

Existence and regularity of multivalued solutions to elliptic equations and systems

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We construct $C^{1,\mu}$ multivalued solutions to more general classes of elliptic equations and systems, including the minimal surface system with small boundary data and the Laplace equation. This extends work of Simon and Wickramasekera in which they construct a large class of $C^{1,\mu}$ multivalued solutions to the minimal surface equation. We use methods for differential equations, which are more general than the specific minimal submanifold approach adopted by Simon and Wickramasekera. We also prove the branch set of the graphs of the solutions are real analytic submanifolds by inductively using Schauder estimates.

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1. Introduction

We consider solutions to elliptic differential equations and systems which are represented by multivalued functions as developed by Almgren in [1]. In [10],

Simon and Wickramasekera constructed a rich class of $C^{1,\mu}$ q -valued solutions to the Dirichlet problem for the minimal surface equation on the cylinder $\mathcal{C} = B_1^2(0) \times \mathbb{R}^{n-2}$. However, the method of Simon and Wickramasekera was specific to the minimal surface equation and does not readily generalize to other elliptic equations or to elliptic systems. We extend the results of [10] by establishing in Theorem 2 the existence of $C^{1,\mu}$ q -valued solutions with small boundary data to the Dirichlet problem for a large class of elliptic systems and in Theorem 3 and Corollary 1 the existence of $C^{1,\mu}$ q -valued solutions (possibly without small data) to the Dirichlet problem for large classes of elliptic equations. In particular, we extend the results of [10] by giving examples of q -valued harmonic functions and branched minimal submanifolds with codimension greater than one. The boundary data of these solutions satisfy a k -fold symmetry condition as in [10]. Our approach uses techniques for differential equations, which have the advantage applying in a more general context than codimension one minimal surfaces.

We also study the regularity of the branch set of minimal immersions. The singular set of minimal submanifolds is known to have Hausdorff dimension at most $n - 2$ in the case of area minimizing n -dimensional integral currents due Almgren [1] and stationary graphs of $C^{1,\mu}$ two-valued functions due to Simon and Wickramasekera [11]. The branch set of the minimal surfaces constructed in [10] and this paper are obviously $C^{1,\mu}$ $(n - 2)$ -dimensional submanifolds. We extend these results by showing that the branch sets of minimal immersions constructed in [10] are locally real analytic $(n - 2)$ -dimensional submanifolds.

The methods of differential equations require adding and multiplying functions. However, it is not generally possible to add or multiply q -valued functions to obtain a q -valued sum or product. To handle this difficulty we consider q -valued functions \tilde{u} on an open set Ω in \mathbb{R}^n each associated with a map $u = (u_1, u_2, \dots, u_q) : \Omega \setminus [0, \infty) \times \{0\} \times \mathbb{R}^{n-2} \rightarrow (\mathbb{R}^m)^q$ such that $\tilde{u}(X) = \{u_1(X), u_2(X), \dots, u_q(X)\}$ as an unordered q -tuple for each $X \in \Omega \setminus [0, \infty) \times \{0\} \times \mathbb{R}^{n-2}$, as we can then add and multiply the corresponding maps u .

To construct q -valued solutions to elliptic equations and systems, we first prove Theorem 1, which establishes the existence of q -valued solutions to the Dirichlet problem for a class of Poisson equations. Using a change of variable $\xi_1 + i\xi_2 = (x_1 + ix_2)^{1/q}$, we transform the Poisson equation of q -valued functions into a singular differential equation of single-valued functions, which we can solve using Fourier analysis and standard elliptic theory. Using the average-free and k -fold symmetry properties of the solution, we obtain a bound on how the solution decays at points on the axis $\{0\} \times \mathbb{R}^{n-2}$

of \mathcal{C} , which implies the Hölder continuity of the gradient of the solution. The existence result for elliptic systems, Theorem 2, then follows from the contraction mapping principle and the existence results for elliptic equations, Theorem 3 and Corollary 1, follow from the Leray-Schauder theory.

The branch set of the graphs of the q -valued solutions \tilde{u} constructed in [10] is the graph of \tilde{u} over $\{0\} \times \mathbb{R}^{n-2}$. Thus the real analyticity of the branch set follows in Theorem 4, which establishes that a q -valued solution $\tilde{u}(x, y)$ (where $x \in \mathbb{R}^2$ and $y \in \mathbb{R}^{n-2}$) to a elliptic equation with real analytic data is real analytic with respect to y in the sense that \tilde{u} locally satisfies bounds of the form $|D_y^\gamma \tilde{u}(x, y)| \leq |\gamma|! C^{|\gamma|}$ for some constant $C \in (0, \infty)$. Note that an analogous regularity result, Theorem 5, also holds for elliptic systems. Rather than proving Theorem 4 by extending an approach of Morrey in [6] using integral kernels, we inductively apply the Schauder estimates. This argument readily yields $C^{1,\mu}$ estimates on derivatives $D_y^\gamma \tilde{u}$ for every multi-index γ , where $\mu \in (0, 1/q)$. More care is needed to obtain the particular type of bound on $D_y^\gamma \tilde{u}(x, y)$ required for real analyticity with respect to y . We obtain such bounds using a modified version of a technique due to Friedman [2] involving majorants.

2. Preliminaries and statement of main results

We adopt the following notation and conventions.

$n \geq 3$, $m \geq 1$, and $q \geq 2$ are fixed integers.

$B_R^l(X_0)$ denotes the open ball of radius R centered at X_0 in \mathbb{R}^l and $B_R(X_0) = B_R^n(X_0)$.

$\mathcal{C} = B_1^2(0) \times \mathbb{R}^{n-2}$ denotes an open cylinder in \mathbb{R}^n .

$X = (x, y)$ denotes a point in \mathbb{R}^n , where $x \in \mathbb{R}^2$ and $y \in \mathbb{R}^{n-2}$. We identify x with the point $re^{i\theta}$ in \mathbb{C} , where $r \in [0, \infty)$ and $\theta \in \mathbb{R}$.

Let $\mathcal{A}_q(\mathbb{R}^m)$ denote the space of unordered q -tuples $\tilde{u} = \{u_1, u_2, \dots, u_q\}$, where $u_1, u_2, \dots, u_q \in \mathbb{R}^m$ and we allow $u_i = u_j$ for $i \neq j$. We define a metric \mathcal{G} on $\mathcal{A}_q(\mathbb{R}^m)$ by

$$\mathcal{G}(\tilde{u}, \tilde{v}) = \min_{\sigma} \left(\sum_{l=1}^q |u_l - v_{\sigma(l)}|^2 \right)^{1/2}$$

for all unordered q -tuple $\tilde{u} = \{u_1, \dots, u_q\}$ and $\tilde{v} = \{v_1, \dots, v_q\}$, where the minimum is taken over all permutations σ of $\{1, \dots, q\}$. A q -valued function

\tilde{u} on a set $\Omega \subseteq \mathbb{R}^n$ is a map $\tilde{u} : \Omega \rightarrow \mathcal{A}_q(\mathbb{R}^m)$ (note that this definition of q -valued functions is equivalent to the definition of Almgren [1]). A q -valued function $\tilde{u} : \Omega \rightarrow \mathcal{A}_q(\mathbb{R}^m)$ is continuous at $X_0 \in \Omega$ if either X_0 is an isolated point of Ω or

$$\lim_{X \rightarrow X_0} \mathcal{G}(\tilde{u}(X), \tilde{u}(X_0)) = 0,$$

where the limit is taken over $X \in \Omega$. $C^0(\Omega; \mathcal{A}_q(\mathbb{R}^m))$ denotes the space of continuous q -valued functions $\tilde{u} : \Omega \rightarrow \mathcal{A}_q(\mathbb{R}^m)$. A q -valued function $\tilde{u} : \Omega \rightarrow \mathcal{A}_q(\mathbb{R}^m)$ is Hölder continuous with exponent $\mu \in (0, 1]$ if

$$[\tilde{u}]_{\mu, \Omega} \equiv \sup_{X, Y \in \Omega, X \neq Y} \frac{\mathcal{G}(\tilde{u}(X), \tilde{u}(Y))}{|X - Y|^\mu} < \infty.$$

A q -valued function $\tilde{u} : \Omega \rightarrow \mathcal{A}_q(\mathbb{R}^m)$ is differentiable at a point X in the interior of Ω if for some $m \times n$ matrices A_1, \dots, A_q ,

$$(2.1) \quad \lim_{h \rightarrow 0} \frac{\mathcal{G}(\tilde{u}(X + h), \{u_l(X) + A_l h\})}{|h|} = 0,$$

in which case we say $D\tilde{u}(X) \equiv \{A_1, \dots, A_q\}$ is the derivative of \tilde{u} at X . For each open set $\Omega \subseteq \mathbb{R}^n$, $C^1(\Omega; \mathcal{A}_q(\mathbb{R}^m))$ denotes the space of q -valued functions $\tilde{u} : \Omega \rightarrow \mathcal{A}_q(\mathbb{R}^m)$ such that $D\tilde{u}$ exists at each point in Ω and \tilde{u} and Du are continuous on Ω . For each $\mu \in (0, 1]$ and open set $\Omega \subseteq \mathbb{R}^n$, $C^{1,\mu}(\Omega; \mathcal{A}_q(\mathbb{R}^m))$ denotes the space of q -valued functions $\tilde{u} \in C^1(\Omega; \mathcal{A}_q(\mathbb{R}^m))$ such that $[D\tilde{u}]_{\mu; \Omega'} < \infty$ for every open set $\Omega' \subset \subset \Omega$.

Let Ω be an open set in \mathbb{R}^n and $\tilde{u} \in C^1(\Omega; \mathcal{A}_q(\mathbb{R}^m))$. We let $\mathcal{B}_{\tilde{u}}$ denote the set of points $X_0 \in \Omega$ such that there is no ball $B_R(X_0) \subseteq \Omega$ on which $\tilde{u} = \{u_1, u_2, \dots, u_q\}$ for some single-valued functions $u_1, u_2, \dots, u_q \in C^1(B_R(X_0); \mathbb{R}^m)$. We say \tilde{u} satisfies

$$D_i(A^i(X, \tilde{u}, D\tilde{u})) + B(X, \tilde{u}, D\tilde{u}) = 0 \text{ weakly in } \Omega \setminus \mathcal{B}_{\tilde{u}}$$

for continuous single-valued functions $A^i, B : \Omega \times \mathbb{R}^m \times \mathbb{R}^{mn} \rightarrow \mathbb{R}$ if for every ball $B_R(X_0) \subseteq \Omega \setminus \mathcal{B}_{\tilde{u}}$, $\tilde{u} = \{u_1, u_2, \dots, u_q\}$ on $B_R(X_0)$ for single-valued functions $u_l \in C^1(B_R(X_0); \mathbb{R}^m)$ such that

$$D_i(A^i(X, u_l, Du_l)) + B(X, u_l, Du_l) = 0 \text{ weakly in } B_R(X_0)$$

for $l = 1, 2, \dots, q$.

Observe that we cannot in general add or multiply multivalued functions. Given two q -valued functions $\tilde{u}, \tilde{v} : \Omega \rightarrow \mathcal{A}_q(\mathbb{R}^m)$, there is no canonical way to pair the elements $u_i(X)$ and $v_i(X)$ of the unordered q -tuples

$\tilde{u}(X) = \{u_1(X), u_2(X), \dots, u_q(X)\}$ and $\tilde{v}(X) = \{v_1(X), v_2(X), \dots, v_q(X)\}$ to obtain a sum

$$(\tilde{u} + \tilde{v})(X) = \{u_1(X) + v_1(X), u_2(X) + v_2(X), \dots, u_q(X) + v_q(X)\}$$

or product

$$(\tilde{u}\tilde{v})(X) = \{u_1(X)v_1(X), u_2(X)v_2(X), \dots, u_q(X)v_q(X)\}.$$

Moreover, for some q -valued functions $\tilde{u}, \tilde{v} \in C^1(\Omega; \mathcal{A}_q(\mathbb{R}^m))$, there is no way to pair the elements $u_i(X)$ and $v_i(X)$ to obtain a sum

$$(\tilde{u} + \tilde{v})(X) = \{u_1(X) + v_1(X), u_2(X) + v_2(X), \dots, u_q(X) + v_q(X)\}$$

that is C^1 on Ω ; for example, consider $\tilde{u}, \tilde{v} \in C^1(\mathbb{R}^2; \mathcal{A}_2(\mathbb{R}))$ given by $\tilde{u}(x_1, x_2) = \{\pm \operatorname{Re}(x_1 + ix_2 - 1)^{3/2}\}$ and $\tilde{v}(x_1, x_2) = \{\pm \operatorname{Re}(x_1 + ix_2 + 1)^{3/2}\}$. In what follows, we develop a theory of multivalued solutions to linear and quasilinear elliptic differential equations, which requires adding and multiplying functions. Rather than working with multivalued functions directly, we will work with functions $u : \Omega \setminus [0, \infty) \times \{0\} \times \mathbb{R}^{n-2} \rightarrow (\mathbb{R}^m)^q$, which we can add and multiply. The class of functions u that we consider take the form $u(X) = (u_1(X), u_2(X), \dots, u_q(X))$ at each $X \in \Omega \setminus [0, \infty) \times \{0\} \times \mathbb{R}^{n-2}$, where $u_l : \Omega \setminus [0, \infty) \times \{0\} \times \mathbb{R}^{n-2} \rightarrow \mathbb{R}^m$ for $l = 1, 2, \dots, q$, and, roughly speaking, satisfy $\lim_{x_2 \uparrow 0} u_l(x_1, x_2, y) = \lim_{x_2 \downarrow 0} u_{l+1}(x_1, x_2, y)$ for $l = 1, 2, \dots, q - 1$ and $\lim_{x_2 \uparrow 0} u_q(x_1, x_2, y) = \lim_{x_2 \downarrow 0} u_1(x_1, x_2, y)$ whenever $(0, x_2, y) \in \Omega$. To each such map u we will associate a q -valued function $\tilde{u} : \Omega \rightarrow \mathcal{A}_q(\mathbb{R}^m)$ such that $\tilde{u}(X) = \{u_1(X), u_2(X), \dots, u_q(X)\}$ for $X \in \Omega \setminus [0, \infty) \times \{0\} \times \mathbb{R}^{n-2}$.

Definition 1. Let Ω be an open set in \mathbb{R}^n and $k \geq 0$ be an integer. $C^{k;q}(\Omega; \mathbb{R}^m)$ denotes the set of maps $u = (u_1, u_2, \dots, u_q) : \Omega \setminus [0, \infty) \times \{0\} \times \mathbb{R}^{n-2} \rightarrow (\mathbb{R}^m)^q$ such that $u_l|_{\Omega \cap \mathbb{R} \times (0, \infty) \times \mathbb{R}^{n-2}}$ extend to C^k functions on $\Omega \cap \mathbb{R} \times [0, \infty) \times \mathbb{R}^{n-2}$ and $u_l|_{\Omega \cap \mathbb{R} \times (-\infty, 0) \times \mathbb{R}^{n-2}}$ extend to C^k functions on $\Omega \cap \mathbb{R} \times (-\infty, 0] \times \mathbb{R}^{n-2}$ for $l = 1, 2, \dots, q$ and

$$\begin{aligned} \lim_{x_2 \uparrow 0} D^\alpha u_l(x_1, x_2, y) &= \lim_{x_2 \downarrow 0} D^\alpha u_{l+1}(x_1, x_2, y) \text{ for } l = 1, 2, \dots, q - 1, \\ \lim_{x_2 \uparrow 0} D^\alpha u_q(x_1, x_2, y) &= \lim_{x_2 \downarrow 0} D^\alpha u_1(x_1, x_2, y), \end{aligned}$$

for all $x_1 \geq 0, y \in \mathbb{R}^{n-2}$, and $|\alpha| \leq k$. Given $u \in C^{k;q}(\Omega; \mathbb{R}^m)$, we let

$$D^\alpha u(0, y) \equiv \lim_{x \rightarrow 0} D^\alpha u_1(x, y)$$

whenever $(0, y) \in \Omega$ and $|\alpha| \leq k$. We let $C^{\infty;q}(\Omega; \mathbb{R}^m) = \bigcap_{k=0}^{\infty} C^{k;q}(\Omega; \mathbb{R}^m)$. $C_c^{k;q}(\Omega; \mathbb{R}^m)$ denotes the set of $u \in C^{k;q}(\Omega; \mathbb{R}^m)$ such that for some $\Omega' \subset\subset \Omega$, $u = 0$ on $(\Omega \setminus \Omega') \setminus [0, \infty) \times \{0\} \times \mathbb{R}^{n-2}$.

Definition 2. Let Ω be an open set in \mathbb{R}^n , $k \geq 0$ be an integer, and $\mu \in (0, 1]$. $C^{k,\mu;q}(\Omega; \mathbb{R}^m)$ denotes the set of maps $u \in C^{k;q}(\Omega; \mathbb{R}^m)$ such that $u_l|_{\Omega \cap \mathbb{R} \times (0, \infty) \times \mathbb{R}^{n-2}}$ extend to $C^{k,\mu}$ functions on $\Omega \cap \mathbb{R} \times [0, \infty) \times \mathbb{R}^{n-2}$ and $u_l|_{\Omega \cap \mathbb{R} \times (-\infty, 0) \times \mathbb{R}^{n-2}}$ extend to $C^{k,\mu}$ functions on $\Omega \cap \mathbb{R} \times (-\infty, 0] \times \mathbb{R}^{n-2}$ for $l = 1, 2, \dots, q$.

To each $u \in C^{k;q}(\Omega; \mathbb{R}^m)$, where $k \in \{0, 1\}$, we associate a unique q -valued function $\tilde{u} \in C^k(\Omega; \mathcal{A}_q(\mathbb{R}^m))$ given by $\tilde{u}(X) = \{u_1(X), u_2(X), \dots, u_q(X)\}$ for $X \in \Omega \setminus [0, \infty) \times \{0\} \times \mathbb{R}^{n-2}$. Of course, more than one $u \in C^{k;q}(\Omega; \mathbb{R}^m)$ may be associated with the same q -valued function \tilde{u} .

Let Ω be an open set in \mathbb{R}^n . Given a set $S \subseteq \Omega$, we define

$$(2.2) \quad \begin{aligned} \inf_S u &\equiv \inf_{X \in S \setminus [0, \infty) \times \{0\} \times \mathbb{R}^{n-2}} \min\{u_1(X), \dots, u_q(X)\}, \\ \sup_S u &\equiv \sup_{X \in S \setminus [0, \infty) \times \{0\} \times \mathbb{R}^{n-2}} \max\{u_1(X), \dots, u_q(X)\}, \end{aligned}$$

for each $u \in C^{0;q}(\Omega; \mathbb{R})$ and we define

$$(2.3) \quad \sup_S |u| \equiv \sup_{X \in S \setminus [0, \infty) \times \{0\} \times \mathbb{R}^{n-2}} \max\{|u_1(X)|, \dots, |u_q(X)|\}.$$

for each $u \in C^{0;q}(\Omega; \mathbb{R}^m)$. Note that if instead $u : \Omega \setminus [0, \infty) \times \{0\} \times \mathbb{R}^{n-2} \rightarrow (\mathbb{R}^m)^q$ is measurable, we can define $\inf_{\Omega} u$ and $\sup_{\Omega} u$ if $m = 1$ and $\sup_S |u|$ by (2.2) and (2.3) by replacing the infimums and supremums with essential infimums and supremums. We say $u \in C^{0;q}(\Omega; \mathbb{R})$ attains its maximum value at $X_0 \in \Omega$ if either $X_0 \in \Omega \setminus [0, \infty) \times \{0\} \times \mathbb{R}^{n-2}$ and

$$\sup_{\Omega} u = \max\{u_1(X_0), u_2(X_0), \dots, u_q(X_0)\}$$

or $X_0 \in \Omega \cap [0, \infty) \times \{0\} \times \mathbb{R}^{n-2}$ and

$$\sup_{\Omega} u = \lim_{X \rightarrow X_0} \max\{u_1(X), u_2(X), \dots, u_q(X)\}.$$

For each integer $k \geq 0$,

$$\|u\|_{C^{k;q}(\Omega)} \equiv \sum_{|\alpha| \leq k} \sup_{\Omega} |D^{\alpha} u|$$

for every $u \in C^{k;q}(\Omega; \mathbb{R}^m)$. For each integer $k \geq 0$ and $\mu \in (0, 1]$,

$$[u]_{\mu;q,\Omega} \equiv \sum_{l=1}^q ([u_l]_{\mu;\Omega \cap \mathbb{R} \times (0,\infty) \times \mathbb{R}^{n-2}} + [u_l]_{\mu;\Omega \cap \mathbb{R} \times (-\infty,0) \times \mathbb{R}^{n-2}})$$

for every $u \in C^{0,\mu;q}(\Omega; \mathbb{R}^m)$ and

$$\|u\|_{C^{k,\mu;q}(\Omega)} \equiv \sum_{|\alpha| \leq k} \sup_{\Omega} |D^\alpha u| + \sum_{|\alpha|=k} \sup_{\Omega} [D^\alpha u]_{\mu;q,\Omega}$$

for every $u \in C^{k,\mu;q}(\Omega; \mathbb{R}^m)$. When $\Omega = B_R(X_0)$ is an open ball, we define

$$\|u\|'_{C^{k;q}(B_R(X_0))} \equiv \sum_{|\alpha| \leq k} R^{|\alpha|} \sup_{B_R(X_0)} |D^\alpha u|$$

for every $u \in C^{k;q}(B_R(X_0); \mathbb{R}^m)$ and

$$\|u\|'_{C^{k,\mu;q}(B_R(X_0))} \equiv \sum_{|\alpha| \leq k} R^{|\alpha|} \sup_{B_R(X_0)} |D^\alpha u| + \sum_{|\alpha|=k} \sup_{\Omega} R^{k+\mu} [D^\alpha u]_{\mu;q,\Omega}.$$

for every $u \in C^{k,\mu;q}(B_R(X_0); \mathbb{R}^m)$. Given a sequence $\{u^{(j)}\}_{j=1,2,3,\dots}$ in $C^{k;q}(\Omega; \mathbb{R}^m)$ and $u \in C^{k;q}(\Omega)$, we say $u^{(j)} \rightarrow u$ in $C^{k;q}(\Omega; \mathbb{R}^m)$ if $\|u^{(j)} - u\|_{C^{k;q}(\Omega)} \rightarrow 0$. Note that if Ω is a bounded open set in \mathbb{R}^n and $u^{(j)} = (u_1^{(j)}, u_2^{(j)}, \dots, u_q^{(j)})$, $j = 1, 2, 3, \dots$, is a sequence in $C^{k,\mu;q}(\Omega; \mathbb{R}^m)$ such that $\sup_j \|u^{(j)}\|_{C^{k,\mu;q}(\Omega)} < \infty$, then by Arzela-Ascoli applied using the sequences $\{u_l^{(j)}\}_{\mu;\Omega \cap \mathbb{R} \times (0,\infty) \times \mathbb{R}^{n-2}}$ and $\{u_l^{(j)}\}_{\mu;\Omega \cap \mathbb{R} \times (-\infty,0) \times \mathbb{R}^{n-2}}$ for $l = 1, 2, \dots, q$, there is a subsequence $\{u^{(j_i)}\}_{i=1,2,3,\dots}$ of $\{u^{(j)}\}_{j=1,2,3,\dots}$ and $u \in C^{k,\mu;q}(\Omega; \mathbb{R}^m)$ such that $u^{(j_i)} \rightarrow u$ in $C^{k;q}(\Omega; \mathbb{R}^m)$ as $i \rightarrow \infty$.

Given open sets $\Omega' \subset\subset \Omega \subseteq \mathbb{R}^n$, $h \in \mathbb{R}$ and $\eta \in \mathbb{R}^{n-2}$ such that $0 < |h\eta| < \text{dist}(\Omega', \partial\Omega)$, and $u \in C^{0;q}(\Omega; \mathbb{R}^m)$, we define

$$\delta_{h,\eta}u = (\delta_{h,\eta}u_1, \delta_{h,\eta}u_2, \dots, \delta_{h,\eta}u_q) \in C^{0;q}(\Omega'; \mathbb{R}^m)$$

by

$$(2.4) \quad \delta_{h,\eta}u_l(x, y) \equiv \frac{u_l(x, y + h\eta) - u_l(x, y)}{h}$$

for all $(x, y) \in \Omega' \setminus [0, \infty) \times \{0\} \times \mathbb{R}^{n-2}$ and $l = 1, 2, \dots, q$.

Definition 3. Let $\Omega \subseteq \mathbb{R}^n$. For $1 \leq p < \infty$, $L^{p;q}(\Omega; \mathbb{R}^m)$ denotes the set of Lebesgue measurable functions $u = (u_1, u_2, \dots, u_q) : \Omega \rightarrow (\mathbb{R}^m)^q$ such that

$$\|u\|_{L^{p;q}(\Omega)} \equiv \left(\int_{\Omega} \sum_{l=1}^q |u_l|^p \right)^{1/p} < \infty.$$

$L^{\infty;q}(\Omega; \mathbb{R}^m)$ denotes the set of Lebesgue measurable functions $u = (u_1, u_2, \dots, u_q) : \Omega \rightarrow (\mathbb{R}^m)^q$ such that $\sup_{\Omega} |u_l| < \infty$ for $l = 1, 2, \dots, q$.

Definition 4. Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $1 \leq p < \infty$ and $k \geq 1$ be an integer. $W^{k,p;q}(\Omega; \mathbb{R}^m)$ denotes the set of $u = (u_1, u_2, \dots, u_q) \in L^{p;q}(\Omega; \mathbb{R}^m)$ such that for every α with $|\alpha| \leq k$ there exists a $v = (v_1, v_2, \dots, v_q) \in L^{p;q}(\Omega; \mathbb{R}^m)$ (depending on α) such that

$$\int_{\Omega} \sum_{l=1}^q u_l D^{\alpha} \zeta_l = (-1)^{|\alpha|} \int_{\Omega} \sum_{l=1}^q v_l \zeta_l$$

for every $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_q) \in C_c^{k;q}(\Omega; \mathbb{R}^m)$. $D^{\alpha}u \equiv v$ denotes the order α weak derivative of u .

To each measurable function $u : \Omega \rightarrow (\mathbb{R}^m)^q$ we associate a measurable q -valued function $\tilde{u} : \Omega \rightarrow \mathcal{A}_q(\mathbb{R}^m)$ given by $\tilde{u}(X) = \{u_1(X), u_2(X), \dots, u_q(X)\}$ for $X \in \Omega$. \tilde{u} is unique up to its values \mathcal{L}^n -a.e. on Ω .

For each integer $k \geq 1$ and $1 \leq p < \infty$,

$$\|u\|_{W^{k,p;q}(\Omega)} \equiv \sum_{|\alpha| \leq k} \sup_{\Omega} \|D^{\alpha}u\|_{L^{p;q}(\Omega)}$$

for every $u \in W^{k,p;q}(\Omega; \mathbb{R}^m)$. Given a sequence $u^{(j)} \in L^{p;q}(\Omega)$, $j = 1, 2, 3, \dots$, and $u \in L^{p;q}(\Omega)$, we say $u^{(j)} \rightarrow u$ in $L^{p;q}(\Omega; \mathbb{R}^m)$ if $\|u^{(j)} - u\|_{L^{p;q}(\Omega)} \rightarrow 0$. Given a sequence $\{u^{(j)}\}_{j=1,2,3,\dots}$ in $W^{k,p;q}(\Omega)$ and $u \in W^{k,p;q}(\Omega)$, we say $u^{(j)} \rightarrow u$ in $W^{k,p;q}(\Omega; \mathbb{R}^m)$ if $\|u^{(j)} - u\|_{W^{k,p;q}(\Omega)} \rightarrow 0$. We let $W_0^{k,p;q}(\Omega; \mathbb{R}^m)$ denote the closure of $C_c^{k;q}(\Omega; \mathbb{R}^m)$ in the Banach space $W^{k,p;q}(\Omega; \mathbb{R}^m)$. Note that if $u^{(j)} = (u_1^{(j)}, u_2^{(j)}, \dots, u_q^{(j)})$, $j = 1, 2, 3, \dots$, is a sequence in $W^{1,2;q}(\Omega; \mathbb{R}^m)$ such that $\sup_j \|u^{(j)}\|_{W^{1,2;q}(\Omega)} < \infty$, then by Rellich's compactness lemma applied to the sequences

$$\{u_l^{(j)}\}_{\mu; \Omega \cap \mathbb{R} \times (0, \infty) \times \mathbb{R}^{n-2}} \quad \text{and} \quad \{u_l^{(j)}\}_{\mu; \Omega \cap \mathbb{R} \times (-\infty, 0) \times \mathbb{R}^{n-2}}$$

for $l = 1, 2, \dots, q$, there is a subsequence $\{u^{(j_i)}\}_{i=1,2,3,\dots}$ of $\{u^{(j)}\}_{j=1,2,3,\dots}$ and $u \in W^{1,2;q}(\Omega; \mathbb{R}^m)$ such that $u^{(j_i)} \rightarrow u$ strongly in $L^{2;q}(\Omega; \mathbb{R}^m)$ as $i \rightarrow \infty$ and $\|Du\|_{L^{2;q}(\Omega)} \leq \liminf_j \|Du^{(j)}\|_{L^{2;q}(\Omega)}$.

Given a set $\Omega \subseteq \mathbb{R}^n$ and $u = (u_1, u_2, \dots, u_q) : \Omega \setminus [0, \infty) \times \{0\} \times \mathbb{R}^{n-2} \rightarrow (\mathbb{R}^m)^q$, there exists $u_a : \Omega \setminus [0, \infty) \times \{0\} \times \mathbb{R}^{n-2} \rightarrow \mathbb{R}^m$ and $u_f = (u_{f,1}, u_{f,2}, \dots, u_{f,q}) : \Omega \setminus [0, \infty) \times \{0\} \times \mathbb{R}^{n-2} \rightarrow (\mathbb{R}^m)^q$ such that

$$(2.5) \quad u_l = u_a + u_{f,l} \text{ for } l = 1, 2, \dots, q, \quad \text{where } u_a = \frac{1}{q} \sum_{j=1}^q u_j.$$

We call u_a the average of u . We say u is average-free if $u_a = 0$ on Ω . u_f is average-free and thus we call u_f the average-free part of u .

The first of our main results concern the existence of solutions to the Dirichlet problem for elliptic differential equations in the cylinder $\mathcal{C} = B_1^2(0) \times \mathbb{R}^{n-2}$. Fix an integer $k \geq 2$ such that k and q are relatively prime. We say $u \in C^{0;q}(\bar{\mathcal{C}}; \mathbb{R}^m)$ is k -fold symmetric if

$$\begin{aligned} u_l(re^{i\theta+i2\pi/k}, y) &= u_l(re^{i\theta}, y) \text{ if } 0 < \theta < 2\pi - 2\pi/k, \quad l = 1, 2, \dots, q, \\ u_1(re^{i\theta+i2\pi/k}, y) &= u_q(re^{i\theta}, y) \text{ if } 2\pi - 2\pi/k < \theta < 2\pi, \\ u_l(re^{i\theta+i2\pi/k}, y) &= u_{l-1}(re^{i\theta}, y) \text{ if } 2\pi - 2\pi/k < \theta < 2\pi, \quad l = 2, 3, \dots, q, \end{aligned}$$

for all $(re^{i\theta}, y) \in \Omega$. We will let \mathbf{R} denote the $n \times n$ matrix such that

$$\mathbf{R}(re^{i\theta}, y) = (re^{i\theta+i2\pi/k}, y).$$

We write

$$\mathbf{R} = (R_j^i)_{i,j=1,\dots,n} = \begin{pmatrix} \cos(2\pi/k) & -\sin(2\pi/k) & 0 & 0 & \dots & 0 \\ \sin(2\pi/k) & \cos(2\pi/k) & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

where R_i^j denotes the entry in the i -th row and j -th column of \mathbf{R} . We say u is periodic with respect to y_j with period $\rho_j > 0$ for $j = 1, 2, \dots, n - 2$ if

$$u_l(x, y + \rho_j e_j) = u_l(x, y)$$

for all $(x, y) \in \Omega \setminus [0, \infty) \times \{0\} \times \mathbb{R}^{n-2}$, $l = 1, 2, \dots, q$, and $j = 1, 2, \dots, n - 2$, where e_1, e_2, \dots, e_{n-2} denotes the standard basis for \mathbb{R}^{n-2} .

We will be interested in the regularity of multivalued solutions up to the boundary of \mathcal{C} . Recall that the continuity of multivalued functions on $\bar{\mathcal{C}}$ is defined above. We say $\tilde{u} : \bar{\mathcal{C}} \rightarrow \mathcal{A}_q(\mathbb{R}^m)$ is differentiable at $X_0 \in \partial\mathcal{C}$ if (2.1) holds when the limit is taken over h such that $X + h \in \bar{\mathcal{C}}$. We let $C^1(\bar{\mathcal{C}}; \mathcal{A}_q(\mathbb{R}^m))$ denote the space of $\tilde{u} : \bar{\mathcal{C}} \rightarrow \mathcal{A}_q(\mathbb{R}^m)$ that are continuously differentiable on $\bar{\mathcal{C}}$. We say $\tilde{u} \in C^{1,\mu}(\bar{\mathcal{C}}; \mathcal{A}_q(\mathbb{R}^m))$ for $\mu \in (0, 1]$ if $\tilde{u} \in C^1(\bar{\mathcal{C}}; \mathcal{A}_q(\mathbb{R}^m))$ and $[D\tilde{u}]_{\mu; B_1^2(0) \times B_\rho^{n-2}(0)} < \infty$ for all $\rho \in (0, \infty)$. We define $C^{k;q}(\bar{\mathcal{C}}; \mathbb{R}^m)$ for integers $k \geq 0$ by Definition 1 with $\bar{\mathcal{C}}$ in place of Ω . We say $u \in C^{k,\mu;q}(\bar{\mathcal{C}}; \mathbb{R}^m)$ for an integer $k \geq 0$ and $\mu \in (0, 1)$ if $u \in C^{k;q}(\bar{\mathcal{C}}; \mathbb{R}^m)$ and $[D^k u]_{\mu;q, B_1^2(0) \times B_\rho^{n-2}(0)} < \infty$ for all $\rho \in (0, \infty)$. Note that given a set $S \subseteq \bar{\mathcal{C}}$, we define $\inf_S u$ and $\sup_S u$ for $u \in C^0(\bar{\mathcal{C}})$ by (2.2) and $\sup_S |u|$ for $u \in C^0(\bar{\mathcal{C}}; \mathbb{R}^m)$ by (2.3).

We will first establish the existence of solutions in $C^{0;q}(\bar{\mathcal{C}}) \cap C^{1,\mu;q}(\mathcal{C})$ to weak Poisson equations:

Theorem 1. *Let $\mu \in (0, 1/q)$ and $k > q$ be an integer such that k and q are relatively prime. Given $f^j = (f_1^j, f_2^j, \dots, f_q^j) \in C^{0,\mu;q}(\bar{\mathcal{C}})$ and $g, \varphi \in C^{0;q}(\bar{\mathcal{C}})$ such that*

$$\begin{aligned}
 (2.6) \quad f_l^j(re^{i\theta+i2\pi/k}, y) &= \sum_{p=1}^n R_p^j f_l^p(re^{i\theta}, y) \\
 &\quad \text{if } 0 < \theta < 2\pi - 2\pi/k, \quad l = 1, 2, \dots, q, \\
 f_1^j(re^{i\theta+i2\pi/k}, y) &= \sum_{p=1}^n R_p^j f_q^p(re^{i\theta}, y) \\
 &\quad \text{if } 2\pi - 2\pi/k < \theta < 2\pi, \\
 f_l^j(re^{i\theta+i2\pi/k}, y) &= \sum_{p=1}^n R_p^j f_{l-1}^p(re^{i\theta}, y) \\
 &\quad \text{if } 2\pi - 2\pi/k < \theta < 2\pi, \quad l = 2, 3, \dots, q,
 \end{aligned}$$

for all $(re^{i\theta}, y) \in \mathcal{C}$, g and φ are k -fold symmetric, and

$$\sup_{\partial\mathcal{C}} |\varphi| + [f]_{\mu;q,\mathcal{C}} + \sup_{\mathcal{C}} |g| < \infty,$$

there is a $u \in C^{0;q}(\bar{\mathcal{C}}) \cap C^{1,\mu;q}(\mathcal{C})$ such that u is k -fold symmetric,

$$\begin{aligned}
 (2.7) \quad \int_{\mathcal{C}} \sum_{l=1}^q D_j u_l D_j \zeta_l &= \int_{\mathcal{C}} \sum_{l=1}^q (f_l^j D_j \zeta_l - g_l \zeta_l) \\
 &\quad \text{for all } \zeta \in C_c^{1;q}(\mathcal{C} \setminus \{0\} \times \mathbb{R}^{n-2}), \\
 u_l &= \varphi_l \quad \text{on } \partial\mathcal{C},
 \end{aligned}$$

for $l = 1, 2, \dots, q$ and

$$\sup_C |u| \leq C \left(\sup_{\partial \bar{C}} |\varphi| + [f]_{\mu; q, C} + \sup_C |g| \right).$$

Moreover, if $f^j, g,$ and φ are periodic with respect to y_i with period $\rho_i > 0$ for $i = 1, 2, \dots, n - 2,$ then u is the unique solution to (2.7) that is periodic with respect to y_i with period ρ_i for $i = 1, 2, \dots, n - 2.$

Note that in the special case where $f^j = 0$ and $g = 0,$ the q -valued function $\tilde{u}(X) = \{u_1(X), u_2(X), \dots, u_q(X)\}$ associated with the solution u obtained in Theorem 1 is a q -valued function in $C^0(\bar{C}; \mathcal{A}_q(\mathbb{R})) \cap C^{1, \mu}(C; \mathcal{A}_q(\mathbb{R}))$ such that $\Delta \tilde{u} = 0$ weakly in $C \setminus \mathcal{B}_{\tilde{u}}.$

To prove Theorem 1, we first assume $f^j, g,$ and φ are periodic with respect to each $y_i,$ as the general result follows by approximation of $f^j, g,$ and $\varphi.$ We use the change of variable $\xi_1 + i\xi_2 = (x_1 + ix_2)^{1/q}$ to transform $u(x_1, x_2, y)$ to a single-valued function $u_0(\xi_1, \xi_2, y)$ defined by $u_0(re^{i\theta}, y) = u_l(r^{1/q}e^{i\theta/q}, y)$ for $r \in [0, 1], \theta \in (2(l - 1)\pi, 2l\pi),$ and $y \in \mathbb{R}^{n-2}.$ The single-valued function u_0 satisfies a singular differential equation which we solve using Fourier series with respect to the y_i variables and the existence theory for single-valued solutions to elliptic equations to solve for the Fourier coefficients as functions of ξ_1 and $\xi_2.$ By linearity, we can assume that $f^j, g,$ and φ are all average-free and therefore the constructed solution u will be average-free and k -fold symmetric. The average-free and k -fold symmetry conditions on u will guarantee that $u(x, y)$ decays sufficiently quickly as x approaches zero to guarantee that $u \in C^{1, \mu; q}(C).$

Using Theorem 1 and the contraction mapping principle, we can construct solutions to quasilinear elliptic systems with small boundary data φ in $C^{1, \mu; q}(\bar{C}; \mathbb{R}^m):$

Theorem 2. *Let $m \geq 1$ be an integer, $\mu \in (0, 1/q),$ and $k > q$ be an integer such that k and q are relatively prime. Let $F_\kappa^i \in C^2(\mathbb{R}^{mn})$ and $G_\kappa \in C^1(\mathbb{R}^m \times \mathbb{R}^{mn})$ be single-valued functions, where \mathbb{R}^{mn} is the space of $m \times n$ matrices, such that $F_\kappa^i(0) = 0, DF_\kappa^i(0) = 0, G_\kappa(0, 0) = 0, DG_\kappa(0, 0) = 0,$ and*

$$F_\kappa^i(P\mathbf{R}) = R_i^j F_\kappa^j(P), \quad G_\kappa(Z, P\mathbf{R}) = G_\kappa(Z, P)$$

for all $P \in \mathbb{R}^{mn}.$ For some $\varepsilon > 0$ depending on $m, n, q, \mu, F_\kappa^i,$ and $G_\kappa,$ if $\varphi \in C^{1, \mu; q}(\bar{C}, \mathbb{R}^m)$ is k -fold symmetric with $\|\varphi\|_{C^{1, \mu; q}(C)} \leq \varepsilon,$ there exists a

solution $u \in C^{1,\mu;q}(\bar{\mathcal{C}}; \mathbb{R}^m)$ to

$$(2.8) \quad \int_{\mathcal{C}} \sum_{l=1}^q D_j u_l^\kappa D_j \zeta_l^\kappa = \int_{\mathcal{C}} \sum_{l=1}^q (F_\kappa^i(Du_l) D_i \zeta^\kappa - G_\kappa(u_l, Du_l) \zeta^\kappa)$$

for all $\zeta \in C_c^{1;q}(\mathcal{C} \setminus \{0\} \times \mathbb{R}^{n-2}; \mathbb{R}^m)$,

$u_l = \varphi_l$ on $\partial\mathcal{C}$ for $l = 1, 2, \dots, q$.

Moreover, u is k -fold symmetric and $\|u\|_{C^{1,\mu;q}(\mathcal{C})} \leq \varepsilon$. The q -valued function $\tilde{u}(X) = \{u_1(X), u_2(X), \dots, u_q(X)\}$ associated with u is a q -valued solution in $C^{1,\mu}(\bar{\mathcal{C}}; \mathcal{A}_q(\mathbb{R}^m))$ to

$$\Delta \tilde{u}^\kappa - D_i F_\kappa^i(D\tilde{u}) - G_\kappa(\tilde{u}, D\tilde{u}) = 0 \text{ weakly in } \mathcal{C} \setminus \mathcal{B}_{\tilde{u}}.$$

In particular, Theorem 2 yields q -valued solutions $\tilde{u} \in C^{1,\mu}(\mathcal{C}; \mathcal{A}_q(\mathbb{R}^m))$ to the minimal surface system in $\mathcal{C} \setminus \mathcal{B}_{\tilde{u}}$. For sufficiently small $\varepsilon > 0$, these solutions to the minimal surface system are stable in the sense that

$$(2.9) \quad \int_{\Sigma_{\tilde{u}}} \left(\sum_{i=1}^n |(D_{\tau_i} \mathbf{X})^\perp|^2 - \sum_{i,j=1}^n |\mathbf{X} \cdot A(\tau_i, \tau_j)|^2 \right) \geq 0$$

for all normal vector fields $\mathbf{X} \in C_c^0(\Sigma_{\tilde{u}}; \mathbb{R}^{n+m}) \cap W^{1,2}(\Sigma_{\tilde{u}}, \mathbb{R}^{n+m})$, where $\Sigma_{\tilde{u}}$ is the graph of \tilde{u} regarded as an immersed submanifold. (2.9) holds true in the case that $\mathbf{X} = 0$ near $\{0\} \times \mathbb{R}^{n-2+m}$ by the convexity of the area functional. To prove (2.9) for general \mathbf{X} , for $\delta \in (0, 1)$ let $\chi_\delta \in C^1([0, \infty))$ be the logarithmic cutoff function given by $\chi_\delta = 0$ on $B_{\delta^2}^2(0) \times \mathbb{R}^{n-2-m}$, $\chi_\delta(x, y, Z) = -\log(|x|/\delta^2)/\log(\delta)$ if $x \in B_\delta^2(0) \setminus B_{\delta^2}^2(0)$, $y \in \mathbb{R}^{n-2}$, and $Z \in \mathbb{R}^m$, and $\chi_\delta = 1$ on $\mathbb{R}^{n+m} \setminus B_\delta^2(0) \times \mathbb{R}^{n-2+m}$. Replace \mathbf{X} by $\chi_\delta \mathbf{X}$ in (2.9) and let $\delta \downarrow 0$ to obtain (2.9) with the original \mathbf{X} . Theorem 2 also yields q -valued solutions to the Euler-Lagrange equations for functionals of the form $\int_{\mathcal{C}} (|Du|^2 + f(Du))$ where $f \in C^3(\mathbb{R}^{mn}; \mathbb{R})$ is a single-valued function such that $Df(0) = 0$, $D^2f(0) = 0$, and $f(P\mathbf{R}_{2\pi/k}) = f(P)$ for all $P \in \mathbb{R}^{mn}$.

Note that Theorem 2 would not be true without the assumption of small boundary data as a consequence of [5], which showed that for some boundary data there are no C^1 single-valued solutions to the Dirichlet problem for the minimal surface system.

We also use Theorem 1 and the Leray-Schauder theory to construct solutions to general quasilinear elliptic equations (without assuming small boundary data):

Theorem 3. *Let $k > q$ be an integer such that k and q are relatively prime. Let $A^i \in C^2(\mathbb{R}^n)$, $B \in C^1(\mathbb{R} \times \mathbb{R}^n)$ be single-valued functions such that*

$$(2.10) \quad A^i(P\mathbf{R}) = R_i^j A^j(P), \quad B(Z, P\mathbf{R}) = B(Z, P).$$

Suppose

$$0 < \lambda(P)|\xi|^2 \leq D_{P_j} A^i(P)\xi_i \xi_j \leq \Lambda(P)|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n$$

for some continuous positive functions λ and Λ , the structure conditions

$$(2.11) \quad B(Z, P) \operatorname{sgn} Z/\lambda(P) \leq \beta_1|P| + \beta_2,$$

$$(2.12) \quad |\Lambda(P)| + |B(Z, P)| \leq \beta_3\lambda(P)|P|^2 \text{ if } |P| \geq 1,$$

for some constants $\beta_1, \beta_2, \beta_3 \in (0, \infty)$, and $B(Z, P)$ is non-increasing in Z for fixed $P \in \mathbb{R}^n$. Let $\varphi \in C^{2;q}(\bar{\mathcal{C}})$ is k -fold symmetric with $\|\varphi\|_{C^{2;q}(\bar{\mathcal{C}})} < \infty$. Then there exists a $u \in C^{1;q}(\bar{\mathcal{C}})$ such that $u \in C^{1,\mu;q}(\bar{\mathcal{C}})$ for every $\mu \in (0, 1/q)$ and

$$(2.13) \quad \int_{\mathcal{C}} \sum_{l=1}^q (A^i(Du_l)D_i \zeta_l - B(u_l, Du_l)\zeta_l) = 0$$

for all $\zeta \in C_c^{1;q}(\mathcal{C} \setminus \{0\} \times \mathbb{R}^{n-2})$,
 $u_l = \varphi_l$ on $\partial\mathcal{C}$ for $l = 1, 2, \dots, q$.

Moreover, u is k -fold symmetric. The q -valued function $\tilde{u}(X) = \{u_1(X), u_2(X), \dots, u_q(X)\}$ associated with u satisfies $\tilde{u} \in C^{1,\mu}(\bar{\mathcal{C}}; \mathcal{A}_q(\mathbb{R}))$ for all $\mu \in (0, 1/q)$ and

$$(2.14) \quad D_i A^i(D\tilde{u}) + B(\tilde{u}, D\tilde{u}) = 0 \text{ weakly in } \mathcal{C} \setminus \mathcal{B}_{\tilde{u}}.$$

Note that to prove Theorem 3 we need a new $C^{1,\tau;q}$ Schauder estimate (Lemma 2 in Section 3) in order to construct a compact map to apply the Leray-Schauder theory.

The proof of Theorem 3 uses the maximum principle to obtain a global gradient estimate. By obtaining interior gradient estimates via [8] and using an approximation argument, we can assume $\varphi \in C^{0;q}(\bar{\mathcal{C}})$. See Section 4 of [8] for other examples of structural conditions on A^i and B that imply interior gradient estimates.

Corollary 1. *Let $k > q$ be an integer such that k and q are relatively prime. Let $A^i \in C^2(\mathbb{R}^n)$, $B \in C^1(\mathbb{R} \times \mathbb{R}^n)$ be single-valued functions satisfying (2.10). Let $v(P) = (1 + |P|^2)^{1/2}$ and $g^{ij}(P) = \delta^{ij} - P_i P_j / (1 + |P|^2)$ and suppose the structure conditions (2.11),*

$$\begin{aligned}
 P_i A^i(P) &\geq v(P) - \gamma_1, & |A(P)| &\leq \gamma_2, & |B(Z, P)| &\leq \gamma_2/v(P), \\
 v(P) D_{P_j} A^i(P) \xi_j \xi_j &\geq g^{ij}(P) \xi_i \xi_j && \text{for all } \xi \in \mathbb{R}^n, \\
 v(P) |D_{P_j} A^i(P) \xi_i \eta_j| &\leq \gamma_2 (g^{ij}(P) \xi_i \xi_j)^{1/2} (g^{ij}(P) \eta_i \eta_j)^{1/2} && \text{for all } \xi, \eta \in \mathbb{R}^n,
 \end{aligned}$$

hold for all $Z \in \mathbb{R}$ and $P \in \mathbb{R}^n$ for some constants $\beta_1, \beta_2, \beta_3 \in (0, \infty)$, $\gamma_1 \in [0, 1)$, and $\gamma_2 \in (0, \infty)$. Also suppose $B(Z, P)$ is non-increasing in Z for fixed $P \in \mathbb{R}^n$. Let $\varphi \in C^{0,q}(\bar{\mathcal{C}})$ is k -fold symmetric with $\sup_{\partial\mathcal{C}} |\varphi| < \infty$. Then there exists a solution $u \in C^{0,q}(\bar{\mathcal{C}}) \cap C^{1,q}(\mathcal{C})$ to (2.13) such that u is k -fold symmetric and $u \in C^{1,\mu;q}(\bar{\mathcal{C}})$ for every $\mu \in (0, 1/q)$. The q -valued function $\tilde{u}(X) = \{u_1(X), u_2(X), \dots, u_q(X)\}$ associated with u is a q -valued solution in $C^0(\bar{\mathcal{C}}; \mathcal{A}_q(\mathbb{R}))$ and $C^{1,\mu}(\mathcal{C}; \mathcal{A}_q(\mathbb{R}))$ for all $\mu \in (0, 1/q)$ to (2.14).

Finally we consider the interior regularity of q -valued solutions to elliptic equations:

Theorem 4. *Let $\tilde{u} \in C^1(B_1(0); \mathcal{A}_q(\mathbb{R}))$ be a q -valued function such that $\mathcal{B}_{\tilde{u}}$ is nonempty, $\mathcal{B}_{\tilde{u}} \subseteq \{0\} \times B_1^{n-2}(0)$, and $\|\tilde{u}\|_{C^1(B_1(0))} \leq 1/2$. Suppose \tilde{u} is a solution to*

$$(2.15) \quad D_i(A^i(X, \tilde{u}, D\tilde{u})) + B(X, \tilde{u}, D\tilde{u}) = 0 \text{ weakly in } B_1(0) \setminus \mathcal{B}_{\tilde{u}},$$

for some locally real analytic single-valued functions $A^i, B : B_1(0) \times (-1, 1) \times B_1^n(0) \rightarrow \mathbb{R}$ and

$$(2.16) \quad (D_j A^i)(X, Z, P) \xi_i \xi_j \geq \lambda |\xi|^2$$

for all $X \in B_1(0)$, $|Z| \leq 1$, $P \in B_1^n(0)$, and $\xi \in \mathbb{R}^n$ for some constant $\lambda > 0$. Then $\tilde{u}(x, y)$ is real analytic in y in the sense that for $B_R(x_0, y_0) \subset\subset B_1(0)$,

$$\sup_{(x,y) \in B_{R/2}(x_0,y_0)} |D_y^\gamma \tilde{u}(x, y)| \leq p! C^p R^{-p} \text{ for } p = |\gamma| \geq 1$$

for some constant $C \in (0, \infty)$ depending on n, q, μ, A^i, B , and $\mathcal{B}_{\tilde{u}}$. Consequently, the branch set of the graph of \tilde{u} is a union of $N \leq q/2$ real analytic, $(n - 2)$ -dimensional submanifolds.

In particular, (4) establishes that the branch sets of the minimal hypersurfaces constructed in [10] are locally real analytic $(n - 2)$ -dimensional submanifolds.

Suppose \tilde{u} is as in the statement of Theorem 4. Then $\tilde{u} = \{u_1, u_2, \dots, u_q\}$ on $B_1(0) \setminus (0, \infty) \times \{0\} \times \mathbb{R}^{n-2}$ for some C^1 single-valued u_1, u_2, \dots, u_q such that $D_i A^i(X, u_l, Du_l) + B(X, u_l, Du_l) = 0$ weakly in $B_1(0) \setminus [0, \infty) \times \{0\} \times \mathbb{R}^{n-2}$ and $\tilde{u} = \{v_1, v_2, \dots, v_q\}$ on $B_1(0) \setminus (-\infty, 0) \times \{0\} \times \mathbb{R}^{n-2}$ for some C^1 single-valued v_1, v_2, \dots, v_q such that $D_i A^i(X, v_l, Dv_l) + B(X, v_l, Dv_l) = 0$ weakly in $B_1(0) \setminus (-\infty, 0] \times \{0\} \times \mathbb{R}^{n-2}$. By unique continuation, we can order v_1, v_2, \dots, v_q so that $u_l = v_l$ on $B_1(0) \cap \mathbb{R} \times (-\infty, 0) \times \mathbb{R}^{n-2}$. Moreover, there exists a permutation σ of $\{1, 2, \dots, q\}$ such that $u_{\sigma(l)} = v_l$ on $B_1(0) \cap \mathbb{R} \times (0, \infty) \times \mathbb{R}^{n-2}$. After reordering u_1, u_2, \dots, u_q , we may assume that

$$\sigma = (1, 2, \dots, i_1)(i_1 + 1, i_1 + 2, \dots, i_2) \cdots (i_{N-1}, i_{N-1} + 1, \dots, q)$$

for some integers i_j so that $(u_{i_{j-1}+1}, u_{i_{j-1}+2}, \dots, u_{i_j}) \in C^{1;i_j-i_{j-1}}(B_1(0))$ for $j = 1, 2, \dots, N$, where $i_0 = 0$ and $i_N = q$. To prove Theorem 4, it suffices to assume $N = 1$ so that $u = (u_1, u_2, \dots, u_q) \in C^{1;q}(B_1(0))$ and show that

$$(2.17) \quad \sup_{(x,y) \in B_{R/2}(x_0,y_0)} |D_y^\gamma u(x,y)| \leq p! C^p R^{-p} \text{ for } p = |\gamma| \geq 1$$

for some constant $C \in (0, \infty)$ depending on n, q, μ, A^i , and B . The branch set of the graph of \tilde{u} is $\{(0, y, u_1(0, y)) : y \in B_1^{n-2}(0)\}$, which is real analytic if u satisfies (2.17).

We can regard Theorem 4 as analogous to the result that single-valued solutions to (2.15) are real analytic. One approach to proving such theorems for single-valued functions due to Morrey (see [6, Sections 5.8 and 6.7] or [7]) is to use integral kernels to show the single-valued solution extends to a holomorphic function on some domain in \mathbb{C}^n . However, we cannot use integral kernels for q -valued functions, so instead we take another approach of inductively using Schauder estimates. To prove Theorem 4, we first show that $D_y^\gamma u \in C^{1,\mu;q}(B_1(0))$ for all γ by an inductive argument involving difference quotients and Schauder estimates. For Theorem 4 we need estimates on $D_y^\gamma u$ of the particular form (2.17), which requires obtaining precise estimates on terms appearing in the Schauder estimates using a modified version of a technique used by Friedman in [2] involving majorants.

By replacing a Schauder estimate for equations (Lemma 7 in Section 3) with a Schauder estimate for elliptic systems (Lemma 8 in Section 3), we obtain a similar result for elliptic systems:

Theorem 5. *Let $\mu \in (0, 1/q)$. There is a $\varepsilon = \varepsilon(n, m, \mu, \nu) > 0$ such that the following is true. Let $\tilde{u} \in C^{1,\mu}(B_1(0); \mathcal{A}_q(\mathbb{R}^m))$, where $\mu \in (0, 1/q)$, such that $\mathcal{B}_{\tilde{u}}$ is nonempty, $\mathcal{B}_{\tilde{u}} \subseteq \{0\} \times B_1^{n-2}(0)$, and $\|\tilde{u}\|_{C^1(B_1(0))} \leq 1/2$. Suppose \tilde{u} is a solution to the non-linear elliptic differential equation*

$$D_i A_\kappa^i(X, \tilde{u}_l, D\tilde{u}_l) + B_\kappa(X, \tilde{u}_l, D\tilde{u}_l) = 0 \text{ weakly in } B_1(0) \setminus \mathcal{B}_{\tilde{u}},$$

where $A_\kappa^i, B_\kappa : B_1(0) \times B_1^m(0) \times B_1^{mn}(0) \rightarrow \mathbb{R}$ are locally real analytic single-valued functions such that

$$|(D_{P_j^\lambda} A_\kappa^i)(X, Z, P) - \delta^{ij} \delta_{\kappa\lambda}| < \varepsilon \text{ for } (X, Z, P) \in B_1(0) \times B_1^m(0) \times B_1^{mn}(0).$$

Then $\tilde{u}(x, y)$ is real analytic in y in the sense that for $B_R(x_0, y_0) \subset\subset B_1(0)$,

$$\sup_{(x,y) \in B_{R/2}(x_0,y_0)} |D_y^\gamma \tilde{u}(x, y)| \leq p! C^p R^{-p} \text{ for } p = |\gamma| \geq 1,$$

for some constant $C \in (0, \infty)$ depending on n, q, μ, A^i, B , and $\mathcal{B}_{\tilde{u}}$. Consequently, the branch set of the graph of \tilde{u} is a union of at most $q/2$ real analytic, $(n - 2)$ -dimensional submanifolds.

In particular, (4) establishes that the branch sets of the minimal submanifolds constructed in Theorem 2 are locally real analytic $(n - 2)$ -dimensional submanifolds provided ε is sufficiently small.

3. Elliptic theory for multivalued functions

The proof of the main results use standard theorems for elliptic differential equation such as the maximum principle and the Schauder estimates. This chapter is concerned with extending those theorems to solutions in the spaces $C^{k;q}$ and $W^{1,2;q}$ discussed in Section 2. We first consider differential equations of the form

$$a_l^{ij} D_{ij} u_l + b_l^i D_i u_l + c_l u_l = (\geq, \leq) f_l \text{ in } \Omega \setminus [0, \infty) \times \{0\} \times \mathbb{R}^{n-2}$$

for $l = 1, 2, \dots, q$, where Ω is an open set in \mathbb{R}^n , $u = (u_1, u_2, \dots, u_q) \in C^{2;q}(\Omega \setminus \{0\} \times \mathbb{R}^{n-2})$, and $a^{ij} = (a_1^{ij}, a_2^{ij}, \dots, a_q^{ij})$, $b^i = (b_1^i, b_2^i, \dots, b_q^i)$, $c =$

$(c_1, c_2, \dots, c_q), f = (f_1, f_2, \dots, f_q) \in C^{0;q}(\Omega)$. We assume the ellipticity condition

$$(3.1) \quad a_l^{ij}(X)\xi_i\xi_j \geq \lambda|\xi|^2 \text{ for } X \in \Omega, \xi \in \mathbb{R}^n, l = 1, 2, \dots, q$$

for some constant $\lambda > 0$. Given $(x, y) \in \Omega$, if $x_1 \leq 0$ then each u_l solves $a_l^{ij}D_{ij}u_l + b_l^iD_iu_l + c_lu_l = (\geq, \leq) f_l$ in $B_{|x|/2}(x, y) \cap \Omega$. Similarly if $x_1 > 0$ and \hat{u}_l are the C^2 single-valued functions defined by $\hat{u}_l = u_l$ on $B_{|x|/2}(x, y) \cap \Omega \cap \mathbb{R} \times (0, \infty) \times \mathbb{R}^{n-2}$ for $l = 1, 2, \dots, q$, $\hat{u}_1 = u_q$ on $B_{|x|/2}(x, y) \cap \Omega \cap \mathbb{R} \times (-\infty, 0) \times \mathbb{R}^{n-2}$, $\hat{u}_l = u_{l-1}$ on $B_{|x|/2}(x, y) \cap \Omega \cap \mathbb{R} \times (-\infty, 0) \times \mathbb{R}^{n-2}$ for $l = 2, 3, \dots, q$, then each \hat{u}_l satisfies an elliptic differential equation on $B_{|x|/2}(x, y) \cap \Omega$. Thus u satisfies standard elliptic estimates on $B_{|x|/2}(x, y) \cap \Omega$. Our first result is a strong maximum principle:

Lemma 1. *Let $u \in C^{0;q}(\bar{\mathcal{C}}) \cap C^{1;q}(\mathcal{C}) \cap C^{2;q}(\mathcal{C} \setminus \{0\} \times \mathbb{R}^{n-2})$, $a^{ij}, b^i, c \in C^{0;q}(\mathcal{C})$ satisfy*

$$(3.2) \quad a_l^{ij}D_{ij}u_l + b_l^iD_iu_l + c_lu_l \geq 0 \text{ in } \mathcal{C} \setminus [0, 1) \times \{0\} \times \mathbb{R}^{n-2}$$

for $l = 1, 2, \dots, q$. Assume (3.1) holds true for some constant $\lambda > 0$ and $c_l \leq 0$ in $\mathcal{C} \setminus [0, 1) \times \{0\} \times \mathbb{R}^{n-2}$ for $l = 1, 2, \dots, q$. Then u does not attain its maximum value in the interior of \mathcal{C} unless u_l all equal the same constant function.

Proof. Assume u_l do not all equal the same constant function. By the strong maximum principle [4, Theorem 3.5] applied locally in $\mathcal{C} \setminus \{0\} \times \mathbb{R}^{n-2}$, u does not attain its maximum value in $\mathcal{C} \setminus \{0\} \times \mathbb{R}^{n-2}$. Suppose u attained its maximum value at $(0, y_0)$ for some $y_0 \in \mathbb{R}^{n-2}$. Then u_1 extends to a C^1 function on $\overline{B_{1/4}(0, 1/4, y_0)}$ that attains its maximum value at $(0, y_0)$, $Du_1(0, y_0) = 0$, and satisfies (3.2), contradicting the Hopf boundary point lemma [4, Lemma 3.4]. □

Next we prove a Schauder estimate that will be needed for the proof of Theorem 3.

Lemma 2. *Let $0 < \mu < \tau < 1/q$ and $B_R(X_0) \subseteq \mathbb{R}^n$. Suppose*

$$u \in C^{1,\tau;q}(\overline{B_R(X_0)}), \quad a^{ij}, f \in C^{0,\mu;q}(\overline{B_R(X_0)})$$

satisfy

$$(3.3) \quad a_l^{ij}D_{ij}u_l = f_l \text{ in } B_R(X_0) \setminus \{0\} \times \mathbb{R}^{n-2}.$$

Assume (3.1) holds true for some constant $\lambda > 0$ and $\|a^{ij}\|'_{C^{0,\mu;q}(B_R(X_0))} \leq \Lambda$ for some constant $\Lambda > 0$. Then

$$(3.4) \quad \|u\|'_{C^{1,\tau;q}(B_{R/2}(X_0))} \leq C \left(R^{-n/2} \|u\|_{L^{2;q}(B_R(X_0))} + R^2 \|f\|'_{C^{0,\mu;q}(B_R(X_0))} \right)$$

for some constant $C = C(n, q, \mu, \tau, \lambda, \Lambda) \in (0, \infty)$.

The proof of Lemma 2 extending Liouville-type result [11, Corollary 2.6].

Lemma 3. *Let $\mu \in (0, 1/q)$ and suppose $u \in C^{1,\mu;q}(\mathbb{R}^n) \cap C^{\infty;q}(\mathbb{R}^n)$ such that*

$$(3.5) \quad \Delta u_l = 0 \text{ in } \mathbb{R}^n \setminus [0, \infty) \times \mathbb{R}^{n-2}$$

and $[Du]_{\mu;q,\mathbb{R}^n} < \infty$. Then $u_l(X) = a + b \cdot X$ for all $X \in \mathbb{R}^n \setminus [0, \infty) \times \{0\} \times \mathbb{R}^{n-2}$ and $l = 1, 2, \dots, q$ for some $a \in \mathbb{R}$ and $b \in \mathbb{R}^n$ independent of l .

Proof. Let u be as in the statement of Lemma 3. $u = u_a + u_f$ where $u_a = \frac{1}{q} \sum_{j=1}^q u_j$ and u_f are as by (2.5) and u_a is an affine function by the Liouville theorem. Thus it suffices to suppose that u is average-free and show that $u = 0$.

For $u \in C^{1;q}(\mathbb{R}^n)$ that is non-zero, average-free, and satisfies (3.5) and $y_0 \in \mathbb{R}^{n-2}$, we define the frequency function of u at $(0, y_0)$ by

$$N_{u,(0,y_0)}(\rho) = \frac{\rho^{2-n} \int_{B_\rho(0,y_0)} \sum_{l=1}^q |Du_l|^2}{\rho^{1-n} \int_{\partial B_\rho(0,y_0)} \sum_{l=1}^q |u_l|^2}$$

for $\rho \in (0, \infty)$. We extend the two identities in [11, Remark 2.3(2)] by either the argument in [11] using the fact that u and Du vanish on $\{0\} \times \mathbb{R}^{n-2}$ or by using a cutoff function argument. We then can extend Lemma 2.2, Remark 2.3(1)(3)(4), and Remark 2.4 of [11] to establish monotonicity and other standard properties of frequency functions for $N_{u,(0,y_0)}$.

Next we extend [11, Lemma 2.5] by showing that for some $\delta = \delta(n, q) \in (0, 1)$, there are no $u \in C^{1;q}(\mathbb{R}^n)$ that are non-zero, are average-free, satisfy (3.5), and are homogeneous degree σ for $\sigma \in [1, 1 + \delta)$. Arguing as in [11] using the fact that Du vanishes on $\{0\} \times \mathbb{R}^{n-2}$, if $u \in C^{1;q}(\mathbb{R}^n)$ is average-free, satisfies (3.5), and is homogeneous degree one then Du_l all equal the same constant function on $S^{n-1} \setminus [0, \infty) \times \{0\} \times \mathbb{R}^{n-2}$. Since u is average-free, this implies $u_l = 0$ on $\mathbb{R}^n \setminus [0, \infty) \times \{0\} \times \mathbb{R}^{n-2}$ for all $l = 1, 2, \dots, q$. By arguing as in [11] we also conclude that if $u^{(j)} \in C^{1;q}(\mathbb{R}^n)$ that are non-zero, are

average-free, satisfy (3.5) with $u^{(j)}$ in place of u , and are homogeneous degree σ_j for $\sigma_j \downarrow 1$, then after passing to a subsequence $u^{(j)}$ converges strongly in $L^{2;q}(B_1(0))$ to $u \in W^{1,2;q}(B_1(0))$ such that u is average free, u satisfies (3.5), Du_l all equal the same constant function, and $\|u\|_{L^{2;q}(B_1(0))} = 1$. But u being average-free implies that Du_l must all equal the zero function and thus u_l all equal the zero function, contradicting $\|u\|_{L^{2;q}(B_1(0))} = 1$.

By extending the proof of [11, Corollary 2.6] there is no nonzero $u \in C^{1,\mu;q}(\mathbb{R}^n)$ such that u is average-free, u satisfies (3.5), and $[Du]_{\mu;q,\mathbb{R}^n} < \infty$ for some $\mu \in (0, \delta)$. Having established Lemma 3 in the case that $\mu \in (0, \delta)$, we can prove Lemma 2 in the special case that $0 < \mu < \tau < \delta$. Using the dimension reduction argument in the proof of [11, Theorem 4.1] and the fact that the homogeneous, average-free $u \in C^{1;q}(\mathbb{R}^2)$ satisfying (3.5) are given by $u_l(re^{i\theta}) = \operatorname{Re}(cr^{1+k/q}e^{ik/q(\theta+2(l-1)\pi)})$ for $r \geq 0, \theta \in [0, 2\pi)$, and $l = 1, 2, \dots, q$ for some constant $c \in \mathbb{C}$ and integer $k \geq q + 1$, we conclude that there are no non-zero, average-free $u \in C^{1;q}(\mathbb{R}^n)$ that satisfies (3.5) is homogeneous degree $\sigma \in [1, 1 + 1/q)$ and thus Lemma 3 holds for all $\mu \in (0, 1/q)$. □

Proof of Lemma 2. We adapt the proof of [11, Lemma 3.2]. We in fact assume $R = 1$ and prove the weaker inequality that for every $\delta > 0$,

$$\begin{aligned}
 [Du]_{\tau;q,B_{1/2}(X_0)} &\leq \delta [Du]_{\tau;q,B_1(X_0)} \\
 &\quad + C \left(\sup_{B_1(0)} |u| + \sup_{B_1(0)} |Du| + \|f\|_{C^{0,\mu;q}(B_1(X_0))} \right)
 \end{aligned}$$

for some constant $C = C(n, q, \mu, \lambda, \Lambda, \delta) \in (0, \infty)$. Then by translating and rescaling, u as in the statement of Lemma 2 satisfies

$$\begin{aligned}
 \rho^{1+\tau} [Du]_{\tau;q,B_{\rho/2}(Y)} &\leq \delta \rho^{1+\tau} [Du]_{\tau;q,B_\rho(Y)} \\
 &\quad + C \left(\sup_{B_\rho(Y)} |u| + \rho \sup_{B_\rho(Y)} |Du| + \rho^2 \|f\|'_{C^{0,\mu;q}(B_\rho(Y))} \right)
 \end{aligned}$$

for all $B_\rho(Y) \subseteq B_R(X_0)$ and (3.4) follows by standard interpolation inequalities.

Suppose instead that for some $\delta > 0$ and every positive integer k , there is a ball $B_1(X_k)$ and $u_k = (u_{k,1}, u_{k,2}, \dots, u_{k,q}) \in C^{2;q}(B_R(X_l) \setminus \{0\} \times \mathbb{R}^{n-2}) \cap C^{1,\tau;q}(B_R(X_k))$ and $a_k^{ij} = (a_{k,1}^{ij}, a_{k,2}^{ij}, \dots, a_{k,q}^{ij}), f_k = (f_{k,1}, f_{k,2}, \dots, f_{k,q}) \in C^{0,\mu;q}(B_R(X_l))$ such that (3.1) and (3.3) hold with $u_{k,l}, a_{k,l}^{ij}$, and $f_{k,l}$ in

place of u_l , a_l^{ij} , and f_l and $\|a_k^{ij}\|_{C^{0,\mu;q}(B_R(X_0))} \leq \Lambda$ but

$$(3.6) \quad [Du_k]_{\tau;q,B_{1/2}(X_k)} > \delta [Du_k]_{\tau;q,B_1(X_k)} + k \left(\sup_{B_1(0)} |u_k| + \sup_{B_1(0)} |Du_k| + \|f_k\|'_{C^{0,\mu;q}(B_1(X_0))} \right)$$

Assume $[Du_k]_{\tau;q,B_{1/2}(X_k)} \leq 2q [Du_{k,1}]_{\mu;B_{1/2}(X_k) \cap \mathbb{R} \times (0,\infty) \times \mathbb{R}^{n-2}}$. Select $Y_k, Y'_k \in B_{1/2}(X_k) \cap \mathbb{R} \times (0,\infty) \times \mathbb{R}^{n-2}$ such that

$$(3.7) \quad \frac{|Du_{k,1}(Y_k) - Du_{k,1}(Y'_k)|}{|Y_k - Y'_k|^\tau} \geq \frac{1}{4q} [Du_k]_{\tau;B_{1/2}(X_k)}$$

and let $\rho_k = |Y_k - Y'_k|$. By (3.6) and (3.7),

$$\frac{1}{4q} [Du_k]_{\tau;B_{1/2}(X_k)} \leq \frac{2}{\rho_k^\tau} \sup_{B_1(X_k)} |Du_k| < \frac{2}{k\rho_k^\tau} [Du_{k,1}]_{\tau;B_{1/2}(X_k)},$$

so $\rho_k^\tau \leq 8/qk$ for all k and thus $\rho_k \rightarrow 0$ as $k \rightarrow \infty$.

Suppose $\text{dist}(\{Y_k, Y'_k\}, \{0\} \times \mathbb{R}^{n-2})/\rho_k \leq c$ for some constant $c \in [1, \infty)$. Then for some $Z_k \in \{0\} \times \mathbb{R}^{n-2}$, $|Y_k - Z_k| \leq 2c\rho_k$. By translating assume $Z_k = 0$. Let $R_k = 1/2\rho_k - 2c > 0$ for k sufficiently large. Rescale letting $\zeta_k = Y_k/\rho_k$ and $\zeta'_k = Y'_k/\rho_k$ and $\hat{u}_k = (\hat{u}_{k,1}, \hat{u}_{k,2}, \dots, \hat{u}_{k,q})$, $\hat{a}_k^{ij} = (\hat{a}_{k,1}^{ij}, \hat{a}_{k,2}^{ij}, \dots, \hat{a}_{k,q}^{ij})$, and $\hat{f}_k = (\hat{f}_{k,1}, \hat{f}_{k,2}, \dots, \hat{f}_{k,q})$ where

$$\begin{aligned} \hat{u}_{k,l}(X) &= \rho_l^{-1-\tau} [Du_k]_{\tau,B_1(X_k)}^{-1} (u_{k,l}(\rho_k X) - u_{k,l}(0) - Du_{k,l}(0) \cdot \rho_k X), \\ \hat{a}_{k,l}^{ij}(X) &= a_{k,l}^{ij}(\rho_k X), \\ \hat{f}_{k,l}(X) &= \rho_l^{1-\tau} [Du_k]_{\tau,B_1(X_k)}^{-1} f_{k,l}(\rho_k X), \end{aligned}$$

for $X \in B_{R_k}(0) \setminus [0, \infty) \times \{0\} \times \mathbb{R}^{n-2}$ and $l = 1, 2, \dots, q$ so that

$$\begin{aligned} \hat{a}_{k,l}^{ij} D_{ij} \hat{u}_{k,l} &= \hat{f}_{k,l} \text{ in } B_{R_k}(0) \setminus \{0\} \times \mathbb{R}^{n-2}, \\ [D\hat{u}_k]_{\tau;q,B_{R_k}(0)} &\leq 1, \quad |D\hat{u}_{k,1}(\zeta_k) - D\hat{u}_{k,1}(\zeta'_k)| \geq \frac{\delta}{4q}. \end{aligned}$$

Since $\{\zeta_k\}$ and $\{\zeta'_k\}$ are bounded, after passing to a subsequence, $\zeta_k \rightarrow \zeta$ and $\zeta'_k \rightarrow \zeta'$ for some points $\zeta, \zeta' \in \mathbb{R}^n$. Since

$$\sup_{B_{R_k}(0)} |\hat{a}_k^{ij}| + \rho_k^{-\mu} [a_k^{ij}]_{\mu;q,B_{R_k}(0)} \leq C\Lambda,$$

after passing to a subsequence $\{\hat{a}_{k,l}^{ij}\}$ converges to some constant \hat{a}^{ij} (independent of l) uniformly on $K \setminus [0, \infty) \times \{0\} \times \mathbb{R}^{n-2}$ as $k \rightarrow \infty$ for every compact subset K of \mathbb{R}^n . By (3.6),

$$\sup_{B_{R_l}(0)} |\hat{f}_k| + \rho^{-\mu}[\hat{f}_k]_{\mu, B_{R_l}(0)} \leq C\rho_k^{1-\tau}/k$$

for some constant $C = C(m, n) \in (0, \infty)$, so after passing to a subsequence $\{\hat{f}_k\}$ converges to zero in $C^{0;q}(\bar{\Omega})$ as $k \rightarrow \infty$ for all $\Omega \subset\subset \mathbb{R}^n$. Since $[D\hat{u}_k]_{\tau;q, B_{R_k}(0)} \leq 1$, after passing to a subsequence, $\{u_k\}$ converges in $C^{1;q}(\bar{\Omega})$ to some $\hat{u} = (\hat{u}_1, \hat{u}_2, \dots, \hat{u}_q) \in C^{1,\tau;q}(\mathbb{R}^n)$ for all $\Omega \subset\subset \mathbb{R}^n$. Moreover, by the interior $C^{2,\mu}$ Schauder estimates for single-valued functions [4, Corollary 6.3] applied locally on $\mathbb{R}^n \setminus \{0\} \times \mathbb{R}^{n-2}$, after passing to a subsequence $\hat{u}_k \rightarrow \hat{u}$ in $C^{2;q}$ on compact subsets of $\mathbb{R}^n \setminus \{0\} \times \mathbb{R}^{n-2}$. Hence \hat{u} satisfies the differential equation $\hat{a}^{ij}D_{ij}\hat{u} = 0$ on $\mathbb{R}^n \setminus \{0\} \times \mathbb{R}^{n-2}$. However, $[D\hat{u}]_{\tau, \mathbb{R}^n} \leq 1$ and $|D\hat{u}_1(\zeta) - D\hat{u}_1(\zeta')| \geq \delta/4q$, which after an affine change of variables contradicts Lemma 3.

Suppose instead that $\text{dist}(\{Y_k, Y'_k\}, \{0\} \times \mathbb{R}^{n-2})/\rho_k$ is unbounded. Assume $\text{dist}(\{Y_k, Y'_k\}, \{0\} \times \mathbb{R}^{n-2})/\rho_k$ tends to infinity and $Y'_k \in B_{1/2}(0) \cap \{0\} \times (0, \infty) \times \mathbb{R}^{n-2}$. For some $R_k \rightarrow \infty$, $R_k < \text{dist}(\{Y_k, Y'_k\}, \{0\} \times \mathbb{R}^{n-2})/\rho_k$ and $R_k < 1/2\rho_k$. Rescale letting $\zeta_k = (Y_k - Y'_k)/\rho_k$ and letting $\hat{u}_k, \hat{a}_k^{ij}$, and \hat{f}_k be the single-valued functions defined by

$$\begin{aligned} \hat{u}_k(X) &= \rho_k^{-1-\tau}[Du_{k,1}]_{\tau, B_1(X_k)}^{-1}(u_k(Y'_k + \rho_k X) - u_k(Y'_k) - Du_k(Y'_k) \cdot \rho_k X), \\ \hat{a}_k^{ij}(X) &= a_{k,1}^{ij}(Y'_k + \rho_k X), \\ \hat{f}_k(X) &= \rho_k^{-1-\tau}[Du_{k,1}]_{\tau, B_1(X_k)}^{-1}f_k(Y'_k + \rho_k X), \end{aligned}$$

for $X \in B_{R_l}(0)$ for large k . Similar to above, after passing to a subsequence, $\{\zeta_k\}$ converges to some ζ , $\{\hat{a}_k^{ij}\}$ converges uniformly on compact subsets of \mathbb{R}^n to some constant \hat{a}^{ij} , $\{\hat{u}_k\}$ converges in C^2 on compact subsets of \mathbb{R}^n to some single-valued function \hat{u} , and $\{\hat{f}_k\}$ converges uniformly to zero. \hat{u} satisfies $\hat{a}^{ij}D_{ij}\hat{u} = 0$ on \mathbb{R}^n , $[D\hat{u}]_{\tau, \mathbb{R}^{n-2}} \leq 1$, and $|D\hat{u}(\zeta) - D\hat{u}(0)| \geq \frac{\delta}{4q}$, which after an affine change of variables contradicts the Liouville theorem for single-valued harmonic functions. □

Next we consider equations of the form

$$\begin{aligned} (3.8) \quad & \int_{\Omega} \sum_{l=1}^q \left((a_l^{ij}D_j u_l + b^i u_l) D_i \zeta_l - (c_l^j D_j u + d_l u_l) \zeta_l \right) \\ & = (\leq, \geq) \int_{\Omega} \sum_{l=1}^q (f_l^i D_i \zeta_l - g_l \zeta_l) \end{aligned}$$

for all $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_q) \in C_c^{1;q}(\Omega \setminus \{0\} \times \mathbb{R}^{n-2})$, where Ω is an open set in \mathbb{R}^n , $u = (u_1, u_2, \dots, u_q) \in W^{1,2;q}(\Omega)$, $a^{ij} = (a_1^{ij}, a_2^{ij}, \dots, a_q^{ij})$, $b^i = (b_1^i, b_2^i, \dots, b_q^i)$, $c^j = (c_1^j, c_2^j, \dots, c_q^j)$, $d = (d_1, d_2, \dots, d_q) \in L^\infty;q(\Omega)$, and $f^i = (f_1^i, f_2^i, \dots, f_q^i)$, $g = (g_1, g_2, \dots, g_q) \in L^{2;q}(\Omega)$. We require ellipticity condition (3.1) to hold for some constant $\lambda > 0$. We claim that (3.8) continues to hold if instead $\zeta \in C_c^{1;q}(\Omega)$. To see this, for every $\delta > 0$ let $\chi_\delta \in C^1(\mathbb{R}^n)$ be a single-valued function such that $0 \leq \chi_\delta \leq 1$, $\chi_\delta = 1$ on $\mathbb{R}^n \setminus B_\delta^2(0) \times \mathbb{R}^{n-2}$, $\chi_\delta = 0$ on $B_{\delta/2}^2(0) \times \mathbb{R}^{n-2}$, and $|D\chi_\delta| \leq 3/\delta$. Replace ζ_l with $\zeta_l \chi_\delta$ in (3.8) to get

$$\begin{aligned} & \int_\Omega \sum_{l=1}^q \left((a_l^{ij} D_j u_l + b^i u_l) D_i \zeta_l - (c_l^j D_j u + d_l u_l) \zeta_l \right) \chi_\delta \\ &= (\leq, \geq) \int_\Omega \sum_{l=1}^q (f_l^i D_i \zeta_l - g_l \zeta_l) \chi_\delta - \int_\Omega \sum_{l=1}^q (a_l^{ij} D_j u_l + b^i u_l - f_l^i) \zeta_l D_i \chi_\delta \end{aligned}$$

and let $\delta \downarrow 0$ to get (3.8) for $\zeta \in C_c^{1;q}(\Omega)$. Using (3.8) and Sobolev inequality Lemma 4 below, the maximum principle [4, Theorem 8.1] and global supremum estimates [4, Theorem 8.16] readily extend to $u \in W^{1,2;q}(\Omega)$ satisfying (3.8) with the \leq sign. Using (3.8) and Sobolev inequality Lemma 4 and Poincaré inequality Lemma 5 below we will extend local Hölder continuity estimates [4, Theorem 8.22] to solutions to (3.8) with the $=$ sign.

Lemma 4. *Let $1 \leq p < n$. Suppose $u \in W_0^{1,p;q}(\mathbb{R}^n)$. Then*

$$\|u\|_{L^{np/(n-p);q}(\mathbb{R}^n)} \leq C \|Du\|_{L^{p;q}(\mathbb{R}^n)}$$

for some $C = C(n, q, p) \in (0, \infty)$.

Proof. By the Sobolev inequality for the single-valued functions $u_l|_{\mathbb{R} \times (0, \infty) \times \mathbb{R}^{n-2}}$ and $u_l|_{\mathbb{R} \times (-\infty, 0) \times \mathbb{R}^{n-2}}$, $l = 1, 2, \dots, q$,

$$\begin{aligned} & \|u\|_{L^{np/(n-p);q}(\mathbb{R}^n)} \\ &= \left(\sum_{l=1}^q (\|u_l\|_{L^{np/(n-p);q}(\mathbb{R} \times (0, \infty) \times \mathbb{R}^{n-2})}^{np/(n-p)} + \|u_l\|_{L^{np/(n-p);q}(\mathbb{R} \times (-\infty, 0) \times \mathbb{R}^{n-2})}^{np/(n-p)}) \right)^{(n-p)/np} \\ &\leq C \left(\sum_{l=1}^q (\|Du_l\|_{L^{p;q}(\mathbb{R} \times (0, \infty) \times \mathbb{R}^{n-2})}^{np/(n-p)} + \|Du_l\|_{L^{p;q}(\mathbb{R} \times (-\infty, 0) \times \mathbb{R}^{n-2})}^{np/(n-p)}) \right)^{(n-p)/np} \\ &\leq C \|Du\|_{L^{p;q}(\mathbb{R}^n)} \end{aligned}$$

for $C = C(n, q, p) \in (0, \infty)$. □

Lemma 5. For $u \in W^{1,2;q}(B_R(0))$,

$$(3.9) \quad \|u - \ell\|_{L^{2;q}(B_R(0))} \leq CR \|Du\|_{L^{2;q}(B_R(0))},$$

for some $C = C(n, q) \in (0, \infty)$ where $\ell = \int_{B_R(0)} \frac{1}{q} \sum_{j=1}^q u_j$.

Remark 1. The reason for stating (3.9) in terms of $W^{1,2;q}$ functions on a ball $B_R(0)$ centered at a point on $\{0\} \times \mathbb{R}^{n-2} = \emptyset$ is that (3.9) fails if we replace $B_R(0)$ with a ball B such that $\overline{B} \cap \{0\} \times \mathbb{R}^{n-2} = \emptyset$. For example, (3.9) fails if $u_1 \equiv -1$, $u_2 \equiv 1$, and $u_l \equiv 0$ for $l \geq 3$ in B .

Proof of Lemma 5. By scaling, we may suppose $R = 1$. Writing $u = u_a + u_f$ for $u_a = \frac{1}{q} \sum_{j=1}^q u_j$ and u_f as by (2.5),

$$\begin{aligned} \|u - \ell\|_{L^{2;q}(B_1(0))}^2 &= \|u_a - \ell\|_{L^{2;q}(B_1(0))}^2 + \|u_f\|_{L^{2;q}(B_1(0))}^2 \\ \|Du\|_{L^{2;q}(B_1(0))}^2 &= \|Du_a\|_{L^{2;q}(B_1(0))}^2 + \|Du_f\|_{L^{2;q}(B_1(0))}^2. \end{aligned}$$

By the Poincaré inequality for single-valued functions, $\|u_a - \ell\|_{L^2(B_1(0))} \leq C \|Du_a\|_{L^2(B_1(0))}$ for some $C = C(n) \in (0, \infty)$, so it suffices to suppose u is average-free.

Suppose that for every integer $j \geq 1$ there are average-free $u^{(j)} \in W^{1,2;q}(B_1(0))$ such that

$$\|u^{(j)}\|_{L^{2;q}(B_1(0))} > j \|Du^{(j)}\|_{L^{2;q}(B_1(0))}.$$

By scaling we may suppose $\|u^{(j)}\|_{L^{2;q}(B_1(0))} = 1$ so that

$$\|u^{(j)}\|_{L^{2;q}(B_1(0))} = 1, \quad \|Du^{(j)}\|_{L^{2;q}(B_1(0))} < 1/j.$$

By Rellich’s lemma, after passing to a subsequence $u^{(j)}$ converges in $L^{2;q}(B_1(0))$ to some average-free $u = (u_1, u_2, \dots, u_q) \in W^{1,2;q}(B_1(0))$ such that $\|u\|_{L^{2;q}(B_1(0))} = 1$ and $\|Du\|_{L^{2;q}(B_1(0))} = 0$. Since $Du_l = 0$ a.e. in $B_1(0)$ for $l = 1, 2, \dots, q$ and $u \in W^{1,2;q}(B_1(0))$, u_l all equal the same constant functions on $B_1(0)$. Since u is average free, $u_l = 0$ a.e. on $B_1(0)$ for $l = 1, 2, \dots, q$, contradicting $\|u\|_{L^{2;q}(B_1(0))} = 1$. □

Lemma 6. *Let $R_0 > 0$. Let $u \in W^{1,2;q}(B_{R_0}(0))$, $a^{ij}, b^i, c^j, d \in L^\infty;q(B_{R_0}(0))$, and $f^j \in L^{s;q}(B_{R_0}(0))$ and $g \in L^{s/2;q}(B_{R_0}(0))$ for $s > n$ such that*

$$\begin{aligned} & \int_{B_{R_0}(0)} \sum_{l=1}^q \left((a_l^{ij} D_j u_l + b^i u_l) D_i \zeta_l - (c_l^j D_j u + d_l u_l) \zeta_l \right) \\ &= \int_{B_{R_0}(0)} \sum_{l=1}^q (f_l^i D_i \zeta_l - g_l \zeta_l) \end{aligned}$$

for all $\zeta \in C_c^{1;q}(B_{R_0}(0) \setminus \{0\} \times \mathbb{R}^{n-2})$. Suppose (3.1) holds for some constant $\lambda > 0$ and

$$\sup_{B_{R_0}(0)} |a^{ij}| \leq \Lambda, \quad R_0 \sup_{B_{R_0}(0)} |b^i| + R_0 \sup_{B_{R_0}(0)} |c^j| + R_0^2 \sup_{B_{R_0}(0)} |d| \leq \nu,$$

for some constants $\Lambda, \nu > 0$. Then for some constants $\mu \in (0, 1/q)$ and $C \in (0, \infty)$ depending on $n, q, s, \lambda, \Lambda$, and ν , u is equal to an element of $C^{0,\mu;q}(B_{R_0/2}(0))$ a.e. in $B_{R_0/2}(0)$ and, taking u to be in $C^{0,\mu;q}(B_{R_0/2}(0))$,

$$\begin{aligned} & R_0^\mu [u]_{\mu;q,B_{R_0/2}(0)} \\ & \leq C \left(\sup_{B_{R_0}(0)} |u| + R^{1-n/s} \|f\|_{L^{s;q}(B_{R_0}(0))} + R^{2-2n/s} \|g\|_{L^{s/2;q}(B_{R_0}(0))} \right). \end{aligned}$$

Proof. First we show that if $B_{4R}(0, y_0) \subset B_{R_0}(0)$ and $\hat{u} \in W^{1,2;q}(B_{4R}(0, y_0))$, $\hat{f}^j \in L^{s;q}(B_{4R}(0, y_0))$, and $\hat{g} \in L^{s/2;q}(B_{4R}(0, y_0))$ for $s > n$ such that $\hat{u}_l \geq 0$ for $l = 1, 2, \dots, q$ and

$$\begin{aligned} (3.10) \quad & \int_{B_{4R}(0,y_0)} \sum_{l=1}^q \left((a_l^{ij} D_j \hat{u}_l + b^i \hat{u}_l) D_i \zeta_l - (c_l^j D_j \hat{u} + d_l \hat{u}_l) \zeta_l \right) \\ & \geq \int_{B_{4R}(0,y_0)} \sum_{l=1}^q (\hat{f}_l^i D_i \zeta_l - \hat{g}_l \zeta_l) \end{aligned}$$

for all $\zeta \in C_c^{1;q}(B_{4R}(0, y_0) \setminus \{0\} \times \mathbb{R}^{n-2})$ such that $\zeta_l \geq 0$ for $l = 1, 2, \dots, q$ then

$$\begin{aligned} (3.11) \quad R^{-n} \int_{B_{2R}(0,y_0)} \sum_{l=1}^q \hat{u}_l & \leq C \left(\inf_{B_R(0,y_0)} \hat{u} + R^{1-n/s} \|\hat{f}\|_{L^{s;q}(B_{4R}(0,y_0))} \right. \\ & \quad \left. + R^{2-2n/s} \|\hat{g}\|_{L^{s/2;q}(B_{4R}(0,y_0))} \right) \end{aligned}$$

for some constant $C = C(n, q, s, \Lambda/\lambda, \nu/\lambda) \in (0, \infty)$. Translate and rescale so that $y_0 = 0$ and $R = 1$. (3.11) follows from standard arguments using (3.10) and Sobolev inequality Lemma 4 such as the proof of Theorem 8.18 of [4] except to prove

$$\left(\int_{B_3(0)} \sum_{l=1}^q \bar{u}_l^p \right) \left(\int_{B_3(0)} \sum_{l=1}^q \bar{u}_l^{-p} \right) \leq C$$

where $\bar{u}_l = \hat{u}_l + \lambda^{-1}(\|\hat{f}\|_{L^{s;q}(B_4(0))} + \|\hat{g}\|_{L^{s/2;q}(C^0(B_4(0)))})$ for $p \in (0, 1)$ and some constant $C = C(n, q, p, s, \Lambda/\lambda, \nu/\lambda) \in (0, \infty)$ we use a standard argument that uses (3.10), Sobolev inequality Lemma 4, and Poincaré inequality Lemma 5 to bound the integrals of $|w - \ell|^k/k^k$, where $w_l = \log(\bar{u}_l)$, for large integers k for some $\ell \in \mathbb{R}$ and that avoids using the John-Nirenberg inequality (see the proof of Theorem 4.15 of [3]).

Rescale so that $R_0 = 1$. Arguing as in the proof of Theorem 8.22 of [4], replacing the weak Harnack inequality [4, Theorem 8.18] with (3.11), we obtain

$$(3.12) \quad \text{osc}_{B_R(0,y_0)} u \leq CR^\mu \left(\sup_{B_{1/2}(0,y_0)} u + K \right) \leq CR^\mu \left(\sup_{B_1(0)} u + K \right),$$

for all $y_0 \in B_{1/2}^{n-2}(0)$, $R \in (0, 1/2]$ and for some constants $\mu \in (0, 1/q)$ and $C \in (0, \infty)$ depending on $n, q, s, \Lambda/\lambda$, and ν/λ , where

$$\begin{aligned} \text{osc}_{B_R(0,y_0)} u &= \sup_{B_R(0,y_0)} u - \inf_{B_R(0,y_0)} u, \\ K &= \lambda^{-1}(\|f\|_{L^{s;q}(B_1(0))} + \|g\|_{L^{s/2;q}(B_1(0))}). \end{aligned}$$

We want to bound $[u_l]_{\mu; B_{1/2}(0) \cap \mathbb{R} \times (0, \infty) \times \mathbb{R}^{n-2}}$ for $l \in \{1, 2, \dots, q\}$ by showing that if $X_1 = (x_1, y_1)$ and $X_2 = (x_2, y_2)$ are distinct points in $B_{1/2}(0) \cap \mathbb{R} \times (0, \infty) \times \mathbb{R}^{n-2}$ then

$$(3.13) \quad |u_l(X_1) - u_l(X_2)| \leq C|X_1 - X_2|^\mu \left(\sup_{B_1(0)} |u| + K \right)$$

for some constant $C = C(n, q, s, \Lambda/\lambda, \nu/\lambda) \in (0, \infty)$. Assume $|x_1| \leq |x_2|$. We consider four cases: (a) $|X_1 - X_2| < |x_2|/2$, (b) $|x_2|/2 \leq |X_1 - X_2| \leq |x_2|$, (c) $x_1 = x_2 = 0$, and (d) $|X_1 - X_2| > |x_2| > 0$. In case (a) (3.13) follows by using the Hölder continuity estimates for single-valued functions [4, Theorem 8.22] to bound $[u]_{\mu; B_{|x_1-x_2|}(X_2)}$, replacing μ with a smaller value if

necessary. In case (b) (3.13) follows by using (3.12) to bound $\text{osc}_{B_{2|x_2|}(0,y_2)} u$ if $|x_2| \leq 1/4$ and (3.13) is obvious if $|x_2| > 1/4$. In case (c) (3.13) follows by using (3.12) to bound $\text{osc}_{B_{|x_1-x_2|/2}(0,\frac{y_1+y_2}{2})} u$. In case (d) (3.13) follows from cases (b) and (c) and the triangle inequality:

$$\begin{aligned} |u_l(X_1) - u_l(X_2)| &\leq |u_l(X_1) - u_l(0, y_1)| + |u_l(0, y_1) - u_l(0, y_2)| \\ &\quad + |u_l(0, y_2) - u_l(X_2)| \\ &\leq 3C|X_1 - X_2|^\mu \left(\sup_{B_1(0)} |u| + K \right). \end{aligned}$$

Similarly we can bound $[u_l]_{\mu, B_{1/2}(0) \cap \mathbb{R} \times (-\infty, 0) \times \mathbb{R}^{n-2}}$ by proving (3.13) when X_1 and X_2 are instead distinct points in $B_{1/2}(0) \cap \mathbb{R} \times (-\infty, 0) \times \mathbb{R}^{n-2}$ \square

Lemma 7. *Let $\mu \in (0, 1/q)$ and $B_R(X_0) \subseteq \mathbb{R}^n$. Suppose $u \in C^{1,\mu;q}(B_R(X_0))$, $a^{ij}, b^i, f^i \in C^{0,\mu;q}(B_R(X_0))$, and $c^j, d, g \in C^{0;q}(B_R(X_0))$ satisfy*

$$\begin{aligned} (3.14) \quad &\int_{B_R(X_0)} \sum_{l=1}^q \left((a_l^{ij} D_j u_l + b_l^i u_l) D_i \zeta_l - (c_l^j D_j u + d_l u_l) \zeta_l \right) \\ &= \int_{B_R(X_0)} \sum_{l=1}^q (f_l^i D_i \zeta_l - g_l \zeta_l) \end{aligned}$$

for all $\zeta \in C_c^{1;q}(B_R(X_0) \setminus \{0\} \times \mathbb{R}^{n-2})$. Suppose (3.1) holds for some constant $\lambda > 0$ and

$$\begin{aligned} (3.15) \quad &\|a^{ij}\|'_{C^{0,\mu;q}(B_R(X_0))} \leq \Lambda, \\ &R \|b^i\|'_{C^{0,\mu;q}(B_R(X_0))} + R \sup_{B_R(X_0)} |c^j| + R^2 \sup_{B_R(X_0)} |d| \leq \nu, \end{aligned}$$

for some constants $\Lambda, \nu > 0$. Then

$$(3.16) \quad \|u\|'_{C^{1,\mu;q}(B_{R/2}(X_0))} \leq C \left(\sup_{B_R(X_0)} |u| + R^{1+\mu} [f]_{\mu;q, B_R(X_0)} + R^2 \sup_{B_R(X_0)} |g| \right)$$

for some constant $C = C(n, q, \mu, \lambda, \Lambda, \nu) \in (0, \infty)$.

Proof. First observe that Lemma 7 holds true in the special case where a_l^{ij} all equal the same constant function, $b^i = 0$, $c^i = 0$, and $d = 0$ by a scaling argument similar to the proof of Lemma 2. (Note that unlike in the proof of Lemma 2, we do not need to show that after passing to a subsequence $\hat{u}_k \rightarrow \bar{u}$ in $C^{2;q}(\Omega)$ for $\Omega \subset \subset \mathbb{R}^n \setminus \{0\} \times \mathbb{R}^{n-2}$.)

Next we prove Lemma 7 in general. Consider any $B_r(X) \subseteq B_R(X_0)$. Suppose $\{0\} \times \mathbb{R}^{n-2} \cap B_r(X) \neq \emptyset$ and let $Z \in \{0\} \times \mathbb{R}^{n-2} \cap B_r(X)$. By (3.14),

$$a^{ij}(Z)D_{ij}u_l = D_i((a^{ij}(Z) - a_l^{ij})D_ju_l - b_l^i u_l + f_l^i) - c_l^j D_ju_l - d_l u_l + g_l$$

in $B_r(X) \setminus \{0\} \times \mathbb{R}^{n-2}$. By Lemma 7 for the operator $a^{ij}(Z)D_{ij}$ and (3.15),

$$(3.17) \quad r^{1+\mu}[Du]_{\mu;q,B_{r/2}(X)} \leq C \left(\frac{r^{1+2\mu}}{R^\mu}[Du]_{\mu,B_r(X)} + \|u\|'_{C^{1;q}(B_R(X_0))} + R^{1+\mu}[f]_{\mu,B_R(X_0)} + R^2 \sup_{B_R(X_0)} |g| \right)$$

for some constant $C = C(n, q, \mu, \lambda, \Lambda, \nu) \in (0, \infty)$. If instead $\{0\} \times \mathbb{R}^{n-2} \cap B_r(X) = \emptyset$, then (3.17) holds by the Schauder estimates for single-valued functions. By (3.17) with $r \leq \varepsilon R$ for $\varepsilon = \varepsilon(n, \mu, \lambda, \Lambda, \nu) > 0$ sufficiently small and by interpolation, we obtain (3.16). □

By slightly modifying the proof of Lemma 7 we obtain the following Schauder estimate for linear systems, whose proof we omit.

Lemma 8. *Let $\mu \in (0, 1/q)$ and $B_R(X_0) \subseteq \mathbb{R}^n$. Suppose $u = (u_1, u_2, \dots, u_q) \in C^{1,\mu;q}(B_R(X_0); \mathbb{R}^m)$, $a_{\kappa\lambda}^{ij} = (a_{\kappa\lambda,1}^{ij}, a_{\kappa\lambda,2}^{ij}, \dots, a_{\kappa\lambda,q}^{ij})$, $b_{\kappa\lambda}^i = (b_{\kappa\lambda,1}^i, b_{\kappa\lambda,2}^i, \dots, b_{\kappa\lambda,q}^i)$, $f_{\kappa}^i = (f_{\kappa,1}^i, f_{\kappa,2}^i, \dots, f_{\kappa,q}^i) \in C^{0,\mu;q}(B_R(X_0))$, and $c_{\kappa\lambda}^j = (c_{\kappa\lambda,1}^j, c_{\kappa\lambda,2}^j, \dots, c_{\kappa\lambda,q}^j)$, $d_{\kappa\lambda} = (d_{\kappa\lambda,1}, d_{\kappa\lambda,2}, \dots, d_{\kappa\lambda,q})$, $g_{\kappa} = (g_{\kappa,1}, g_{\kappa,2}, \dots, g_{\kappa,q}) \in C^{0;q}(B_R(X_0))$ satisfy*

$$\begin{aligned} & \int_{B_R(X_0)} \sum_{l=1}^q \left((a_{\kappa\lambda,l}^{ij} D_j u_l^\lambda + b_{\kappa\lambda,l}^i u_l^\lambda) D_i \zeta_l^\kappa - (c_{\kappa\lambda,l}^j D_j u_l^\lambda + d_{\kappa\lambda,l} u_l^\lambda) \zeta_l^\kappa \right) \\ &= \int_{B_R(X_0)} \sum_{l=1}^q (f_{\kappa,l}^i D_i \zeta_l^\kappa - g_{\kappa,l} \zeta_l^\kappa) \end{aligned}$$

for all $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_q) \in C_c^{1;q}(B_R(X_0) \setminus \{0\} \times \mathbb{R}^{n-2}; \mathbb{R}^m)$. Suppose

$$\begin{aligned} & R^\mu [a_{\kappa\lambda}^{ij}]_{\mu;q,B_R(X_0)} + R \|b_{\kappa\lambda}^i\|'_{C^{0,\mu;q}(B_R(X_0))} \\ & \quad + R \sup_{B_R(X_0)} |c_{\kappa\lambda}^j| + R^2 \sup_{B_R(X_0)} |d_{\kappa\lambda}| \leq \nu, \end{aligned}$$

for some constants $\Lambda, \nu > 0$. For some $\varepsilon = \varepsilon(n, m, \mu, \nu) > 0$, if

$$\|a_{\kappa\lambda}^{ij} - \delta^{ij} \delta_{\kappa\lambda}\|_{C^0(B_R(X_0))} \leq \varepsilon,$$

where δ^{ij} and $\delta_{\kappa\lambda}$ denote Kronecker deltas, then

$$\|u\|'_{C^{1,\mu;q}(B_{R/2}(X_0))} \leq C \left(\sup_{B_R(X_0)} |u| + R^{1+\mu} [f]_{\mu;q,B_R(X_0)} + R^2 \sup_{B_R(X_0)} |g| \right)$$

for some constant $C = C(n, m, q, \mu, \nu) \in (0, \infty)$.

To prove Theorem 1 and Theorem 2 in the case of non-periodic data we need a global estimate supremum estimate on u that is independent of $\rho_1, \dots, \rho_{n-2}$.

Lemma 9. *Let $\mu \in (0, 1/q)$. Suppose $u \in C^{1,\mu;q}(\bar{\mathcal{C}})$, $f^i \in C^{0,\mu;q}(\bar{\mathcal{C}})$, and $g \in C^{0;q}(\bar{\mathcal{C}})$ satisfy*

$$(3.18) \quad \int_{\mathcal{C}} \sum_{l=1}^q D_i u_l D_i \zeta_l = \int_{\mathcal{C}} \sum_{l=1}^q (f_l^i D_i \zeta_l - g_l \zeta_l)$$

for all $\zeta \in C_c^{1;q}(\mathcal{C} \setminus \{0\} \times \mathbb{R}^{n-2})$. Then

$$\sup_{\mathcal{C}} |u| \leq C \left(\sup_{\partial\mathcal{C}} |u| + [f]_{\mu;q,\bar{\mathcal{C}}} + \sup_{\mathcal{C}} |g| \right)$$

for some constant $C = C(n, q, \mu) \in (0, \infty)$ independent of $\rho_1, \dots, \rho_{n-2}$.

Proof. Suppose instead for every integer $k \geq 1$ there are $u_k = (u_{k,1}, u_{k,2}, \dots, u_{k,q}) \in C^{1,\mu;q}(\bar{\mathcal{C}})$, $f_k^i = (f_{k,1}^i, f_{k,2}^i, \dots, f_{k,q}^i) \in C^{0,\mu;q}(\bar{\mathcal{C}})$, and $g_k = (g_{k,1}, g_{k,2}, \dots, g_{k,q}) \in C^{0;q}(\bar{\mathcal{C}})$ such that (3.18) holds with $u_{k,l}$, $f_{k,l}^i$, and $g_{k,l}$ in place of u_l , f_l^i , and g_l but

$$(3.19) \quad \sup_{\mathcal{C}} |u_k| > k \left(\sup_{\partial\mathcal{C}} |u_k| + [f_k]_{\mu;q,\bar{\mathcal{C}}} + \sup_{\mathcal{C}} |g_k| \right).$$

Assume $\sup_{\mathcal{C}} |u_k| = 1$, $|u_{k,1}(\xi_k, 0)| = 1$ for some $\xi_k \in B_1^2(0) \setminus [0, 1] \times \{0\}$, and $f_{k,l}(0, 0) = 0$ for all $k = 1, 2, 3, \dots$ and $l = 1, 2, \dots, q$. Now (3.19) becomes

$$(3.20) \quad \sup_{\partial\mathcal{C}} |u_k| + [f_k]_{\mu;q,\bar{\mathcal{C}}} + \sup_{\mathcal{C}} |g_k| < 1/k.$$

After passing to a subsequence $\{\xi_k\}$ converges to some $\hat{\xi} \in B_1^2(0)$. By (3.20), $f_{k,l} \rightarrow 0$ uniformly on $K \setminus [0, \infty) \times \{0\} \times \mathbb{R}^{n-2}$ for every compact set $K \subset \bar{\mathcal{C}}$ and $g_{k,l} \rightarrow 0$ uniformly on $\bar{\mathcal{C}} \setminus [0, 1] \times \{0\} \times \mathbb{R}^{n-2}$. By the interior Schauder

estimates Lemma 7 and (3.20), after passing to a subsequence $\{u_k\}$ converges in $C^{1;q}$ on compact subsets of \mathcal{C} to some $\hat{u} = (\hat{u}_1, \hat{u}_2, \dots, \hat{u}_q)$. Given $(x_0, y_0) \in \partial\mathcal{C}$, u decomposes into q single-valued solutions to Poisson equations on $\mathcal{C} \cap B_{1/2}(x_0, y_0)$, so by local estimates [4, Theorem 8.27] and (3.20) \hat{u} extends to a continuous function on $\bar{\mathcal{C}}$ with $\hat{u}_l = 0$ on $\partial\mathcal{C}$ for $l = 1, 2, \dots, q$. Now \hat{u} satisfies

$$\begin{aligned} \Delta \hat{u}_l &= 0 \text{ in } \mathcal{C} \setminus [0, 1) \times \{0\} \times \mathbb{R}^{n-2} \text{ for } l = 1, 2, \dots, q, \\ \hat{u}_l &= 0 \text{ on } \partial\mathcal{C} \text{ for } l = 1, 2, \dots, q. \end{aligned}$$

Note that $\hat{u} \in C^{\infty;q}(\mathcal{C} \setminus \{0\} \times \mathbb{R}^{n-2})$ by elliptic regularity. Since $\sup_{\mathcal{C}} |\hat{u}| \leq 1$ and $|\hat{u}(\hat{\xi}, 0)| = 1$, \hat{u} has attains its maximum value of 1 at $(\hat{\xi}, 0)$. However, $\hat{u} = 0$ on $\partial\mathcal{C}$, so \hat{u} in fact attains an interior maximum at $(\hat{\xi}, 0)$, contradicting strong maximum principle Lemma 1. □

Combining the global supremum estimates similar to [4, Theorem 8.16], Lemma 9, Lemma 7, and the local boundary Schauder estimates for single-valued solutions [4, Section 8.11], we obtain global Schauder estimates:

Lemma 10. *Let $\mu \in (0, 1/q)$ and $B_R(X_0) \subseteq \mathbb{R}^n$. Suppose $u, \varphi \in C^{1,\mu;q}(\bar{\mathcal{C}})$, $a^{ij}, b^i, f^i \in C^{0,\mu;q}(\bar{\mathcal{C}})$, and $c, d, g \in C^{0;q}(\bar{\mathcal{C}})$ satisfy*

$$\int_{\mathcal{C}} \sum_{l=1}^q \left((a_l^{ij} D_j u_l + b^i u_l) D_i \zeta_l - (c_l^j D_j u + d_l u_l) \zeta_l \right) = \int_{\mathcal{C}} \sum_{l=1}^q (f_l^i D_i \zeta_l - g_l \zeta_l)$$

for all $\zeta \in C_c^{1;q}(\mathcal{C} \setminus \{0\} \times \mathbb{R}^{n-2})$ and $u_l = \varphi_l$ on $\partial\mathcal{C}$ for $l = 1, 2, \dots, q$. Suppose (3.1) holds for some constant $\lambda > 0$,

$$\|a^{ij}\|_{C^{0,\mu;q}(\mathcal{C})} \leq \Lambda, \quad \|b^i\|_{C^{0,\mu;q}(\mathcal{C})} + \sup_{\mathcal{C}} |c^j| + \sup_{\mathcal{C}} |d| \leq \nu,$$

for some constants $\Lambda, \nu > 0$, and

$$\int_{\mathcal{C}} \sum_{l=1}^q \left(-b_l^j D_j \zeta_l + d_l \zeta_l \right) \leq 0$$

for all $\zeta \in C_c^{1;q}(\mathcal{C} \setminus \{0\} \times \mathbb{R}^{n-2})$ such that $\zeta_l \geq 0$ for $l = 1, 2, \dots, q$. Then

$$\|u\|_{C^{1,\mu;q}(\bar{\mathcal{C}})} \leq C \left([f]_{\mu;q,\mathcal{C}} + \sup_{\mathcal{C}} |g| + \|\varphi\|_{C^{1,\mu;q}(\bar{\mathcal{C}})} \right).$$

for some constant $C = C(n, q, \mu, \lambda, \Lambda, \nu, \rho_1, \dots, \rho_{n-2} \in (0, \infty)$. Moreover, in the special case that (3.18) holds, the constant C is independent of $\rho_1, \dots, \rho_{n-2}$.

4. Existence of solutions to a Poisson equation

We now want to prove Theorem 1, which recall involves finding a solution $u = (u_1, u_2, \dots, u_q) \in C^{0;q}(\bar{\mathcal{C}}) \cap C^{1,\mu;q}(\mathcal{C})$ to (2.7) given $f^i = (f_1^i, f_2^i, \dots, f_q^i) \in C^{0,\mu;q}(\bar{\mathcal{C}})$ and $g = (g_1, g_2, \dots, g_q), \varphi = (\varphi_1, \varphi_2, \dots, \varphi_q) \in C^{0;q}(\bar{\mathcal{C}})$ as in the statement of Theorem 1. Note that by the weak maximum principle analogous to [4, Theorem 8.1], in the case that $u, f^i, g,$ and φ are periodic with respect to y_i for $i = 1, 2, \dots, n - 2$ there is at most one solution u to (2.7). To solve (2.7) we will first assume that $f^j, g,$ and φ are periodic with respect to y_i with period ρ_i for $i = 1, 2, \dots, n - 2$. Let $u_a, f_a, g_a,$ and φ_a denote the average parts of $u, f, g,$ and φ respectively and $u_f, f_f, g_f,$ and φ_f denote the average-free parts of $u, f, g,$ and φ respectively as defined by (2.5). By linearity, it suffices to use the existence theory for single-valued functions [4, Theorem 8.34] to solve for u_a such that $\Delta u_a = D_j f_a^j + g_a$ weakly in \mathcal{C} and $u_a = \varphi_a$ on $\partial\mathcal{C}$ and then solve for u_f such that $\Delta u_f = D_j f_f^j + g_f$ weakly in \mathcal{C} and $u_f = \varphi_f$ on $\partial\mathcal{C}$. Thus we may suppose $f^j, g,$ and φ are average-free and find an average-free solution u to (2.7).

For simplicity, we will first assume $f_l^j = 0$ in $\bar{\mathcal{C}}$ for $l = 1, 2, \dots, q$. To solve (2.7), we will use the change of variables $\xi_1 + i\xi_2 = (x_1 + ix_2)^{1/q}$. Under this change of variables, $u, g,$ and φ transform into the continuous single-valued function $u_0, g_0,$ and φ_0 on $\bar{\mathcal{C}}$ given by

$$\begin{aligned} u_0(re^{i\theta}, y) &= u_l(r^q e^{iq\theta}, y), & g_0(re^{i\theta}, y) &= g_l(r^q e^{iq\theta}, y), \\ \varphi_0(re^{i\theta}, y) &= \varphi_l(r^q e^{iq\theta}, y), \end{aligned}$$

for $r \in [0, 1], 2(l - 1)\pi/q < \theta < 2l\pi/q,$ and $y \in \mathbb{R}^{n-2},$ and $l = 1, 2, \dots, q$ and (2.7) transforms into

$$\begin{aligned} \Delta_\xi u_0 + q^2 |\xi|^{2q-2} \Delta_y u_0 &= q^2 |\xi|^{2q-2} g_0 \text{ weakly in } \mathcal{C} \setminus \{0\} \times \mathbb{R}^{n-2}, \\ u_0 &= \varphi_0 \text{ on } \partial\mathcal{C}. \end{aligned}$$

We will assume that g_0 and φ_0 are smooth on $\bar{\mathcal{C}}$ and $\varphi_0 = 0$ in $B_{1/2}^2(0) \times \mathbb{R}^{n-2}$. Since g and φ are periodic with respect to y_j with period ρ_j for $j = 1, 2, \dots, n - 2,$ g_0 and φ_0 are periodic with respect to y_j with period ρ_j for $j = 1, 2, \dots, n - 2$. Thus for each $z = (z_1, z_2, \dots, z_{n-2}) \in \mathbb{Z}^{n-2}$ we can

define Fourier coefficients $g_{0,z}$ and $\varphi_{0,z}$ of u_0 , g_0 , and φ_0 with respect to y by

$$g_{0,z}(\xi) = \int_{\mathbb{R}^{n-2}} e^{-i2\pi \sum_{j=1}^{n-2} z_j y_j / \rho_j} g_0(\xi, y) dy,$$

$$\varphi_{0,z}(\xi) = \int_{\mathbb{R}^{n-2}} e^{-i2\pi \sum_{j=1}^{n-2} z_j y_j / \rho_j} \varphi_0(\xi, y) dy$$

for $\xi \in B_1^2(0)$ and solve for the Fourier coefficient $u_{0,z}$ of u satisfying

$$(4.1) \quad \Delta_\xi u_{0,z} - q^2(z_1^2/\rho_1^2 + z_2^2/\rho_2^2 + \dots + z_{n-2}^2/\rho_{n-2}^2)|\xi|^{2q-2}u_{0,z} \\ = q^2|\xi|^{2q-2}g_{0,z} \text{ in } B_1^2(0),$$

$$u_0 = \varphi_0 \text{ on } \partial\mathcal{C}.$$

for each $z \in \mathbb{Z}^{n-2}$. By standard elliptic theory [4, Theorems 8.14] there exists a unique solution $u_{0,z} \in C^\infty(\overline{B_1^2(0)})$ to (4.1) for every $z \in \mathbb{Z}^{n-2}$.

Fix $z \in \mathbb{Z}^{n-2}$ and define $u_z = (u_{1,z}, u_{2,z}, \dots, u_{q,z}) : \overline{B_1^2(0)} \setminus [0, 1] \times \{0\} \rightarrow \mathbb{R}^q$ by

$$u_{l,z}(re^{i\theta}) = u_{0,z}(r^{1/q}e^{i(\theta+2(l-1)\pi)/q})$$

for $r \in [0, 1]$, $\theta \in (0, 2\pi)$, and $l = 1, 2, \dots, q$. We will show $u_z \in C^{1,\mu;q}(\overline{B_1(0)})$ using the average-free and k -fold symmetry assumptions. Since g and φ are average-free,

$$(4.2) \quad \sum_{l=0}^{q-1} g_{0,z}(re^{i(\theta+2\pi l/q)}) = \sum_{l=0}^{q-1} \varphi_{0,z}(re^{i(\theta+2\pi l/q)}, y) = 0$$

for all $r \in [0, 1]$ and $\theta \in [0, 2\pi)$. Since g and φ have k -fold symmetry and k and q are relatively prime,

$$(4.3) \quad g_{0,z}(re^{i(\theta+2\pi/k)}) = g_{0,z}(re^{i\theta}), \quad \varphi_{0,z}(re^{i(\theta+2\pi/k)}) = \varphi_{0,z}(re^{i\theta})$$

for all $r \in [0, 1]$ and $\theta \in [0, 2\pi)$. By the maximum principle, (4.2), and (4.3),

$$(4.4) \quad \sum_{l=0}^{q-1} u_{0,z}(re^{i(\theta+2\pi l/q)}) = 0, \quad u_{0,z}(re^{i(\theta+2\pi/k)}, y) = u_{0,z}(re^{i\theta}, y),$$

for all $r \in [0, 1]$ and $\theta \in [0, 2\pi)$. By (4.1), the degree $2q - 1$ Taylor polynomial of $u_{0,z}$ is harmonic and thus takes the form

$$\operatorname{Re} \left(\sum_{j=0}^{2q-1} c_j (\xi_1 + i\xi_2)^j \right)$$

for some $c_j \in \mathbb{C}$. By (4.4), $c_j = 0$ for $j = 0, 1, 2, \dots, q$. By the Schauder estimates,

$$\begin{aligned} \sum_{j=0}^2 |\xi|^j |D^j u_{0,z}(\xi)| &\leq \sup_{B_1(0)} |D^{q+1} u_{0,z}| |\xi|^{q+1} \\ &\leq C (\|g_{0,z}\|_{C^q(B_1^2(0))} + \|\varphi_{0,z}\|_{C^{q+2}(\partial B_1^2(0))}) |\xi|^{q+1} \end{aligned}$$

for $\xi \in \overline{B_1^2(0)}$ for some constant $C = C(n, q, \mu, z, \rho_1, \dots, \rho_{n-2}) \in (0, \infty)$. By the change of variable $\xi_1 + i\xi_2 = (x_1 + ix_2)^{1/q}$,

$$\sum_{j=0}^2 |x|^j |D^j u_{l,z}(x)| \leq C (\|g_{0,z}\|_{C^q(B_1^2(0))} + \|\varphi_{0,z}\|_{C^{q+2}(\partial B_1^2(0))}) |x|^{1+1/q}$$

for all $x \in B_1^2(0)$ and thus $u_z \in C^{1,1/q;q}(B_1^2(0))$ for all $z \in \mathbb{Z}^{n-2}$.

For each integer $\nu \geq 1$, we define partial sums of the Fourier series of u_0 , g_0 , and φ_0 by

$$\begin{aligned} u_0^{(\nu)}(\xi, y) &= \sum_{|z| \leq \nu} u_{0,z}(\xi) e^{i2\pi \sum_{j=1}^{n-2} z_j y_j / \rho_j}, \\ g_0^{(\nu)}(\xi, y) &= \sum_{|z| \leq \nu} g_{0,z}(\xi) e^{i2\pi \sum_{j=1}^{n-2} z_j y_j / \rho_j}, \\ \varphi_0^{(\nu)}(\xi, y) &= \sum_{|z| \leq \nu} \varphi_{0,z}(\xi) e^{i2\pi \sum_{j=1}^{n-2} z_j y_j / \rho_j}, \end{aligned}$$

and define $u^{(\nu)} = (u_1^{(\nu)}, u_2^{(\nu)}, \dots, u_q^{(\nu)})$, $g^{(\nu)} = (g_1^{(\nu)}, g_2^{(\nu)}, \dots, g_q^{(\nu)})$, and $\varphi^{(\nu)} = (\varphi_1^{(\nu)}, \varphi_2^{(\nu)}, \dots, \varphi_q^{(\nu)})$ in $C^{0;q}(\overline{\mathcal{C}})$ by

$$\begin{aligned} u_l^{(\nu)}(re^{i\theta}, y) &= u_0^{(\nu)}(r^{1/q} e^{i(\theta+2(l-1)\pi)/q}, y), \\ g_l^{(\nu)}(re^{i\theta}, y) &= g_0^{(\nu)}(r^{1/q} e^{i(\theta+2(l-1)\pi)/q}, y), \\ \varphi_l^{(\nu)}(re^{i\theta}, y) &= \varphi_0^{(\nu)}(r^{1/q} e^{i(\theta+2(l-1)\pi)/q}, y), \end{aligned}$$

for $r \in [0, 1]$, $\theta \in (0, 2\pi)$, $y \in \mathbb{R}^{n-2}$, and $l = 1, 2, \dots, q$. Since

$$u_l^{(\nu)}(x, y) = \sum_{z \in \mathbb{Z}^{n-2}} u_{l,z}(x) e^{i2\pi \sum_{j=1}^{n-2} z_j y_j / \rho_j},$$

for all $x \in \overline{B_1^2(0)} \setminus [0, 1] \times \{0\}$ and $y \in \mathbb{R}^{n-2}$ and $u_z \in C^{1,1/q;q}(B_1^2(0))$ for all $z \in \mathbb{Z}^{n-2}$, $u^{(\nu)} \in C^{1,1/q;q}(\overline{\mathcal{C}})$. Since g_0 is smooth,

$$\sup_{\xi \in B_1^2(0)} |g_{0,z}(\xi)| \leq C(1 + |z_1|)^{-2}(1 + |z_2|)^{-2} \cdots (1 + |z_{n-2}|)^{-2} \|g_0\|_{C^2(\overline{\mathcal{C}})}.$$

Thus $g_0^{(\nu)}$ converges uniformly to g_0 on $\overline{\mathcal{C}}$ as $\nu \rightarrow \infty$. Hence $g^{(\nu)}$ converges to g in $C^{0;q}(\overline{\mathcal{C}})$ as $\nu \rightarrow \infty$. Similarly $\varphi_0^{(\nu)}$ converges to φ_0 in $C^2(\overline{\mathcal{C}})$ and thus, since $\varphi_0 = 0$ in $B_{1/2}^2(0) \times \mathbb{R}^{n-2}$, $\varphi^{(\nu)}$ converges to φ in $C^{2;q}(\overline{\mathcal{C}})$ as $\nu \rightarrow \infty$. By the Schauder estimate of Lemma 10, for every $\mu \in (0, 1/q)$,

$$\begin{aligned} \|u^{(\nu)}\|_{C^{1,\mu;q}(\overline{\mathcal{C}})} &\leq C \left(\sup_{\mathcal{C}} |g^{(\nu)}| + \|\varphi^{(\nu)}\|_{C^{1,\mu;q}(\overline{\mathcal{C}})} \right) \\ &\leq C \left(\sup_{\mathcal{C}} |g| + \|\varphi\|_{C^{1,\mu;q}(\overline{\mathcal{C}})} + 1 \right) \end{aligned}$$

for some constant $C = C(n, q, \mu) \in (0, \infty)$ independent of ν . After passing to a subsequence, $\{u^{(\nu)}\}$ converges to some u in $C^{1;q}(\overline{\mathcal{C}})$ such that $u \in C^{1,\mu;q}(\overline{\mathcal{C}})$ for all $\mu \in (0, 1/q)$ and u satisfies (2.7).

Consider the case where $f^j \neq 0$ and g_0 and φ_0 are merely continuous. We will construct functions $f^{(\nu)} \in C^{\infty;q}(\overline{\mathcal{C}}; \mathbb{R}^n)$ and $g^{(\nu)}, \varphi^{(\nu)} \in C^{\infty;q}(\overline{\mathcal{C}})$ approximating f, g , and φ as follows. Extend f to an element of $C_c^{0,\mu;q}(\mathbb{R}^n; \mathbb{R}^n)$ such that $[f]_{\mu;q,\mathbb{R}^n} \leq C[f]_{\mu;q,\mathcal{C}}$ for some $C = C(n, \mu) \in (0, \infty)$. For each $\delta > 0$, let $\chi_\delta \in C^\infty(\mathbb{R}^2)$ be a single-valued function such that $0 \leq \chi_\delta \leq 1$, $\chi_\delta = 1$ on $\mathbb{R}^2 \setminus B_\delta^2(0)$, $\chi_\delta = 0$ on $B_{\delta/2}^2(0)$, and $|D\chi_\delta| \leq 3/\delta$ and extend χ_δ to a function $\chi_\delta(x, y)$ of $x \in \mathbb{R}^2$ and $y \in \mathbb{R}^{n-2}$ that is independent of y . Since $f \in C_c^{0,\mu;q}(\mathbb{R}^n; \mathbb{R}^n)$ and $f_l = 0$ on $\{0\} \times \mathbb{R}^{n-2}$ for $l = 1, 2, \dots, q$ because f is average free, observe that $\chi_\delta f = (\chi_\delta f_1, \chi_\delta f_2, \dots, \chi_\delta f_q)$ is in $C_c^{0,\mu;q}(\mathbb{R}^n; \mathbb{R}^n)$ with

$$\begin{aligned} \chi_\delta f &\rightarrow f \text{ in } C^0(B_2^2(0) \times \mathbb{R}^{n-2}) \text{ as } \delta \downarrow 0, \\ [\chi_\delta f]_{\mu;q,\mathbb{R}^n} &\leq C[f]_{\mu;q,\mathbb{R}^n} \text{ for } C = C(n, \mu) \in (0, \infty). \end{aligned}$$

Select a smooth spherically symmetric mollifier $\psi \in C_c^\infty(B_1(0))$ and let

$$\psi_\sigma(X) = \sigma^{-n} \psi(X/\sigma)$$

for $X \in B_\sigma(0)$ and $\sigma > 0$. Let $(x_1, x_2, y) \in \mathcal{C} \setminus B_{2^{-\nu-2}}^2(0) \times \mathbb{R}^{n-2}$. If $x_1 \leq 0$ then we define $f_l^{(\nu)}(x_1, x_2, y)$ to be the value of the convolution of f_l and $\psi_{2^{-\nu-3}}$ at (x_1, x_2, y) . If $x_1 > 0$ and $x_2 > 0$ then we define $f_l^{(\nu)}(x_1, x_2, y)$ to be the value of the convolution of \hat{f}_l and $\psi_{2^{-\nu-3}}$ at (x_1, x_2, y) , where $\hat{f}_l = f_l$ for $l = 1, 2, \dots, q$ on $\mathcal{C} \cap (0, \infty)^2 \times \mathbb{R}^{n-2}$ and $\hat{f}_1 = f_q$ and $\hat{f}_l = f_{l-1}$ for $l = 2, 3, \dots, q$ on $\mathcal{C} \cap (0, \infty) \times (-\infty, 0) \times \mathbb{R}^{n-2}$. If $x_1 > 0$ and $x_2 < 0$ then we define $f_l^{(\nu)}(x_1, x_2, y)$ to be the value of the convolution of \hat{f}_l and $\psi_{2^{-\nu-3}}$ at (x_1, x_2, y) , where now $\hat{f}_l = f_l$ for $l = 1, 2, \dots, q$ on $\mathcal{C} \cap (0, \infty) \times (-\infty, 0) \times \mathbb{R}^{n-2}$ and $\hat{f}_l = f_{l+1}$ for $l = 1, 2, \dots, q-1$ and $\hat{f}_q = f_1$ on $\mathcal{C} \cap (0, \infty)^2 \times \mathbb{R}^{n-2}$. Define $f_l^{(\nu)} = 0$ on $B_{2^{-\nu-2}}^2(0) \times \mathbb{R}^{n-2}$. Then

$$f^{(\nu)} \rightarrow f \text{ in } C^{0;q}(\bar{\mathcal{C}}), \quad [f^{(\nu)}]_{\mu;q,\mathcal{C}} \leq C[f]_{\mu;q,\mathcal{C}} \text{ for } C = C(n, \mu) \in (0, \infty).$$

Define $g_0^{(\nu)}$ by convolution of g_0 with smooth spherically symmetric mollifiers such that $g_0^{(\nu)} \rightarrow g_0$ uniformly on $\bar{\mathcal{C}}$ and then define $g^{(\nu)}$ by

$$g_l^{(\nu)}(re^{i\theta}, y) = g_l^{(\nu)}(r^q e^{iq\theta}, y) \text{ if } 2(l-1)\pi/q < \theta < 2l\pi/q$$

for $r \in [0, 1]$, $y \in \mathbb{R}^{n-2}$, and $l = 1, 2, \dots, q$. Assume $\varphi = 0$ on $B_{1/2}^2(0) \times \mathbb{R}^{n-2} \setminus [0, 1/2) \times \{0\} \times \mathbb{R}^{n-2}$ and define $\varphi^{(\nu)}$ similarly via convolution of φ_0 with smooth spherically symmetric mollifiers. Let $u^{(\nu)} = (u_1^{(\nu)}, u_2^{(\nu)}, \dots, u_q^{(\nu)}) \in C^{1,\mu}(\bar{\mathcal{C}})$ to be the solution to (2.7) with $u_l^{(\nu)}$, 0 , $\operatorname{div} f_l^{(\nu)} + g_l^{(\nu)}$, and $\varphi_l^{(\nu)}$ in place of u_l , f_l , g_l , and φ_l respectively. By global supremum estimates similar to [4, Theorem 8.16], $\{u^{(\nu)}\}$ is Cauchy in $C^{0;q}(\bar{\mathcal{C}})$ and thus $\{u^{(\nu)}\}$ converges to some u in $C^{0;q}(\bar{\mathcal{C}})$. By the local Schauder estimates of Lemma 7 after passing to a subsequence $\{u^{(\nu)}\}$ converges to u in $C^{1;q}(\bar{\Omega})$ for all $\Omega \subset \subset \mathcal{C}$ and $u \in C^{1,\mu;q}(\mathcal{C})$. Therefore u is a solution to (2.7).

To solve (2.7) in the case that f , g , and φ are not periodic with respect to each y_i , approximate f , g , and φ uniformly on compact subsets of $\bar{\mathcal{C}}$ by $f^{(\nu)} \in C^{0,\mu;q}(\bar{\mathcal{C}})$, $g^{(\nu)}, \varphi^{(\nu)} \in C^{0;q}(\bar{\mathcal{C}})$ such that (2.6) holds with $f^{(\nu)}$ in place of f , $g^{(\nu)}$ and $\varphi^{(\nu)}$ are k -fold symmetric, $f^{(\nu)}$, $g^{(\nu)}$, and $\varphi^{(\nu)}$ are periodic with respect to each y_j with period ρ_ν (independent of j) such that $\rho_\nu \rightarrow \infty$ as $\nu \rightarrow \infty$, and

$$(4.5) \quad [f^{(\nu)}]_{\mu;q,\mathcal{C}} \leq C[f]_{\mu;q,\mathcal{C}}, \quad \sup_{\mathcal{C}} |g^{(\nu)}| \leq \sup_{\mathcal{C}} |g|, \quad \sup_{\partial\mathcal{C}} |\varphi^{(\nu)}| \leq \sup_{\partial\mathcal{C}} |\varphi|,$$

where $C = C(n, \mu) \in (0, \infty)$. Let $u^{(\nu)} \in C^{0;q}(\bar{\mathcal{C}}) \cap C^{1,\mu;q}(\mathcal{C})$ solve (2.7) with $u_l^{(\nu)}$, $f_l^{(\nu)}$, $g_l^{(\nu)}$, and $\varphi_l^{(\nu)}$ in place of u_l , f_l , g_l , and φ_l respectively. By Lemma 9, local Schauder estimates Lemma 7, and (4.5) we have local $C^{1,\mu;q}$ estimates

on $u^{(\nu)}$ that are independent of ν and thus after passing to a subsequence $\{u^{(\nu)}\}$ converges to some $u \in C^{1,\mu;q}(\mathcal{C})$ in $C^{1;q}(\Omega)$ for all $\Omega \subset\subset \mathcal{C}$. By using [4, Theorem 8.27] and (4.5), we can establish uniform modulus of continuity estimates on $u^{(\nu)}$ at points on $\partial\mathcal{C}$ that are independent of ν and thus u extends to an element of $C^{0;q}(\bar{\mathcal{C}})$ such that $u = \varphi$ on $\partial\mathcal{C}$. Therefore u is a solution to (2.7).

5. Existence of solutions to nonlinear systems

In this section we will prove Theorem 2, which recall involves finding the unique solution $u = (u_1, u_2, \dots, u_q) \in C^{1,\mu;q}(\bar{\mathcal{C}}; \mathbb{R}^m)$ to (2.8) given F_κ^j, G_κ , and $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_q) \in C^{1,\mu;q}(\bar{\mathcal{C}})$ as in the statement of Theorem 2. First we consider the case where u and φ are periodic with respect to y_j with period $\rho_j > 0$ for $j = 1, 2, \dots, n - 2$. Let \mathcal{V} denote the space of $u \in C^{1,\mu;q}(\bar{\mathcal{C}}; \mathbb{R}^m)$ that has k -fold symmetry and is periodic with respect to y_j with period ρ_j for $j = 1, 2, \dots, n - 2$. By Theorem 1, we can define $T : \mathcal{V} \rightarrow \mathcal{V}$ by letting $u = Tv$ if $u = (u_1, u_2, \dots, u_q), v = (v_1, v_2, \dots, v_q) \in \mathcal{V}$ satisfy

$$\int_{\mathcal{C}} \sum_{l=1}^q D_j u_l^\kappa D_j \zeta_l^\kappa = \int_{\mathcal{C}} \sum_{l=1}^q (F_\kappa^j(Dv_l) D_j \zeta^\kappa - G_\kappa(v_l, Dv_l) \zeta^\kappa)$$

for all $\zeta \in C_c^{1;q}(\mathcal{C} \setminus \{0\} \times \mathbb{R}^{n-2}; \mathbb{R}^m)$,
 $u_l = \varphi_l$ on $\partial\mathcal{C}$ for $l = 1, 2, \dots, q$.

Let $\varepsilon > 0$ to be determined and choose arbitrary $v = (v_1, v_2, \dots, v_q), v' = (v'_1, v'_2, \dots, v'_q) \in \mathcal{V}$ such that $\|v\|_{C^{1,\mu;q}(\bar{\mathcal{C}})} \leq \varepsilon$ and $\|v'\|_{C^{1,\mu;q}(\bar{\mathcal{C}})} \leq \varepsilon$. Let $u = (u_1, u_2, \dots, u_q) = Tv$ and $u' = (u'_1, u'_2, \dots, u'_q) = Tv'$. By the Schauder estimate Lemma 10,

$$(5.1) \quad \|u\|_{C^{1,\mu;q}(\bar{\mathcal{C}})} \leq C \left([F(Dv)]_{\mu,\mathcal{C}} + \sup_{\mathcal{C}} |G(v, Dv)| + \|\varphi\|_{C^{1,\mu;q}(\bar{\mathcal{C}})} \right),$$

$$\|u - u'\|_{C^{1,\mu;q}(\bar{\mathcal{C}})} \leq C \left([F(Dv) - F(Dv')]_{\mu;q,\mathcal{C}} + \sup_{\mathcal{C}} |G(v, Dv) - G(v', Dv')| \right),$$

for some constant $C \in (0, \infty)$ depending on n, m, q , and μ and independent of $\rho_1, \dots, \rho_{n-2}$, where $F(Dw) = (F(Dw_1), F(Dw_2), \dots, F(Dw_q))$ and $G(w, Dw) = (G(w_1, Dw_1), G(w_2, Dw_2), \dots, G(w_q, Dw_q))$ for $w = (w_1, w_2, \dots, w_q) \in \mathcal{V}$. Note that C being independent of $\rho_1, \dots, \rho_{n-2}$ is necessary for later removing the condition that φ be periodic with respect to y_j . Since

$F \in C^2(\mathbb{R}^{mn})$ with $DF(0) = 0$,

$$\begin{aligned}
 (5.2) \quad [F(Dv)]_{\mu;q,C} &\leq C \sup_{|P| \leq \varepsilon} |DF(P)| [Dv]_{\mu;q,C} \leq C\varepsilon^2 \\
 [F(Dv) - F(Dv')]_{\mu;q,C} &\leq C \sup_{|P| \leq \varepsilon} |D^2F(P)| ([Dv]_{\mu;q,C} + [Dv']_{\mu;q,C}) \\
 &\quad \times \sup_C |Dv - Dv'| \\
 &\quad + C \sup_{|P| \leq \varepsilon} |DF(P)| [Dv - Dv']_{\mu;q,C} \\
 &\leq C\varepsilon \|v - v'\|_{C^{1,\mu;q}(\bar{C})}
 \end{aligned}$$

for some constants $C \in (0, \infty)$ depending on n, m , and $\sup_{|Z|+|P| \leq 1} |D^2F(P)|$. Since $G \in C^1(\mathbb{R}^m \times \mathbb{R}^{mn})$ with $G(0) = 0$,

$$\begin{aligned}
 (5.3) \quad \sup_C |G(v, Dv)| &\leq \sup_{|Z|+|P| \leq \varepsilon} |G(P)| \leq C \sup_{|Z|+|P| \leq \varepsilon} |DG(P)| \varepsilon, \\
 \sup_C |G(v, Dv) - G(v', Dv')| &\leq C \sup_{|Z|+|P| \leq \varepsilon} |DG(P)| \|v - v'\|_{C^{1;q}(\bar{C})}
 \end{aligned}$$

for some constant $C = C(n, m, q) \in (0, \infty)$. Combining (5.1), (5.2), and (5.3) and using the fact that $DG(0) = 0$, for some $\varepsilon > 0$ depending on n, m, q, μ, F , and G ,

$$\|u\|_{C^{1,\mu;q}(\bar{C})} \leq \varepsilon, \quad \|u - u'\|_{C^{1,\mu;q}(\bar{C})} \leq \frac{1}{2} \|v - v'\|_{C^{1,\mu;q}(\bar{C})}.$$

Therefore by the contraction mapping principle, there exists a fixed point $u_0 \in \mathcal{V}$ of T with $\|u\|_{C^{1,\mu;q}(\bar{C})} \leq \varepsilon$. In other words, u satisfies (2.8).

To remove the condition that φ is periodic with respect to y_j , approximate $\varphi^{(\nu)}$ in $C^{1;q}(\bar{C}; \mathbb{R}^m)$ by $\varphi^{(\nu)} \in C^{1,\mu;q}(\bar{C}; \mathbb{R}^m)$ such that $\varphi^{(\nu)}$ is k -fold symmetric and is periodic with respect to each y_j with period ρ_ν where $\rho^{(\nu)} \rightarrow \infty$, $\varphi^{(\nu)} \rightarrow \varphi$ in $C^{1;q}$ on compact subsets of \bar{C} , and $\|\varphi^{(\nu)}\|_{C^{1,\mu;q}(\bar{C})} \rightarrow \|\varphi\|_{C^{1,\mu;q}(\bar{C})}$. Let $u^{(\nu)} \in C^{1,\mu;q}(\bar{C}; \mathbb{R}^m)$ be the unique solution to (2.8) with $u^{(\nu)}$ and $\varphi^{(\nu)}$ in place of u and φ respectively. Since $\|u^{(\nu)}\|_{C^{1,\mu;q}(\bar{C})} \leq \varepsilon$, after passing to a subsequence $\{u^{(\nu)}\}$ converges in $C^{1;q}(B_1^2(0) \times B_\rho^{n-2}(0); \mathbb{R}^m)$ for all $\rho \in (0, \infty)$ to u satisfying (2.8) with the original φ .

6. Existence of solutions to nonlinear equations

In this section we will prove Theorem 3, which recall involves finding a solution $u = (u_1, u_2, \dots, u_q) \in C^{1,\tau;q}(\bar{C}; \mathbb{R}^m)$ for every $\tau \in (0, 1/q)$ to (2.13) given

A^i , B , and $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_q) \in C^{2;q}(\bar{\mathcal{C}})$ as in the statement of Theorem 3. The proof uses the Leray-Schauder theory. For now assume φ is periodic with respect to y_j with period ρ_j for $j = 1, 2, \dots, n - 2$. Rewrite (2.13) as

$$(D_{P_j} A^i)(Du_l) D_{ij} u_l + B(u_l, Du_l) = 0 \text{ in } \mathcal{C} \setminus \{0\} \times \mathbb{R}^{n-2} \text{ for } l = 1, 2, \dots, q,$$

$$u_l = \varphi_l \text{ on } \partial\mathcal{C} \text{ for } l = 1, 2, \dots, q.$$

Let \mathcal{V} denote the space of $u \in C^{1,\mu;q}(\bar{\mathcal{C}})$ that are periodic with respect to y_j with period ρ_j for $j = 1, 2, \dots, n - 2$ and have k -fold symmetry. Define $T : \mathcal{V} \rightarrow \mathcal{V}$ by $u = Tv$ if $u = (u_1, u_2, \dots, u_q), v = (v_1, v_2, \dots, v_q) \in \mathcal{V}$ satisfy

$$(6.1) \quad (D_{P_j} A^i)(Dv_l) D_{ij} v_l + B(v_l, Dv_l) = 0 \text{ in } \mathcal{C} \setminus [0, 1] \times \{0\} \times \mathbb{R}^{n-2}$$

$$\text{for } l = 1, 2, \dots, q,$$

$$u_l = \varphi_l \text{ on } \partial\mathcal{C} \text{ for } l = 1, 2, \dots, q.$$

The existence of a unique $u \in \mathcal{V} \cap C^{2;q}(\mathcal{C} \setminus \{0\} \times \mathbb{R}^{n-2})$ satisfying (6.1) will follow from Lemma 11 below. By Lemma 11, Schauder estimate Lemma 2, and local boundary Schauder estimates for single-valued solutions [4, Section 6.2], T is in fact a continuous map from \mathcal{V} into $C^{1,\tau;q}(\bar{\mathcal{C}})$ for every $\tau \in (\mu, 1/q)$, so by Arzela-Ascoli T is compact. We will need to show that for some constants $\mu \in (0, 1/q)$ and $C \in (0, \infty)$ depending only on n, A^i, B , and $\|\varphi\|_{C^{2;q}(\bar{\mathcal{C}})}$, if $u \in C^{1,\mu;q}(\bar{\mathcal{C}})$ and $\sigma \in [0, 1]$ satisfies

$$(6.2) \quad \int_{\mathcal{C}} \sum_{l=1}^q (A^i(Du_l) D_i \zeta_l - \sigma B(u_l, Du_l) \zeta_l) = 0 \text{ for all } \zeta \in C_c^{1;q}(\mathcal{C} \setminus \{0\} \times \mathbb{R}^{n-2}),$$

$$u_l = \sigma \varphi_l \text{ on } \partial\mathcal{C} \text{ for } l = 1, 2, \dots, q,$$

then $\|u\|_{C^{1,\mu;q}(\bar{\mathcal{C}})} \leq C$. Then by the Leray-Schauder theory, there exists a fixed point $u \in \mathcal{V}$ of T . In other words, the u solves (2.13). Note that $u = Tu \in C^{1,\tau;q}(\bar{\mathcal{C}})$ for all $\tau \in (0, 1/q)$.

Lemma 11. *Let $1 < \mu \leq \tau < 1/q$ and $a^{ij} = (a_1^{ij}, a_2^{ij}, \dots, a_q^{ij}), g = (g_1, g_2, \dots, g_q) \in C^{0,\mu;q}(\bar{\mathcal{C}})$ and $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_q) \in C^{1,\tau;q}(\bar{\mathcal{C}})$. Suppose a^{ij}, g , and φ are periodic with respect to y_l with period ρ_l for $l = 1, 2, \dots, n - 2$, $a^{ij}(\mathbf{R}X) = a^{i'j'}(X) R_{i'}^i R_{j'}^j$ for all $X \in \bar{\mathcal{C}}$, and*

$$a^{ij}(X) \xi_i \xi_j \geq \lambda |\xi|^2 \text{ for } X \in \bar{\mathcal{C}}, \xi \in \mathbb{R}^n, \quad \|a^{ij}\|_{C^{0,\mu;q}(\bar{\mathcal{C}})} \leq \Lambda,$$

for some constants $\lambda, \Lambda > 0$. Then there exists a unique $u = (u_1, u_2, \dots, u_q) \in C^{0;q}(\bar{\mathcal{C}}) \cap C^{1,\tau;q}(\mathcal{C}) \cap C^{2;q}(\mathcal{C} \setminus \{0\} \times \mathbb{R}^{n-2})$ such that

$$(6.3) \quad \begin{aligned} a_l^{ij} D_{ij} u_l &= g_l \text{ in } \mathcal{C} \setminus [0, 1] \times \{0\} \times \mathbb{R}^{n-2} \text{ for } l = 1, 2, \dots, q, \\ u_l &= \varphi_l \text{ on } \partial\mathcal{C} \text{ for } l = 1, 2, \dots, q. \end{aligned}$$

Moreover, u is k -fold symmetric and periodic with respect to y_l with period ρ_l for $l = 1, 2, \dots, n - 2$.

Proof. For now suppose $a^{ij}, \varphi \in C^{\infty;q}(\bar{\mathcal{C}})$. By replacing u with $u - \varphi$, we may suppose $\varphi = 0$. Re-write (6.3) as

$$(6.4) \quad \begin{aligned} \int_{\mathcal{C}} \sum_{l=1}^q (a_l^{ij} D_j u_l D_i \zeta_l + D_i a_l^{ij} D_j u_l \zeta_l) \\ = - \int_{\mathcal{C}} \sum_{l=1}^q g_l \zeta_l \text{ for all } \zeta \in C_c^{1;q}(\mathcal{C} \setminus \{0\} \times \mathbb{R}^{n-2}), \\ u_l = 0 \text{ on } \partial\mathcal{C} \text{ for } l = 1, 2, \dots, q. \end{aligned}$$

We will solve (6.4) using the method of continuity. Let \mathcal{W} denote the space of $u \in C^{1,\tau;q}(\bar{\mathcal{C}})$ such that u is periodic with respect to y_j with period ρ_j for $j = 1, 2, \dots, n - 2$, u has k -fold symmetry, and $u_l = 0$ on $\partial\mathcal{C}$ for $l = 1, 2, \dots, q$. Define the family $\{L_t\}_{t \in [0,1]}$ of weak linear operators on \mathcal{W} by

$$L_t u_l = (1 - t) \Delta u_l + t(D_i(a_l^{ij} D_j u_l) - (D_i a_l^{ij}) D_j u_l)$$

and consider

$$(6.5) \quad \begin{aligned} L_t u_l &= D_j f_l^j + g_l \text{ weakly in } \mathcal{C} \setminus \{0\} \times \mathbb{R}^{n-2}, \\ u_l &= 0 \text{ on } \partial\mathcal{C} \end{aligned}$$

for $f^j = (f_1^j, f_2^j, \dots, f_q^j) \in C^{0,\tau;q}(\bar{\mathcal{C}})$ and $g = (g_1, g_2, \dots, g_q) \in C^{0;q}(\bar{\mathcal{C}})$ such that f^j and g are periodic with respect to y_l with period ρ_l for $l = 1, 2, \dots, n - 2$, f^j satisfies (2.6), and g has k -fold symmetry. By Theorem 1, for $t = 0$ we can find a unique weak solution $u \in \mathcal{W}$ to (6.5). Suppose we can find a unique solution $u \in \mathcal{W}$ to (6.5) for $t = s$ for some $s \in [0, 1]$. Then (6.5) can be rewritten as

$$\begin{aligned} L_s u_l &= D_j f_l^j + g + (L_s - L_t) u_l \text{ in } \mathcal{C} \setminus \{0\} \times \mathbb{R}^{n-2}, \\ u_l &= 0 \text{ on } \partial\mathcal{C}. \end{aligned}$$

Define a map $U : \mathcal{W} \rightarrow \mathcal{W}$ by $u = U(v)$ for $u = (u_1, u_2, \dots, u_q), v = (v_1, v_2, \dots, v_q) \in \mathcal{W}$ if

$$\begin{aligned} L_s u_l &= D_j f_l^j + g + (L_s - L_t)v_l \text{ in } \mathcal{C} \setminus \{0\} \times \mathbb{R}^{n-2}, \\ u_l &= 0 \text{ on } \partial\mathcal{C}. \end{aligned}$$

Choose arbitrary $v = (v_1, v_2, \dots, v_q), v' = (v'_1, v'_2, \dots, v'_q) \in \mathcal{W}$ and let $u = (u_1, u_2, \dots, u_q) = U(v)$ and $u' = (u'_1, u'_2, \dots, u'_q) = U(v')$. Since

$$\begin{aligned} L_s(u_l - u'_l) &= (L_s - L_t)(v_l - v'_l) \\ &= (s - t)(-\Delta(v_l - v'_l) + D_i(a_l^{ij} D_j(v_l - v'_l)) \\ &\quad - (D_i a_l^{ij}) D_j(v_l - v'_l)), \end{aligned}$$

by Schauder estimate Lemma 10

$$\begin{aligned} &\|u - u'\|_{C^{1,\tau,q}(\bar{\mathcal{C}})} \\ &\leq C|s - t| \left([Dv - Dv']_{\tau;q,\mathcal{C}} + [a^{ij} D_j(v - v')]_{\tau;q,\mathcal{C}} + \sup_{\mathcal{C}} |D_i a^{ij}| |Dv_l - Dv'_l| \right) \\ &\leq C|s - t| \|v - v'\|_{C^{1,\tau,q}(\bar{\mathcal{C}})}, \end{aligned}$$

where $C \in (0, \infty)$ depends only on $n, q, \tau, \lambda, \Lambda, \|a^{ij}\|_{C^{1,q}(\bar{\mathcal{C}})}$, and $\rho_1, \dots, \rho_{n-2}$. So if $|s - t| < 1/2C$, then U is a contraction mapping and we can solve (6.5) for t with $|s - t| < 1/2C$. By dividing $[0, 1]$ into intervals of length less than $1/2C$ and applying this result we conclude that we can solve (6.5) for all $t \in [0, 1]$, in particular for $t = 1$. This gives us a $u \in \mathcal{W}$ satisfying (6.4). By elliptic regularity, if $g \in C^{0,\mu;q}(\bar{\mathcal{C}})$ then $u \in C^{2,\mu}(\mathcal{C} \setminus \{0\} \times \mathbb{R}^{n-2})$ and thus u satisfies (6.3).

To solve (6.3) for general a^{ij} and φ , approximate a^{ij} and φ by approximating their average parts by convolution with smooth, spherically symmetric mollifiers and approximate their average-free parts using the same construction as in the proof of Theorem 1 to approximate f by elements of $C^{\infty;q}(\bar{\mathcal{C}})$ to get $a_\nu^{ij} \in C^{\infty;q}(\bar{\mathcal{C}})$ converging to a^{ij} uniformly on $\mathcal{C} \setminus [0, 1] \times \{0\} \times \mathbb{R}^{n-2}$ and $\varphi^{(\nu)} \in C^{\infty;q}(\bar{\mathcal{C}})$ converging to φ uniformly on $\mathcal{C} \setminus [0, 1] \times \{0\} \times \mathbb{R}^{n-2}$. Let $u^{(\nu)} \in C^{1,\tau;q}(\bar{\mathcal{C}})$ solve (6.3) with $u^{(\nu)}, a_\nu^{ij}, g$, and $\varphi^{(\nu)}$ in place of u, a^{ij}, g , and φ . By an extension of [4, Theorem 3.7] proven using maximum principle Lemma 1,

$$\sup_{\mathcal{C}} |u^{(\nu)}| \leq \sup_{\partial\mathcal{C}} |\varphi| + \frac{C}{\lambda} \sup_{\mathcal{C}} |g|$$

for some constant $C = C(n) \in (0, \infty)$. Thus by the Schauder estimate Lemma 2, after passing to a subsequence $\{u^{(\nu)}\}$ converges in $C^{1;q}$ on compact subsets of \mathcal{C} to $u \in C^{1,\tau;q}(\mathcal{C})$. By the local Schauder estimates [4, Corollary 6.3], after passing to a subsequence $\{u^{(\nu)}\}$ converges to some u in $C^{2;q}$ on compact subsets of $\mathcal{C} \setminus \{0\} \times \mathbb{R}^{n-2}$ and thus $a_l^{ij} D_{ij} u_l = g_l$ in $\mathcal{C} \setminus [0, 1) \times \{0\} \times \mathbb{R}^{n-2}$ for $l = 1, 2, \dots, q$. By local barriers [4, Section 6.3], we can establish uniform modulus of continuity estimates for $u^{(\nu)}$ at points on $\partial\mathcal{C}$ that are independent of ν and thus u extends to a continuous function on $\bar{\mathcal{C}}$ such $u_l = \varphi_l$ on $\partial\mathcal{C}$. \square

Suppose that $u \in C^{1;q}(\bar{\mathcal{C}})$ satisfies (6.2). We want to bound $\|u\|_{C^{1,\mu;q}(\bar{\mathcal{C}})}$ for some $\mu \in (0, 1/q)$. By extending [4, Theorem 10.3] using maximum principle Lemma 1 and by (2.11),

$$(6.6) \quad \sup_{\bar{\mathcal{C}}} |u| \leq M_0 \quad \text{where} \quad M_0 = \sup_{\partial\mathcal{C}} |\varphi| + C\beta_2$$

for some constant $C = C(\beta_1) \in (0, \infty)$, where β_1 and β_2 as in (2.11). By a standard argument involving local barriers [4, Corollary 14.3] along $\partial\mathcal{C}$ using structure condition (2.12),

$$(6.7) \quad \sup_{\partial\mathcal{C}} |Du| \leq M_1,$$

for some $M_1 \in (0, \infty)$ depends on n , $\|\varphi\|_{C^{2;q}(\bar{\mathcal{C}})}$, β_1 , β_2 , and β_3 , where β_3 is as in (2.12).

We want to show $u \in W^{2,2;q}(\Omega)$ for all $\Omega \subset\subset \mathcal{C}$. By replacing ζ_l with $D_p\zeta$ for a single-valued function $\zeta \in C_c^{1;q}(\mathcal{C} \setminus \{0\} \times \mathbb{R}^{n-2})$ and $p \in \{1, 2, \dots, n\}$ in the first equation in (2.13) and integrating by parts,

$$(6.8) \quad \int_{\mathcal{C}} \sum_{l=1}^q (D_{P_j} A^i(Du_l) D_{j_p} u_l D_i \zeta - D_{P_j} B(u_l, Du_l) D_{j_p} u_l \zeta - D_Z B(u_l, Du_l) D_p u_l \zeta) = 0$$

for all $\zeta \in C_c^{1;q}(\mathcal{C} \setminus \{0\} \times \mathbb{R}^{n-2})$. Let $\zeta = D_p u \eta^2 \chi_\delta^2$ in (6.8), where $\eta \in C_c^1(\mathcal{C})$ is the single-valued cutoff function such that $0 \leq \eta \leq 1$, $\eta = 1$ on $B_{R/2}(X_0)$, $\eta = 0$ on $\mathbb{R}^n \setminus B_R(X_0)$, and $|D\eta| \leq 3/R$ and χ_δ is the function such that $0 \leq \chi_\delta \leq 1$, $\chi_\delta(x, y) = 1$ if $|x| \geq \delta$, $\chi_\delta(x, y) = 0$ if $|x| \leq \delta/2$, and $|D\chi_\delta| \leq 3/\delta$. By a standard computation using the fact that $D_{P_j} A^i(P) \xi_i \xi_j \geq \lambda(P) |\xi|^2$ for all

$P \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$ and using Cauchy's inequality,

$$\int_{B_{R/2}(X_0)} \sum_{l=1}^q |D^2 u_l|^2 \chi_\delta^2 \leq C \int_{B_R(X_0)} (\chi_\delta^2 + |D\chi_\delta|^2) \leq C$$

for some constants $C \in (0, \infty)$ depending on n, R, A^i, B, M_0 , and $\sup_C |Du|$ and independent of δ . Letting $\delta \downarrow 0$, $\|D^2 u\|_{L^2(B_{R/2}(X_0))} \leq C$ for all $B_R(X_0) \subset C$.

Now let $v(X) = (|Du_1(X)|^2, \dots, |Du_q(X)|^2) \in C^{1;q}(C \setminus \{0\} \times \mathbb{R}^{n-2})$. By replacing ζ_l with $\sum_{p=1}^n D_p(D_p u \zeta_l)$ in the first equation in (2.13) and using integration by parts, we get

$$\begin{aligned} \int_C \sum_{l=1}^q (D_{P_j} A^i(Du_l) D_j v_l D_i \zeta_l + 2D_{P_j} A^i(Du_l) D_{p_j} u_l D_{i_p} u_l \zeta_l \\ - D_{P_j} B(u_l, Du_l) D_j v_l \zeta_l - 2D_Z B(u_l, Du_l) v_l \zeta_l) = 0 \end{aligned}$$

for all $\zeta \in C_c^{1;q}(C \setminus \{0\} \times \mathbb{R}^{n-2})$. Since $D_{P_j} A^i(Du_l) D_{i_p} u_l D_{j_p} u_l \geq 0$,

$$\begin{aligned} \int_C \sum_{l=1}^q (D_{P_j} A^i(Du_l) D_j v_l D_i \zeta_l \\ - D_{P_j} B(u_l, Du_l) D_j v_l \zeta_l - 2D_Z B(u_l, Du_l) v_l \zeta_l) \leq 0 \end{aligned}$$

for all $\zeta \in C_c^{1;q}(C \setminus \{0\} \times \mathbb{R}^{n-2})$ such that $\zeta_l \geq 0$ in $C \setminus \{0\} \times \mathbb{R}^{n-2}$ for $l = 1, 2, \dots, q$. By the weak maximum principle similar to [4, Theorem 8.1]

$$\sup_C |Du| \leq \sup_{\partial C} |Du| \leq M_1,$$

where M_1 is the constant from (6.7).

By the interior Hölder continuity estimate Lemma 6 applied to (6.8) and the boundary Hölder continuity estimates for single-valued functions [4, Section 13.1] we obtain

$$[u]_{\mu;q,\bar{C}} \leq C$$

for some constant $C \in (0, \infty)$ and $\mu \in (0, 1/q)$ depending on $n, q, \beta_1, \beta_2, \beta_3, \bar{\lambda}, \sup_{B_{M_1}^n(0)} |DPA^i|, \sup_{(-M_0, M_0) \times B_{M_1}^n(0)} |DPB|, \sup_{(-M_0, M_0) \times B_{M_1}^n(0)} |DZB|,$

and $\|\varphi\|_{C^{2;q}(\bar{\mathcal{C}})}$. Therefore we have shown that if u satisfies (6.2), then

$$(6.9) \quad \|u\|_{C^{1,\mu;q}(\bar{\mathcal{C}})} \leq C$$

for some constant $C \in (0, \infty)$ and $\mu \in (0, 1/q)$ depending on $n, q, \beta_1, \beta_2, \beta_3, \bar{\lambda}, \sup_{B_{M_1}^n(0)} |D_P A^i|, \sup_{(-M_0, M_0) \times B_{M_1}^n(0)} |D_P B|, \sup_{(-M_0, M_0) \times B_{M_1}^n(0)} |D_Z B|,$ and $\|\varphi_0\|_{C^{2;q}(\bar{\mathcal{C}})}$.

To solve (2.13) in the case that $\varphi \in C^{2;q}(\bar{\mathcal{C}})$ with $\|\varphi\|_{C^{2;q}(\bar{\mathcal{C}})} < \infty$ and φ is not periodic, approximate φ in $C^{2;q}(\bar{\mathcal{C}})$ by $\varphi^{(\nu)}$ that are k -fold symmetric and periodic with respect to each y_j with period ρ_ν such that $\rho_\nu \rightarrow \infty$ as $\nu \rightarrow \infty$. Let $u^{(\nu)} \in C^{1,\mu;q}(\bar{\mathcal{C}})$ solve (2.13) with $u^{(\nu)}$ and $\varphi^{(\nu)}$ in place of u and φ respectively. By (6.9), after passing to a subsequence $\{u^{(\nu)}\}$ converges in $C^{1;q}(\mathcal{C})$ to a solution u to (2.13) with the original φ . By Schauder estimate Lemma 2 and local boundary Schauder estimates for single-valued solutions [4, Section 6.2] for every $\tau \in (0, 1/q)$ we have uniformly local $C^{1,\tau;q}$ estimates on $u^{(\nu)}$ that are independent of ν and thus $u \in C^{1,\tau;q}(\bar{\mathcal{C}})$.

For Corollary 1, we need to obtain an interior gradient estimate without using $\varphi \in C^{2;q}(\bar{\mathcal{C}})$. We will do so by extending an interior gradient estimate due to Simon [8, Theorem 1] to solutions $u \in C^{1;q}(\mathcal{C})$. This requires using cutoff function arguments to handle of singularity of u along $\{0\} \times \mathbb{R}^{n-2}$. For example, the analogue of (2.11) of [8] is

$$(6.10) \quad \sum_{l=1}^q \int_{\{v_l \geq \tau\}} ((1 - \tau/v_l)\mathcal{E}_l^2 + v_l D_{P_j} A^i(Du_l) D_i \omega_l D_j \omega_l) v_l \chi(v_l) \zeta^2 \leq 8n(1 + c_\chi)^2 \sum_{l=1}^q \int_{\{v_l \geq \tau\}} (\Lambda(v) \beta_1^2 \zeta^2 + 2\bar{\mu}_l |D\zeta|^2) v_l \chi(v_l)$$

for all $\zeta \in C_c^{1;q}(\mathcal{C} \setminus \{0\} \times \mathbb{R}^{n-2})$, where $v_l = (1 + |Du_l|^2)^{1/2}$, $g_l^{ij} = \delta_{ij} - D_i u_l D_j u_l / (1 + |Du_l|^2)$, $\mathcal{E}_l^2 = v_l^{-1} D_{P_j} A^i(Du_l) g_l^{kk'} D_{ik} u_l D_{j k'} u_l$, $\bar{\mu}_l$ satisfies

$$|v_l D_{P_j} A^i(Du_l) \xi_j \eta_i| \leq (\bar{\mu}_l |\eta|^2)^{1/2} (v_l D_{P_j} A^i(Du_l) \xi_i \xi_j)^{1/2}$$

on $\mathcal{C} \setminus [0, 1] \times \{0\} \times \mathbb{R}^{n-2}$ for all $\xi, \eta \in \mathbb{R}^n$ and χ and Λ are single-valued functions as in [8]. Since $u \in W^{2,2;q}(\Omega)$ for all $\Omega \subset\subset \mathcal{C}$, we can show (6.10) holds for any $\zeta \in C_c^{0;q}(\mathcal{C}) \cap W_0^{1,2;q}(\mathcal{C})$ by replacing ζ by $\zeta \psi_\delta$ in (6.10) for $\delta > 0$, where $\psi_\delta \in C^\infty(\mathcal{C})$ is the logarithmic cutoff function defined by $\psi_\delta(x, y) = 0$ if $|x| \leq \delta^2$, $\psi_\delta(x, y) = -\log(|x|/\delta^2)/\log(\delta)$ if $\delta^2 < |x| < \delta$, and $\psi_\delta(x, y) = 1$ if $|x| \geq \delta$, and then letting $\delta \downarrow 0$. The arguments of [8] only require using key integral inequalities, in particular analogues of (2.1) and (2.11), with test

functions in $C_c^{0;q}(\mathcal{C}) \cap W_0^{1,2;q}(\mathcal{C})$ and thus [8, Theorem 1] follows. Note that the analogue of (2.1) for test functions h vanishing along $\{0\} \times \mathbb{R}^{n-2}$ follows from the first variation of area formula for the closure of the graph of u as an immersed submanifold in $\mathcal{C} \times \mathbb{R} \setminus \{0\} \times \mathbb{R}^{n-1}$ (see the proof of [9, Theorem 18.6]) and then the analogue of (2.1) holds for any $h \in C_c^{0;q}(\mathcal{C}) \cap W_0^{1,2;q}(\mathcal{C})$ by the logarithmic cutoff function argument.

Now we will solve (2.13) in the case that $\varphi \in C^{0;q}(\bar{\mathcal{C}})$ has k -fold symmetry and $\sup_{\mathcal{C}} |\varphi| < \infty$. Assume $\varphi = 0$ in $B_{1/2}^2(0) \times \mathbb{R}^{n-2} \setminus [0, 1/2) \times \{0\} \times \mathbb{R}^{n-2}$. Approximate φ uniformly on compact subsets of $\bar{\mathcal{C}}$ by $\varphi^{(\nu)}$ that are k -fold symmetric and periodic with respect to each y_j with period ρ_ν such $\rho_\nu \rightarrow \infty$ as $\nu \rightarrow \infty$. Let $u^{(\nu)} \in C^{1,1/2q;q}(\bar{\mathcal{C}})$ solve (2.13) with $u^{(\nu)}$ and $\varphi^{(\nu)}$ in place of u and φ . By (6.6), the interior gradient estimate of [8, Theorem 1], and Lemma 6, $\sup_{\nu} \|u^{(\nu)}\|_{C^{1,\mu;q}(\Omega)} < \infty$ for all $\Omega \subset\subset \mathcal{C}$ for some $\mu \in (0, 1/2q]$ depending on n, A^i, B , and $\sup_{\partial\mathcal{C}} |\varphi|$, so after passing to a subsequence $\{u^{(\nu)}\}$ converges in $C^{1;q}$ on compact subsets of \mathcal{C} to some $u \in C^{1,\mu;q}(\mathcal{C})$. By Schauder estimate Lemma 2 for every $\tau \in (0, 1/q)$ we have uniformly local interior $C^{1,\tau;q}$ estimates on $u^{(\nu)}$ that are independent of ν and thus $u \in C^{1,\tau;q}(\mathcal{C})$. Using local barriers [4, Theorem 14.15], we obtain uniform modulus of continuity estimates on $u^{(\nu)}$ at points on $\partial\mathcal{C}$ that are independent of ν and thereby conclude that u extends to an element of $C^{0;q}(\bar{\mathcal{C}})$ such that $u_l = \varphi_l$ on $\partial\mathcal{C}$ for $l = 1, 2, \dots, q$. Therefore u solves (2.13) with the original φ .

7. Regularity of q -valued solutions

Recall from Section 2 that to prove Theorem 4, it suffices to prove (2.17) for $u = (u_1, u_2, \dots, u_q) \in C^{1;q}(B_1(0))$ such that $\|u\|_{C^{1;q}(B_1(0))} \leq 1/2$ and u satisfies the elliptic equation

$$(7.1) \quad \begin{aligned} D_i(A^i(X, u_l, Du_l)) + B(X, u_l, Du_l) &= 0 \\ \text{in } B_1(0) \setminus [0, \infty) \times \{0\} \times B_1^{n-2}(0) \end{aligned}$$

for given locally real analytic single-valued functions $A^i(X, Z, P)$ and $B(X, Z, P)$ on $B_1(0) \times (-1, 1) \times B_1^n(0)$. Using arguments similar to those from Section 6 we can show that $u \in W^{2,2;q}(\Omega)$ for all $\Omega \subset\subset B_1(0)$ and by Lemma 6 using the fact that $D_k u$ is a satisfies

$$\begin{aligned} D_i(D_{P_j} A^i(X, u_l, Du_l) D_j D_k u_l + D_Z A^i(X, u_l, Du_l) D_k u_l + D_{X_k} A^i(X, u_l, Du_l)) \\ + D_{P_j} B(X, u_l, Du_l) D_j D_k u_l + D_Z B(X, u_l, Du_l) D_k u_l + D_{X_k} B(X, u_l, Du_l) = 0 \end{aligned}$$

in $B_1(0) \setminus [0, \infty) \times \{0\} \times B_1^{n-2}(0)$, we can conclude that $u \in C^{1,\mu;q}(B_1(0))$ for some $\mu \in (0, 1/q)$. The first step to proving Theorem 4 is to establish that $D_y^\gamma u \in C^{1,\mu;q}(B_1(0))$ for all γ .

Lemma 12. *Let $\mu \in (0, 1/q)$. Suppose $u \in C^{1,\mu;q}(B_1(0))$ such that $\|u\|_{C^{1,q}(B_1(0))} \leq 1/2$ and u satisfies (7.1) for given smooth single-valued functions $A^i, B : B_1(0) \times (-1, 1) \times B_1^n(0) \rightarrow \mathbb{R}$ and assume (2.16) holds. Then $D_y^\gamma u \in C^{1,\mu;q}(B_1(0))$ for all γ .*

The proofs of Lemma 12 and Theorem 4 require applying D_y^γ to (7.1). Observe that

$$(7.2) \quad \begin{aligned} D_y^\gamma(A^i(X, u_l, Du_l)) &= (D_{P_j} A^i)(X, u_l, Du_l) D_j D_y^\gamma u_l \\ &\quad + F_\gamma^i(X, u_l, \{DD_y^\beta u_l\}_{|\beta| \leq |\gamma|-1}), \\ D_y^\gamma(B(X, u_l, Du_l)) &= (D_{P_j} B)(X, u_l, Du_l) D_j D_y^\gamma u_l \\ &\quad + G_\gamma(X, u_l, \{DD_y^\beta u_l\}_{|\beta| \leq |\gamma|-1}), \end{aligned}$$

on $B_1(0) \setminus [0, \infty) \times \{0\} \times B_1^{n-2}(0)$ for $l = 1, 2, \dots, q$ for some functions F_γ^i and G_γ . To simplify notation, let $a^{ij} = (a_1^{ij}, a_2^{ij}, \dots, a_q^{ij})$, $b^j = (b_1^j, b_2^j, \dots, b_q^j)$, $f_\gamma^i = (f_{\gamma,1}^i, f_{\gamma,2}^i, \dots, f_{\gamma,q}^i)$, and $g_\gamma = (g_{\gamma,1}, g_{\gamma,2}, \dots, g_{\gamma,q})$ where

$$(7.3) \quad \begin{aligned} a_l^{ij} &= (D_{P_j} A^i)(X, u_l, Du_l), & f_{\gamma,l}^i &= -F_\gamma^i(X, u_l, \{DD_y^\beta u_l\}_{|\beta| \leq |\gamma|-1}), \\ b_l^j &= (D_{P_j} B)(X, u_l, Du_l), & g_{\gamma,l} &= -G_\gamma(X, u_l, \{DD_y^\beta u_l\}_{|\beta| \leq |\gamma|-1}), \end{aligned}$$

on $B_1(0) \setminus [0, \infty) \times \{0\} \times B_1^{n-2}(0)$. We can express f_γ^i as

$$(7.4) \quad \begin{aligned} f_{\gamma,l}^i &= \sum c_{\alpha,j,\beta} (D_{(y,Z,P)}^\alpha A^i)(X, u_l, Du_l) \\ &\quad \times \prod_{k \leq |\alpha_Z|} D_y^{\beta_{Z,k}} u_l \prod_{k \leq |\alpha_P|} D_y^{\beta_{P,k}} D_{j_k} u_l \end{aligned}$$

on $B_1(0) \setminus [0, \infty) \times \{0\} \times B_1^{n-2}(0)$, where $\alpha = (\alpha_y, \alpha_Z, \alpha_P)$ and $D_{(y,Z,P)}^\alpha = D_y^{\alpha_y} D^{\alpha_Z} D^{\alpha_P}$ and the sum is taken over all nonzero multi-indices α , $\beta_{Z,k}$, and $\beta_{P,k}$ and $1 \leq j_k \leq n$ such that

$$(7.5) \quad \alpha_y + \sum_{j \leq |\alpha_Z|} \beta_{Z,j} + \sum_{k \leq |\alpha_P|} \beta_{P,k} = \gamma$$

and $|\beta_{P,k}| < p$ and the coefficients $c_{\alpha,j,\beta}$ are positive integers depending on $\alpha, j_1, \dots, j_{|\alpha|}$, and $\beta_1, \dots, \beta_{|\alpha|}$. Note that in (7.4) and (7.5) assume the convention that sums over $j \leq 0$ equal zero and products over $j \leq 0$ equal one.

We can write a similarly express g_γ as

$$(7.6) \quad g_\gamma = \sum c_{\alpha,j,\beta}(D_{(y,Z,P)}^\alpha B)(X, u, Du) \prod_{k \leq |\alpha_Z|} D_y^{\beta_{Z,k} u} \prod_{k \leq |\alpha_P|} D_y^{\beta_{P,k}} D_{j_k} u$$

on $B_1(0) \setminus [0, \infty) \times \{0\} \times B_1^{n-2}(0)$, where the sum is taken over all nonzero multi-indices α , $\beta_{Z,k}$, and $\beta_{P,l}$ and $1 \leq j_k \leq n$ such that (7.5) holds and $|\beta_{P,k}| < p$. By (7.2), applying D_y^γ to (7.1) yields

$$(7.7) \quad \begin{aligned} D_i(a^{ij} D_j D_y^\gamma u_l) + b^j D_j D_y^\gamma u_l &= D_i f_{\gamma,l}^i + g_{\gamma,l} \\ &\text{in } B_1(0) \setminus [0, \infty) \times \{0\} \times B_1^{n-2}(0). \end{aligned}$$

Proof of Lemma 12. Given $\eta \in \mathbb{R}^{n-2} \setminus \{0\}$, for each $h \neq 0$ let $\delta_{h,\eta}$ be the difference quotient defined by (2.4) and let $D_{(0,\eta)}$ denote the derivative in the direction $(0, \eta)$. We will show that $D_{(0,\eta)} D^\gamma u \in C^{1,\mu;q}(B_1(0))$ for every $\eta \in \mathbb{R}^{n-2} \setminus \{0\}$ by induction on $|\gamma|$. This follows by a standard difference quotient argument where we use the Schauder estimates Lemma 7 to obtain uniform local $C^{1,\mu;q}$ bounds on $\delta_{h,\eta} D_y^\gamma u$ that are independent of h . The key to the proof is the fact that such difference quotients $\delta_{h,\eta}$ of u and its derivatives are well-defined.

First we show $D_{(0,\eta)} u \in C^{1,\mu;q}(B_1(0))$ for every $\eta \in \mathbb{R}^{n-2} \setminus \{0\}$. Let $B_R(x_0, y_0) \subset\subset B_1(0)$ and suppose $|h\eta| \leq R/4$. By applying $\delta_{h,\eta}$ to (7.1) and using Schauder estimate Lemma 7,

$$\begin{aligned} R \|D\delta_{h,\eta} u\|'_{C^{0,\mu;q}(B_{R/4}(x_0,y_0))} &\leq C|\eta| \left(\sup_{B_{R/2}(x_0,y_0)} |\delta_{h,\eta} u| + 1 \right) \\ &\leq C|\eta| \left(\sup_{B_R(x_0,y_0)} |D_y u| + 1 \right) \end{aligned}$$

for some constant $C \in (0, \infty)$ depending on $n, q, \mu, A^i, B, \|u\|_{C^{1,\mu}(B_R(x_0,y_0))}$, and R and independent of h . So given any sequence of $h_j \rightarrow 0$, we can pass to a subsequence $\{\delta_{h_{j'},\eta} u\}$ that converges in $C^{1;q}$ on compact subsets of $B_1(0)$ with a limit in $C^{1,\mu;q}(B_1(0))$. But $\delta_{h,\eta} u \rightarrow D_{(0,\eta)} u$ pointwise, so in fact $\delta_{h,\eta} u \rightarrow D_{(0,\eta)} u$ in $C^{1;q}$ on compact subsets of $B_1(0)$ as $h \rightarrow 0$ and $D_{(0,\eta)} u \in C^{1,\mu;q}(B_1(0))$.

Now let $|\gamma| \geq 1$. We will show that $D_{(0,\eta)} D_y^\gamma u \in C^{1,\mu}(B_1(0))$ assuming $D_y^\beta u \in C^{1,\mu}(B_1(0))$ whenever $|\beta| < |\gamma|$. Let $B_R(x_0, y_0) \subset\subset B_1(0)$ and suppose $|h\eta| \leq R/4$. Recall that applying D_y^γ to (7.1) yields (7.7). By applying $\delta_{h,\eta}$ to (7.7) and using the Schauder estimates Lemma 7, if $B_R(x_0, y_0) \subset\subset$

$B_1(0)$ and $|h\eta| \leq R/4$,

$$\begin{aligned} & R\|D\delta_{h,\eta}D_y^\gamma u\|'_{C^{0,\mu;q}(B_{R/4}(x_0,y_0))} \\ & \leq C \left(\sup_{B_{R/2}(x_0,y_0)} |\delta_{h,\eta}D_y^\gamma u| + \|DD_y^\gamma u\|'_{C^{0,\mu;q}(B_{R/2}(x_0,y_0+h\eta))} \right. \\ & \quad \left. + R\|\delta_{h,\eta}f_\gamma\|'_{C^{0,\mu;q}(B_{R/2}(x_0,y_0))} + R^2 \sup_{B_{R/2}(x_0,y_0)} |\delta_{h,\eta}g_\gamma| \right) \\ & \leq C|\eta| \left(\sup_{B_R(x_0,y_0)} |D_yD_y^\gamma u| + \|DD_y^\gamma u\|'_{C^{0,\mu;q}(B_R(x_0,y_0))} \right. \\ & \quad \left. + R\|D_yf_\gamma\|'_{C^{0,\mu;q}(B_R(x_0,y_0))} + R^2 \sup_{B_R(x_0,y_0)} |D_yg_\gamma| \right) \end{aligned}$$

for some constant $C \in (0, \infty)$ depending on $n, q, \mu, A^i, B, \|u\|_{C^{1,\mu}(B_R(x_0,y_0))}$, and R and independent of h . So given any sequence of $h_j \rightarrow 0$, we can pass to a subsequence $\{\delta_{h_j,\eta}D_y^\gamma u\}$ that converges in $C^{1;q}$ on compact subsets of $B_1(0)$ with a limit in $C^{1,\mu;q}(B_1(0))$. But $\delta_{h,\eta}D_y^\gamma u \rightarrow D_{(0,\eta)}D_y^\gamma u$ pointwise, so in fact $\delta_{h,\eta}D_y^\gamma u \rightarrow D_{(0,\eta)}D_y^\gamma u$ in $C^{1;q}$ on compact subsets of $B_1(0)$ as $h \rightarrow 0$ and $D_{(0,\eta)}D_y^\gamma u \in C^{1,\mu;q}(B_1(0))$. \square

Recall from the beginning of this section that applying D_y^γ to (7.1) yields (7.7). By the Schauder estimate Lemma 7 applied to (7.7),

$$(7.8) \quad \begin{aligned} & \|D_y^\gamma u\|'_{C^{1,\mu;q}(B_{R/2p}(X_1))} \\ & \leq C \left(\sup_{B_{R/p}(X_1)} |D_y^\gamma u| + (R/p)^{1+\mu} [f_\gamma]_{\mu;q,B_{R/p}(X_1)} + (R/p)^2 \sup_{B_{R/p}(X_1)} |g_\gamma| \right) \end{aligned}$$

for all $B_R(X_1) \subset\subset B_1(0)$ and some constant $C \in (0, \infty)$ depending on $n, q, \mu, \lambda, \sup_{B_R(X_1) \times (-1,1) \times B_1^n(0)} |DA|$, and $\sup_{B_R(X_1) \times (-1,1) \times B_1^n(0)} |DB|$. Since f_γ^i and g_γ can be expressed in terms of $u, Du, DD_y u, \dots, DD_y^{p-1}u$, we can prove (2.17) by inductively computing bounds on $\|DD_y^\gamma u\|'_{C^{0,\mu;q}(B_{R/2p}(X_1))}$. The difficulty is bounding $[f_\gamma]_{\mu;q,B_{R/p}(X_1)}$ and $\sup_{B_{R/p}(X_1)} |g_\gamma|$ in order to obtain the necessary estimates (2.17) on $D_y^\gamma u$. We accomplish this using a modified version of a technique used by Friedman in [2]. Since the estimate on $\sup_{B_{R/p}(X_1)} |g_\gamma|$ is easier to obtain, we will obtain that estimate first.

Lemma 13. *Let $p \geq 5$ be a positive integer, $K_0, K, H_0 \geq 1$ be constants, and $B_R(X_1) \subset\subset B_1(0)$. For some constants $C \in (0, \infty)$ and $H \geq 1$ depending on n, K_0, K , and H_0 and independent of p the following holds true.*

Suppose $u \in C^{1;q}(B_1(0))$ satisfies $\|u\|_{C^{1;q}(B_1(0))} \leq 1/2$ and (7.1) for given smooth single-valued functions $A^i, B : B_1(0) \times (-1, 1) \times B_1^n(0) \rightarrow \mathbb{R}$. Suppose for any multi-index $\alpha = (\alpha_X, \alpha_Z, \alpha_P)$ with $|\alpha| = k$, where we let $D^\alpha = D_X^{\alpha_X} D_Z^{\alpha_Z} D_P^{\alpha_P}$,

$$(7.9) \quad |D_{(X,Z,P)}^\alpha B(X, Z, P)| \leq \begin{cases} K_0 K^k R^{-1-|\alpha_X|-|\alpha_Z|} & \text{if } k = 1, 2, 3, \\ (k-3)! K_0 K^k R^{-1-|\alpha_X|-|\alpha_Z|} & \text{if } 4 \leq k \leq p, \end{cases}$$

for $X \in B_R(X_1)$, $|Z| \leq 1/2$, and $|P| \leq 1/2$ and for any multi-index β with $|\beta| = s < p$,

$$(7.10) \quad \begin{aligned} & \frac{s}{R} \sup_{B_{R/p}(X_1)} |D_y^\beta u| + \sup_{B_{R/p}(X_1)} |DD_y^\beta u| \\ & \leq \begin{cases} H_0 R^{-s} & \text{if } s = 0, 1, 2, \\ (s-2)! H_0 H^{s-3} R^{-s} & \text{if } 3 \leq s < p. \end{cases} \end{aligned}$$

Then g_γ defined by (7.2) and (7.3) satisfies

$$R \sup_{B_{R/p}(X_1)} |g_\gamma| \leq C(p-2)! H^{p-3} R^{-p}.$$

Proof. Suppose we had a function $\Psi(y_1, \dots, y_{n-2}, Z, P_1, \dots, P_n)$ such that $\Psi(0, 0, 0) = 0$ and

$$(7.11) \quad |D_y^{\alpha_y} D_Z^{\alpha_Z} D_P^{\alpha_P} B(X, Z, P)| \leq D_y^{\alpha_y} D_{z_0}^{\alpha_Z} D_{(z_1, \dots, z_n)}^{\alpha_P} \Psi(0, 0, 0)$$

for $X \in B_R(X_1)$, $|Z| \leq \sup_{B_R(X_1)} |u|$, and $|P| \leq \sup_{B_R(X_1)} |Du|$ and for all nonnegative integers α_Z and multi-indices α_y and α_P such that $1 \leq |\alpha_y| + \alpha_Z + |\alpha_P| \leq p$. Further suppose we had functions $v_j(y_1, \dots, y_{n-2})$, $j = 0, 1, \dots, n$, such that $v_0(0) = 0$ and

$$(7.12) \quad \sup_{B_{R/p}(X_1)} |D_y^\beta u| \leq D_y^\beta v_0(0)$$

for $0 < |\beta| \leq p$ and for $j = 1, \dots, n$, $v_j(0) = 0$, $D_y^\gamma v_j(0) \geq 0$, and

$$(7.13) \quad \sup_{B_{R/p}(X_1)} |D_y^\beta D_j u| \leq D_y^\beta v_j(0)$$

for $0 < |\beta| < p$. Recall that g_γ can be expressed as in (7.6); that is,

$$g_{\gamma,l} = \sum c_{\alpha,j,\beta} (D_{(y,Z,P)}^\alpha B)(X, u_l, Du_l) \prod_{k \leq |\alpha_Z|} D_y^{\beta_{Z,k}} u_l \prod_{k \leq |\alpha_P|} D_y^{\beta_{P,k}} D_{j_k} u_l$$

for $l = 1, 2, \dots, q$, where the sum is taken over $\alpha = (\alpha_y, \alpha_Z, \alpha_P)$, $\beta_{Z,k}$, $\beta_{P,l}$, and $1 \leq j_k \leq n$ such that (7.5) holds and $|\beta_{P,l}| < p$ and the $c_{\alpha_y, \alpha_Z, \alpha_P, j, \beta}$ are the positive integers from (7.4). We also have

$$(7.14) \quad D_y^\gamma(\Psi(y, v)) = \sum c_{\alpha, j, \beta} (D_{(y, Z, P)}^\alpha \Psi)(y, v) \times \prod_{k \leq |\alpha_Z|} D_y^{\beta_{Z,k}} v_0 \prod_{k \leq |\alpha_P|} D_y^{\beta_{P,k}} v_{j_k}$$

where the sum is taken over (7.5) and the coefficients $c_{\alpha, j, \beta}$ are the same as above. Here $v = (v_0, v_1, \dots, v_n)$ and $\Psi(y, v) = \Psi(y_1, \dots, y_{n-2}, v_0, v_1, \dots, v_{n-2})$. Comparing (7.6) and (7.14) using (7.11), (7.12), and (7.13),

$$(7.15) \quad \sup_{B_{R/p}(X_1)} |g_\gamma| \leq D_y^\gamma(\Psi(y, v(y)))|_{y=0}.$$

To construct v_0, v_1, \dots, v_n and Ψ , first we simplify the setup by letting $v_0(y) = Rv(R^{-1}(y_1 + \dots + y_{n-2}))$ and $v_1(y) = \dots = v_n(y) = v(R^{-1}(y_1 + \dots + y_{n-2}))$ for some function $v(\xi)$ and replacing $\Psi(y_1, \dots, y_{n-2}, Z, P_1, \dots, P_n)$ with $R^{-1}\Psi(R^{-1}(y_1 + \dots + y_{n-2}), R^{-1}Z + P_1 + \dots + P_n)$ for some function $\Psi(\xi, \zeta)$. We choose

$$\Psi(\xi, \zeta) = \sum_{k=1}^2 \frac{1}{k!} K_0 K^k (\xi + \zeta)^k + \sum_{k=3}^p \frac{1}{k(k-1)(k-2)} K_0 K^k (\xi + \zeta)^k,$$

$$v(\xi) = H_0 \xi + \frac{1}{2} H_0 \xi^2 + \sum_{s=3}^p \frac{1}{s(s-1)} H_0 H^{s-3} \xi^s.$$

It is easy to check (7.11), (7.12), and (7.13) using (7.9) and (7.10).

For functions $f(\xi)$ and $g(\xi)$, let $f \ll_p g$ denote that $|D_\xi^s f(0, 0)| \leq D_\xi^s g(0, 0)$ for $1 \leq s \leq p$. We claim that

$$(7.16) \quad (\xi + (1+n)v(\xi))^k \ll_p c^{k-1} (1 + (1+n)H_0)^k \times \left(\xi^k + \xi^{k+1} + \sum_{s=k+2}^p \frac{1}{(s-k)^2} H^{s-k-2} \xi^s \right)$$

for $k = 1, 2, \dots, p$ for some constant $c \geq 1$ independent of k . We can see this by induction on k . (7.16) obviously holds for $k = 1$. Let $k \geq 2$. Assuming (7.16) holds with $k - 1$ in place of k and multiplying $(\xi + (1+n)v(\xi))$ by

$$(\xi + (1 + n)v(\xi))^{k-1},$$

$$(7.17) \quad (\xi + (1 + n)v(\xi))^k \ll_p 2c^{k-2}(1 + (1 + n)H_0)^k \times \left(\xi^k + \xi^{k+1} + \sum_{s=k+2}^{p^k} \sum_{j=2}^{s-k} \frac{1}{(j-1)^2(s-j-k+1)^2} H^{s-k-2} \xi^s \right).$$

Since

$$(7.18) \quad \sum_{j=2}^{s-k} \frac{1}{(j-1)^2(s-j-k+1)^2} = \sum_{j=2}^{s-k} \frac{1}{(s-k)^2} \left(\frac{1}{j-1} + \frac{1}{s-j-k+1} \right)^2 \leq \sum_{j=2}^{s-k} \frac{1}{(s-k)^2} \frac{4}{(j-1)^2} \leq \frac{2\pi^2}{3(s-k)^2},$$

(7.16) follows from (7.17) provided we choose $c = 4\pi^2/3$. Combining the definition of Ψ and (7.16), for $p \geq 5$,

$$\Psi(\xi, (n + 1)v(\xi)) \ll_p \sum_{k=1}^2 \frac{1}{k!} c^{k-1} (1 + (1 + n)H_0)^k K_0 K^k \times \left(\xi^k + \xi^{k+1} + \sum_{s=k+2}^{p^k} \frac{1}{(s-k)^2} H^{s-k-2} \xi^s \right) + \sum_{k=3}^p \frac{1}{(k-2)^3} c^{k-1} (1 + (1 + n)H_0)^k K_0 K^k \times \left(\xi^k + \xi^{k+1} + \sum_{s=k+2}^{p^k} \frac{1}{(s-k)^2} H^{s-k-2} \xi^s \right).$$

It follows that

$$\begin{aligned} & \left. \frac{1}{p!} R \frac{\partial^p}{\partial \xi^p} \Psi(\xi, (n+1)v(\xi)) \right|_{\xi=0} \\ & \leq \sum_{k=1}^2 \frac{1}{k!(p-k)^2} c^{k-1} (1 + (1+n)H_0)^k K_0 K^k H^{p-k-2} \\ & \quad + \sum_{k=3}^{p-2} \frac{1}{(k-2)^3(p-k)^2} c^{k-1} (1 + (1+n)H_0)^k K_0 K^k H^{p-k-2} \\ & \quad + \frac{1}{(p-3)^3} c^{p-2} (1 + (1+n)H_0)^{p-1} K_0 K^{p-1} \\ & \quad + \frac{1}{(p-2)^3} c^{p-1} (1 + (1+n)H_0)^p K_0 K^p \end{aligned}$$

Let $H = \max\{cK(1 + (1+n)H_0), 1\}$ so that, using a computation similar to (7.18), we have

$$\left. \frac{1}{p!} \frac{\partial^p}{\partial \xi^p} \Psi(\xi, (n+1)v(\xi)) \right|_{\xi=0} \leq \frac{C}{(p-2)^2} K_0 K^3 (1 + (1+n)H_0)^3 H^{p-3}$$

for some constant $C \in (0, \infty)$ independent of p . Thus by (7.15),

$$R \sup_{B_{R/p}(X_1)} |g_\gamma| \leq C(p-2)! H^{p-3} R^{-p}$$

for some constant $C = C(n, K_0, K, H_0) \in (0, \infty)$ independent of p . □

Lemma 14. *Let $p \geq 5$ be a positive integer, $K_0, K, H_0 \geq 1$ be constants, and $B_R(X_1) \subset\subset B_1(0)$. For some constants $C \in (0, \infty)$ and $H \geq 1$ depending on $n, K_0, K,$ and H_0 and independent of p the following holds true. Suppose $u \in C^{1,\mu;q}(B_1(0))$, where $\mu \in (0, 1/q)$, satisfies $\|u\|_{C^{1,q}(B_1(0))} \leq 1/2$ and (7.1) for given smooth single-valued functions $A^i, B : B_1(0) \times (-1, 1) \times B_1^n(0) \rightarrow \mathbb{R}$. Suppose for any multi-index $\alpha = (\alpha_X, \alpha_Z, \alpha_P)$ with $|\alpha| = k$,*

$$(7.19) \quad |D_{(X,Z,P)}^\alpha A(X, Z, P)| \leq \begin{cases} K_0 K^k R^{-|\alpha_X| - |\alpha_Z|} & \text{if } k = 1, 2, 3, \\ (k-3)! K_0 K^k R^{-|\alpha_X| - |\alpha_Z|} & \text{if } 4 \leq k \leq p, \end{cases}$$

for $X \in B_R(X_1)$, $|Z| \leq 1/2$, and $|P| \leq 1/2$ and for any multi-index β with $|\beta| = s < p$,

$$(7.20) \quad \begin{aligned} & \frac{s}{R} \sup_{B_{R/p}(X_1)} |D_y^\beta u| + \sup_{B_{R/p}(X_1)} |DD_y^\beta u| + \left(\frac{R}{s}\right)^\mu [DD_y^\beta u]_{\mu; q, B_{R/p}(X_1)} \\ & \leq \begin{cases} H_0 R^{-s} & \text{if } s = 0, 1, 2, \\ (s-2)! H_0 H^{s-3} R^{-s} & \text{if } 3 \leq s < p. \end{cases} \end{aligned}$$

Then f_γ^i defined by (7.2) and (7.3) satisfies

$$(R/p)^\mu [f_\gamma^i]_{\mu, B_{R/p}(X_1)} \leq C(p-2)! H^{p-3} R^{1-p}.$$

Proof. We will use a similar argument as for Lemma 13, except now we need to compute a Hölder coefficient. To this we will introduce an auxiliary parameter t such that derivatives of Ψ and v with respect to t bound to Hölder coefficients of expressions involving A^i and u . The basic idea is to use the fact that the sum, product, and chain rules for computing derivatives with respect to t are similar to sum, product, and composition rules for computing Hölder coefficients.

Suppose we had a function $\Psi(y_1, \dots, y_{n-2}, t, Z, P_1, \dots, P_n)$ such that $\Psi(0, 0, 0, 0) = 0$ and

$$(7.21) \quad \begin{aligned} & |D_y^{\alpha_y} D_Z^{\alpha_Z} D_P^{\alpha_P} A^i(X, Z, P)| \leq D_y^{\alpha_y} D_{z_0}^{\alpha_Z} D_{(z_1, \dots, z_n)}^{\alpha_P} \Psi(0, 0, 0, 0), \\ & (2R/p)^{1-\mu} |D_y D_y^{\alpha_y} D_Z^{\alpha_Z} D_P^{\alpha_P} A^i(X, Z, P)| \leq D_t D_y^{\alpha_y} D_{z_0}^{\alpha_Z} D_{(z_1, \dots, z_n)}^{\alpha_P} \Psi(0, 0, 0, 0), \end{aligned}$$

for $X \in B_R(X_1)$, $|Z| \leq \sup_{B_R(X_1)} |u|$, $|P| \leq \sup_{B_R(X_1)} |Du|$, and $1 \leq |\alpha_y| + \alpha_Z + |\alpha_P| \leq p$ and we had functions $v_j(y_1, \dots, y_{n-2}, t)$, $j = 0, 1, \dots, n$, such that $v_0(0, 0) = 0$,

$$(7.22) \quad \begin{aligned} & \sup_{B_{R/p}(X_1)} |D_y^\beta u| \leq D_y^\beta v_0(0, 0) \quad \text{for } 0 < |\beta| \leq p, \\ & [D_y^\beta u]_{\mu; q, B_{R/p}(X_1)} \leq D_t D_y^\beta v_0(0, 0) \quad \text{for } 0 \leq |\beta| \leq p, \end{aligned}$$

and $v_j(0, 0) = 0$, $D_y^\gamma v_j(0, 0) \geq 0$,

$$(7.23) \quad \begin{aligned} & \sup_{B_{R/p}(X_1)} |D_y^\beta D_j u| \leq D_y^\beta v_j(0, 0) \quad \text{for } 0 < |\beta| < p, \\ & [D_y^\beta D_j u]_{\mu; q, B_{R/p}(X_1)} \leq D_t D_y^\beta v_0(0, 0) \quad \text{for } 0 \leq |\beta| < p, \end{aligned}$$

for $j = 1, \dots, n$. Recall that f_γ^i can be expressed as in (7.4) and compute that

$$\begin{aligned}
 (7.24) \quad [f_\gamma^i]_{\mu; q, B_{R/p}(X_1)} &\leq \sum c_{\alpha, j, \beta} \left([(D_{(y, Z, P)}^\alpha A^i)(X, u, Du)]_{\mu; q, B_{R/p}(X_1)} \right. \\
 &\quad \times \prod_{k \leq |\alpha_Z|} \sup |D_y^{\beta_{Z, k}} u| \prod_{k \leq |\alpha_P|} \sup |D_y^{\beta_{P, k}} D_{j_k} u| \\
 &\quad + \sup |D_{(X, Z, P)}^\alpha A^i| \sum_{k \leq |\alpha_Z|} [D_y^{\beta_{Z, k}} u]_\mu \\
 &\quad \times \prod_{l \neq k} \sup |D_y^{\beta_{Z, l}} u| \prod_{k \leq |\alpha_P|} \sup |D_y^{\beta_{P, k}} D_{j_k} u| \\
 &\quad + \sup |D_{(X, Z, P)}^\alpha A^i| \prod_{k \leq |\alpha_Z|} \sup |D_y^{\beta_{Z, k}} u| \\
 &\quad \left. \times \sum_{k \leq |\alpha_P|} [D_y^{\beta_{P, k}} D_{j_k} u]_\mu \prod_{l \neq k} \sup |D_y^{\beta_{Z, l}} D_{j_l} u| \right),
 \end{aligned}$$

where $(D_{(y, Z, P)}^\alpha A^i)(X, u, Du) = ((D_{(y, Z, P)}^\alpha A^i)(X, u_l, Du_l))_{l=1, 2, \dots, q}$, the supremums of derivatives of A^i are taken over $X \in B_R(X_1)$, $|Z| \leq \sup_{B_R(X_1)} |u|$, and $|P| \leq \sup_{B_R(X_1)} |Du|$, and the supremums and the Hölder coefficients of derivatives of u are taken over $B_{R/p}(X_1)$. Moreover,

$$\begin{aligned}
 (7.25) \quad &[(D_{(y, Z, P)}^\alpha A^i)(X, u, Du)]_{\mu; q, B_{R/p}(X_1)} \\
 &\leq (2R/p)^{1-\mu} \sup_{B_{R/p}(X_1)} |(D_y D_{(y, Z, P)}^\alpha A^i)(X, u, Du)| \\
 &\quad + \sup_{B_{R/p}(X_1)} |(D_Z D_{(y, Z, P)}^\alpha A^i)(X, u, Du)| [u]_{\mu; q, B_{R/p}(X_1)} \\
 &\quad + \sum_{k=1}^n \sup_{B_{R/p}(X_1)} |(D_{P_k} D_{(y, Z, P)}^\alpha A^i)(X, u, Du)| [D_k u]_{\mu; q, B_{R/p}(X_1)}
 \end{aligned}$$

We also have

$$D_y^\gamma(\Psi(y, t, v)) = \sum c_{\alpha, j, \beta} (D_{(y, Z, P)}^\alpha \Psi)(y, t, v) \prod_{k \leq |\alpha_Z|} D_y^{\beta_{Z, 1}} v_0 \prod_{k \leq |\alpha_P|} D_y^{\beta_{P, k}} v_{j_k}$$

where the sum is taken over (7.5) and the coefficients $c_{\alpha, j, \beta}$ are as in (7.4). Here $v = (v_0, v_1, \dots, v_n)$ and $\Psi(y, t, v) = \Psi(y_1, \dots, y_{n-2}, t, v_0, v_1, \dots, v_{n-2})$.

Thus

$$\begin{aligned}
 (7.26) \quad D_t D_y^\gamma (\Psi(y, t, v(y, t))) &= \sum c_{\alpha, j, \beta} \left(D_t ((D_{(y, Z, P)}^\alpha \Psi)(y, t, v)) \right. \\
 &\quad \times \prod_{k \leq |\alpha_Z|} D_y^{\beta_{Z, k}} v_0 \prod_{k \leq |\alpha_P|} D_y^{\beta_{P, k}} v_{j_k} \\
 &\quad + (D_{(y, Z, P)}^\alpha \Psi)(y, t, v) \sum_{k \leq |\alpha_Z|} D_t D_y^{\beta_{Z, k}} v_0 \\
 &\quad \times \prod_{l \neq k} D_y^{\beta_{Z, l}} v_0 \prod_{k \leq |\alpha_P|} D_y^{\beta_{P, k}} v_{j_k} \\
 &\quad + (D_{(y, Z, P)}^\alpha \Psi)(y, t, v) \prod_{k \leq |\alpha_Z|} D_y^{\beta_{Z, k}} v_0 \\
 &\quad \left. \times \sum_{k \leq |\alpha_P|} D_t D_y^{\beta_{P, k}} v_{j_k} \prod_{l \neq k} D_y^{\beta_{P, l}} v_{j_l} \right)
 \end{aligned}$$

where

$$\begin{aligned}
 (7.27) \quad D_t ((D_{(y, Z, P)}^\alpha \Psi)(y, t, v)) &\leq (D_t D_{(y, Z, P)}^\alpha \Psi)(y, t, v) \\
 &\quad + (D_Z D_{(y, Z, P)}^\alpha \Psi)(y, t, v) D_t v_0 \\
 &\quad + \sum_{k=1}^n (D_{P_k} D_{(y, Z, P)}^\alpha \Psi)(y, t, v) D_t v_k.
 \end{aligned}$$

Comparing (7.24) to (7.26) and (7.25) to (7.27) using (7.21), (7.22), and (7.23),

$$(7.28) \quad [f_\gamma^i]_{\mu; q, B_{R/p}(X_1)} \leq D_t ((D_{(y, Z, P)}^\alpha \Psi)(y, t, v(t, y))) \Big|_{y=0, t=0}.$$

To construct v_0, v_1, \dots, v_n and Ψ , first we simplify the setup by letting $v_0(y, t) = Rv(R^{-1}(y_1 + \dots + y_{n-2}), (R/p)^{-\mu}t)$ and $v_1(y, t) = \dots = v_n(y, t) = v(R^{-1}(y_1 + \dots + y_{n-2}), (R/p)^{-\mu}t)$ for some function $v(\xi, \tau)$ and replacing $\Psi(y_1, \dots, y_{n-2}, t, Z, P_1, \dots, P_n)$ with

$$\Psi(R^{-1}(y_1 + \dots + y_{n-2}), (R/p)^{-\mu}t, R^{-1}Z + P_1 + \dots + P_n)$$

for some function $\Psi(\xi, \tau, \zeta)$. We choose

$$\begin{aligned} \Psi(\xi, \tau, \zeta) &= K_0 K \tau + \sum_{k=1}^2 \frac{1}{k!} (1 + K \tau) K_0 K^k (\xi + \zeta)^k \\ &\quad + \sum_{k=3}^p \frac{1}{k(k-1)(k-2)} K_0 (1 + (k-2)K\tau) K^k (\xi + \zeta)^k \\ v(\xi, \tau) &= (1 + \tau) \left(H_0 \xi + \frac{1}{2} H_0 \xi^2 + \sum_{s=3}^p \frac{1}{s(s-1)} H_0 H^{s-3} \xi^s \right). \end{aligned}$$

It is easy to check (7.21), (7.22), and (7.23) using (7.19) and (7.20). Note that to verify (7.22) in the case $|\beta| = p$ we use (7.20) with $|\beta| = s = p - 1$ and choose $H \geq 5/3$.

For functions $f(\xi, \tau)$ and $g(\xi, \tau)$, let $f \ll_{p,1} g$ denote that $|D_\xi^s f(0, 0)| \leq D_\xi^s g(0, 0)$ and $|D_t D_\xi^s f(0, 0)| \leq D_t D_\xi^s g(0, 0)$ for $1 \leq s \leq p$. By the (7.16), for $k = 1, 2, \dots, p$,

$$(7.29) \quad (\xi + (1 + n)v(\xi, \tau))^k \ll_{p,1} c^{k-1} \bar{H}_0^k (1 + k\tau) \times \left(\xi^k + \xi^{k+1} + \sum_{s=k+2}^{p^k} \frac{1}{(s-k)^2} H^{s-k-2} \xi^s \right)$$

where $c \geq 1$ is a constant independent of k and $\bar{H}_0 = 1 + (n + 1)H_0$. By the definition of Ψ and (7.29), for $p \geq 5$,

$$\begin{aligned} \Psi(\xi, \tau, \zeta) &\ll_{p,1} K_0 K \tau + \sum_{k=1}^2 \frac{1}{k!} c^{k-1} (1 + 2kK\tau) \bar{H}_0^k K_0 K^k \\ &\quad \times \left(\xi^k + \xi^{k+1} + \sum_{s=k+2}^{p^k} \frac{1}{(s-k)^2} H^{s-k-2} \xi^s \right) \\ &\quad + \sum_{k=3}^p \frac{1}{k(k-1)(k-2)} c^{k-1} (1 + 2kK\tau) \bar{H}_0^k K_0 K^k \\ &\quad \times \left(\xi^k + \xi^{k+1} + \sum_{s=k+2}^{p^k} \frac{1}{(s-k)^2} H^{s-k-2} \xi^s \right) \end{aligned}$$

It follows that

$$\begin{aligned} & \left. \frac{1}{p!} \frac{\partial^{p+1}}{\partial \tau \partial \xi^p} \Psi(\xi, (n+1)v(\xi)) \right|_{\xi=0, \tau=0} \\ & \leq \sum_{k=1}^2 \frac{4}{k!(p-k)^2} c^{k-1} \bar{H}_0^k K_0 K^{k+1} H^{p-k-2} \\ & \quad + \sum_{k=3}^{p-2} \frac{2}{(k-2)^2(p-k)^2} c^{k-1} K_0 \bar{H}_0^k K^{k+1} H^{p-k-2} \\ & \quad + \sum_{k=p-1}^p \frac{2}{(k-2)^2} c^{k-1} \bar{H}_0^k K_0 K^{k+1} \end{aligned}$$

Suppose H satisfies $cK\bar{H}_0 \leq H$ so that, using a computation similar to (7.18), we have

$$(R/p)^\mu \left. \frac{1}{p!} \frac{\partial^{p+1}}{\partial \tau \partial \xi^p} \Psi(\xi, (n+1)v(\xi)) \right|_{\xi=0, \tau=0} \leq \frac{C}{(p-3)^2} K_0 K^4 \bar{H}_0^3 H^{p-3} R^{-p}$$

for some constant $C \in (0, \infty)$ independent of p . Thus by (7.28),

$$(R/p)^\mu [f_\gamma^i]_{\mu; q, B_{R/p}(X_1)} \leq C(p-2)! H_0^3 H^{p-3} R^{-p}$$

for some constant $C = C(n, K_0, K, H_0) \in (0, \infty)$ independent of p . □

Lemma 15. *Let $\mu \in (0, 1/q)$ and $u \in C^{1, \mu; q}(B_1(0))$ such that $\|u\|_{C^{1, \mu; q}(B_1(0))} \leq 1/2$ and u is a solution to (7.1) for some locally real analytic single-valued functions $A^i, B : B_1(0) \times (-1, 1) \times B_1^n(0) \rightarrow \mathbb{R}$ such that (2.16) holds for some constant $\lambda > 0$. Let $K_0, K, H_0 \geq 1$ and $B_{R_0}(X_0) \subset\subset B_1(0)$. For some $H = H(n, q, \mu, \lambda, K_0, K, H_0) \geq 1$ the following holds. Suppose that for every multi-index $\alpha = (\alpha_X, \alpha_Z, \alpha_P)$ with $|\alpha| = k$,*

$$\begin{aligned} & |D_{(X,Z,P)}^\alpha A(X, Z, P)| + R_0 |D_{(X,Z,P)}^\alpha B| \\ & \leq \begin{cases} K_0 K^k R_0^{-|\alpha_X| - |\alpha_Z|} & \text{if } k = 1, 2, 3, \\ (k-3)! K_0 K^k R_0^{-|\alpha_X| - |\alpha_Z|} & \text{if } 4 \leq k \leq p, \end{cases} \end{aligned}$$

for $X \in B_{R_0}(X_0)$, $|Z| \leq 1/2$, and $|P| \leq 1/2$. Further suppose that whenever $B_R(X_1) \subseteq B_{R_0}(X_0)$, for every multi-index β with $|\beta| = s$,

$$(7.30) \quad \frac{s}{R} \|D_y^\beta u\|'_{C^{1, \mu; q}(B_{R/2s}(X_1))} \leq \begin{cases} H_0 R^{-s} & \text{if } s = 1, 2, \\ (s-2)! H_0 H^{s-2} R^{-s} & \text{if } 3 \leq s < p. \end{cases}$$

Then for every $B_R(X_1) \subseteq B_{R_0}(X_0)$ and multi-index γ with $|\gamma| = p$,

$$(R/p)^{-1} \|D_y^\gamma u\|'_{C^{1,\mu;q}(B_{R/2p}(X_1))} \leq (p-2)! H_0 H^{p-2} R^{-p}.$$

Proof. First we will apply Lemma 13 and Lemma 14 in order to bound the terms $[f_\gamma]_{\mu;q,B_{R/p}(X_1)}$ and $\sup_{B_{R/p}(X_1)} |g_\gamma|$ in (7.8). We need to check (7.10) and (7.20). For every $X \in B_{R/p}(X_1) \setminus [0, \infty) \times \{0\} \times \mathbb{R}^{n-2}$, $B_{(p-1)R/p}(X) \subseteq B_{R_0}(X_0)$, so by (7.30)

$$\begin{aligned} (7.31) \quad & (R/s)^{-1} |D_y^\beta u(X)| + |DD_y^\beta u(X)| + (R/s)^\mu [DD_y^\beta u]_{\mu;q,B_{(p-1)R/2ps}(X)} \\ & \leq 2^\mu \|D_y^\beta u\|'_{C^{1,\mu;q}(B_{(p-1)R/2ps}(X))} \\ & \leq 2^\mu (s-2)! H_0 H^{s-2} \left(\frac{p}{(p-1)R} \right)^s \\ & \leq 2^\mu e (s-2)! H_0 H^{s-2} R^{-s}. \end{aligned}$$

Now let $X = (x_1, x_2, y)$, $X' = (x'_1, x'_2, y')$ $\in B_{R/p}(X_1)$ with $x_2 > 0$ and $x'_2 > 0$. If $|X - X'| < (p-1)R/ps$, then $\hat{X} = (X + X')/2 \in B_{R/p}(X_1)$ and $X, X' \in B_{(p-1)R/p}(\hat{X}) \subset B_{R_0}(X_0)$, so by (7.31)

$$\begin{aligned} (7.32) \quad & \left(\frac{R}{s} \right)^\mu \frac{|DD_y^s u_l(X) - DD_y^s u_l(X')|}{|X - X'|^\mu} \leq \left(\frac{R}{s} \right)^\mu [DD_y^s u]_{\mu,B_{(p-1)R/2ps}(\hat{X})} \\ & \leq (s-2)! H_0 \frac{2^\mu e}{R^{s+\mu}} \end{aligned}$$

for $l = 1, 2, \dots, q$. If $|X - X'| \geq (p-1)R/ps$, then by using (7.31) to bound $|DD_y^s u_l(X)|$ and $|DD_y^s u_l(Y)|$,

$$(7.33) \quad \left(\frac{R}{s} \right)^\mu \frac{|DD_y^s u_l(X) - DD_y^s u_l(X')|}{|X - X'|^\mu} \leq 2^{1+2\mu} e (s-2)! H_0 R^{-s-\mu}$$

for $l = 1, 2, \dots, q$. By the same computations, (7.32) and (7.33) also hold if instead $x_2 < 0$ and $x'_2 < 0$. By (7.31), (7.32), and (7.33), (7.10) and (7.20) hold with $2^{2+2\mu} H_0$ in place of H_0 when $s \geq 2$. By a similar argument (7.10) and (7.20) hold with $2^{2+2\mu} H_0$ in place of H_0 when $s = 1, 2$.

Now by Lemma 13 and Lemma 14,

$$(7.34) \quad (R/p)^\mu [f_\gamma^i]_{\mu;q,B_{R/p}(X_1)} + (R/p) \sup_{B_{R/p}(X_1)} |g_\gamma| \leq C(p-2)! H_0^3 H^{p-3} R^{-p}.$$

for some constant $C = C(n, q, \mu, K_0, K) \in (0, \infty)$. Thus by (7.8) and (7.34),

$$\begin{aligned} & (R/p)^{-1} \|D_y^\gamma u\|'_{C^{1,\mu;q}(B_{R/2p}(X_1))} \\ & \leq C (p(p-3)!H_0H^{p-3}R^{-p} + (p-2)!H^{p-3}R^{-p}) \\ & \leq C(p-2)!H_0H^{p-2}R^{-p} \left(\frac{p}{(p-2)H} + \frac{1}{H_0H} \right) \\ & \leq (p-2)!H_0H^{p-2}R^{-p} \end{aligned}$$

where $C \in (0, \infty)$ denotes constants depending on $n, q, \mu, \lambda, K_0, K$, and H_0 and independent of p and H is large enough that Lemmas 13 and 14 hold and $H \geq \max\{4C, 2C/H_0\}$. □

To complete the proof of Theorem 4, let $B_R(x_0, y_0) \subset\subset B_1(0)$. Since A^i and B are real analytic and by the proof of Lemma 12, there are constants $K, K_0, H_0 \geq 1$ such that for any multi-index $\alpha = (\alpha_X, \alpha_Z, \alpha_P)$ with $|\alpha| = k$

$$\begin{aligned} |D_{(X,Z,P)}^\alpha A^i| + (R/2)|D_{(X,Z,P)}^\alpha B| & \leq K_0K^k(R/2)^{-|\alpha_X|-|\alpha_Z|} \quad \text{for } k = 2, 3, \\ |D_{(X,Z,P)}^\alpha A^i| + (R/2)|D_{(X,Z,P)}^\alpha B| & \leq (k-3)!K_0K^k(R/2)^{-|\alpha_X|-|\alpha_Z|} \quad \text{for } k \geq 4, \end{aligned}$$

for $X \in B_R(x_0, y_0)$, $|Z| \leq 1/2$, $|P| \leq 1/2$ and for any multi-index β with $|\beta| = s$

$$\frac{2s}{R} \|D_y^\beta u\|'_{C^{1,\mu;q}(B_{R/4s}(x,y))} \leq H_0(R/2)^{-s}$$

whenever $(x, y) \in B_{R/2}(x_0, y_0)$ and $s = 1, 2, 3, 4$. By the Lemma 15 and induction, for some H sufficiently large depending on n, q, K_0, K , and H_0 ,

$$\frac{2s}{R} \|D_y^\beta u\|'_{C^{1,\mu;q}(B_{R/4s}(x,y))} \leq (s-2)!H_0H^{s-2}(R/2)^{-s}$$

whenever $(x, y) \in B_{R/2}(x_0, y_0)$ and $s \geq 5$ and in particular (2.17) holds true with $C = H_0H$.

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