Communications in
Analysis and Geometry
Volume 27, Number 2, 465–489, 2019

Inverse curvature flow in anti-de Sitter-Schwarzschild manifold

Siyuan Lu

In this paper, we consider inverse hessian quotient curvature flow with star-shaped initial hypersurface in anti-de Sitter-Schwarzschild manifold. We prove that the solution exists for all time and the second fundamental form converges to identity exponentially fast.

1. Introduction

Curvature flows of compact hypersurfaces in Riemannian manifolds have been extensively studied in the last 30 years. In the case of Euclidean space, for contracting flow, Huisken [13] considered

$$\dot{X} = -H\nu, \tag{1.1}$$

where H is the mean curvature and ν is the outward unit normal of the hypersurface. He proved that the solution exists for all time and the normalized flow converges to a round sphere if the initial hypersurface is convex.

This result was later generalized by Andrews [1] for a large class of curvature flow. More specifically, Andrews considered

$$\dot{X} = -F\nu, \tag{1.2}$$

where F is a concave function of homogeneous degree one, evaluated at the principal curvature.

For expanding flow, Gerhardt [7] and Urbas [20] considered

$$\dot{X} = \frac{\nu}{F}, \tag{1.3}$$

where F is a concave function of homogeneous degree one, evaluated at the principal curvature. They proved that the solution exists for all time and

Research of the author was supported in part by Schulich Graduate Fellowship.

the normalized flow converges to a round sphere if the initial hypersurface is star-shaped and lies in a certain convex cone.

A natural question is whether these results remain true if the ambient space is no longer Euclidean space. For contraction flow (1.1) and (1.2), Huisken [14] and Andrews [2] generalized their results to certain ambient space respectively.

The case of expanding flow (1.3) is in fact more subtle as the assumption on initial hypersurface is weaker. In the case of space form, Gerhardt [8, 9] proved the solution exists for all time and the second fundamental form converges in hyperbolic space and sphere space, see also earlier work by Ding [6]. More recently, Brendle-Hung-Wang [3] and Scheuer [19] proved that the same results hold in anti-de Sitter-Schwarzschild manifold and a class of warped product manifold for inverse mean curvature flow, which is

$$\dot{X} = \frac{\nu}{H}. \tag{1.4}$$

However, as pointed out by Neves [17] and Hung-Wang [15], for inverse mean curvature flow, the rescaled hypersurface is not necessarily a round sphere in anti-de Sitter-Schwarzschild manifold and in hyperbolic space.

Inverse curvature flows can be used to prove various inequalities. Guan-Li [10] generalized Alexandrov-Fenchel inequalities for star-shaped k-convex hypersurface in Euclidean space using inverse curvature flow (1.3) in Euclidean space. Recently, Brendle-Hung-Wang [3] generalized Alexandrov-Fenchel inequality for $k = 1$ (which they call Minkowski inequality) in anti-de Sitter-Schwarzschild manifold by inverse mean curvature flow (1.4). The inequality was further used to prove a Penrose inequality in General Relativity in [4]. More recently, Li-Wei-Xiong [16] and Ge-Wang-Wu[12] generalized the hyperbolic Alexandrov-Fenchel inequality using inverse curvature flow (1.3) in hyperbolic space.

Motivated by the results above, we consider inverse curvature flow in anti-de Sitter-Schwarzschild manifold. The anti-de Sitter-Schwarzschild manifold is a manifold $N = \mathbb{S}^n \times [s_0, \infty)$ equipped with the following Riemannian metric

$$\bar{g} = \frac{1}{1 - ms^{1-n} + s^2} ds^2 + s^2 g_{\mathbb{S}^n},$$

where s_0 is the unique positive solution of the equation $1 - ms^{1-n} + s^2 = 0$. By a change of variable, we have

$$\bar{g} = dr^2 + \phi^2(r) g_{\mathbb{S}^n},$$

where ϕ satisfies $\phi' = \sqrt{1 - m\phi^{1-n} + \phi^2}$.

The anti-de Sitter-Schwarzschild manifold is thus a special case of warped product manifolds. Moreover, the sectional curvature of $(N, \bar{g})$ approach -1 near infinity exponentially fast and the scalar curvature is of constant $-n(n+1)$. This feature will play an essential role in the proof of our theorem.

To state our theorem, we need the following definition of Garding's Γ_k cone $\Gamma_k = \{(\kappa_i) \in \mathbb{R}^n | \sigma_j > 0, 0 \leq j \leq k\}$, where σ_j is the j-th elementary symmetric function. We say a hypersurface is k-convex if the principal curvature $(\kappa_i) \in \Gamma_k$.

We now state our main theorem:

Theorem 1.1. *Let Σ_0^n be a star-shaped, k-convex closed hypersurface in N^{n+1}, where N^{n+1} is an anti-de Sitter-Schwarzschild manifold, consider the evolution equation*

$$\dot{X} = \frac{\nu}{F}, \tag{1.5}$$

where ν is the outward unit normal and $F = n\frac{C_n^{k-1}}{C_n^k}\frac{\sigma_k}{\sigma_{k-1}}$ which is evaluated at the principal curvature of Σ_t. Then the solution exists for all time t, and the second fundamental form satisfies

$$|h_j^i - \delta_j^i| \leq Ce^{-\frac{2}{n}t},$$

where C depends on the Σ_0, n, k.

The organization of the paper is as follows: in section 2, we give some preliminaries about warped product space and anti-de Sitter-Schwarzschild manifold, we also prove the C^0 estimate. In section 3, we derive the evolution equations and give the C^1 estimate. In section 4 and 5, we estimate the bound for F and the principal curvature respectively. In section 6, we prove that the second fundamental form converges to identity.

After submitting the paper, we have learned that Chen-Mao [5] independently proved the main theorem above.

2. Preliminaries

In this section, we give some basic properties of hypersurface in warped product space. Let N^{n+1} be a warped product space, with the metric

$$g^N := ds^2 = dr^2 + \phi^2(r)\sigma_{ij}, \tag{2.1}$$

where σ_{ij} is the standard metric of $\mathbb{S}^n$.

Define

$$\Phi(r) = \int_0^r \phi(\rho)d\rho, \quad V = \phi(r)\frac{\partial}{\partial r}.$$

We state some well-known lemmas, see [11] with some modification.

Lemma 2.1. *The vector field V satisfies $D_i V_j = \phi'(r) g^N_{ij}$, where D is the covariant derivative with respect to the metric g^N.*

Lemma 2.2. *Let Σ^n be a closed hypersurface in N^{n+1} with induced metric g, then $\Phi|_\Sigma$ satisfies,*

$$\nabla_i \nabla_j \Phi = \phi'(r) g_{ij} - h_{ij} \langle V, \nu \rangle,$$

where ∇ is the covariant derivative with respect to g, ν is the outward unit normal and h_{ij} is the second fundamental form of the hypersurface.

We now state the Gauss and Codazzi equation,

$$R_{ijkl} = \bar{R}_{ijkl} + (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.2}$$

$$\nabla_k h_{ij} - \nabla_j h_{ik} = \bar{R}_{\nu ijk}, \tag{2.3}$$

and the interchanging formula

$$\begin{aligned} \nabla_i \nabla_j h_{kl} = \nabla_k \nabla_l h_{ij} - h_l^m (h_{im} h_{kj} - h_{ij} h_{mk}) - h_j^m (h_{mi} h_{kl} - h_{il} h_{mk}) \\ + h_l^m \bar{R}_{ikjm} + h_j^m \bar{R}_{iklm} + \nabla_k \bar{R}_{ilj\nu} + \nabla_i \bar{R}_{jkl\nu}. \end{aligned} \tag{2.4}$$

Define the support function $u = \langle V, \nu \rangle$, and we have

Lemma 2.3.

$$\begin{aligned} \nabla_i u &= g^{kl} h_{ik} \nabla_l \Phi, \\ \nabla_i \nabla_j u &= g^{kl} \nabla_k h_{ij} \nabla_l \Phi + \phi' h_{ij} - (h^2)_{ij} u + g^{kl} \nabla_l \Phi \bar{R}_{\nu jki}, \end{aligned}$$

where $(h^2)_{ij} = g^{kl} h_{ik} h_{jl}$ and $\bar{R}_{\nu jki}$ is the curvature of ambient space.

Proof. We only need to prove the equality at one point, thus we have $g_{ij} = \delta_{ij}$ and $\nabla_i u = D_i \langle V, \nu\rangle = \langle D\Phi, D_i\nu\rangle = h_{ik}D_k\Phi$.

$$\begin{aligned}\nabla_i\nabla_j u &= \nabla_i h_{jk}\nabla_k\Phi + h_{jk}\nabla_i\nabla_k\Phi \\ &= \nabla_i h_{jk}\nabla_k\Phi + h_{jk}(\phi' g_{ik} - h_{ik}u) \\ &= \left(\nabla_k h_{ij} + \bar{R}_{\nu jki}\right)\nabla_k\Phi + \phi' h_{ij} - (h^2)_{ij}u,\end{aligned}$$

where Codazzi equation (2.3) is used in the last equality, thus by the tensorial property, we have the lemma. □

As to the curvature, we have the following curvature estimates, for proof, we refer readers to [3].

Lemma 2.4. *The sectional curvature satisfies*

$$\begin{aligned}\bar{R}(\partial_i, \partial_j, \partial_k, \partial_l) &= \phi^2\left(1 - \phi'^2\right)(\sigma_{ik}\sigma_{jl} - \sigma_{il}\sigma_{jk}), \\ \bar{R}(\partial_i, \partial_r, \partial_j, \partial_r) &= -\phi\phi''\sigma_{ij},\end{aligned}$$

where ∂_i is the standard frame on $\mathbb{S}^n$ and σ_{ij} is the standard metric of $\mathbb{S}^n$.

Now, back to our case that N is an anti-de Sitter-Schwarzschild manifold.

Lemma 2.5. *Let N be an anti-de Sitter-Schwarzschild manifold, we have*

$$\phi(r) = \sinh(r) + \frac{m}{2(n+1)}\sinh^{-n}(r) + O(\sinh^{-n-2}(r)), \tag{2.5}$$

and

$$\begin{aligned}\bar{R}_{\alpha\beta\gamma\mu} &= -\delta_{\alpha\gamma}\delta_{\beta\mu} + \delta_{\alpha\mu}\delta_{\beta\gamma} + O(e^{-(n+1)r}), \\ \bar{\nabla}_\rho\bar{R}_{\alpha\beta\gamma\mu} &= O(e^{-(n+1)r}),\end{aligned}$$

where $\{e_\alpha\}$ is an orthonormal frame in N.

We also need the following two lemmas regarding to σ_k. These two lemmas are well known, for completeness, we add the proof here.

Lemma 2.6. *Let $F = n\frac{C_n^{k-1}}{C_n^k}\frac{\sigma_k}{\sigma_{k-1}}$ and $(\lambda_i) \in \Gamma_k$, we have*

$$\sum_i F^{ii}\lambda_i^2 \geq \frac{F^2}{n}.$$

Proof. We first consider the term $\sigma_l^{ii}\lambda_i^2$, we have

$$\sigma_l^{ii}\lambda_i^2 = \sigma_1\sigma_l - (l+1)\sigma_{l+1}. \tag{2.6}$$

Let $G = \frac{\sigma_k}{\sigma_{k-1}}$, by (2.6) and Newton-MacLaurin inequality, we have

$$\begin{aligned}
\sum_i G^{ii}\lambda_i^2 &= \sum_i \left(\frac{\sigma_k^{ii}}{\sigma_{k-1}} - \frac{\sigma_k\sigma_{k-1}^{ii}}{\sigma_{k-1}^2}\right)\lambda_i^2 \\
&= \frac{\sigma_1\sigma_k - (k+1)\sigma_{k+1}}{\sigma_{k-1}} - \frac{\sigma_k\left(\sigma_1\sigma_{k-1} - k\sigma_k\right)}{\sigma_{k-1}^2} \\
&= \frac{k\sigma_k^2 - (k+1)\sigma_{k-1}\sigma_{k+1}}{\sigma_{k-1}^2} \\
&\geq \frac{k\sigma_k^2}{(n-k+1)\sigma_{k-1}^2} \\
&= \frac{C_n^{k-1}}{C_n^k}\left(\frac{\sigma_k}{\sigma_{k-1}}\right)^2.
\end{aligned}$$

Thus

$$\sum_i F^{ii}\lambda_i^2 \geq n\left(\frac{C_n^{k-1}}{C_n^k}\right)^2\left(\frac{\sigma_k}{\sigma_{k-1}}\right)^2 = \frac{F^2}{n}.$$

□

Lemma 2.7. *Let $F = n\frac{C_n^{k-1}}{C_n^k}\frac{\sigma_k}{\sigma_{k-1}}$ and $(\lambda_i) \in \Gamma_k$, we have*

$$n \leq \sum_i F^{ii} \leq nk.$$

Proof. Let $G = \frac{\sigma_k}{\sigma_{k-1}}$, we have

$$\begin{aligned}
\sum_i G^{ii} &= \sum_i \left(\frac{\sigma_k^{ii}}{\sigma_{k-1}} - \frac{\sigma_k\sigma_{k-1}^{ii}}{\sigma_{k-1}^2}\right) \\
&= (n-k+1) - (n-k+2)\frac{\sigma_k\sigma_{k-2}}{\sigma_{k-1}^2} \\
&\geq \frac{n-k+1}{k},
\end{aligned}$$

by Newton-Maclaurin inequality.

For the second inequality,

$$\sum_i G^{ii} = \sum_i \left(\frac{\sigma_k^{ii}}{\sigma_{k-1}} - \frac{\sigma_k \sigma_{k-1}^{ii}}{\sigma_{k-1}^2} \right)$$
$$= (n-k+1) - (n-k+2)\frac{\sigma_k \sigma_{k-2}}{\sigma_{k-1}^2}$$
$$\leq n-k+1,$$

as $(\lambda_i) \in \Gamma_k$. The lemma then follows. □

Since the initial hypersurface is star-shaped, we can consider it as a graph on $\mathbb{S}^n$, i.e. $X = (x, r)$ where x is the coordinate on $\mathbb{S}^n$, r is the radius, by taking derivatives, we have

$$X_i = \partial_i + r_i \partial_r, \qquad (2.7)$$
$$g_{ij} = r_i r_j + \phi^2 \sigma_{ij},$$

and

$$\nu = \frac{1}{v}\left(-\frac{r^i}{\phi^2}\partial_i + \partial_r \right), \qquad (2.8)$$

where ν is the unit normal vector, $v = (1 + \frac{|\nabla r|^2}{\phi^2})^{\frac{1}{2}}$, note that all the derivatives are on $\mathbb{S}^n$.

Thus

$$\frac{dr}{dt} = \frac{1}{Fv}, \quad \dot{x}^i = -\frac{r^i}{\phi^2 F v},$$

and we have

$$\frac{\partial r}{\partial t} = \frac{dr}{dt} - r_j \dot{x}^j = \frac{v}{F}. \qquad (2.9)$$

By a direct computation, c.f. (2.6) in [6] we have

$$h_{ij} = \frac{1}{v}\left(-r_{ij} + \phi\phi' \sigma_{ij} + \frac{2\phi' r_i r_j}{\phi} \right). \qquad (2.10)$$

Now we consider a function

$$\varphi = \int_{r_0}^{r} \frac{1}{\phi} \qquad (2.11)$$

We have

$$\varphi_i = \frac{r_i}{\phi}, \quad \varphi_{ij} = \frac{r_{ij}}{\phi} - \frac{\phi' r_i r_j}{\phi^2}. \tag{2.12}$$

If we write everything in terms of φ, we have

$$\frac{\partial \varphi}{\partial t} = \frac{v}{\phi F}, \quad v = (1 + |D\varphi|^2)^{\frac{1}{2}}, \tag{2.13}$$

and

$$g_{ij} = \phi^2(\varphi_i \varphi_j + \sigma_{ij}), \quad g^{ij} = \phi^{-2}\left(\sigma^{ij} - \frac{\varphi^i \varphi^j}{v^2}\right). \tag{2.14}$$

Moreover,

$$\begin{aligned} h_{ij} &= \frac{\phi}{v}\left(\phi'(\sigma_{ij} + \varphi_i\varphi_j) - \varphi_{ij}\right), \\ h^i_j &= g^{ik}h_{kj} = \frac{\phi'}{\phi v}\delta^i_j - \frac{1}{\phi v}\tilde{\sigma}^{ik}\varphi_{kj}, \end{aligned} \tag{2.15}$$

where $\tilde{\sigma}^{ij} = \sigma^{ij} - \frac{\varphi^i\varphi^j}{v^2}$.

We now give the C^0 estimate.

Lemma 2.8. *Let $\bar{r}(t) = \sup_{\mathbb{S}^n} r(\cdot, t)$ and $\underline{r}(t) = \inf_{\mathbb{S}^n} r(\cdot, t)$, then we have*

$$\begin{aligned} \phi(\bar{r}(t)) &\le e^{t/n}\phi(\bar{r}(0)), \\ \phi(\underline{r}(t)) &\ge e^{t/n}\phi(\underline{r}(0)). \end{aligned} \tag{2.16}$$

Proof. Recall that $\frac{\partial r}{\partial t} = \frac{v}{F}$, where F is a normalized operator on (h^i_j). At the point where the function $r(\cdot, t)$ attains its maximum, we have $\nabla r = 0, (r_{ij}) \le 0$, from (2.12), we deduce that $\nabla\varphi = 0, (\varphi_{ij}) \le 0$ at the maximum point. From (2.15), we have $(h^i_j) \ge \left(\frac{\phi'}{\phi}\delta^i_j\right)$, where we may assume (g_{ij}) and (h_{ij}) is diagonalized if necessary. Since F is homogeneous of degree 1, and $F(1, \cdots, 1) = n$, we have

$$v^2 = 1 + |\nabla\varphi|^2 = 1, \quad F(h^i_j) \ge \frac{\phi'}{\phi}F(\delta^i_j) = \frac{n\phi'}{\phi}.$$

Thus

$$\frac{d}{dt}\bar{r}(t) \le \frac{\phi(\bar{r}(t))}{n\phi'(\bar{r}(t))},$$

i.e.

$$\frac{d}{dt}\log\phi(\bar{r}(t)) \leq \frac{1}{n},$$

which yields the first inequality. Similarly, we can prove the second inequality. The lemma is now proved. □

3. Evolution equations and C^1 estimate

Before we go on with the estimate, let us derive some evolution equations first.

$$\dot{g}_{ij} = \frac{2h_{ij}}{F}, \quad \dot{\nu} = \frac{g^{ij}F_i e_j}{F^2}, \tag{3.1}$$

$$\dot{h}^i_j = -\frac{1}{F}h^i_k h^k_j - \nabla^i\nabla_j\left(\frac{1}{F}\right) - \frac{1}{F}\bar{R}^i_{\nu j\nu}. \tag{3.2}$$

Together with the interchanging formula (2.4), we have

$$\begin{aligned}\dot{h}^i_j = &-\frac{1}{F}h^i_k h^k_j + \frac{F^{pq,rs}h_{pq}{}^i h_{rsj}}{F^2} - \frac{2F^{pq}h_{pq}{}^i F^{rs}h_{rsj}}{F^3} - \frac{1}{F}\bar{R}^i_{\nu j\nu} \\ &+ \frac{g^{ki}F^{pq}}{F^2}\Big(h_{kj,pq} - h^m_q(h_{km}h_{pj} - h_{kj}h_{mp}) \\ &\qquad - h^m_j(h_{mk}h_{pq} - h_{kq}h_{mp}) \\ &\qquad + h^m_q\bar{R}_{kpjm} + h^m_j\bar{R}_{kpqm} + \nabla_p\bar{R}_{kqj\nu} + \nabla_k\bar{R}_{jpq\nu}\Big),\end{aligned} \tag{3.3}$$

where $F^{ij} = \frac{\partial F}{\partial h_{pq}}$ and $F^{pq,rs} = \frac{\partial^2 F}{\partial h_{pq}\partial h_{rs}}$.

For support function $u = \langle \phi\partial_r, \nu\rangle = \frac{\phi}{v}$, we have

$$\dot{u} = \frac{\phi'}{F} + \frac{\phi g^{ij}F_i r_j}{F^2}. \tag{3.4}$$

Now, we need to consider the curvature term. By Lemma 2.4, (2.7) and (2.8), we have

$$\begin{aligned}\bar{R}_{k\nu j\nu} = &\left(\frac{1}{v^2}\delta_{kj} + \frac{2r_k r_j}{\phi^2 v^2} + \frac{r_k r_j|\nabla r|^2}{\phi^4 v^2}\right)(-\phi\phi'') \\ &+ \frac{(|\nabla r|^2\delta_{kj} - r_k r_j)}{\phi^2 v^2}(1 - \phi'^2),\end{aligned} \tag{3.5}$$

$$\bar{R}_{\nu jnk} = \frac{r_n \delta_{jk}}{v}\left(-\phi\phi'' - (1-(\phi')^2)\right) + \frac{r_k \delta_{jn}}{v}\left(\phi\phi'' + (1-\phi'^2)\right).$$

Note that $g^{mn} = \phi^{-2}\left(\sigma_{mn} - \frac{r^m r^n}{v^2\phi^2}\right)$, thus

$$g^{mn}\nabla_m \Phi \bar{R}_{\nu jnk} = \left(\frac{|\nabla r|^2 \delta_{jk} - r_j r_k}{\phi v^3}\right)\left(-\phi\phi'' - (1-\phi'^2)\right). \tag{3.6}$$

Lemma 3.1. *Along the flow, $|\dot{\varphi}| \leq C$, where C only depends on Σ_0, n, k.*

Proof. By (2.13) and (2.15), we have

$$\frac{\partial \varphi}{\partial t} = \frac{v^2}{F(\phi'\delta_{ij} - \tilde{\sigma}^{ik}\varphi_{kj})} = \frac{1}{G}.$$

Let $G^{ij} = \frac{\partial G}{\partial \varphi_{ij}}$, $G^k = \frac{\partial G}{\partial \varphi_k}$, then

$$G^{ij} = -\frac{1}{v^2} F^i_l \tilde{\sigma}^{lj}.$$

Thus

$$\frac{\partial \dot{\varphi}}{\partial t} = -\frac{\dot{G}}{G^2} = \frac{1}{v^2 G^2}\left(F^i_l \tilde{\sigma}^{lj}\dot{\varphi}_{ij} - v^2 G^k \dot{\varphi}_k - F^i_i \phi\phi'' \dot{\varphi}\right).$$

By maximum principle, we conclude that $|\dot{\varphi}|$ is bounded above. □

Lemma 3.2. *Along the flow, $|\nabla\varphi| \leq C$, where C only depends on Σ_0, n, k. In addition, if F is bounded above, we have $|\nabla\varphi| \leq Ce^{-\alpha t}$, where α only depends on* $\sup F$ *and n.*

Proof. By (2.13) and (2.15), we have

$$\frac{\partial \varphi}{\partial t} = \frac{v^2}{F(\phi'\delta_{ij} - \tilde{\sigma}^{ik}\varphi_{kj})} = \frac{1}{G}.$$

Let $G^{ij} = \frac{\partial G}{\partial \varphi_{ij}}$, $G^k = \frac{\partial G}{\partial \varphi_k}$, then

$$G^{ij} = -\frac{1}{v^2} F^i_l \tilde{\sigma}^{lj}.$$

Let $\omega = \frac{1}{2}|\nabla\varphi|^2$, we have

$$\frac{\partial \omega}{\partial t} = -\frac{\varphi^k}{G^2}\nabla_k G = \frac{1}{v^2 G^2}\left(F^i_l \tilde{\sigma}^{lj}\varphi^k \varphi_{ijk} - v^2 G^k \omega_k - 2F^i_i \phi\phi'' \omega\right).$$

We want to write the term $\tilde{\sigma}^{lj}\varphi_{ijk}$ in terms of second derivative of ω. Note that

$$\begin{aligned}\omega_{ij} &= \varphi_{kij}\varphi^k + \varphi_{ki}\varphi^k_j \\ &= \varphi_{ijk}\varphi^k + (\sigma_{ij}\sigma_{kp} - \sigma_{ik}\sigma_{jp})\varphi^p\varphi^k + \varphi_{ki}\varphi^k_j \\ &= \varphi_{ijk}\varphi^k + \sigma_{ij}|\nabla\varphi|^2 - \varphi_i\varphi_j + \varphi_{ki}\varphi^k_j,\end{aligned}$$

and

$$\tilde{\sigma}^{lj}\left(\sigma_{ij}|\nabla\varphi|^2 - \varphi_i\varphi_j\right) = \delta^l_i|\nabla\varphi|^2 - \varphi_i\varphi^l.$$

Thus we have

$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{1}{v^2G^2}\Big(F^i_l\tilde{\sigma}^{lj}\omega_{ij} - F^i_i|\nabla\varphi|^2 + F^i_l\varphi_i\varphi^l - v^2G^k\omega_k - 2F^i_i\phi\phi''\omega\Big) \\ &\quad - \frac{1}{v^2G^2}F^i_l\tilde{\sigma}^{lj}\varphi_{ki}\varphi^k_j.\end{aligned}$$

Note that $-F^i_i|\nabla\varphi|^2 + F^i_l\varphi_i\varphi^l \le 0$ and $-F^i_l\tilde{\sigma}^{lj}\varphi_{ki}\varphi^k_j \le 0$, thus by the maximum principle, we have

$$\omega(\cdot,t) \le \sup\omega_0.$$

More precisely, if $F \le C$, consider the test function $\tilde{\omega} = \omega e^{\lambda t}$, thus at the maximum point of $\tilde{\omega}$, we have

$$\begin{aligned}0 &\le \frac{\partial\omega}{\partial t}e^{\lambda t} + \lambda\omega e^{\lambda t} \le \omega e^{\lambda t}\left(\frac{-2F^i_i\phi\phi''}{v^2G^2} + \lambda\right) \\ &= \omega e^{\lambda t}\left(\frac{-2F^i_i(h^i_j)\phi''}{\phi F^2(h^i_j)} + \lambda\right) \\ &\le \omega e^{\lambda t}\left(\frac{-2n\phi''}{\phi F^2(h^i_j)} + \lambda\right) \le 0,\end{aligned}$$

if $0 < \lambda \le \frac{2n}{\sup^2 F} \le \frac{2n\phi''}{\phi\sup^2 F}$, we have used Lemma 2.7 in last line. By maximum principle,

$$|\nabla\varphi| \le Ce^{-\alpha t},$$

where $0 < \alpha \le \frac{n}{\sup^2 F}$. □

4. Bound for F

Lemma 4.1. *Along the flow, $F \le C$, where C only depends on Σ_0, n, k.*

Proof. By (3.2), we have

$$\begin{aligned}\dot{F} &= F_i^j \left(-\frac{1}{F} h_k^i h_j^k - \nabla^i \nabla_j \left(\frac{1}{F}\right) - \frac{1}{F} \bar{R}^i_{\nu j \nu}\right) \\ &= F_i^j \left(-\frac{1}{F} h_k^i h_j^k + \frac{\nabla^i \nabla_j F}{F^2} - 2\frac{\nabla^i F \nabla_j F}{F^3} - \frac{1}{F} \bar{R}^i_{\nu j \nu}\right).\end{aligned}$$

By Lemma 2.6, we have

$$\dot{F} \le -\frac{F}{n} + F_i^j \left(\frac{\nabla^i \nabla_j F}{F^2} - 2\frac{\nabla^i F \nabla_j F}{F^3} - \frac{1}{F} \bar{R}^i_{\nu j \nu}\right).$$

By Lemma 2.4, we know that $\bar{R}^i_{\nu j \nu}$ is uniformly bounded, together with Lemma 2.7, we have

$$-F_i^j \bar{R}^i_{\nu j \nu} \le C \sum_i F^{ii} \le C.$$

It follows that

$$\dot{F}^2_{max} \le -\frac{2}{n} F^2_{max} + C,$$

which gives

$$F^2_{max} \le C.$$

□

Lemma 4.2. *Along the flow, $F \ge c$, where c only depends on Σ_0, n, k.*

Proof. Consider the function $-\log F - \log \tilde{u}$, where $\tilde{u} = u e^{-t/n}$, by Lemma 2.8, $\tilde{u}$ is uniformly bounded. At the maximum point, we have

$$\begin{aligned}&-\frac{F_i}{F} - \frac{u_i}{u} = 0, \quad -\frac{F_{ij}}{F} + \frac{F_i F_j}{F^2} - \frac{u_{ij}}{u} + \frac{u_i u_j}{u^2} \le 0, \\ &-\frac{F_i^j}{F} \dot{h}_j^i - \frac{\dot{u}}{u} + \frac{1}{n} \ge 0.\end{aligned}$$

By (3.2), (3.4) and the critical equation, we have

$$
\begin{aligned}
0 &\le -\frac{F_i^j}{F}\left(-\frac{1}{F}h_k^i h_j^k - \nabla^i\nabla_j\left(\frac{1}{F}\right) - \frac{1}{F}\bar{R}^i_{\nu j\nu}\right) - \frac{\phi'}{Fu} - \frac{\phi g^{ij}F_i r_j}{F^2 u} + \frac{1}{n}\\
&= \frac{F_i^j}{F^2}\left(h_k^i h_j^k + \bar{R}^i_{\nu j\nu}\right) + \frac{g^{ki}F_i^j}{F^2}\left(-\frac{F_{kj}}{F} + 2\frac{F_k F_j}{F^2}\right) - \frac{\phi'}{Fu} - \frac{\phi g^{ij}F_i r_j}{F^2 u} + \frac{1}{n}\\
&\le \frac{F_i^j}{F^2}\left(h_k^i h_j^k + \bar{R}^i_{\nu j\nu}\right) + \frac{g^{ki}F_i^j}{F^2}\frac{u_{kj}}{u} - \frac{\phi'}{Fu} - \frac{\phi g^{ij}F_i r_j}{F^2 u} + \frac{1}{n},
\end{aligned}
$$

By Lemma 2.3, we have

$$
\begin{aligned}
0 \le{}& \frac{F_i^j}{F^2}\left(h_k^i h_j^k + \bar{R}^i_{\nu j\nu}\right)\\
&+ \frac{g^{ki}F_i^j}{F^2 u}\left(g^{mn}h_{kjm}\phi r_n + \phi' h_{kj} - (h^2)_{kj}u + g^{mn}\nabla_m\Phi\bar{R}_{\nu jnk}\right)\\
&- \frac{\phi'}{Fu} - \frac{\phi g^{ij}F_i r_j}{F^2 u} + \frac{1}{n}\\
={}& \frac{F_i^j\bar{R}^i_{\nu j\nu}}{F^2} + \frac{g^{ki}F_i^j}{F^2 u}g^{mn}\nabla_m\Phi\bar{R}_{\nu jnk} + \frac{1}{n},
\end{aligned}
$$

By (3.5) and (3.6), we have

$$
\begin{aligned}
0 \le{}& \frac{g^{ki}F_i^j}{F^2}\Bigg(\left(\frac{1}{v^2}\delta_{kj} + \frac{2r_k r_j}{v^2\phi^2} + \frac{r_k r_j|\nabla r|^2}{v^2\phi^4}\right)(-\phi\phi'')\\
&+ \frac{(|\nabla r|^2\delta_{kj} - r_k r_j)}{v^2\phi^2}(1-\phi'^2)\\
&+ \left(\frac{|\nabla r|^2\delta_{jk} - r_j r_k}{v^2\phi^2}\right)\left(-\phi\phi'' - (1-\phi'^2)\right)\Bigg) + \frac{1}{n}\\
={}& \frac{g^{ki}F_i^j}{F^2}\left(\delta_{kj} + \frac{r_k r_j}{\phi^2}\right)(-\phi\phi'') + \frac{1}{n}\\
\le{}& -\frac{g^{ij}F_i^j}{F^2}\phi\phi'' + \frac{1}{n} \le -\frac{C}{F} + \frac{1}{n},
\end{aligned}
$$

we have used the Lemma 2.7 in last line. Now we conclude that F is bounded below. □

Remark 4.3. For the lower bound, we only need the first inequality of Lemma 2.7, which is satisfied by a class of concave functions with homogeneous degree one, for example $F = \sigma_k^{1/k}$.

5. Bound for principal curvature

Lemma 5.1. *Along the flow, $|\kappa_i| \le C$, where κ_i is the principal curvature of Σ_t and C only depends on Σ_0, n, k.*

Proof. Define $\tilde{u} = ue^{-t/n}$, consider the test function $\log(\eta) - \log(\tilde{u})$, where

$$\eta = \sup\{h_{ij}\xi^i\xi^j : g_{ij}\xi^i\xi^j = 1\}.$$

Without loss of generality, we suppose that at the maximum point $\eta = h_1^1$, and we have

$$\frac{\dot{h}_1^1}{h_1^1} - \frac{\dot{u}}{u} + \frac{1}{n} \ge 0, \tag{5.1}$$

and

$$\frac{h_{1i}^1}{h_1^1} - \frac{u_i}{u} = 0, \quad \frac{h_{1ij}^1}{h_1^1} \le \frac{u_{ij}}{u}. \tag{5.2}$$

By (3.3), (3.4) and the critical equation, we have

$$\begin{aligned} 0 \le \frac{1}{h_1^1}\Bigg(&- \frac{1}{F}h_k^1 h_1^k + \frac{F^{pq,rs}h_{pq}{}^1 h_{rs1}}{F^2} - \frac{2F^{pq}h_{pq}{}^1 F^{rs}h_{rs1}}{F^3} - \frac{1}{F}\bar{R}_{\nu 1\nu}^1 \\ &+ \frac{g^{k1}F^{pq}}{F^2}\big(h_{k1,pq} - h_q^m(h_{km}h_{p1} - h_{k1}h_{mp}) \\ &- h_1^m(h_{mk}h_{pq} - h_{kq}h_{mp}) + h_q^m\bar{R}_{kp1m} + h_1^m\bar{R}_{kpqm} \\ &+ \nabla_p\bar{R}_{kq1\nu} + \nabla_k\bar{R}_{1pq\nu}\big)\Bigg) \\ - \frac{\phi'}{Fu} - \frac{\phi g^{ij}F_i r_j}{F^2 u} + \frac{1}{n}. \end{aligned} \tag{5.3}$$

Consider the term $\frac{F^{pq}}{F^2}\frac{h_{1,pq}^1}{h_1^1}$, by (5.2) and Lemma 2.3, we have

$$\begin{aligned} \frac{F^{pq}}{F^2}\frac{h_{1,pq}^1}{h_1^1} &\le \frac{F^{pq}}{F^2}\frac{u_{pq}}{u} \\ &= \frac{F^{pq}}{F^2 u}\left(g^{kl}h_{pqk}\Phi_l + \phi' h_{pq} - (h^2)_{pq}u + g^{kl}\nabla_l\Phi\bar{R}_{\nu pkq}\right). \end{aligned} \tag{5.4}$$

Insert (5.4) into (5.3), together with the concavity of F, yields

$$
\begin{aligned}
(5.5)\quad 0 \le \frac{1}{h_1^1}\Big(-\frac{1}{F}h_k^1 h_1^k - \frac{1}{F}\bar{R}^1_{\nu 1\nu} + \frac{g^{k1}F^{pq}}{F^2}\big(-h_1^m h_{mk} h_{pq} + h_q^m \bar{R}_{kp1m} \\
+ h_1^m \bar{R}_{kpqm} + \nabla_p \bar{R}_{kq1\nu} + \nabla_k \bar{R}_{1pq\nu}\big)\Big) \\
+ \frac{g^{kl}F^{pq}}{F^2 u}\nabla_l \Phi \bar{R}_{\nu pkq} + \frac{1}{n}.
\end{aligned}
$$

Using the fact $1-\phi'^2+\phi\phi''\ge 0$, together with (3.6)

$$
(5.6)\qquad g^{kl}\nabla_l\Phi\bar{R}_{\nu pkq} = \left(\frac{|\nabla r|^2\delta_{pq} - r_p r_q}{v^3\phi}\right)\left(-\phi\phi'' - (1-\phi'^2)\right) \le 0.
$$

Thus we have

$$
\begin{aligned}
(5.7)\quad 0 \le \frac{1}{h_1^1}\Big(-\frac{2}{F}h_k^1 h_1^k - \frac{1}{F}\bar{R}^1_{\nu 1\nu} \\
+ \frac{g^{k1}F^{pq}}{F^2}\big(h_q^m \bar{R}_{kp1m} + h_1^m \bar{R}_{kpqm} + \nabla_p \bar{R}_{kq1\nu} + \nabla_k \bar{R}_{1pq\nu}\big)\Big) + \frac{1}{n}.
\end{aligned}
$$

By Lemma 2.5, all terms involving curvature terms of the ambient space are uniformly bounded, i.e.

$$
h_q^m \bar{R}_{kp1m} + h_1^m \bar{R}_{kpqm} + \nabla_p \bar{R}_{kq1\nu} + \nabla_k \bar{R}_{1pq\nu} \le Ch_1^1 + C.
$$

By Lemma 2.7 and Lemma 4.2,

$$
\frac{g^{k1}F^{pq}}{F^2}\big(h_q^m \bar{R}_{kp1m} + h_1^m \bar{R}_{kpqm} + \nabla_p \bar{R}_{kq1\nu} + \nabla_k \bar{R}_{1pq\nu}\big) \le Ch_1^1 + C.
$$

Plug into (5.7), together with Lemma 4.1 yields

$$
0 \le -Ch_1^1 + C,
$$

i.e. $h_1^1 \le C$. The lemma is now proved.

□

Corollary 5.2. *The solution of the inverse curvature flow exists for all time.*

Proof. We have established up to C^2 a priori estimate, by Lemma 5.1, F is uniformly elliptic, by Evans-Krylov theorem, we have $C^{2,\alpha}$ estimate, together with Schauder estimate, we have all the high order estimates, the corollary now follows. □

6. Asymptotic behaviour of second fundamental form

In this section, we consider the asymptotic behaviour of second fundamental form, the test function was first considered by Scheuer in [18].

Lemma 6.1.

$$\limsup_{t\to\infty}\sup_i \kappa_i \le 1,$$

where κ_i is the principal curvature of Σ_t.

Proof. Let us consider the test function $w = (\log\eta - \log\tilde{u} + r - \log 2)\,t$, where

$$\eta = \sup\{h_{ij}\xi^i\xi^j : g_{ij}\xi^i\xi^j = 1\}.$$

Noting that

$$(-\log\tilde{u} + r - \log 2)\,t = (\log v - \log\phi + r - \log 2)\,t.$$

By Lemma 3.2 and Lemma 2.5, we have

$$t\log v \le C, \quad \phi \ge \frac{e^r}{2} - Ce^{-r},$$

Thus

$$(-\log\phi + r - \log 2)\,t \le t\log\frac{e^r}{e^r - Ce^{-r}} \le t\log\left(1 + Ce^{-2r}\right) \le C,$$

i.e.

$$(-\log\tilde{u} + r - \log 2)\,t \le C. \tag{6.1}$$

Similarly,

$$(-\log\tilde{u} + r - \log 2)\,t \ge -C. \tag{6.2}$$

Without loss of generality, we suppose that at the maximum point of w, say (x_0, t_0), $\eta = h_1^1$, and we have

$$0 \le \left(\frac{\dot{h}_1^1}{h_1^1} - \frac{\dot{u}}{u} + \dot{r}\right)t + \left(\log h_1^1 - \log\tilde{u} + r - \log 2\right), \tag{6.3}$$

and

$$\begin{aligned}\frac{h^1_{1i}}{h^1_1} - \frac{u_i}{u} + r_i = 0, \\ \frac{h^1_{1ij}}{h^1_1} - \frac{h^1_{1i}h^1_{1j}}{(h^1_1)^2} - \frac{u_{ij}}{u} + \frac{u_iu_j}{u^2} + r_{ij} \leq 0.\end{aligned} \tag{6.4}$$

By (2.9), (3.3), (3.4) and the critical equation, we have

$$\begin{aligned}0 \leq \frac{t_0}{h^1_1}\Big(&- \frac{1}{F}h^1_k h^k_1 + \frac{F^{pq,rs}h_{pq}{}^1 h_{rs1}}{F^2} - \frac{2F^{pq}h_{pq}{}^1 F^{rs}h_{rs1}}{F^3} - \frac{1}{F}\bar{R}^1_{\nu 1\nu} \\ &+ \frac{g^{k1}F^{pq}}{F^2}\big(h_{k1,pq} - h^m_q(h_{km}h_{p1} - h_{k1}h_{mp}) - h^m_1(h_{mk}h_{pq} - h_{kq}h_{mp}) \\ &+ h^m_q\bar{R}_{kp1m} + h^m_1\bar{R}_{kpqm} + \nabla_p\bar{R}_{kq1\nu} + \nabla_k\bar{R}_{1pq\nu}\big)\Big) \\ &- \frac{t_0}{u}\left(\frac{\phi'}{F} + \frac{\phi g^{ij}F_i r_j}{F^2}\right) + \frac{vt_0}{F} + \big(\log h^1_1 - \log\tilde{u} + \tilde{r} - \log 2\big),\end{aligned}$$

i.e.

$$\begin{aligned}0 \leq \frac{t_0}{h^1_1}\Big(&- \frac{2}{F}h^1_k h^k_1 - \frac{1}{F}\bar{R}^1_{\nu 1\nu} + \frac{g^{k1}F^{pq}}{F^2}\big(h_{k1,pq} + h^m_q h_{k1}h_{mp} \\ &+ h^m_q\bar{R}_{kp1m} + h^m_1\bar{R}_{kpqm} + \nabla_p\bar{R}_{kq1\nu} + \nabla_k\bar{R}_{1pq\nu}\big)\Big) \\ &- \frac{t_0}{u}\left(\frac{\phi'}{F} + \frac{\phi g^{ij}F_i r_j}{F^2}\right) + \frac{vt_0}{F} + C.\end{aligned} \tag{6.5}$$

Consider the term $\frac{F^{pq}}{F^2}\frac{h^1_{1,pq}}{h^1_1}$, by (2.10), Lemma 2.3 and the critical equation we have

$$\begin{aligned}\frac{F^{pq}}{F^2}\frac{h^1_{1,pq}}{h^1_1} &\leq \frac{F^{pq}}{F^2}\left(\frac{u_{pq}}{u} + \frac{h^1_{1p}h^1_{1q}}{(h^1_1)^2} - \frac{u_pu_q}{u^2} - r_{pq}\right) \\ &= \frac{F^{pq}}{F^2u}\left(g^{kl}h_{pqk}\Phi_l + \phi' h_{pq} - (h^2)_{pq}u + g^{kl}\nabla_l\Phi\bar{R}_{\nu pkq}\right) \\ &\quad + \frac{F^{pq}}{F^2}\left(h_{pq}v - \phi\phi'\delta_{pq} - \frac{2\phi' r_pr_q}{\phi}\right) \\ &\quad + \frac{F^{pq}}{F^2}\left(\frac{h^1_{1p}h^1_{1q}}{(h^1_1)^2} - \frac{u_pu_q}{u^2}\right).\end{aligned} \tag{6.6}$$

Plug into (6.5), we have

$$0 \leq \frac{t_0}{h_1^1}\Big(- \frac{2}{F} h_k^1 h_1^k - \frac{1}{F}\bar{R}_{\nu 1 \nu}^1 + \frac{g^{k1}F^{pq}}{F^2}\big(h_q^m \bar{R}_{kp1m} + h_1^m \bar{R}_{kpqm} + \nabla_p \bar{R}_{kq1\nu} + \nabla_k \bar{R}_{1pq\nu}\big)\Big) + \frac{t_0 g^{kl}F^{pq}}{F^2 u}\nabla_l \Phi \bar{R}_{\nu pkq} - \frac{t_0 F^{pq}}{F^2}\left(\phi\phi'\delta_{pq} + \frac{2\phi' r_p r_q}{\phi}\right) + \frac{t_0 F^{pq}}{F^2}\left(\frac{h_{1p}^1 h_{1q}^1}{(h_1^1)^2} - \frac{u_p u_q}{u^2}\right) + \frac{2vt_0}{F} + C. \tag{6.7}$$

By Lemma 2.5 and Lemma 3.2, we have

$$\begin{aligned} &\frac{g^{k1}F^{pq}}{F^2}\big(h_q^m \bar{R}_{kp1m} + h_1^m \bar{R}_{kpqm} + \nabla_p \bar{R}_{kq1\nu} + \nabla_k \bar{R}_{1pq\nu}\big) \\ =& \frac{F_p^p}{F^2}(-h_p^p + h_1^1) + O(e^{-\alpha t_0}) \\ =& -\frac{1}{F} + \frac{F_p^p}{F^2}h_1^1 + O(e^{-\alpha t_0}). \end{aligned}$$

Similarly,

$$-\frac{1}{F}\bar{R}_{\nu 1\nu}^1 = \frac{1}{F} + O(e^{-\alpha t_0}), \quad \frac{g^{kl}F^{pq}}{F^2 u}\nabla_l \Phi \bar{R}_{\nu pkq} = O(e^{-\alpha t_0}).$$

Plug into (6.7), we have

$$\begin{aligned} 0 \leq& \frac{t_0}{h_1^1}\left(-\frac{2}{F}h_k^1 h_1^k + \frac{F_p^p}{F^2}h_1^1\right) + \frac{2vt_0}{F} + C \\ &- \frac{t_0 F^{pq}}{F^2}\left(\phi\phi'\delta_{pq} + \frac{2\phi' r_p r_q}{\phi}\right) + \frac{t_0 F^{pq}}{F^2}\left(\frac{h_{1p}^1 h_{1q}^1}{(h_1^1)^2} - \frac{u_p u_q}{u^2}\right). \end{aligned}$$

By the critical equation, we have

$$\frac{F^{pq}}{F^2}\left(\frac{h_{1p}^1 h_{1q}^1}{(h_1^1)^2} - \frac{u_p u_q}{u^2}\right) = \frac{F^{pq}}{F^2}\left(-\frac{2u_p r_q}{u} + r_p r_q\right).$$

Since

$$\nabla_i u = g^{kl}h_{ik}\nabla_l \Phi = g^{kl}h_{ik}\phi r_l,$$

together with Lemma 3.2, we have

$$\frac{t_0 F^{pq}}{F^2}\left(\frac{h^1_{1p}h^1_{1q}}{(h^1_1)^2} - \frac{u_p u_q}{u^2}\right) \leq C.$$

Thus

$$\begin{aligned} 0 \leq \frac{t_0}{h^1_1}\left(-\frac{2}{F}h^1_k h^k_1 + \frac{F^p_p}{F^2}h^1_1\right) + \frac{2vt_0}{F} + C \\ - \frac{t_0 F^{pq}}{F^2}\left(\phi\phi'\delta_{pq} + \frac{2\phi' r_p r_q}{\phi}\right). \end{aligned}$$

Again by Lemma 3.2 and the relation $\phi' = \phi + O(1)$, we have

$$\begin{aligned} 0 &\leq \frac{t_0}{h^1_1}\Big(- \frac{2}{F}h^1_k h^k_1 + \frac{F^p_p}{F^2}h^1_1\Big) + \frac{2t_0}{F} + C - \frac{t_0 F^p_p}{F^2} \\ &= -\frac{2t_0}{F}h^1_1 + \frac{2t_0}{F} + C. \end{aligned}$$

Thus

$$h^1_1 - 1 \leq \frac{C}{t_0}.$$

We have

$$w \leq t_0 \log\left(1 + \frac{C}{t_0}\right) + t_0\left(-\log\tilde{u} + \tilde{r} - \log 2\right) \leq C.$$

Thus

$$\big(\log h^1_1 - \log\tilde{u} + \tilde{r} - \log 2\big)\, t \leq C,$$

for any t, together with (6.2), we have

$$\limsup_{t\to\infty}\sup_i \kappa_i(t,\cdot) \leq 1.$$

□

Lemma 6.2. *Along the flow, $F \geq n - Cte^{-2\alpha t}$, where C and α only depend on Σ_0, n, k.*

Proof. Consider the test function $w = \frac{v}{F}$, thus $\dot{\varphi} = \frac{1}{G} = \frac{w}{\phi}$, we have

$$\frac{\partial w}{\partial t} = \phi \frac{\partial \dot{\varphi}}{\partial t} + \phi\phi' \dot{\varphi}^2.$$

Let $G^{ij} = \frac{\partial G}{\partial \varphi_{ij}}$, $G^k = \frac{\partial G}{\partial \varphi_k}$, then

$$G^{ij} = -\frac{1}{v^2} F^i_l \tilde{\sigma}^{lj}.$$

Similar to Lemma 3.1, we have

$$\begin{aligned}
\frac{\partial w}{\partial t} &= \frac{\phi}{v^2 G^2}\left(F^i_l \tilde{\sigma}^{lj} \dot{\varphi}_{ij} - v^2 G^k \dot{\varphi}_k - F^i_i \phi\phi'' \dot{\varphi}\right) + \frac{\phi'}{\phi} w^2 \\
&= \frac{w^2}{v^2 \phi}\left(F^i_l \tilde{\sigma}^{lj}\left(\frac{w}{\phi}\right)_{ij} - v^2 G^k \left(\frac{w}{\phi}\right)_k - F^i_i \phi'' w\right) + \frac{\phi'}{\phi} w^2 \\
&= \frac{w^2}{v^2 \phi^2}\left(F^i_l \tilde{\sigma}^{lj} w_{ij} - \frac{2}{\phi} F^i_l \tilde{\sigma}^{lj} w_i \phi_j - v^2 G^k w_k\right) \\
&\quad + \frac{w^2}{v^2 \phi^2}\left(\frac{2w}{\phi^2} F^i_l \tilde{\sigma}^{lj} \phi_i \phi_j - \frac{w}{\phi} F^i_l \tilde{\sigma}^{lj} \phi_{ij} + \frac{v^2 w}{\phi} G^k \phi_k\right) \\
&\quad + \frac{\phi'}{\phi} w^2 - \frac{F^i_i \phi''}{v^2 \phi} w^3.
\end{aligned}$$

First, note that w is bounded by our previous estimate, thus we only need to consider the second line.

By Lemma 3.2, We have

$$\frac{2w}{\phi^2} F^i_l \tilde{\sigma}^{lj} \phi_i \phi_j \le C e^{(\frac{2}{n} - \alpha)t}.$$

Now by (2.12)

$$\begin{aligned}
\phi_{ij} &= \phi' r_{ij} + \phi'' r_i r_j \\
&= \phi\phi' \varphi_{ij} + \phi\left(\phi'^2 + \phi\phi''\right) \varphi_i \varphi_j.
\end{aligned}$$

Thus

$$\begin{aligned}
-F^i_l \tilde{\sigma}^{lj} \phi_{ij} &= -\phi\phi' F^i_l \tilde{\sigma}^{lj} \varphi_{ij} - \phi\left(\phi'^2 + \phi\phi''\right) F^i_l \tilde{\sigma}^{lj} \varphi_i \varphi_j \\
&\le -\phi\phi' F^i_l \left(\phi' \delta^l_i - \phi v h^l_i\right) + C e^{(\frac{3}{n} - 2\alpha)t}.
\end{aligned}$$

By Lemma 2.7,

$$\begin{aligned}-F_l^i\tilde{\sigma}^{lj}\phi_{ij} &\le -\phi\phi'\left(n\phi'-\phi vF\right)+Ce^{(\frac{3}{n}-2\alpha)t}\\ &=\phi\phi'\left(v^2\frac{\phi}{w}-n\phi'\right)+Ce^{(\frac{3}{n}-2\alpha)t},\end{aligned}$$

i.e.

$$-\frac{w}{\phi}F_l^i\tilde{\sigma}^{lj}\phi_{ij}\le\phi\phi' v^2-n\phi'^2 w+Ce^{(\frac{2}{n}-2\alpha)t}.$$

Now, consider G^k, we have

$$G^k=\frac{F_k^i\varphi_{ij}}{v^2}\frac{\varphi^i}{v^2}-2\frac{F_l^i\varphi_{ij}}{v^2}\frac{\varphi^l\varphi^j\varphi^k}{v^4}-2\frac{F}{v^4}\varphi^k.$$

Since h_j^i is bounded, by (2.15), we have $|\varphi_{ij}|\le Ce^{\frac{t}{n}}$, thus

$$\begin{aligned}G^k\phi_k &=\phi\phi' G^k\varphi_k\\ &=\phi\phi'\left(\frac{F_k^i\varphi_{ij}}{v^4}\varphi^j\varphi_k-2\frac{F_l^i\varphi_{ij}}{v^6}\varphi^l\varphi^j|\nabla\varphi|^2-2\frac{F}{v^4}|\nabla\varphi|^2\right)\\ &\le Ce^{(\frac{3}{n}-2\alpha)t}.\end{aligned}$$

Put all together, we have

$$\begin{aligned}\frac{\partial w}{\partial t} &\le\frac{w^2}{v^2\phi^2}\left(F_l^i\tilde{\sigma}^{lj}w_{ij}-\frac{2}{\phi}F_l^i\tilde{\sigma}^{lj}w_i\phi_j-v^2G^kw_k\right)\\ &\quad+\frac{w^2}{v^2\phi^2}\left(\phi\phi' v^2-n\phi'^2w\right)+\frac{\phi'}{\phi}w^2-\frac{F_i^i\phi''}{v^2\phi}w^3+Ce^{-2\alpha t}\\ &\le\frac{w^2}{v^2\phi^2}\left(F_l^i\tilde{\sigma}^{lj}w_{ij}-\frac{2}{\phi}F_l^i\tilde{\sigma}^{lj}w_i\phi_j-v^2G^kw_k\right)\\ &\quad+2\frac{\phi'}{\phi}w^2-n\frac{\phi\phi''+\phi'^2}{v^2\phi^2}w^3+Ce^{-2\alpha t}.\end{aligned}$$

By Lemma 3.2 and Lemma 2.5, we have

$$\frac{d}{dt}w_{max}\le 2w_{max}^2-2nw_{max}^3+Ce^{-2\alpha t}.$$

By Lemma 6.1, we have $w_{max}\ge\frac{1}{n}$, thus

$$\frac{d}{dt}w_{max}\le\frac{2}{n^2}-\frac{2}{n}w_{max}+Ce^{-2\alpha t}.$$

Thus

$$w_{max} \leq \frac{1}{n} + Cte^{-2\alpha t}.$$

It follows that

$$F \geq n - Cte^{-2\alpha t}.$$

□

Put Lemma 6.1 and Lemma 6.2 together, we have

Corollary 6.3.

$$\lim_{t\to\infty} |h^i_j - \delta^i_j| = 0.$$

Now let us compute the convergence rate, we have the following lemma.

Lemma 6.4.

$$|h^i_j - \delta^i_j| \leq O(e^{-\frac{2}{n}t}).$$

Proof. Consider the test function

$$G = \frac{1}{2}\sum_{ij} \left(h^i_j - \delta^i_j\right) \left(h^j_i - \delta^j_i\right) e^{\lambda t}.$$

We have

$$\dot{G} = \sum_{ij} \dot{h^i_j} \left(h^j_i - \delta^j_i\right) e^{\lambda t} + \lambda G.$$

For each t, G attains maximum at some point x_0, at x_0

$$\sum_{ij} h^i_{jk} \left(h^j_i - \delta^j_i\right) = 0,$$
$$\sum_{ij} h^i_{jkl} \left(h^j_i - \delta^j_i\right) + h^i_{jk} h^j_{il} \leq 0.$$

Thus

$$\begin{aligned}\dot{G} = \Bigg(&- \frac{1}{F} h^i_k h^k_j + \frac{F^{pq,rs} h_{pq}{}^i h_{rsj}}{F^2} - \frac{2F^{pq} h_{pq}{}^i F^{rs} h_{rsj}}{F^3} - \frac{1}{F}\bar{R}^i_{\nu j\nu} \\ &+ \frac{g^{ki} F^{pq}}{F^2} \big(h_{kj,pq} - h^m_q(h_{km}h_{pj} - h_{kj}h_{mp}) - h^m_j(h_{mk}h_{pq} - h_{kq}h_{mp}) \\ &+ h^m_q \bar{R}_{kpjm} + h^m_j \bar{R}_{kpqm} + \nabla_p \bar{R}_{kqj\nu} + \nabla_k \bar{R}_{jpq\nu}\big)\Bigg) \left(h^j_i - \delta^j_i\right) e^{\lambda t} + \lambda G.\end{aligned}$$

By the critical equation, we have

$$\begin{aligned}\dot{G} \leq \Bigg(&- \frac{1}{F} h_k^i h_j^k + \frac{F^{pq,rs} h_{pq}{}^i h_{rsj}}{F^2} - \frac{2F^{pq} h_{pq}{}^i F^{rs} h_{rsj}}{F^3} - \frac{1}{F} \bar{R}^i_{\nu j \nu} \\ &+ \frac{g^{ki} F^{pq}}{F^2} \big(- h_q^m (h_{km} h_{pj} - h_{kj} h_{mp}) - h_j^m (h_{mk} h_{pq} - h_{kq} h_{mp}) \\ &+ h_q^m \bar{R}_{kpjm} + h_j^m \bar{R}_{kpqm} + \nabla_p \bar{R}_{kqj\nu} + \nabla_k \bar{R}_{jpq\nu} \big) \Bigg) \left(h_i^j - \delta_i^j \right) e^{\lambda t} \\ &- \frac{F^{pq}}{F^2} h^i_{jp} h^j_{iq} e^{\lambda t} + \lambda G.\end{aligned}$$

By Corollary 6.3, all the terms involving the derivatives of h_j^i can be controlled by $-\frac{F^{pq}}{F^2} h^i_{jp} h^j_{iq}$, thus

$$\begin{aligned}\dot{G} \leq \Bigg(&- \frac{1}{F} h_k^i h_j^k - \frac{1}{F} \bar{R}^i_{\nu j \nu} + \frac{g^{ki} F^{pq}}{F^2} \big(- h_q^m (h_{km} h_{pj} - h_{kj} h_{mp}) \\ &- h_j^m (h_{mk} h_{pq} - h_{kq} h_{mp}) + h_q^m \bar{R}_{kpjm} + h_j^m \bar{R}_{kpqm} \\ &+ \nabla_p \bar{R}_{kqj\nu} + \nabla_k \bar{R}_{jpq\nu} \big) \Bigg) \left(h_i^j - \delta_i^j \right) e^{\lambda t} + \lambda G.\end{aligned}$$

Without loss of generality, we may assume

$$g_{ij} = \delta_{ij}, \quad h_{ij} = \kappa_i \delta_{ij}, \quad \kappa_1 \leq \cdots \leq \kappa_n,$$

together with Lemma 2.5, Lemma 3.2 and Lemma 6.1, we have

$$\begin{aligned}\dot{G} &\leq \left(- \frac{1}{F} \kappa_i^2 + \frac{F^{pp}}{F^2} \left(\kappa_i \kappa_p^2 - \kappa_i^2 \kappa_p + \kappa_i \right) + c e^{-\frac{2}{n} t} \right) (\kappa_i - 1) \, e^{\lambda t} + \lambda G \\ &= \left(- \frac{2}{F} \left(\kappa_i^2 - \kappa_i \right) + \frac{F^{pp}}{F^2} \kappa_i \left(\kappa_p - 1 \right)^2 + c e^{-\frac{2}{n} t} \right) (\kappa_i - 1) \, e^{\lambda t} + \lambda G \\ &\leq \left(- \frac{4}{F} \kappa_i + \lambda + 2 \frac{F^{pp}}{F^2} \kappa_i |\kappa_i - 1| \right) G + c \left(\kappa_i - 1 \right) e^{(-\frac{2}{n} + \lambda) t}.\end{aligned}$$

Thus if we choose λ small enough, we conclude that G is bounded, i.e. $|h_j^i - \delta_j^i| = O(e^{-\frac{\lambda}{2} t})$ for small λ.

Now if we choose $\tilde{G} = \sup \frac{1}{2} |h_j^i - \delta_j^i|^2 e^{\frac{4t}{n}}$, we have

$$\begin{aligned}\dot{\tilde{G}} &\leq \left(- \frac{4}{F} \kappa_i + \frac{4}{n} + 2 \frac{F^{pp}}{F^2} \kappa_i |\kappa_i - 1| \right) \tilde{G} + c e^{-\frac{\lambda t}{2}} \\ &\leq c e^{-\frac{\lambda t}{2}} \tilde{G} + c e^{-\frac{\lambda t}{2}}.\end{aligned}$$

Write $\sqrt{\tilde{G}} = f$, we have

$$\dot{f} \leq ce^{-\frac{\lambda t}{2}} f + ce^{-\frac{\lambda t}{2}}.$$

It follows that $f \leq C$. The lemma is now proved. □

Acknowledgement. The author would like to express gratitude to his supervisor Professor Pengfei Guan for consistent support and encouragement.

References

[1] B. Andrews, *Contraction of convex hypersurfaces in Euclidean space*, Calc. Var. Partial Differential Equations **2** (1994), no. 2, 151–171.

[2] B. Andrews, *Contraction of convex hypersurfaces in Riemannian spaces*, J. Differential Geom. **39** (1994), no. 2, 407–431.

[3] S. Brendle, P.-K. Huang, and M.-T. Wang, *A Minkowski inequality for hypersurfaces in the anti-de Sitter-Schwarzschild manifold*, Comm. Pure Appl. Math. **69** (2016), no. 1, 124–144.

[4] S. Brendle and M.-T. Wang, *A Gibbons-Penrose inequality for surfaces in Schwarzschild spacetime*, Comm. Math. Phys. **330** (2014), no. 1, 33–43.

[5] L. Chen and J. Mao, *Non-parametric inverse curvature flows in the AdS-Schwarzschild manifold*, J. Geom. Anal. **28** (2018), no. 2, 921–949.

[6] Q. Ding, *The inverse mean curvature flow in rotationally symmetric spaces*, Chin. Ann. Math. **32B** (2011), 27–44.

[7] C. Gerhardt, *Flow of nonconvex hypersurfaces into spheres*, J. Differential Geometry **32** (1990), 299–314.

[8] C. Gerhardt, *Inverse curvature flows in hyperbolic space*, J. Differential Geometry **89** (2011), 487–527.

[9] C. Gerhardt, *Curvature flows in the sphere*, J. Differential Geom. **100** (2015), no. 2, 301–347.

[10] P. Guan and J. Li, *The quermassintegral inequalities for k-convex star-shaped domains*, Adv. Math. **221** (2009), no. 5, 1725–1732.

[11] P. Guan and J. Li, *A mean curvature type flow in space forms*, International Mathematics Research Notices **2015** (2015), no. 13, 4716–4740.

[12] Y. Ge, G. Wang, and J. Wu, *Hyperbolic Alexandrov-Fenchel quermass-integral inequalities II*, J. Differential Geom. **98** (2014), no. 2, 237–260.

[13] G. Huisken, *Gerhard Flow by mean curvature of convex surfaces into spheres*, J. Differential Geom. **20** (1984), no. 1, 237–266.

[14] G. Huisken, *Contracting convex hypersurfaces in Riemannian manifolds by their mean curvature*, Invent. Math. **84** (1986), no. 3, 463–480.

[15] P.-K. Hung and M.-T. Wang, *Inverse mean curvature flows in the hyperbolic* 3*-space revisited*, Calc. Var. Partial Differential Equations **54** (2015), no. 1, 119–126.

[16] H. Li, Y. Wei and C. Xiong, *A geometric inequality on hypersurface in hyperbolic space*, Adv. Math. **253** (2014), 152–162.

[17] A. Neves, *Insufficient convergence of inverse mean curvature flow on asymptotically hyperbolic manifolds*, J. Differential Geom. **84** (2010), no. 1, 191–229.

[18] J. Scheuer, *Non-scale-invariant inverse curvature flows in hyperbolic space*, Calc. Var. Partial Differential Equations **53** (2015), no. 1-2, 91–123.

[19] J. Scheuer, *The inverse mean curvature flow in warped cylinders of non-positive radial curvature*, Adv. Math. **306** (2017), 1130–1163.

[20] J. Urbas, *On the expansion of starshaped hypersurfaces by symmetric functions of their principal curvatures*, Math. Z. **205** (1990), 355–372.

Department of Mathematics and Statistics, McGill University
805 Sherbrooke O, Montreal, Quebec, Canada, H3A 0B9
Current address:
Department of Mathematics and Statistics, McMaster University
1280 Main Street West, Hamilton, ON, Canada, L8S 4K1
E-mail address: siyuan.lu@mcmaster.ca

Received December 28, 2016