

A finite dimensional approach to Donaldson’s J-flow

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Consider a projective manifold with two distinct polarisations L_1 and L_2 . From this data, Donaldson has defined a natural flow on the space of Kähler metrics in $c_1(L_1)$, called the J-flow. The existence of a critical point of this flow is closely related to the existence of a constant scalar curvature Kähler metric in $c_1(L_1)$ for certain polarisations L_2 .

Associated to a quantum parameter $k \gg 0$, we define a flow over Bergman type metrics, which we call the J-balancing flow. We show that in the quantum limit $k \rightarrow +\infty$, the rescaled J-balancing flow converges towards the J-flow. As corollaries, we obtain new proofs of uniqueness of critical points of the J-flow and also that these critical points achieve the absolute minimum of an associated energy functional.

We show that the existence of a critical point of the J-flow implies the existence of J-balanced metrics for $k \gg 0$. Defining a notion of Chow stability for linear systems, we show that this in turn implies the linear system $|L_2|$ is asymptotically Chow stable. Asymptotic Chow stability of $|L_2|$ implies an analogue of K-semistability for the J-flow introduced by Lejmi-Székelyhidi, which we call J-semistability. We prove also that J-stability holds automatically in a certain numerical cone around L_2 , and that if L_2 is the canonical class of the manifold that J-semistability implies K-stability. Eventually, this leads to new K-stable polarisations of surfaces of general type.

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1. Introduction

Let M be a smooth projective manifold equipped with two ample line bundles L_1, L_2 and of complex dimension $n \geq 2$. Fix Hermitian metrics $h_1 \in \text{Met}(L_1)$, $h_2 \in \text{Met}(L_2)$ such that the curvatures $\omega = c_1(h_1)$, and $\chi = c_1(h_2)$ are both Kähler forms. Given this data, Donaldson introduced a flow of Kähler metrics called the J-flow [14]. This flow is given by the following parabolic PDE in the (smooth) ω -potentials ϕ_t :

$$(1) \quad \frac{\partial \phi_t}{\partial t} = \gamma - \frac{\chi \wedge (\omega + \sqrt{-1} \partial \bar{\partial} \phi_t)^{n-1}}{(\omega + \sqrt{-1} \partial \bar{\partial} \phi_t)^n},$$

where γ is the topological constant $\frac{\int_M c_1(L_2) c_1(L_1)^{n-1}}{\int_M c_1(L_1)^n}$ called the *J-constant*. A *critical metric* for Donaldson's J-flow is a solution at time $+\infty$ of the flow, i.e. a Kähler metric $\omega + \sqrt{-1} \partial \bar{\partial} \phi \in c_1(L_1)$ solution of

$$(2) \quad \chi \wedge (\omega + \sqrt{-1} \partial \bar{\partial} \phi)^{n-1} = \gamma (\omega + \sqrt{-1} \partial \bar{\partial} \phi)^n.$$

Donaldson's motivation for studying this flow is its relation to an infinite dimensional moment map picture. Soon after its introduction, the work of Chen, together with a refinement due to Song-Weinkove, proved that if $L_2 = K_M$ is the canonical class of M , then the existence of a critical point of the J-flow implies properness of the Mabuchi functional for (M, L_1) [7, 43]. Properness of the Mabuchi functional is conjecturally equivalent to the existence of a constant scalar curvature Kähler (which we abbreviate to cscK) metric on L_1 . Indeed, the existence of solution of the J-flow is equivalent to properness of a functional, which we denote I_{μ_J} , due to work of Chen and Collins-Székelyhidi [7, 10]. When $L_2 = K_M$ the functional I_{μ_J} is a component of the Mabuchi functional, which explains the relation between the cscK problem and the J-flow. Since then there has been much work in providing numerical criteria for the existence of critical metrics of the J-flow, in order to better understand the existence of cscK metrics [8, 43, 49], under the assumption K_M is ample. Chen has also very recently proposed a program to prove the existence of cscK metrics using a continuity method, in which the metric at $t = 0$ is a critical metric of the J-flow [9]. We later discuss these motivations for studying the J-flow in further detail.

We now give an overview of the main results of this paper. As is now well understood, one can quantise metrics on L_1 using metrics on the finite dimensional vector spaces $H^0(M, L_1^k)$ with quantum parameter k , called Bergman metrics. In the present work we study a flow on the space of

Bergman metrics, which we call the J-balancing flow. Critical points of the J-balancing flow are called J-balanced metrics, and these fit naturally into a finite dimensional moment map picture. Our main result is that in the quantum limit $k \rightarrow +\infty$, the J-balancing flow converges to the J-flow.

Theorem 1.1. *Fix $T > 0$ and let $\omega_k(t)$ be the solution of the J-balancing flow, for $t \in [0, T]$. Then as $k \rightarrow \infty$, the sequence $\omega_k(t)$ converges in C^∞ to the solution of the J-flow as $k \rightarrow \infty$. Furthermore, the convergence is C^1 in the variable t . Assuming there is a critical point of the J-flow, the convergence holds for all $t > 0$.*

By showing J-balanced metrics are unique, we obtain the following Corollary, which is an analogue of Donaldson's quantisation proof of uniqueness of cscK metrics [15].

Corollary 1.2. *Critical metrics of the J-flow are unique.*

This was first proven by Chen using the strict convexity of the $I_{\mu, J}$ functional along certain geodesics in the space of Kähler metrics on L_1 [8]. Similarly, we recover the fact that J-balanced metrics achieve the absolute minimum of an associated functional.

Corollary 1.3. *Critical metrics of the J-flow achieve the absolute minimum of the $I_{\mu, J}$ functional.*

This is analogous to Donaldson's proof that cscK metrics achieve the absolute minimum of the Mabuchi functional [16].

Our next results relate the existence of a critical metric to algebro-geometric notions of stability. We define a notion of *Chow stability* for a linear system on a polarised manifold, and by relating this notion of stability with the existence of J-balanced metrics we obtain the following.

Theorem 1.4. *The existence of a critical metric of the J-flow implies the linear system $|L_2|$ is asymptotically Chow stable.*

This is an analogue of Donaldson's result proving the existence of a cscK metric implies asymptotic Chow stability [15]. The first result relating the J-flow to algebro-geometric stability is due to Lejmi-Székelyhidi [23]. They proved the existence of a critical point implies an analogue of K-stability in this case, which we call J-stability. Since asymptotic Chow stability of $|L_2|$ implies J-semistability, we obtain a new proof of the semistability part

of their result. We remark that they then use a perturbation argument to prove strict J-stability.

Corollary 1.5. *If (M, L_1, L_2) admits a critical metric of the J-flow, then it is J-semistable.*

The existence of a cscK metric is conjecturally equivalent to the algebro-geometric notion of K-stability; this is called the Yau-Tian-Donaldson conjecture. Through Chen's work one would therefore expect that when $L_2 = K_M$, J-stability is naturally related to K-stability [7]. We show that this is indeed the case. We extend Lejmi-Székelyhidi's definition of J-stability to the case where L_2 is an arbitrary (not necessarily ample) line bundle, and our results hold in this generality.

Theorem 1.6. *If (M, L_1, K_M) is J-semistable, then (M, L_1) is K-stable.*

Similarly, analogously to the work giving numerical criteria for the existence of critical metrics of the J-flow, we give a numerical criterion for J-stability.

Theorem 1.7. *Suppose $\gamma L_1 - L_2$ is nef, with $\gamma > 0$. Then (M, L_1, L_2) is J-stable.*

Here we have used additive notation for line bundles. Note that while $\gamma > 0$ is automatic when L_2 is ample, our result holds for L_2 arbitrary provided $\gamma > 0$. Combining the previous two results gives the following, which was also proven by the first author using a different method [13, Theorem 1.7].

Corollary 1.8. *Let γ be the J-constant for (M, L_1, K_M) , and suppose $\gamma > 0$. If $\gamma L_1 - K_M$ is nef, then (M, L_1) is K-stable.*

We emphasise that in contrast to the corresponding analytic results due to Chen, Weinkove and Song-Weinkove proving properness of the Mabuchi functional [8, 43, 49], we do *not* need to assume that K_M is ample. We expect however that a similar phenomenon occurs for the Mabuchi functional, see Conjecture 4.1.

We can strengthen the above results when M is a surface by a more delicate analysis of the J-stability condition.

Theorem 1.9. *If M is a projective surface satisfying $\frac{4}{3}\gamma L_1 - L_2$ is nef, then (M, L_1, L_2) is J-semistable provided $\gamma > 0$. In particular if $L_2 = K_M$, then M is K-stable.*

This gives the most general currently known criterion proving K-stability of polarised surfaces with $\gamma > 0$.

Finally, building on work of Sano and Seyyadali [41, 42], we prove that J-balanced metrics are the fixed points of a natural dynamical system. This gives a variational approach to the existence of such metrics and provides a natural interpretation of the energy functionals associated to the J-flow that appeared in [8, 43].

Theorem 1.10. *Assume there exists a critical metric of the J-flow. Then for each $k \gg 0$, there is a natural map*

$$T_{k,\chi} : \text{Met}(L_1^k) \rightarrow \text{Met}(L_1^k)$$

defining a dynamical system which has the J-balanced metric as its (unique) fixed attractive point.

This also gives an algorithm to compute critical metrics, however we do not pursue this direction in the present work.

1.1. The J-flow from symplectic geometry

The J-flow is very natural from the point of view of symplectic geometry as we now briefly explain. Consider M a compact symplectic manifold and ω, χ are symplectic forms. If one considers the manifold $\text{Diff}(M)$ of diffeomorphisms $f : M \rightarrow M$ homotopic to the identity and equipped with a natural symplectic form

$$(3) \quad \Upsilon_{\chi,\omega}(a, b) = \int_M \chi(a, b) \frac{\omega^n}{n!},$$

then there is a naturally associated moment map picture [14]. Here $a, b \in \Gamma(f^*(TM))$ as we identify the tangent space of $\text{Diff}(M)$ at f to the space of smooth sections of $f^*(TM)$. The group \mathcal{G} of exact ω -symplectomorphisms of M acts on $\text{Diff}(M)$ and preserves $\Upsilon_{\chi,\omega}$. Since we can identify the Lie algebra $\text{Lie}(\mathcal{G})$ with the set

$$\left\{ f \in C^\infty(M, \mathbb{R}), \int_M f \omega^n = 0 \right\},$$

we can express simply the associated moment map $\mu_J : \text{Diff}(M) \rightarrow \text{Lie}(\mathcal{G})^*$ for the group action as

$$(4) \quad \mu_J(f) = \frac{f^*(\chi) \wedge \omega^{n-1}}{\omega^n} - \gamma.$$

This moment map induces also a gradient flow f_t of the function $\|\mu_J(f_t)\|^2$. Assuming that M has an integrable almost-complex structure compatible with ω (in the sequel M will always be Kähler), the J -flow is just the gradient flow expressed using $(f_t^*)^{-1}(\omega)$ on M , fixing the complex structure and varying the form within the Kähler class. From [8], it is known long time existence of Donaldson's J -flow for all time, that is (1) admits a smooth solution for all $t \geq 0$. If it exists, Chen moreover shows that the critical metric is actually unique.

1.2. CscK metrics and the J -flow

While the Yau-Tian-Donaldson conjecture predicts the existence of a cscK metric is equivalent to K-stability, in practice it seems just as difficult to check K-stability in specific examples as it is to check the existence of a cscK metric. Thus, it is beneficial to give more explicit criteria that apply in concrete examples. For example, one can ask if given such a Kähler class that admits a cscK metric (for example a Kähler-Einstein metric, where many examples are known), if it is possible to describe nearby Kähler classes that have also cscK metrics. One way of achieving this goal is by understanding properties of the Mabuchi energy; recall an important conjecture of Mabuchi-Tian asserts that the existence of a cscK metric should be equivalent to the properness of the Mabuchi energy (a small modification is needed if M admits holomorphic vector fields).

From the above discussion, it is therefore natural to search for Kähler classes with proper Mabuchi energy around a Kähler class that is endowed with a cscK metric. In this direction, it was first proven by Chen that if $[\chi] \in -c_1(M) < 0$ and there exists a critical metric of the J -flow in the class $[\omega]$, then the Mabuchi energy for $[\omega]$ is bounded from below, see [7, 50]. This was strengthened by Song and Weinkove who proved the properness of the Mabuchi energy [43]. Thus Mabuchi-Tian's conjecture predicts the existence of a cscK metric in such $[\omega]$.

Donaldson observed that a necessary condition for the existence of a critical metric is the following inequality on the Chern classes

$$(5) \quad n\gamma[\omega] - [\chi] = n\gamma c_1(L_1) - c_1(L_2) > 0.$$

In dimension 2, it is proved in [8, 49] that the condition is sufficient, the problem being proved equivalent to solve a Monge-Ampère equation in [8] (see also the study of the convergence of the J-flow in [49]). However, the condition is *not* sufficient in higher dimensions, see [23]. Ideally, one could hope that an optimal Chern inequality similar to (5) (or more generally an algebro-geometric condition) would imply the existence of critical metrics of the J-flow, and that this condition would be relatively simple to check, at least for general type manifolds, in view of [39, Section 4]. Keeping in mind the Yau-Tian-Donaldson conjecture, one can expect that the right algebro-geometric condition for the existence of a critical metric would provide the K-stability of the class.

The problem to find a good algebro-geometric condition equivalent to the existence of a critical metric is subtle. For instance, it is known from [43, 50] that if there exists a Kähler metric $\omega' = \omega + \sqrt{-1}\partial\bar{\partial}\psi \in c_1(L_1)$ satisfying

$$(6) \quad (n\gamma\omega' - (n - 1)\chi) \wedge (\omega')^{n-2} \wedge u \wedge \bar{u} > 0$$

for all $(1, 0)$ -form u , then there is convergence of the flow towards a critical metric. Conversely if there is convergence such a metric ω' does exist. Unfortunately, it seems pretty hard to check (6) in practice. For instance it is not even clear whether it depends on the class only (and not on the forms) as pointed out in [23].

The most general result using the J-flow to prove properness of the Mabuchi functional for certain Kähler classes is due to Li-Shi-Yao [24]. We remark that they are also able to use J-flow techniques to prove properness of the Mabuchi functional for certain Kähler classes also when $c_1(M)$ is *positive*, i.e. M is Fano (see [12] for a similar result proven by a direct analysis of the Mabuchi functional).

Very recently, Lejmi and Székelyhidi [23] have made an important contribution to the subject by proposing an algebro-geometric condition (that we call J-stability, see Section 4) using the technology of deformation to the normal cone and modelled on the K-stability theory. Their conjecture has been proven in the toric case [10].

1.3. Organisation of the paper

In Section 2, we introduce a notion of *J-balanced* embeddings for X in certain projective spaces depending on a quantum parameter $k \gg 0$. This notion of J-balanced embedding is based on the symplectic formalism of moment maps

and has an algebro-geometric interpretation that will become clear in Section 4. By pulling-back the Fubini-Study metric, such embeddings provide algebraic metrics that we call *J-balanced* metrics. We also introduce certain flows on the space of Bergman metrics of level k , called J-balancing flows, that converge towards J-balanced metrics if they do exist. More precisely, we prove that at the quantum limit $k \rightarrow +\infty$, the J-balancing flows converge towards Donaldson's J-flow (Theorem 2.4, Theorem 2.7).

In Section 3, building on the work of Sano and Seyyedali [41, 42], we prove that J-balanced metrics when they do exist, are actually attractive fixed points of a natural dynamical system (Theorem 3.11, Corollary 3.18). This gives a variational approach to the existence of such metrics and provides a natural interpretation of the energy functionals associated to the J-flow that appeared in [8, 43].

In Section 4, we introduce a generalisation of the notion of Chow stability adapted to our context and prove that the existence of J-balanced metrics implies the stability in that sense (Theorem 4.19). In particular, our Chow stability condition is a necessary condition for the existence of a critical metric to the J-flow (Corollary 4.20). We describe how it is related to the notion of J-stability of Lejmi-Székelyhidi. Building on the blowing-up formalism of Wang and Odaka for expressing weights, we are able to relate the notion of J-stability and the classical notion of K-stability, giving a simple criterion for J-stability which corresponds to (weaker) algebraic version of a result of Song-Weinkove [43] (Theorem 4.32, Theorem 4.41). Eventually we relate J-semistability to K-stability (Theorem 4.36).

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2. Finite dimensional approach to the J-flow

Given the Hermitian metric $h \in \text{Met}(L_1^k)$ with positive curvature, one can consider the Hilbertian map

$$\text{Hilb}_\chi = \text{Hilb}_{k,\chi} : \text{Met}(L_1^k) \rightarrow \text{Met}(H^0(L_1^k))$$

such that

$$\text{Hilb}_\chi(h) = \frac{1}{\gamma} \int_M \langle \cdot, \cdot \rangle_h \chi \wedge c_1(h)^{n-1}$$

is the L^2 metric induced by the fibrewise h and the volume form $\chi \wedge c_1(h)^{n-1}$. On the other hand, one can consider the injective Fubini-Study maps $FS = FS_k : \text{Met}(H^0(L_1^k)) \rightarrow \text{Met}(L_1^k)$, such that for $H \in \text{Met}(H^0(L_1^k))$, $\{s_i\}$ an H -orthonormal basis of $H^0(X, L_1^k)$ and for all $p \in X$,

$$\sum_{i=1}^{\dim H^0(X, L_1^k)} |s_i(p)|_{FS_k(H)}^2 = \frac{\dim H^0(X, L_1^k)}{\text{Vol}_{L_1}(X)},$$

which means that we fix pointwisely the metric $FS_k(H) \in \text{Met}(L^k)$. The curvature of $FS(H)$ is the pull-back of the Fubini-Study metric living in the projective space, using the embedding defined by the H -orthonormal basis $\{s_i\}$.

We also define the map

$$T_{k,\chi} = FS \circ \text{Hilb}_\chi.$$

Definition 2.1 (J-balanced metric). A fixed point h_k of the map

$$T_{k,\chi} : \text{Met}(L_1^k) \rightarrow \text{Met}(L_1^k)$$

is called a J-balanced metric at level k .

Let us denote in the sequel $N = N_k = \dim H^0(L_1^k) - 1$. We introduce a moment map setting in finite dimensions. Let us consider first $\mu_{FS} : \mathbb{P}^N \rightarrow \sqrt{-1}\text{Lie}(U(N+1))$ which is a moment map for the $U(N+1)$ action and the Fubini-Study metric ω_{FS} on \mathbb{P}^N . Given homogeneous unitary coordinates,

one sets explicitly $\mu_{FS} = (\mu_{FS})_{\alpha,\beta}$ as

$$(7) \quad (\mu_{FS}([z_0, \dots, z_N]))_{\alpha,\beta} = \frac{z_\alpha \bar{z}_\beta}{\sum_i |z_i|^2}.$$

Then, given a holomorphic embedding $\iota : M \hookrightarrow \mathbb{P}H^0(L_1^k)^*$, and the Fubini Study form ω_{FS} on the projective space, define

$$(8) \quad \mu_{k,\chi}(\iota) = \frac{1}{\gamma} \int_M \mu_{FS}(\iota(p)) \chi \wedge \iota^*(\omega_{FS}^{n-1})(p).$$

An alternative, but equivalent, point of view is to consider $\mu_{k,\chi}$ as a map from the space of holomorphic bases of $\mathfrak{s} = \{s_i\}$ of $H^0(L_1^k)$ as a basis determines a unique Hermitian inner product H for which it is orthonormal. Then, this inner product induces a Fubini-Study metric $\omega_{FS,\mathfrak{s}}$ and thus we can consider

$$\mu_{k,\chi}(\mathfrak{s}) = \frac{1}{\gamma} \int_M \mu_{FS}([s_1(p), \dots, s_N(p)]) \chi \wedge \omega_{FS,\mathfrak{s}}^{n-1}(p).$$

Later we will be interested in special Hermitian metrics H (associated to a particular basis) and so we shall write $\mu_{k,\chi}(H)$.

Proposition 2.2. *The map $\mu_{k,\chi}$ is a moment map for the $U(N + 1)$ action over the space of all bases of $H^0(L_1^k)$ with respect to the symplectic structure defined by Equation (9).*

Proof. We follow essentially the techniques developped in [18, Theorem 8.5.1]. Given $A \in Lie(GL(N + 1))$, we denote \hat{A} the induced action on \mathbb{P}^N and $\hat{A}^\perp = \hat{A} - \hat{A}|_{TM}$ where $\hat{A}|_{TM}$ stands for the orthogonal projection on TM with respect to the Fubini-Study metric. On the space of bases identified to $GL(N + 1)$, we have a natural symplectic structure ϖ defined at \mathfrak{s} by

$$(9) \quad \begin{aligned} \varpi(\hat{A}, \hat{B}) &= \frac{1}{\gamma} \int_M \omega_{FS,\mathfrak{s}}(\hat{A}^\perp, \hat{B}^\perp) \chi \wedge \omega_{FS,\mathfrak{s}}^{n-1} \\ &+ \frac{1}{n\gamma} \int_M g_{FS,\mathfrak{s}} \left(\chi, i_{\hat{A}|_{TM}} \omega_{FS,\mathfrak{s}} \wedge i_{\hat{B}|_{TM}} \omega_{FS,\mathfrak{s}} \right) \chi \wedge \omega_{FS,\mathfrak{s}}^{n-1}, \end{aligned}$$

where $A, B \in Lie(GL(N + 1))$ and $g_{FS} = \omega_{FS,\mathfrak{s}}(\cdot, J\cdot)$ is the associated metric to the Kähler form $\omega_{FS,\mathfrak{s}}$. We shall use the following fact (see for instance the proof of [18, Lemma 3.2.1]): for any 1-forms α, β and Kähler forms ω, χ ,

one has the identity

$$\frac{1}{n}g(\chi, \alpha \wedge \beta)\omega^n = \omega(\alpha, \beta) \wedge \chi \wedge \omega^{n-1} - (n-1)\alpha \wedge \beta \wedge \chi \wedge \omega^{n-2},$$

where g is the associated metric to ω . Hence we can compute

$$\begin{aligned} \gamma\varpi(\hat{A}, \hat{B}) &= \int_M \omega_{FS,s}(\hat{A}^\perp, \hat{B}^\perp) \chi \wedge \omega_{FS,s}^{n-1} \\ &\quad + \int_M \omega_{FS,s} \left(i_{\hat{A}|_{TM}} \omega_{FS,s}, i_{\hat{B}|_{TM}} \omega_{FS,s} \right) \chi \wedge \omega_{FS,s}^{n-1} \\ &\quad - (n-1) \int_M i_{\hat{A}|_{TM}} \omega_{FS,s} \wedge i_{\hat{B}|_{TM}} \omega_{FS,s} \wedge \chi \wedge \omega_{FS,s}^{n-2}, \\ &= \int_M \omega_{FS,s}(\hat{A}, \hat{B}) \chi \wedge \omega_{FS,s}^{n-1} \\ &\quad - (n-1) \int_M i_{\hat{A}|_{TM}} \omega_{FS,s} \wedge i_{\hat{B}|_{TM}} \omega_{FS,s} \wedge \chi \wedge \omega_{FS,s}^{n-2}, \\ &= \int_M \omega_{FS,s}(\hat{A}, \hat{B}) \chi \wedge \omega_{FS,s}^{n-1} \\ &\quad - (n-1) \int_M \partial\mu_{FS}(\hat{A}) \wedge \bar{\partial}\mu_{FS}(\hat{B}) \wedge \chi \wedge \omega_{FS,s}^{n-2}, \\ &= \int_M \omega_{FS,s}(\hat{A}, \hat{B}) \chi \wedge \omega_{FS,s}^{n-1} \\ &\quad + (n-1) \int_M \text{tr}(\mu_{FS}A) \bar{\partial}\partial\mu_{FS}(\hat{B}) \wedge \chi \wedge \omega_{FS,s}^{n-2}, \\ &= \left\langle d \int_M \mu_{FS} \chi \wedge \omega_{FS,s}^{n-1}(\hat{B}), A \right\rangle. \end{aligned}$$

During the computation we used the fact that the embedding given by \mathbf{s} is holomorphic and μ_{FS} is a moment map on the projective space. Moreover $\mu_{k,\chi}$ is Ad -equivariant as the integral of the Ad -equivariant moment map μ_{FS} . □

Now $SU(N+1)$ acts isometrically on the spaces of holomorphic bases with the moment map given by

$$\mathbf{s} \mapsto -\sqrt{-1} \left(\mu_{k,\chi}(\mathbf{s}) - \frac{\text{tr}(\mu_{k,\chi}(\mathbf{s}))}{N+1} \text{Id}_{N+1} \right) \in \sqrt{-1} \text{Lie}(SU(N+1)).$$

In the Bergman space of metrics $GL(N+1)/U(N+1)$, we have a preferred metric associated and this is precisely a J-balanced metric.

Definition 2.3 (J-balanced embedding). The embedding ι is J-balanced if and only if

$$\mu_{k,\chi}^0(\iota) := \mu_{k,\chi}(\iota) - \frac{\text{tr}(\mu_{k,\chi}(\iota))}{N+1} \text{Id}_{N+1} = 0.$$

A J-balanced embedding corresponds (up to $SU(N+1)$ -isomorphisms) to a J-balanced metric $\iota^* \omega_{FS}$ by pull-back of the Fubini-Study metric from $\mathbb{P}H^0(L_1^k)^*$ so our two definitions of J-balanced metric and embedding actually agree. A notion of balanced basis can also be derived in an obvious way using previous discussion. Note that for $H \in \text{Met}(H^0(L_1^k))$, it also makes sense to consider $\mu_{k,\chi}(h)$ where $h = FS(H) \in \text{Met}(L_1^k)$, i.e when h belongs to the space of *Bergman* type fibrewise metric that we identify with $\mathcal{B} = \mathcal{B}_k$.

On the other hand, seen as a Hermitian matrix, $\mu_{k,\chi}^0(\iota)$ induces a vector field on \mathbb{P}^N . We are lead to study the following flow

$$\frac{d\iota(t)}{dt} = -\mu_{k,\chi}^0(\iota(t)),$$

and we call this flow the J-balancing flow. To fix the starting point of this flow, we choose a Kähler metric $\omega = \omega(0)$ and we construct a sequence of Hermitian metrics $h_k(0)$ such that $\omega_k(0) := c_1(h_k(0))$ converges smoothly to $\omega(0)$ providing a sequence of embeddings $\iota_k(0)$ for $k \gg 0$. For technical reasons, we decide to rescale this flow by considering the following ODE,

$$(10) \quad \frac{d\iota_k(t)}{dt} = -k^2 \mu_{k,\chi}^0(\iota_k(t)),$$

which we call the *rescaled J-balancing flow*. In the following Subsections 2.1, 2.2, 2.3, we will study the behaviour of the sequence of Kähler metrics

$$\omega_k(t) = \frac{1}{k} \iota_k(t)^*(\omega_{FS})$$

as k tends to infinity.

2.1. The limit of the rescaled J-balancing flow

In this section, we assume that the sequence $\omega_k(t)$ is convergent and we want to relate its limit to Equation (1).

Theorem 2.4. *Suppose that for each $t \in \mathbb{R}_+$, the metric $\omega_k(t)$ induced by Equation (10) converges in smooth topology to a metric ω_t and that this*

convergence is C^1 in $t \in \mathbb{R}_+$. Then the limit ω_t is a solution to Donaldson’s J-flow (1) starting at $\omega_0 = \lim_{k \rightarrow \infty} \omega_k(0)$.

The proof is similar to [5, Theorem 3]. The only difference is that we are dealing with orthonormal basis of holomorphic sections $\{s_i\}$ of L_1^k with respect to $Hilb_{k,\chi}(h^k)$. But in that case, the asymptotic of the Bergman function stands as

$$(11) \quad \sum_{i=1}^{N_k+1} |s_i|_{h^k}^2 = k^n \frac{\gamma \omega^n}{\chi \wedge \omega^{n-1}} + O(k^{n-1})$$

where $\omega = c_1(h)$ thanks to the following proposition, see [47] (we also refer to [6, 51] and [3] where the first term of the asymptotic expansion is identified).

Proposition 2.5 (Catlin-Tian-Yau-Zelditch expansion). *Let (M, L) be a projective polarized manifold. Let $h \in \text{Met}(L)$ be a metric such that its curvature $c_1(h) = \omega > 0$ is a Kähler form. Assume Ω to be a smooth volume form. Then we have the following asymptotic expansion for $k \rightarrow \infty$,*

$$\sum_{i=1}^{N+1} |s_i|_{h^k}^2 = k^n \frac{\omega^n}{\Omega} + O(k^{n-1}),$$

where $\{s_i\}$ is an orthonormal basis of $H^0(L^k)$ with respect to the L^2 inner product $\int_M h^k(\cdot, \cdot) \Omega = Hilb_{\Omega}(h^k)$. Here by $O(k^{n-1})$, we mean that for $r \geq 0$

$$\left\| \sum_{i=1}^{N+1} |s_i|_{h^k}^2 - k^n \frac{\omega^n}{\Omega} \right\|_{C^r} \leq c_r k^{n-1}$$

where c_r remains bounded if h varies in a compact set (in smooth topology) in the space of Hermitian metrics with positive curvature.

In particular, the potentials $\beta_k = -k \text{tr}(\mu_{k,\chi}^0 \mu_{FS})$ converge in smooth topology to the potential

$$(12) \quad 1 - \frac{\chi \wedge \omega^{n-1}}{\gamma \omega^n}$$

when $k \rightarrow +\infty$. We also have

Proposition 2.6. *Let $h(t) \in \text{Met}(L_1)$ be a path of Hermitian metrics on L_1 with $c_1(h(t)) > 0$. Let us consider $h_k(t) = FS(Hilb_{k,\chi}(h(t)^k))^{1/k}$ the path*

of induced Bergman metrics. Then $\frac{\partial h_k(t)}{\partial t}$ converges to $\frac{\partial h(t)}{\partial t}$ as $k \rightarrow +\infty$ in C^∞ topology. This convergence is uniform if $h(t)$ belongs to a compact set in the space of positively curved Hermitian metrics on L_1 .

Proof. This is obtained easily by a simple modification of the proof of [5, Proposition 2.3] or [17, End of Section 1.4.1 and Theorem 9]. \square

Proof of Theorem 2.4. Let's write $\omega_t = \omega + \sqrt{-1}\partial\bar{\partial}\phi_t$. By assumption, $\dot{\phi}_t$ is continuous and can be normalized to be unique by demanding that it has vanishing integral. Consider the potential $\beta_k(t)$ induced by the embedding $\iota_k(t)$ given by the rescaled J-balancing flow. The integral of $\beta_k(t)$ is zero. Therefore, with Proposition 2.6, this sequence of potentials converges to $\dot{\phi}_t$. Hence, together with (12), the theorem is proved. \square

2.2. Convergence result for the rescaled J-balancing flow

Theorem 2.7. *Fix $T > 0$. For any $t \in [0, T]$, the sequence $\omega_k(t)$ converges in C^∞ topology to the solution of Donaldson's J-flow (1) with $\phi_0 = 0$ and $\omega = \lim_{k \rightarrow \infty} \omega_k(0)$. Furthermore, the convergence is C^1 in the variable t . If there is a critical metric, then there is convergence for all $t > 0$.*

The last part of the theorem is a consequence of the long time existence of the flow and the fact that when there is a critical metric, the J-flow converges towards this critical metric [43, Theorem 1.1, (i) \Leftrightarrow (ii)]. Thus the metrics involved in the J-flow belong to a compact set in the space of smooth Kähler metrics when there is a critical metric. The proof of Theorem 2.7 will occupy Subsections 2.2.2, 2.2.3, 2.2.4, and 2.2.5.

2.2.1. The Q_k operator. In this section, we recall the following important technical result, see [25, Theorem 1], [26, Section 6]. Note that the C^r estimate below holds for any $f \in C^\infty(M, \mathbb{R})$.

Theorem 2.8. *Let us consider $h \in \text{Met}(L)$ with positive curvature on an ample line bundle L , and $\omega = c_1(h)$ the induced Kähler form, Ω a smooth positive volume form and $\{s_a\}$ orthonormal basis of $H^0(L^k)$ with respect to $\text{Hilb}_\Omega(h^k)$. Then the operator on $C^\infty(M, \mathbb{R})$ given by*

$$Q_k(f)(p) = \frac{1}{k^n} \int_M \sum_{a,b} \langle s_a, s_b \rangle_{h^k}(q) \langle s_a, s_b \rangle_{h^k}(p) f(q) \Omega(q),$$

approximates the operator $\frac{\omega^n}{\Omega} \exp(-\frac{\Delta}{4\pi k})$ in the following sense. For any $r \in \mathbb{N}^$, there exists $C > 0$ such that for all k sufficiently large and any function*

$f \in C^\infty(M, \mathbb{R})$, one has

$$(13) \quad \left\| \left(\frac{\Delta}{k}\right)^r \left(Q_k(f) - \frac{\omega^n}{\Omega} \exp\left(-\frac{\Delta}{4\pi k}\right) f\right) \right\|_{L^2} \leq \frac{C}{k} \|f\|_{L^2}$$

$$(14) \quad \left\| Q_k(f) - \frac{\omega^n}{\Omega} f \right\|_{C^r} \leq \frac{C}{k} \|f\|_{C^{r+2}}$$

where the norms are taken with respect to the induced Kähler form obtained from the fibrewise metric on the polarisation L and Δ is the Laplace operator for the induced Kähler metric. The estimate is uniform when the metric varies in a compact set of smooth Hermitian metrics with positive curvature.

2.2.2. First order approximation. We know that from any starting point $\omega = \omega_0$, there exists a solution

$$\omega_t = \omega + \sqrt{-1} \partial \bar{\partial} \phi_t$$

to the J-flow for $t > 0$. We can write $\omega_t = c_1(h_t)$ where h_t is a sequence of Hermitian metrics on the line bundle L_1 . Furthermore, we can construct a natural sequence of Bergman metrics

$$\hat{h}_k(t) = FS(Hilb_\chi(h_t^k))^{1/k}$$

by pulling back the Fubini-Study metric using sections which are orthonormal with respect to the inner product

$$\frac{1}{k^n} \frac{1}{\gamma} \int_M h_t(\cdot, \cdot)^k \chi \wedge c_1(h)^{n-1}.$$

Using Proposition 2.5, we obtain the asymptotic behavior

$$\hat{h}_k(t) = \left(\frac{\gamma k^n c_1(h_t)^n}{\chi \wedge c_1(h_t)^{n-1}} + O\left(\frac{1}{k}\right) \right)^{1/k} h_t$$

for $k \gg 1$. Thus, the sequence $\hat{h}_k(t)$ converges to h_t as $k \rightarrow \infty$.

On the other hand, the rescaled J-balancing flow provides a sequence of metrics $\omega_k(t) = c_1(h_k(t))$ which are solutions to (10). Note that by construction, we fix $h_k(0) = \hat{h}_k(0)$ for the starting point of the rescaled J-balancing flow.

In this section, we wish to evaluate the distance between the two metrics $h_k(t)$ and $\hat{h}_k(t)$. Since we are dealing with algebraic metrics, we have the

(rescaled) metric on Hermitian matrices given by

$$d_k(H_0, H_1) = \left(\frac{\text{tr} (H_0 - H_1)^2}{k^2} \right)^{\frac{1}{2}} \quad \text{on } \text{Met}(H^0(L_1^k))$$

which induces a metric on $\text{Met}(L_1)$, that we denote by dist_k .

Proposition 2.9. *One has for $t \in [0, T]$,*

$$\text{dist}_k(h_k(t), \hat{h}_k(t)) \leq \frac{C}{k},$$

for some constant $C > 0$ independent of k and t .

Proof. Let us consider $e^{\phi(t)}h_0$ a family of Hermitian metrics with positive curvature, and denote

$$\omega_t = c_1(e^{\phi(t)}h_0).$$

The infinitesimal change at t in the L^2 inner product induced by this path and the induced volume form is given by

$$\hat{U}_{\alpha,\beta}(t) = \frac{1}{\gamma k^n} \int_M \langle s_\alpha, s_\beta \rangle \left(\left(k\dot{\phi}(t) + \Delta_{\omega_t}\dot{\phi}(t) \right) \chi \wedge \omega_t^{n-1} - \tilde{\Delta}_{\omega_t}\dot{\phi}(t)\omega_t^n \right)$$

where Δ_{ω_t} is the Laplacian with respect to ω_t and $\tilde{\Delta}_{\omega_t}$ is given by the Laplacian-type operator

$$\tilde{\Delta}_{\omega_t}u = \frac{1}{n} \omega_t^{k\bar{j}} \omega_t^{i\bar{l}} \chi_{i\bar{j}} \partial_k \partial_{\bar{l}} u.$$

Here $\{s_\alpha\}$ is an orthonormal basis of $H^0(L_1^k)$ with respect to the L^2 -inner product

$$\frac{1}{\gamma k^n} \int_M e^{k\phi(t)} \chi \wedge \omega_t^{n-1}.$$

The formula is obtained by noticing that the variation occurs with respect to the fibrewise metric and the induced volume form. Now, if furthermore $\phi(t)$ is a solution to the J-flow, this infinitesimal change is given at $\hat{h}_k(t)$ as

$$\hat{U}_{\alpha,\beta}(t) = \frac{1}{\gamma k^n} \int_M \langle s_\alpha, s_\beta \rangle \left(k \left(1 - \frac{\chi \wedge \omega_t^{n-1}}{\gamma \omega_t^n} \right) + O(1) \right) \chi \wedge \omega_t^{n-1}$$

with $\{s_\alpha\}$ satisfy the same assumption as above.

On another hand, the tangent (at the same point $\hat{h}_k(t)$) to the rescaled J-balancing flow (10) is given directly by the moment map $\mu_{k,\chi}^0$, and we write the infinitesimal change of the L^2 metric as

$$U_{\alpha,\beta}(t) = \frac{k^2}{\gamma k^n} \int_M \left(\frac{\delta_{\alpha\beta}}{N+1} - \frac{\langle s_\alpha, s_\beta \rangle}{\sum_{i=1}^{N+1} |s_i|^2} \right) \chi \wedge \omega_{FS}^{n-1},$$

where s_i are L^2 orthonormal with respect to the L^2 inner product induced by $h(t)^k$ and $\chi \wedge \omega_t^{n-1}$. Again, using the fact that $\omega_{FS} = k\omega_t + O(k^{-1})$ and from Proposition 2.5, one has asymptotically

$$U_{\alpha,\beta}(t) = \hat{U}_{\alpha,\beta}(t) + \frac{1}{k^n} \int_M \langle s_\alpha, s_\beta \rangle O(1) \chi \wedge \omega_t^{n-1}.$$

Here the term $O(1)$ stands implicitly for a (smooth) function which is bounded independently of the variables t and k . Thus, one has

$$\frac{\text{tr} (\hat{U}_{\alpha,\beta}(t) - U_{\alpha,\beta}(t))^2}{k^2} = \left\langle \frac{1}{k} O(1), Q_k \left(\frac{1}{k} O(1) \right) \right\rangle_{L^2}.$$

We can use Theorem 2.8, Inequality (13) to obtain that

$$\frac{\text{tr} (\hat{U}_{\alpha,\beta}(t) - U_{\alpha,\beta}(t))^2}{k^2} = O(k^{-2}).$$

This shows that $d_k(\hat{U}_{\alpha,\beta}(t), U_{\alpha,\beta}(t)) = O(1/k)$. If we denote by $\tilde{h}_k(t)$ the rescaled J-balancing flow passing through $\hat{h}_k(t_0)$ at $t = t_0$, we have just proved that $\tilde{h}_k(t)$ and $\hat{h}_k(t)$ are tangent up to an error term in $O(1/k)$ at $t = t_0$. On the other hand, it is clear that $\tilde{h}_k(t)$ and $h_k(t)$ are close when $t \rightarrow \infty$, because they are obtained through the gradient flow of the same moment map and this gradient flow is distance decreasing (see also [8, Theorem 1]). Thus $\text{dist}(\tilde{h}_k(t), h_k(t)) = O(1/k)$. This finally proves the result. □

2.2.3. Higher order approximation. In this section, we only describe the main differences with [5, Section 4.2]. The key operator appearing in the linearisation of the problem is actually,

$$\eta \rightarrow \mathfrak{L}_t(\eta) = \frac{\partial \eta}{\partial t} - \tilde{\Delta} \eta.$$

We mean that it is sufficient to solve inductively equations of the form $\mathfrak{L}_t(\eta_i) = \gamma_{i,0}(\eta_1, \dots, \eta_{i-1})$ where $\gamma_{i,0}$ is smooth. By the standard parabolic

theory, a smooth solution η_t of

$$\{\mathfrak{L}_t(\eta) = \xi, \eta(0) = 0, \xi \in C^\infty(M, \mathbb{R})\}$$

exists for all time $t \geq 0$. Using this remark, it is easy to modify the arguments of [5, Theorem 4] in order to obtain the following result.

Theorem 2.10. *Fix $T > 0$. Given solution ϕ_t for $t \in [0, T]$ to Donaldson’s J -flow (1) and $k \gg 0$, there exist functions η_1, \dots, η_m , $m \geq 1$, such that the deformation of ϕ_t given by the potential*

$$\psi(k, t) = \phi_t + \sum_{j=1}^m \frac{1}{k^j} \eta_j(t)$$

satisfies

$$\text{dist}_k(h_k(t), \bar{h}_k(t)) \leq \frac{C}{k^{m+1}}$$

and

$$\text{dist}_k\left(\frac{\partial h_k(t)}{\partial t}, \frac{\partial \bar{h}_k(t)}{\partial t}\right) \leq \frac{C}{k^m}.$$

Here $\bar{h}_k(t) = FS(\text{Hilb}_\chi(h_0^k e^{k\psi(k,t)}))^{1/k} \in \text{Met}(L_1)$ is the induced Bergman metric from the potential ψ , $h_k(t) \in \text{Met}(L_1)$ is the metric obtained by the rescaled J -balancing flow (10), and C is a positive constant independent of k and t .

2.2.4. L^2 estimates in finite dimension. In this section we provide analogues of Proposition 4.3, Lemma 4.1 and Corollary 4.1 of [5], where μ_Ω is replaced by the moment map $\mu_{k,\chi}$. This is because all the proof of these results depend only on the integrand of the expression for the moment map $\mu_{k,\chi}$ given in (8).

Fix

$$H_A = \sum_{i,j} A_{ij}(s_i, s_j) = \text{tr}(A\mu_{FS}) \in C^\infty(M, \mathbb{R}),$$

where $A = (A_{ij})$ is a Hermitian matrix, $\{s_i\}$ is a basis of $H^0(L_1^k)$, and (\cdot, \cdot) denotes the fibrewise Fubini-Study inner-product induced by the basis $\{s_i\}$.

We start this section by recalling the notion of R -boundedness in C^r topology (see [15, Section 3.2]). The purpose of this definition is to avoid constants depending on k in the forthcoming estimates. Let us fix a reference

metric $\omega_{ref} \in c_1(L_1)$. We denote $\tilde{\omega}_{ref} = k\omega_{ref}$ the induced metric in $kc_1(L_1)$. We say that another metric $\tilde{\omega} \in kc_1(L_1)$ has R -bounded geometry in C^r if

$$\tilde{\omega} > \frac{1}{R}\tilde{\omega}_{ref} \quad \text{and} \quad \|\tilde{\omega} - \tilde{\omega}_{ref}\|_{C^r(\tilde{\omega}_{ref})} < R.$$

Moreover, we say that a basis $\{s_i\}$ of $H^0(L_1^k)$ is R -bounded if the Fubini-Study metric induced by the embedding of M in $\mathbb{P}H^0(L_1^k)^*$ associated to $\{s_i\}$ has R -bounded geometry.

Proposition 2.11. *There exists $C > 0$ independent of k , such that for any basis $\{s_i\}$ of $H^0(L_1^k)$ with R -bounded geometry in C^r and any Hermitian matrix A ,*

$$\|H_A\|_{C^r} \leq C\|\mu_{k,\chi}(\iota)\|_{op}\|A\|$$

where ι is the embedding induced by $\{s_i\}$.

In the above proposition, the constant C depends on the parameters $(M, L_1, r, \chi, \omega_{ref})$.

Lemma 2.12. *Let us fix $r \geq 2$. Assume that for all $t \in [0, T]$, the family of basis $\{s_i\}(t)$ of $H^0(L_1^k)$ have R -bounded geometry. Let us define by $h(t)$ the family of Bergman metrics induced by $\{s_i\}(t)$. Then the induced family of Fubini-Study metrics $\tilde{\omega}(t)$ satisfy*

$$\|\tilde{\omega}(0) - \tilde{\omega}(T)\|_{C^{r-2}} < C \sup_t \|\mu_{k,\chi}(\iota(t))\|_{op} \int_0^T \text{dist}(h(s), h(0)) ds,$$

and also

$$\begin{aligned} \left\| \frac{\partial \tilde{\omega}}{\partial t}(0) - \frac{\partial \tilde{\omega}}{\partial t}(T) \right\|_{C^{r-2}} &< C^* \sup_t \|\mu_{k,\chi}(\iota(t))\|_{op} \int_0^T \text{dist} \left(\frac{\partial h}{\partial s}(s), \frac{\partial h}{\partial s}(0) \right) ds \\ &+ C^* \sup_t \|d\mu_{k,\chi}(\iota(t))\|_{op} \int_0^T \text{dist}(h(s), h(0)) ds, \end{aligned}$$

where C, C^* are uniform constants in k .

Corollary 2.13. *Let $\tilde{\omega}_k$ be a sequence of metrics with $R/2$ -bounded geometry in C^{r+2} such that the norms $\|\mu_{k,\chi}(\tilde{\omega}_k)\|_{op}$ are uniformly bounded. Then, there is a constant $C > 0$ independent of k such that if $\tilde{\omega}$ has $\text{dist}_k(\tilde{\omega}, \tilde{\omega}_k) < C$, then $\tilde{\omega}$ has R -bounded geometry in C^r .*

2.2.5. Projective estimates. We collect here some projective estimates, following the lines of [17, Section 5].

Proposition 2.14. *Let h be a Hermitian metric on L_1 with curvature $\omega = c_1(h) > 0$. Consider the sequence $h_k = FS(Hilb(h)) \in \text{Met}(L_1^k)$ of Bergman metrics, approximating h after renormalisation, thanks to Proposition 2.5. Let us call*

$$\mathfrak{J}_{k,\chi} = \frac{1}{\gamma} \int_M \langle s_i, s_j \rangle_{h^k} \chi \wedge \omega^{n-1},$$

where $\{s_i\}$ is a basis of holomorphic sections of $H^0(L_1^k)$ orthonormal with respect to $Hilb(h)$. Then, as $k \rightarrow +\infty$,

$$\|\mu_{k,\chi}(h_k) - \mathfrak{J}_{k,\chi}\|_{op} \rightarrow 0$$

and the convergence is uniform for ω lying in a compact subset of Kähler metrics in $c_1(L)$.

Proof. Because of the asymptotic expansion (Proposition 2.5), we have

$$\mu_{k,\chi}(h_k) = \frac{1}{\gamma} \int_M \langle s_i, s_j \rangle_{h^k} (1 + O(1/k)) \chi \wedge \omega_{FS}^{n-1}.$$

Then, we apply [15, Lemma 28] that gives that for the operator norm,

$$\left\| \frac{1}{\gamma} \int_M \langle s_i, s_j \rangle_{h^k} (1 + O(1/k)) \chi \wedge \omega_{FS}^{n-1} \right\|_{op} \leq \left\| \frac{\chi \wedge \omega_{FS}^{n-1}}{\gamma \omega_{FS}^n} O(1/k) \right\|_{L^\infty}.$$

The uniformity of the convergence is given by the uniformity of the expansion in the asymptotics. □

In the sequel we fix a point $b \in \mathcal{B}$.

Lemma 2.15. *For any pair of Hermitian matrices $A, B \in T_b\mathcal{B}$, denote \hat{A}, \hat{B} the induced vector field on \mathbb{P}^N . One has*

$$\text{tr}(Bd\mu_{k,\chi}(A)) = \frac{1}{\gamma} \int_M (\hat{A}, \hat{B}) \chi \wedge \omega_{FS}^{n-1} - \partial H_B \wedge \bar{\partial} H_A \wedge \chi \wedge \omega_{FS}^{n-1},$$

where (\cdot, \cdot) denotes the Fubini-Study form induced on the tangent vectors.

Proof. This is contained in the proof of Proposition 2.2 but for the sake of clearness let us provide the proof. We have, using the fact that μ_{FS} is a

moment map,

$$\begin{aligned} \text{tr}(Bd\mu_{k,\chi}(A)) &= \frac{1}{\gamma} \int_M \text{tr}(Bd\mu_{FS}(A))\chi \wedge \omega_{FS}^{n-1} \\ &\quad + \frac{1}{\gamma} \int_M \text{tr}(B\mu_{FS})L_{\hat{A}}(\chi \wedge \omega_{FS}^{n-1}) \\ &= \frac{1}{\gamma} \int_M (\hat{A}, \hat{B})\chi \wedge \omega_{FS}^{n-1} \\ &\quad - \frac{1}{\gamma} \int_M \text{tr}(B\mu_{FS})\partial\bar{\partial}\text{tr}(A\mu_{FS}) \wedge \chi \wedge \omega_{FS}^{n-1} \\ &= \frac{1}{\gamma} \int_M (\hat{A}, \hat{B})\chi \wedge \omega_{FS}^{n-1} \\ &\quad - \partial\text{tr}(B\mu_{FS}) \wedge \bar{\partial}\text{tr}(A\mu_{FS}) \wedge \chi \wedge \omega_{FS}^{n-1}. \end{aligned}$$

□

By the fact that μ_{FS} is a moment map, we have the following simple lemma.

Lemma 2.16. *Let $A, B \in T_b\mathcal{B}$. Pointwise over $(\mathbb{P}^N)^*$, one has*

$$H_A H_B + (\hat{A}, \hat{B}) = \text{tr}(AB\mu_{FS}).$$

The next lemma is deduced from Lemmas 2.15 and 2.16.

Lemma 2.17. *For any Hermitian matrices $A, B \in T_b\mathcal{B}$,*

$$\text{tr}(Bd\mu_{k,\chi}(A)) + \langle H_A, H_B \rangle_{L^2_1(M, \frac{1}{\gamma}\chi \wedge \omega_{FS}^{n-1})} = \text{tr}(AB\mu_{k,\chi}),$$

where the $L^2_1(M, \frac{1}{\gamma}\chi \wedge \omega_{FS}^{n-1})$ -norm is computed with respect to the volume form $\frac{1}{\gamma}\chi \wedge \omega_{FS}^{n-1}$ and the gradient induced by χ .

We can derive from the two previous results the following corollaries.

Lemma 2.18. *For any Hermitian matrix $A \in T_b\mathcal{B}$,*

$$\|H_A\|_{L^2_1(M, \frac{1}{\gamma}\chi \wedge \omega_{FS}^{n-1})}^2 \leq \|A\|^2 \|\mu_{k,\chi}\|_{op}.$$

Proof. From the last lemma,

$$\|H_A\|_{L^2_1(M, \frac{1}{\gamma}\chi \wedge \omega_{FS}^{n-1})}^2 = \text{tr}(A^2 \mu_{k,\chi}) - \text{tr}(Ad\mu_{k,\chi}(A)).$$

Now, by Lemma 2.15, $\text{tr}(Ad\mu_{k,\chi}(A)) \geq 0$. The conclusion follows from the fact that $\text{tr}(A^2 \mu_{k,\chi}) \leq \|A\|^2 \|\mu_{k,\chi}\|_{op}$. \square

Lemma 2.19. *For any Hermitian matrix $A \in T_b\mathcal{B}$,*

$$\|d\mu_{k,\chi}(A)\|_{op} \leq \|d\mu_{k,\chi}(A)\| \leq 2\|A\| \|\mu_{k,\chi}\|_{op}.$$

Proof. From Lemma 2.17, one has

$$\begin{aligned} \|d\mu_{k,\chi}(A)\|^2 &= \text{tr}(d\mu_{k,\chi}(A)^2) \\ &= \text{tr}(Ad\mu_{k,\chi}(A)\mu_{k,\chi}) - \langle H_A, H_{d\mu_{k,\chi}(A)} \rangle_{L^2_1(M, \frac{1}{\gamma}\chi \wedge \omega_{FS}^{n-1})} \\ &\leq \|A\| \|d\mu_{k,\chi}(A)\| \|\mu_{k,\chi}\|_{op} - \langle H_A, H_{d\mu_{k,\chi}(A)} \rangle_{L^2_1(M, \frac{1}{\gamma}\chi \wedge \omega_{FS}^{n-1})}. \end{aligned}$$

By Cauchy-Schwarz,

$$\begin{aligned} &|\langle H_A, H_{d\mu_{k,\chi}(A)} \rangle_{L^2_1(M, \frac{1}{\gamma}\chi \wedge \omega_{FS}^{n-1})}| \\ &\leq \|H_A\|_{L^2_1(M, \frac{1}{\gamma}\chi \wedge \omega_{FS}^{n-1})} \|H_{d\mu_{k,\chi}(A)}\|_{L^2_1(M, \frac{1}{\gamma}\chi \wedge \omega_{FS}^{n-1})}, \end{aligned}$$

and the previous lemma, we conclude the proof. \square

Finally, we obtain as a consequence of our work the following proposition.

Proposition 2.20. *Let $b_0, b_1 \in \mathcal{B}$. Then,*

$$\|\mu_{k,\chi}(b_1)\|_{op} \leq e^{2\text{dist}_k(b_0, b_1)} \|\mu_{k,\chi}(b_0)\|_{op}.$$

Proof. The proof is similar to the proof of [5, Proposition 4.5], using Lemma 2.19. \square

We have now all the ingredients to proceed to the proof of the main result of this section.

Proof of Theorem 2.7. The only difference with the proof of [5, Theorem 1, p.26] is that here we need to estimate $\|\mathfrak{J}_{k,\chi}\|_{op}$ which is bounded from above by $\sup_M \frac{\chi \wedge \omega^{n-1}}{\gamma \omega^n}$ using [15, Lemma 28]. This latter term is also bounded along the J-flow by a maximum principle argument. The other main ingredient of

the proof is the uniformity in the evolving metrics, which is ensured by the fact that we are working in finite time or that we have smooth convergence.

For the sake of the clarity, we now provide a complete proof. Using Theorem 2.10, for any $m > 0$, we have obtained a sequence of Kähler metrics

$$\omega(k; t) = c_1(h_0 e^{\psi(k,t)})$$

such that $\omega(k; t)$ converges, when $k \rightarrow +\infty$ and in smooth sense, towards the solution $\omega_t = c_1(h_0 e^{\phi_t})$ to the J-flow.

Moreover, one has, for k large enough and with $\bar{h}_k(t) \in \mathcal{B}$ the Bergman metric associated to $h_0 e^{\psi(k,t)} \in \text{Met}(L)$, the estimate

$$(15) \quad \text{dist}_k(h_k(t), \bar{h}_k(t)) \leq \frac{C}{k^{m+1}},$$

where $h_k(t)$ is the metric induced by the rescaled J-balancing flow. Consequently, in order to get the C^0 convergence in t , all what we need to show is that

$$(16) \quad \|\omega_k(t) - c_1(\bar{h}_k(t))\|_{C^r(\omega_t)} \rightarrow 0.$$

The idea is to consider the geodesic in the Bergman space between these two points.

Firstly, we will get that along the geodesic from $\bar{h}_k(t)$ to $h_k(t)$ in \mathcal{B} , $\|\mu_{k,\chi}\|_{op}$ is controlled uniformly if we can apply Proposition 2.20. This requires to prove that $\bar{h}_k(t)$ is at a uniformly bounded distance of $h_k(t)$ and that $\|\mu_{k,\chi}(\bar{h}_k(t))\|_{op}$ is bounded in k . But, this comes from the fact that one can choose precisely $m \geq n + 1$ in Inequality (15) and one can apply Proposition 2.14.

Secondly, we show that the points along this geodesic have R -bounded geometry. This is a consequence of Corollary 2.13, applied with the reference metric ω_t to the sequence $c_1(\bar{h}_k(t))$. On one side, $\|\mu_\Omega(\bar{h}_k(t))\|_{op}$ is under control as we have just seen. On another side, $c_1(\bar{h}_k(t))$ are convergent to ω_t in C^∞ topology (hence in C^{r+4} topology), thus they have $R/2$ -bounded geometry. Given $m \geq n + 2$, one obtains, thanks to Corollary 2.13 and inequality (15), that all the metrics along the geodesic from $\bar{h}_k(t)$ to $h_k(t)$ have R -bounded geometry in C^{r+2} .

Thirdly, we are exactly under the conditions of Lemma 2.12. It follows that, by renormalising the metrics in the Kähler class $c_1(L)$ and by (15),

that

$$\begin{aligned} \|k\omega_k(t) - kc_1(\bar{h}_k(t))\|_{C^r(k\omega_t)} &\leq C\|\mu_{k,\chi}(\bar{h}_k(t))\|_{op}k^{n+2}\text{dist}_k(h_k(t), \bar{h}_k(t)), \\ \|\omega_k(t) - c_1(\bar{h}_k(t))\|_{C^r(\omega_t)} &\leq C\|\mu_{k,\chi}(\bar{h}_k(t))\|_{op}k^{n+2-m-1+r/2}, \end{aligned}$$

where we have used that the geodesic path from 0 to 1 is just a line. Here $C > 0$ is a constant that does not depend on k . If we choose $m > r/2 + 1 + n$, we get the expected convergence in C^r topology, i.e Inequality (16). Of course, this reasoning works to get the uniform C^0 convergence in t for $t \in \mathbb{R}_+$, because all the Kähler metrics ω_t that we are using are uniformly equivalent under our assumptions, and because we have uniformity of the expansion in Proposition 2.5 and Theorem 2.8.

We now prove that one has C^1 convergence in t of the flows $\omega_k(t)$. Again, we need to show the C^1 convergence of $\omega_k(t)$ to $c_1(\bar{h}_k(t))$, because we already know the convergence of $c_1(\bar{h}_k(t))$ to ω_t by Proposition 2.6. We are under the conditions of Lemma 2.12 by what we have just proved above. So we have, using again that our path is a geodesic,

$$\begin{aligned} &\left\| k \frac{\partial \omega_k(t)}{\partial t} - k \frac{\partial c_1(\bar{h}_k(t))}{\partial t} \right\|_{C^r} \\ &\leq C^* \|\mu_{k,\chi}(\bar{h}_k(t))\|_{op}k^{n+2}\text{dist}_k\left(\frac{\partial h_k(t)}{\partial t}, \frac{\partial \bar{h}_k(t)}{\partial t}\right) \\ &\quad + C^* \|d\mu_{k,\chi}(\bar{h}_k(t))\|_{op}k^{n+2}\text{dist}_k(h_k(t), \bar{h}_k(t)). \end{aligned}$$

Here the C^r norm is computed with respect to $k\omega_t$. If we apply Lemma 2.19, Theorem 2.10 (2nd inequality), we can bound from above the RHS of the last inequality, and get

$$\begin{aligned} \left\| \frac{\partial \omega_k(t)}{\partial t} - \frac{\partial c_1(\bar{h}_k(t))}{\partial t} \right\|_{C^r(\omega_t)} &\leq C' \|\mu_{k,\chi}(\bar{h}_k(t))\|_{op}k^{n+2-m-r/2} \\ &\quad + C'' \|\mu_{k,\chi}(\bar{h}_k(t))\|_{op}k^{n+2+r/2}k^{-m-1}k^{-m-1} \\ &\leq C''' k^{n+2-m-r/2}. \end{aligned}$$

Finally, we choose $m > r/2 + n + 2$ to obtain C^1 convergence. This completes the proof of Theorem 2.7. □

2.3. Convergence result for J-balanced metrics

The previous results are uniform if one assumes that the J-flow is convergent (and so T can be chosen $T = +\infty$). Thus a direct corollary of Theorem 2.7, the long time existence and convergence of the J-flow is the following.

Corollary 2.21. *Consider (M, L_1, L_2) a polarised manifold by L_1, L_2 such that there exists a critical metric solution of (2). Then for k sufficiently large, there exists a sequence of J-balanced metrics on $\text{Met}(L_1^k)$ obtained as the limit of the rescaled J-balancing flow at time $t = +\infty$. Furthermore, the sequence of J-balanced metrics converges in smooth topology towards the critical metric when $k \rightarrow +\infty$.*

This is an analogue of the main result of [15]. Of course a more direct proof inspired from [15] could be used to derive Corollary 2.21. This would involve to the operator obtained from linearising the Bergman function close to the critical point ω_∞ i.e explicitly

$$\phi \mapsto \tilde{\Delta}_{\omega_\infty} \phi$$

This operator is a uniformly elliptic second order operator. Its kernel consists of constant functions. Eventually, the existence of J-balanced metrics can be seen as a necessary condition for the existence of critical metrics. As a consequence of our work, we recover the uniqueness of the critical metrics.

3. Variational approach to the rescaled J-balancing flow

3.1. Convexity along geodesics

Let us consider the functional

$$J_\chi : \text{Met}(L_1) \rightarrow \mathbb{R}$$

on the space of smooth Hermitian metrics with positive curvature on L_1 defined up to an additive function by

$$(17) \quad \frac{dJ_\chi(h_t)}{dt} = \frac{1}{\gamma} \int_M \dot{\phi}_t \chi \wedge c_1(h_t)^{n-1},$$

where $h_t = e^{-\phi_t} h_0$ is a smooth path in $\text{Met}(L_1)$. Setting

$$\omega_t = c_1(h_t) = \omega_0 + \sqrt{-1} \partial \bar{\partial} \phi_t,$$

a direct computation gives

$$\frac{d^2 J_\chi(h_t)}{dt^2} = \frac{1}{\gamma} \int_M \ddot{\phi}_t \chi \wedge \omega_t^{n-1} - \dot{\phi}_t \tilde{\Delta}_{\omega_t} \dot{\phi}_t \omega_t^n + \dot{\phi}_t \Delta_{\omega_t} \dot{\phi}_t \chi \wedge \omega_t^{n-1}.$$

We shall use the same notation J_χ as above for the induced functional defined on $\text{Met}(L_1^k)$ for $k > 0$. Consider now the following functional $I_{\mu_{k,\chi}^0} : \mathcal{B}_k \rightarrow \mathbb{R}$ on the Bergman space defined by

$$I_{\mu_{k,\chi}^0}(H) = J_\chi \circ FS(H) + \frac{\text{Vol}_{L_1}(M)}{N+1} \log \det(H),$$

where $H \in \mathcal{B}_k$. It is clear that the derivative of $J_\chi \circ FS$ at a point $H \in \mathcal{B}_k$ is given by

$$\frac{1}{\gamma} \sum_{i,j} \int_M (\delta H)_{i,j} \langle s_i, s_j \rangle_{FS(H)} \chi \wedge c_1(FS(H))^{n-1},$$

where $\{s_i\}$ is an orthonormal basis of holomorphic sections of L_1^k with respect to H . Thus a J-balanced metric H is a critical point of the functional $I_{\mu_{k,\chi}^0}$.

The functional $I_{\mu_{k,\chi}^0}$ is the integral of the moment map $\mu_{k,\chi}^0$ (or Kempf-Ness function), in the sense of [30]. In particular it is decreasing along the rescaled J-balancing flow. Furthermore, due to Kempf-Ness theory and its convexity, its properness on $SL(N+1)$ is equivalent to the existence of a (unique) critical point which turns out to be a J-balanced metric (see [30, Proposition 3.5] and [19, Sections 4 and 8], [46]).

One can ask at that stage what is the analogue of $I_{\mu_{k,\chi}^0}$ for the infinite dimensional space of Kähler potentials. Let us consider $\omega, \omega_\phi = \omega + \sqrt{-1} \partial \bar{\partial} \phi$ two Kähler metrics in $c_1(L_1)$. We define the functional

$$I_{\mu_J}(\omega, \omega_\phi) = \int_0^1 \int_M \dot{\phi}_t \left(\frac{1}{\gamma} \chi \wedge \omega_{\phi_t}^{n-1} - \omega_{\phi_t}^n \right) dt,$$

for ω_{ϕ_t} a smooth Kähler path from ω to ω_ϕ . The functional I_{μ_J} is well defined and independent of the chosen path. Remark that this functional appeared also in [43] where it is called \hat{J} . Moreover $I_{\mu_J}(\omega, \cdot)$ vanishes at ω . Without loss of generality, as it is defined up to a constant, we can assume that J_χ also vanishes at ω .

Lemma 3.1. *The functional J_χ is strictly convex on the $C^{1,1}$ geodesics of the space $\text{Met}(L_1)$ of Kähler potentials in $c_1(L_1)$.*

Proof. See [8, Proposition 2.1]. □

We sum up the main properties of I_{μ_J} in the next proposition.

Proposition 3.2. *The functional I_{μ_J} is strictly convex on the $C^{1,1}$ geodesics of the space $\text{Met}(L_1)$ of Kähler potentials in $c_1(L_1)$. Along Donaldson's J-flow, the functionals I_{μ_J} and J_χ are equal and decreasing. The functionals I_{μ_J} satisfies the cocyclicity property*

$$I_{\mu_J}(\omega, \omega_{\phi_0}) + I_{\mu_J}(\omega_{\phi_0}, \omega_{\phi_1}) = I_{\mu_J}(\omega, \omega_{\phi_1})$$

for $\omega_{\phi_0}, \omega_{\phi_1}$ Kähler forms in the Kähler class $[\omega]$. I_{μ_J} is the integral of the moment map of μ_J defined by (4).

Proof. It is well known that the functional

$$I^{\text{AYM}}(\omega, \omega_\phi) = - \int_0^1 \int_M \dot{\phi}_t \omega_{\phi_t}^n dt$$

(often called Aubin-Yau-Mabuchi energy) is affine along geodesics of the space of Kähler potentials. Therefore the convexity is just a consequence of Lemma 3.1. Moreover, I_{μ_J} and J_χ are equal since $\int_M \dot{\phi}_t \omega_{\phi_t}^n$ vanishes and they agree at ω . Furthermore, along the flow, they are decreasing by definition. The cocyclicity property can be proved following the lines of [27, Theorem 2.3]. □

Lemma 3.3. *The functional $J_\chi \circ FS$ is convex along geodesics of \mathcal{B}_k .*

Proof. Firstly, let us explain formally why the result holds. A result of Phong-Sturm asserts that the geodesics in the the space of Kähler potentials can be approximated by the image via the FS map of geodesics from the Bergman space \mathcal{B}_k (see [1, 2, 38]). Now, from Lemma 3.1, the functional J_χ is strictly convex on the geodesics in the space of Kähler potentials which leads to the convexity of $J_\chi \circ FS$ along geodesics of \mathcal{B}_k .

We proceed by a direct computation to prove the lemma. A geodesic in \mathcal{B}_k is just a line. Given A a Hermitian matrix and a Hamiltonian function $H_A = \text{tr}(A\mu_{FS})$ for the corresponding action and the 1-parameter group of embeddings $\iota_t = t^A \circ \iota$, one needs to evaluate the derivative with respect to

t of the quantity

$$\frac{1}{\gamma} \int_M \iota_t^*(H_A)\chi \wedge \iota_t^*(\omega_{FS}^{n-1}),$$

but this is equal, up to the factor $\frac{1}{\gamma}$, to

$$\begin{aligned} &= \int_M |\nabla h_A|^2 \chi \wedge \omega_{FS}^{n-1} - 2 \int_M h_A \partial \bar{\partial} h_A \wedge \chi \wedge \omega_{FS}^{n-1}, \\ &= \int_M |\nabla h_A|^2 \chi \wedge \omega_{FS}^{n-1} - 2 \int_M \partial h_A \wedge \bar{\partial} h_A \wedge \chi \wedge \omega_{FS}^{n-1}. \end{aligned}$$

Now, the last term is

$$\int_M \partial h_A \wedge \bar{\partial} h_A \wedge \chi \wedge \omega_{FS}^{n-1} = \int_M |\partial h_A|_{TM}^2 \chi \wedge \omega_{FS}^{n-1} - \frac{1}{n} \int_M |\partial h_A|_{TM,\chi}^2 \omega_{FS}^n.$$

Here $|\partial h_A|_{TM}^2 = \frac{1}{2} |\nabla h_A|_{TM}^2$ is the norm of the tangential part to M of ∂h_A and $|\partial h_A|_{TM,\chi}^2$ is the norm with respect to χ . Consequently, the derivative we are looking for is given by

$$\int_M |\nabla h_A|_{\perp}^2 \chi \wedge \omega_{FS}^{n-1} + \frac{1}{n} \int_M |\partial h_A|_{TM,\chi}^2 \omega_{FS}^n \geq 0,$$

where $|\nabla h_A|_{\perp}^2$ stands for the norm of the normal component. Therefore, we have obtained the required convexity. \square

Corollary 3.4. *The functional $I_{\mu_J} \circ FS$ is convex along geodesics of \mathcal{B}_k .*

Proof. This is a consequence of the previous lemma and the fact that the functional $I^{\text{AYM}} \circ FS$ is convex along geodesics in the Bergman space, cf. [16, Proposition 1]. \square

Now, using the fact that log det is linear on geodesics, we also get

Corollary 3.5. *The functional $I_{\mu_{k,\chi}^0}$ is convex along geodesics of \mathcal{B}_k . It has at most one critical point. A J -balanced metric is an absolute minimum of the functional $I_{\mu_{k,\chi}^0}$ and is in particular unique.*

3.2. Iterates of the maps $\text{Hilb}_{\chi} \circ FS$ and $FS \circ \text{Hilb}_{\chi}$

In this section, we investigate the iterates of the map $T_{k,\chi}$.

Lemma 3.6. *Consider $h_0 \in \text{Met}(L_1)$, $h = e^{-\phi}h_0 \in \text{Met}(L_1)$. Then*

$$\frac{1}{\gamma} \int_M \phi \chi \wedge c_1(h)^{n-1} \leq J_\chi(h) - J_\chi(h_0) \leq \frac{1}{\gamma} \int_M \phi \chi \wedge c_1(h_0)^{n-1}.$$

Proof. If one defines $h_t = e^{-t\phi}h_0$, and

$$f(t) = \frac{1}{\gamma} \int_M t\phi\chi \wedge c_1(h_0)^{n-1} - (J_\chi(h_t) - J_\chi(h_0)),$$

then $f(0) = f'(0) = 0$ and furthermore along the considered path,

$$\begin{aligned} f''(t) &= -(n-1) \frac{1}{\gamma} \int_M \phi\chi \wedge c_1(h_t)^{n-2} \wedge \sqrt{-1}\partial\bar{\partial}\phi, \\ &= (n-1)\sqrt{-1} \frac{1}{\gamma} \int_M \partial\phi \wedge \bar{\partial}\phi \wedge \chi \wedge c_1(h_t)^{n-2}, \end{aligned}$$

which is non-negative. Thus $f(t) \geq 0$ at $t = 1$ which provides one inequality. By symmetry, we get the result. One can perform a direct computation. For instance, when $n = 2$, $J_\chi(h) - J_\chi(h_0)$ can be written as $\frac{1}{2\gamma} \int_M \phi \chi \wedge (c_1(h_0) + c_1(h))$. \square

Define for $h \in \text{Met}(L_1^k)$, $H \in \text{Met}(H^0(L_1^k))$,

$$\begin{aligned} \hat{P}(h, H) &= \log \sum_{i=1}^{N+1} \|S_i\|_{\text{Hilb}_\chi(h)}^2 - \log(N+1) \\ &\quad + \log \det(H) + \frac{N+1}{\text{Vol}_{L_1}(M)} J_\chi(h) \end{aligned}$$

where $\{S_i\}$ form an orthonormal basis with respect to H . Then it is not difficult to check that

$$\hat{P}(FS(H), H) = \frac{N+1}{\text{Vol}_{L_1}(M)} I_{\mu_{\chi,k}^0}(H).$$

Lemma 3.7. *For any metrics h, H , one has*

$$\hat{P}(h, H) \geq \hat{P}(FS(H), H).$$

Proof. One checks that if we define $h = e^{-\phi}FS(H)$, then

$$\begin{aligned} \hat{P}(h, H) - \hat{P}(FS(H), H) &= \log \left(\frac{N + 1}{\gamma \text{Vol}_{L_1}(M)} \int_M e^{-\phi} \chi \wedge c_1(h)^{n-1} \right) \\ &\quad + \frac{N + 1}{\text{Vol}_{L_1}(M)} (J_\chi(h) - J_\chi(FS(H))), \\ &\geq \frac{N + 1}{\gamma \text{Vol}_{L_1}(M)} \int_M -\phi \chi \wedge c_1(h)^{n-1} \\ &\quad + \frac{N + 1}{\text{Vol}_{L_1}(M)} (J_\chi(h) - J_\chi(FS(H))), \\ &\geq 0, \end{aligned}$$

using Lemma 3.6. □

Lemma 3.8.

$$\hat{P}(h, H) \geq \hat{P}(h, \text{Hilb}_\chi(h)).$$

Proof. This is a consequence of the arithmetic-geometric inequality. □

Assume the existence of $h_{bal} \in \text{Met}(L_1^k)$ J-balanced metric and $H_{bal} \in \text{Met}(H^0(L_1^k))$ J-balanced metric on the Bergman space. Then for any $h \in \text{Met}(L_1^k)$, and $H \in \text{Met}(H^0(L_1^k))$ one has

$$\begin{aligned} \hat{P}(h, H) &\geq \hat{P}(FS(H), H), \\ &= \frac{1}{\text{Vol}_{L_1}(M)} I_{\mu_{\chi, k}^0}(H), \\ &\geq \frac{1}{\text{Vol}_{L_1}(M)} I_{\mu_{\chi, k}^0}(H_{bal}), \\ &= \hat{P}(FS(H_{bal}), H_{bal}), \\ &= \hat{P}(h_{bal}, H_{bal}). \end{aligned}$$

In particular it gives that

$$I_{\mu_{\chi, k}^0}(H) \geq I_{\mu_{\chi, k}^0}(H_{bal}).$$

Moreover $\hat{P}(h, \text{Hilb}_\chi(h)) \geq \hat{P}(h_{bal}, \text{Hilb}_\chi(h_{bal}))$.

Thus the functional on $\text{Met}(L_1^k)$ defined by

$$\hat{I}_k(h) := \frac{\text{Vol}_{L_1}(M)}{N+1} \hat{P}(h, \text{Hilb}_\chi(h)) = J_\chi(h) + \frac{\text{Vol}_{L_1}(M)}{N+1} \log \det \text{Hilb}_\chi(h)$$

satisfies

$$\hat{I}_k(h) \geq \hat{I}_k(h_{bal}).$$

We will see soon that this new functional has a geometric interpretation. A direct consequence of Lemmas 3.7 and 3.8 is the following corollary.

Corollary 3.9. *In our setting, the following hold:*

- 1) *A J-balanced metric on $\text{Met}(L_1^k)$ (resp. $\text{Met}(H^0(L_1^k))$) is a minimum of the functional \hat{I}_k (resp. $I_{\mu_{\chi,k}^0}$).*
- 2) *The map $FS \circ \text{Hilb}_\chi$ decreases the functional \hat{I}_k while the map $\text{Hilb}_\chi \circ FS$ decreases the functional $I_{\mu_{\chi,k}^0}$.*
- 3) *The functional \hat{I}_k is bounded from below if and only if the functional $I_{\mu_{\chi,k}^0}$ is bounded from below.*

We now explain the asymptotic behavior of the functional \hat{I}_k , by studying the term

$$\log \det \text{Hilb}_\chi(h_t)$$

where $h_t = h e^{-k\phi_t}$ (with $\|\phi_t\|_{C^\infty} = O(1)$ when $k \rightarrow \infty$) is a path in $\text{Met}(L_1^k)$. Then, we can write $\text{Hilb}_\chi(h_t) = \langle s_i, s_j \rangle_{\text{Hilb}(h_t)}$ where $\{s_i\}$ is an orthonormal basis with respect to $\text{Hilb}_\chi(h)$. Thus its derivative at $t = 0$ is given by the derivative of $\sum_i \|s_i\|_{\text{Hilb}_\chi(h_t)}^2$ and because of the variation of the volume form, this is written as

$$-\frac{1}{\gamma} \int_M k \dot{\phi} \sum_i |s_i|_h^2 \chi \wedge c_1(h)^{n-1} + (n-1) \sum_i |s_i|_h^2 \chi \wedge c_1(h)^{n-1} \wedge \sqrt{-1} \partial \bar{\partial} \dot{\phi},$$

Together with (11) and the fact that the second term in the integrand is negligible compared to the first one when $k \rightarrow +\infty$ (by uniformity of the Bergman expansion in C^2 topology), we obtain when $k \rightarrow +\infty$ that

$$\frac{d}{dt} \Big|_{t=0} \left(\frac{\text{Vol}_{L_1}(M)}{N+1} \log \det \text{Hilb}_\chi(h e^{-k\phi_t}) \right) = -k \int_M \dot{\phi} c_1(h)^n + O(1),$$

where we have used that $N = \text{Vol}_{L_1}(M)k^n + O(k^{n-1})$. This leads to the following conclusion for \hat{I}_k .

Corollary 3.10. *Over compact subsets of $\text{Met}(L_1)$, the functionals \hat{I}_k and I_{μ_J} are equivalent, up to a normalisation, i.e*

$$\frac{1}{k} \hat{I}_k(h^k) = I_{\mu_J}(h) + O(1/k)$$

for $h \in \text{Met}(L_1)$.

Corollary 2.21 and our last result show that a critical metric solution of (2) is actually an absolute minimum of I_{μ_J} . Of course this is conceptually a consequence of I_{μ_J} being the integral of the moment map μ_J . Formally, this can be seen from the facts that the J-flow decreases I_{μ_J} , and from the analytic study of the flow (convergence to a unique critical metric in smooth topology).

Finally, following the techniques of [41], we obtain that if there is a J-balanced metrics of order k , then the iterates of $\text{Hilb}_\chi \circ FS$ on $\text{Met}(H^0(L_1^k))$ will converge towards this metric.

Theorem 3.11. *Assume that there exists $H_\infty \in \text{Met}(H^0(L_1^k))$ J-balanced. For any $H_0 \in \text{Met}(H^0(L_1^k))$, denote from now*

$$H_l = \text{Hilb}_\chi \circ FS(H_{l-1}),$$

for $l \geq 1$. Then, up to a positive constant r ,

$$H_l \rightarrow rH_\infty$$

as $l \rightarrow +\infty$.

For the sake of clarity, we give the details of the proof which consists in an easy modification of [41, 42], which is not surprising since it is a purely finite dimensional problem and our setting is very close. We will decompose the proof into several lemmas starting with the following definition.

Definition 3.12. Let $\{s_i\}$ be a basis of $H^0(L_1^k)$. Using this basis, we can view elements of $\text{Met}(H^0(L_1^k))$ as Hermitian matrices $(N + 1) \times (N + 1)$. A subset $\mathcal{U} \subset \text{Met}(H^0(L_1^k))$ is bounded if there exists a number $R > 1$ satisfying the following. For any $H \in \mathcal{U}$, there exists a constant $\gamma_H > 0$ so that the smallest and largest eigenvalues of H satisfy

$$\frac{\gamma_H}{R} \leq \min \frac{|H(\zeta)|}{|\zeta|} \leq \max \frac{|H(\zeta)|}{|\zeta|} \leq \gamma_H R.$$

With the notations of the the previous definition, we have the following obvious proposition due to the fact that the closure of bounded sets are compact in finite dimension.

Proposition 3.13. *Any bounded sequence H_k has a subsequence H_{n_k} such that $\frac{1}{\gamma_{n_k}}H_{n_k}$ converges in $\text{Met}(H^0(L_1^k))$.*

Lemma 3.14. *[42, Lemma 3.2] The set \mathcal{U} is bounded if and only if there exists a number $R > 1$ so that for any $H \in \mathcal{U}$, we have*

$$\frac{1}{R} \leq \min \frac{|\tilde{H}(\zeta)|}{|\zeta|} \leq \max \frac{|\tilde{H}(\zeta)|}{|\zeta|} \leq R,$$

where $\tilde{H} = \frac{1}{\det(H)^{\frac{1}{N+1}}}H$.

Proof. We include the proof for the sake of clarity. Without loss of generality, we can assume $H(s_i, s_j)$ is diagonal with entries e^{λ_i} , $\lambda_1 \leq \dots \leq \lambda_{N+1}$. Assuming \mathcal{U} bounded, we obtain $\gamma_H \leq Re^{\lambda_i}$ and $\gamma_H \geq \frac{1}{R}e^{\lambda_i}$. Thus, $e^{\lambda_{N+1}} \leq R^2e^{\lambda_i}$ and $e^{\lambda_1} \geq R^{-2}e^{\lambda_i}$ for all $i = 1, \dots, N + 1$, which gives

$$\det(H)^{-1/(N+1)}e^{\lambda_{N+1}} = \left(\prod_i e^{\lambda_{N+1}-\lambda_i} \right)^{1/(N+1)} \leq R^2.$$

Similarly, we obtain $\det(H)^{-1/(N+1)}e^{\lambda_1} \geq \frac{1}{R^2}$. □

Lemma 3.15. *Under the assumption of the theorem, if the sequence H_l is bounded in $\text{Met}(H^0(L_1^k))$, then the sequence $\det(H_l)$ is convergent and*

$$\det(H_{l+1}H_l^{-1}) \rightarrow 1$$

as $l \rightarrow +\infty$.

Proof. From Lemma 3.8, we deduce that the sequence $\log \det(H_l)$ is decreasing. From Lemma 3.7, we deduce that the sequence $J_\chi \circ FS(H_l)$ is also decreasing. Since $I_{\mu_{\chi,k}^0}(H_l)$ is decreasing and bounded from below, $\log \det(H_l)$ is bounded and converges. □

Lemma 3.16. *Assume the sequence H_l is bounded in $\text{Met}(H^0(L_1^k))$. Let $H \in \text{Met}(H^0(L_1^k))$ and $\{s_i^l\}_i$ be an orthonormal basis with respect to H_l so that the matrix $H(s_i^l, s_j^l)$ is diagonal. Then,*

$$\lim_{l \rightarrow +\infty} \|s_i^l\|_{\text{Hilb}_\chi(FS(H_l))}^2 = \frac{\text{Vol}_{L_1}(M)}{N + 1}.$$

Proof. Let us consider \hat{s}_i^l another basis, orthonormal with respect to H_l and so that $H_{l+1}(\hat{s}_i^l, \hat{s}_j^l)$ is diagonal. From the previous lemma, we deduce that $\lim_{l \rightarrow +\infty} \det(H_{l+1}(\hat{s}_i^l, \hat{s}_i^l)) = 1$. We have always that

$$\text{tr}(\text{Hilb}_\chi(FS(H))H^{-1}) = N + 1,$$

so we get $\text{tr}(H_{l+1}(\hat{s}_i^l, \hat{s}_i^l)) = N + 1$ for all l . It is not difficult to check that the arithmetic-geometric inequality implies

$$H_{l+1}(\hat{s}_i^l, \hat{s}_i^l) \rightarrow 1$$

as $l \rightarrow +\infty$. Now, we write

$$s_i^l = \sum_{j=1}^{N+1} a_{ij}^l \hat{s}_j^l,$$

with $(a_{ij}^l) \in U(N + 1)$. The matrix a_{ij}^l converges when $l \rightarrow +\infty$ up to taking a subsequence. For the limit $(a_{ij}^\infty) \in U(N + 1)$ we have

$$H_{l+1}(s_i^l, s_j^l) \rightarrow \sum_{j=1}^{N+1} |a_{ij}^\infty|^2 = 1,$$

which means that

$$\lim_{l \rightarrow +\infty} \frac{N + 1}{\gamma \text{Vol}_{L_1}} \int_M |s_i^l|_{FS(H_l)}^2 \chi \wedge c_1(FS(H_l))^{n-1} = 1,$$

as expected. □

Lemma 3.17. *If the sequence H_l is bounded, then for any Hermitian metric $H \in \text{Met}(H^0(L_1^k))$ and $\epsilon > 0$, we have*

$$(18) \quad I_{\mu_{\chi,k}^0}(H) \geq I_{\mu_{\chi,k}^0}(H_l) - \epsilon$$

for l sufficiently large.

Proof. Fix $\{s_i^l\}$ an orthonormal basis with respect to H_l such that $H(s_i^l, s_j^l)$ is diagonal with entries $e^{\lambda_i^l}$. Define $f(t) = I_{\mu_{\chi,k}^0}(H_t)$ where H_t is the matrix of entries $e^{t\lambda_i^l}$, so that $H|_{t=0} = H_l$ and $H|_{t=1} = H$. By convexity of $I_{\mu_{\chi,k}^0}$ in

the Bergman space, we have $f(1) - f(0) \geq f'(0)$. By definition, one gets

$$\begin{aligned} f'(0) &= \int_M \frac{d}{dt} \Big|_{t=0} (FS(H_t)) \frac{\chi \wedge c_1(FS(H_t))^n}{\gamma} - \frac{\text{Vol}_{L_1}(M)}{N+1} \sum_i \lambda_i^l, \\ &= \int_M \sum_{i=1}^{N+1} \lambda_i^l |s_i^l|_{FS(H_t)}^2 \frac{\chi \wedge c_1(FS(H_t))^n}{\gamma} - \frac{\text{Vol}_{L_1}(M)}{N+1} \sum_i \lambda_i^l. \end{aligned}$$

Let us assume that λ_i^l are bounded. Then we can apply Lemma 3.16 and obtain that $f'(0) \rightarrow 0$ when $l \rightarrow +\infty$, which provides eventually (18). It remains to show that λ_i^l are all bounded when l varies. Using the fact that $\{e^{-\lambda_i^l/2} s_i^l\}_i$ is an orthonormal basis for H and Lemma 3.14, we have the existence of $R > 1$ such that

$$\frac{1}{R} < \frac{H_l(s_i^l, s_i^l) e^{-\lambda_i}}{\det(H_l)^{1/(N+1)}} = \frac{e^{-\lambda_i}}{\det(H_l)^{1/(N+1)}} < R$$

from which we deduce that λ_i^l is bounded if and only if $\det(H_l)$ is bounded. But this is the case by Lemma 3.15. □

Proof of Theorem 3.11. As we have already seen in the previous section, the J-balanced metric H_∞ is unique up to normalisation. To normalise our metrics, we choose to work in the space

$$\{\tilde{H} = (\det H)^{-1/(N+1)} H, H \in \text{Met}(H^0(L_1^k))\} \subset \text{Met}(H^0(L_1^k)).$$

Furthermore, as integral of the moment map $\mu_{\chi,k}^0$, the functional $I_{\mu_{\chi,k}^0}$ is proper and bounded from below. The sequence $I_{\mu_{\chi,k}^0}(H_l)$ is decreasing, and thus $J_\chi((\det H_l)^{-1/(N+1)} FS(H_l))$ is also bounded. By properness, it follows that

$$\tilde{H}_l = (\det H_l)^{-1/(N+1)} FS(H_l)$$

is bounded. This forces this sequence to converge as we now see.

Suppose not. Then we can at least take a non convergent subsequence \tilde{H}_{l_k} which always remain at a distance ϵ of the J-balanced metric H_∞ . But \tilde{H}_{l_k} is bounded and its image by $I_{\mu_{\chi,k}^0}$ converges to the minimum of the functional $I_{\mu_{\chi,k}^0}$, up to taking a subsequence that we denote $\tilde{H}_{l_{k_m}}$. In fact, we have obtained from the previous results that for any bounded sequence H_l , $I_{\mu_{\chi,k}^0}(H_l)$ converges to the minimum of the functional $I_{\mu_{\chi,k}^0}$, see Lemma 3.17. Therefore, $\tilde{H}_{l_{k_m}}$ converges and its limit is actually a J-balanced metric from Corollary 3.5. This is a contradiction with the fact that all the terms \tilde{H}_{l_k} are at distance ϵ of H_∞ .

From Lemma 3.15, we get that $\log(\det(H_l))$ is bounded and decreasing. This allows us to conclude that H_l is convergent to rH_∞ . \square

From Corollary 2.21 we obtain the following result.

Corollary 3.18. *Assume the existence of a critical metric solution ω_∞ of (1). Then for all $k \gg 0$, the map*

$$\text{Hilb}_\chi \circ FS : \text{Met}(H^0(L_1^k)) \rightarrow \text{Met}(H^0(L_1^k))$$

(respectively $T_{k,\chi} = FS \circ \text{Hilb}_\chi : \text{Met}(L_1^k) \rightarrow \text{Met}(L_1^k)$) defines a dynamical system that has a fixed (unique) attractive point, the J-balanced metric H_{bal} (respectively h_{bal} with $FS(H_{bal}) = h_{bal}$). Furthermore,

$$\left\| \omega_\infty - \frac{1}{k} c_1(FS(H_{bal})) \right\|_{C^\infty} = O\left(\frac{1}{k}\right).$$

4. Algebro-geometric aspects

In this section we describe a notion of *Chow stability* for a linear system $|L_2|$ in a fixed polarised variety (M, L_1) , which is related to Geometric Invariant Theory (GIT). Our main result is then that the existence of a J-balanced metric implies this notion of Chow stability for $|L_2|$. It follows that the existence of a critical point of the J-flow implies asymptotic Chow stability of $|L_2|$. Our definition of asymptotic Chow stability is motivated by Lejmi-Székelyhidi’s notion of J-stability (see Definition 4.15), which is an analogue of K-stability for the J-flow. Indeed we show that asymptotic Chow semistability of $|L_2|$ implies J-semistability, just as asymptotic Chow semistability of a polarised variety implies K-semistability.

Before discussing the notions of stability relevant to us, we recall some standard GIT as motivation. Let $G \subset SL(n + 1, \mathbb{C})$ act on a scheme $M \subset \mathbb{P}^n$, and let $x \in M$ have lift $\hat{x} \in \mathbb{C}^{n+1}$. We say the point x is *stable* in the sense of GIT if the orbit $G \cdot \hat{x}$ is closed and x has finite stabiliser. There is a weaker notion of polystability which allows higher dimensional stabiliser, but we will not need this. Roughly speaking, a quotient $M // G$ parameterising polystable orbits exists. To understand whether or not the point is represented in the quotient, one therefore seeks to understand the *stability* of the point.

The Hilbert-Mumford criterion gives a numerical way of checking this stability. Let $\lambda : \mathbb{C}^* \hookrightarrow G$ be a one-parameter subgroup. Denote the limit $x_0 = \lim_{t \rightarrow 0} \lambda(t) \cdot x$, so that x_0 is a fixed point of λ . There is therefore a \mathbb{C}^* -action on the line above x_0 , which acts with weight $w = -\mu(\lambda, x)$. By this we

mean that, for \hat{x}_0 any lift of x_0 , we have $\lambda(t).\hat{x}_0 = t^w \hat{x}_0$. Then the Hilbert-Mumford criterion states that x is stable if and only if $\mu(x, \lambda) > 0$ for all one-parameter subgroups λ . We will later need the following properties of the weight where one fixes a one-parameter subgroup λ and varies x .

Lemma 4.1. *Let M be an irreducible subvariety of \mathbb{P}^n . Fix a one-parameter subgroup $\lambda : \mathbb{C}^* \hookrightarrow G$. Then the weight $\mu(\lambda, x)$ is constant outside a Zariski closed subset of M . This general value is equal to the largest weight of any point in X .*

Proof. Both of these properties are consequences of the following alternative characterisation of the weight of a one-parameter subgroup. Diagonalise the \mathbb{C}^* -action as $\text{diag}(t^{\lambda_0} \cdots t^{\lambda_n})$ with $\lambda_0 \leq \cdots \leq \lambda_n$, and set $l = \min\{i \mid x_i \neq 0\}$. Then $\mu(\lambda, x) = -\lambda_l$. Remark that $-\lambda_l \geq -\lambda_{l+1} \geq \cdots \geq -\lambda_n$.

We now use the hypothesis that M is irreducible. The intersection of a hyperplane with the irreducible variety X is either empty, or of codimension one in M . The subset of M with weight *not equal* to λ_0 is the intersection of M with the hyperplane $\{x_0 = 0\}$; by irreducibility this is either empty or of codimension one. In the latter case, the general point of M has weight λ_0 and we are done. On the other hand if it is empty, we continue this process until we find the smallest i such that $X \cap \{x_i = 0\} \neq \emptyset$ (remark that M must intersect some hyperplane section). Then the general point of M will have weight $-\lambda_i$. By construction, $-\lambda_i$ is the largest weight of any point in M . □

Remark 4.2. It is essential here that M is irreducible; if $M = V(xy) \subset \mathbb{C}^2$ and λ is given as $\text{diag}(t^a, t^b)$, then clearly no such general weight exists for $a \neq b$.

One can apply GIT to *moduli spaces* of varieties. Our discussion roughly follows [40]. Let $Y \subset \mathbb{P}^n$ be a fixed subvariety. There are two natural moduli spaces in which one can naturally consider Y as a point. The first is the Hilbert scheme of subschemes of \mathbb{P}^n with the same Hilbert polynomial, which essentially parameterises ideal sheaves. The second is the Chow variety, whose definition uses intersection theory and compactifies by adding cycles at the boundary. Both moduli spaces naturally embed in certain Grassmanians constructed from \mathbb{P}^n , and so we will have corresponding GIT problems. To calculate stability of points in each moduli space, we will need to understand the line above a point under these embeddings into Grassmanians.

We first consider the Hilbert scheme, for which we refer to [20, Theorem 1.4] for the detailed construction. For our purposes we only need to understand how one obtains a point in a certain projective space from a given scheme, thus our construction omits many important details. The exact sequence of sheaves on \mathbb{P}^n

$$0 \rightarrow \mathcal{I}_Y \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_Y \rightarrow 0$$

induces by twisting with $\mathcal{O}_{\mathbb{P}^n}(1)$ an exact sequence for sufficiently large K

$$(19) \quad 0 \rightarrow H^0(\mathbb{P}^n, \mathcal{I}_Y(K)) \rightarrow S^K H^0(\mathbb{P}^n, \mathcal{O}(1)) \rightarrow H^0(Y, \mathcal{O}_Y(K)) \rightarrow 0.$$

If one fixes a Hilbert polynomial, one can choose K *independently* of $Y \subset \mathbb{P}^n$ (see for example [28, Lecture 14] or [20, Theorem 1.5]); in particular the previous sequence is exact for *all* such Y .

The Hilbert scheme of subschemes of \mathbb{P}^n with Hilbert polynomial $p(K) = \dim H^0(Y, \mathcal{O}_Y(K))$ is embedded via this exact sequence as a subscheme of $\text{Grass}(S^K H^0(\mathbb{P}^n, \mathcal{O}(1)), p(K))$, with the point corresponding to Y given by the vector subspace $H^0(\mathbb{P}^n, \mathcal{I}_Y(K))$. In turn these Grassmanians naturally embed into projective spaces using the Plücker embedding, explicitly we have embeddings

$$\text{Hilb} \hookrightarrow \text{Grass}(S^K H^0(\mathbb{P}^n, \mathcal{O}(1)), p(K)) \hookrightarrow \mathbb{P}(\Lambda^{p(K)}(S^K H^0(\mathbb{P}^n, \mathcal{O}(1)))).$$

The $SL(n+1, \mathbb{C})$ action on \mathbb{P}^n naturally induces an action on these projective spaces, and so for each K we have an associated GIT problem.

Definition 4.3. We say that $Y \subset \mathbb{P}^n$ is *Hilbert stable* if the corresponding point in the Hilbert scheme is GIT stable embedded as a subscheme of projective space using the Plücker embedding for $K \gg 0$.

We now turn to the Chow variety, for which we refer to [20, Section 1.3] for the detailed construction and definition. Again we only need to understand how, in the construction of the Chow variety, one associates a point in a certain projective space from a given scheme, so we omit many important details in our discussion. We denote by m the dimension of $Y \subset \mathbb{P}^n = \mathbb{P}(V)$ and d the degree of Y , i.e. $(\mathcal{O}_{\mathbb{P}^n}(1)|_Y)^m = d$. Let Z be the set of $(n-m-1)$ -dimensional planes intersecting Y nontrivially, so that $Z \subset \text{Grass}(n-m, n+1)$. We denote the Plücker embedding of this Grassmanian

as

$$(20) \quad Pl : \text{Grass}(n - m, n + 1) \hookrightarrow \mathbb{P}(\Lambda^{n-m}V).$$

One can show that Z is of codimension one in the Grassmanian, and that $Z = V(f)$ for some section $f \in H^0(\text{Grass}(n - m, n + 1), \mathcal{O}(d))$ unique up to scaling. One therefore has a corresponding point $[f] \in \mathbb{P}(H^0(\text{Grass}(n - m, n + 1), \mathcal{O}(d)))$, called the Chow point. The Chow variety has a natural compactification obtained by adding limit cycles at the boundary. Again the $SL(n + 1, \mathbb{C})$ action on \mathbb{P}^n induces one on $\mathbb{P}(H^0(\text{Grass}(n - m, n + 1), \mathcal{O}(d)))$.

Definition 4.4. We say $Y \subset \mathbb{P}^n$ is *Chow stable* if its Chow point $[f]$ is GIT stable under the induced action of $SL(n + 1, \mathbb{C})$.

The Hilbert-Mumford criterion states it is enough to show a corresponding weight is positive for each one-parameter subgroup $\lambda : \mathbb{C}^* \hookrightarrow SL(n + 1, \mathbb{C})$. Fixing some one-parameter subgroup λ , the limit $Y_0 = \lim_{t \rightarrow 0} \lambda(t).Y$ is naturally a point in the same Hilbert scheme and Chow variety as Y , which has a corresponding Hilbert-Mumford weight. Here one considers Y_0 as the limit scheme in the Hilbert scheme, and as the limit cycle in the Chow variety. Hence Y is Hilbert stable or Chow stable if and only if these weights are strictly positive for each one-parameter subgroup.

Considering instead a polarised variety (M, L_1) , one naturally has a sequence of embeddings $M \hookrightarrow \mathbb{P}(H^0(X, L_1^r)^*)$ for each r sufficiently large. Hence one can define asymptotic Hilbert stability (respectively asymptotic Chow stability) to mean the point corresponding to M is GIT stable in the appropriate Hilbert scheme (respectively Chow variety) for k sufficiently large. We will need an extension of the above discussion to subvarieties of M .

Definition 4.5 (Twisted stability). Let $Y \subset M$ be a subvariety. We say that Y is *M -twisted asymptotically Hilbert stable (respectively Chow stable)* if it is Hilbert stable (respectively Chow stable) under the natural embeddings $Y \hookrightarrow \mathbb{P}(H^0(M, L_1^r)^*)$ for all $r \gg 0$.

Remark 4.6. Clearly setting $Y = M$ recovers the usual definition of asymptotic Chow and Hilbert stability.

In what follows we will explicitly calculate the weight of the one-parameter subgroup $\mu(\lambda, Y)$ in each moduli space using data arising only

from the action on $\mathbb{P}(H^0(M, L_1^r)^*)$ itself. We first need the following definition, which is a geometrisation of the one-parameter subgroups considered above.

Definition 4.7. A *test configuration* $(\mathcal{X}, \mathcal{L})$ for a polarised variety (M, L_1) is a variety \mathcal{X} together with

- a proper flat morphism $\pi : \mathcal{X} \rightarrow \mathbb{C}$,
- a \mathbb{C}^* -action on \mathcal{X} covering the natural action on \mathbb{C} ,
- and an equivariant relatively very ample line bundle \mathcal{L} on \mathcal{X}

such that the fibre $(\mathcal{X}_t, \mathcal{L}_t)$ over $t \in \mathbb{C}$ is isomorphic to (M, L_1^r) for one, and hence all, $t \in \mathbb{C}^*$ and for some $r > 0$. We call r the *exponent* of the test configuration.

For each subvariety $Y \subset X$, by taking the closure of Y under the \mathbb{C}^* -action, one naturally obtains a test configuration for (Y, L_1) , where we have abused notation by writing L_1 as the restriction of L_1 to Y . We denote this induced test configuration by $(\mathcal{Y}, \mathcal{L})$, with similar abuse of notation.

Proposition 4.8. [40, Proposition 3.7] *A test configuration of exponent r is equivalent to a one-parameter subgroup of $GL(h(r), \mathbb{C})$, where $h(r)$ is the Hilbert polynomial of (M, L_1) . Given a one-parameter subgroup λ , the corresponding test configuration is $\mathcal{X}_t = \lambda(t).X$ with the line bundle the restriction of the $\mathcal{O}(1)$ from the projective space. Conversely, for each $K > 0$ one can equivariantly embed a given test configuration $(\mathcal{X}, \mathcal{L})$ of exponent r into $\mathbb{P}(H^0(X, L^{rK})^*)$ such that the test configuration is realised by some one-parameter subgroup.*

We are now in a position to give a numerical criterion for M -twisted asymptotic Hilbert and Chow stability. In order to do so, we introduce the following notation.

Let $(\mathcal{X}, \mathcal{L})$ be a test configuration. As the \mathbb{C}^* -action on $(\mathcal{X}, \mathcal{L})$ fixes the central fibre, there is an induced \mathbb{C}^* -action on $(\mathcal{X}_0, \mathcal{L}_0)$ and hence on $H^0(\mathcal{X}_0, \mathcal{L}_0^K)$ for each K . We denote the Hilbert polynomial and total weight of this action respectively by

$$\begin{aligned} h(K) &= a_0 K^n + a_1 K^{n-1} + O(K^{n-2}), \\ w(K) &= b_0 K^{n+1} + b_1 K^n + O(K^{n-1}). \end{aligned}$$

By asymptotic Riemann-Roch and flatness of the test configuration, we have intersection-theoretic formulas for a_0, a_1 , as

$$a_0 = r^n \frac{L^n}{n!}, \quad a_1 = r^{n-1} \frac{K_M \cdot L^{n-1}}{2(n-1)!}.$$

For $(\mathcal{Y}_0, \mathcal{L}_0)$ denote the corresponding Hilbert and weight polynomials by

$$\begin{aligned} \hat{h}(K) &= \hat{a}_0 K^m + O(K^{m-1}), \\ \hat{w}(K) &= \hat{b}_0 K^{m+1} + O(K^m), \end{aligned}$$

where m is the dimension of Y .

Theorem 4.9. *Let (M, L_1) be a polarised variety and $Y \subset M$ a subvariety. Then Y is M -twisted asymptotically Hilbert stable if and only if, for all $r \gg 0$, each test configuration of exponent r has normalised weight (setting $k = rK$)*

$$\hat{w}_{r,k} = \hat{w}(k)rh(r) - kw(r)\hat{h}(k) > 0$$

for $k \gg 0$.

The normalised weight $\hat{w}_{r,k}$ is a polynomial in k and r of degree $m + 1$ in k , write $\hat{w}_{r,k} = \sum_{i=0}^{m+1} e_i(r)k^i$. Then Y is M -twisted asymptotically Chow stable if and only if for all $r \gg 0$ we have $e_{m+1}(r) > 0$.

Proof. We first prove with the Hilbert stability statement. The Hilbert-Mumford criterion together with Proposition 4.8 imply X -twisted asymptotic Hilbert stability is equivalent to the asymptotic Hilbert weight of each test configuration being strictly positive, provided it lies in $SL(h(r), \mathbb{C})$ rather than $GL(h(r), \mathbb{C})$. The proof is in two steps. The first is to modify a fixed test configuration to lie in $SL(h(r), \mathbb{C})$, and the second is to explicitly calculate the Hilbert weight asymptotically.

For clarity we consider a more general situation. Let λ be a one-parameter subgroup of $GL(N, \mathbb{C})$ diagonalised as $\text{diag}(t^{\lambda_0} \cdots t^{\lambda_N})$, with total weight $w = \sum_i \lambda_i$. If $w = 0$, then λ lies in $SL(N, \mathbb{C})$; assume this is not the case. Then one gets a new one-parameter subgroup by subtracting $\frac{w}{N}$ from each weight, provided each $\lambda_i - \frac{w}{N}$ remains an integer. As this is not necessarily the case, we first multiply each λ_i by N to produce a new one-parameter subgroup with total weight wN , and subtract w from each weight to produce a one-parameter subgroup of $SL(N, \mathbb{C})$.

There are two natural operations one can perform on a test configuration which leave the total space \mathcal{X} intact but modifies the weight. The first is

pulling back the test configuration over a finite map $\mathbb{C} \rightarrow \mathbb{C}$ given by the map $t \rightarrow t^d$, which modifies the total weight $w(k)$ by multiplication by d . The second is by adding a constant τ to the weights, which modifies the total weight by adding $\tau kh(k)$. We use these operations to ensure the test configuration induces one-parameter subgroups of $SL(h(r), \mathbb{C})$, following the previous paragraph. Firstly, we pull-back over a finite map $t \rightarrow t^{rh(r)}$, so that the total weight is $w(r)rh(r)$. Next we add $-rw(r)$ to the weights, ensuring the test configuration now lies in $SL(h(r), \mathbb{C})$.

From the exact sequence (19), the line above the limit scheme \mathcal{Y}_0 is given as

$$\Lambda^{max} H^0(\mathcal{Y}_0, \mathcal{L}_0^K) \otimes \Lambda^{max} S^K H^0(M, L_1^r)^*.$$

Since the one-parameter subgroup we consider lies in $SL(h(r))$, the induced action on $\Lambda^{max} S^K H^0(X, L^r)^*$ has weight zero. We therefore need only calculate the change in weight for the action on $\Lambda^{max} H^0(\mathcal{Y}_0, \mathcal{L}_0^K)$. The finite cover $t \rightarrow t^{rh(r)}$ of the test configuration modifies this weight by $\hat{w}(rK) \rightarrow \hat{w}(rK)rh(r)$, while adding $-rw(r)$ to the weights modifies the weight by

$$\hat{w}(rK)rh(r) \rightarrow \hat{w}(rK)rh(r) - (rw(r))K\hat{h}(rK).$$

We conclude by recalling $k = rK$.

The Chow statement follows since for *fixed* r , the Chow weight is the leading order term in the polynomial which determines the Hilbert weight [29, Lemma 2.11] [40, Theorem 3.9]. □

Remark 4.10. Theorem 4.9 extends the well known criterion for asymptotic Hilbert and Chow stability obtained by setting $M = Y$, see for example [40, Theorem 3.9] whose proof we have followed above. The reason that we have discussed Hilbert stability in this section is primarily its use in the above proof. Indeed, to show the X -twisted asymptotic Chow weight is a polynomial, we first showed the X -twisted asymptotic Hilbert weight was a polynomial and used that the Chow weight is the leading order term in the Hilbert weight.

We now focus on the setting relevant to the J-flow, which is a minor modification of the above. Here we fix a polarised variety (M, L_1) together with an auxiliary ample line bundle L_2 . The notion of stability we then define is not a bona fide GIT notion, instead we formally adapt the definition of stability using the Hilbert-Mumford criterion to incorporate the linear

system associated to L_2 . For the moment we assume that $|L_2|$ is an arbitrary linear system.

Lemma 4.11. *Let $M \subset \mathbb{P}^n$ be a projective variety together with a linear system $|L_2|$. For each one-parameter subgroup $\lambda \hookrightarrow SL(n + 1, \mathbb{C})$, the Chow weight of λ for $D \in |L_2|$ is constant outside a Zariski closed subset of $|L_2|$. We define the Chow weight of λ for $|L_2|$ to equal this general value.*

Proof. The Chow weight is the GIT weight of the one-parameter subgroup in an appropriate Chow variety. Note that the linear system $|L_2|$ is a projective space, hence irreducible, and the image of an irreducible variety is irreducible. Therefore Lemma 4.1 implies the Chow weight of λ for $D \in |L_2|$ is constant outside a Zariski closed subset of $|L_2|$, as required. □

Remark 4.12. To the authors' knowledge, this phenomenon was first noticed by Székelyhidi in the study of twisted Kähler-Einstein metrics [45].

Definition 4.13. Let $M \subset \mathbb{P}^n$ be a projective variety together with a linear system $|L_2|$. We say that $|L_2|$ is *Chow stable* if for each one-parameter subgroup $\lambda \hookrightarrow SL(n + 1, \mathbb{C})$, the Chow weight of λ for $|L_2|$ is strictly positive.

This immediately implies that the *asymptotic* Chow weight of a linear system for a fixed test configuration is constant outside a Zariski closed subset of $|L_2|$. Note that, as the M -twisted asymptotic Chow weight for a fixed divisor D is a polynomial in r , it is determined by finitely many values, and hence its general value is also constant outside a Zariski closed subset of $|L_2|$.

Definition 4.14. We say that a linear system $|L_2|$ is *asymptotically Chow stable* if for all $r \gg 0$, the asymptotic Chow weight of each test configuration of exponent r is strictly positive. This is equivalent to the linear system being Chow stable in $\mathbb{P}(H^0(M, L_1^r)^*)$ for $r \gg 0$.

The following definition is equivalent to one due to Lejmi-Székelyhidi [23]. Indeed our definition of Chow stability of a linear system is motivated by their work.

Definition 4.15. We define the J-weight $J_{L_2}(\mathcal{X}, \mathcal{L})$ of a test configuration $(\mathcal{X}, \mathcal{L})$ to equal $1/a_0$ times the leading order term of the asymptotic Chow

weight of the linear system $|L_2|$. We say that (M, L_1, L_2) is *J-stable* (resp. *J-semistable*) if $J_{L_2}(\mathcal{X}, \mathcal{L}) > 0$ (resp. ≥ 0) for all non-trivial test configurations with normal total space. Explicitly, the J-weight of a test configuration is

$$J_{L_2}(\mathcal{X}, \mathcal{L}) = \frac{\hat{b}_0 a_0 - b_0 \hat{a}_0}{a_0}.$$

Indeed, one obtains this by simply expanding the polynomial $e_{m+1}(r)$ defined in Theorem 4.9; the J-weight is simply $1/a_0$ times the leading order term in r of this polynomial.

Remark 4.16. Our assumption that the total space \mathcal{X} of the test configuration is normal is to exclude certain pathological test configurations which normalise to the trivial test configuration, and necessarily have J-weight zero. These test configurations are alternatively characterised as having *norm* zero [13, Theorem 1.3] [4].

For comparison we state the definition of K-stability.

Definition 4.17. Let (X, L) be a polarised normal variety. We define the Donaldson-Futaki invariant of a test configuration to be (a positive constant times) the leading order term in its asymptotic Chow weight. Explicitly, we define

$$\text{DF}(\mathcal{X}, \mathcal{L}) = \frac{b_0 a_1 - b_1 a_0}{a_0}.$$

We say that (X, L) is K-stable if $\text{DF}(\mathcal{X}, \mathcal{L}) > 0$ for all non-trivial test configurations with normal total space.

Proposition 4.18. *Assume the linear system $|L_2|$ is asymptotically Chow semistable. Then (M, L_1, L_2) is J-semistable.*

Proof. This is immediate as the J-weight of a test configuration is the leading order term in the asymptotic Chow weight for the linear system $|L_2|$. \square

4.1. J-balanced metrics and Chow stability

In this section we work with a smooth projective variety M embedded in a fixed projective space \mathbb{P}^n , and a very ample line bundle L_2 . By Bertini's Theorem, since $|L_2|$ is basepoint free, a general element $D \in |L_2|$ is smooth. The main result of this section is the following.

Theorem 4.19. *If (M, L_1, L_2) admits a J-balanced metric, then $|L_2|$ is Chow stable.*

We conjecture that the two are actually equivalent, however technical issues prevent us from proving this. From Corollary 2.21, we can relate the existence of solutions of the J-flow to algebraic geometry as follows.

Corollary 4.20. *Suppose there exists a fixed point of the J-flow, i.e a critical metric solution of (2). Then $|L_2|$ is asymptotically Chow stable.*

Our proof is reminiscent of the Kempf-Ness Theorem. The existence of a J-balanced embedding for (M, L_1, L_2) is equivalent to the properness of a certain functional on a space of metrics. Taking a one-parameter subgroup as in the definition of Chow stability of a linear system, we show that properness of the this functional along this one-parameter subgroup forces the Chow weight to be strictly positive.

To prove Theorem 4.19, we need some preliminary results.

Theorem 4.21. *[23] Let M be a smooth projective n -dimensional variety together with a very ample line bundle L_2 . Let $\alpha \in c_1(L_2)$ be a positive $(1, 1)$ -form. Then there is a smooth signed measure μ on the projective space $|L_2|$ such that*

$$\alpha = \int_{D \in |L_2|} [D] d\mu$$

holds in the weak sense, i.e. for all smooth $(n - 1, n - 1)$ -forms β we have

$$\int_M \alpha \wedge \beta = \int_{D \in |L_2|} \left(\int_D \beta \right) d\mu.$$

The above Theorem allows us to replace our smooth $(1, 1)$ -form χ with an integral over the linear system $|L_2|$. To work with the relevant functional for a fixed divisor, we use a result due to Phong-Sturm [37]. In order to state this result, we define the corresponding functional and a certain norm on sections of line bundles over Grassmannians.

Definition 4.22. Consider $D \in |L_2|$ a smooth divisor in M . We define the functional I_D^{AYM} over the space of $Met(L_1)$ variationally by

$$\frac{d}{dt} I_D^{\text{AYM}}(\phi(t)) = -\frac{1}{\text{Vol}_{L_1}(D)} \int_D \dot{\phi}_t \omega_{\phi_t}^{n-1},$$

taking the value zero on $\phi = 0$. Here

$$\text{Vol}_{L_1}(D) = \int_D c_1(L_1)^{n-1} = \int_M c_1(L_1)^{n-1} \cdot c_1(L_2)$$

is the volume of D .

Comparing this to the functional relevant to the existence of J-balanced metric (see Section 3.1), defined variationally by

$$(21) \quad \frac{d}{dt} J_\chi(\phi(t)) = \frac{1}{\gamma} \int_M \dot{\phi}(t) \chi \wedge \omega_\phi^{n-1},$$

where $\chi \in c_1(L_2)$ is a smooth positive $(1, 1)$ -form, we obtain the following Corollary of Theorem 4.21.

Lemma 4.23. *The J-balancing functional satisfies*

$$J_\chi(\phi) = -\text{Vol}_{L_1}(M) \int_{D \in |L_2|} I_D^{\text{AYM}}(\phi) d\mu,$$

where $d\mu$ is chosen as in Theorem 4.21.

Here we may integrate only over the smooth elements $D \in |L_2|$, since the complement has measure zero (by Bertini’s Theorem) this does not affect the value of the integral.

In the previous section, in defining stability of projective varieties, we have fixed a basis of projective space and acted on the variety itself by one-parameter subgroups. An alternative, but equivalent, point of view that we will now take up is to *fix a variety* and *vary the basis* of projective space.

Let $\omega = \omega_{FS}$ be the Fubini-Study metric of projective space. For $\sigma \in SL(n + 1, \mathbb{C})$, define

$$(22) \quad \phi_\sigma = \log \left(\frac{|\sigma \cdot x|^2}{|x|^2} \right),$$

and also

$$\omega_\sigma = \omega + i\partial\bar{\partial}\phi_\sigma.$$

In this way we can consider both $J_\chi(\phi)$ and $I_D^{\text{AYM}}(\phi)$ as functionals on $SL(n + 1, \mathbb{C})$. By abuse of notation, we denote these functionals as $J_\chi(\sigma)$ and $I_D^{\text{AYM}}(\sigma)$ for $\sigma \in SL(n + 1, \mathbb{C})$.

We now define a norm on the space of sections over certain Grassmannians, following Phong-Sturm.

Definition 4.24. [37, Section 4] Let $f \in H^0(\text{Grass}(n - m, n + 1), \mathcal{O}(d))$ be the Chow point of a degree d variety in \mathbb{P}^n . Denote by $\omega_{Gr} = Pl^*(\omega_{FS})$, where we recall from equation (20) that Pl is the Plücker embedding of the Grassmannian. We define the norm of f as

$$\log \|f\|^2 = \frac{1}{\text{Vol}(Gr)} \int_{Gr} \log \frac{|f(z)|^2}{|Pl(z)|^2} \omega_{Gr}^r,$$

where $\text{Vol}(Gr) = \int_{Gr} \omega_{Gr}^{m+1}$ is the volume of the Grassmannian and $r + 1 = (n - m)(m + 1)$ is its dimension.

Theorem 4.25. [37, Theorem 5] Let $D \subset \mathbb{P}^n$ be a smooth projective variety of dimension m . Denoting by f the Chow point of Y , we have, where V is the volume of D with respect to the Fubini-Study metric

$$(23) \quad -V(m + 1)I_D^{AYM}(\sigma) = \log \frac{\|\sigma \cdot f\|^2}{\|f\|^2}.$$

Corollary 4.26. Suppose the functional

$$-I_D^{AYM}(\sigma) : SL(n + 1, \mathbb{C}) \rightarrow \mathbb{R}$$

is proper. Let $\sigma(t) \hookrightarrow SL(n + 1, \mathbb{C})$ be a one-parameter subgroup. Then the Chow weight of D with respect to $\sigma(t)$ is strictly positive.

Proof. The properness of the functional means that $\|\sigma \cdot f\|^2 \rightarrow \infty$ as $\|\sigma\|^2 \rightarrow \infty$. Applying this to our one-parameter subgroup, this implies the orbit $\sigma_t \cdot f$ is closed.

To relate the closedness of the orbit to the Chow weight, we use the Hilbert-Mumford criterion. Let f_0 be the limit $\lim_{t \rightarrow \infty} \sigma_t \cdot f$, which is a fixed point of the one-parameter subgroup. Write \hat{f}_0 for a lift of f_0 . Defining w by $\lambda(t) \cdot \hat{x}_0 = t^w \hat{x}_0$, the Chow weight is $-w$. If $w > 0$, then 0 lies in the closure of the orbit of the one-parameter subgroup, hence the orbit is not closed. Similarly if $w = 0$ then the action is trivial on the line above x_0 , and so the orbit of a lift of x cannot be closed. We conclude that the orbit being closed implies the Chow weight is positive. □

We can now prove Theorem 4.19.

Proof of Theorem 4.19. By Lemma 4.23, we have

$$J_\chi(\phi) = -\text{Vol}_{L_1}(M) \int_{D \in |T|} I_D^{\text{AYM}}(\phi).$$

For $\sigma \in SL(n + 1, \mathbb{C})$ this implies

$$(24) \quad J_\chi(\sigma) = -\text{Vol}_{L_1}(M) \int_{D \in |L_2|} I_D^{\text{AYM}}(\sigma).$$

By hypothesis, the functional $J_\chi(\varphi)$ is proper. Fix a one-parameter subgroup $\sigma(t) \hookrightarrow SL(n + 1, \mathbb{C})$. By Lemma 4.1, the Chow weight of this one-parameter subgroup is constant outside a Zariski closed subset of $|L_2|$. In the integral over $|L_2|$ in equation (24), we ignore the union of the non-smooth D together with the subset on which the Chow weight is not equal to the general value. Remark this set has measure zero, and hence this does not affect the value of the integral.

For the one-parameter subgroup $\sigma(t)$, one of two things must therefore happen. Either the general Chow weight is positive, or the general Chow weight is non-positive. Remark that by Lemma 4.1, in the latter case the Chow weight is non-positive for *each* $D \in |L_2|$. If the general Chow weight is positive for each one-parameter subgroup, then by definition the linear system $|L_2|$ is Chow stable and we are done. We therefore wish to show that if each Chow weight is non-positive, the functional $J_\chi(\sigma(t))$ is not proper.

We now consider the latter case. We use the fact that a C^2 convex function $f(t) : \mathbb{R} \rightarrow \mathbb{R}$ is proper if and only if

$$\lim_{t \rightarrow \infty} \frac{d}{dt} f'(t) > 0,$$

where the value infinity is allowed. Applying this in our situation along the one-parameter subgroup $\sigma(t)$, by Corollary 4.26 we see that the Chow weight for each $D \in |L_2|$ being non-positive implies for smooth D that

$$(25) \quad - \lim_{t \rightarrow \infty} \frac{d}{dt} (I_D^{\text{AYM}})'(\sigma(t)) \leq 0.$$

We now relate the limit derivatives of the two functionals. For *fixed* t , we have

$$J_\chi(\sigma(t)) = -\text{Vol}_{L_1}(M) \int_{D \in |L_2|} I_D^{\text{AYM}}(\sigma(t)) d\mu$$

and thus $I_D^{\text{AYM}}(\sigma(t))$ is integrable. Furthermore, for all D , $t \mapsto I_D^{\text{AYM}}(\sigma(t))$ can be differentiated and we can express its derivative using the notation of

(22) as

$$-I_D^{\text{AYM}}(\sigma(t))' = \frac{1}{\text{Vol}_L(D)} \int_D \frac{x^* \sigma(t)^* (\zeta^* + \zeta) \sigma(t) x}{\|\sigma(t)x\|^2} \omega_{\sigma(t)}^{n-1}$$

where $x = (1, z_1, \dots, z_N)$ and as explained in [37]. Here $\sigma(t) = e^{\zeta t} \sigma_0$ with ζ traceless Hermitian. Consequently, if r_ζ is the spectral radius of ζ , we have

$$|I_D^{\text{AYM}}(\sigma(t))'| \leq 2r_\zeta \text{Vol}_L(D)$$

and $D \mapsto \text{Vol}_L(D)$ is integrable over $|L_2|$. Hence a variant of Lebesgue dominated convergence theorem asserts that

$$\frac{d}{dt} J_\chi(\sigma(t)) = -\text{Vol}_{L_1}(M) \int_{D \in |L_2|} \left(\frac{d}{dt} I_D^{\text{AYM}}(\sigma(t)) \right) d\mu.$$

Now, for almost all $D \in |L_2|$, $\frac{d}{dt}(I_D^{\text{AYM}})'(\sigma(t))$ converges simply when t tends to $+\infty$, so together with above properties, another variant of Lebesgue dominated convergence theorem gives

$$\begin{aligned} \lim_{t \rightarrow +\infty} \frac{d}{dt} J_\chi(\sigma(t)) &= -\lim_{t \rightarrow \infty} \text{Vol}_{L_1}(M) \int_{D \in |L_2|} \left(\frac{d}{dt} I_D^{\text{AYM}}(\sigma(t)) \right) d\mu, \\ &= -\text{Vol}_{L_1}(M) \int_{D \in |L_2|} \left(\lim_{t \rightarrow +\infty} \frac{d}{dt} I_D^{\text{AYM}}(\sigma(t)) \right) d\mu. \end{aligned}$$

Equation (25) then implies

$$\lim_{t \rightarrow \infty} \frac{d}{dt} J_\chi(\sigma(t)) \leq 0,$$

contradicting properness. This completes the proof. □

4.2. The J-flow and K-stability

In this section we study J-stability, as in Definition 4.15. In this definition one takes a projective variety M with two ample line bundles L_1, L_2 , and associates a weight called the J-weight $J_{L_2}(\mathcal{X}, \mathcal{L})$ to each test configuration $(\mathcal{X}, \mathcal{L})$. J-stability then requires that $J_{L_2}(\mathcal{X}, \mathcal{L}) > 0$ for each non-trivial test configuration with \mathcal{X} normal. In order to study J-stability, we reduce the class of test configurations needed to those which are blow-ups along flag ideals.

Remark 4.27. To clarify notation in this section, for a line bundle L we denote rL its r th tensor power. We denote intersection numbers as $L^n = \int_M c_1(L)^n$ and so on. We also use additive notation for the tensor product of line bundles.

Definition 4.28. A flag ideal \mathcal{I} is a coherent ideal sheaf on $M \times \mathbb{C}$ of the form $\mathcal{I} = I_0 + (t)I_1 + \dots + (t^N)$, where t is the coordinate on \mathbb{C} and $I_0 \subset \dots \subset I_{N-1} \subset \mathcal{O}_M$ are a sequence of coherent ideal sheaves on M corresponding to subschemes $Z_0 \supset Z_1 \supset \dots \supset Z_{N-1}$ of M .

Denote by $\tilde{\mathcal{B}}$ the blow-up of \mathcal{I} on $X \times \mathbb{C}$, i.e.

$$\pi : \tilde{\mathcal{B}} = \text{Bl}_{\mathcal{I}} M \times \mathbb{C} \rightarrow M \times \mathbb{C},$$

denote by $\mathcal{L}_1, \mathcal{L}_2$ the pullbacks of L_1, L_2 to $\tilde{\mathcal{B}}$ and let $\mathcal{O}(-E) = \pi^{-1}\mathcal{I}$ be the exceptional divisor of the blow-up. The map $\tilde{\mathcal{B}} \rightarrow \mathbb{C}$ is flat, and the natural \mathbb{C}^* -action on $X \times \mathbb{C}$ lifts to $\tilde{\mathcal{B}}$. It follows that $(\tilde{\mathcal{B}}, r\mathcal{L}_1 - E)$ is a test configuration of exponent r for (X, L_1) provided $r\mathcal{L}_1 - E$ is relatively ample. Remark that each such test configuration has a canonical compactification obtained by blowing up \mathcal{I} on $M \times \mathbb{P}^1$; we denote this test configuration by \mathcal{B} and by abuse of notation denote $\mathcal{L}_1, \mathcal{L}_2, E$ the corresponding line bundles and divisors on \mathcal{B} . The following Proposition then states that it is enough to check J-stability with respect to these test configurations, provided one allows $r\mathcal{L}_1 - E$ to be relatively *semi*-ample.

Proposition 4.29. *Let M be a normal projective variety with ample line bundles L_1, L_2 . Then (M, L_1, L_2) is J-stable is equivalent to the fact that*

$$J_{L_2}(\mathcal{B}, r\mathcal{L}_1 - E) > 0$$

for all flag ideals \mathcal{I} with $\mathcal{B} = \text{Bl}_{\mathcal{I}} M \times \mathbb{P}^1$ normal and $r\mathcal{L}_1 - E$ relatively semi-ample over \mathbb{P}^1 . Recall the J-constant of (M, L_1, L_2) is defined as

$$\gamma = \frac{L_2 \cdot L_1^{n-1}}{L_1^n}.$$

On such blow-ups the J-weight defined by Definition 4.15 is given by the formula

$$(26) \quad J_{L_2}(\mathcal{B}, r\mathcal{L}_1 - E) = (r\mathcal{L}_1 - E)^n \cdot \left(-\frac{n}{n+1} \gamma r^{-1} (r\mathcal{L}_1 - E) + \mathcal{L}_2 \right),$$

up to multiplication by a positive dimensional constant.

Proof. By [32, Corollary 3.11], given an arbitrary test configuration $(\mathcal{X}, \mathcal{H})$, there exists a flag ideal \mathcal{I} and a \mathbb{C}^* -equivariant map

$$\psi : (\tilde{\mathcal{B}}, r\mathcal{L}_1 - E) \rightarrow (\mathcal{X}, \mathcal{H})$$

with $\tilde{\mathcal{B}}$ normal and $r\mathcal{L}_1 - E = \psi^*\mathcal{H}$ relatively semi-ample over \mathbb{C} . Since the map ψ is \mathbb{C}^* -equivariant, the total weight $\text{wt } H^0(\mathcal{X}_0, k\mathcal{H}_0)$ is equal to the total weight $\text{wt } H^0(\tilde{\mathcal{B}}_0, k(r\mathcal{L}_1 - E)_0)$. Similarly given a divisor $D \subset M$, denote by $\mathcal{D} \subset \mathcal{X}$ the closure under the \mathbb{C}^* -action. If one denotes by $\tilde{\mathcal{B}}_D$ the proper transform of $D \times \mathbb{C}$, then similarly by [35, Proposition 3.5] we have

$$\text{wt } H^0(\mathcal{D}_0, k\mathcal{H}_0) = \text{wt } H^0(\tilde{\mathcal{B}}_{D,0}, k(r\mathcal{L}_1 - E)|_0).$$

To obtain a formula for the J-weight, we need to calculate the b_0 and \hat{b}_0 , where the latter is for a general divisor $D \in |T|$. We first of all compactify the above semi-test configuration as detailed above. We denote by \mathcal{B} and \mathcal{B}_D the corresponding test configurations over \mathbb{P}^1 . For the b_0 term, by [32, Theorem 3.2] or [48, Proposition 17] we have

$$\begin{aligned} \text{wt}(H^0(\mathcal{B}_0, k(r\mathcal{L}_1 - E)|_0^k)) &= \chi(\mathcal{B}, k(r\mathcal{L}_1 - E)) + O(k^{n-1}), \\ &= \frac{(r\mathcal{L}_1 - E)^{n+1}}{(n + 1)!} k^{n+1} + O(k^n), \end{aligned}$$

using asymptotic Riemann-Roch for normal varieties [32, Lemma 3.5]. Similarly for arbitrary $D \in |L_2|$ we have

$$\begin{aligned} \text{wt}(H^0(\mathcal{B}_{D,0}, k(r\mathcal{L}_1 - E)|_0)) &= \chi(\mathcal{B}_D, k(r\mathcal{L}_1 - E)) + O(k^{n-2}), \\ &= \frac{(r\mathcal{L}_1 - E)^n \cdot \mathcal{B}_D}{n!} k^n + O(k^{n-1}), \\ &= \frac{(r\mathcal{L}_1 - E)^n \cdot (\mathcal{L}_2 + (\mathcal{B}_D - \mathcal{L}_2))}{n!} k^n + O(k^{n-1}). \end{aligned}$$

Here \mathcal{L}_2 is the line bundle associated to the *total* transform of $D \times \mathbb{P}^1$, while \mathcal{B}_D corresponds to the *proper* transform. Remark that these are equal for general D , indeed they are not equal if and only if the flag \mathcal{I} has a component contained in D . It follows that

$$\hat{b}_0 = \frac{(r\mathcal{L}_1 - E)^n \cdot \mathcal{L}_2}{n!}.$$

Summing up, we see that

$$(n + 1)!J_{L_2}(\mathcal{B}, r\mathcal{L}_1 - E) = (r\mathcal{L}_1 - E)^n \cdot \left(-\frac{n}{n + 1}\gamma r^{-1}(r\mathcal{L}_1 - E) + \mathcal{L}_2 \right),$$

using

$$\gamma = nr \frac{\hat{a}_0}{a_0}. \quad \square$$

Using the above, we can extend the definition of J-stability to the case where L_2 is an arbitrary line bundle, which will be useful in applications. By Proposition 4.29, it follows that when L_2 is ample, this definition is equivalent to Definition 4.15.

Definition 4.30. Let (M, L_1, L_2) be a normal projective variety M with an ample line bundle L_1 and an auxiliary line bundle L_2 , not necessarily ample. We say that (M, L_1, L_2) is *J-stable* if for each flag ideal \mathcal{I} , the corresponding J-weight given in equation (26) is strictly positive.

In order to use this formalism, we need the following positivity properties of certain intersection numbers.

Lemma 4.31. [34, Proposition 4.3, Theorem 2.6] [31, Equation (3)] [11, Lemma 3.7]

With all notation as above, let R be a nef divisor on M , and denote $p^*R = \mathcal{R}$ where $p : \mathcal{B} \rightarrow M$ is the natural morphism induced by the blow-up map. Then the following positivity properties of intersection numbers hold.

- (i) $(r\mathcal{L}_1 - E)^n \cdot \mathcal{R} \leq 0$,
- (ii) $(r\mathcal{L}_1 - E)^n \cdot E > 0$,
- (iii) $(r\mathcal{L}_1 - E)^n \cdot (r\mathcal{L}_1 + nE) > 0$.

We can now apply this blowing-up formalism.

Theorem 4.32. Suppose that $\gamma L_1 - L_2$ is nef, with $\gamma > 0$. Then (M, L_1, L_2) is J-stable.

Proof. We use the blowing-up formalism of Proposition 4.29. Let \mathcal{I} be a flag ideal with corresponding blow-up \mathcal{B} . The J-weight is given as

$$\begin{aligned} J_{L_2}(\mathcal{B}, r\mathcal{L}_1 - E) &= (r\mathcal{L}_1 - E)^n \cdot \left(-\frac{n}{n+1} \gamma r^{-1} (r\mathcal{L}_1 - E) + \mathcal{L}_2 \right), \\ &= (r\mathcal{L}_1 - E)^n \cdot \left(\frac{1}{n+1} \gamma r^{-1} (r\mathcal{L}_1 + nE) + (-\gamma\mathcal{L}_1 + \mathcal{L}_2) \right). \end{aligned}$$

By Lemma 4.31 (iii), we have

$$(r\mathcal{L}_1 - E)^n \cdot (r\mathcal{L}_1 + nE) > 0,$$

while Lemma 4.31 (i) together with the assumption that $\gamma L_1 - L_2$ is nef gives

$$(r\mathcal{L}_1 - E)^n \cdot (-\gamma\mathcal{L}_1 + \mathcal{L}_2) \geq 0.$$

Combining these proves the result. □

Remark 4.33. Since $\gamma = \frac{L_1^{n-1} \cdot L_2}{L_1^n}$, the assumption $\gamma > 0$ is automatic when L_2 is ample, or even effective. The above result holds for general L_2 however using Definition 4.30, provided $\gamma > 0$.

Remark 4.34. In the case L_2 is ample, a result of Weinkove [50] states that if

$$\gamma L_1 - \frac{n-1}{n} L_2$$

is also ample, then the I_{μ_J} functional is bounded. Note that this is a weaker assumption than we made in Theorem 4.32; however we do not assume L_2 is ample. It would be interesting to directly prove J-stability under Weinkove’s hypothesis.

We now prove a link between J-stability and K-stability (as in Definition 4.17). This is the algebro-geometric analogue of the relationship between existence of solutions to the J-flow and coercivity of the Mabuchi functional, due to Chen [7]. Our result will make use of certain measures of singularities of projective varieties.

Definition 4.35. Let M be a normal \mathbb{Q} -Gorenstein variety. Let $\pi : Y \rightarrow M$ be an arbitrary birational map with Y normal. We can then write

$$K_Y - \pi^*K_M \equiv \sum a(E_i, M)E_i.$$

We say that M is *Kawamata log terminal* if $a(E_i, M) > -1$ for all E_i with M normal. By [21, Lemma 3.13] it suffices to check this property for $Y \rightarrow M$ a resolution of singularities. In particular, smooth varieties are Kawamata log terminal.

Theorem 4.36. *Suppose (M, L_1, K_M) is J -semistable, with M Kawamata log terminal. Then (M, L_1) is K -stable.*

Proof. A result of Odaka [32, Corollary 3.11] states that, similarly to Proposition 4.29, to check K -stability it suffices to show the Donaldson-Futaki invariant of each semi-test configuration given in Proposition 4.29 is strictly positive. The proof is a comparison of the Donaldson-Futaki invariant and J -weight. Indeed, letting \mathcal{B} be a blow-up along a flag ideal as above, we have

$$J_{K_M}(\mathcal{B}, r\mathcal{L}_1 - E) = (r\mathcal{L}_1 - E)^n \cdot \left(-\frac{n}{n+1} \gamma r^{-1} (r\mathcal{L}_1 - E) + \mathcal{K}_M \right),$$

while the corresponding Donaldson-Futaki invariant is given by Odaka as

$$\text{DF}(\mathcal{B}, r\mathcal{L}_1 - E) = (r\mathcal{L}_1 - E)^n \cdot \left(-\frac{n}{n+1} \gamma r^{-1} (r\mathcal{L}_1 - E) + \mathcal{K}_M + K_{\mathcal{B}/M \times \mathbb{P}^1} \right).$$

Here we have denoted \mathcal{K}_M the pullback of K_M to \mathcal{B} . Hence

$$\text{DF}(\mathcal{B}, r\mathcal{L}_1 - E) = J_{K_M}(\mathcal{B}, \mathcal{L}_1^r - E) + (r\mathcal{L}_1 - E)^n \cdot K_{\mathcal{B}/M \times \mathbb{P}^1}.$$

The term $K_{\mathcal{B}/M \times \mathbb{P}^1} = K_{\mathcal{B}} - \pi^*K_{M \times \mathbb{P}^1}$ is an exceptional divisor of the blow-up $\pi : \mathcal{B} \rightarrow M \times \mathbb{P}^1$, since \mathcal{B} is normal the intersection numbers make sense. Hence to show K -stability, by Lemma 4.31 (ii), it suffices to show that $K_{\mathcal{B}/M \times \mathbb{P}^1}$ is effective. This follows by inversion of adjunction [22, Theorem 5.50]. Indeed, M being Kawamata log terminal implies that $X \times \mathbb{P}^1$ is purely log terminal, hence the discrepancy term is effective. \square

The following Corollary is our motivation for extending the Definition of J -stability to the case with L_2 not necessarily ample.

Corollary 4.37. *Suppose that (M, L_1) is a Kawamata log terminal variety with $\gamma > 0$ which satisfies*

$$\gamma L_1 - K_M \geq 0,$$

i.e. the difference is nef. Then (M, L_1) is K-stable.

Proof. This is immediate by combining Theorems 4.32 and 4.36. □

Remark 4.38. This Corollary should be compared to Weinkove’s work [50] in the smooth case with K_M ample, which proves that the Mabuchi functional for (M, L_1) is proper provided

$$\gamma L_1 - \frac{n-1}{n} K_M$$

is ample. While we need a stronger ampleness criterion for L_1 , we do *not* need to assume K_M is ample, merely that $\gamma > 0$.

Remark 4.39. Corollary 4.37 was also obtained by the first author through a direct analysis of the K-stability condition [13, Theorem 1.7]. When $L_1 = K_M$, i.e. M is a canonically polarised variety, this result is due to Odaka [31].

When M is a surface we can improve these results, using another intersection theoretic Lemma. This is essentially a strengthening of Lemma 4.31 (iii) for surfaces.

Lemma 4.40. *Suppose M has dimension 2. Then*

$$(r\mathcal{L}_1 - E)^2 \cdot (r\mathcal{L}_1 + E) \geq 0.$$

Proof. The intersection number expands as

$$(r\mathcal{L}_1 - E)^2 \cdot (r\mathcal{L}_1 + E) = r^3 \mathcal{L}_1^3 - r^2 \mathcal{L}_1^2 \cdot E - r\mathcal{L}_1 \cdot E^2 + E^3.$$

As \mathcal{L} is the pullback of a line bundle from the surface M , we have $\mathcal{L}_1^3 = 0$.

Recall that the flag ideal is of the form $\mathcal{I} = I_0 + (t)I_1 + \dots + (t^N)$. As in the proof of [31, Theorem 2.6], we can assume that the flag ideal has support

$$s = \dim \text{Supp}(\mathcal{O}_{M \times \mathbb{P}^2} / \mathcal{I}) \leq 1.$$

Indeed otherwise, the flag idea satisfies $I_0 = \mathcal{O}_M$ and dividing by a power of t does not change the blow-up and hence the Donaldson-Futaki invariant,

but ensures $s \leq 1$. Then provided $s \leq 1$, we have $\mathcal{L}_1^2.E = 0$ by the projection formula, as in [33, Lemma 3.5].

It follows that

$$(27) \quad (r\mathcal{L}_1 - E)^2.(r\mathcal{L}_1 + E) = -r\mathcal{L}_1.E^2 + E^3,$$

$$(28) \quad \quad \quad = -E^2.(r\mathcal{L}_1 - E).$$

But this latter term is non-negative by [31, Lemma 2.8 (i)] (setting $i = 1$ in the notation of that Lemma), concluding the proof. \square

Using the above Lemma we can strengthen Theorem 4.32 for surfaces.

Theorem 4.41. *Suppose M has dimension 2 satisfying $\frac{4}{3}\gamma L_1 - L_2$ is nef, and $\gamma > 0$. Then (M, L_1, L_2) is J-semistable.*

Proof. We follow the proof of Theorem 4.32 using Lemma 4.40. Letting \mathcal{I} be a flag ideal as above, the J-weight is given as

$$\begin{aligned} J_{L_2}(\mathcal{B}, r\mathcal{L}_1 - E) &= (r\mathcal{L}_1 - E)^2. \left(-\frac{2}{3}\gamma r^{-1}(r\mathcal{L}_1 - E) + \mathcal{L}_2 \right), \\ &= (r\mathcal{L}_1 - E)^2. \left(\frac{2}{3}\gamma r^{-1}(r\mathcal{L}_1 + E) + \left(-\frac{4}{3}\gamma\mathcal{L}_1 + \mathcal{L}_2\right) \right) \end{aligned}$$

By Lemma 4.40 we have

$$(r\mathcal{L}_1 - E)^2.(r\mathcal{L}_1 + E) \geq 0,$$

while Lemma 4.31 (i) together with the hypothesis of the Theorem ensures

$$(r\mathcal{L}_1 - E)^2. \left(-\frac{4}{3}\gamma\mathcal{L}_1 + \mathcal{L}_2 \right) \geq 0. \quad \square$$

Remark 4.42. It is important to note that in the above result we only prove J-semistability, it would be interesting to prove J-stability assuming the line bundle $\frac{4}{3}\gamma L_1 - L_2$ is actually ample. This does not follow directly from our method.

We immediately obtain the following Corollary, by Theorem 4.36. This is the most general currently known result for K-stability of surfaces of general type.

Corollary 4.43. *Let (M, L_1) be a polarised Kawamata log terminal surface satisfying $\frac{4}{3}\gamma L_1 - K_M$ is nef and $\gamma > 0$. Then (M, L_1) is K-stable.*

Remark 4.44. Panov and Ross have proved slope stability of surfaces with ample canonical class under the weaker assumption $2\gamma L_1 - K_M$ is ample [36, Example 5.8]. However by [36, Example 7.8], slope stability is a strictly weaker condition than K-stability, i.e. K-stability implies slope stability but there are examples of slope stable polarised varieties which are not K-stable.

Corollaries 4.43 and 4.37, together with the link between J-stability and the I_{μ_J} functional lead us to the following conjecture. We remark that the definition of the I_{μ_J} functional makes sense with L_2 arbitrary, taking a not necessarily positive $(1, 1)$ -form $\chi \in c_1(L_2)$.

Conjecture 4.1. *Let M be a smooth n -dimensional variety with an ample line bundle L_1 and an arbitrary line bundle L_2 . Suppose $\gamma > 0$ and*

$$\gamma L_1 - \frac{n-1}{n} L_2$$

is ample. Then the I_{μ_J} functional is proper. In particular if $L_2 = K_M$, then (X, L_1) has proper Mabuchi functional.

We emphasise again that in the above Conjecture we do *not* assume L_2 is positive just that $\gamma > 0$. In the case L_2 is positive the above Conjecture follows from Weinkove’s work [50]. In the case of minimal surface of general type (i.e with big and nef canonical bundle), the Conjecture is proved in [44].

References

- [1] R. Berman and J. Keller, *Bergman geodesics*, in: Complex Monge-Ampère equations and geodesics in the space of Kähler metrics, Vol. 2038 of Lecture Notes in Math., pp. 283–302, Springer, Heidelberg (2012).
- [2] B. Berndtsson, *Positivity of direct image bundles and convexity on the space of Kähler metrics*, J. Differential Geom. **81** (2009), no. 3, 457–482.
- [3] T. Bouche, *Convergence de la métrique de Fubini-Study d’un fibré linéaire positif*, Ann. Inst. Fourier (Grenoble) **40** (1990), no. 1, 117–130.
- [4] S. Boucksom, T. Hisamoto, and M. Jonsson, *Uniform K-stability, Duistermaat-Heckman measures and singularities of pairs*, arXiv: math.AG/1504.06568, (2015).

- [5] H.-D. Cao and J. Keller, *On the Calabi problem: a finite-dimensional approach*, J. Eur. Math. Soc. (JEMS) **15** (2013), no. 3, 1033–1065.
- [6] D. Catlin, *The Bergman kernel and a theorem of Tian*, in: Analysis and Geom. in Several Complex Var. (Katata, 1997), Trends Math., Birkhäuser (1999).
- [7] X. X. Chen, *On the lower bound of the Mabuchi energy and its application*, Internat. Math. Res. Notices (2000), no. 12, 607–623.
- [8] X. X. Chen, *A new parabolic flow in Kähler manifolds*, Comm. Anal. Geom. **12** (2004), no. 4, 837–852.
- [9] X. X. Chen, *On the existence of constant scalar curvature Kähler metric: a new perspective*, arXiv:math.DG/1506.06423, (2015).
- [10] T. Collins and G. Székelyhidi, *Convergence of the J-flow on toric manifolds*, arXiv:math.DG/1412.4809, to appear in J. Differential Geom. (2014).
- [11] R. Dervan, *Alpha invariants and K-stability for general polarizations of Fano varieties*, Int. Math. Res. Not. IMRN (2015), no. 16, 7162–7189.
- [12] R. Dervan, *Alpha invariants and coercivity of the Mabuchi functional on Fano manifolds*, Ann. Fac. Sci. Toulouse Math. (6) **25** (2016), no. 4, 919–934.
- [13] R. Dervan, *Uniform stability of twisted constant scalar curvature Kähler metrics*, Int. Math. Res. Not. IMRN (2016), no. 15, 4728–4783.
- [14] S. K. Donaldson, *Moment maps and diffeomorphisms*, Asian J. Math. **3** (1999), no. 1, 1–15. Sir Michael Atiyah: A Great Mathematician of the Twentieth Century.
- [15] S. K. Donaldson, *Scalar curvature and projective embeddings. I*, J. Differential Geom. **59** (2001), no. 3, 479–522.
- [16] S. K. Donaldson, *Scalar curvature and projective embeddings. II*, Q. J. Math. **56** (2005), no. 3, 345–356.
- [17] J. Fine, *Calabi flow and projective embeddings*, J. Differential Geom. **84** (2010), no. 3, 489–523. With an appendix by Kefeng Liu and Xiaonan Ma.
- [18] P. Gauduchon, *Calabi’s extremal Kähler metrics: An elementary introduction.*, preprint, 316 pages.

- [19] V. Georgoulas, J. W. Robbin, and D. A. Salamon, *The moment-weight inequality and the Hilbert-Mumford criterion*, [arXiv:math.SG/1311.0410](https://arxiv.org/abs/1311.0410), (2013).
- [20] J. Kollár, *Rational Curves on Algebraic Varieties*, Vol. 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*, Springer-Verlag, Berlin (1996).
- [21] J. Kollár, *Singularities of pairs*, in: *Algebraic Geometry — Santa Cruz 1995*, Vol. 62 of *Proc. Sympos. Pure Math.*, pp. 221–287, Amer. Math. Soc., Providence, RI (1997).
- [22] J. Kollár and S. Mori, *Birational Geometry of Algebraic Varieties*, Vol. 134 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge (1998). With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.
- [23] M. Lejmi and G. Székelyhidi, *The J-flow and stability*, *Adv. Math.* **274** (2015), 404–431.
- [24] H. Li, Y. Shi, and Y. Yao, *A criterion for the properness of the K-energy in a general Kähler class*, *Math. Annalen* **361** (2015), no. 1-2, 135–156.
- [25] K. Liu and X. Ma, *A remark on: “Some numerical results in complex differential geometry”* [by S. K. Donaldson, [arXiv:math/0512625](https://arxiv.org/abs/math/0512625)], *Math. Res. Lett.* **14** (2007), no. 2, 165–171.
- [26] X. Ma and G. Marinescu, *Berezin-Toeplitz quantization on Kähler manifolds*, *J. Reine Angew. Math.* **662** (2012), 1–56.
- [27] T. Mabuchi, *K-energy maps integrating Futaki invariants*, *Tohoku Math. J. (2)* **38** (1986), no. 4, 575–593.
- [28] D. Mumford, *Lectures on Curves on An Algebraic Surface*, with a section by G. M. Bergman, *Annals of Mathematics Studies*, no. 59, Princeton University Press, Princeton, N.J. (1966).
- [29] D. Mumford, *Stability of Projective Varieties*, *L’Enseignement Mathématique*, Geneva (1977). Lectures given at the “Institut des Hautes Études Scientifiques”, *Monographie de l’Enseignement Mathématique*, no. 24.
- [30] I. Mundet i Riera, *A Hitchin-Kobayashi correspondence for Kähler fibrations*, *J. Reine Angew. Math.* **528** (2000), 41–80.

- [31] Y. Odaka, *The Calabi conjecture and K-stability*, Int. Math. Res. Not. IMRN (2012), no. 10, 2272–2288.
- [32] Y. Odaka, *A generalization of the Ross-Thomas slope theory*, Osaka J. Math. **50** (2013), no. 1, 171–185.
- [33] Y. Odaka, *The GIT stability of polarized varieties via discrepancy*, Ann. of Math. (2) **177** (2013), no. 2, 645–661.
- [34] Y. Odaka and Y. Sano, *Alpha invariant and K-stability of \mathbb{Q} -Fano varieties*, Adv. Math. **229** (2012), no. 5, 2818–2834.
- [35] Y. Odaka and S. Sun, *Testing log K-stability by blowing up formalism*, Ann. Fac. Sci. Toulouse Math. (6) **24** (2015), no. 3, 505–522.
- [36] D. Panov and J. Ross, *Slope stability and exceptional divisors of high genus*, Math. Ann. **343** (2009), no. 1, 79–101.
- [37] D. H. Phong and J. Sturm, *Stability, energy functionals, and Kähler-Einstein metrics*, Comm. Anal. Geom. **11** (2003), no. 3, 565–597.
- [38] D. H. Phong and J. Sturm, *The Monge-Ampère operator and geodesics in the space of Kähler potentials*, Invent. Math. **166** (2006), no. 1, 125–149.
- [39] J. Ross, *Unstable products of smooth curves*, Invent. Math. **165** (2006), no. 1, 153–162.
- [40] J. Ross and R. Thomas, *A study of the Hilbert-Mumford criterion for the stability of projective varieties*, J. Algebraic Geom. **16** (2007), no. 2, 201–255.
- [41] Y. Sano, *Numerical algorithm for finding balanced metrics*, Osaka J. Math. **43** (2006), no. 3, 679–688.
- [42] R. Seyyedali, *Numerical algorithm for finding balanced metrics on vector bundles*, Asian J. Math. **13** (2009), no. 3, 311–321.
- [43] J. Song and B. Weinkove, *On the convergence and singularities of the J-flow with applications to the Mabuchi energy*, Comm. Pure Appl. Math. **61** (2008), no. 2, 210–229.
- [44] J. Song and B. Weinkove, *The degenerate J-flow and the Mabuchi energy on minimal surfaces of general type*, Univ. Iagel. Acta Math. (2013), no. 50, [2012 on articles], 89–106.

- [45] G. Székelyhidi, *A remark on conical Kähler-Einstein metrics*, Math. Res. Lett. **20** (2013), no. 3, 581–590.
- [46] R. P. Thomas, *Notes on GIT and symplectic reduction for bundles and varieties*, in Surveys in Differential Geometry. Vol. X, Vol. 10 of Surv. Differ. Geom., pp. 221–273, Int. Press, Somerville, MA (2006).
- [47] G. Tian, *On a set of polarized Kähler metrics on algebraic manifolds*, J. Differential Geom. **32** (1990), no. 1, 99–130.
- [48] X. Wang, *Height and GIT weight*, Math. Res. Lett. **19** (2012), no. 4, 909–926.
- [49] B. Weinkove, *Convergence of the J-flow on Kähler surfaces*, Comm. Anal. Geom. **12** (2004), no. 4, 949–965.
- [50] B. Weinkove, *On the J-flow in higher dimensions and the lower boundedness of the Mabuchi energy*, J. Differential Geom. **73** (2006), no. 2, 351–358.
- [51] S. Zelditch, *Asymptotics of holomorphic sections of powers of a positive line bundle*, in: Séminaire sur les Équations aux Dérivées Partielles, pp. 1997–1998, Exp. No. XXII, 12, École Polytech., Palaiseau (1998).

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