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Graphs with large girth and nonnegative curvature dimension condition

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In this paper, we classify unweighted graphs satisfying the curvature dimension condition $\mathrm{CD}(0, \infty)$ whose girth are at least five.

1. introduction

In Riemannian geometry, there are various geometric curvature notions, such as sectional curvature, Ricci curvature and scalar curvature, derived from the Riemann curvature tensor. Of particular interest, curvature bounds usually impose many topological and geometric constraints for underlying manifolds. Even in the non-smooth setting, there are generalizations of curvature bounds, e.g. sectional curvature on Alexandrov spaces, see [8, 9], and Ricci curvature on metric measure spaces [33, 42, 43], from which many geometric consequences can be derived accordingly.

Many authors attempted to define appropriate curvature conditions on discrete metric spaces, e.g. graphs, in order to resemble some geometric properties of Riemannian curvature bounds.

One is so-called combinatorial curvature introduced by [17, 22, 41]. The idea is to properly embed a graph into a Riemannian manifold, in particular a surface, and to define the curvature bound of the graph from that of the ambient space. In this way, one can derive some global geometric properties of the graph via the embedding, see [5, 6, 10, 11, 13, 18, 20, 24, 26, 27, 44, 45].

Ollivier [38] used L^1-Wasserstein distance for the space of probability measures on graphs to define a curvature notion mimicking the Ricci curvature on manifolds. Interesting results can be obtained from the optimal transport strategy, see [4, 7, 23, 39, 40]. Lin, Lu and Yau [30] modified Ollivier's definition and [31] gave a classification of Ricci flat graphs with girth at least five. Maas [35] identified the heat flow and the gradient flow of the Boltzmann-Shannon type entropy by introducing a Riemannian structure on the space of probability measures on graphs. Erbar and Maas [14] defined the generalized Ricci curvature via the convexity of the entropy functional and derived many functional inequalities under this curvature assumption.

From a different strategy, one can define curvature dimension conditions via the so-called Γ-calculus for general Markov semigroups, where Γ is the "carré du champ" operator, see [2, Definition 1.4.2]. In particular, the curvature bound is defined via a Bochner type inequality using the iterated Γ operator, denoted by Γ_2, see Definition 2.3 in this paper. For the diffusion semigroup, curvature dimension conditions were initiated in Bakry and Émery [1], and for the non-diffusion case, e.g. graphs, introduced by Lin and Yau [32]. Later, variants of curvature dimension conditions were introduced to obtain important analytic results, see e.g. [3, 12, 15, 16, 19, 19, 21, 28, 29, 36, 37].

We introduce the setting of graphs and refer to Section 2 for details. Let (V, E) be an undirected, connected, locally finite simple graph with the set of vertices V and the set of edges E. Without loss of generality, we exclude the trivial graph consisting of a single vertex. Two vertices x, y are called neighbors if $\{x, y\} \in E$, denoted by $x \sim y$. The combinatorial degree of a vertex $x \in V$ is the number of its neighbors, denoted by d_x. We assign a weight m_x to each vertex x and a weight μ_{xy} to each edge $\{x, y\}$, and refer to the quadruple $G = (V, E, m, \mu)$ as a *weighted graph*. The graph G is called *unweighted* if $\mu \equiv 1$ on E. For any $x \in V$, we denote $\mu_x := \sum_{y\sim x} \mu_{xy}$.

We are mostly interested in functions defined on V, and denote by $C(V)$ the set of all such functions. For any weighted graph G, there is an associated *Laplacian* operator, $\Delta : C(V) \to C(V)$, defined as

$$\Delta f(x) = \frac{1}{m_x} \sum_{y\sim x} \mu_{xy}(f(y) - f(x)), \quad f \in C(V), x \in V. \tag{1}$$

One can see that the weights μ and m play the essential role in the definition of Laplacian. Given the weight μ on E, typical choices of m are of interest:

- In case of $m_x = \mu_x$ for all $x \in V$, we call the associated Laplacian the *normalized* Laplacian.
- In case of $m \equiv 1$ on V, the Laplacian is called *physical* (or combinatorial) Laplacian.

Moreover, if the graph is unweighted, the corresponding Laplacian is called unweighted normalized (i.e. $\mu \equiv 1$ on E and $m \equiv \mu$ on V) or unweighted physical Laplacian (i.e. $\mu \equiv 1$ on E and $m \equiv 1$ on V) respectively. For simplicity, we also call the graph *unweighted normalized* or *unweighted physical* graph accordingly.

We denote by $\ell^p(V, m)$ or simply ℓ^p_m, the space of ℓ^p summable functions on the discrete measure space (V, m) and by $\|\cdot\|_{\ell^p_m}$ the ℓ^p norm of a function.

Define the weighted vertex degree $\mathrm{D} : V \to [0,\infty)$ by

$$\mathrm{D}_x = \frac{1}{m_x}\sum_{y\sim x}\mu_{xy}, \qquad x \in V.$$

It is well known, see e.g. [25], that the Laplacian associated with the graph G is a bounded operator on ℓ^2_m if and only if $\sup_{x\in V}\mathrm{D}_x < \infty$.

The curvature dimension condition $\mathrm{CD}(K,n)$, for $K \in \mathbb{R}$ and $n \in (0,\infty]$, on graphs was introduced by [32], which serves as the combination of a lower bound K for the Ricci curvature and an upper bound n for the dimension, see Definition 2.4. To verify the $\mathrm{CD}(K,n)$ condition, we adopt the following crucial identity for general Laplacians, analogous to the Bochner identity on Riemannian manifolds, which was first proved in [32], see also [34], for normalized Laplacians.

Proposition 1.1. *For any function f and $x \in V$,*

$$\begin{aligned}\Gamma_2(f)(x) = {} & \frac{1}{4}|D^2 f|^2(x) + \frac{1}{2}(\Delta f(x))^2 \\ & - \frac{1}{4}\sum_{y\sim x}\frac{\mu_{xy}}{m_x}(\mathrm{D}_x + \mathrm{D}_y)|f(y)-f(x)|^2,\end{aligned} \tag{2}$$

where

$$|D^2 f|^2(x) := \sum_{\substack{y,z\in V\\ y\sim x, z\sim y}}\frac{\mu_{xy}\mu_{yz}}{m_x m_y}|f(x)-2f(y)+f(z)|^2.$$

Note that the summation over the terms with $z \neq x$ in $|D^2 f|^2(x)$ is a discrete analogue of the squared norm of the Hessian of a function f in the Riemannian setting.

The girth of a vertex x, denoted by $\mathrm{Gir}(x)$, is defined as the minimal length of cycles passing through x, and the girth of a graph is the minimal girth of vertices, see Definition 2.1. Inspired by the work [31], we classify the unweighted graphs with large girth and satisfying the $\mathrm{CD}(0,\infty)$ condition. By definition, the curvature condition at a vertex is determined by the local structure, in particular, the ball of radius two centered at the vertex, denoted by B_2. The key observation is that if the girth of a vertex is large, B_2 is essentially a tree, see Proposition 2.2, which is intuitively non-positively curved. By using the Bochner type identity (2), one obtains the sufficient and necessary condition for $\mathrm{CD}(0,\infty)$ in that case, see Corollary 2.6, which yields the following classifications.

Theorem 1.2. *Let* G *be an unweighted normalized graph with* $\inf_{x\in V} \mathrm{d}_x \geq 2$ *and* $\mathrm{Gir}(x_0) \geq 5$ *for some* $x_0 \in V$*. Then* G *satisfies the* $\mathrm{CD}(0, \infty)$ *condition if and only if* G *is either the infinite line* $P_{\mathbb{Z}}$ *or the cycle graphs* C_n *for* $n \geq 5$*, see Figure 1.*

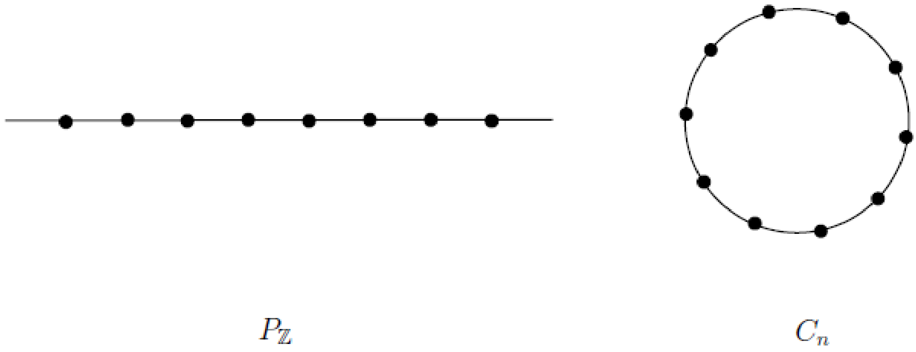


Figure 1: Theorem 1.2.

It is remarkable that if all vertex degrees are at least two, to derive the classification we only assume the girth of a vertex is large. For the general case below, a stronger assumption that the girth of the whole graph is large is needed.

Theorem 1.3. *Let* G *be an unweighted normalized graph with girth at least* 5*. Then* G *satisfies the* $\mathrm{CD}(0, \infty)$ *condition if and only if* G *is one of the following:*

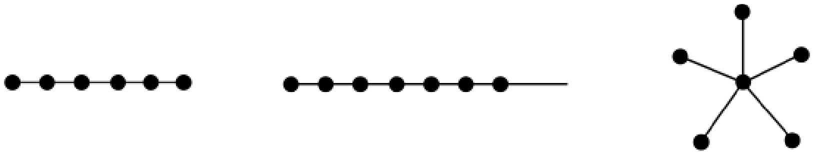
(a) The path graphs P_k $(k \geq 1)$*, the cycle graphs* C_n $(n \geq 5)$*, the infinite line* $P_{\mathbb{Z}}$*, or the infinite half line* $P_{\mathbb{N}}$*, see Figure 2.*

(b) The star graphs Star_n $(n \geq 3)$*, or* Star_3^i $(1 \leq i \leq 3)$*,*

where Star_3^i *is the* 3*-star graph with* i *edges added,* $1 \leq i \leq 3$*, see Figure 3.*

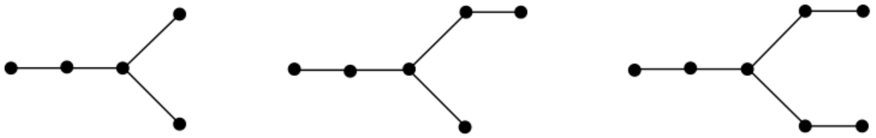
For physical Laplacians, we also obtain the classification results, see Section 4. Note that, similar results for physical Laplacians have been obtained in Cushing, Liu and Peyerimhoff [12, Corollary 6.9].

The organization of the paper is as follows: In next section, we introduce the definitions for graphs, Γ-calculus, and criteria for curvature dimension



$P_k, k = 5$ $P_{\mathbb{N}}$ $Star_5$

Figure 2: Theorem 1.3.



$Star_3^1$ $Star_3^2$ $Star_3^3$

Figure 3: Theorem 1.3.

conditions for graphs with large girth. In Section 3, we study normalized graphs and prove the classification results, Theorem 1.2 and 1.3. The last section is devoted to physical Laplacians.

2. Graphs

2.1. Combinatorial and weighted graphs

Let (V, E) be a (finite or infinite) undirected graph with the set of vertices V and the set of edges E, i.e. two-elements subsets in V. The graph is called simple if there is no self-loops and multiple edges. The graph is called locally finite, if the combinatorial degree $\mathrm{d}_x < \infty$ for any $x \in V$. We say a vertex x is a pending vertex if $\mathrm{d}_x = 1$. For any subsets $A, B \subset V$, we denote by $E(A,B) := \{\{x,y\} \in E : x \in A, y \in B\}$ the set of edges between A and B. For vertices x and y, a *walk* from x to y is given by

$$x = x_0 \sim x_1 \sim \cdots \sim x_k = y,$$

where $x_i \in V$ for $1 \leq i \leq k-1$ and k is the length of the walk. A graph is said to be connected if for any $x, y \in V$ there is a walk from x to y. The minimal length of walks from x to y is called the (combinatorial) distance between them, denoted by $d(x,y)$. In this paper, we only consider undirected, connected, locally finite simple graphs.

A *cycle* of length k, $k > 2$, is a walk,

$$x_0 \sim x_1 \sim \cdots \sim x_k = x_0,$$

satisfying $x_i \neq x_j$ for all $0 \leq i < j \leq k-1$. A graph is called a tree if it contains no cycles.

Definition 2.1. The girth of a vertex x in (V,E), denoted by $\mathrm{Gir}(x)$, is defined as the minimal length of cycles passing through x. (If there is no cycle passing through x, then we define $\mathrm{Gir}(x) = \infty$.) The girth of a graph is defined as $\inf_{x \in V} \mathrm{Gir}(x)$.

For any $x \in V$, $r \in \mathbb{N}_0$, we denote by $B_r(x) := \{y \in V : d(y,x) \leq r\}$ the ball of radius r centered at x, and by $S_r(x) := \{y \in V : d(y,x) = r\}$ the corresponding sphere. For our purposes, we define a graph, denoted by $\widehat{B_2}(x)$, consisting of the set of vertices in $B_2(x)$ and the set of edges $\{\{x,y\} \in E : x \in B_1(x) \text{ or } y \in B_1(y)\}$. That is, $\widehat{B_2}(x)$ is obtained by removing edges in $E(S_2(x), S_2(x))$ from the induced subgraph $B_2(x)$. The following proposition is elementary and useful.

Proposition 2.2. *For a graph (V,E) and $x \in V$, $\widehat{B_2}(x)$ is a tree if and only if* $\mathrm{Gir}(x) \geq 5$.

Proof. $\Longrightarrow$: Suppose $\widehat{B_2}(x)$ is a tree and $\mathrm{Gir}(x) \leq 4$. Let $C = \{x_i\}$ be a cycle of minimal length passing through x of length ≤ 4. Then C is a cycle in $\widehat{B_2}(x)$ which contradicts to that $\widehat{B_2}(x)$ is a tree.

$\Longleftarrow$: Conversely, suppose that $\mathrm{Gir}(x) \geq 5$ and $\widehat{B_2}(x)$ is not a tree, then there is a cycle $C = \{x_i\}$ in $\widehat{B_2}(x)$. We divide it into two cases:

Case 1. If the cycle C contains no vertices in $S_2(x)$, then it is included in $B_1(x)$. The cycle C passes through an edge $e = (x_j, x_{j+1})$ in $E(S_1(x), S_1(x))$. (Otherwise it will be contained in the graph $\widehat{B_1}(x)$ obtained by removing edges $E(S_1(x), S_1(x))$ from the induced subgraph $B_1(x)$ which is a tree. A contradiction.) Now we have a cycle $\{x, x_j, x_{j+1}\}$ of length 3 which contradicts to $\mathrm{Gir}(x) \geq 5$.

Case 2. $S_2(x) \cap C \neq \emptyset$. Let $x_k \in S_2(x)$ for some k and, without loss of generality, denote $x_{k-1} \sim x_k \sim x_{k+1}$ in C. Since in $\widehat{B_2}(x)$ there is no edges connecting vertices in $S_2(x)$, the consecutive neighbors of x_k in C, x_{k-1} and x_{k+1}, are hence contained in $S_1(x)$. Thus, $\{x, x_{k-1}, x_k, x_{k+1}\}$ is a cycle of length 4. A contradiction.

□

As mentioned in the introduction, for a combinatorial graph (V, E), we assign weights on the set of vertices V and edges E respectively, $m : V \to (0, \infty)$ and $\mu : E \to (0, \infty)$, to obtain a weighted graph $G = (V, E, m, \mu)$. In the following we always write G abbreviately for a weighted graph. For convenience, we extend the function μ on E to the total set $V \times V$, $\mu : V \times V \to [0, \infty)$, by

$$(x, y) \mapsto \begin{cases} \mu_{xy}, & \text{for } x \sim y, \\ 0, & \text{for } x \not\sim y. \end{cases}$$

So that we may write for $x \in V$,

$$\sum_{y} \mu_{xy} f(x, y) = \sum_{y \sim x} \mu_{xy} f(x, y)$$

in the following context. The Laplacian of a weighted graph G is defined as in (1) which can be identified with the generator of a standard Dirichlet form associated to the weighted graph G, see [25].

2.2. Gamma calculus

We introduce the Γ-calculus and curvature dimension conditions on graphs following [32].

Given $f : V \to \mathbb{R}$ and $x, y \in V$, we denote by $\nabla_{xy} f := f(y) - f(x)$ the difference of the function f on the vertices x and y. First we define two natural bilinear forms associated to the Laplacian Δ.

Definition 2.3. The gradient form Γ, called the "carré du champ" operator, is defined by, for $f, g \in C(V)$ and $x \in V$,

$$\begin{aligned}\Gamma(f,g)(x) &= \frac{1}{2}(\Delta(fg) - f\Delta g - g\Delta f)(x)\\ &= \frac{1}{2m(x)}\sum_y \mu_{xy}\nabla_{xy}f\nabla_{xy}g.\end{aligned}$$

For simplicity, we write $\Gamma(f) := \Gamma(f,f)$. Moreover, the iterated gradient form, denoted by Γ_2, is defined as

$$\Gamma_2(f,g) = \frac{1}{2}(\Delta\Gamma(f,g) - \Gamma(f,\Delta g) - \Gamma(g,\Delta f)),$$

and we write $\Gamma_2(f) := \Gamma_2(f,f) = \frac{1}{2}\Delta\Gamma(f) - \Gamma(f,\Delta f)$.

Now we prove the Bochner type identity on graphs.

Proof of Proposition 1.1. For any function f and $x \in V$,

$$\begin{aligned}\Delta\Gamma(f)(x) &= \sum_y \frac{\mu_{xy}}{m_x}\Gamma(f)(y) - \mathrm{D}_x\Gamma(f)(x)\\ &= \sum_{y,z}\frac{\mu_{xy}\mu_{yz}}{2m_xm_y}(\nabla_{yz}f)^2 - \mathrm{D}_x\Gamma(f)(x)\\ &= \sum_{y,z}\frac{\mu_{xy}\mu_{yz}}{2m_xm_y}\left[(\nabla_{yz}f)^2 - (\nabla_{xy}f)^2\right]\\ &\quad + \sum_y \frac{\mu_{xy}}{2m_x}(\mathrm{D}_y - \mathrm{D}_x)(\nabla_{xy}f)^2\\ &= (I) + \sum_y \frac{\mu_{xy}}{2m_x}(\mathrm{D}_y - \mathrm{D}_x)(\nabla_{xy}f)^2.\end{aligned}$$

For the first term on the right hand side of the equation, we write

$$\begin{aligned}(I) &= \sum_{y,z}\frac{\mu_{xy}\mu_{yz}}{2m_xm_y}\left\{(\nabla_{yz}f - \nabla_{xy}f)^2 + 2\nabla_{xy}f(\nabla_{yz}f - \nabla_{xy}f)\right\}\\ &= \frac{1}{2}|D^2f|^2(x) + \sum_y \frac{\mu_{xy}}{m_x}\nabla_{xy}f\Delta f(y) - \sum_y \frac{\mu_{xy}}{m_x}\mathrm{D}_y(\nabla_{xy}f)^2\\ &= \frac{1}{2}|D^2f|^2(x) + 2\Gamma(f,\Delta f) + (\Delta f(x))^2 - \sum_y \frac{\mu_{xy}}{m_x}\mathrm{D}_y(\nabla_{xy}f)^2.\end{aligned}$$

Combining the above equations, we prove the proposition. $\square$

Now we can introduce curvature dimension conditions on graphs.

Definition 2.4. Let $K \in \mathbb{R}, n \in (0,\infty]$. We say a graph G satisfies the $\mathrm{CD}(K,n)$ condition at $x \in V$, denoted by $\mathrm{CD}(K,n,x)$, if for any $f \in C(V)$,

$$\Gamma_2(f)(x) \geq \frac{1}{n}(\Delta f(x))^2 + K\Gamma(f)(x). \tag{3}$$

A graph is said to satisfy $\mathrm{CD}(K,n)$ condition if the above inequality holds for all $x \in V$.

2.3. Criteria for curvature dimension conditions

By Proposition 1.1, we have the following criterion for the curvature dimension condition of a vertex with large girth.

Theorem 2.5. *If* $\mathrm{Gir}(x) \geq 5$, *then* $\mathrm{CD}(K,n,x)$ *holds if and only if for any* $f \in C(V)$,

$$(1-\frac{2}{n})(\Delta f(x))^2 \geq \sum_y \frac{\mu_{xy}}{m_x}\left(\frac{\mathrm{D}_x+\mathrm{D}_y}{2} - \frac{2\mu_{xy}}{m_y} + K\right)(f(y)-f(x))^2. \tag{4}$$

Proof. Since the terms $\Delta f, \Gamma(f)$ and $\Gamma_2(f)$ are all invariant by adding a constant to f, it suffices to check the curvature conditions at $x \in V$ for functions f satisfying $f(x) = 0$. Note that the right hand side of (3) only depends on the values of f on $B_1(x)$. Set $W_f := \{g \in C(V) : g|_{B_1(x)} = f|_{B_1(x)}\}$. It suffices to prove the following

$$\begin{aligned}\inf_{g\in W_f}\Gamma_2(g) = &\sum_y \frac{\mu_{xy}^2}{m_x m_y} f(y)^2 + \frac{1}{2}(\Delta f(x))^2 \\ &- \frac{1}{4}\sum_y \frac{\mu_{xy}}{m_x}(\mathrm{D}_x+\mathrm{D}_y)f(y)^2.\end{aligned} \tag{5}$$

By the formula in (2), it suffices to minimize $|D^2 g|^2(x)$ under the same constraints. Note that $\widehat{B_2}(x)$ is a tree by Proposition 2.2, i.e. for any $z \in$

$S_2(x)$ there is a unique path from z to x. For $g \in W_f$,

$$\begin{aligned}|D^2g|^2(x) &= \sum_{y\in V}\frac{\mu_{xy}}{m_xm_y}\sum_{z\in S_2(x)\cup\{x\}}\mu_{yz}|g(z)-2f(y)|^2\\ &= \sum_{y\in V}\frac{\mu_{xy}}{m_xm_y}\left(\sum_{z\in S_2(x)}\mu_{yz}|g(z)-2f(y)|^2+\mu_{yx}|2f(y)|^2\right).\end{aligned}$$

The first equality follows from the fact that the nontrivial terms in the summation are all in the form $x\sim y\sim z$, and hence $z\in S_2(x)\cup\{x\}$. Then it is easy to see that the infimum over $g\in W_f$ is attained by setting $g(z)=2f(y)$ for any $z\in S_2(x)$ where y is the unique vertex in $S_1(x)$ such that $x\sim y\sim z$. This proves (5) and hence the theorem. □

For the curvature conditions at $x\in V$, it suffices to verify the inequality (4) for all functions f with $f(x)=0$. Note that the inequality only involves the values of f on $S_1(x)$. From now on, we label the vertices in $S_1(x)$ as $\{y_1,\dots,y_M\}$ where $M=\mathrm{d}_x$. Any function f on $S_1(x)$ can be understood as an M-tuple

$$(Y_1,\dots,Y_M):=(f(y_1),\dots,f(y_M)),$$

and the space of functions on $S_1(x)$ is identified with an M-dimensional vector space $\mathbb{R}^M$ indexed by the vertices of $S_1(x)$.

For any $x,y\in V$ with $x\sim y$, we denote

$$\alpha_{xy}:=\frac{m_x}{\mu_{xy}}\left(\frac{\mathrm{D}_x+\mathrm{D}_y}{2}-\frac{2\mu_{xy}}{m_y}\right), \tag{6}$$

which will be a key quantity in our argument, see the corollary below.

Corollary 2.6. *If* $\mathrm{Gir}(x)\geq 5$, *then* $\mathrm{CD}(0,\infty,x)$ *holds if and only if*

$$\left(\sum_{y_i\sim x}\frac{\mu_{xy_i}}{m_x}Y_i\right)^2\geq\sum_{y_i\sim x}\frac{\mu_{xy_i}}{m_x}\left(\frac{\mathrm{D}_x+\mathrm{D}_{y_i}}{2}-\frac{2\mu_{xy_i}}{m_{y_i}}\right)Y_i^2,\quad\forall\ Y_i\in\mathbb{R},$$

or equivalently,

$$\left(\sum_{y_i\sim x}Y_i\right)^2\geq\sum_{y_i\sim x}\alpha_{xy_i}Y_i^2,\quad\forall\ Y_i\in\mathbb{R}, \tag{7}$$

where α_{xy_i} *is defined in* (6).

Proof. The first inequality is equivalent to (4) for $K = 0, n = \infty$. The second one follows from the first one by setting $Y_i' = \frac{\mu_{xy_i}}{m_x} Y_i$ for all $y_i \sim x$ and rename Y_i' as Y_i. □

Corollary 2.7. *Let x be a pending vertex, i.e.* $\mathrm{d}_x = 1$, *in a weighted graph G. Then* $\mathrm{CD}(0, \infty, x)$

1) always holds for normalized Laplacian, and

2) holds for unweighted physical Laplacian if and only if $\mathrm{d}_y \leq 5$, *for $y \sim x$.*

Proof. By the inequality (7), $\mathrm{CD}(0, \infty, x)$ is equivalent to

$$\frac{m_x}{m_y}\left(\frac{1}{2}\frac{\mu_y}{\mu_x} - 2\right) \leq \frac{1}{2},$$

where $y \sim x$. This proves the corollary. □

The following calculus lemma will be useful in our setting.

Lemma 2.8. *Let $\{a_i\}_{1 \leq i \leq N}, a_i \geq 0$, and $c > 0$. The inequality,*

$$\left(Y + \sum_{i=1}^{N} Y_i\right)^2 + \sum_{i=1}^{N} a_i Y_i^2 \geq cY^2,$$

cannot hold for all $Y, Y_i \in \mathbb{R}$ $(1 \leq i \leq N)$ if one of the following holds:

1) $c \geq 1$.

2) $c > 0$ *and* $a_j = 0$ *for some* $j \in \{1, \ldots, N\}$.

Proof. Suppose it holds for all Y and Y_i.

(1) $c \geq 1$. For any $t \in \mathbb{R}$, setting $Y_i = tY$, $1 \leq i \leq N$, we have

$$Y^2(1 + Nt)^2 + \sum_i a_i t^2 Y^2 \geq cY^2.$$

This yields $2Nt + (N^2 + \sum a_i)t^2 \geq c - 1 \geq 0$. It is not true for

$$t \in \left(-2N(N^2 + \sum a_i)^{-1}, 0\right).$$

(2) $c > 0$ and $a_j = 0$ for some j. Let $Y_i = 0$ for all $i \neq j$ and $1 \leq i \leq N$. Then

$$(Y + Y_j)^2 \geq cY^2.$$

This yields a contradiction by setting $Y = -Y_j \neq 0$. □

Lemma 2.9. *If* $\mathrm{Gir}(x) \geq 5$ *and* $\mathrm{CD}(0, \infty, x)$ *hold for a vertex* x *in a weighted graph* G. *Suppose that* $\mathrm{d}_x \geq 2$, *then*

$$\alpha_{xy} < 1, \quad \forall\ y \sim x.$$

Proof. Without loss of generality, consider $y = y_1 \sim x$. Then setting $Y_i = 0$, for all $i \geq 2$ in (7), we have

$$\alpha_{xy_1} \leq 1.$$

Suppose that $\alpha_{xy_1} = 1$. For the terms on the right hand side of (7), we eliminate those with positive coefficients on the right hand side, and move those with negative coefficients to the left hand side. This reduces to the case (1) in Lemma 2.8. This yields a contradiction and proves the strict inequality. □

For any $x \in V$, we define

$$Q_x := \{y \in V : y \sim x, \alpha_{xy} > 0\} \quad \text{and} \quad q_x := \sharp Q_x.$$

In fact the set Q_x consists of those neighbors of x which contribute positive terms on the right hand side of (7).

Lemma 2.10. *If* $\mathrm{Gir}(x) \geq 5$ *and* $\mathrm{CD}(0, \infty, x)$ *hold for a vertex* x *in a weighted graph* G. *Then* $q_x \leq 1$.

Proof. Suppose that $q_x \geq 2$. Without loss of generality, pick $y_1, y_2 \in Q_x$. By setting $Y_j = 0$ for all $3 \leq j \leq M$ in (7), we have

$$(Y_1 + Y_2)^2 \geq \alpha_{xy_1} Y_1^2 + \alpha_{xy_2} Y_2^2 \geq \alpha_{xy_1} Y_1^2, \quad \forall\ Y_1, Y_2 \in \mathbb{R}.$$

This is impossible by (2) in Lemma 2.8. □

3. Normalized Laplacians

In this section, we consider the curvature dimension conditions for normalized Laplacians. For unweighted normalized Laplacians, a corollary of Theorem 2.5 reads as follows.

Corollary 3.1. *Let* G *be an unweighted normalized graph, and for some* $x \in V$, $\mathrm{Gir}(x) \geq 5$. *Then* $\mathrm{CD}(0, \infty, x)$ *is equivalent to*

$$\left(\sum_{y_i \sim x} Y_i\right)^2 \geq \mathrm{d}_x \sum_{y_i \sim x} \left(1 - \frac{2}{\mathrm{d}_{y_i}}\right) Y_i^2, \quad \forall\ Y_i \in \mathbb{R}. \tag{8}$$

In this setting, $\alpha_{xy} = \mathrm{d}_x \left(1 - \frac{2}{\mathrm{d}_y}\right)$ for any $y \sim x$ and $Q_x = \{y \sim x : \mathrm{d}_y \geq 3\}$ for all $x \in V$. By Lemma 2.10, we know that $q_x \leq 1$ if $\mathrm{Gir}(x) \geq 5$ and $\mathrm{CD}(0, \infty, x)$ hold.

Lemma 3.2. *Let G be an unweighted normalized graph. For some $x \in V$, $\mathrm{d}_x \geq 2$, $\mathrm{Gir}(x) \geq 5$ and $\mathrm{CD}(0, \infty, x)$ hold. If $q_x = 1$, then $\mathrm{d}_x = 2$, $\mathrm{d}_{y_1} = 3$ and $\mathrm{d}_{y_2} = 1$ where $y_i \sim x, i = 1, 2$.*

Proof. Without loss of generality, let $y_1 \in Q_x$, i.e. $\mathrm{d}_{y_1} \geq 3$, and $S_1(x) = \{y_1, \ldots, y_M\}$ with $M = \mathrm{d}_x$, where $\mathrm{d}_{y_i} \leq 2$, for all $i \neq 1$. By Lemma 2.9, $\alpha_{xy_1} < 1$. By this inequality,

$$\mathrm{d}_{y_1} < 2\left(1 - \frac{1}{\mathrm{d}_x}\right)^{-1} \leq 4,$$

which yields that $\mathrm{d}_{y_1} = 3$, and

$$\mathrm{d}_x < \left(1 - \frac{2}{\mathrm{d}_{y_1}}\right)^{-1} = 3,$$

which implies $\mathrm{d}_x = 2$. Hence $S_1(x) = \{y_1, y_2\}$.

For the case of $\mathrm{d}_{y_2} = 2$, by (8),

$$(Y_1 + Y_2)^2 \geq \frac{2}{3} Y_1^2, \quad \forall\, Y_1, Y_2 \in \mathbb{R},$$

which is impossible by setting $Y_1 = -Y_2 \neq 0$, or (2) in Lemma 2.8.

For the case of $\mathrm{d}_{y_2} = 1$, by (8),

$$(Y_1 + Y_2)^2 + 2Y_2^2 \geq \frac{2}{3} Y_1^2, \quad \forall\, Y_1, Y_2 \in \mathbb{R},$$

which is true and proves the lemma. □

Combining Corollary 2.7, Lemma 2.10 with Lemma 3.2, we have the following.

Lemma 3.3. *Let G be an unweighted normalized Laplacian with $\mathrm{Gir}(x) \geq 5$ for some $x \in V$. Then $\mathrm{CD}(0, \infty, x)$ holds if and only if we have the following cases:*

1) $\mathrm{d}_x = 1$, *and* d_y *is arbitrary for* $y \sim x$.

2) $\mathrm{d}_x = 2$, *and either* (a) $\mathrm{d}_{y_1} = 3$ *and* $\mathrm{d}_{y_2} = 1$, *or* (b) $\mathrm{d}_{y_1} \leq 2$ *and* $\mathrm{d}_{y_2} \leq 2$, *where* $y_i \sim x, i = 1, 2$.

3) $\mathrm{d}_x \geq 3$, *and* $\mathrm{d}_y \leq 2$ *for all* $y \sim x$.

Proof. $\Longrightarrow$: The case (1) follows from Corollary 2.7. For $q_x = 0$, i.e. $\mathrm{d}_y \leq 2$ for all $y \sim x$, (8) always holds. For $q_x = 1$, we apply Lemma 3.2. Hence, we obtain the cases (2) and (3).

$\Longleftarrow$: It is easy to check case by case. □

Now we are ready to classify the graph with large girth which has non-negative curvature dimension condition. The first case is that the degree of all vertices are at least two.

Proof of Theorem 1.2. $\Longrightarrow$: Applying Lemma 3.3 at the vertex x_0, noting that $\inf_{x \in V} \mathrm{d}_x \geq 2$, we have $\mathrm{d}_{x_0} \geq 2, \mathrm{d}_y = 2$ for all $y \sim x_0$.

We claim that $\mathrm{d}_{x_0} = 2$. Suppose not, i.e. $\mathrm{d}_{x_0} \geq 3$. Pick a neighbor of x_0, say y_1. Noting that $\mathrm{d}_{y_1} = 2$ and $\mathrm{Gir}(x_0) \geq 5$, we have $\mathrm{Gir}(y_1) \geq 5$. Now we may apply Lemma 3.3 to y_1, and obtain that $\mathrm{d}_{x_0} = 2$. A contradiction.

Hence $\mathrm{d}_{x_0} = 2, \mathrm{d}_y = 2$ for all $y \sim x_0$. This yields that $\mathrm{Gir}(y) \geq 5$ for $y \sim x_0$. Using the same argument at y, we have that $\mathrm{d}_z = 2$ for $z \sim y$. Continuing this process, by the connectedness of the graph we conclude that $\mathrm{d}_x = 2, \forall x \in V$.

$\Longleftarrow$: This is obvious.

This proves the theorem. □

Next we classify the general cases without any restrictions on vertex degrees.

Proof of Theorem 1.3. $\Longrightarrow$: We claim that $W_3 := \sharp\{x \in V : \mathrm{d}_x \geq 3\} \leq 1$. Suppose it is not true. Let x_1, x_2 be two distinct vertices with $\mathrm{d}_{x_i} \geq 3, i = 1, 2$. Then by the connectedness, there is a walk connecting them, $x_1 = z_0 \sim z_1 \sim \cdots \sim z_N = x_2$ with $N \geq 1$. Applying Lemma 3.3 at the vertex x_1, we have $\mathrm{d}_{z_1} \leq 2$ and $N \geq 2$, which implies that $\mathrm{d}_{z_1} = 2$ since z_1 lies in the walk connecting x_1 and x_2. Applying Lemma 3.3 at the vertex z_1, we obtain that $\mathrm{d}_{x_1} = 3$ and $\mathrm{d}_{z_2} = 1$. This yields a contradiction since z_2 lies in the walk and proves the claim.

For the case of $W_3 = 0$, we get the classification (a) in the theorem.

Let $W_3 = 1$ and $\mathrm{d}_{x_0} \geq 3$ for $x_0 \in V$. By Lemma 3.3, $\mathrm{d}_y \leq 2$ for any $y \sim x_0$. We divide it into cases:

Case 1. $\mathrm{d}_{x_0} \geq 4$. Applying Lemma 3.3 at any $y \sim x$, we have $\mathrm{d}_y = 1$. Hence the graph is Star_n, $n \geq 4$.

Case 2. $\mathrm{d}_{x_0} = 3$. For any $y \sim x_0$ satisfying $\mathrm{d}_y = 2$, applying Lemma 3.3 to y, we get $\mathrm{d}_z = 1$ for $z \sim y$ $(z \neq x)$. Hence we obtain Star_3, or Star_3^i, $i = 1, 2, 3$. This gives the classification (b).

$\Longleftarrow$: It is easy to check case by case.

This proves the theorem. $\square$

4. Physical Laplacians

In this section, we consider unweighted physical Laplacians and have a corollary of Theorem 2.5 as follows.

Corollary 4.1. *Let G be an unweighted physical graph and for $x \in V$ $\mathrm{Gir}(x) \geq 5$. Then $\mathrm{CD}(0, \infty, x)$ is equivalent to*

$$\left(\sum_{y_i \sim x} Y_i\right)^2 \geq \sum_{y_i \sim x} \left(\frac{\mathrm{d}_x + \mathrm{d}_y}{2} - 2\right) Y_i^2, \quad \forall\, Y_i \in \mathbb{R}. \tag{9}$$

In this setting, $\alpha_{xy} = \frac{\mathrm{d}_x + \mathrm{d}_y}{2} - 2$ for any $y \sim x$ and $Q_x = \{y \in V : y \sim x, \mathrm{d}_x + \mathrm{d}_y \geq 5\}$ for all $x \in V$.

Lemma 4.2. *Let G be an unweighted physical graph and for $x \in V$, $\mathrm{Gir}(x) \geq 5$. Then $\mathrm{CD}(0, \infty, x)$ holds if and only if we have the following cases:*

1) $\mathrm{d}_x = 1$, *and* $\mathrm{d}_y \leq 5$ *for* $y \sim x$.

2) $\mathrm{d}_x = 2$, *and* $\mathrm{d}_{y_i} \leq 2$ *for* $y_i \sim x, i = 1, 2$.

3) $\mathrm{d}_x = 3$, *and* $\mathrm{d}_{y_i} = 1$ *for all* $y_i \sim x, 1 \leq i \leq 3$.

Proof. $\Longrightarrow$: Without loss of generality, we may assume $\mathrm{d}_x \geq 2$ by Corollary 2.7 and denote $S_1(x) = \{y_1, \ldots, y_M\}$ with $M = \mathrm{d}_x$. By Lemma 2.10, $q_x \leq 1$. We divide it into cases:

Case 1. $q_x = 1$. Without loss of generality, let $y_1 \in Q_x$. By Lemma 2.9, we have

$$5 \leq \mathrm{d}_x + \mathrm{d}_{y_1} < 6,$$

which yields $\mathrm{d}_x + \mathrm{d}_{y_1} = 5$. Then $\mathrm{d}_x = 2, 3$ or 4. The case $\mathrm{d}_x = 4$ can be excluded by $\mathrm{d}_x + \mathrm{d}_{y_2} \leq 4$ since $q_x = 1$.

Suppose $\mathrm{d}_x = 2$. Then $\mathrm{d}_{y_1} = 3, \mathrm{d}_{y_2} \le 2$. For $\mathrm{d}_{y_2} = 2$, the equation (9) reads as

$$(Y_1 + Y_2)^2 \ge \frac{1}{2} Y_1^2, \quad \forall\, Y_1, Y_2 \in \mathbb{R},$$

which is impossible. For $\mathrm{d}_{y_2} = 1$, we have

$$(Y_1 + Y_2)^2 \ge \frac{1}{2} Y_1^2 - \frac{1}{2} Y_2^2, \quad \forall\, Y_1, Y_2 \in \mathbb{R},,$$

which is also wrong.

Suppose $\mathrm{d}_x = 3$. Then $\mathrm{d}_{y_1} = 2, \mathrm{d}_{y_2} = \mathrm{d}_{y_3} = 1$. Then the equation (9) has the form

$$(Y_1 + Y_2 + Y_3)^2 \ge \frac{1}{2} Y_1^2, \quad \forall\, Y_1, Y_2, Y_3 \in \mathbb{R},$$

which is excluded by (2) in Lemma 2.8.

Case 2. $q_x = 0$. That is, $\mathrm{d}_x + \mathrm{d}_{y_i} \le 4$ for all $y_i \sim x$. This implies that $\mathrm{d}_x \le 3$. For the case of $\mathrm{d}_x = 2$, we have $\mathrm{d}_{y_i} \le 2$, for $i = 1, 2$. This gives the case (2) in the lemma. For the case of $\mathrm{d}_x = 3$, $\mathrm{d}_{y_i} = 1$, for $1 \le i \le 3$. That is the case (3).

$\Longleftarrow$: It is easy to check case by case.

This proves the lemma. □

By this lemma, following the arguments as in Theorem 1.2 and 1.3, one can prove the following results for unweighted physical Laplacians. We omit the proofs here.

Theorem 4.3. *Let G be an unweighted physical graph with* $\inf_{x \in V} \mathrm{d}_x \ge 2$ *and* $\mathrm{Gir}(x_0) \ge 5$ *for some $x_0 \in V$. Then G satisfies the* $\mathrm{CD}(0, \infty)$ *condition if and only if G is either the infinite line $P_{\mathbb{Z}}$ or the cycle graphs C_n for $n \ge 5$.*

Theorem 4.4. *Let G be an unweighted physical graph with girth at least five. Then G satisfies the* $\mathrm{CD}(0, \infty)$ *condition if and only if G is one of the following:*

1) P_k $(k \ge 1)$, C_n $(n \ge 5)$, $P_{\mathbb{Z}}$, *or* $P_{\mathbb{N}}$.

2) Star_3.

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