminus \partial M$). Define $\iota_{\text{int}}(p)$ as the supremum of $r > 0$ such that if

$$\gamma : [0, t_\gamma] \longrightarrow M$$

is a normal geodesic with $\gamma(0) = p$, then it is minimizing from 0 to $\min\{t_\gamma, r\}$, where t_γ is the first time that γ intersects ∂M.

We define the interior injectivity radius of M as

$$\iota_{\text{int}} := \inf\left\{\iota_{\text{int}}(p) : p \in M \setminus \partial M\right\}.$$

Definition 4.2. For a Riemannian manifold with boundary M, and $p \in \partial M$, define $\iota_\partial(p)$ as the supremum $r > 0$ such that any minimizing geodesic γ issuing from p normally to ∂M uniquely minimizes distance to ∂M up to distance r (i.e., $\gamma(0) = p$ and $\text{dist}(\gamma(r), \partial M) = r$).

Define $i_\partial(M)$ the boundary injectivity radius of M (as opposed to the injectivity radius of the boundary) as

$$\iota_\partial(M) = \inf\{i_\partial(p) : p \in \partial M\}.$$

The *injectivity radius* ι_M of the surface is defined as

$$\iota_M = \min\left\{\iota_\partial, \iota_{\text{int}}\right\}.$$

From the definition of the injectivity radius for a surface with boundary, and the Klingenberg estimates for the injectivity radius of a compact surface of positive curvature (see Theorem 1.114 in [11] and [24, §6]), one can conclude that in the case of a surface with boundary, we have an estimate from below for the injectivity radius ι_M of a surface with boundary given by

$$\iota_M \geq \min\left\{\text{Foc}(\partial M), \frac{1}{2}l, \frac{c}{\sqrt{K_+}}\right\},$$

where $\text{Foc}(\partial M)$ is the focal distance of ∂M, l is the lenght of the shortest geodesic meeting ∂M at its two endpoints at a right angle, and $c > 0$ is a universal constant. For the benefit of the reader, let us recall the definition of the *focal distance* of ∂M:

Let ν be the normal bundle of ∂M, and denote by ν^- denote the bundle of inward pointing normal vectors. Then we can define the exponential map

$$\exp : \nu^- \longrightarrow M,$$

as

$$\exp(p) = \gamma_p(1),$$

where γ_p is a geodesic starting at p whose velocity vector is normal to ∂M and points inwards. For a compact surface, there is a $\delta > 0$ for which this map is well defined when restricted to normal vectors to ∂M of length at most δ, so we will think of this map as defined over this subset of ν^-, which we will denote by $\nu^-(\delta)$. We say that $p \in M$ is a focal point of ∂M if p is a critical point of the exponential map. The *focal distance* is the minimal distance of a focal point of ∂M to ∂M.

Since it is also well known from comparison geometry that (see [24, §6])

$$\mathrm{Foc}(\partial M) \geq \frac{1}{\sqrt{K_+}} \arctan\left(\frac{\sqrt{K_+}}{\alpha_+}\right) \quad \left(\geq \frac{\pi}{2\sqrt{K_+}} \quad \text{if} \quad \alpha_+ = 0\right),$$

our estimate on the injectivity radius reduces to

$$\iota_M \geq \min\left\{\frac{1}{2}l, \frac{c}{\sqrt{K_+}}\right\},$$

for a new constant c.

Our wish now is to show that along the Ricci flow (1), with initial and boundary data satisfying the requirements of Theorem 1.1, on any finite interval $(0, T)$ of time where it is defined there is a constant $\kappa > 0$, which may depend on T but which is otherwise independent of time, such that at any time $t \in (0, T)$

$$\iota_{(M,g(t))} \geq \frac{\kappa}{\sqrt{R_{\max}(t)}} \quad \text{where} \quad R_{\max}(t) = \max_{p \in M} R_g(p, t). \tag{8}$$

The desired estimate is then a consequence of the following estimate, which is an analogue of Klingenberg's Lemma for surfaces of positive scalar curvature and convex boundary.

Proposition 4.1. *Let (M, g) be a compact surface with boundary. Assume that the scalar curvature of M satisfies $0 < R \leq 2K_+$, $K_+ \geq 1$, and that the geodesic curvature of the boundary satisfies $0 \leq k_g \leq \alpha_+$. Let l be the length of the shortest geodesic in M whose both endpoints are orthogonal to the boundary. There is a constant $\kappa := \kappa(\alpha_+) > 0$ such that $l \geq \dfrac{\kappa}{\sqrt{K_+}}$.*

Proof. For this proof we may assume $k_g > 0$, as we can deal with the case $k_g \geq 0$ by considering, instead of M,

$$M(\epsilon) = \{p \in M : \rho(p) \geq \epsilon\},$$

where ρ is the distance function to ∂M; for any $\epsilon > 0$ small enough, if $R > 0$ and $k_g \geq 0$, it is not difficult to show that $M(\epsilon)$ has boundary with strictly positive geodesic curvature. Indeed, recall that the geodesic curvature k_g of a level curve of ρ (see [12, Proposition 6.2]) satisfies

$$\frac{\partial k_g}{\partial \rho} = \frac{R}{2} + k_g^2.$$

The proposition will be proved in this case by noticing that a geodesic hitting the boundary of $M(\epsilon)$ orthogonally at both of its endpoints is at most 2ϵ shorter than a geodesic with the same property in M, for ϵ as small as desired.

Now, let l be the length of the shortest geodesic that hits the boundary of M orthogonally. Let $z_0 = \dfrac{\alpha_+}{2C\sqrt{K_+}}$, $C > 0$ a constant to be chosen, in Theorem 4.1 and Corollary 4.1. By Corollary 4.1 and Klingenberg's injectivity radius estimate applied to $\hat{M}$, the extension of M given by Theorem 4.1, we have that

$$2l + \frac{\alpha_+}{C\sqrt{K_+}} \geq \frac{c'}{\sqrt{K_+ + C\sqrt{K_+}}},$$

where c' is a universal constant. Hence we have an estimate for l:

$$l \geq \frac{c'}{2\sqrt{K_+ + C\sqrt{K_+}}} - \frac{\alpha_+}{2C\sqrt{K_+}} \geq \frac{c'}{2\sqrt{1 + C\sqrt{K_+}}} - \frac{\alpha_+}{2C\sqrt{K_+}},$$

and to get the last inequality we have used that $K_+ \geq 1$. If $\alpha_+ \leq \frac{c'}{4}$, we can choose $C = 1$. If $\alpha_+ > \frac{c'}{4}$, choose $C > 0$ so that

$$\frac{c'}{2\sqrt{1 + C\sqrt{K_+}}} - \frac{\alpha_+}{2C\sqrt{K_+}} \geq \frac{\alpha_+}{2C\sqrt{K_+}},$$

by taking, for instance, $C = \dfrac{4\alpha_+^2 + \sqrt{16\alpha_+^4 + 16(c'\alpha_+)^2}}{2(c')^2}$. This shows the proposition. □

4.3. Compactness for Ricci flows

In this section we shall state the compactness result, due to Gianniotis [18], we need in order to prove Theorem 1.1, and then we will use it to prove a compactness theorem for Ricci flows in surfaces with boundary.

For the convenience of the reader, in this section we shall adopt Gianniotis' notation, which is described below. In particular, notice that his i_b is our i_∂ as defined above, and that g^T is the restriction of g to ∂M. Following [18], first we fix local coordinates around a point, defined onto $B(0,r)$, the open ball in $\mathbb{R}^d$ of radius r centered at the origin. Then we define the following quantities in $Q_r = B(0,r)\times\left[0,r^2\right]$, $0<\alpha<1$ (the derivatives are defined with respect to the chart, and the norms with respect to the Euclidean metric in the chart; see [18, §2]).

$$\langle u\rangle_{\alpha,x} = \sup_{(x,t)\,,\,(x',t)\,\in\,\overline{Q}_r} \frac{|u(x,t)-u(x',t)|}{|x-x'|^{\mu}},$$

$$\langle u\rangle_{\alpha,t} = \sup_{(x,t)\,,\,(x',t)\,\in\,\overline{Q}_r} \frac{|u(x,t)-u(x,t')|}{|t-t'|^{\alpha}}.$$

For an integer j define

$$\langle u\rangle^{(j)}_{Q_r} = \sum_{2s+|\beta|=j} \sup_{Q_r} \left|\partial_t^s\partial_x^\beta u(x,t)\right|.$$

Given $l=m+\alpha$, m a positive integer, large enough and which is fixed from now on, define

$$|u|^*_{l,r} = \sum_{2v+|\beta|=m} r^l \left\langle \partial_t^v\partial_x^\beta u\right\rangle_{\alpha,x} + \sum_{0<l-2v-|\beta|<2} r^l \left\langle \partial_t^v\partial_x^\beta u\right\rangle_{\frac{l-2v-|\beta|}{2},t} + \sum_{j=0}^{m} r^j \langle u\rangle^{(j)}_{Q_r}. \tag{9}$$

Before we state Gianniotis' Compactness Theorem, we must introduce the definition of Λ-controlled boundary (a definition also due to Gianniotis: see Definition 3.1 in [18]; notice also that Gianniotis considers the possiblity of noncomplete manifolds and Ricci flows, for us, *all manifolds and Ricci flows are complete*).

Definition 4.3. A (complete) Ricci flow on a $(d+1)$-dimensional manifold with boundary $(M, g(t))$, $t \in (a, b]$, has a boundary with Λ-controlled conformal class and mean curvature in the interval $(a, b]$, if there is a smooth one parameter family $\gamma(t)$ of metrics on ∂M such that

(1) $\left[g^T(t)\right] = [\gamma(t)]$ and $\Lambda^{-2}\gamma(t) \leq g^T(t) \leq \Lambda^2\gamma(t)$ for all $t \in (a, b]$.

(2) For every $(\overline{x}, \overline{t}) \in \partial M \times (a, b]$, set $\tilde{\gamma}(s) = \gamma\left(s + \overline{t} - r^2\right)$ and $\overline{\mathcal{H}}(s) = \mathcal{H}\left(g\left(s + \overline{t} - r^2\right)\right)$, where $\mathcal{H}(g)$ denotes the mean curvature of the boundary with respect to the metric g. We require that for any $r \leq \rho_\Lambda \leq \Lambda^{-1}$ there exists $\gamma\left(\overline{t}\right)$-harmonic coordinates $u : U \longrightarrow B(0, r)$ around $\overline{x}$ such that

(a) $Q^{-1}\delta \leq \tilde{\gamma}(s) \leq Q\delta$, in $B(0, r)$ and $s \in \left[0, r^2\right]$,

(b) $|\tilde{\gamma}_{\alpha\beta}|^*_{l,r} \leq Q$, where $\alpha, \beta = 1, 2, \ldots, d$,

(c) $\left|\overline{\mathcal{H}}(s)\right|^*_{l-1,r} \leq Q$.

Here δ denotes the Euclidean metric. Such triplet $(M, g(t), \gamma(t))$ will be called a Ricci flow with Λ-controlled boundary in $(a, b]$.

Gianniotis' Compactness Theorem (Theorem 4.1 in [18]) can be now stated as follows.

Theorem 4.2. *Let (M_k, p_k) be a sequence of pointed manifolds with compact boundary, and $(g_k(t), \gamma_k(t))$ be complete Ricci flows on M_k, $t \in (a, b]$ with Λ-controlled boundary in $(a, b]$. Assume*

(1) $|Rm(g_k)|_{g_k} \leq K$ in $M_k \times (a, b]$,

(2) $|\mathcal{A}(g_k)|_{g_k^T} \leq K$ in $\partial M_k \times (a, b]$ (here $\mathcal{A}$ denotes the second fundamental form of the boundary),

(3) $\iota_{b, g_k(0)} \geq \iota_0$,

for all k. Then there is a pointed manifold with boundary (M_∞, p_∞), a (complete) Ricci flow $g_\infty(t)$ on M_k and a family of metrics $\gamma_\infty(t)$ such that up to a subsequence

$$(M_k, g_k(t), \gamma_k(t), p_k) \to (M_\infty, g_\infty(t), \gamma_\infty(t), p_\infty),$$

in the C^{m-3} topology.

Notice that in the statement of the previous theorem we can assume that the k-th flow is defined in an interval $a_k < 0 < b_k$ with either $a_k \to -\infty$, or $b_k \to \infty$ (or both), since a diagonal procedure can be used to extend the result to these cases.

The careful reader must have noticed that Gianniotis states in his theorem that the limit manifold has boundary (in the discussion below we assume $a < 0 < b$), as this is the case he seems to be most interested in, so he seems to be tacitly assuming that either $p_k \in \partial M_k$ or $d_{g_k(0)}(p_k, \partial M_k)$ remains uniformly bounded from above (or at least along a subsequence), where $d_{g_k(0)}(p_k, \partial M_k)$ represents the distance of p_k to ∂M_k with respect to the metric $g_k(0)$. However, it might happen that $r_k = d_{g_k(0)}(p_k, \partial M_k) \to \infty$. In this case, we show in the next two paragraphs that the limit is a complete Ricci flow defined in a noncompact manifold without boundary.

Consider the metric space $\mathcal{X}_k = \overline{B}_k\left(p_k, \frac{r_k}{2}\right)$, the closed ball of radius $r_k/2$ centered at p_k in $(M_k, g_k(0))$, with the metric generated by the Riemannian structure $g_k(0)$. Assuming uniform bounds on the curvature of the M_k's, the sequence $(\mathcal{X}_k, p_k)$ converges in the pointed Gromov-Hausdorff sense to a complete pointed metric space $(\mathcal{X}, p)$ (Fact 4, page 174 in [31]). We show that, assuming a uniform lower bound $\iota_0 > 0$ on the injectivity radius of each M_k with respect to $g_k(0)$, and uniform bounds on the curvature and its covariant derivatives (with respect to $g_k(0)$ and independent of k), this metric space is a manifold.

Indeed, given $x \in \mathcal{X}$, there is a sequence $x_k \in \mathcal{X}_k$ such that $x_k \to x$ (in the Gromov-Hausdorff sense, see page 307 in [30]), and for which, by our assumption on a uniform lower bound on the injectivity radius, there exist normal coordinates around x_k for k large enough, say $\varphi_k : B(0, \iota_0) \longrightarrow \mathcal{X}_k$ (here $B(0, \iota_0)$ is the open ball of $\mathbb{R}^{d+1}$ of radius ι_0 centered at the origin). The assumed curvature bounds in Theorem 4.2 (which, by Shi's estimates, gives bounds on the covariant derivatives of the curvature) imply that the sequence of φ_k's converges to a $\varphi : B(0, \iota_0) \longrightarrow \mathcal{X}$ which is a chart around x in $\mathcal{X}$, and also that the transition functions between charts converge to smooth transition functions in the limit. Also, we have, that along a subsequence, the Riemannian metrics $g_k(0)$ converge to a metric $g_\infty(0)$ in $\mathcal{X}$. Finally, again by the curvature bounds, applying the arguments in Section 2 of [20], we have a complete (noncompact and boundaryless) Ricci flow $g_\infty(t)$ defined in $\mathcal{X} \times (a, b]$ to which a subsequence of the $(M_k, g_k(t), p_k)$ converges.

We have then the following compactness theorem.

Theorem 4.3. *Let (M_n, p_n) be a sequence of pointed surfaces with compact boundary, and $g_n(t)$ be complete Ricci flows on M_n, $t \in (a, b]$, $a < 0 < b$. Assume*

(1) $|R_{g_n}|_{g_n} \leq K$ in $M_n \times (a, b]$,

(2) k_{g_n}, the geodesic curvature of ∂M_n with respect to g_n and the outward unit normal, only depends on t (i.e., it is constant in space in $\partial M_n \times (a, b]$),

(3) for every nonnegative integer j, $\left|\frac{d^j k_{g_n}}{dt^j}\right| \leq K_j$ on $\partial M_n \times (a, b]$,

(4) the length of the boundary with respect to $g_n(0)$ *is at least* λ,
(5) $\iota_{M,g_n(0)} \geq \iota_0$,
for all n. *Then there is a pointed surface* (M_∞, p_∞), *a complete Ricci flow* $g_\infty(t)$ *on* M_n *such that up to a subsequence*

$$(M_n, g_n(t), p_n) \to (M_\infty, g_\infty(t), p_\infty),$$

in the C^{m-3} *topology.*

Proof. The proof is an application of Gianniotis' Compactness Theorem and the comments made right before its statement. Indeed, if

$$\liminf_{n\to\infty} d_{g_n(0)}(p_n, \partial M_n) \to \infty,$$

then we can form a blow up limit and it will be a noncompact complete Ricci flow without boundary.

Otherwise, we can apply Giannotis' Compactness Theorem directly. First, notice that hypothesis (2) in Theorem 4.2 follows from hypothesis (3) of the statement of the theorem and from the fact that the second fundamental form of the boundary of a surface is given by $k_g g^T$, where k_g is the geodesic curvature of the boundary, and g^T is the metric of the surface restricted to the boundary. So we must verify that in the case of a surface, as long as the geodesic curvature of the boundary remains constant in space and we have uniform bounds on the curvature, the Ricci flow is Λ-controlled. Indeed, we can take γ as the restriction of g to ∂M, i.e., $\gamma = g^T$. Given a point $\overline{x} \in \partial M$ at time t let $U \subset \partial M$ be an open neighborhood of $\overline{x}$ and $w : (-r, r) \longrightarrow \partial M$ be an arclength parametrization of U with $w(0) = \overline{x}$. It is clear that $u = w^{-1}$ are γ-harmonic coordinates around $\overline{x}$. Notice that if the length of the boundary is uniformly bounded from below by λ' on $(a, b]$, then we can define these γ-harmonic coordinates for all $r \leq \lambda'/4$ (so we can take $\Lambda^{-1} = \min\{\lambda'/4, 1\}$ to satisfy Definition 4.3). In the coordinates thus defined, it is not difficult to show that γ is the Euclidean metric, so (a) in Definition 4.3 holds with $Q = 1$, and it is immediate that (b) also holds. Moreover, from (3) and (4) we also have (c) (with a Q conveniently defined, and which only depends on the K_j's, $j = 0, 1, 2, 3, \ldots, m+1$, α, and λ', see equation (9) above).

To finish the proof, we only need to show how to obtain a uniform lower bound on the length of ∂M_n with respect to $g_n(t)$, for $t \in (a, b]$, independent of both t and n (that is, we must show that λ' exists). To this end, notice that if in a given (finite) time interval $I = (a, b]$ at $t_0 \in I$ we have such lower bound on the length of the boundary, a uniform bound on the curvature gives a lower bound on the length of the boundary on the whole time interval I, and this lower bound depends only on the bound on the curvature, on the lower bound on the length of ∂M_n with respect to g_n at $t = t_0$, and on a bound on the length of the time interval (in this case $b - a$); since we are assuming that we have a uniform bound from below for the length of ∂M_n with respect to $g_n(0)$ (namely λ), the theorem is proved. □

4.4. Application to blow-up limits of the Ricci flow

Recall that a blow-up limit is constructed as follows: if $(0, T)$, $0 < T < \infty$, is the maximal interval of existence for a solution to (1), we pick a sequence of times $t_j \to T$ and a sequence of points such that

$$\lambda_j := R_g(p_j, t_j) = \max_{M \times [0, t_j]} R_g(x, t),$$

and then we define the dilations

$$g_j(t) := \lambda_j g\left(t_j + \frac{t}{\lambda_j}\right), \quad -\lambda_j t_j < t < \lambda_j (T - t_j).$$

By the procedure just described, from a solution to (1) we can define a sequence $(M_n, g_n(t), p_n)$ of pointed Ricci flows. So assume that we have a solution to (1), which satisfies the hypothesis of Theorem 1.1. As proved above, this solution will blow up in finite time; we will show that we can construct blow-up limits, via an application of Theorem 4.3:

The bound on the curvature is obvious from the construction. The needed control over the injectivity radius of the surface and its boundary injectivity radius at $t = 0$ are provided by the results in Section 4.1. On the other hand when the boundary is convex, a bound from below on ι_M gives a bound from below on the length of the boundary (and hence we have such control at $t = 0$). Further, the smoothness of ψ $(= k_g)$ on $\partial M \times [0, \infty)$, the fact that it rescales as

$$k_{\lambda_j g} = \frac{k_g}{\sqrt{\lambda_j}},$$

and that we are assuming $\lambda_j \to \infty$, gives hypotheses (2) and (3) of the statement of Theorem 4.3.

Therefore, applying Theorem 4.3, we have that there is solution to the Ricci flow $(M_\infty, g_\infty(t), p_\infty)$, defined on an interval $-\infty < t < \Omega$, such that

$$(M_n, g_n(t), p_n) \to (M_\infty, g_\infty(t), p_\infty),$$

on any compact interval of time in the C^{m-3} topology for any m, and hence, using a diagonal procedure, smoothly.

5. Proof of Theorem 1.1

As shown in the previous section, given $(M, g(t))$ a solution to (1) in a maximal time interval $0 < t < T < \infty$, which satisfies the hypothesis of Theorem 1.1, we can produce blow-up limits.

In our case, we can classify the possible blow-up limits we may obtain. We have the following result which, with minor modifications, is essentially proved in [12].

Proposition 5.1. *Let $(M, g(t))$, M a compact surface with boundary, be a solution to (1). Let $(0, T)$, $T < \infty$, be the maximal interval of existence of $g(t)$. Assume that there is an $\epsilon > 0$ such that for all $0 < t < T$, $R_g > -\epsilon$, and that k_g is bounded. There are two possible blow-up limits for $(M, g(t))$ as $t \to T$. If the blow-up limit is compact, then it is a homotetically shrinking round hemisphere with totally geodesic boundary. If the blow-up limit is non compact then it is (or its double is) a cigar soliton.*

Proof. Just notice that any blow-up limit of $(M, g(t))$ as $t \to T$, will have nonnegative scalar curvature, which is strictly positive at one point, a totally geodesic boundary, or no boundary at all, and will be defined in an interval of time $(-\infty, \Omega)$ (and Ω could be ∞). Then, by doubling the manifold if needed (which as discussed in Appendix B.1, gives a smooth solution to the Ricci flow), everything reduces to the boundaryless case, and [21, Thm. 26.1 and Thm. 26.3] can be applied to give the proposition. □

Now we proceed with the proof of Theorem 1.1. In what follows, we let (M, g_0) be a compact surface with boundary of positive scalar curvature, and such that ∂M has nonnegative geodesic curvature, ψ be as described in the statement of Theorem 1.1, and we let $(M, g(t))$ be the solution to (1) associated to the initial data g_0 and the boundary data ψ. By Proposition 2.2 we have $R > 0$ and R blows up in finite time, say T, and as discussed in the last paragraph of the previous section, we are allowed to take blow-up limits of $(M, g(t))$ as $t \to T$. Hence, from Proposition 5.1, and the monotonicity formula in §3 (Theorem 3.2), it follows that along a sequence of times $t_k \to T$, it holds that

$$\lim_{k\to\infty} \frac{R_{\max}(t_k)}{R_{\min}(t_k)} = 1, \tag{10}$$

where

$$R_{\max}(t) = \max_{p\in M} R_g(p,t) \quad \text{and} \quad R_{\min}(p,t) = \min_{p\in M} R_g(p,t).$$

Indeed, as is the case of closed surfaces, the monotonicity formula provided by Theorem 3.2 (as long as $\psi \geq 0$ and $\psi' \leq 0$) precludes the cigar as a blow up limit (see [12, § 7.1] and [10, Corollary D.48]). To see this more clearly, double the blow-up limit: it is an ancient solution to the Ricci flow (now without boundary), and hence it is smooth (see the discussion in Appendix B.1); so we have two possible scenarios. The obtained ancient solution is a Type I solution, in which case it must be a sphere; or it is a Type II solution. In this case, as the curvature of the solution assumes its maximum at an origin, it must be the cigar. But then, this origin must be located at the boundary of the original blow up limit (before doubling) for otherwise the curvature would assume two maximums, which does not happen in the cigar; so this blow-up type II limit is half a cigar (as cut along a radial geodesic) if it has a boundary, or just a cigar (if the limit is boundaryless). Hence, we can apply the arguments in [12, § 7.1, pp. 45-46] and define a family of functions ϕ in half the cigar (restricting the definition of ϕ given for the whole cigar in the obvious way) or in the whole cigar (if the limit is boundaryless) to show that the functional $\mathcal{W}$ has as its infimum $-\infty$, a contradiction with the fact that the original Ricci flow (the one we extracted the limit from) cannot have this infimum to be $-\infty$ as $\mathcal{W}$ is monotone along the Ricci flow (Theorem 3.2), and at time $t = 0$ this infimum is finite. Therefore the only possible blow-up limit is the round hemisphere with totally geodesic boundary, which implies (10).

The following interesting estimate on the evolution of the area $A(t) := A_{g(t)}(M)$ of M under the Ricci flow (1), with initial data g_0, can now be proved.

Proposition 5.2. *There are constants $c_1, c_2 > 0$ such that*

$$c_1(T-t) \leq A(t) \leq c_2(T-t).$$

Proof. Since in our case any blow up limit is compact, we must have

$$\lim_{t\to T} A(t) = 0.$$

Since $R > 0$, and $\int_{\partial M} k_g \, ds_g$ is nonincreasing, by the Gauss-Bonnet Theorem we have the inequalities

$$-2\pi \leq \frac{dA}{dt} \leq -c,$$

and the result follows by integration. □

As a consequence of the previous proposition and from the normalization (2) we can immediatly conclude the following.

Corollary 5.1. *The normalized flow exists for all time.*

Proof. The normalized flow exists up to time

$$\lim_{t \to T^-} \int_0^t \frac{1}{A(\tau)} \, d\tau = \infty,$$

by Proposition 5.2. □

Also, an estimate for the maximum of the scalar curvature can be deduced.

Proposition 5.3. *There are constants $c_1, c_2 > 0$ such that*

$$\frac{c_1}{T-t} \leq R_{\max}(t) \leq \frac{c_2}{T-t}.$$

Proof. By the Gauss-Bonnet theorem and the fact that, under the hypothesis of Theorem 1.1, $\displaystyle\int_{\partial M} k_g \, ds_g$ is nonincreasing, we have that

$$\int_M R_{\max}(t) \; dA_g \geq C,$$

and from Proposition 5.2, there is a $c' > 0$ such that

$$c' R_{\max}(t) \, (T-t) \geq C,$$

so the left inequality follows.

To show the other inequality we proceed by contradiction. Assume that there is no constant $c_2 > 0$ for which

$$R_{\max}(t) \le \frac{c_2}{T-t}$$

holds. Then we can find a sequence of times $t_j \to T$ such that

$$R_{\max}(t_j)(T-t_j) \to \infty,$$

and hence along this sequence the blow-up limit would not be compact, as it would have infinite area due to Proposition 5.2, and this would contradict the arguments given right after Proposition 5.1. □

Corollary 5.3 shows that along any sequence of times we can take a blow-up limit since for any sequence of times, the curvature is blowing up at maximal rate (i.e. $\sim \frac{1}{T-t}$). By the arguments following Proposition 5.1 this blow-up limit is a round homotetically shrinking hemisphere. This proves the following theorem.

Theorem 5.1. *Let (M, g_0) be a compact surface with boundary, of positive scalar curvature and such that the geodesic curvature of ∂M is nonnegative, and let ψ be as in the statement of Theorem 1.1. Then, the solution to the Ricci flow (1) with initial condition g_0 blows up in finite time T, and for the scalar curvature R we have that*

$$\lim_{t\to T} \frac{R_{\max}(t)}{R_{\min}(t)} = 1.$$

As a consequence, under the corresponding normalized flow we obtain that

$$\tilde{R}_{\max}\left(\tilde{t}\right) - \tilde{R}_{\min}\left(\tilde{t}\right) \to 0 \quad as \quad \tilde{t} \to \infty.$$

So far we have shown that, in the normalized flow, as $\tilde{t} \to \infty$ the curvature uniformizes. Now we argue why given any sequence of times $\tilde{t}_n \to \infty$, we can find a subsequence along which the metric is converging smoothly. Indeed, to each time $\tilde{t}_n$ corresponds a t_n in the unnormalized flow; by the estimates given in Corollary 5.3, we can produce a blow-up limit, taking as time origins each of the t_n. Hence, $\lambda_n g(t_n)$, where $\lambda_n = \max_{p\in M} R(p, t_n)$, converges smoothly to a metric $\overline{g}$ of constant curvature. Observe that $\tilde{g}\left(\tilde{t}_n\right) = c_n \lambda_n g(t_n)$. As $\tilde{g}\left(\tilde{t}_n\right) = \frac{1}{A(t_n)} g(t_n)$, where $A(t_n)$ is the area of the surface with respect to the metric $g(t_n)$, and by our previous arguments $1/A(t_n) \sim$

λ_n, we can infer that $c_n > 0$ is a bounded sequence (which is also bounded away from 0). Therefore, we can choose a sequence $n_k \to \infty$ such that $c_{n_k} \to c$. It is clear then that along this subsequence $\tilde{g}\left(\tilde{t}_{n_k}\right) \to c\overline{g}$ smoothly, and it is clear that $c\overline{g}$ has constant curvature.

Thus we have proved that for any sequence $\tilde{t}_n \to \infty$, there is a subsequence of times $\tilde{t}_{n_k} \to \infty$ such that $\tilde{g}\left(\tilde{t}_{n_k}\right)$ is converging to a metric of constant curvature (these metrics may be different according to the sequence considered). Notice that the metrics $\tilde{g}(t)$ have their maximum curvatures uniformly bounded from below, by Propositions 5.2 and 5.3 (this precludes the flat cylinder as a possible limit), and hence the curvature approaches a positive constant -along any sequence of times. To be able to conclude that these limit metrics are isometric to that of a standard hemisphere, we must also show that the geodesic curvature of the boundary approaches 0. Let us now conclude the proof of Theorem 1.1.

Finishing the proof of Theorem 1.1. The only part of the statement that has not been proved in the previous discussion is that regarding the behavior of the geodesic curvature. Notice that

$$\frac{c}{T-t} \leq \phi(t) \leq \frac{C}{T-t},$$

where ϕ is the normalizing factor defined in the introduction, and hence

$$T-t \leq Te^{-c\tilde{t}},$$

which shows that

$$k_{\tilde{g}} \leq C\sqrt{T-t}\psi \leq ce^{-c\tilde{t}}\psi.$$

Since $\psi \geq 0$ and $\psi' \leq 0$, it remains bounded, and the theorem follows. □

6. Proof of Theorem 1.2

In this section we let $g(t)$ be the solution to (1) in the two-ball D, with initial and boundary data as described in the hypotheses of Theorem 1.2. It is clear, from uniqueness, that this solution is also rotationally symmetric. Let $A(t) := A_{g(t)}(D)$ be the area of D with respect to $g(t)$. Given the fact that

$$\frac{dA}{dt} = -4\pi + 2\int_{\partial M} k_g \, ds_g,$$

it is not difficult to conclude that there are positive constants c, C such that

$$-c \leq \frac{d}{dt}A \leq -C < 0.$$

Therefore, to show that the normalized flow does exists for all time, all we must prove is that $A(t) \to 0$ as $t \to T$, as this would show that $A(t) \sim T - t$.

First notice that the scalar curvature of the solution to (1), with initial data g_0 and boundary data ψ as in Theorem 1.2, blows up in finite time by Proposition 2.3. Let us denote by $\mathcal{R}(t)$ the radius of D with respect to $g(t)$; by comparison geometry we have that at any time $\mathcal{R} \geq \frac{\pi}{2\sqrt{R_{\max}(t)}}$, so the boundary injectivity radius is conveniently bounded from below. Using Hamilton's arguments (see [8, § 5]), it can be shown that for points at distance at least $\frac{1}{4}\mathcal{R}$ from the boundary, the injectivity radius is also conveniently bounded from below. The fact that we have good lower bounds on the length of the boundary is given by Corollary 1.2 (c) in [1]. Finally, as the geodesic curvature of the boundary remains constant in space, it is not difficult to show (as we did before) that for any $0 < T' < T$, the boundary is Λ controlled on $(0, T']$.

The arguments in the previous paragraph show that we can take a blow-up limit of $(M, g(t))$ as $t \to T$. This blow-up limit might be compact, and in this case it is a round hemisphere, and as we did before, it can be shown then that $A(t) \to 0$ as $t \to T$, and we would be done. If this blow-up limit is non compact it must be the cigar. In this case, it is not difficult to prove that we can take as an origin for the blow up limit the center of D (otherwise, in the limit surface there would be a radial geodesic along which the curvature is constant, and this does not happen in the cigar). To study this case we define the following quantities which depend on $g(t)$:

$$I(r,t) = \frac{L_g(\partial D_r)^2}{A_g(D_r)}, \quad \overline{I}(t) = \inf_{0 \leq r \leq \mathcal{R}} I(r,t),$$

where $D_r \subset D$ is the geodesic ball of radius r centered at the center of D, $L_g(\partial D_r)$ is the length of ∂D_r and $A_g(D_r)$ is the area of D_r both with respect to the metric $g(t)$. There are two cases to be considered. The first case to be considered is when there is a $\delta > 0$ such that this infimum is attained at $r(t) \in [0, \mathcal{R})$ on the time interval $(T - \delta, T)$. If $r = 0$, then $I = 4\pi$; if $r > 0$, we have the following formula.

Lemma 6.1. *If $\overline{I}$ at time t is attained at $r \in (0, \mathcal{R})$ then I satisfies an evolution equation*

$$\frac{\partial}{\partial t} \log I(r,t) = \frac{\partial^2}{\partial r^2} \log I(r,t) + \frac{1}{A_g(D_r)} (4\pi - I(r,t)). \tag{11}$$

Proof. We let $L = L_g(\partial D_r)$, $A = A_g(D_r)$, k be the geodesic curvature of ∂D_r and K be the Gaussian curvature of D. We have the following set of formulas

$$\frac{\partial L}{\partial r} = \int_{\partial D_r} k\, ds = kL, \quad \frac{\partial^2 L}{\partial r^2} = -\int_{\partial D_r} K\, ds = \frac{\partial L}{\partial t},$$

and

$$\frac{\partial A}{\partial r} = L, \quad \frac{\partial^2 A}{\partial r^2} = \int_{\partial D_r} k\, ds, \quad \frac{\partial A}{\partial t} = -4\pi + 2\int_{\partial D_r} k\, ds.$$

Notice that at the value of r where the infimum is attained we have

$$0 = \frac{\partial}{\partial r} \log I = \frac{2}{L}\frac{\partial L}{\partial r} - \frac{1}{A}\frac{\partial A}{\partial r}, \quad \text{so we have,} \quad \frac{2}{L}\frac{\partial L}{\partial r} = \frac{1}{A}\frac{\partial A}{\partial r}.$$

Formula (11) now follows from all these identities by a straightforward calculation. □

Clearly Lemma 6.1, under the assumption that $\overline{I}$ is always attained at $r \in (0, \mathcal{R})$, precludes the fact that $\overline{I} \to 0$ (even along a subsequence of times), since it does imply that $\overline{I}$ increases if $\overline{I} < 4\pi$. Therefore, the blow-up limit in this case cannot be the cigar, hence it is a round hemisphere, and we would be done. We are left with one more possibility: $\overline{I}$ is reached at $r = \mathcal{R}$ for a sequence of times of times $t_k \to T$; if this is so, then since for the cigar $\overline{I} = 0$, if $A(t) \not\to 0$ as $t \to T$, we must have $L(\partial D) \to 0$ as $t \to T$. Under the hypotheses of Theorem 1.2, using the Maximum Principle, it is not difficult to show that the scalar curvature remains uniformly bounded from below on $(0, T)$, and therefore $\mathcal{R}$ remains uniformly bounded above throughout the flow in $0 < t < T < \infty$. But then we have the following lemma (see [12]).

Lemma 6.2. *Let g_k be a sequence of rotationally symmetric metrics on the two-ball D. Assume that there is a constant $\epsilon > 0$ such that $R_{g_k} \geq -\epsilon$ and $k_{g_k} \geq -\epsilon$, and that the radius of D with respect with this sequence of metrics is uniformly bounded from above by $\rho > 0$. Then if the length of the boudary of D with respect to g_k, $L_{g_k}(\partial D)$, goes to 0 as $k \to \infty$, then $A_{g_k}(D) \to 0$.*

Proof. By rescaling we may assume that $\epsilon = 1$. Hence a comparison argument shows that,

$$A_{g_k}(D) \leq 2\pi \int_0^\rho L_{g_k}(\partial D)\, e^r\, dr \leq 2\pi e^\rho \rho L_{g_k}(\partial D),$$

and the conclusion of the lemma follows. □

The previous lemma shows that we must have $A(t) \to 0$ as $t \to T$. This proves that, in any case, $A(t) \to 0$ as $t \to T$ and the theorem follows.

Appendix A. Derivative estimates

In this section of the appendix we shall show how to bound the derivatives of the conformal factor in terms of bounds on the the curvature and of the geodesic curvature. We will make use of the of the following result of Gianniotis [18, Thm. 1.2] (in fact, we will only need the case $j = 1$ of the theorem, which might be of interest to the reader).

Theorem A.1. *Let $(M, g(t), \gamma(t))$, $t \in [0, T]$, be a complete Ricci flow with Λ-controlled boundary in $(0, T]$. Suppose*

(1) $|Rm(g(t))|_{g(t)} \leq K$ in M and $|\mathcal{A}(g(t))|_{g^T(t)} \leq K$ on ∂M for all t.

(2) $i_{b,g(0)} \geq i_0$.

For any $j = 1, 2, \ldots, m-2$ and $\tau > 0$, there exists a constant

$$C = C(n, \tau, T, \Lambda, l, j, K, i_0) > 0$$

such that for any $t \in [\tau, T]$

$$\begin{aligned} \left|\nabla^j Rm(g(t))\right| &\leq C \quad \text{in} \quad M, \\ \left|\nabla^{j+1} \mathcal{A}(g(t))\right| &\leq C \quad \text{in} \quad \partial M. \end{aligned}$$

Gianniotis' Theorem applies to the problems studied in this paper, since we are assuming constant in space geodesic curvature throughout the evolution, and, as discussed before, from this fact (assuming bounds on the curvature) follows that the boundary is Λ-controlled.

Let us write as usual $g = e^u g_0$ for a solution on the Ricci flow on the surface. We shall show how to bound derivatives of u when restricted to a Fermi chart. So, let (U, φ) be a Fermi chart, namely, we let $U = (-\epsilon, \epsilon) \times$

$[0,\delta)$

$$\varphi : U \longrightarrow M,$$

where if $p = \varphi(x,s)$, then s is the distance from p to the boundary. In Fermi coordinates the metric is written as

$$g = dx^2 + f(x,s)^2 \ ds^2.$$

Since the Laplace operator is written as

$$\Delta = e^{-u} \frac{1}{\sqrt{|g_0|}} \partial_i \left(g_0^{ij} \sqrt{|g_0|} \partial_j \right),$$

in Fermi coordinates we have

$$\Delta_{g_0} = \frac{\partial^2}{\partial s^2} - k_{g_0}(x,s) \frac{\partial}{\partial s} + \frac{f_x}{f} \frac{\partial}{\partial x} + \frac{1}{f^2} \frac{\partial^2}{\partial x^2},$$

where k_{g_0} represents the geodesic curvature of the curve at distance s from the boundary with respect to $\dfrac{\partial}{\partial s}$.

The main idea we shall employ to obtain estimates on u is to use classical regularity results (Schauder estimates) for parabolic equations with oblique boundary conditions ([26, Chapter IV]). To be able to use these results and start a bootstrapping argument, we need to have uniform control in time over the Hölder norm of the partial derivatives of u in the Fermi chart. Our main tool to obtain this control will be Theorem 4.1 in [27], so we must verify the structure conditions imposed as assumptions in this theorem for the following elliptic operator

$$\Delta_{g_0} u - R_{g_0} - Re^u,$$

with boundary condition

$$-\frac{\partial u}{\partial s} - 2k_g e^{\frac{u}{2}} - 2k_{g_0}.$$

In order to verify these structure conditions, we must have control over R, R_{g_0}, and their first derivative in the chart, and on k_{g_0}, k_g, and certain Hölder norms of their first derivatives in the chart. We will assume bounds on the C^k seminorms of g_0 in a Fermi chart (U,φ) (this takes care of the needed control on R_{g_0} and k_{g_0}), and we will also are assume bounds on k_g and its derivatives on $\partial M \times (0,T)$, and on R and ∇R (via Theorem A.1) on $M \times (0,T)$. Notice that the bound given for ∇R, once we have control over

u and g_0, gives control over $\partial_i R$, the derivatives of R, in the chart. Indeed, If in the chart we have that

$$\Lambda^{-2}\delta_{ij} \leq g_{0ij} \leq \Lambda^2 \delta_{ij},$$

and $|u| < M$, then

$$|\partial_i R| \leq \left| e^u \left(g_0\right)_{ij} \nabla^j R \right| \leq \Lambda^2 e^M \left|\nabla R\right|.$$

These assumptions are enough to verify the structure conditions in Theorem 4.1 in [27]. We leave the verification of these structure conditions to the interested reader.

As before, we shall use the notation

$$\|g\|_{C^k(U,\varphi)} = \sup_{x\in U} \sum_{|\beta|\leq k} \left|\partial^\beta g\left(x\right)\right|$$

to denote the C^k-norm of the metric g written in local coordinates, and a few times for a function u

$$\|u\|_{C^k_\tau(U,\varphi)}$$

to denote the same C^k-norm with $t = \tau$ fixed (as u depends also on t). The Hölder spaces $C^{k,\alpha}\left(U, \varphi\right)$ are defined as usual.

We will also make use of the parabolic Hölder spaces $H^{(l)} = H^{l,\frac{l}{2}}$ (as defined in pages 6-9 in [26]; we defined similar norms before in this very same paper, but for the convenience of the reader, and to adopt the notation in [26], let us do it again):

Let $Q_T = \Omega \times (0, T)$, $\rho_0 > 0$ and $0 < \alpha < 1$ fixed. Define

$$\langle u\rangle^{(\alpha)}_{x,Q_T} = \sup_{\substack{(x,t),\,(x',t)\,\in\,\overline{Q}_T,\\ |x-x'|\,<\,\rho_0}} \frac{\left|u\left(x,t\right) - u\left(x',t\right)\right|}{\left|x-x'\right|^\alpha},$$

$$\langle u\rangle^{(\alpha)}_{t,Q_T} = \sup_{\substack{(x,t),\,(x',t)\,\in\,\overline{Q}_T,\\ |t-t'|\,<\,\rho_0}} \frac{\left|u\left(x,t\right) - u\left(x,t'\right)\right|}{\left|t-t'\right|^\alpha}.$$

For an integer j define

$$\langle u\rangle^{(j)}_{Q_T} = \sum_{2r+|\beta|=j} \sup_{Q_T} \left|\partial_t^r \partial_x^\beta u\left(x,t\right)\right|.$$

Given $l+\alpha$, l a positive integer as before and $0<\alpha<1$, define

$$\text{(A.1)}\quad \|u\|_{H^{(l+\alpha)}(\overline{Q}_T)} = \sum_{2r+|\beta|=l} \left\langle \partial_t^r \partial_x^\beta u \right\rangle_{x,Q_T}^{(\alpha)} + \sum_{0<l+\alpha-2r-|\beta|<2} \left\langle \partial_t^r \partial_x^\beta u \right\rangle_{t,Q_T}^{\left(\frac{l+\alpha-2r-|\beta|}{2}\right)} + \sum_{j=0}^{l} \langle u\rangle_{Q_T}^{(j)}.$$

The dependence of these norms on $\rho_0>0$ is not important in our case, as different choices produce equivalent norms. We can then define the space $H^{(l+\alpha)}\left(\overline{Q}_T\right)$ as the space of all functions with finite norm (A.1). Finally the set $H^{(l+\alpha)}\left(Q_T\right)$ as the space of functions belonging to $H^{(l+\alpha)}\left(\overline{Q'}\right)$ for any subdomain Q' such that $\overline{Q'}\subset Q$. In our notation (for these parabolic spaces) we will suppress the dependence on φ.

We are ready to state an proof the following:

Theorem A.2. *Write $g=e^u g_0$ on $M\times[0,T)$. Let (U,φ) be a Fermi chart with respect to g_0 of size (ϵ,δ). Assume that u and R are uniformly bounded on $U\times(0,T)$, and that we have bounds for all integers $l>0$ and $0<\alpha<1$ on $\|k_g\|_{H^{l+\alpha}(((-\epsilon,\epsilon)\times\{0\})\times(0,T))}$ (recall $k_g=\psi$ in (1)). Let τ be such that $0<\tau<T$, and U' be an open subset of U such that $\overline{U'}\subset U$. Then for each integer $k>0$ there is a C_k which depends only on the bound on R, on the bounds on k_g and its derivatives, on bounds on $\|g_0\|_{C^l(U,\varphi)}$, on U' and on τ and T, such that*

$$\left|\partial^k u\,(x,t)\right| \le C_k \quad \text{on} \quad U'\times[\tau,T),$$

where $U'=\left(-\frac{\epsilon}{2},\frac{\epsilon}{2}\right)\times\left[0,\frac{\delta}{2}\right)$ and ∂ represents partial differentiation in the chart.

Proof. Using the chart, we shall work now in the set $U\subset\mathbb{R}^2_+$. Define the following sets

$$U^k = \left(-\epsilon\left(\frac{1}{2}+\frac{1}{2^{k+1}}\right), \epsilon\left(\frac{1}{2}+\frac{1}{2^{k+1}}\right)\right)\times\left[0,\delta\left(\frac{1}{2}+\frac{1}{2^{k+1}}\right)\right),$$

$$\partial' U^k = \left(-\epsilon\left(\frac{1}{2}+\frac{1}{2^{k+1}}\right), \epsilon\left(\frac{1}{2}+\frac{1}{2^{k+1}}\right)\right)\times\{0\},$$

and

$$W^k = U^k\times(\tau_k,T), \quad \tau_k=\tau\left(\frac{1}{2}-\frac{1}{2^{k+1}}\right).$$

Fix $\tau_1 > 0$. Then for any $t \geq \tau_1$, having control over R_{g_0} and ∂R_{g_0} (as we are assuming this control over the metric g_0), and over R and ∇R (and this control is uniform in t for $0 < \tau_1 \leq t < T$ once we have fixed τ_1) from the elliptic equation

$$\Delta_{g_0} u - R_{g_0} = Re^u$$

with oblique boundary condition

$$\frac{\partial u}{\partial \eta_{g_0}} = 2k_g e^{\frac{u}{2}} - 2k_{g_0},$$

we can obtain an estimate on the $C^{1,\alpha}$ norm of u at any time $t > 0$ (see Theorem 4.1 in [27] and the paragraph right after the statement of the theorem). To be more precise, we obtain, for any $t \geq \tau_1 > 0$, a bound (uniform in t)

$$\begin{aligned} &\|\partial u\|_{C^\alpha_t(U^1,\varphi)} \\ &\leq C_{1,\alpha}\left(U, U^1, \mu_0, \mu_1, \|k_g\|_{H^{(1+\alpha)}(((-\epsilon,\epsilon)\times\{0\})\times(0,T))}, \|g_0\|_{C^{2,\alpha}(U,\varphi)}, \tau_1, T\right), \end{aligned}$$

were μ_0 is a bound on R and μ_1 is a bound on ∂R on $U \times (\tau_1, T)$, From Theorem A.1, we can even suppress the dependence on μ_1 as we can give it in terms of μ_0, and on τ_1 and T. Also, from the bounds on R, and equation

$$\frac{\partial u}{\partial t} = -R,$$

we find a bound on $\frac{\partial u}{\partial t}$. This shows that $u \in H^{(1+\alpha)}\left(W^1\right)$, and that bounds in this parabolic Hölder space are controlled by bounds on R and on g_0 and its derivatives.

Now we use standard parabolic estimates for the problem (in particular Theorem 10.1 in [26, Chapter IV]: For the relevant definitions please consult also [26, Chapter I-§1])

$$\begin{cases} \dfrac{\partial u}{\partial t} = e^{-u}\left(\Delta_{g_0} u - R_{g_0}\right) & \text{in} \quad U \times (0,T) \\ \dfrac{\partial u}{\partial \eta} = 2k_g e^{\frac{u}{2}} - 2k_{g_0} & \text{on} \quad ((-\epsilon,\epsilon)\times\{0\}) \times (0,T)\,. \end{cases}$$

As we have $H^{(1+\alpha)}$ bounds on the boundary terms (here we use again our assumptions on k_g), this gives us a bound on $\|u\|_{H^{(2+\alpha)}(W^2)}$ and of course this bound depends on all the previous bounds, namely

$$\begin{aligned}&\|u\|_{H^{(2+\alpha)}(W^2)}\\ &\leq C\left(W^1, W^2, \|u\|_{H^{(1+\alpha)}(W^1)}, \|k_g\|_{H^{(1+\alpha)}((\partial' U^1)\times(\tau_1,T))}, \|g_0\|_{C^{2,\alpha}(U,\varphi)}\right)\end{aligned}$$

From this bound on $\|u\|_{H^{(2+\alpha)}(W^2)}$ we can obtain bounds on $\|u\|_{H^{3+\alpha}(W^3)}$, and this bounds also depend only on R. We can continue this process (bootstrapping), by repeated applications of Theorem 10.1 in in [26, Chapter IV], to obtain bounds on higher Hölder norms of u. Indeed, given a bound on $\|u\|_{H^{(n+\alpha)}(W^n)}$, [26, Theorem 10.1, Chapter IV] gives a bound

$$\begin{aligned}&\|u\|_{H^{(n+2+\alpha)}(W^{n+1})}\\ &\leq C\left(W^n, W^{n+1}, \|u\|_{H^{(n+\alpha)}(W^n)}, \|k_g\|_{H^{(n+1+\alpha)}((\partial' U^n)\times(\tau_n,T))}, \|g_0\|_{C^{n+2,\alpha}(U,\varphi)}\right).\end{aligned}$$

So we finally can bound any number of derivatives of u in terms of bounds on R, on k_g and its derivatives, and on g_0 and its derivatives in the chart (U,φ) (in ever smaller domains, but this is not a problem, as all these domains contain $U' \times [\tau, T)$). This proves the Theorem.

□

Appendix B. On the double of a surface

Here we discuss the regularity of the metric of the double of a manifold. Given M a manifold with boundary, we define its double as follows:

We let $M_j = M \times \{j\}$, $j = 0, 1$, and define an equivalence relation

$$(m, i) \sim (m, j)$$

if $i = j$, or if $i \neq j$ and $m \in \partial M$. Then the double is the set

$$\tilde{M} = (M_0 \cup M_1) / \sim$$

endowed with the quotient topology. To give a smooth structure to $\tilde{M}$, given a chart

$$\psi : U \subset \mathbb{R}^n_+ \longrightarrow M$$

we define a chart on $\tilde{M}$ as follows. If $U \cap \{x^n = 0\} = \emptyset$, we obtain two charts by defining

$$\tilde{\psi}_j = (\psi(x), j).$$

If $U \cap \{x^n = 0\} \neq \emptyset$, let $U^* \subset \mathbb{R}^n_-$ be defined as

$$U^* = \left\{x : \left(x^1, \ldots, x^{n-1}, -x^n\right) \in U\right\},$$

and define

$$\tilde{\psi} : U \cup U^* \longrightarrow \tilde{M}$$

as

$$\tilde{\psi}\left(x^1, \ldots, x^n\right) = \begin{cases} (\psi(x), 0) & \text{if} \quad x^n \geq 0 \\ (\psi(x), 1) & \text{if} \quad x^n \leq 0 \end{cases}$$

This gives a smooth structure to $\tilde{M}$. We call $\tilde{M}$ the double of M.

However, when M is also endowed with a metric, the regularity of the double metric is a different matter, and in general it might not be smooth. Let us first review how to double the metric. We shall explain the construction near the boundary, being relatively obvious how the construction proceeds away from it.

Pick a Fermi chart on M,

$$\varphi : (-\epsilon, \epsilon) \times [0, \delta) \longrightarrow M,$$

and as explained above, construct the double chart. In this new chart the metric is written as

$$g = ds^2 + f(x, s)^2 \, dx^2, \quad \text{if} \quad s \geq 0,$$

and

$$g = ds^2 + f(x, -s)^2 \, dx^2, \quad \text{if} \quad s \leq 0.$$

We shall write

$$g_{11} = 1, \quad g_{22} = f^2, \quad g_{12} = 0 = g_{21}.$$

Notice that at ∂M, g is the Euclidean metric (i.e. $f(x, 0) = 1$). So we have obtained at least a continuous metric on the double. However, if we assume that the original metric is smooth we can show the following:

Proposition B.1. *Assume that $k_g = 0$, then the metric on the double is at least $C^{2,1}$. Here $C^{2,1}$ is the space of functions twice differentiable and whose second derivatives are locally (i.e., in the chart) Lipschitz continuous.*

Proof. Let (U,φ) be a Fermi chart around a point $p\in\partial M$. Then the metric is written as a 2×2 positive definite matrix g_{ij}, as shown above. All we must prove is that when doubled, each of these functions is $C^{2,1}$. The fact that $k_g=0$ implies that the metric is C^2 (see [16], the argument in pp 43-44), and as we already know that the g_{ij}'s are smooth away from ∂M (being the original metric smooth), the proposition will follow from Lemma B.1. $\square$

Before stating and proving Lemma B.1, let us introduce some notation. Given a point $x=\left(x^1,x^2\right)$, define

$$\|x\|=\sqrt{\left(x^1\right)^2+\left(x^2\right)^2},$$

and for a fixed $r>0$ we let

$$B_+=\left\{x\,|\,\|x\|<r,\quad x^2\geq 0\right\},$$

and in the same way define B_-. Given $x=\left(x^1,x^2\right)$ define $x^*=\left(x^1,-x^2\right)$. For a pair of functions $f_+:B_+\longrightarrow\mathbb{R}$ and $f_-:B_-\longrightarrow\mathbb{R}$ which coincide over $\left\{x^2=0\right\}$ define

$$f_+\cup f_-:B_+\cup B_-\longrightarrow\mathbb{R}$$

as

$$f_+\cup f_-\left(x\right)=f_\pm\left(x\right)\quad\text{if}\quad x\in B_\pm.$$

Then we have the following elementary lemma.

Lemma B.1. *Let $f_+:B_+\longrightarrow\mathbb{R}$ and $f_-:B_-\longrightarrow\mathbb{R}$ be Lipschitz . Asume that $f_+=f_-$ on $B_+\cap\left\{x^2=0\right\}$. Then $f_+\cup f_-$ is also Lipschitz.*

Proof. Let $x=\left(x^1,x^2\right)\in B_-$ and $y=\left(y^1,y^2\right)\in B_+$. Then we have the following inequalities

$$\begin{aligned}
\left|\tilde f\left(x\right)-\tilde f\left(y\right)\right| &= \left|\tilde f\left(x^1,x^2\right)-\tilde f\left(y^1,y^2\right)\right|\\
&= \left|f\left(x^1,0\right)-f\left(x^1,x^2\right)\right|+\left|f\left(x^1,0\right)-f\left(y^1,0\right)\right|\\
&\quad+\left|f\left(y^1,0\right)-f\left(y^1,y^2\right)\right|\\
&\leq C\left|x^2\right|+C\left|y^2\right|+C\left|y^1-x^1\right|\\
&= C\left(-x^2+y^2\right)+C\left|x^1-y^1\right|\\
&= C\left|y^2-x^2\right|+C\left|x^1-y^1\right|\\
&\leq 2C\left\|y-x\right\|.
\end{aligned}$$

$\square$

Regarding the procedure and regularity results described above, the reader is advised to consult the references [28, 35] (again, we thank the referee who pointed out these references).

B.1. An application: doubling and regularity for the Ricci flow

Let g be a solution to the Ricci flow on $M \times (A, B)$ with $k_g \equiv 0$. Let $\tilde{M}$ be the double of M, then from the solution g we obtain in an obvious way a solution $\tilde{g}$ of the Ricci flow on $\tilde{M} \times (A, B)$. Let us show that $\tilde{g}$ is smooth. Pick $t_0 \in (A, B)$; by Proposition B.1 we know that $\tilde{g}\,(t_0)$ is $C^{2,1}$, and smooth away from ∂M (in this case we shall refer as ∂M to the subset of $\tilde{M}$ corresponding to the equivalence class of $\partial M \times \{0\}$).

Consider $p \in \partial M \subset \tilde{M}$. Let $R_{\tilde{g}(t_0)}$ be the curvature of the double at time $t = t_0$. By our considerations, $R_{\tilde{g}(t_0)}$ is Lipschitz. Then the equation

$$\Delta_{\tilde{g}(t_0)} u_0 = R_{\tilde{g}(t_0)},$$

has a $C^{2,\alpha}$ solution on a perhaps even smaller neighbourhood of p (see [15, Theorem 2.3]). Notice that $e^{u_0}\tilde{g}\,(0)$ is flat, and hence in a coordinate system around p we have that $e^{u_0}\tilde{g}\,(0)$ can be written as the Euclidean metric, that we will denote again by g_E (another way of proving this is by using the existence of isothermal coordinates, see [6]). Hence, we have a solution to the Ricci flow $\tilde{g}$ with initial condition $\tilde{g}\,(t_0)$, so we have a solution, in a small neighbourhood of p, to (writing $\tilde{g} = e^u g_E$)

$$\frac{\partial u}{\partial t} = e^{-u}\Delta_{g_E} u, \quad u\,(\cdot, t_0) = u_0,$$

and u_0 is at least $C^{2,\alpha}$, for any $0 < \alpha < 1$. Therefore by parabolic regularity ([26, Theorem 10.1, Chapter IV]), u is smooth, and so is the metric $\tilde{g} = e^u g_E$, since g_E is smooth. This means that the solution to the Ricci flow becomes smooth for $t > t_0$. This justifies, at least in the case of surfaces the doubling procedure: the solution to the Ricci flow becomes smooth instantaneously, so all the results on long time behaviour proved for smooth solutions apply. In particular, if we double an ancient solution along its totally geodesic boundary, we obtain a smooth ancient solution to the Ricci flow.

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Departamento de Matemáticas, Universidad de los Andes
Bogotá DC, Colombia
E-mail address: jcortiss@uniandes.edu.co
E-mail address: a.murcia256@uniandes.edu.co

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