Communications in
Analysis and Geometry
Volume 27, Number 4, 791–807, 2019

On the nature of isolated asymptotic singularities of solutions of a family of quasi-linear elliptic PDE's on a Cartan-Hadamard manifold

Leonardo Bonorino and Jaime Ripoll

Let M be a Cartan-Hadamard manifold with sectional curvature satisfying $-b^2 \leq K \leq -a^2 < 0,\ b \geq a > 0$. Denote by $\partial_\infty M$ the asymptotic boundary of M and by $\bar{M} := M \cup \partial_\infty M$ the geometric compactification of M with the cone topology. We investigate here the following question: Given a finite number of points $p_1, \ldots, p_k \in \partial_\infty M,$ if $u \in C^1(M) \cap C^0\left(\bar{M} \backslash \{p_1, \ldots, p_k\}\right)$ satisfies a PDE $\mathcal{Q}(u) = 0$ in M and if $u|_{\partial_\infty M \backslash \{p_1,\ldots,p_k\}}$ extends continuously to p_i, $i = 1, \ldots, k$, can one conclude that $u \in C^0\left(\bar{M}\right)$? When $\dim M = 2$, for $\mathcal{Q}$ belonging to a linearly convex space of quasi-linear elliptic operators $\mathcal{S}$ of the form

$$\mathcal{Q}(u) = \operatorname{div}\left(\frac{\mathcal{A}(|\nabla u|)}{|\nabla u|}\nabla u\right) = 0,$$

where $\mathcal{A}$ satisfies some structural conditions, then the answer is yes provided that $\mathcal{A}$ has a certain asymptotic growth. This condition includes, besides the minimal graph PDE, a class of minimal type PDEs.

In the hyperbolic space $\mathbb{H}^n$, $n \geq 2$, we are able to give a complete answer: we prove that $\mathcal{S}$ splits into two disjoint classes of minimal type and $p-$Laplacian type PDEs, $p > 1$, where the answer is yes and no respectively. These two classes are determined by the asymptotic behaviour of $\mathcal{A}$. Regarding the class where the answer is negative, we obtain explicit solutions having an isolated non removable singularity at infinity.

1. Introduction

Let M be a Cartan-Hadamard n-dimensional manifold (complete, connected, simply connected Riemannian manifold with non-positive sectional curvature). It is well-known that M can be compactified with the so called cone topology by adding a sphere at infinity, also called the asymptotic boundary of M; we refer to [5] for details. In the sequel, we will denote by $\partial_\infty M$ the sphere at infinity and by $\bar{M} = M \cup \partial_\infty M$ the compactification of M.

We recall that the asymptotic Dirichlet problem of a PDE $\mathcal{Q}(u) = 0$ in M for a given asymptotic boundary data $\psi \in C^0(\partial_\infty M)$ consists in finding a solution $u \in C^0\left(\bar{M}\right)$ of $Q(u) = 0$ in M such that $u|_{\partial_\infty M} = \psi$, determining the uniqueness of u as well.

The asymptotic Dirichlet problem for the Laplacian PDE has been studied during the last 30 years and there is a vast literature in this case. More recently, it has been studied in a larger class of PDEs which include the $p-$Laplacian PDE, $p > 1$,

$$\Delta_p u = \operatorname{div} |\nabla u|^{p-2} \nabla u = 0,$$

see [8], and the minimal graph PDE,

$$\mathcal{M}(u) = \operatorname{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 0, \tag{1}$$

see [7], [10], case that we are specially interested in the present work. We note that div and ∇ are the divergence and the gradient in M and it is worth to mention that the graph

$$G(r) = \{(x, u(x)) \mid x \in M\}$$

of u is a minimal surface in $M \times \mathbb{R}$ if and only if u satisfies (1).

Presently it is known that the asymptotic Dirichlet problem can be solved in any Cartan-Hadamard manifold under hypothesis on the growth of the sectional curvature that includes the ones with negatively pinched curvature, for any given continuous data at infinity, and on a large class of PDEs that includes both $p-$Laplacian and minimal graph PDEs (see [2], [3], [11]).

A natural question related to the asymptotic Dirichlet problem concerns the existence or not of solutions with isolated singularities at $\partial_\infty M$. We investigate this problem on the following class $\mathcal{S}$ of quasi-linear elliptic operators:

$$\mathcal{Q}(u) = \operatorname{div}\left(\frac{\mathcal{A}(|\nabla u|)}{|\nabla u|}\nabla u\right) = 0, \tag{2}$$

where $\mathcal{A} \in C[0,\infty) \cap C^1(0,\infty)$ satisfies the following conditions:

$$\left.\begin{array}{l} \mathcal{A}(0) = 0, \mathcal{A}'(s) > 0 \text{ for } s > 0; \\ \mathcal{A}(s) \le C(s^{p-1}+1) \text{ for some } C > 0, \text{ some } p \ge 1 \text{ and any } s > 0; \\ \text{there exist positives } q, \delta_0 \text{ and } \bar{D} \text{ s.t. } \mathcal{A}(s) > \bar{D}s^q \text{ for } s \in [0,\delta_0]. \end{array}\right\} \tag{3}$$

This class of operators, as the authors know, was first introduced and studied regarding the solvability of the asymptotic Dirichlet problem in [11]; it includes well known geometric operators as the p-laplacian, for $p > 1$, ($\mathcal{A}(s) = s^{p-1}$) and the minimal graph operator ($\mathcal{A}(s) = s/\sqrt{1+s^2}$). Note that $\mathcal{S}$ is linearly convex that is, any two elements $\mathcal{Q}_1, \mathcal{Q}_2$ of $\mathcal{S}$ are homotopic in $\mathcal{S}$ by the line segment $t\mathcal{Q}_1 + (1-t)\mathcal{Q}_2$, $0 \le t \le 1$.

As we shall see, the nature of an isolated asymptotic singularity of $\mathcal{Q}$ depends on the asymptotic behavior of $\mathcal{A}$ and can change drastically accordingly to it. It is worth to mention at this point that this behavior of $\mathcal{A}$ is closely related to the existence or not of "Scherk type" solutions of (2) (see the beginning of the next section). Minimal Scherk surfaces play a fundamental role on the theory of minimal surfaces in Riemannian manifolds (a well known breakthrough result using Scherk minimal surfaces were obtained by P. Collin and H. Rosenberg in [4]).

In our first three results we are concerned with removable singularities. We first show that isolated singularities are removable if $n = 2$, M has negatively pinched curvature and $\mathcal{A}$ satisfies

$$\int_0^\infty \mathcal{A}^{-1}(K_0(\cosh(ar))^{-1})\,dr = +\infty,$$

for some $K_0 > 0$. Since $\mathcal{A}^{-1}(t) \le ct^{1/q}$ holds for small t, due to (3), the change of variable $t = K_0(\cosh(ar))^{-1}$ implies that this condition is equivalent to

$$\int_0^{K_0} \frac{\mathcal{A}^{-1}(t)}{\sqrt{K_0 - t}}\,dt = +\infty. \tag{4}$$

Precisely, we prove:

Theorem 1.1. *Suppose that M is a $2-$dimensional Cartan-Hadamard manifold with sectional curvature satisfying $-b^2 \leq K \leq -a^2 < 0$, $b \geq a > 0$. Given a finite number of points $p_1, \ldots, p_k \in \partial_\infty M$, if*

$$m \in C^1(M) \cap C^0\left(\bar{M} \backslash \{p_1, \ldots, p_k\}\right)$$

is a weak solution of (2) *in M, $\mathcal{A}(s)$ satisfies* (3) *and* (4), *and $m|_{\partial_\infty M \backslash \{p_1,\ldots,p_k\}}$ extends continuously to p_i, $i = 1, \ldots, k$, then $m \in C^0\left(\bar{M}\right)$.*

We observe that condition (4) fails if $K_0 < \sup \mathcal{A}$. Hence, (4) implies that $\mathcal{A}$ is bounded and $K_0 = \sup \mathcal{A}$. This happens, for instance, if $\mathcal{A}(s) = s/\sqrt{1+s^2}$. Therefore, we have

Corollary 1.2. *Suppose that M is a $2-$dimensional Cartan-Hadamard manifold with sectional curvature satisfying $-b^2 \leq K \leq -a^2 < 0$, $b \geq a > 0$. Given a finite number of points $p_1, \ldots, p_k \in \partial_\infty M$, if*

$$m \in C^\infty(M) \cap C^0\left(\bar{M} \backslash \{p_1, \ldots, p_k\}\right)$$

is a solution of the minimal surface equation and if $m|_{\partial_\infty M \backslash \{p_1,\ldots,p_k\}}$ extends continuously to p_i, $i = 1, \ldots, k$, then $m \in C^0\left(\bar{M}\right)$.

We observe that a similar problem can obviously be posed to solutions of (2) on a bounded C^0 domain Ω of $\mathbb{R}^2$. In the minimal case, this is an old problem. From a classical result of R. Finn [6], it follows that if u, as in the above theorem, with M replaced by Ω, ∂_∞ by ∂, is a solution of the minimal graph equation (1) and if there is a solution $v \in C^\infty(\Omega) \cap C^0\left(\bar{\Omega}\right)$ of (1) such that

$$u|_{\partial\Omega \backslash \{p_1,\ldots,p_n\}} = v|_{\partial\Omega \backslash \{p_1,\ldots,p_n\}},$$

then $u = v$ and hence u extends continuously through the singularities. If the Dirichlet problem $\mathcal{M}(u) = 0$ on Ω is not solvable for the continuous boundary data $\phi := u|_{\partial\Omega}$ then the result is false, a known fact on the classical minimal surface theory (see [9], Chapter V, Section 3). We remark that even if the Dirichlet problem is not solvable there might exist smooth compact minimal surfaces which boundary is the graph of ϕ if ϕ and the domain are regular enough (see [1]).

Although under the hypothesis of Corollary 1.2 there exists a solution $v \in C^\infty(M) \cap C^0\left(\bar{M}\right)$ of (1) such that $u|_{\partial_\infty M \backslash \{p_1,\ldots,p_n\}} = v|_{\partial_\infty M \backslash \{p_1,\ldots,p_n\}}$, we felt necessary to use a different approach from Finn's. First because the boundedness of the domain is fundamental to the arguments used in [6].

Secondly, because it is not clear that the asymptotic Dirichlet problem for the PDE (2), under the conditions (3), is solvable for any continuous boundary data given at infinity.

Our proof relies heavily on the asymptotic properties of a $2-$dimensional Cartan-Hadamard manifold M with negatively pinched sectional curvature. It is fundamentally based on the fact that a point p of the asymptotic boundary of M is an isolated point of the asymptotic boundary of a domain U such that $M \setminus U$ is convex. This property allows the construction of suitable barriers at infinity. Although the existence of U in the $n = 2$ dimensional case is trivial (for example, a domain which boundary are two geodesics asymptotic to p), we don't know if such an U exists in M if $n \geq 3$. Nevertheless, it is possible in the special case of the hyperbolic space to give an ad hoc proof of Theorem 1.1 using the symmetries of the space. Precisely, our result in $\mathbb{H}^n$ reads:

Theorem 1.3. *Let $\mathbb{H}^n$ be the hyperbolic space of constant sectional curvature -1. Given a finite number of points $p_1, \ldots, p_k \in \partial_\infty \mathbb{H}^n$, if $m \in C^1(\mathbb{H}^n) \cap C^0\left(\bar{\mathbb{H}^n} \setminus \{p_1, \ldots, p_k\}\right)$ is a weak solution of* (2) *in $\mathbb{H}^n$, $\mathcal{A}(s)$ satisfies* (3) *and* (4)*, and if $m|_{\partial_\infty \mathbb{H}^n \setminus \{p_1, \ldots, p_k\}}$ extends continuously to p_i, $i = 1, \ldots, k$, then $m \in C^0\left(\bar{\mathbb{H}^n}\right)$.*

Finally, in the next last result, we prove the existence of a class of solutions of (2) in $\mathbb{H}^n$ admitting a non removable isolated asymptotic singularity. Note that this class contains the $p-$Laplacian PDE, $p > 1$.

Theorem 1.4. *Suppose that* (3) *holds and $\mathcal{A}(s)$ is unbounded. Given a point $p_1 \in \partial_\infty \mathbb{H}^n$, there exists a solution $m \in C^2(\mathbb{H}^n) \cap C^0\left(\bar{\mathbb{H}^n} \setminus \{p_1\}\right)$ of* (2) *in $\mathbb{H}^n$, such that $m = 0$ on $\partial_\infty \mathbb{H}^n \setminus \{p_1\}$ and $\limsup_{x \to p_1} m = +\infty$.*

2. Proof of the theorems

We begin by constructing Scherk type supersolutions to the equation (2), which are fundamental to prove the nonexistence of true asymptotic singularities.

Lemma 2.1. *Let γ be some geodesic of M, let U be one of the connected component of $M \setminus \gamma$ and $\delta > 0$. If $\mathcal{A}$ satisfies* (3) *and* (4)*, then there exists a* *solution of*

$$\begin{cases} \operatorname{div}\left(\dfrac{\mathcal{A}(|\nabla u|)}{|\nabla u|}\nabla u\right) \leq 0 & \text{in } U \\ u = +\infty & \text{on } \gamma \\ u = \delta & \text{in int } \partial_\infty U. \end{cases}$$

Proof. Let $d : U \to \mathbb{R}$ be defined by $d(x) = dist(x, \gamma)$ and $g : (0, +\infty) \to \mathbb{R}$ be defined by

$$g(d) = \delta + \int_d^\infty \mathcal{A}^{-1}\left(\frac{K_0}{\cosh(at)}\right) dt,$$

where $K_0 = \sup \mathcal{A}$. Observe that according to [11], $g(d)$ is well defined and finite for all $d > 0$, and $v(x) := g(d(x))$ is a supersolution of (2). Moreover, $g(d) \to \delta$ as $d \to +\infty$ and, therefore, $g(d(x)) \to \delta$ as $x \to p \in \text{int } \partial_\infty U$. That is, $v = \delta$ on $\text{int } \partial_\infty U$. Finally, making the change of variable $z = K_0(\cosh(at))^{-1}$, we can prove that condition (4) implies that $g(d) \to +\infty$ as $d \to 0$. Hence $v(x) = g(d(x)) \to +\infty$ as $x \to x_0 \in \gamma$, completing the proof of the lemma. □

2.1. Proof of Theorem 1.1

We first claim that m is bounded: for each p_i, consider a geodesic Γ_i such that the asymptotic boundary of one of the connected components of $M \setminus \Gamma_i$, say X_i, does not contain p_j for $j \neq i$. Assume also that $p_i \in \text{int } \partial_\infty X_i$. Since $\Gamma_i(\pm\infty) \notin \{p_1, \ldots p_n\}$, m is continuous at $\Gamma_i(\pm\infty)$ and therefore it is bounded on Γ_i. Let $S_i = \sup_{\Gamma_i} m$ for $i \in \{1, \ldots n\}$, $S_0 = \sup m|_{\partial_\infty M \setminus \{p_1, \ldots, p_n\}}$ and

$$S = \max\{S_0, S_1, \ldots, S_n\}.$$

In $M \setminus \{X_1 \cup \cdots \cup X_n\}$, m is bounded from above and from below by some constant, since m is continuous in $\overline{M \setminus \{X_1 \cup \cdots \cup X_n\}}$ with the cone topology. To prove that $m \leq S$ in X_i, take a sequence of geodesics β_k such that the ending points $\beta_k(+\infty)$ and $\beta_k(-\infty)$ converge to p_i. Let Y_k be the connected component of $M \setminus \beta_k$ whose the asymptotic boundary does not contain p_i. Observe that $M \setminus X_i \subset Y_k$ for large k and $\cup Y_k = M$. Let w_k be the supersolution of (2) given by Lemma 2.1 such that w_k is $+\infty$ on β_k and S at $\partial_\infty Y_k \setminus \{\beta_k(\pm\infty)\}$. Hence $w_k \geq S$ and therefore $w_k \geq m$ on $\Gamma_i = \partial X_i$, $w_k = S \geq m$ on $\partial_\infty(Y_k \cap X_i)$ and $w_k = +\infty > m$ on $\beta_k = \partial Y_k$. Thus, for $\varepsilon > 0$, $w_k + \varepsilon > m$ in some neighborhood of $\partial(Y_k \cap X_i) \cup \partial_\infty(Y_k \cap X_i)$.

That is, for some compact $F \subset Y_k \cap X_i$, $w_k + \varepsilon > m$ in $Y_k \cap X_i \backslash F$. By a Comparison Principle (for instance, see Lemma 2.2 of [11] or Lemma 2.1 of [2]), $w_k + \varepsilon \geq m$ in F and, therefore, $w_k + \varepsilon \geq m$ in $Y_k \cap X_i$ for large k. Since ε is arbitrary, it follows that $w_k \geq m$ in $Y_k \cap X_i$ for large k. For any given $x \in M$, $x \in Y_k$ for large k. Hence, using that $w_k(x) \to S$ (this is a consequence of $w_k(x) = g(dist(x, \beta_k))$, according to Lemma 2.1, and $dist(x, \beta_k) \to +\infty$ as $k \to +\infty$), we have $m(x) \leq S$ in X_i. In a similar way, we can conclude that m is bounded from below, proving the claim.

Assume that $m \leq S$. Denote by ϕ the continuous extension of $m|_{\partial_\infty M \backslash \{p_1, \ldots, p_n\}}$ to $\partial_\infty M$. Let $p \in \{p_1, \ldots, p_n\}$. Adding a constant to ϕ we may assume wlg that $\phi(p) = 0$. Let $0 < \delta \leq S$ be given. We will prove that $K := \limsup_{x \to p} m(x) \leq \delta$. By contradiction assume that $K > \delta$.

By the continuity of ϕ, there exists an open connected neighborhood $\mathcal{O} \subset \partial_\infty M$ of p such that $\phi(q) \leq \delta$ for all $q \in \mathcal{O}$. Moreover, we may assume that $\mathcal{O}$ does not contain another point p_i except p.

Let γ be a geodesic such that $\gamma(\infty) = p$. Set $\gamma = \gamma(\mathbb{R})$. Choose a point $q_0 \in \gamma$ and a geodesic α_0 orthogonal to γ at q_0 such that $\alpha_0(\pm\infty) \in \mathcal{O}$. Let γ_i, $i \in \{1, 2\}$, be the geodesics with ending points at p and $q_1 := \alpha_0(\infty)$ and p and $q_2 := \alpha_0(-\infty)$, respectively. Denote by U_i the connected component of $M \setminus \gamma_i$ that does not contain α_0. Let Sh_i be the solution constructed in Lemma 2.1 to the problem

$$\begin{cases} \operatorname{div}\left(\dfrac{\mathcal{A}(|\nabla u|)}{|\nabla u|}\nabla u\right) \leq 0 & \text{in } U_i \\ u = +\infty & \text{on } \gamma_i \\ u = \delta & \text{in int } \partial_\infty U_i. \end{cases}$$

Again, by a Comparison Principle, $m < Sh_i$. Let c_i be the level set of Sh_i

$$c_i = \left\{ x \in U_i \ : \ Sh_i(x) = \frac{K}{2} + \frac{\delta}{2} \right\}$$

and

$$V_i = \left\{ x \in U_i \ : \ Sh_i(x) < \frac{K}{2} + \frac{\delta}{2} \right\}.$$

Hence $m < K/2 + \delta/2$ on V_i. Let A be the connected component of $M \setminus \alpha_0$ containing p on its asymptotic boundary and set $V = A \backslash (V_1 \cup V_2)$.

Now, let W be a neighborhood of p such that the asymptotic boundary of $W \cap V$ is $\{p\}$. Observe that for $R > 0$ and any point z on the boundary of $W \cap V$ there exist a ball of radius R, $B_R \subset M \backslash (W \cap V)$ such that $B_R \cap \overline{W \cap V} = \{z\}$. We consider $R = 1$.

Since p is an ending point of both γ_1 and γ_2 it follows from the very definition of asymptotic geometric boundary (see [5]) that the distance between any point of $W \cap V$ and the geodesic γ_i is bounded by some constant. This property still holds if we consider the curve c_i instead γ_i, since these two curves are equidistant. Then there is $\rho > 0$ be such that

$$\operatorname{dist}(x, V_i) < \rho \qquad \text{for any } x \in W \cap V.$$

That is, for any $x \in W \cap V$, there is a ball B_ρ centered at some point of $\partial(V_1 \cup V_2) \cap W$ s.t. $x \in B_\rho$.

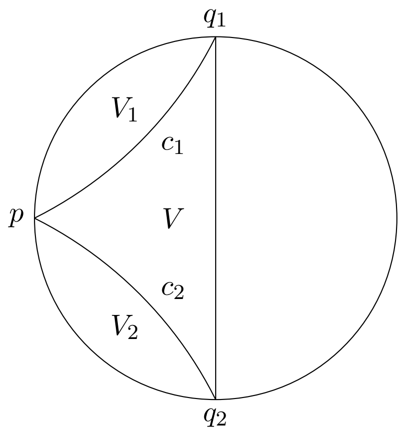

Fig. 1

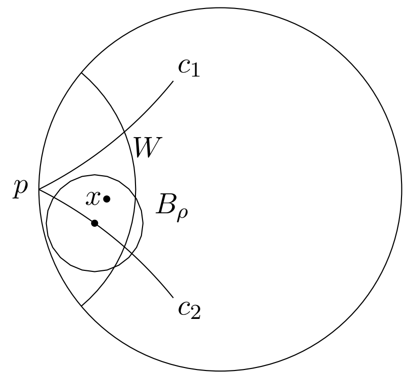

Fig. 2

Lemma 2.2. *There exist h_0 and h_1 depending only on b, ρ, K and δ, satisfying*

$$\delta < h_1 < h_0 < K/2 + \frac{\delta}{2}$$

such that, for any $y \in M$, the Dirichlet problem in the annulus $B_{2\rho+1}(y) \backslash \overline{B_1(y)}$

$$\begin{cases} \operatorname{div}\left(\dfrac{\mathcal{A}(|\nabla u|)}{|\nabla u|}\nabla u\right) = 0 & \text{in } B_{2\rho+1}(y)\backslash\overline{B_1(y)} \\ u = \delta & \text{on } \partial B_1(y) \\ u = h_0 & \text{on } \partial B_{2\rho+1}(y) \end{cases}$$

has a supersolution $w_y(x)$ and $w_y(x) \leq h_1$ if $\operatorname{dist}(x, y) < \rho + 1$.

Proof. Let $f : [1, \infty) \to \mathbb{R}$ be the function defined by

$$f(r) = \delta + \int_1^r \mathcal{A}^{-1}\left(\frac{\sinh(b\,\alpha)}{\sinh(bs)}\right)\, ds,$$

where $0 < \alpha \leq 1$. Hence $f(1) = \delta$ and, choosing α sufficiently small, $f(2\rho + 1) < K/2 + \delta/2$. Let $h_0 = f(2\rho + 1)$. Observe that if $r = r(\tilde{x})$ is the distance in $\mathbb{H}^2(-b^2)$ from $\tilde{x}$ to a fixed point, then the graphic of f is a radially symmetric surface, solution of (2) in the hyperbolic plane with constant negative sectional curvature $-b^2$, that is, f satisfies

$$\mathcal{A}'(f'(r))f''(r) + \mathcal{A}(f'(r))b \coth br = 0.$$

Moreover, from the Comparison Laplacian Theorem

$$\Delta d(x) \leq \Delta r(\tilde{x}) = b \coth br,$$

where $d(x) = dist(x, y)$ and $\tilde{x} \in \mathbb{H}^2(-b^2)$ is a point such that $d(x) = r(\tilde{x})$. Then, using these two relations and that $f' > 0$, we conclude that $w_y(x) := f(d(x))$ is a supersolution of (2) in M.

Since $f(1) = \delta$ and $f(2\rho + 1) = h_0$, $w_y(x)$ satisfies the required boundary conditions. Finally defining $h_1 := f(\rho + 1)$, $w_y(x) \leq h_1 < h_0$ in $B_{\rho+1}(y)$. $\square$

Let ε be a positive real satisfying $h_0 - h_1 - (K - \delta)/2 \leq \varepsilon < h_0 - h_1$ and $W_0 \subset W$ be a neighborhood of p s.t.

$$m < K + \varepsilon \quad \text{in} \quad W_0.$$

Let $\tilde{W} \subset W_0$ be a neighborhood of p s.t.

$$\mathrm{dist}(\partial W_0, \tilde{W}) > 3\rho + 2.$$

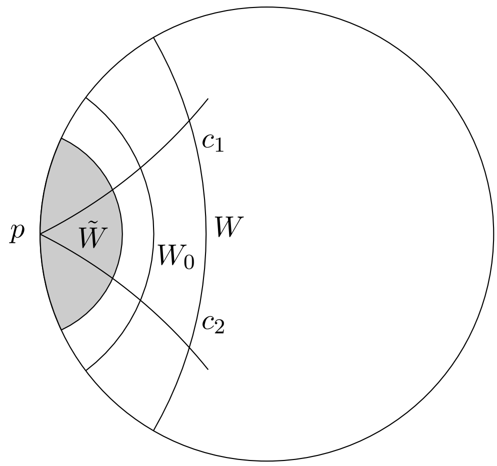

Fig. 3

We claim that

$$m < K + \varepsilon - h_0 + h_1 < K$$

in $\tilde{W}$.

Indeed: Let $x \in \tilde{W}$ and assume first that $x \in V$. As observed above, there is some $z \in \partial(V_1 \cup V_2)$, say $z \in \partial V_1$, s.t.

$$x \in B_\rho(z)$$

and there is $y \in V_1$ s.t.

$$\partial B_1(y) \cap \overline{W \cap V} = \{z\}.$$

Therefore

$$\operatorname{dist}(x, y) < \rho + 1.$$

Using triangular inequality and that $\operatorname{dist}(\partial W_0, \tilde{W}) > 3\rho + 2$, we have

$$B_{2\rho+1}(y) \subset B_{3\rho+2}(x) \subset W_0.$$

Let w_y be the solution associated to the annulus $B_{2\rho+1}(y) \backslash B_1(y)$ given by Lemma 2.2. Define

$$w = w_y + K + \varepsilon - h_0$$

Then, using that $B_1(y) \subset V_1$,

$$w = \delta + K + \varepsilon - h_0 > K + \delta + \varepsilon - \frac{K}{2} - \frac{\delta}{2} > \frac{K}{2} + \frac{\delta}{2} > m \quad \text{on} \quad \partial B_1(y)$$

and, from $B_{2\rho+1}(y) \subset W_0$,

$$w = h_0 + K + \varepsilon - h_0 = K + \varepsilon > m \quad \text{on} \quad \partial B_{2\rho+1}(y).$$

From the comparison principle,

$$m < w \quad \text{in} \quad \mathrm{B}_{2\rho+1}(\mathrm{y}) \backslash \mathrm{B}_1(\mathrm{y})$$

and, therefore

$$m < w_y + K + \varepsilon - h_0 < h_1 + K + \varepsilon - h_0 \quad \text{in} \quad B_{\rho+1}(y) \backslash B_1(y).$$

Since $\mathrm{dist}(x, y) < \rho + 1$, then $x \in B_{\rho+1}(y)$. Hence, using that $x \notin V_1 \cup V_2$, we have $x \in B_{\rho+1}(y) \backslash B_1(y)$. In this case, $m(x) < h_1 + K + \varepsilon - h_0$. Finally, if $x \in V_1 \cup V_2$, the definition of ε implies that $m(x) < K/2 + \delta/2 \leq K + \varepsilon - h_0 + h_1$ proving the claim.

To conclude the proof of the theorem, note that $\nu := -\varepsilon + h_0 - h_1 > 0$, since $\varepsilon < h_0 - h_1$. Then

$$K + \varepsilon - h_0 + h_1 = K - \nu$$

and, from the above claim,

$$m < K - \nu < K \quad \text{in} \quad \tilde{W}.$$

Hence $\limsup_{x \to p} m(x) \leq K - \nu < K$ leading a contradiction.

2.2. Proof of Theorem 1.3.

Proof. First we introduce the terminology of totally geodesic hyperball as a domain in $\mathbb{H}^n$ whose boundary is a totally geodesic hypersphere of $\mathbb{H}^n$.

The proof that m is bounded follows the same idea as in Theorem 1.1 replacing the geodesics Γ_i and β_k by totally geodesic hyperspheres H_i and Λ_k respectively and considering the same S. To build a supersolution w_k in Y_k (the connected component of $\mathbb{H}^n \backslash \Lambda_k$ that does not contain p_i) such that $w_k = +\infty$ on Λ_k, we use the same construction as in Lemma 2.1, that is, we consider

$$g(d) = S + \int_d^\infty \mathcal{A}^{-1} \left(\frac{K_0}{(\cosh(at))^{n-1}} \right) dt,$$

that is well defined and finite for all $d > 0$. The function $w_k(x) := g(d(x))$, where $d(x) = dist(x, \Lambda_k)$, is a supersolution according to [11]. Moreover it satisfies $w_k(x) = +\infty$ for $x \in \Lambda_k$ since $g(0) = +\infty$ as a result of (4). Using this w_k, we conclude in the same way as in Theorem 1.1 that m is bounded from above by S. In the same way, m is bounded from below. Now we prove that m is continuous at $p \in \{p_1, \ldots, p_k\}$. Denote by ϕ the continuous extension of $m|_{\partial_\infty M \setminus \{p_1,\ldots,p_k\}}$ to $\partial_\infty M$. Adding a constant to ϕ we may assume wlg that $\phi(p) = 0$. Hence we have to prove that

$$\lim_{x \to p} m(x) = 0.$$

Let

$$K = \limsup_{x \to p} m(x).$$

We will show that, for any $\delta > 0$, it follows that $K \leq \delta$. Since $v \leq S$, it follows that $K \leq S$. Suppose that $K > \delta$. Let V_j be a decreasing sequence of neighborhood of p such that

$$\bigcap \overline{V}_j = \{p\}, \quad \sup_{x \in V_j} m(x) < K + 1/j \quad \text{and} \quad \phi \leq \frac{\delta}{2} \quad \text{on} \quad \partial_\infty V_j$$

We can suppose that each V_j is a totally geodesic hyperball.

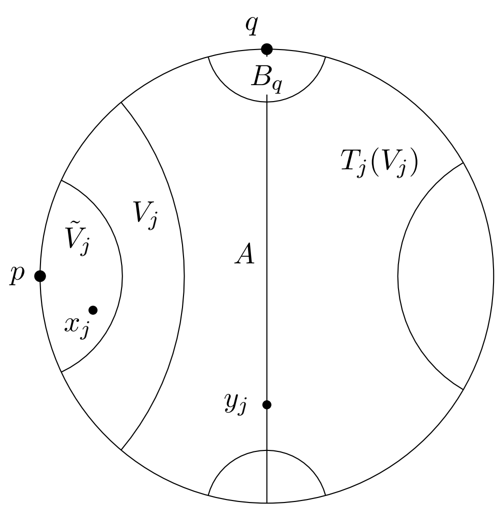

Fig. 4

For each j, let $\tilde{V}_j \subset V_j$ be a totally geodesic hyperball such that $p \in \operatorname{int} \partial_\infty \tilde{V}_j$,

$$dist(\partial\tilde{V}_j, \partial V_j) \geq j \quad \text{and} \quad \sup_{x\in\tilde{V}_j} m(x) > K - 1/j.$$

Then there exists a sequence (x_j) that satisfies $x_j \in \tilde{V}_j$ and

$$K - 1/j < m(x_j) < K + 1/j.$$

Denote $A = V_1$. It is well known that there exists an isometry $T_j : \mathbb{H}^n \to \mathbb{H}^n$ that preserves p, $T_j(\tilde{V}_j) \supset A$ and $y_j := T_j(x_j) \in \partial A$. Since $T_j(V_j)$ and $T_j(\tilde{V}_j)$ are totally geodesic hyperballs and $T_j(V_j) \supsetneq T_j(\tilde{V}_j) \supset A$, we have that $\partial_\infty A \subset \operatorname{int} \partial_\infty T_j(V_j)$ for any j. Observe that

$$u_j = m \circ T_j^{-1}$$

is a solution of (2) and satisfies

$$\sup_{T_j(V_j)} u_j < K + 1/j \quad \text{and} \quad u_j(y_j) > K - 1/j. \tag{5}$$

Moreover $\tilde{V}_j \subset V_j \subset A \subset T_j(\tilde{V}_j)$ implies that

$$\begin{aligned} dist(\partial T_j(V_j), A) &\geq dist(\partial T_j(V_j), T_j(\tilde{V}_j)) \\ &= dist(\partial V_j, \tilde{V}_j) \geq j \to \infty. \end{aligned}$$

Observe that $T_j(V_j)$ is a totally geodesic hyperball and

$$u_j \leq \frac{\delta}{2} \quad \text{on} \quad \partial_\infty(T_j(V_j))\backslash\{p\},$$

since $u_j = m \circ T_j^{-1}$ and $m = \phi \leq \delta/2$ on $V_j\backslash\{p\}$. Using that $A \subset T_j(V_j)$ and $p \notin \partial_\infty(\mathbb{H}^n\backslash A)$, we have that $\partial_\infty A \cap \partial_\infty(\mathbb{H}^n\backslash A) \subset \partial_\infty T_j(V_j)\backslash\{p\}$ and, therefore, $u_j \leq \delta/2$ on $\partial_\infty A \cap \partial_\infty(\mathbb{H}^n\backslash A)$. For $q \in \partial_\infty A \cap \partial_\infty(\mathbb{H}^n\backslash A)$, let B_q be a totally geodesic hyperball, neighborhood of q, disjoint with V_2 such that $B_q \subset T_j(V_j)$ for any j. (This is possible since (V_j) is a decreasing sequence, $\partial T_j(V_j)$ is a totally geodesic hypersphere, $dist(\partial T_j(V_j), A) \to \infty$ and, then some neighborhood of $\partial_\infty A \subset \operatorname{int} \partial_\infty T_j(V_j)$ for any j). In the same way as we did in the beginning, we can find supersolutions w_q of

$$\begin{cases} \operatorname{div}\left(\dfrac{\mathcal{A}(|\nabla u|)}{|\nabla u|}\nabla u\right) = 0 & \text{in } B_q \\ u = +\infty & \text{on } \partial B_q \\ u = \delta/2 & \text{on int } \partial_\infty B_q. \end{cases}$$

Since $u_j \leq w_q = \delta/2$ on int $\partial_\infty B_q$, the comparison principle implies that $u_j \leq w_q$ in B_q. Let $\tilde{B}_q \subset B_g$ be the hyperball with boundary equidistant to ∂B_q, for which $w_q < \delta$ in $\tilde{B}_q$. Hence $u_j < \delta$ in $\tilde{B}_q$ and, therefore, $u_j < \delta$ in $\tilde{B}$ for any j, where

$$\tilde{B} = \bigcup_{q \in \partial_\infty A \cap \partial_\infty(\mathbb{H}^n \backslash A)} \tilde{B}_q.$$

Observe that $\tilde{B}$ is a neighborhood of $\partial_\infty A \cap \partial_\infty(\mathbb{H}^n \backslash A)$ and $\partial A \backslash \tilde{B}$ is compact.

Now we prove that there exist $\nu > 0$ and $j_0 \in \mathbb{N}$ such that $u_j(y) \leq K - \nu$ for any $j \geq j_0$ and $y \in \partial A$ contradicting $u_j(y_j) > K - 1/j$ and $y_j \in \partial A$.

Let y be some point of $\tilde{B}$ such that the ball of radius 1 centered at y, $B_1(y)$, is contained in $\tilde{B}$. Due to the fact that $\partial A \backslash \tilde{B}$ is compact, there exists $\rho > 0$ such that the ball of radius $\rho + 1$, $B_{\rho+1}(y)$, contains $\partial A \backslash \tilde{B}$. Henceforth, we proceed as in Theorem 1.1, using Lemma 2.2. This lemma also holds in $\mathbb{H}^n$ and to prove it we define $f : [1, \infty) \to \mathbb{R}$ by

$$f(r) = \delta + \int_1^r \mathcal{A}^{-1}\left(\frac{\sinh^{n-1}(\alpha)}{\sinh^{n-1}(s)}\right) ds \quad \text{with} \quad 0 < \alpha \leq 1,$$

that satisfies

$$\mathcal{A}'(f'(r))f''(r) + \mathcal{A}(f'(r))(n-1)\coth r = 0,$$

and apply the same argument, obtaining a supersolution (indeed a solution) $w_y(x) = f(d(x))$. Then, we can consider h_0 and h_1 as in Lemma 2.2 and define $w = w_y + K + \varepsilon - h_0$, where ε satisfies $h_0 - h_1 - (K - \delta)/2 \leq \varepsilon < h_0 - h_1$. Take j_0 such that $1/j_0 < \varepsilon$ and $B_{2\rho+1}(y) \subset T_{j_0}(V_{j_0})$. From (5),

$$\sup_{\partial A} u_j \leq \sup_{T_j(V_j)} u_j < K + 1/j < K + \varepsilon \quad \text{for} \quad j \geq j_0.$$

Hence, following the same computation as in Theorem 1.1, w is a supersolution that satisfies $w \geq u_j$ in $B_{2\rho+1}(y) \backslash \overline{B_1(y)}$ for any $j \geq j_0$. Moreover $w <$

$h_1 + K + \varepsilon - h_0$ in $B_{\rho+1}(y)\backslash B_1(y) \supset \partial A\backslash\tilde{B}$. In $\partial A \cap \tilde{B}$, we also have $u_j < \delta < h_1 + K + \varepsilon - h_0$. Thus, defining $\nu = h_0 - h_1 - \varepsilon > 0$, it follows that

$$u_j < K - \nu \quad \text{in} \quad \partial A \quad \text{for} \quad j \geq j_0.$$

But this contradicts $u_j(y_j) > K - 1/j$ for any j. Therefore $K = 0$. In a similar way $\liminf_{x\to p} m(x) \geq 0$ completing the proof. □

2.3. Proof of Theorem 1.4

Proof. The idea is to build solutions that are constant along horospheres for which the asymptotic boundary is p_1. For that, let B_1 be some horoball such that the asymptotic boundary is p_1, $H_1 = \partial B_1$ and $d(x)$ the distance with sign given by

$$d(x) = \begin{cases} dist(x, H_1) & \text{if } x \in B_1 \\ -dist(x, H_1) & \text{if } x \notin B_1. \end{cases}$$

We search solutions of the form $m(x) = g(d(x))$, where $g : \mathbb{R} \to \mathbb{R}$ is a positive increasing function. From (2), we have that g must satisfy

$$\mathcal{A}'(g'(d))g''(d) + \mathcal{A}(g'(d))\Delta d = 0.$$

Since $d(x)$ is the distance (with sign) between x and the horosphere H_1, then $\Delta d(x) = -(n-1)$. Therefore

$$\mathcal{A}'(g'(d))g''(d) - (n-1)\mathcal{A}(g'(d)) = 0. \tag{6}$$

To find a solution to this equation, note first that $\mathcal{A}^{-1}(t)$ is defined for any $t > 0$, since $\mathcal{A}$ is unbounded. Hence we can consider the function

$$g_0(d) = \int_{-\infty}^{d} \mathcal{A}^{-1}(e^{(n-1)s})\, ds$$

for all $d \in \mathbb{R}$. This integral converges at $-\infty$ since condition (3) implies that $\mathcal{A}^{-1}(t) \leq (t/\bar{D})^{1/q}$ for $\mathcal{A}^{-1}(t) \in [0, \delta_0]$. Observe that g_0 is positive, increasing, satisfies equation (6), converges to 0 as $d \to -\infty$ and diverges to $+\infty$

as $d \to +\infty$, because $\mathcal{A}^{-1}$ is increasing. Therefore

$$m(x) = g_0(d(x))$$

is a C^2 solution of (2) that satisfies $m(x_k) \to +\infty$ if $x_k \to p_1$ with $d(x_k) \to +\infty$. Moreover, using that $d(x) \to -\infty$ as $x \to p \in \partial_\infty \mathbb{H}^n \backslash \{p_1\}$, it follows that $m(x) \to 0$ proving the result. $\square$

References

[1] T. Bourni, *$C^{1,\alpha}$ theory for the prescribed mean curvature equation with Dirichlet data*, Journal of Geom Analysis **21** (2011), 982–1035.

[2] Jb. Casteras, I. Holopainen, and J. Ripoll, *On the asymptotic Dirichlet problem for the minimal hypersurface equation in a Hadamard manifold*, Potential Analysis **47** (2017), 485-501.

[3] Jb. Casteras, I. Holopainen, and J. Ripoll, *Asymptotic Dirichlet problem for A-harmonic and minimal graph equations in Cartan-Hadamard manifolds*, to appear in Communications in Analysis and Geometry **27** (2019). `arXiv:1501.05249`.

[4] P. Collin and H. Rosenberg, *Construction of harmonic diffeomorphisms and minimal graphs*, Annals of Mathematics, second series **172** (2010), no. 3, 1879–1906.

[5] P. Eberlein and B. O'Neill, *Visibility manifolds*, Pacific J. Math. **46** (1973), 45–109.

[6] R. Finn, *New estimates for equations of minimal surface type*, Archive for Rational Mechanics and Analysis **14** (1963), 337–375.

[7] J. A. Gálvez and H. Rosenberg, *Minimal surfaces and harmonic diffeomorphisms from the complex plane onto certain Hadamard surfaces*, American Journal of Mathematics **132** (2010), no. 5, 1249–1273.

[8] I. Holopainen, *Asymptotic Dirichlet problem for the $p-$Laplacian in Cartan-Hadamard manifolds*, Proceedings of the AMS **130** (2002), no. 11, 3393–3400.

[9] J. C. C. Nitsche, Lectures on Minimal Surfaces, Cambridge University Press, Vol. 1, (1989).

[10] J. Ripoll and M. Telichevesky, *Complete minimal graphs with prescribed asymptotic boundary on rotationally symmetric Hadamard surfaces*, Geometriae Dedicata **161** (2012), 277–283.

[11] J. Ripoll and M. Telichevesky, *Regularity at infinity of Hadamard manifolds with respect to some elliptic operators and applications to asymptotic Dirichlet problems*, Transactions of the American Mathematical Society (Online) **367** (2015), 1523–1541.

Instituto de Matemática e Estatística
Universidade Federal do Rio Grande do Sul
Av. Bento Gonçalves 9500
91509-900 Porto Alegre-RS, Brazil
E-mail address: leonardo.bonorino@ufrgs.br
E-mail address: jaime.ripoll@ufrgs.br

Received May 25, 2016
Accepted March 23, 2017