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# Minimal diffeomorphism between hyperbolic surfaces with cone singularities

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We prove the existence of a minimal diffeomorphism isotopic to the identity between two hyperbolic cone surfaces $(\Sigma, g_1)$ and $(\Sigma, g_2)$ when the cone angles of $g_1$ and $g_2$ are different and smaller than $\pi$. When the cone angles of $g_1$ are strictly smaller than the ones of $g_2$, this minimal diffeomorphism is unique.



## 1. Introduction

A diffeomorphism $f : (M, g_1) \longrightarrow (N, g_2)$ between two Riemannian manifolds is called **minimal** if its graph $\Gamma$ is a minimal submanifold of $(M \times N, g_1 \oplus g_2)$ (that is its mean curvature vector field vanishes everywhere). Minimal diffeomorphisms between hyperbolic surfaces have been studied by R. Schoen [20] (see also F. Labourie [10]). He proved that for any two hyperbolic metrics $g_1$ and $g_2$ on $\Sigma$, there exists a unique minimal diffeomorphism $\Psi : (\Sigma, g_1) \longrightarrow (\Sigma, g_2)$ isotopic to the identity. Such a minimal diffeomorphism is also area-preserving and so its graph is a Lagrangian submanifold of $\big(\Sigma \times \Sigma, \omega_1 \oplus (-\omega_2)\big)$ (where $\omega_i$ is the area form associated to $g_i$); we call such a map a **minimal Lagrangian** diffeomorphism.

The minimal Lagrangian diffeormorphism $\Psi$ is related to harmonic maps. It is well-known (see [17, 27]) that, given a conformal struture $\mathfrak{c}$ and a hyperbolic metric $g$ on $\Sigma$, there exists a unique harmonic diffeomorphism $u : (\Sigma, \mathfrak{c}) \longrightarrow (\Sigma, g)$ isotopic to the identity and $g$ is characterized by the Hopf differential $\Phi(u)$ of $u$ (see Section 2 for definitions). It is proved in [20] that for each pair $g_1$ and $g_2$ of hyperbolic metrics on $\Sigma$, there exists a unique conformal structure $\mathfrak{c}$ such that $\Phi(u_1) + \Phi(u_2) = 0$ where $u_i : (\Sigma, \mathfrak{c}) \longrightarrow (\Sigma, g_i)$ is the unique harmonic map isotopic to the identity ($i = 1, 2$). Moreover, $u_2 \circ u_1^{-1}$ is minimal Lagrangian and isotopic to the identity.

For an angle $\theta \in (0, 2\pi)$, consider the metric obtained by gluing an angular sector of angle $\theta$ between two half-lines in the hyperbolic disk by a rotation. This metric is called **local model for hyperbolic metric with cone singularity of angle** $\theta$. For a marked surface $\Sigma_\mathfrak{p} := \Sigma \setminus \mathfrak{p}$ where $\mathfrak{p} = (p_1, \ldots, p_n) \subset \Sigma$ and for $\alpha := (\alpha_1, \ldots, \alpha_n) \in \left(0, \frac{1}{2}\right)^n$ such that $\chi(\Sigma) + \sum_{i=1}^n (\alpha_i - 1) < 0$ (in particular, $\Sigma_\mathfrak{p}$ can be a punctured sphere), one can construct the Fricke space $\mathscr{F}_\alpha(\Sigma_\mathfrak{p})$ with cone singularities of angle $\alpha$ as the moduli space of marked hyperbolic metrics on $\Sigma_\mathfrak{p}$ with cone singularities of angle $2\pi\alpha_i$ at the $p_i$ (see Section 2 for the construction).

In a previous paper [23], we proved the existence of a unique minimal Lagrangian diffeomorphism isotopic to the identity for each pair of points $g_1, g_2 \in \mathscr{F}_\alpha(\Sigma_\mathfrak{p})$ (that is when the cone angles of $g_1$ and $g_2$ are equal). The proof of this result used the deep connections between three dimensional anti-de Sitter (AdS) geometry and hyperbolic surfaces: we showed the existence of a unique area-maximizing surface in some AdS singular space-time, and realized the minimal Lagrangian map as the Gauss lift of the maximal surface (see [23] for more details).

In this paper, we address the question of the existence and uniqueness of minimal diffeomorphism between hyperbolic cone surfaces with different cone angles. In particular, we prove:

**Main Theorem.** *Given $\alpha, \alpha' \in \left(0, \frac{1}{2}\right)^n$, $g_1 \in \mathscr{F}_\alpha(\Sigma_\mathfrak{p})$ and $g_2 \in \mathscr{F}_{\alpha'}(\Sigma_\mathfrak{p})$, there exists a minimal diffeomorphism $\Psi : (\Sigma_\mathfrak{p}, g_1) \longrightarrow (\Sigma_\mathfrak{p}, g_2)$ isotopic to the identity. If moreover for all $i \in \{1, \ldots, n\}$, $\alpha_i < \alpha'_i$ then $\Psi$ is unique.*

Note that we just recover the existence part of the result of [23], but neither uniqueness nor the Lagrangian property.

The proof of Main Theorem is totally different from the proof in [23]. Here, we study the energy functional over $\mathscr{T}(\Sigma_\mathfrak{p})$, the Teichmüller space of $\Sigma_\mathfrak{p}$. In [7], J. Gell-Redman proved the existence of a unique harmonic diffeomorphism isotopic to the identity from a conformal surface to a negatively curved surface with cone singularities of angles less than $\pi$. So, given a hyperbolic metric $g$ with cone singularites of angle $2\pi\alpha_i$ at the $p_i$, we can define the energy functional $\mathscr{E}_g : \mathscr{T}(\Sigma_{\mathfrak{p}}) \longrightarrow \mathbb{R}$ which associates to a conformal structure on $\Sigma_{\mathfrak{p}}$ the energy of the unique harmonic diffeomorphism $u : (\Sigma_{\mathfrak{p}}, \mathfrak{c}) \longrightarrow (\Sigma_{\mathfrak{p}}, g)$ provided by [7]. This functional only depends on the class of $g$ in $\mathscr{F}_\alpha(\Sigma_{\mathfrak{p}})$.

The strategy of the proof is the following. We first prove that the energy functional $\mathscr{E}_g$ is proper and that its gradient at a point $\mathfrak{c}$ is given by the Hopf differential of the harmonic map $u : (\Sigma_{\mathfrak{p}}, \mathfrak{c}) \longrightarrow (\Sigma_{\mathfrak{p}}, g)$ (up to a multiplicative constant). Then, given two hyperbolic metric with cone singularities $g_1 \in \mathscr{F}_\alpha(\Sigma_{\mathfrak{p}})$ and $g_2 \in \mathscr{F}_{\alpha'}(\Sigma_{\mathfrak{p}})$, we show that critical points of $\mathscr{E}_{g_1} + \mathscr{E}_{g_2}$ correspond to minimal diffeomorphisms.

In the classical case (namely, without conical singularities), Schoen proved that such a minimal diffeomorphism is also Lagrangian. In particular, uniqueness of the minimal diffeomorphism follows from stability of minimal Lagrangian submanifold of Kähler-Einstein manifold (as studied by [8, 13]). In our case, the Lagrangian property generally fails and we have to prove stability "by hand".

The paper is organized as follow. In Section 2, we define surfaces with cone singularities and construct the moduli space $\mathscr{F}_\alpha(\Sigma_{\mathfrak{p}})$ of marked hyperbolic cone surfaces.

In Section 3, we define and study the energy functional. We prove the properness and explicit the gradient at a point.

In Section 4, we prove the Main Theorem. We construct a minimal diffeomorphism from $(\Sigma_{\mathfrak{p}}, g_1)$ to $(\Sigma_{\mathfrak{p}}, g_2)$ for each local critical point of $\mathscr{E}_{g_1} + \mathscr{E}_{g_2}$. We prove stability of minimal graphs in $(\Sigma_{\mathfrak{p}} \times \Sigma_{\mathfrak{p}}, g_1 \oplus g_2)$ with a maximum principle on elliptic PDE satisfied by the harmonic diffeomorphisms.

It would be interesting to study the possible connections between the minimal map of the Main Theorem and AdS geometry. In particular, we expect that this minimal map should be related to some "maximal" surface in some AdS manifold with spin particles (as introduced in [1] in the Minkowski case). We leave this question for a future work.

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## 2. Fricke space with cone singularities

In this Section, we construct the moduli space $\mathscr{F}_\alpha(\Sigma_\mathfrak{p})$ of marked hyperbolic metrics with cone singularities on $\Sigma_\mathfrak{p}$ when the cone angles are smaller than $\pi$. By definition, $\mathscr{F}_\alpha(\Sigma_\mathfrak{p})$ is the quotient of the infinite dimensional space $\mathscr{M}_\alpha^{-1}(\Sigma_\mathfrak{p})$ of hyperbolic cone metric on $\Sigma_\mathfrak{p}$ by the action of the group $\mathscr{D}_0(\Sigma_\mathfrak{p})$ of diffeomorphisms isotopic to the identity.

Given a infinitesimal deformation $h$ of a hyperbolic metric $g_0 \in \mathscr{M}_\alpha^{-1}(\Sigma_\mathfrak{p})$, we associate a unique deformation $h_0$ which is transverse to the orbit of $\mathscr{D}_0(\Sigma_\mathfrak{p})$ and differs from $h$ by the action of an element in $\mathscr{D}_0(\Sigma_\mathfrak{p})$. This is achieved by considering a Bianchi gauge. We then show that the deformation $h_0$ obtained in that way is the real part of holomorphic quadratic differential on $\Sigma_\mathfrak{p}$ with at most simple poles at $\mathfrak{p}$.

This construction provides an identification between the tangent space of $\mathscr{F}_\alpha(\Sigma_\mathfrak{p})$ at $[g_0]$ and the space of meromorphic quadratic differential on $\Sigma$ with at most simple pole at $\mathfrak{p}$. Riemann-Roch Theorem implies that $\mathscr{F}_\alpha(\Sigma_\mathfrak{p})$ has complex dimension $3g-3+n$.

### 2.1. Function spaces

In order to define metrics with cone singularities, we need to introduce the Hölder-based (b-Hölder) and weighted b-Hölder spaces. We refer to [9, Section 2.6.2] for more details about these spaces.

Fix $R>0$ and $\alpha \in (0,1)$ and set $D := \{z \in \mathbb{C},\ |z|<R\}$. From now and so on, we will use the following notations

$$z = \rho e^{i\theta}, \quad r = \frac{\rho^\alpha}{\alpha}.$$

**Definition 2.1.** For $\gamma \in (0,1)$, the space $\mathscr{C}_b^{0,\gamma}(D)$ of b-Hölder $(0,\gamma)$ functions on $D$ consists of these functions $f : D \longrightarrow \mathbb{C}$ such that

$$\sup_{z,z' \in D} \frac{|f(z)-f(z')|}{|\theta-\theta'|^\gamma + \frac{|r-r'|^\gamma}{|r+r'|^\gamma}} < +\infty.$$

For $k \geq 0$, we set

$$\mathscr{C}_b^{k,\gamma}(D) := \{f,\ (r\partial_r)^i(\partial_\theta)^j f \in \mathscr{C}_b^{0,\gamma}(D),\ i+j \leq k\},$$

and finally, for $\nu > 0$,

$$r^\nu \mathscr{C}_b^{k,\gamma}(D) := \{f,\ r^{-\nu} f \in \mathscr{C}_b^{k,\gamma}(D)\}.$$

Note that for instance, for $a \in \mathbb{C}$ with $\Re(a) > 0$ and $\psi \in \mathscr{C}^\infty$, the function $f(z) = r^a \psi(\theta)$ is in $r^\nu \mathscr{C}_b^{k,\gamma}(D)$ for $\nu < \Re(a)$ and all $k$.

### 2.2. Local model of cone singularity

**Definition 2.2.** A metric $g$ on $D = \{z \in \mathbb{C},\ |z| < R\}$ has a conical singularity of angle $2\pi\alpha$ at the origin if $g$ has the following form

$$g = ce^{2\mu}|z|^{2(\alpha-1)}|dz|^2,$$

where $c > 0$ and $\mu \in r^\nu \mathscr{C}_b^{1,\gamma}(D)$ for some $\nu > 0$ and $\gamma \in (0,1)$.

Equivalently (see [9, 11]), a metric on $D$ has a conical singularity on angle $2\pi\alpha$ at the origin if, in polar coordinates $(r,\theta)$, with $z = \rho e^{i\theta}$ and $r = \frac{\rho^\alpha}{\alpha}$, the metric $g$ has the following form

$$g = dr^2 + \alpha^2 r^2 d\theta^2 + h$$

where

$$h = adr^2 + brdrd\theta + cr^2 d\theta^2,\ a,b,c \in r^\nu \mathscr{C}_b^{1,\gamma}(D).$$

**From now and so on, all the cone angles will be considered strictly smaller than $\pi$, that means we only consider the case** $\alpha \in \left(0, \frac{1}{2}\right)$.

The main example of metric with cone singularity is the following:

**Example 2.3.** Let $\mathbb{H}^2 := (\mathbb{D}^2, g_p)$ be the unit disk equipped with the Poincaré metric. Cut $\mathbb{D}^2$ along two half-lines making an angle $2\pi\alpha$ intersecting at the center 0 of $\mathbb{D}^2$ and define $\mathbb{H}^2_\alpha$ as the space obtained by gluing the boundary of the angular sector of angle $2\pi\alpha$ by a rotation fixing 0.

Topologically, $\mathbb{H}^2_\alpha = \mathbb{D}^2 \setminus \{0\}$ and the induced metric $g_\alpha$ (which is not complete) is hyperbolic outside 0 and carries a conical singularity of angle $2\pi\alpha$ at 0. We call $\mathbb{H}^2_\alpha = (\mathbb{D}^*, g_\alpha)$ the **hyperbolic disk with cone singularity of angle** $2\pi\alpha$.

In conformal coordinates, we have the well-known expression:

$$g_p = \frac{4}{(1-|\widetilde{z}|^2)^2}|d\widetilde{z}|^2.$$

Using the coordinates $\widetilde{z} = \frac{1}{\alpha} z^\alpha$, we obtain:

$$g_\alpha = \frac{4|z|^{2(\alpha-1)}}{(1-\alpha^{-2}|z|^{2\alpha})^2}|dz|^2 = 4e^{2\mu}|z|^{2(\alpha-1)}|dz|^2,$$

where $\mu = -\ln\left(1-\alpha^{-2}|z|^{2\alpha}\right)$. Note that $\mu \in r^{\nu}\mathscr{C}_b^{1,\gamma}(\mathbb{D}^2)$ for $\nu < 2\alpha$.

In polar coordinates $(r,\theta)$, the metric $g_\alpha$ expresses as:

$$g_\alpha = dr^2 + \alpha^2 \sinh^2 r d\theta^2 = dr^2 + \alpha^2 r^2 d\theta^2 + h,$$

where $h = (\sinh^2 r - r^2)d\theta^2$. Note that so $r^{-2}(\sinh^2 r - r^2) \in r^{\nu}\mathscr{C}_b^{1,\gamma}$.

### 2.3. Surfaces with cone singularities

Let $\Sigma$ be a closed oriented surface, $\mathfrak{p} = (p_1, \ldots, p_n) \subset \Sigma$ be a set of points. Denote by $\Sigma_\mathfrak{p} := \Sigma \setminus \mathfrak{p}$ and let $\alpha := (\alpha_1, \ldots, \alpha_n) \in \left(0, \frac{1}{2}\right)^n$ be such that $\chi(\Sigma_\mathfrak{p}) - \sum_{i=1}^n (\alpha_i - 1) < 0$. By a theorem of McOwen and Troyanov [12, 25], this condition implies the existence of hyperbolic metrics with cone singularities of angle $\alpha$. Note also that $\Sigma_\mathfrak{p}$ can be a sphere with at least 3 punctures.

**Definition 2.4.** A metric $g$ on $\Sigma_\mathfrak{p}$ is a metric with cone singularities of angle $\alpha$ if $g$ is a $\mathscr{C}^2$ metric on each compact $K \subset \Sigma_\mathfrak{p}$ and for each puncture $p_i \in \mathfrak{p}$, the exists a holomorphic chart $z : U_i \longrightarrow D \subset \mathbb{C}$ where $U_i$ is a neighborhood of $p_i$ and $g_{|U_i}$ has the form of Definition 2.2.

We denote by $\mathscr{M}_\alpha(\Sigma_\mathfrak{p})$ the space of such metrics.

**Remark 2.5.** The space $\mathscr{M}_\alpha(\Sigma_\mathfrak{p})$ is a positive cone in the vector space of $\mathscr{C}^2$ sections of the bundle of symmetric 2-tensors (where the regularity around the puncture $p_i$ is $r^{\nu}\mathscr{C}_b^{1,\gamma}$). In particular, given a metric $g_0 \in \mathscr{M}_\alpha(\Sigma_\mathfrak{p})$, a vector $h \in T_{g_0}\mathscr{M}_\alpha(\Sigma_\mathfrak{p})$ is a $\mathscr{C}^2$ symmetric 2-tensor such that in a neighborhood of each puncture, $h$ has the following expression (see Subsection 2.2):

$$h = a dr^2 + b r dr d\theta + c r^2 d\theta^2,\ a, b, c \in r^{\nu}\mathscr{C}_b^{1,\gamma}.$$

We have the following map

$$\begin{array}{rccc} K: & \mathscr{M}_\alpha(\Sigma_\mathfrak{p}) & \longrightarrow & \mathscr{C}^0(\Sigma_\mathfrak{p}) \\ & g & \longmapsto & K(g) \end{array},$$

where $K(g)$ is the Gauss curvature of the metric $g$.

**Definition 2.6.** The space $\mathscr{M}_\alpha^{-1}(\Sigma_\mathfrak{p})$ of hyperbolic metrics with cone singularities of angle $\alpha$ is $K^{-1}(-1)$.

Note that, given $g \in \mathscr{M}_\alpha(\Sigma_\mathfrak{p})$, for each puncture $p \in \mathfrak{p}$, there is a local holomorphic chart around $p$ such that the expression of $g$ in that chart is given by $g_\alpha$, where $g_{\alpha_i}$ is the hyperbolic metric with cone singularity of angle $2\pi\alpha_i$ described in Example 2.3.

We now define the group of diffeomorphisms we will consider.

**Definition 2.7.** Let $\mathscr{D}_0(\Sigma_\mathfrak{p})$ be the group of diffeomorphisms $\psi$ of $\Sigma = \Sigma_\mathfrak{p} \cup \mathfrak{p}$ isotopic to the identity (in the isotopy class fixing each $p_i \in \mathfrak{p}$) so that, for each compact $K \subset \Sigma_\mathfrak{p}$, $\psi_{|K}$ is of class $\mathscr{C}^3$ and, for each marked point $p_i \in \mathfrak{p}$, the expression of $\psi$ in a holomorphic chart centered at $p_i$ has the form

$$\psi(z) = \lambda z + f(z),\ f \in r^{1+\nu}\mathscr{C}_b^{2,\gamma},\ \lambda \in \mathbb{C}^*.$$

**Proposition 2.8.** *For each $g \in \mathscr{M}_\alpha(\Sigma_\mathfrak{p})$ and $\psi \in \mathscr{D}_0(\Sigma_\mathfrak{p})$, the pull-back metric $\psi^* g$ is in $\mathscr{M}_\alpha(\Sigma_\mathfrak{p})$.*

*Proof.* Away from the singular points, $\psi$ is $\mathscr{C}^3$ and $g$ is $\mathscr{C}^2$ so in particular, $\psi^* g$ is $\mathscr{C}^2$.

In a holomorphic chart around a point $p \in \mathfrak{p}$ corresponding to an angle $2\pi\alpha$, $\psi$ has the expression

$$\psi(z) = \lambda z + r^{1+\nu} f(z),$$

where $f \in \mathscr{C}_b^{2,\gamma}$ and $\lambda \in \mathbb{C}^*$. Writing $\lambda = r_0 e^{i\theta_0}$ One gets the following expression

$$\begin{aligned} d\psi &= \big(r_0 + (1+\nu) r^\nu f + r^\nu (r\partial_r) f\big) dr + (1 + r^{1+\nu}\partial_\theta f) d\theta \\ &= (r_0 + \eta) dr + (1 + r\xi) d\theta \end{aligned}$$

where $\eta, \xi \in r^\nu \mathscr{C}_b^{1,\gamma}$.

In particular, if $g$ has local expression

$$g = dr^2 + \alpha^2 r^2 d\theta^2 + h,\ h = a dr^2 + b r dr d\theta + c r^2 d\theta^2,\ a, b, c \in r^\nu \mathscr{C}^{1,\gamma},$$

then we have

$$\psi^* g = r_0^2 dr^2 + \alpha^2 |\psi(z)|^2 d\theta^2 + h',$$

where $h' = (2r_0\eta + \eta^2) dr^2 + \alpha^2 |\psi(z)|^2 (2r\xi + r^2\xi^2) d\theta^2 + \psi^* h$.

Using $|\psi(z)| = r_0 r + \chi$ where $\chi \in r^{1+\nu}\mathscr{C}_b^{2,\gamma}$, and setting $\hat{r} = r_0 r$, one obtains that the pull-back metric has the form

$$\psi^* g = d\hat{r}^2 + \alpha^2 \hat{r}^2 d\theta^2 + h'',$$

where $h'' = a' d\hat{r}^2 + b' \hat{r} d\hat{r} d\theta + c' \hat{r}^2 d\theta^2,\ a', b', c' \in r^\nu \mathscr{C}_b^{1,\gamma}$. □

It follows that the space $\mathscr{D}_0(\Sigma_\mathfrak{p})$ acts by pull-back on $\mathscr{M}_\alpha^{-1}(\Sigma_\mathfrak{p})$ and the quotient space $\mathscr{F}_\alpha(\Sigma_\mathfrak{p}) := \mathscr{M}_\alpha^{-1}(\Sigma_\mathfrak{p})/\mathscr{D}_0(\Sigma_\mathfrak{p})$ is a smooth manifold called the **Fricke space with cone singularities of angles** $\alpha$.

One of the main reason to impose the cone angles being smaller than $\pi$ is the following proposition. It will be of main importance when studying deformations of hyperbolic cone metrics.

**Proposition 2.9.** *Let $g \in \mathscr{M}_\alpha^{-1}(\Sigma_\mathfrak{p})$ be a hyperbolic metric with cone singularities on $\Sigma_\mathfrak{p}$, where $\alpha \in \left(0, \frac{1}{2}\right)^n$. The distance between two singularities is bounded from below by a positive constant depending only on the angles.*

*Proof.* Let $p_1, p_2 \in \mathfrak{p}$ be two singularities of $(\Sigma_\mathfrak{p}, g)$ corresponding to the cone angle $2\pi\alpha_1$ and $2\pi\alpha_2$ respectively. Let $\beta$ the shortest geodesic joining $p_1$ to $p_2$ and denote by $\delta_i$ the geodesic from $p_i$ making an angle $\pi\alpha_i$ with $\beta$.

Given a closed disk $D$ with boundary $\partial D$ and an embedding $\iota : D \hookrightarrow \Sigma = \Sigma_\mathfrak{p} \cup \mathfrak{p}$ such that $\mathfrak{p} \cap \iota(D) = \{p_1, p_2\}$, denote by $\gamma$ the unique geodesic homotopic to $\iota(\partial D)$.

We have the following:

**Lemma 2.10.** *The distance between $\beta$ and $\gamma$ is strictly positive.*

*Proof.* We first claim that if the distance between $\gamma$ and $\beta$ is zero, then the image of $\gamma$ and $\beta$ coincides. In fact, for $\epsilon > 0$ small enough, consider the $\epsilon$-neighborhood $U_\epsilon$ of $\beta$ and let $\psi : U_\epsilon \longrightarrow U_\epsilon$ be the isometric involution given by the reflection along $\beta \cup \delta_1 \cup \delta_2$.

By uniqueness of the geodesic in the homotopy class of $\iota(\partial D)$, the intersection of $U_\epsilon$ with the image of $\gamma$ has to be fixed by $\psi$. In particular, if $\gamma$ intersects $\beta$, the intersection has to be tangent to $\beta$ and so the image of $\gamma$ and $\beta$ coincide.

Let $V$ be one of the connected component of $U_\epsilon \setminus \{\beta \cup \delta_1 \cup \delta_2\}$ and send $V$ isometrically to $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2,\ y > 0\}$, sending $\beta$ to the imaginary axis. The vector field $N = \partial_x$ restricts to a Jacobi field along $\beta$ and so the curve $\varphi_N^t(\gamma)$ (where $\varphi_N$ is the flow generated by $N$) is a geodesic in $V$. The condition on the angle implies that $\pi\alpha_i < \frac{\pi}{2}$ and so for $t$ small enough, $\varphi_N^t(\gamma)$ is strictly shorter than $\gamma$ (see Figure 1). In particular, the distance between $\beta$ and $\gamma$ is strictly positive. $\square$

Consider the connected component $S$ of $\Sigma \setminus \gamma$ containing $p_1$ and $p_2$, and cut it along $\beta$, $\delta_1$ and $\delta_2$. The remaining surfaces are two isometric hyperbolic quadrilaterals (see Figure 2). When the length of $\gamma$ tends to

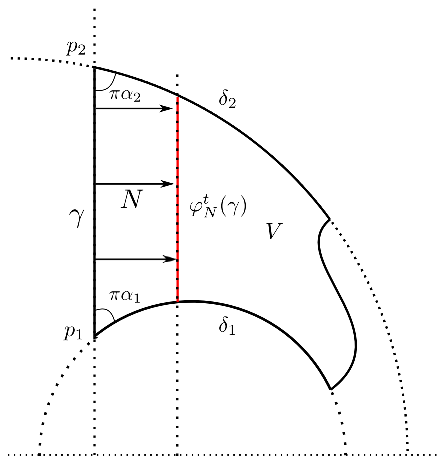


Figure 1: The geodesic $\beta_\epsilon$.

zero, each quadrilateral tends to a hyperbolic triangle of angles $\pi\alpha_1$, $\pi\alpha_2$ and 0. In such a triangle, the length on $\beta$ satisfies

$$\cosh(l(\beta)) = \frac{1+\cos(\pi\alpha_1)\cos(\pi\alpha_2)}{\sin(\pi\alpha_1)\sin(\pi\alpha_2)}.$$

It corresponds to the lower bound for the distance between two hyperbolic cone singularities of angles $2\pi\alpha_1$ and $2\pi\alpha_2$. □

Applying this result to the universal cover of $(\Sigma_\mathfrak{p}, g)$, one gets a lower bound for the injectivity radius of the singular points on a hyperbolic cone surface. In particular, there exists a neighborhood $V_i$ of each $p_i \in \mathfrak{p}$ such that, the restriction of any $g \in \mathscr{M}^{-1}_\alpha(\Sigma_\mathfrak{p})$ to $V_i$ is isometric to a neighborhood of the origin in $\mathbb{H}^2_{\alpha_i}$.

From now and so on, we fix a cylindrical coordinates system $(r_i, \theta_i)$ : $V_i \to \mathbb{H}^2_{\alpha_i}$ centered at $p_i$ for each $i \in \{1, \dots, n\}$. Proposition 2.9 implies that, up to a gauge, we can always assume that for each $i \in \{1, \dots, n\}$, every metric $g \in \mathscr{M}^{-1}_\alpha(\Sigma_\mathfrak{p})$ has the following expression:

$$g_{|V_i} = dr_i^2 + \alpha_i^2 \sinh^2 r_i d\theta_i^2.$$

We get the following Corollary:

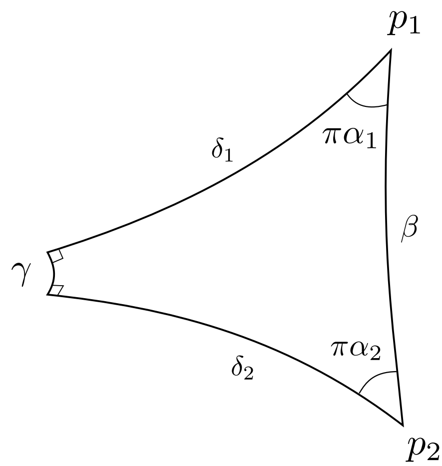


Figure 2: Hyperbolic quadrilateral.

**Corollary 2.11.** *Let $g_0 \in \mathscr{M}_\alpha^{-1}(\Sigma_\mathfrak{p})$ and let $\widetilde{h} := \frac{d}{dt}_{|t=0} g_t$ be a deformation of $g_0$. There exists a vector field $v \in Lie(\mathscr{D}_0(\Sigma_\mathfrak{p}))$ (the Lie algebra of $\mathscr{D}_0(\Sigma_\mathfrak{p})$), so that*

$$\widetilde{h} = h + \mathscr{L}_v g_0, \text{ and } h_{|V_i} = 0 \quad \forall i \in \{1, \dots, n\}.$$

*Here $\mathscr{L}_v g_0$ is the Lie derivative of $g$ in the direction $v$ and the $V_i$ are defined as in Proposition 2.9. We call such a $h$ a* ***normalized deformation****.*

**Analysis on hyperbolic cone manifolds.** Let $(\Sigma_\mathfrak{p}, g)$ be a hyperbolic surface with cone singularities of angle $\alpha \in \left(0, \frac{1}{2}\right)^n$. It is not obvious that classical results of geometric analysis on Riemannian manifolds (as integration by parts) extend to hyperbolic cone surfaces.

In this section, we study differential operators on vector bundles over $(\Sigma_\mathfrak{p}, g)$ in the framework of unbounded operators. For the convenience of the reader, we recall here basic facts about unbounded operators between Hilbert spaces. A good reference for the subject is [18].

**Unbounded operators.** Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two Hilbert spaces with scalar product $\langle ., . \rangle_1$ and $\langle ., . \rangle_2$ respectively.

**Definition 2.12.** An *unbounded operator* is a linear map

$$T : \mathscr{D}(T) \subset \mathscr{H}_1 \longrightarrow \mathscr{H}_2$$

where $\mathscr{D}(T)$ is a linear subset of $\mathscr{H}_1$ called the domain of $T$.

**Example 2.13.** Let $I \subset \mathbb{R}$ be an interval and $D$ an order $n \in \mathbb{N}$ linear differential operator. We see $D : \mathscr{C}_0^\infty(I) \subset L^2(I) \longrightarrow L^2(I)$ as an unbounded operator (here $\mathscr{C}_0^\infty(I)$ is the space of $\mathscr{C}^\infty$ real valued functions over $I$ with compact support).

Of course, one notes that in this example, $\mathscr{C}_0^\infty(I)$ is probably not the largest set (with respect to the inclusion) where $D$ can be defined. This motivates the following definitions:

**Definition 2.14.** Let $T_1$ and $T_2$ two unbounded operators from $\mathscr{H}_1$ to $\mathscr{H}_2$. We say that $T_1$ **extends** $T_2$ (and we denote by $T_2 \subset T_1$) if $\mathscr{D}(T_2) \subset \mathscr{D}(T_1)$ and $T_{1|_{\mathscr{D}(T_2)}} = T_2$.

We have the important notion of closed and closable operators:

**Definition 2.15.** An unbounded operator $T$ is **closed** if its graph $\mathscr{G}(T)$ is closed in $\mathscr{H}_1 \oplus \mathscr{H}_2$. $T$ is called **closable** if the closure of $\mathscr{G}(T)$ in $\mathscr{H}_1 \oplus \mathscr{H}_2$ is the graph of an unbounded operator $\overline{T}$. In this case, $\overline{T}$ is called the closure of $T$.

Using the scalar products of $\mathscr{H}_1$ and $\mathscr{H}_2$, we can define the adjoint of an unbounded operator with dense domain:

**Definition 2.16.** Let $T : \mathscr{D}(T) \subset \mathscr{H}_1 \longrightarrow \mathscr{H}_2$ be an unbounded operator such that $\mathscr{D}(T)$ is dense in $\mathscr{H}_1$. We define the **adjoint** of $T$ as the unbounded operator $T^* : \mathscr{D}(T^*) \subset \mathscr{H}_2 \longrightarrow \mathscr{H}_1$ where:

$$\mathscr{D}(T^*) := \{y \in \mathscr{H}_2, \text{ there exists } u \in \mathscr{H}_1 \text{ such that} \\ \langle Tx, y\rangle_2 = \langle x, u\rangle_1,\ \forall x \in \mathscr{D}(T)\}.$$

As $\mathscr{D}(T)$ is dense, $u$ is uniquely defined and we set $T^* y := u$.

The following proposition can be found in [18, Theorem 1.8]:

**Proposition 2.17.** *If $T : \mathscr{D}(T) \subset \mathscr{H}_1 \longrightarrow \mathscr{H}_2$ has dense domain and $\mathscr{D}(T^*)$ is dense in $\mathscr{H}_2$, then $T$ is closable with $\overline{T} = T^{**}$.*

Determining the domain of an adjoint operator is generally difficult. Hence we have the notion of a formal adjoint:

**Definition 2.18.** Let $T : \mathscr{D}(T) \subset \mathscr{H}_1 \longrightarrow \mathscr{H}_2$ be an unbounded operator with dense domain. We say that an operator $T^t : \mathscr{D}(T^t) \subset \mathscr{H}_2 \longrightarrow \mathscr{H}_1$ is a **formal adjoint** of $T$ is for all $x \in \mathscr{D}(T),\ y \in \mathscr{D}(T^t)$ we have $\langle Tx, y\rangle_2 = \langle x, T^t y\rangle_1$.

**Remark 2.19.** By Riesz' theorem, $y \in \mathscr{D}(T^*)$ if and only if the application $x \longmapsto \langle Tx, y\rangle$ is continuous on $\mathscr{D}(T)$. Given $y \in \mathscr{D}(T^t)$, the functional $x \longmapsto \langle Tx, y\rangle$ is continuous on $\mathscr{D}(T)$, so $\mathscr{D}(T^t) \subset \mathscr{D}(T^*)$ and $T^t \subset T^*$. So $T^*$ extends every formal adjoint of $T$.

The following result is of main importance for us [16, Theorem 13.13]:

**Proposition 2.20.** *If $T : \mathscr{D}(T) \subset \mathscr{H}_1 \longrightarrow \mathscr{H}_2$ is closed with dense domain, then the operator $T^*T + Id$ is invertible and its inverse $A := (T^*T + Id)^{-1} : \mathscr{H}_1 \longrightarrow \mathscr{H}_1$ is continuous with $\|A\| \leq 1$.*

**Application to geometric analysis on cone surfaces.** Let $(\Sigma_{\mathfrak{p}}, g)$ be a hyperbolic cone surface (recall that the cone angles are supposed strictly smaller than $\pi$). Denote by $T^{(r,0)}\Sigma_{\mathfrak{p}} := \bigotimes_{i=1}^{r} T^*\Sigma_{\mathfrak{p}}$ the bundle of $(r,0)$-tensors on $\Sigma_{\mathfrak{p}}$ and by $S^r\Sigma_{\mathfrak{p}} \subset T^{(r,0)}\Sigma_{\mathfrak{p}}$ the sub-bundle of symmetric $r$-tensors.

The cone metric $g$ on $\Sigma_{\mathfrak{p}}$ induces a metric on $T^{(r,0)}\Sigma_{\mathfrak{p}}$ and $S^rT\Sigma_{\mathfrak{p}}$ that we still denote by $g$. In the sequel, we will consider the Hilbert spaces $L^2\left(T^{(r,0)}\Sigma_{\mathfrak{p}}\right)$ of $L^2$ sections of $T^{(r,0)}\Sigma_{\mathfrak{p}}$ endowed with the scalar product

$$\langle \eta, \mu\rangle_{T^{(r,0)}} := \int_{\Sigma_{\mathfrak{p}}} g(\eta, \mu)\mathrm{vol}_g.$$

For $k \in \mathbb{N}$, we denote by $\mathscr{C}_0^k\left(T^{(r,0)}\Sigma_{\mathfrak{p}}\right)$ (respectively $\mathscr{C}^k\left(T^{(r,0)}\Sigma_{\mathfrak{p}}\right)$) the space of sections of $T^{(r,0)}\Sigma_{\mathfrak{p}}$ which are $\mathscr{C}^k$ with compact support (respectively $\mathscr{C}^k$). We use similar notations for $S^r\Sigma_{\mathfrak{p}}$.

Note that $\mathscr{C}_0^\infty\left(T^{(r,0)}\Sigma_{\mathfrak{p}}\right) \subset L^2\left(T^{(r,0)}\Sigma_{\mathfrak{p}}\right)$ is a dense subset.

We need some results of integration by parts in cone manifolds. Some good references for this theory are [4, 14].

**Divergence and covariant derivative.** We denote by $\mathring{\nabla}$ the covariant derivative associated to $g$. We see $\mathring{\nabla}$ as an unbounded operator:

$$\mathring{\nabla} : \mathscr{D}(\nabla) := \mathscr{C}_0^1\left(T^{(r,0)}\Sigma_{\mathfrak{p}}\right) \subset L^2\left(T^{(r,0)}\Sigma_{\mathfrak{p}}\right) \longrightarrow L^2\left(T^{(r+1,0)}\Sigma_{\mathfrak{p}}\right).$$

Because $\mathscr{D}(\mathring{\nabla})$ is dense in $L^2\left(T^{(r,0)}\Sigma_{\mathfrak{p}}\right)$, $\mathring{\nabla}$ admits an adjoint denoted $\nabla^*$. Define the operator $\nabla^t$ by

$$\nabla^t : \mathscr{D}(\nabla^t) = \mathscr{C}_0^1\left(T^{(r+1,0)}\Sigma_{\mathfrak{p}}\right) \subset L^2\left(T^{(r+1,0)}\Sigma_{\mathfrak{p}}\right) \longrightarrow L^2\left(T^{(r,0)}\Sigma_{\mathfrak{p}}\right),$$

where

$$\nabla^t\eta(X_1,\ldots,X_r) = -\sum_{i=1}^{2}(\nabla_{e_i}\eta)(e_i,X_1,\ldots,X_r),$$

for $(e_1,e_2)$ an orthonormal framing of $T\Sigma_{\mathfrak{p}}$.

Stokes' formula for compactly supported tensors implies that $\nabla^t$ is a formal adjoint of $\mathring{\nabla}$. In particular, $\nabla^t \subset \nabla^*$, so $\nabla^*$ has dense domain and $\mathring{\nabla}$ is closable (by Proposition 2.17). We denote by $\nabla := \nabla^{**}$ its closure.

The divergence operator $\delta : \mathscr{D}(\delta) \subset L^2\left(S^r\Sigma_{\mathfrak{p}}\right) \longrightarrow L^2\left(S^{r-1}\Sigma_{\mathfrak{p}}\right)$ is the restriction of $\nabla^*$ to the sections of $S^r\Sigma_{\mathfrak{p}}$. Its adjoint $\delta^*$ is just the composition of $\nabla$ with the symmetrization.

In particular, for $r = 1$, the decomposition of $(2,0)$-tensors into symmetric and anti-symmetric part gives

$$\nabla = \delta^* + \frac{1}{2}d$$

where $d$ is the usual differential acting on 1-forms.

Similarly, one checks that $\delta^* : \mathscr{D}(\delta^*) \subset L^2(\Sigma_{\mathfrak{p}}) \longrightarrow L^2\left(S^1\Sigma_{\mathfrak{p}}\right)$ is just the differential $d$.

We have a result of integration by parts for covariant tensors on $(\Sigma_{\mathfrak{p}}, g)$. The proof is analogous to the proof of [15, Theorem 1.4.3], however, as it is a central result in what follows, we include it.

**Proposition 2.21.** *For all $f \in \mathscr{C}^1(\Sigma_{\mathfrak{p}}) \cap \mathscr{D}(d)$ and $u \in \mathscr{C}^1(S^1\Sigma_{\mathfrak{p}}) \cap \mathscr{D}(\delta)$, we have*

$$\langle df, u\rangle_{S^1} = \langle f, \delta u\rangle_{L^2(\Sigma_{\mathfrak{p}})}.$$

*Proof.* Let us prove the result when $(\Sigma_{\mathfrak{p}}, g)$ contains a unique cone singularity $p$ of angle $2\pi\alpha$. To prove the result in the general case, we just apply the following computation to each puncture.

Fix cylindrical coordinates $(r,\theta) \in (0,R) \times \mathbb{R}/2\pi\alpha\mathbb{Z}$ in a neighborhood of $p$ so that

$$g_{|V} = dr^2 + \sinh^2 r d\theta^2.$$

For $t \in (0,r)$, denote by $U_t := \{(r,\theta) \in V, r < t\}$.

For $f \in \mathscr{D}(d) \cap \mathscr{C}^1(\Sigma_{\mathfrak{p}})$ and $u \in \mathscr{D}(\delta) \cap \mathscr{C}^1(S^1\Sigma_{\mathfrak{p}})$, Stokes' formula gives:

$$\int_{\Sigma \setminus U_t} \big(g(df,u) - f\delta u\big) dv_g = \int_{\partial U_t} f.u(\partial_r) dv_{g_{|\partial U_t}}$$

where $\partial_r$ is the unit vector field normal to $\partial U_t$.

As $t$ tends to 0, the left hand side tends to $\langle df, u\rangle_{S^1} - \langle f, \delta u\rangle_{L^2(\Sigma_{\mathfrak{p}})}$. Denote by $I_t$ the right hand side. By Cauchy-Schwarz inequality,

$$|I_t| \leq \int_{\partial U_t} |f||u(\partial_r)| dv_{g_{|\partial U_t}} \leq \left(\int_{\partial U_t} |f|^2 dv_{g_{|\partial U_t}}\right)^{1/2} \left(\int_{\partial U_t} |u(\partial_r)|^2 dv_{g_{|\partial U_t}}\right)^{1/2}.$$

When $f \neq 0$, $|f|$ is differentiable and $\partial_r|f| = \pm\partial_r f$. When $f = 0$, we set $\partial_r|f| = 0$. It follows that $\partial_r|f|$ is the partial derivative of $|f|$ is the sense of distributions. In fact, for all $t, a \in (0,r)$ and $\theta$ fixed, we have

$$|f(t,\theta)| - |f(a,\theta)| = \int_a^t \partial_r |f(r,\theta)| dr.$$

In particular, as $|\partial_r|f|| \leq |\partial_r f|$, we get

$$|f(t,\theta)| \leq |f(a,\theta)| + \int_t^a |\partial_r f| dr.$$

So

$$|f(t,\theta)|^2 \leq 2|f(a,\theta)|^2 + 2\left(\int_t^a |\partial_r f| dr\right)^2.$$

Applying Cauchy-Schwarz, we obtain

$$\begin{aligned}\left(\int_t^a |\partial_r f| dr\right)^2 &\leq \int_t^a \frac{dr}{r} \int_t^a r|\partial_r f|^2 dr \\ &\leq \left|\ln\left(\frac{t}{a}\right)\right| \int_t^a r|\partial_r f|^2 dr.\end{aligned}$$

Finally, we get

$$\begin{aligned}\int_{\partial U_t} |f|^2 dv_{g_{|\partial U_t}} &\leq 2\int_{\partial U_t} |f(a,\theta)|^2 dv_{g_{|\partial U_t}}\\ &\quad + \int_{\partial U_t}\left(2\left|\ln\left(\frac{t}{a}\right)\right|\int_t^a r|\partial_r f|^2 dr\right) dv_{g_{|\partial U_t}}\\ &\leq 2t\int_{\theta=0}^{2\pi\alpha} |f(a,\theta)|^2 d\theta\\ &\quad + 2\left|\ln\left(\frac{t}{a}\right)\right|\int_{\partial U_t}\left(\int_t^a r|\partial_r f|^2 dr\right) dv_{g_{|\partial U_t}}\\ &\leq 2t\int_{\theta=0}^{2\pi\alpha} |f(a,\theta)|^2 d\theta + 2t\left|\ln\left(\frac{t}{a}\right)\right|\int_{\theta=0}^{2\pi\alpha}\int_t^a |\partial_r f|^2 r dr d\theta\\ &\leq 2t\int_{\theta=0}^{2\pi\alpha} |f(a,\theta)|^2\, d\theta + 2t\left|\ln\left(\frac{t}{a}\right)\right|\int_{U_a} |\partial_r f|^2 dv_g\\ &= O(t\ln t).\end{aligned}$$

Now, as $u \in L^2(S^2\Sigma_{\mathfrak{p}})$,

$$\int_0^a\left(\int_{\partial U_t} |u(\partial_r)|^2 dv_{g_{|\partial U_t}}\right) \leq \int_0^a\left(\int_{\partial U_t} |u|^2 dv_{g_{|\partial U_t}}\right) = \int_{U_a} |u|^2 dv_g < +\infty,$$

that is, the function $t \longmapsto \int_{\partial U_t} |u|^2$ is integrable on $(0,a)$. As the function $(t\ln t)^{-1}$ is not integrable in 0, there exists a sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \to 0$ such that

$$\int_{\partial U_{t_n}} |u|^2 dv_{g_{|\partial U_t}} = o\big((t_n \ln t_n)^{-1}\big).$$

It follows that $\lim\limits_{n\to\infty} I_{t_n} = 0$. □

We will use the following corollary later.

**Corollary 2.22.** *If $f \in \mathscr{D}(\delta\circ d)\cap\mathscr{C}^2(\Sigma_{\mathfrak{p}})$ satisfies $(\delta d + \lambda Id)f = 0$ for $\lambda >$ 0, then $f = 0$.*

*Proof.* Let $\lambda \geq 0$ and $f \in \mathscr{D}(\delta\circ d)\cap\mathscr{C}^2(\Sigma_{\mathfrak{p}})$ such that

$$(\delta d + \lambda\mathrm{Id})f = 0.$$

Taking the scalar product with $f$, and using Proposition 2.21, we get:

$$\langle\delta df + \lambda f, f\rangle_{L^2(\Sigma_{\mathfrak{p}})} = \|df\|^2_{S^1} + \lambda\|f\|^2_{L^2(\Sigma_{\mathfrak{p}})} = 0,$$

and so $f = 0$. □

### 2.4. Deformations of hyperbolic cone metrics

We want to understand the deformations of a hyperbolic cone metric $g_0 \in \mathscr{M}_\alpha^{-1}(\Sigma_\mathfrak{p})$ up to the action of diffeomorphisms in $\mathscr{D}_0(\Sigma_\mathfrak{p})$.

Given a smooth path $(g_t)_{t\in(-\epsilon,\epsilon)}$ in $\mathscr{M}_\alpha^{-1}(\Sigma_\mathfrak{p})$ with $g_{t=0} = g_0$, the infinitesimal deformation $h := \frac{d}{dt}_{|t=0} g_t \in T_{g_0}\mathscr{M}_\alpha^{-1}(\Sigma_\mathfrak{p})$ is a symmetric 2-tensor on $\Sigma_\mathfrak{p}$.

The fact that $(g_t)_{t\in(-\epsilon,\epsilon)} \subset \mathscr{M}_\alpha^{-1}(\Sigma_\mathfrak{p})$ implies that

$$\frac{d}{dt}_{|t=0} K(g_t) = dK_{g_0}(h) = 0$$

where $K(g_t)$ is the Gauss curvature of the metric $g_t$.

The linearized operator $dK_{g_0}$ has the following well-known expression (see for instance [24, Formula 1.5, p.33]):

$$dK_{g_0}(h) = \delta d(\mathrm{tr}_{g_0} h) + \delta\delta h + \frac{1}{2}\mathrm{tr}_{g_0} h, \tag{1}$$

where $\mathrm{tr}_{g_0}$ denote the trace with respect to $g_0$. It follows that infinitesimal deformations of $g_0$ in $\mathscr{M}_\alpha^{-1}(\Sigma_\mathfrak{p})$ correspond to vectors $h \in T_{g_0}\mathscr{M}_\alpha(\Sigma_\mathfrak{p})$ that are solution to (1).

Given a smooth path $(\psi_t)_{t\in(-\epsilon,\epsilon)} \subset \mathscr{D}_0(\Sigma_\mathfrak{p})$ with $\psi_0 = \mathrm{Id}$, the pull-back metrics $\psi_t^* g_0$ are in $\mathscr{M}_\alpha^{-1}(\Sigma_\mathfrak{p})$. Moreover, we have

$$\frac{d}{dt}_{|t=0} \psi_t^* g_0 = \mathscr{L}_X g_0 = 2\delta^* u,$$

where $\mathscr{L}_X$ is the Lie derivative along the the vector field $X$ generating $\psi_t$ and $u$ is the 1-form dual to $X$. A deformation $h \in T_{g_0}\mathscr{M}_\alpha^{-1}(\Sigma_\mathfrak{p})$ is called a **trivial deformation** if $h = 2\delta^* u$ for some 1-form $u$ dual to a vector field $X \in \mathrm{Lie}(\mathscr{D}_0(\Sigma_\mathfrak{p}))$.

It follows that the deformations of $[g_0] := \mathscr{D}_0(\Sigma_\mathfrak{p}).g_0 \in \mathscr{F}_\alpha(\Sigma_\mathfrak{p})$ correspond to solutions of (1) up to trivial deformations.

The end of this subsection is devoted to the proof of the following:

**Proposition 2.23.** *Given a hyperbolic metric with cone singularities $g_0 \in \mathscr{M}_\alpha^{-1}(\Sigma_\mathfrak{p})$ and an infinitesimal deformation $h \in T_{g_0}\mathscr{M}_\alpha^{-1}(\Sigma_\mathfrak{p})$, there exists a unique traceless and divergence free symmetric 2-tensor $h_0 \in T_{g_0}\mathscr{M}_\alpha^{-1}(\Sigma_\mathfrak{p})$ which is transverse to the orbit $\mathscr{D}_0(\Sigma_\mathfrak{p}).g_0$ and differs from $h$ by a trivial deformation.*

*Proof.* First note that, given an infinitesimal deformation

$$h := \frac{d}{dt}_{|t=0} g_t,$$

one can always find a family of diffeomorphisms $\psi_t$ such that the expression on $\psi_t^* g_t$ is constant on the neighborhood $V_i$ of each marked point $p_i$ (see Corollary 2.11). It means that, if we want to consider infinitesimal deformations $h$ up to the action of $\mathscr{D}_0(\Sigma_{\mathfrak{p}})$, we can restrict ourselves to deformations vanishing in a neighborhood of the $p_i$. These deformations belong to $\mathscr{C}_0^2\left(S^2\Sigma_{\mathfrak{p}}\right)$ and are called **normalized deformations**.

A classical way to get rid of trivial deformations is to impose a gauge on $h$. Here we will consider a Bianchi gauge (as in [3]).

Let $\beta$ be the operator acting on symmetric 2-tensors by

$$\beta(\eta) := \delta\eta + \frac{1}{2} d(\mathrm{tr}_{g_0}\eta).$$

**Lemma 2.24.** *Given a normalized deformation $h \in T_{g_0}\mathscr{M}_\alpha^{-1}(\Sigma_{\mathfrak{p}})$, there exists a unique 1-form $u \in \mathscr{D}(\nabla\nabla^*)$ such that*

$$\beta(h - 2\delta^* u) = 0. \tag{2}$$

*Proof.* We want to solve the equation

$$2(\beta \circ \delta^*)u = \beta h,$$

where

$$2(\beta \circ \delta^*)u = 2\delta\delta^* u + d(\mathrm{tr}_{g_0}\delta^* u).$$

Using the decomposition of $\nabla$ into symmetric and anti-symmetric part, namely $\nabla = \delta^* + \frac{1}{2}d$, we obtain

$$\mathrm{tr}_{g_0}(\delta^* u) = \mathrm{tr}_{g_0}(\nabla u) = -\nabla^* u = -\delta u.$$

So

$$\begin{aligned} 2(\beta \circ \delta^*)u &= 2\delta\left(\nabla u - \frac{1}{2}du\right) - d\delta u \\ &= 2\nabla^*\nabla u - \Delta u, \end{aligned}$$

where $\Delta = d\delta + \delta d$ is the Hodge Laplacian.

The Weitzenböck formula gives a relation between $\nabla^*\nabla$ and $\Delta$ (see [2, Chapter 1.]):

$$\Delta = \nabla^*\nabla - \mathrm{Id}.$$

Note that here we used the fact that $g_0$ is hyperbolic.

Substituting, we finally obtain

$$2(\beta \circ \delta^*)u = (\nabla^*\nabla + \mathrm{Id})u.$$

It follows from Proposition 2.20 that the operator $\nabla^*\nabla + \mathrm{Id} : \mathscr{D}(\nabla^*\nabla) \subset L^2\big(S^1\Sigma_{\mathfrak{p}}\big) \longrightarrow L^2\big(S^1\Sigma_{\mathfrak{p}}\big)$ is invertible. Moreover, because $h \in \mathscr{C}^2_0\big(S^2\Sigma_{\mathfrak{p}}\big)$, $\beta h \in \mathscr{C}^1_0\big(S^1\Sigma_{\mathfrak{p}}\big) \subset L^2\big(S^1\Sigma_{\mathfrak{p}}\big)$ and the equation $2(\beta \circ \delta^*)u = \beta h$ admits a unique solution. $\square$

So far, we just know that $\delta^* u$ is $L^2$. In particular, we will need to have a control on the behavior of $h_0 = h - 2\delta^* u$ at the punctures in order to prove that $h_0 \in T_{g_0}\mathscr{M}_\alpha^{-1}(\Sigma_{\mathfrak{p}})$. This is achieved in the following lemmas.

**Lemma 2.25.** *On each compact $K \subset \Sigma_{\mathfrak{p}}$, the solution $u$ to equation (2) is smooth.*

*Proof.* Let $K \subset \Sigma_{\mathfrak{p}}$ be a compact. We can assume, without loss of generality, that $K$ is isometric to a closed hyperbolic disk $D_r \subset \mathbb{H}^2$ of radius $r$. Denote by $\varphi : D_r \longrightarrow K \subset \Sigma_{\mathfrak{p}}$ the isometry.

By uniqueness of the Levi-Civita connection, the pull-back of the covariant derivative $\nabla$ by $\varphi$ is the covariant derivative associated to the Levi-Civita connection on $\mathbb{H}^2$. It follows that the pull-back 1-form $\varphi^* u$ satisfies

$$(\nabla^*_{\mathbb{H}^2}\nabla_{\mathbb{H}^2} + \mathrm{Id})\varphi^* u = \beta(\varphi^* h).$$

The operator on the left hand side is elliptic of order two and the right hand side is $\mathscr{C}^1$. We can thus apply Schauder estimates and we obtain that $u$ is $\mathscr{C}^3$. $\square$

**Lemma 2.26.** *The symmetric 2-tensors $h_0 := h - 2\delta^* u$ (where $u$ is the solution to equation (2)) has zero trace and is divergence free.*

*Proof.* Because $u$ is smooth on $\Sigma_{\mathfrak{p}}$, the flow of the vector field dual to $u$ is a diffeomorphism of $\Sigma_{\mathfrak{p}}$ (which may not be in $\mathscr{D}_0(\Sigma_{\mathfrak{p}})$). In particular,

$$dK_{g_0}(h) = dK_{g_0}(h_0 - 2\delta^* u) = dK_{g_0}(h_0) = 0.$$

That is,

$$\delta d(\mathrm{tr}_{g_0} h_0) + \delta\delta h_0 + \frac{1}{2}\mathrm{tr}_{g_0} h_0 = 0.$$

On the other hand, $\beta(h_0) = \delta h_0 + \frac{1}{2} d(\mathrm{tr}_{g_0} h_0) = 0$ implies that $h_0$ satisfies

$$\delta d(\mathrm{tr}_{g_0} h_0) + \mathrm{tr}_{g_0} h_0 = 0.$$

The previous lemma implies that $\mathrm{tr}_{g_0} h_0$ is at least $\mathscr{C}^2(\Sigma_{\mathfrak{p}})$, so by Corollary 2.22, $\mathrm{tr}_{g_0} h_0 = 0$.

Finally, the condition $\beta(h_0) = 0$ becomes $\delta h_0 = 0$. □

The next lemma is classical:

**Lemma 2.27.** *The symmetric 2-tensors $h_0$ is the real part of a meromorphic quadratic differential on $\Sigma$ with at most simple poles at $\mathfrak{p}$.*

*Proof.* For $(dx, dy)$ an orthonormal framing of $T^*\Sigma_{\mathfrak{p}}$, write

$$h_0 = u(x, y)dx^2 - v(x, y)(dxdy + dydx) + w(x, y)dy^2.$$

The condition $tr_{g_0} h_0 = 0$ implies $w(x, y) = -u(x, y)$. Write $(\partial_x, \partial_y)$ the framing dual to $(dx, dy)$. Let us explicit the divergence-free condition:

$$\begin{aligned} 0 &= \delta h_0(\partial_x) \\ &= -(\nabla_{\partial_x} h_0)(\partial_x, \partial_x) - (\nabla_{\partial_y} h_0)(\partial_y, \partial_x) \\ &= -\partial_x u + \partial_y v. \end{aligned}$$

In the same way, we get:

$$0 = \delta h_0(\partial_y) = \partial_x v + \partial_y u.$$

These are the Cauchy-Riemann equations. It implies in particular that $\varphi = u + iv$ is holomorphic on $\Sigma_{\mathfrak{p}}$.

Now, for $z = x + iy,\ dz = dx + idy$, set $\phi = \varphi(z)dz^2$. It is a holomorphic quadratic differential on $\Sigma_{\mathfrak{p}}$ such that $h_0 = \Re(\phi)$. It follows that $\phi$ is meromorphic on $\Sigma$ with possible poles at the $p_i \in \mathfrak{p}$.

We claim that, as $h_0 = \Re(\phi) \in L^2\left(S^2\Sigma_{\mathfrak{p}}\right)$, the poles of $\phi$ at the $p_i$ are at most simple. In fact, let $p \in \mathfrak{p}$ be a cone singularity of angle $2\pi\alpha$, $z$ be a local holomorphic coordinates around $p$ and

$$\phi(z) = \left(\frac{\lambda}{z^n} + f(z)\right) dz^2$$

for $\lambda \in \mathbb{C}^*$, $n \geq 0$ and $f$ meromorphic so that $z^n f(z) \underset{z\to 0}{\longrightarrow} 0$.

It follows from Proposition 2.9 that around $p$, each lifting $g_0 \in \mathscr{M}_\alpha^{-1}(\Sigma_{\mathfrak{p}})$ of $[g_0] \in \mathscr{F}_\alpha(\Sigma_{\mathfrak{p}})$ is isometric to the expression $g_\alpha$ given in Definition 2.2. In particular,

$$\phi\overline{\phi} = \left(O(|z|^{-2n}\right)|dz|^4,$$

so

$$g_0(\phi,\overline{\phi})(z) = O\left(|z|^{2(2-2\alpha-n)}\right).$$

It follows,

$$g_0(\phi,\overline{\phi})dv_{g_0} = O\left(|z|^{2(1-\alpha-n)}\right)|dz|^2.$$

As $\alpha \in \left(0,\frac{1}{2}\right)$, $g_0(\phi,\overline{\phi})dv_{g_0}$ is integrable in 0 if and only if $n \leq 1$. In particular, $h_0$ is in $L^2$ only when $n \leq 1$. $\square$

**Lemma 2.28.** *The symmetric 2-tensor $h_0$ is tangent to the space of hyperbolic metric $\mathscr{M}_\alpha^{-1}(\Sigma_{\mathfrak{p}})$ at $g_0$.*

*Proof.* We want to prove that $h_0 \in T_{g_0}\mathscr{M}_\alpha(\Sigma_{\mathfrak{p}})$. By Remark 2.5, we have to show that, for each puncture $p \in \mathfrak{p}$ of angle $2\pi\alpha$ and local polar coordinate system $(r,\theta)$, $h_0$ has the following expression

$$h_0 = adr^2 + brdrd\theta + cdr^2d\theta^2,\ a,b,c \in r^\nu\mathscr{C}_b^{2,\gamma}.$$

By Lemma 2.27, $h_0 = \frac{1}{2}(\phi+\overline{\phi})$, where $\phi(z) = \varphi(z)dz^2$, $\varphi(z) = \frac{\lambda}{z} + f(z)$, $\lambda \in \mathbb{C}$ and $f$ is holomorphic.

As usual, we set $z = \rho e^{i\theta}$ and $r = \frac{\rho^\alpha}{\alpha}$. We have

$$dz^2 = e^{2i\theta}d\rho^2 + 2i\rho e^{2i\theta}d\rho d\theta - \rho^2 e^{2i\theta}d\theta^2.$$

In particular, writting

$$h_0 = \frac{1}{2}(\phi+\overline{\phi}) = adr^2 + brdrd\theta + cr^2d\theta^2,$$

one obtains

$$\begin{cases} adr^2 = \Re\left(\varphi(z)e^{2i\theta}\right)d\rho^2 \\ brdrd\theta = 2\Re\big(i\rho\varphi(z)e^{2i\theta}\big)d\rho d\theta \\ cr^2d\theta^2 = -\Re\big(\rho^2\varphi(z)e^{2i\theta}\big)d\theta^2. \end{cases}$$

Using

$$r = \frac{\rho^\alpha}{\alpha},\ \rho = (\alpha r)^{\frac{1}{\alpha}},\ d\rho = (\alpha r)^{\frac{1}{\alpha}-1}dr,$$

one gets

$$\begin{aligned} a &= \Re\big(\varphi(z)e^{2i\theta}\big)(\alpha r)^{2(\frac{1}{\alpha}-1)} \\ &= \Re\left(\frac{\overline{z}\lambda e^{2i\theta}}{\rho^2} + e^{2i\theta}f(z)\right)(\alpha r)^{2(\frac{1}{\alpha}-1)} \\ &= \Re\left(\frac{\lambda e^{i\theta}}{\rho}\right)(\alpha r)^{2(\frac{1}{\alpha}-1)} + O\left(r^{2(\frac{1}{\alpha}-1)}\right) \\ &= \Re(\lambda e^{i\theta})(\alpha r)^{\frac{1}{\alpha}-2} + O\left(r^{2(\frac{1}{\alpha}-1)}\right). \end{aligned}$$

$$\begin{aligned} b &= 2\Re\big(i\rho\varphi(z)e^{2i\theta}\big)(\alpha r)^{\frac{1}{\alpha}-1}r^{-1} \\ &= 2\Re\left(i\rho\frac{\lambda\overline{z}e^{2i\theta}}{\rho^2} + i\rho e^{2i\theta}f(z)\right)(\alpha r)^{\frac{1}{\alpha}-1}r^{-1} \\ &= 2\alpha^{\frac{1}{\alpha}-1}\Re(i\lambda e^{i\theta})r^{\frac{1}{\alpha}-2} + O\left(r^{2(\frac{1}{\alpha}-1)}\right). \end{aligned}$$

$$c = -\Re(\lambda e^{i\theta})(\alpha r)^{\frac{1}{\alpha}-2} + O\left(r^{2(\frac{1}{\alpha}-1)}\right).$$

But $\frac{1}{\alpha} > 2$, so in particular, for $\nu < \frac{1}{\alpha} - 2$, $a, b, c \in r^\nu \mathscr{C}_b^{1,\gamma}$. □

This complete the proof of the proposition. □

In particular, one can associate a meromorphic quadratic differential on $\Sigma$ with at most simple pole at $\mathfrak{p}$ to each deformation of $g_0 \in \mathscr{M}_\alpha^{-1}(\Sigma_\mathfrak{p})$. On the other hand, the real part of any (non-zero) meromorphic quadratic differential is a deformation of $g_0$ transverse to $\mathscr{D}_0(\Sigma_\mathfrak{p}).g_0$. We thus get the following:

**Proposition 2.29.** *The tangent space to $\mathscr{F}_\alpha(\Sigma_\mathfrak{p})$ at $[g_0]$ is canonically identified with the space of meromorphic quadratic differential on $\Sigma = \Sigma_\mathfrak{p} \cup \mathfrak{p}$ with at most simple poles at $\mathfrak{p}$.*

### 2.5. A Weil-Petersson metric on $\mathscr{F}_{\alpha}(\Sigma_{\mathfrak{p}})$

Here we describe the Weil-Petersson metric on the Fricke space with cone singularities. This metric was first introduced and studied by Schumacher and Trapani in [21].

Let $h, k \in T_{[g_0]}\mathscr{F}_{\alpha}(\Sigma_{\mathfrak{p}})$. Fix a lift $g_0 \in \mathscr{M}_{\alpha}^{-1}(\Sigma_{\mathfrak{p}})$ of $[g_0]$. It follows from the above construction that there exists a unique lift $\hat{h}, \hat{k} \in T_{g_0}\mathscr{M}_{\alpha}^{-1}(\Sigma_{\mathfrak{p}})$ of $h$ and $k$ respectively which are divergence-free 2-symmetric tensors of zero trace. We call such a lifting a **horizontal lifting**. Define:

$$\frac{1}{8}\langle h, k\rangle_{\mathrm{WP}_{\alpha}} := \langle \hat{h}, \hat{k}\rangle_{S^2}.$$

In the case of a closed surface, it was proved by Fischer and Tromba [6, Theorem (0.8)] that this $L^2$ metric coincides with the Weil-Petersson metric. We call $\langle ., .\rangle_{\mathrm{WP}_{\alpha}}$ the **Weil-Petersson metric with cone singularities of angle** $\alpha$.

Note that in [21], the author proved that $\langle ., .\rangle_{\mathrm{WP}_{\alpha}}$ is a Kähler metric and studied its curvature.

**Uniformization.** Here, we recall a fundamental result proved by R.C. McOwen [12] and independently M. Troyanov [25]. Let $\mathscr{T}(\Sigma_{\mathfrak{p}})$ be the Teichmüller space of $\Sigma_{\mathfrak{p}}$, that is the moduli space of marked conformal structures on $\Sigma_{\mathfrak{p}}$. We have

**Theorem 2.1 (McOwen, Troyanov).** *Given $\mathfrak{c} \in \mathscr{T}(\Sigma_{\mathfrak{p}})$, there exists a unique $g \in \mathscr{F}_{\alpha}(\Sigma_{\mathfrak{p}})$ in the conformal class $\mathfrak{c}$ as long as $\chi(\Sigma) + \sum_{i=1}^{n}(\alpha_i - 1) < 0$ (where $\Sigma = \Sigma_{\mathfrak{p}} \cup \mathfrak{p}$).*

This theorem provides a family of identification $\Theta_{\alpha} : \mathscr{T}(\Sigma_{\mathfrak{p}}) \longrightarrow \mathscr{F}_{\alpha}(\Sigma_{\mathfrak{p}})$ for each $\alpha \in \mathbb{R}^n_{>0}$ such that $\chi(\Sigma_{\mathfrak{p}}) + \sum_{i=1}^{n}(\alpha_i - 1) < 0$. In particular, one can define a family $(\Theta_{\alpha}^{*}\langle ., .\rangle_{\mathrm{WP}_{\alpha}})_{\alpha\in(0,\frac{1}{2})^n}$ of Weil-Petersson metric on $\mathscr{T}(\Sigma_{\mathfrak{p}})$.

## 3. Energy functional on $\mathscr{T}(\Sigma_{\mathfrak{p}})$

In this section, we define and study the energy functional on the Teichmüller space of $\Sigma_{\mathfrak{p}}$. Its definition is based on the following theorem, which is a particular case of the main theorem of [7]:

**Theorem 3.1 (Gell-Redman).** *Given two hyperbolic metrics $g, g_0 \in \mathscr{M}_{\alpha}^{-1}(\Sigma_{\mathfrak{p}})$ with cone singularities of angle $\alpha$, there exists a unique diffeomorphism $u \in \mathscr{D}_0(\Sigma_{\mathfrak{p}})$ such that $u : (\Sigma_{\mathfrak{p}}, g) \longrightarrow (\Sigma_{\mathfrak{p}}, g_0)$ is harmonic.*

Recall that a harmonic map $f : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds is a critical point of the energy, where the energy of $f$ is defined as follow:

$$E(f) := \int_M e(f) dv_g,$$

and $e(f) = \frac{1}{2}\|df\|^2$ is called the **energy density of** $f$. Here, $df$ is seen as a section of $T^*M \otimes f^*TN$ with the metric $g^* \otimes f^*h$ ($g^*$ stands for the metric on $T^*M$ dual to $g$).

Note that, when $\dim M = 2$, the energy functional only depends on the conformal class $\mathfrak{c}$ of the metric $g$. We denote by $u_{\mathfrak{c},g_0}$ the harmonic diffeomorphism isotopic to the identity from $(\Sigma_{\mathfrak{p}}, \mathfrak{c})$ to $(\Sigma_{\mathfrak{p}}, g_0)$.

Moreover, a complex structure $J_{\mathfrak{c}}$ on $\Sigma_{\mathfrak{p}}$ is canonically associated to $\mathfrak{c}$. It allows us to split each symmetric 2-tensor on $\Sigma_{\mathfrak{p}}$ into its $(2, 0)$, $(1, 1)$ and $(0, 2)$ part.

**Definition 3.1.** Let $u : (\Sigma_{\mathfrak{p}}, \mathfrak{c}) \longrightarrow (\Sigma_{\mathfrak{p}}, g_0)$ be a diffeomorphism. The Hopf differential $\Phi(u)$ of $u$ is the quadratic form define by

$$\Phi(u) := (u^* g_0)^{(2,0)}.$$

We have the following (see [7, Section 5.1])

**Proposition 3.2 (J. Gell-Redman).** *Let $g_0 \in \mathscr{M}_\alpha^{-1}(\Sigma_{\mathfrak{p}})$ be a hyperbolic metric with cone singularities and $\mathfrak{c}$ be a conformal structure on $\Sigma_{\mathfrak{p}}$. A diffeomorphism $u : (\Sigma_{\mathfrak{p}}, \mathfrak{c}) \longrightarrow (\Sigma_{\mathfrak{p}}, g_0)$ is harmonic if and only if its Hopf differential $\Phi(u)$ is holomorphic on $\Sigma_{\mathfrak{p}}$ with at most simple poles at $\mathfrak{p}$.*

**Local expressions.** Let $u : (\Sigma_{\mathfrak{p}}, g) \longrightarrow (\Sigma_{\mathfrak{p}}, g_0)$ be a diffeomorphism, $z = x + iy$ be a local holomorphic coordinates on $(\Sigma, g)$. Set $g = \rho^2(z)|dz|^2$ and $g_0 = \sigma^2(u)|du|^2$. As usual, write $u = u^1 + iu^2$ and

$$\begin{cases} \partial_z = \frac{1}{2}(\partial_1 - i\partial_2), & \overline{\partial}_z = \frac{1}{2}(\partial_1 + i\partial_2) \\ dz = dx_1 + idx_2, & d\overline{z} = dx_1 - idx_2 \\ \partial_u = \frac{1}{2}(\partial_{u^1} - i\partial_{u^2}), & \overline{\partial}_u = \frac{1}{2}(\partial_{u^1} + i\partial_{u^2}). \end{cases}$$

We have the following expression:

$$\begin{aligned} du &= \sum_{i,j=0}^{2} \partial_i u^j dx_i \otimes \partial_{u^j} \\ &= \partial_z u dz \partial_u + \partial_z \overline{u} dz \overline{\partial}_u + \overline{\partial}_z u d\overline{z} \partial_u + \overline{\partial}_z \overline{u} d\overline{z} \overline{\partial}_u. \end{aligned}$$

It follows that

$$\begin{aligned}\Phi(u) &= u^*g_0(\partial_z, \partial_z)dz^2 \\ &= g_0\big(du(\partial_z), du(\partial_z)\big)dz^2 \\ &= \sigma^2(u)\partial_z u \partial_z \overline{u} dz^2.\end{aligned}$$

Moreover, for $g^{ij}$ the coefficient of the metric dual to $g$,

$$\begin{aligned}e(u) &= \frac{1}{2}\sum_{\alpha,\beta,i,j=0}^{2} g^{ij} g_{0\alpha\beta}\partial_i u^\alpha \partial_j u^\beta \\ &= \rho^{-2}(z)\sigma^2(u)\left(|\partial_z u|^2 + |\overline{\partial}_z u|^2\right).\end{aligned}$$

In particular, we have

$$\begin{aligned}(u^*g_0)^{(1,1)} &= \left(u^*g_0(\partial_z, \overline{\partial}_z) + u^*g_0(\overline{\partial}_z, \partial_z)\right)|dz|^2 \\ &= 2g_0\big(du(\partial_z), du(\overline{\partial}_z)\big)|dz|^2 \\ &= \sigma^2(u)(|\partial_z u|^2 + |\overline{\partial}_z u|^2)|dz|^2 \\ &= \rho^2(z)e(u)|dz|^2.\end{aligned}$$

Note that we get the following equation for each section $\xi$ of $T^*\Sigma_{\mathfrak{p}} \otimes u^*T\Sigma_{\mathfrak{p}}$ with the metric $g^* \otimes u^*g$:

$$\|\xi\|^2 = 4\rho^2|\langle \xi(\partial_z), \xi(\overline{\partial}_z)\rangle|, \tag{3}$$

where $\langle ., . \rangle$ is the scalar product with respect to the metric $g_0$.

Finally, noting that the framing $(dz\partial_u, dz\overline{\partial}_u, d\overline{z}\partial_u, d\overline{z}\overline{\partial}_u)$ of $(T^*\Sigma_{\mathfrak{p}} \otimes u^*T\Sigma_{\mathfrak{p}}, g^* \otimes u^*g_0)$ is orthogonal and each vector has norm $\rho^{-1}(z)\sigma(u)$, we get the following expression for the Jacobian $J(u)$ of $u$:

$$\begin{aligned}J(u) &= \det{}_{g^*\otimes u^*g_0}\begin{pmatrix} \partial_z u & \partial_z \overline{u} \\ \overline{\partial}_z u & \overline{\partial}_z \overline{u}\end{pmatrix} \\ &= \rho^{-2}(z)\sigma^2(u)\left(|\partial_z u|^2 - |\overline{\partial}_z u|^2\right).\end{aligned}$$

In particular, we have the following expression:

$$u^*g_0 = \Phi(u) + \rho^2(z)e(u)|dz|^2 + \overline{\Phi(u)}.$$

Thus $\Phi(u)$ measures the difference of the conformal class of $u^*g_0$ with $\mathfrak{c}$.

**Energy functional.** Fixing $g_0 \in \mathscr{M}_\alpha^{-1}(\Sigma_\mathfrak{p})$, we define the energy functional $\widetilde{\mathscr{E}}_{g_0}$ on the space of conformal structures of $\Sigma_\mathfrak{p}$ by:

$$\widetilde{\mathscr{E}}_{g_0}(\mathfrak{c}) := E(u_{\mathfrak{c},g_0}),$$

where $u_{\mathfrak{c},g_0} : (\Sigma_\mathfrak{p}, \mathfrak{c}) \longrightarrow (\Sigma_\mathfrak{p}, g_0)$ is the unique harmonic diffeomorphism isotopic to the identity.

**Proposition 3.3.** *The energy functional $\widetilde{\mathscr{E}}_{g_0}$ descends to a functional $\mathscr{E}_{g_0}$ on $\mathscr{T}(\Sigma_\mathfrak{p})$.*

*Proof.* For each diffeomorphism isotopic to the identity $f \in \mathscr{D}_0(\Sigma_\mathfrak{p})$, $f : (\Sigma_\mathfrak{p}, f^*\mathfrak{c}) \longrightarrow (\Sigma_\mathfrak{p}, \mathfrak{c})$ is holomorphic and $E$ is invariant under holomorphic mapping (see [5, Proposition p.126]), that is $E(u_{\mathfrak{c},g_0}) = E(f^*u_{\mathfrak{c},g_0})$. Moreover, $f^*u_{\mathfrak{c},g_0} = u_{f^*\mathfrak{c},g_0}$. In fact,

$$f^*u_{\mathfrak{c},g_0} : (\Sigma_\mathfrak{p}, f^*\mathfrak{c}) \longrightarrow (\Sigma_\mathfrak{p}, g_0)$$

is harmonic. So, as $f \in \mathscr{D}_0(\Sigma_\mathfrak{p})$ is isotopic to the identity, uniqueness of the harmonic diffeomorphism implies $f^*u_{\mathfrak{c},g_0} = u_{f^*\mathfrak{c},g_0}$. So $\widetilde{\mathscr{E}}_{g_0}$ is $\mathscr{D}_0(\Sigma_\mathfrak{p})$-invariant and descends to a functional $\mathscr{E}_{g_0}$ on $\mathscr{T}(\Sigma_\mathfrak{p})$. $\square$

**Remark 3.4.** The same argument shows that $\mathscr{E}_{g_0}$ only depends on the class of $g_0$ in $\mathscr{F}_\alpha(\Sigma_\mathfrak{p})$.

Now, we are going to prove the following main result:

**Theorem 3.2.** *The energy functional $\mathscr{E}_{g_0} : \mathscr{T}(\Sigma_\mathfrak{p}) \longrightarrow \mathbb{R}$ is proper. Moreover, its Weil-Petersson gradient at $[g] \in \mathscr{T}(\Sigma_\mathfrak{p})$ is given by $-2\Re(\Phi(u_{[g],g_0})) \in T_{[g]}\mathscr{T}(\Sigma_\mathfrak{p})$.*

## 3.1. Properness of $\mathscr{E}_{g_0}$

Recall that (Proposition 2.9), for each $g \in \mathscr{F}_\alpha(\Sigma_\mathfrak{p})$ and $i \in \{1, \ldots, n\}$, there exists a neighborhood $V_i = \{x \in \Sigma_\mathfrak{p},\ d(x, p_i) < R_i\}$ of $p_i$ such that

$$g_{|V_i} = dr_i^2 + \sinh^2 r_i d\theta_i^2$$

where $(\rho_i, \theta_i)$ are fixed cylindrical coordinates on $V_i$. We can choose the $V_i$ such that $V_i \cap V_j = \emptyset$ whenever $i \neq j$. We denote $V := \bigcup_{i=1}^n V_i$. We need an important result, corresponding to Mumford's compactness theorem for the case of hyperbolic surfaces with cone singularities. The proof is an extension of Tromba's proof in the classical case [24].

**Proposition 3.5.** *Let $(g_k)_{k\in\mathbb{N}} \subset \mathscr{M}_\alpha^{-1}(\Sigma_\mathfrak{p})$ be such that, the length of every closed geodesic $\gamma^k \subset (\Sigma_\mathfrak{p} \setminus V, g_k)$ is uniformly bounded from below by $l > 0$. There exists $g \in \mathscr{M}_\alpha^{-1}(\Sigma_\mathfrak{p})$ and a sequence $(f_k)_{k\in\mathbb{N}} \subset Diff(\Sigma_\mathfrak{p})$ such that*

$$f_k^* g_k \underset{\mathscr{C}^2}{\longrightarrow} g.$$

*Proof.* Let $(g_k)_{k\in\mathbb{N}}$ be as above. It follows that there exists $\rho > 0$ such that, for each $k \in \mathbb{N}$ and $x \in \Sigma_\mathfrak{p} \setminus V$, the injectivity radius of $x$ is bigger than $\rho$ (for example, take $\rho = \min\{l, r_1, \dots, r_n\}$).

Fix $R > 0$ such that $R < \frac{1}{2}\rho$. As the area of $(\Sigma_\mathfrak{p} \setminus V, g_k)$ is independent of $k$, there exists $N > 0$ such that for each $k \in \mathbb{N}$, $N$ is the maximum number of disjoint disks of radius $\frac{R}{2}$ in $\Sigma_\mathfrak{p}$.

That is, for each $k \in \mathbb{N}$, there exists $\left(x_1^k, \dots, x_N^k\right) \subset \Sigma_\mathfrak{p} \setminus V$ such that $D_{\frac{R}{2}}\left(x_1^k\right), \dots, D_{\frac{R}{2}}\left(x_N^k\right)$, $V_1, \dots, V_n$ are disjoints (here $D_{\frac{R}{2}}(x_i^k) \subset \Sigma_\mathfrak{p}$ is the disk of center $x_i^k$ and radius $\frac{R}{2}$) and $D_R(x_1^k), \dots, R_R(x_N^k), V_1, \dots, V_n$ is a covering of $\Sigma_p$.

For each $i, j \in \{1, \dots, N\}$ with $D_R(x_i^k) \cap D_R(x_j^k) \neq \emptyset$, note that $x_i^k \in D_{2R}(x_j^k)$, $x_j^k \in D_{2R}(x_i^k)$ and, as $2R < \rho$, there exists isometries $\Psi_i^k$ and $\Psi_j^k$ sending $D_{2R}(x_i^k)$ (resp. $D_{2R}(x_j^k)$) to the disk $B$ of radius $2R$ centered at $0$ in $\mathbb{H}^2$.

It follows that the map $\tau_{ij}^k := \Psi_i^k \circ (\Psi_j^k)^{-1}$ is a positive local isometry of $\mathbb{H}^2$ which uniquely extend to $\tau_{ij}^k \in \mathrm{PSL}(2,\mathbb{R})$. Moreover, for each $k$,

$$\tau_{ij}^k(\Psi_j^k(x_i^k)) = \Psi_i^k(x_j^k) \in B,$$

that is $(\tau_{ij}^k)_{k\in\mathbb{N}}$ is compact. So $(\tau_{ij}^k)_{k\in\mathbb{N}}$ admits a convergent subsequence whose limit is denoted by $\tau_{ij}$.

For each $i \in \{1, \dots, N\}$ and $j \in \{1, \dots, n\}$ with $D_{2R}(x_i^k) \cap V_j \neq \emptyset$, there exists an isometry $\Psi_i^k : D_{2R}(x_i^k) \longrightarrow B \subset \mathbb{H}^2$ and $\psi_j : V_j \longrightarrow \mathbb{H}^2_{\alpha_j}$. As $\psi_i(D_{2R}(x_i^k) \cap V_j)$ is a simply connected subset of $\mathbb{H}^2_{\alpha_j}$, it is isometric to a subset of $B \subset \mathbb{H}^2$ by an isometry denoted $\Phi_j$.

Pick-up a point $y^k \in D_{2R}(x_i^k) \cap V_j$. The map $\alpha_{ij}^k := \Phi_j \circ \psi_j \circ (\Psi_i^k)^{-1}$ (see Figure 3) is a positive local isometry of $\mathbb{H}^2$ which uniquely extends to an element of $\mathrm{PSL}(2,\mathbb{R})$. Moreover, $\alpha_{ij}^k$ sends $\Psi_i^k(y)$ to $\Phi \circ \psi_j(y)$ which are both in the compact set $\overline{B} \subset \mathbb{H}^2$ (the closure of $B$). Then, by the same argument as before, $\alpha_{ij}^k \longrightarrow \alpha_{ij} \in \mathrm{PSL}(2,\mathbb{R})$ (up to a subsequence).

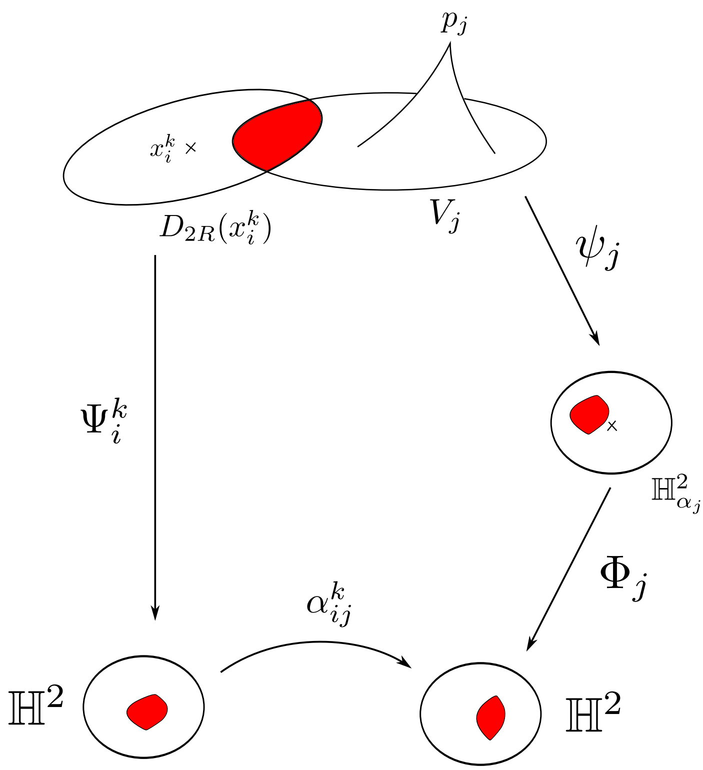


Figure 3: The map $\alpha_{ij}^k$.

Now, define

$$M := (B_1 \sqcup \cdots \sqcup B_N \sqcup \psi_1(V_1) \sqcup \cdots \sqcup \psi_n(V_n)) \,/ \sim,$$

where $B_i = B \subset \mathbb{H}^2$ for each $i$ and $\sim$ identifies:

- $x_i \in B_i$ with $x_j \in B_j$ whenever $\tau_{ij}$ exists and $\tau_{ij}(x_j) = x_i$.
- $x_i \in B_i$ with $x_j \in \psi_j(V_j)$ whenever $\alpha_{ij}$ exists and $\alpha_{ij}(x_i) = \Phi(x_j)$.

Obviously, $M$ is an hyperbolic surface with cone singularities and defines a point $g \in \mathscr{M}_\alpha^{-1}(\Sigma_{\mathfrak{p}})$.

Now, we claim that there exist diffeomorphisms $f_k : M \longrightarrow (\Sigma_{\mathfrak{p}}, g_k)$ with $f_k(B_j) \subset D_R(x_j^k)$, $f_k(V_i) \subset V_i$ and such that

$$\Psi_j^k \circ f_k \xrightarrow[\mathscr{C}^2]{} id \text{ on each } B_j, \quad \text{and } \psi_i \circ f_k \xrightarrow[\mathscr{C}^2]{} id \text{ on each } \mathbb{H}^2_{\alpha_i}.$$

The proof of this claim is exactly analogous to the proof of [24, Lemma C4 p.188] and will not be repeated here.

Hence, on each $B_j$, we have

$$f_k^* \Psi_j^{k*} g_P \underset{\mathscr{C}^2}{\longrightarrow} g_P,$$

(where $g_P$ is the Poincaré metric) and on each $V_i$

$$f_k^* \psi_i^* g_{\alpha_i} \underset{\mathscr{C}^2}{\longrightarrow} g_{\alpha_i}.$$

But, as $\Psi_j^k$ and $\psi_i$ are isometries, we get:

$$f_k^* g_k \underset{\mathscr{C}^2}{\longrightarrow} g. \qquad \square$$

Now we are able to prove the properness of $\mathscr{E}_{g_0}$. Let $(\mathfrak{c}_k)_{k\in\mathbb{N}} \subset \mathscr{T}(\Sigma_{\mathfrak{p}})$ such that $(\mathscr{E}_{g_0}(\mathfrak{c}_k))_{k\in\mathbb{N}}$ is convergent. For each $k \in \mathbb{N}$, choose a point $g_k \in \mathscr{M}_\alpha^{-1}(\Sigma_{\mathfrak{p}})$ such that the conformal class of $g_k$ is $\mathfrak{c}_k$. It follows that $E(u_{g_k,g_0}) \leq K$ for all $k \in \mathbb{N}$.

Let $\gamma$ be a simple closed curve in $\Sigma_{\mathfrak{p}} \setminus V$. For $k \in \mathbb{N}$, let $\gamma_k$ be the unique geodesic isotopic to $\gamma$ for the metric $g_k$. Note that there exists no geodesic homotopic to a cone point on a hyperbolic surface. In fact, if $\gamma$ would be such a geodesic, consider the surface obtained by taking two times the connected component of $\Sigma \setminus \gamma$ containing the cone point and glue them along $\gamma$. The remaining surface would be a hyperbolic sphere with two punctures, but it is well-know that such a hyperbolic surface does not exist.

So $\gamma$ is not homotopic to $\partial V_i$ for some $i \in \{1, \dots, n\}$, so by [24, Theorem 3.2.4] we have:

$$l(\gamma_k) > \frac{C}{K}$$

for some constant $C > 0$.

So $(l(\gamma_k))_{k\in\mathbb{N}}$ is bounded from below and we can use Proposition 3.5 and we get a family $(f_k)_{k\in\mathbb{N}} \subset \mathrm{Diff}(\Sigma_{\mathfrak{p}})$ such that $f_k^* g_k \underset{\mathscr{C}^2}{\longrightarrow} g$.

For all $k \in \mathbb{N}$, denote by $u_k : (\Sigma_{\mathfrak{p}}, \mathfrak{c}_k) \longrightarrow (\Sigma_{\mathfrak{p}}, g_0)$ the harmonic diffeomorphism isotopic to the identity. By [24, Lemma 3.2.3], the sequence $(u_k)_{k\in\mathbb{N}}$ is equicontinuous. It follows that the classes of $(f_k)_{k\in\mathbb{N}}$ in $\mathrm{Diff}(\Sigma_{\mathfrak{p}})/\mathrm{Diff}_0(\Sigma_{\mathfrak{p}})$ takes only a finite set of values. In fact, as

$$E(u_{\mathfrak{c}_k,g_0}) = E(u_{f_k^*\mathfrak{c}_k,g_0}) = E(f_k^* u_{\mathfrak{c}_k,g_0}) < K,$$

the sequence $(f_k^* u_k)_{k\in\mathbb{N}}$ is equicontinuous and admits a convergent subsequence by Arzelá-Ascoli. As $\mathrm{Diff}(\Sigma_{\mathfrak{p}})/\mathrm{Diff}_0(\Sigma_{\mathfrak{p}})$ is discrete, there exists a

$N \in \mathbb{N}$ such that, for $k$ bigger than $N$, $[f_k] \in \mathrm{Diff}_0(\Sigma_{\mathfrak{p}})$ is constant. It follows that, up to a subsequence, $([f_k^* \mathfrak{c}_k])_{k\in\mathbb{N}}$ converges in $\mathscr{T}(\Sigma_{\mathfrak{p}})$.

### 3.2. Weil-Petersson gradient of $\mathscr{E}_{g_0}$

Let $\mathfrak{c} \in \mathscr{T}(\Sigma_{\mathfrak{p}})$. We are going to use real coordinates $(x, y)$ on $(\Sigma_{\mathfrak{p}}, \mathfrak{c})$. From now on, denote by $\partial_1 := \partial_x$ and $\partial_2 := \partial_y$ and by $(dx_1, dx_2)$ the dual framing. Denote by $u := u_{\mathfrak{c},g_0}$ and fix $\widetilde{g} \in \mathscr{M}_\alpha^{-1}(\Sigma_{\mathfrak{p}})$ such that the conformal class of $\widetilde{g}$ is $\mathfrak{c}$. In local coordinates, we have the following expression:

$$du = \sum_{i,j,\alpha,\beta=1}^{2} \partial_i u^\alpha dx_i \otimes \partial_{u^\alpha},$$

where $(u^1, u^2)$ are the coordinates of $u$ on $(\Sigma_{\mathfrak{p}}, g_0)$. Assume that $(u^1, u^2)$ are isothermal coordinates for $g_0$, so

$$g_0 = \sum_{\alpha,\beta=1}^{2} \sigma^2(u)\delta_{\alpha\beta} du^\alpha du^\beta,$$

(here $\delta_{\alpha\beta}$ is the Kronecker symbol). Writing $\widetilde{g}$ in coordinates and using the Einstein convention, we have the following expression:

$$E(u) = \frac{1}{2}\int_{\Sigma_{\mathfrak{p}}} \|du\|^2 dv_{\widetilde{g}} = \frac{1}{2}\int_{\Sigma_{\mathfrak{p}}} \sigma^2 \delta_{\alpha\beta}\widetilde{g}^{ij}\partial_i u^\alpha \partial_j u^\beta vol_{\widetilde{g}}.$$

Here, $vol_{\widetilde{g}}$ is the volume form of $(\Sigma_{\mathfrak{p}}, \widetilde{g})$ and $\widetilde{g}^{ij}$ are the coefficients of the metric dual to $\widetilde{g}$ in $T^*\Sigma_{\mathfrak{p}}$.

For $h \in T_{\mathfrak{c}}\mathscr{T}(\Sigma_{\mathfrak{p}})$, denote by $\widetilde{h}$ the horizontal lift of $d\Theta_\alpha(h)$ in $T_{\widetilde{g}}\mathscr{M}_\alpha^{-1}(\Sigma_{\mathfrak{p}})$ (recall that $\Theta_\alpha$ is the application given by the uniformization). So $\widetilde{h}$ is a zero trace divergence-free symmetric $2-$tensor on $(\Sigma_{\mathfrak{p}}, \widetilde{g})$.

We are going to compute the differential of $\widetilde{\mathscr{E}}_{g_0}$ at $\widetilde{g}$ in the direction $\widetilde{h}$. Note that the differential of $\widetilde{g} \longmapsto (\widetilde{g}^{ij})$ is given by $\widetilde{h} \longmapsto (-\widetilde{h}^{ij})$ and the differential of $\widetilde{g} \longmapsto vol_{\widetilde{g}}$ is $\widetilde{h} \longmapsto (\frac{1}{2}\mathrm{tr}_{\widetilde{g}}\widetilde{h})vol_{\widetilde{g}}$. So one gets:

$$\begin{aligned} d\widetilde{\mathscr{E}}_{g_0}(\widetilde{g})(\widetilde{h}) = &-\frac{1}{2}\int_{\Sigma_{\mathfrak{p}}} \sigma^2\widetilde{h}^{ij}\partial_i u^\alpha \partial_j u^\alpha vol_{\widetilde{g}} \\ &+\frac{1}{4}\int_{\Sigma_{\mathfrak{p}}} \sigma^2\widetilde{g}^{ij}\partial_i u^\alpha \partial_j u^\alpha (tr_{\widetilde{g}}\widetilde{h})vol_{\widetilde{g}} + R(\widetilde{h}), \end{aligned}$$

where the term $R(\widetilde{h})$ is obtained by fixing $\widetilde{g}$ and $dvol_{\widetilde{g}}$ and varying the rest. It follows that $R(\widetilde{h})$ correspond to the first order variation of $E(u)$ in the direction $\widetilde{h}$. But as $u$ is harmonic, $R(\widetilde{h}) = 0$.

Moreover, the second term is zero because we have chosen a horizontal lift of $h$, hence $tr_{\widetilde{g}}\widetilde{h} = 0$.

Writing $u = u^1 + iu^2$ and using the fact that $\widetilde{h}^{11} = -\widetilde{h}^{22}$ and $\widetilde{h}^{12} = \widetilde{h}^{21}$ (see Section 2), we get the following expression:

$$\begin{aligned} d\mathscr{E}_{g_0}(\widetilde{g})(\widetilde{h}) &= -\frac{1}{2}\int_{\Sigma_{\mathfrak{p}}} \sigma^2 \left(\widetilde{h}^{11}\left(|\partial_1 u|^2 - |\partial_2 u|^2\right) + 2\widetilde{h}^{12}\Re(\partial_1 u \partial_2 \overline{u})\right) vol_{\widetilde{g}} \\ &= \langle \widetilde{h}, \varphi\rangle_{S^2(\Sigma_{\mathfrak{p}})}, \end{aligned}$$

where

$$\varphi = -\frac{1}{2}\sigma^2(u)\left((|\partial_1 u|^2 - |\partial_2 u|^2)(dx^2 - dy^2) + 2\Re(\partial_1 u\partial_2\overline{u})(dxdy + dydx)\right).$$

Note that, by definition, $\varphi$ is the Weil-Petersson gradient $\nabla\mathscr{E}(\mathfrak{c})$ of $\mathscr{E}$ at the point $\mathfrak{c} \in \mathscr{T}(\Sigma_{\mathfrak{p}})$. On the other hand,

$$\begin{aligned} &\Re(\Phi(u)) = \Re(\sigma^2(u)\partial_z u \partial_z \overline{u} dz^2) \\ &= \Re\left(\frac{1}{4}\sigma^2(u)(\partial_1 u - i\partial_2 u)(\partial_1\overline{u} - i\partial_2\overline{u})(dx^2 - dy^2 + i(dxdy + dydx))\right) \\ &= \frac{1}{4}\sigma^2(u)\left((|\partial_1 u|^2 - |\partial_2 u|^2)(dx^2 - dy^2) + 2\Re(\partial_1 u\partial_2\overline{u})(dxdy + dydx)\right). \end{aligned}$$

So $\nabla\mathscr{E}(\mathfrak{c}) = -2\Re(\Phi(u))$.

## 4. Minimal diffeomorphisms between hyperbolic cone surfaces

In this section, we prove the Main Theorem by studying the PDE satisfied by harmonic diffeomorphisms.

### 4.1. Existence

**Proposition 4.1.** *For each $\alpha, \alpha' \in \left(0, \frac{1}{2}\right)^n$, $g_1 \in \mathscr{F}_\alpha(\Sigma_{\mathfrak{p}})$ and $g_2 \in \mathscr{F}_{\alpha'}(\Sigma_{\mathfrak{p}})$, there exists a minimal diffeomorphism $\Psi : (\Sigma_{\mathfrak{p}}, g_1) \longrightarrow (\Sigma_{\mathfrak{p}}, g_2)$ isotopic to the identity.*

*Proof.* Let $g_1 \in \mathscr{F}_\alpha(\Sigma_\mathfrak{p})$, $g_2 \in \mathscr{F}_{\alpha'}(\Sigma_\mathfrak{p})$ and consider $M := (\Sigma_\mathfrak{p} \times \Sigma_\mathfrak{p}, g_1 \oplus g_2)$. Given a conformal structure $\mathfrak{c} \in \mathscr{T}(\Sigma_\mathfrak{p})$, one can consider the map

$$f_\mathfrak{c} := (u_1, u_2) : (\Sigma_\mathfrak{p}, \mathfrak{c}) \longrightarrow M,$$

where $u_i : (\Sigma_\mathfrak{p}, \mathfrak{c}) \longrightarrow (\Sigma_\mathfrak{p}, g_i)$ is the harmonic diffeomorphism isotopic to the identity $(i = 1, 2)$.

Clearly, $E(f_\mathfrak{c}) = E(u_1) + E(u_2)$ (where $E(f) = \frac{1}{2}\int_{\Sigma_\mathfrak{p}} \|df\|^2 dv_g$). From Section 3, the functional $\mathscr{E} := \mathscr{E}_{g_1} + \mathscr{E}_{g_2} : \mathscr{T}(\Sigma_\mathfrak{p}) \longrightarrow \mathbb{R}$ is proper. Let $\mathfrak{c}_0$ be a critical point of $\mathscr{E}$, so the map

$$\Psi := f_{\mathfrak{c}_0} : \ (\Sigma, \mathfrak{c}_0) \longrightarrow M$$

is a harmonic immersion. We claim that $\Psi$ is also conformal. In fact, $\Psi = (u_1, u_2)$, so

$$\begin{aligned}\Psi^*(g_1 \oplus g_2) &= u_1^* g_1 \oplus u_2^* g_2 \\ &= \Phi(u_1) + \Phi(u_2) + \rho^2(z)(e(u_1) + e(u_2))|dz|^2 + \overline{\Phi(u_1)} + \overline{\Phi(u_2)},\end{aligned}$$

where $z$ is a local holomorphic coordinates on $(\Sigma_\mathfrak{p}, \mathfrak{c}_0)$ such that $\Theta_\alpha(\mathfrak{c}_0) = \rho^2(z)|dz|^2$ (where $\Theta_\alpha : \mathscr{T}(\Sigma_\mathfrak{p}) \longrightarrow \mathscr{F}_\alpha(\Sigma_\mathfrak{p})$ is defined in Subsection 2.5).

Now, as $\mathfrak{c}_0$ is a minimum of $\mathscr{E}$, $\nabla\mathscr{E}(\mathfrak{c}_0) = -2\Re(\Phi(u_1) + \Phi(u_2)) = 0$, so $\Phi(u_1) + \Phi(u_2) = 0$ and $\Psi$ is conformal. It follows that $\Psi$ is a conformal harmonic immersion, hence $\Psi(\Sigma_\mathfrak{p})$ is a minimal surface in $M$ (see [5, Proposition p. 119]).

Denoting by $p_i : M \longrightarrow \Sigma_\mathfrak{p}$ the projection on the $i$-th factor $(i = 1, 2)$ and $\Gamma = \Psi(\Sigma_\mathfrak{p})$, we get that $u_i = p_{i_{|\Gamma}}$ and $\Gamma = \mathrm{graph}(p_{2_{|\Gamma}} \circ p_{1_{|\Gamma}}^{-1})$. It follows that

$$p_{2_{|\Gamma}} \circ p_{1_{|\Gamma}}^{-1} : (\Sigma_\mathfrak{p}, g_1) \longrightarrow (\Sigma_\mathfrak{p}, g_2)$$

is a minimal diffeomorphism isotopic to the identity. □

**Remark 4.2.** For $\Psi : (\Sigma, g_1) \longrightarrow (\Sigma, g_2)$ a minimal diffeomorphism as in Proposition 4.1, the induced metric $g_\Gamma$ on $\Gamma = \mathrm{graph}(\Psi)$ carries conical singularities of angle $\beta = (\beta_1, \dots, \beta_n)$ where $\beta_i = 2\pi \min(\alpha_i, \alpha_i')$.

In fact, suppose $\alpha \le \alpha'$ and normalize the metrics $g_1$ and $g_2$ so that $\Psi = \mathrm{id}$, and choosing conformal coordinates $z$ in a neighborhood of $(p_i, p_i) \in \Gamma$, one has the following expression:

$$\begin{aligned} g_\Gamma &= \mathrm{id}^* g_1 + \mathrm{id}^* g_2 \\ &= \left( c e^{2\mu} |z|^{2(\alpha-1)} + c' e^{2\mu'} |z|^{2(\alpha'-1)} \right) |dz|^2 \\ &= c e^{2\mu} |z|^{2(\alpha-1)} \left( 1 + \frac{c'}{c} e^{2(\mu'-\mu)} |z|^{2(\alpha'-\alpha)} \right) |dz|^2 \\ &= c e^{2\hat{\mu}} |z|^{2(\alpha-1)} |dz|^2 \end{aligned}$$

where $\hat{\mu} = \mu + \ln\left(1 + \frac{c'}{c} e^{2(\mu'-\mu)} |z|^{2(\alpha'-\alpha)}\right) \in r^\nu \mathscr{C}_b^{1,\gamma}$. So, by Definition 2.2, $g_\Gamma$ carries a conical singularity of angle $2\pi\alpha$ at $(p_i, p_i)$.

### 4.2. Uniqueness

Before proving the rest of the Main Theorem, let us recall some results about the harmonic diffeomorphisms provided by [7]. We use the same notations as in the proof above. Let $z$ be conformal coordinates on $\Gamma$ such that

$$g_\Gamma = \rho^2(z)|dz|^2, \;\; g_i = \sigma_i^2(u_i(z))|du_i|^2.$$

The natural complex structure on $\Gamma$ provides a decomposition of vector-valued 1-forms on $\Gamma$ into their $\mathbb{C}$-linear and $\mathbb{C}$-antilinear part. In particular, for $i = 1, 2$, we get:

$$\frac{1}{\sqrt{2}} du_i = \partial u_i + \overline{\partial} u_i,$$

where $\partial u_i \in \Omega^{1,0}(\Gamma, u_i^* T\Sigma_\mathfrak{p} \otimes \mathbb{C})$, $\overline{\partial} u_i \in \Omega^{0,1}(\Gamma, u_i^* T\Sigma_\mathfrak{p} \otimes \mathbb{C})$. It follows that

$$e(u_i) = \frac{1}{2}\|du_i\|^2 = \|\partial u_i\|^2 + \|\overline{\partial} u_i\|^2,$$

which in coordinates gives

$$\begin{cases} \|\partial u_i\|^2(z) = \rho^{-2}(z)\sigma_i^2(u_i(z))|\partial_z u_i|^2 \\ \|\overline{\partial} u_i\|^2(z) = \rho^{-2}(z)\sigma_i^2(u_i(z))|\overline{\partial}_z u_i|^2. \end{cases}$$

Then we have the following expressions (cf. Section 3):

$$\begin{cases} \|\Phi(u_i)\| = \|\partial u_i\| \|\overline{\partial} u_i\| \\ e(u_i) = \|\partial u_i\|^2 + \|\overline{\partial} u_i\|^2 \\ J(u_i) = \|\partial u_i\|^2 - \|\overline{\partial} u_i\|^2. \end{cases}$$

Note that, as $u_i$ is orientation preserving, $J(u_i) > 0$ and in particular $\|\partial u_i\| \neq 0$.

It is well-known that these functions satisfy a Bochner type identities everywhere it is defined (see [19])

$$\begin{cases} \Delta \ln \|\partial u_i\| = \|\partial u_i\|^2 - \|\overline{\partial} u_i\|^2 - 1 \\ \Delta \ln \|\overline{\partial} u_i\| = -\|\partial u_i\|^2 + \|\overline{\partial} u_i\|^2 - 1, \end{cases} \tag{4}$$

where $\Delta = \Delta_{g_\Gamma} = \delta\delta^*$.

Note that, as $\Phi(u_i)$ is holomorphic outside $\mathfrak{p}$, the singularities of $\ln \|\overline{\partial} u_i\|$ on $\Sigma_{\mathfrak{p}}$ are isolated and have the form $c \ln r$ for some $c > 0$. In fact, as $J(u_i) > 0$, $\|\partial u_i\| \neq 0$. Because $\|\Phi(u_i)\| = \|\partial u_i\| \|\overline{\partial} u_i\|$, the singularities of $\ln \|\overline{\partial} u_i\|$ correspond to zeros of $\Phi(u_i)$.

Remind that, in a local holomorphic chart around a puncture $p$ of angle $2\pi\alpha$,

$$u_i(z) = \lambda_i z + r^{1+\nu} f_i(z),$$

where $\lambda_i \in \mathbb{C}^*$, $f \in \mathscr{C}_b^{2,\gamma}(U)$.

Using

$$\begin{cases} \partial_z = \frac{1}{2z}(r\partial_r - i\partial_\theta) \\ \overline{\partial}_z = \frac{1}{2\overline{z}}(r\partial_r + i\partial_\theta) \end{cases}$$

we get that

$$\begin{cases} \partial_z u_i = \lambda_i + r^\epsilon L(f_i) \\ \overline{\partial}_z u_i = r^\epsilon \overline{L}(f_i) \end{cases}$$

where

$$\begin{cases} L = \frac{r}{2z}\big((1+\epsilon)Id + \partial_r - i\partial_\theta\big) \\ \overline{L} = \frac{r}{2\overline{z}}\big((1+\epsilon)Id + \partial_r + i\partial_\theta\big). \end{cases}$$

Let $\alpha$ (resp. $\alpha'$) be the cone angle of the singularity of $g_1$ (resp. $g_2$) at $p$. So, from subsection 2.2, there exists some bounded non vanishing functions $c_1$ and $c_2$ so that

$$\begin{cases} \sigma_1^2(u_1) = c_1^2 |u_1|^{2(\alpha-1)} \\ \sigma_2^2(u_2) = c_2^2 |u_2|^{2(\alpha'-1)}. \end{cases}$$

It follows that

$$\begin{cases} \|\partial u_1\|^2 = \rho^{-2}(z) c_1^2 |\lambda_1 z + r^{1+\epsilon} f_1|^{2(\alpha-1)} |\lambda_1 + r^\epsilon L(f_1)|^2 \\ \qquad\qquad = \rho^{-2}(z) c_1^2 |\lambda_1|^{2\alpha} r^{2(\alpha-1)} \left(1 + O(r^\epsilon)\right) \\ \|\overline{\partial} u_1\|^2 = \rho^{-2}(z) c_1^2 |\lambda_1|^{2(\alpha-1)} r^{2(\alpha-1)+2\epsilon} |\overline{L}(f_1)|^2 (1 + O(r^\epsilon)). \end{cases} \tag{5}$$

**Proposition 4.3.** *If $\alpha_i < \alpha_i'$ for all $i \in \{1, \dots, n\}$, the minimal diffeomorphism $\Psi : (\Sigma_{\mathfrak{p}}, g_1) \longrightarrow (\Sigma_{\mathfrak{p}}, g_2)$ of Proposition 4.1 is unique.*

The proof follows from the stability of $\Gamma$.

**Lemma 4.4.** *Under the same conditions as in Proposition 4.3, a minimal graph $\Gamma \in (\Sigma_{\mathfrak{p}} \times \Sigma_{\mathfrak{p}}, g_1 \oplus g_2)$ is stable.*

*Proof.* Let $\Gamma$ be a minimal graph in $(\Sigma_{\mathfrak{p}} \times \Sigma_{\mathfrak{p}}, g_1 \oplus g_2)$, and denote by $u_i$ the $i^{th}$ projection from $\Gamma$ to $(\Sigma, g_i)$ (for $i = 1, 2$). As $\Gamma$ is minimal, the $u_i$ are harmonic and $\Phi(u_1) + \Phi(u_2) = 0$.

Stability of minimal graph in products of surfaces has been studied for the classical case in [26]. We have the following lemma:

**Lemma 4.5.** *Let $\Gamma$ be a minimal graph in $(\Sigma_{\mathfrak{p}} \times \Sigma_{\mathfrak{p}}, g_1 \oplus g_2)$, then the second variation of the area functional under a deformation of $\Gamma$ fixing its intersection with the singular loci is given by:*

$$A'' = E_2'' - 4 \int_\Gamma \frac{\|\Phi'(u_2)\|^2}{e(u_1) + e(u_2)} vol_\Gamma, \tag{6}$$

*where $E_2''$ is the second variation of the energy of $u_2$ and $\Phi'(u_2)$ is the variation of the Hopf differential of $u_2$.*

*Proof.* By definition, the area of $\Gamma$ is given by:

$$A = \int_\Gamma \left(\det(u_1^* g_1 \oplus u_2^* g_2)\right)^{1/2} |dz|^2.$$

But we have:

$$\begin{aligned}
\det(u_1^* g_1 \oplus u_2^* g_2) &= \det\big(\rho^2(e(u_1) + e(u_2))|dz|^2 + 2\Re(\Phi(u_1) + \Phi(u_2))\big) \\
&= \det\big(\rho^2(e(u_1) + e(u_2))(dx^2 + dy^2) + 2\Re(\phi(u_1) \\
&\qquad + \phi(u_2))(dx^2 - dy^2) - 2\Im(\phi(u_1) \\
&\qquad + \phi(u_2))(dxdy + dydx)\big) \\
&= \rho^4(e(u_1) + e(u_2))^2 - 4|\phi(u_1) + \phi(u_2)|^2,
\end{aligned}$$

where $\Phi(u_i) = \phi(u_i)dz^2$. It follows that

$$A = \int_\Gamma \left(e(u_1) + e(u_2))^2 - 4\|\Phi(u_1) + \Phi(u_2)\|^2\right)^{1/2} dv_\Gamma.$$

Writing

$$a := (e(u_1) + e(u_2))^2 - 4\|\Phi(u_1) + \Phi(u_2)\|^2,$$

we get

$$A = \int_\Gamma a^{1/2} dv_\Gamma.$$

Recall that, for $i = 1, 2$, we have

$$E(u_i) = \int_{\Sigma_\mathfrak{p}} e(u_i) dv_\Gamma.$$

Denote by $v_{1,t}$ and $v_{2,t}$ be the variations of $u_1$ and $u_2$ respectively corresponding to a variation $\Gamma_t$ of $\Gamma$. Set $\psi_i := \frac{d}{dt}_{|t=0} v_{i,t}$ which is a section of $u_i^* T\Sigma_\mathfrak{p}$. Denote by $\nabla^{u_i}$ the pull-back by $u_i$ of the Levi-Civita connection on $(\Sigma_\mathfrak{p}, g_i)$. In particular, we have:

$$\frac{d}{dt}_{|t=0} dv_{i,t} = \nabla^{u_i} \psi_i.$$

Now we have:

$$A''(\Gamma) = \frac{d^2}{dt^2}_{|t=0} \int_\Gamma a_t^{1/2} dv_\Gamma = \frac{1}{2} \int_\Gamma \left( a^{-1/2} a'' - \frac{1}{2} a^{-3/2} a'^2 \right) dv_\Gamma.$$

But

$$\begin{aligned}
a' &= \frac{d}{dt}_{|t=0} \left( (e(v_{1,t}) + e(v_{2,t}))^2 - 4(\|\Phi(v_{1,t}) + \Phi(v_{2,t})\|^2 \right) \\
&= 2(e(u_1) + e(u_2))(e'(u_1) + e'(u_2)) - 8\langle \Phi'(u_1) + \Phi'(u_2), \Phi(u_1) + \Phi(u_2) \rangle \\
&= 2(e(u_1) + e(u_2))(e'(u_1) + e'(u_2)),
\end{aligned}$$

and

$$\begin{aligned}
a'' &= \frac{d^2}{dt^2}_{|t=0} \left( (e(v_{1,t}) + e(v_{2,t}))^2 - 4(\|\Phi(v_{1,t}) + \Phi(v_{2,t})\|^2 \right) \\
&= 2(e'(u_1) + e'(u_2))^2 + 2(e(u_1) + e(u_2))(e''(u_1) + e''(u_2)) \\
&\quad - 8\|\Phi'(u_1) + \Phi'(u_2)\|^2.
\end{aligned}$$

Hence,

$$a^{-1/2} a'' - \frac{1}{2} a^{-3/2} a'^2 = 2(e''(u_1) + e''(u_2)) - 8 \frac{\|\Phi'(u_1) + \Phi'(u_2)\|^2}{e(u_1) + e(u_2)}.$$

It follows

$$A''(\Gamma) = E''(u_1) + E''(u_2) - 4\int_\Gamma \frac{\|\Phi'(u_1)+\Phi'(u_2)\|^2}{e(u_1)+e(u_2)} dv_\Gamma.$$

Now, as pointed out in [26], such a variation can be realized as a variation of $u_2$ only since the variation of $u_1$ can be interpreted as a change of coordinates which does not change the area functional. So, setting $\psi_1 = 0$, we get the formula. □

Writing $w_i := \ln \dfrac{\|\partial u_i\|}{\|\overline{\partial} u_i\|}$ and using equation (4), we obtain:

$$\begin{aligned}
\Delta w_i &= \Delta \ln \|\partial u_i\| - \Delta \ln \|\overline{\partial} u_i\| \\
&= 2\|\partial u_i\|^2 - 2\|\overline{\partial} u_i\|^2 \\
&= 2\|\Phi\| \left( \frac{\|\partial u_i\|}{\|\overline{\partial} u_i\|} - \left( \frac{\|\partial u_i\|}{\|\overline{\partial} u_i\|} \right)^{-1} \right) \\
&= 4\|\Phi\| \sinh w_i,
\end{aligned}$$

where $\|\Phi\| = \|\Phi(u_1)\| = \|\Phi(u_2)\|$. That is, $w_1$ and $w_2$ satisfy the same equation. Note that, outside $\mathfrak{p}$, the singularities of $w_1$ and $w_2$ are the same. In fact, singularities of $w_i$ correspond to zeros of $\|\partial u_i\|$ (as $J(u_i) = \|\partial u_i\|^2 - \|\overline{\partial} u_i\|^2 > 0$). But as $\|\Phi(u_1)\| = \|\partial u_1\| \|\overline{\partial} u_1\| = \|\partial u_2\| \|\overline{\partial} u_2\|$, the zeros of $\|\partial u_1\|$ and $\|\partial u_2\|$ are the same. In particular, $w_2 - w_1$ is a regular function on $\Sigma_{\mathfrak{p}}$ satisfying:

$$\Delta(w_2 - w_1) = 4\|\Phi\|(\sinh w_2 - \sinh w_1). \tag{7}$$

Let us study the behavior of $w_1 - w_2$ at a singularity $p \in \mathfrak{p}$. Using the same notation as above, the norm of the Hopf differentials satisfy:

$$\begin{aligned}
\rho^2(z)\|\Phi(u_1)\|(z) &= \sigma_1^2(u_1)|\partial_z u_1||\partial_z \overline{u}_1| \\
&= c_1^2 |\lambda_1 z + r^{1+\epsilon} f|^{2(\alpha-1)} |\lambda_i + r^\epsilon L(f_1)||r^\epsilon \overline{L}(f_1)| \\
&= c_1^2 |\overline{L}(f_1)||\lambda_i|^{2\alpha-1} r^{2(\alpha-1)+\epsilon}(1+O(r^\epsilon))
\end{aligned}$$

and

$$\begin{aligned}
\rho^2(z)\|\Phi(u_2)\|(z) &= \sigma_1^2(u_2)|\partial_z u_2||\partial_z \overline{u}_2| \\
&= c_2^2 |\lambda_2 z + r^{1+\epsilon} f_2|^{2(\alpha'-1)} |\lambda_2 + r^\epsilon L(f_2)||r^\epsilon \overline{L}(f_2)| \\
&= c_2^2 |\overline{L}(f_2)||\lambda_i|^{2\alpha'-1} r^{2(\alpha'-1)+\epsilon}(1+O(r^\epsilon)).
\end{aligned}$$

Hence, using $\|\Phi(u_1)\| = \|\Phi(u_2)\|$,

$$\left|\frac{\overline{L}(f_1)}{\overline{L}(f_2)}\right| = r^{2(\alpha'-\alpha)}C,$$

where $C$ is a non-vanishing bounded function. Now, using equation (5), we obtain:

$$w_i = \ln\left(\frac{|\lambda_i|}{r^\epsilon|\overline{L}(f_i)|}(1+O(r^\epsilon))\right) = \ln\left(\frac{|\lambda_i|}{r^\epsilon|\overline{L}(f_i)|}\right) + O(r^\epsilon).$$

In particular,

$$w_2 - w_1 = 2(\alpha-\alpha')\ln r + C', \tag{8}$$

where $C'$ is a bounded function. As $\alpha - \alpha' > 0$, $w_2 - w_1$ tends to $-\infty$ at the singularities.

So we can apply the maximum principle to equation (7), and we obtain that $w_2 \leq w_1$. Using $\|\Phi(u_1)\| = \|\Phi(u_2)\| = \|\Phi\|$, we finally obtain:

$$\|\partial u_2\| \leq \|\partial u_1\|.$$

Let us consider the function $f(x) = x + \|\Phi\|^2 x^{-1}$ defined on $\mathbb{R}_{>0}$. Its derivative is $f'(x) = 1 - \|\Phi\|^2 x^{-2}$, so $f$ is increasing for $x \geq \|\Phi\|$. As $J(u_2) > 0$,

$$\|\partial u_2\|^2 \geq \|\partial u_2\|\|\overline{\partial} u_2\| = \frac{\|\Phi\|}{2}.$$

Applying $f$ to $\|\partial u_2\|^2 \leq \|\partial u_1\|^2$, we get

$$e(u_2) \leq e(u_1).$$

So, from equation (6), we obtain:

$$A'' \geq E_2'' - 2\int_\Omega \frac{\|\Phi'(u_2)\|^2}{e(u_2)} vol_\Gamma.$$

Let $\psi := \frac{d}{dt}_{|t=0} v_t$ be a deformation of $u_2$ (so $\psi$ is a section of $u_2^* T\Sigma_{\mathfrak{p}}$). We have the following expression (see e.g [22, Equation 2]):

$$E''(u_2) = \int_\Gamma \left(\langle \nabla^{u_2}\psi, \nabla^{u_2}\psi\rangle - tr_{g_\Gamma} R^{g_2}(du_2, \psi, \psi, du_2)\right) dv_\Gamma,$$

where $R^{g_2}$ is the curvature tensor on $(\Sigma_{\mathfrak{p}}, g_2)$, $\nabla^{u_2}$ is the pull-back by $u_2$ of the Levi-Civita connection on $(\Sigma_{\mathfrak{p}}, g_2)$ and the scalar product is taken with respect to the metric $g_\Gamma^* \otimes u_2^* g_2$ on $T^*\Gamma \otimes u_2^* T\Sigma_{\mathfrak{p}}$. Computing $\Phi'$, we get:

$$\begin{aligned}\Phi' &= \frac{d}{dt}_{|t=0} v_t^* g_2(\partial_z, \partial_z) dz^2 \\ &= \frac{d}{dt}_{|t=0} g_2(dv_t(\partial_z), dv_t(\partial_z)) dz^2 \\ &= 2g_2(\nabla^{u_2}\psi(\partial_z), du_2(\partial_z)) dz^2.\end{aligned}$$

That is

$$\|\Phi'\|^2 = 4\sigma^2(u_2) |\langle \nabla^{u_2}\psi(\partial_z), du_2(\partial_z)\rangle|^2,$$

(where $\langle .,. \rangle$ is the scalar product with respect to $g_2$). By Cauchy-Schwarz and equation (3), we get

$$\begin{aligned}\|\Phi'\|^2 &\leq 4\sigma^2(u) \left|\langle \nabla^{u_2}\psi(\partial_z), \overline{\nabla^{u_2}\psi(\partial_z)}\rangle\right| \left|\langle du_2(\partial_z), \overline{du_2(\partial_z)}\rangle\right| \\ &\leq \frac{1}{4}\|\nabla^{u_2}\psi\|^2 \|du_2\|^2.\end{aligned}$$

Hence,

$$\int_\Gamma \frac{\|\Phi'\|^2}{e(u_2)} vol_\Gamma \leq \frac{1}{2}\int_\Gamma \langle \nabla^u \psi, \nabla^u\psi\rangle vol_\Gamma.$$

Finally, we obtain:

$$A'' \geq -\int_\Gamma tr_{g_\Gamma} R^{g_2}(du, \psi, \psi, du) dv_\Gamma.$$

But as the sectional curvature of $(\Sigma_{\mathfrak{p}}, g_2)$ is $-1$, the right-hand side of the last equation is strictly positive (for a non zero $\psi$). So $\Gamma$ is strictly stable. $\square$

Now, using the classical estimates (see [5, Proposition p.126] or the proof of lemma 4.5),

$$\mathrm{Area}(\Gamma) \leq E(\Psi)$$

and equality holds if and only if $\Psi$ is a minimal immersion. It follows from the stability of $\Gamma$ that the critical points of $\mathscr{E}_{g_1} + \mathscr{E}_{g_2}$ can only be minima. But a proper function whose unique extrema are minima with non-degenerate Hessian admits a unique minimum. So $\Psi$ is the unique minimal diffeomorphism isotopic to the identity.

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