Communications in
Analysis and Geometry
Volume 27, Number 6, 1355–1379, 2019

3-manifolds admitting locally large distance 2 Heegaard splittings

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It is known that every closed, orientable 3-manifold admits a Heegaard splitting. By Thurston's Geometrization conjecture, proved by Perelman, a 3-manifold admitting a Heegaard splitting of distance at least 3 is hyperbolic. So what about 3-manifolds admitting distance at most 2 Heegaard splittings?

Inspired by the construction of hyperbolic 3-manifolds in [Qiu, Zou and Guo, Pacific J. Math. 275 (2015), no. 1, 231-255], we introduce the definition of a locally large geodesic in curve complex and also a locally large distance 2 Heegaard splitting. Then we prove that if a 3-manifold admits a locally large distance 2 Heegaard splitting, then it is either a hyperbolic 3-manifold or an amalgamation of a hyperbolic 3-manifold and a small Seifert fiber space along an incompressible torus. After examining those non hyperbolic cases, we give a sufficient and necessary condition to determine a hyperbolic 3-manifold admitting a locally large distance 2 Heegaard splitting.

1. Introduction

In 1898, Heegaard [8] introduced a Heegaard splitting for a closed, orientable, triangulated 3-manifold, i.e., there is a closed, orientable surface cutting this manifold into two handlebodies. Later, Moise [16] proved that every closed, orientable 3-manifold admits a triangulation. So each closed orientable 3-manifold admits a Heegaard splitting.

For a closed orientable 3-manifold M, Kneser and Milnor proved that it is the connected sum of a unique collection of finitely many irreducible 3-manifolds and $S^2 \times S^1$. It is known that Heegaard splittings of $S^2 \times S^1$ are standard. Therefore we only concern Heegaard splittings of an irreducible 3-manifold.

One astonishing result by Haken [4] is that every Heegaard splitting of a reducible 3-manifold is reducible, i.e., there is an essential simple closed curve in Heegaard surface bounding two essential disks for both of these two sides simultaneously. So if all Heegaard splittings of a 3-manifold are reducible, then this manifold is reducible. Therefore for any irreducible 3-manifold, there is at least one irreducible Heegaard splitting. Later, Casson and Gordon [1] defined a weakly reducible Heegaard splitting and proved that if a 3-manifold admits a weakly reducible and irreducible Heegaard splitting, then there is a closed incompressible surface in it, i.e., it is Haken. Both of these two phenomena drive people to think how Heegaard splittings reflect 3-manifolds.

For classifying 3-manifolds, Thurston [28] introduced the Geometrization conjecture (the Haken version proved by Thurston [28] and the full version proved by Perelman [19–21]) as follows. For any closed, irreducible, orientable 3-manifold, there are finitely many (possibly zero) disjoint, non isotopic essential tori so that after cutting it along those tori, each piece has one of eight geometries (among all of those eight geometries, one is hyperbolic, another one is solvable and the left six pieces are realized by Seifert fiber spaces). In those eight geometries, Seifert fiber spaces have been completely classified. Moreover all of their irreducible Heegaard splittings are either vertical or horizontal, see [17]. Cooper and Scharlemann [2] studied all irreducible Heegaard splittings of a solvmanifold. And there are some works on Heegaard splittings of some typical 3-manifolds, such as Lens space, surface $\times S^1$ etc.

With the curve complex defined by Harvey [6], Hempel [9] introduced an index-Heegaard distance for studying a Heegaard splitting. Mainly Heegaard distance is defined to be the length of a shortest geodesic in the curve complex which connects two vertices representing the boundary curves of two essential disks from different sides. Then he proved that all Heegaard splittings of a Seifert fiber space have distance at most two; if a 3-manifold contains an essential torus, then its all Heegaard splittings have distance at most two. This result is also proved and extended by Hartshorn [5] and Scharlemann [24]. Then by the Geometrization theorem, if a 3-manifold admits a Heegaard splitting with Heegaard distance at least three, then it is hyperbolic. So a question arises.

Question 1.1. What dose a 3-manifold look like if it only admits distance at most two Heegaard splittings?

Before stating some results related to Question 1.1, let us introduce a remark.

Remark 1.2. By the definition, for a distance two Heegaard splitting, there are an essential simple closed curve and a pair of essential disks from different handlebodies so that this curve is disjoint from those two essential disks' boundary curves. It seems that this Heegaard splitting is simple and so is the 3-manifold. However, things for distance two Heegaard splittings are complicated because a 3-manifold admitting a distance two Heegaard splitting could be a Seifert fiber space or hyperbolic or contains an essential torus, see [9, 22, 23, 27].

Thompson [27] studied all distance at most two and genus two Heegaard splittings and found that even for genus two Heegaard splittings, those 3-manifolds could be very complicated. Later, Rubinstein and Thompson [23] extended this result to genus at least three cases. In [22, 29], the authors studied the curve complex and introduced the definition of a locally large geodesic. In the proof of Theorem 1.3 in [22], they found that the locally large property of a geodesic forces every realizing Heegaard distance geodesic to share some vertex γ in common. So if the resulted 3-manifold contains an essential torus T^2, then T^2 intersects this Heegaard surface in some essential simple closed curves, which are all isotopic to γ. Thus T^2 intersects that Heegaard surface in fixed essential simple closed curves. If the resulted 3-manifold contains no essential torus, by Geometrization theorem, it is either a small Seifert fiber space or a hyperbolic 3-manifold. Since a small Seifert fiber space is well understood and so are all of its Heegaard splittings, we may understand this 3-manifold. Under this circumstance, it seems it is not hard to understand the corresponding 3-manifold. So we introduce the definition of a locally large distance two Heegaard splitting.

We denote a length two geodesic in the curve complex realizing Heegaard distance by $\mathcal{G} = \{\alpha, \gamma, \beta\}$, where both α and β bound essential disks from two sides of the Heegaard surface and γ is disjoint from both α and β. It is known that there is a nonseparating essential simple closed curve disjoint from α and β. So we may assume γ is represented by a nonseparating essential simple closed curve. Suppose that S is the Heegaard surface. Then the geodesic $\mathcal{G}$ is locally large if in $S_\gamma = \overline{S - \gamma}$, $d_{\mathcal{S}_\gamma}(a, b) \geq 11$, for any pair of a and b disjoint from γ, where both a and b bound essential disks in different sides of S respectively. Moreover, a Heegaard splitting is locally large if there is a locally large geodesic realizing its Heegaard distance. With the definition of a locally large Heegaard splitting, we have the following result.

Theorem 1.3. *If a closed orientable 3-manifold M admits a locally large distance two Heegaard splitting $V \cup_S W$, then M is either a hyperbolic 3-manifold or an amalgamation of a hyperbolic 3-manifold and a small Seifert fiber space along an incompressible torus.*

Note 1.4. A 3-manifold is small if there is no embedded essential closed orientable surface in it.

The following facts are well-known to experts (see, for example [26]).

(1) a Seifert fiber space does not admit a complete hyperbolic structure;

(2) a solvmanifold does not admit a complete hyperbolic structure;

(3) an amalgamation of a complete hyperbolic 3-manifold and a small Seifert fiber space along a torus is not one of those eight geometries.

Thus by Theorem 1.3,

Corollary 1.5. *Neither a solvmanifold nor a Seifert fiber space admits a locally large distance two Heegaard splitting.*

The most important geometry in the Geometrization Theorem is hyperbolic geometry. Thus giving a sufficient condition for a hyperbolic 3-manifold is critical in studying its Heegaard splittings. Although the example in Section 3 shows that there is a case where the manifold M in Theorem 1.3 contains an essential torus, to give a sufficient condition for a hyperbolic 3-manifold, we only need to eliminate all possible essential tori in it. For this purpose, we introduce two definitions.

An essential simple closed curve γ in $\partial_+ V$ is a co-core if there is an essential disk in V so that its boundary curve intersects γ in one point. A once punctured torus in S is called a torus domain in V (resp. W) if it is essential in S and its boundary curve bounds a disk in V (resp. W). The proof of Theorem 1.3 implies that under the locally large condition of γ, the non hyperbolic case happens if γ is not only contained in two torus domains in both V and W but also not a co-core for both of these two sides of S. Then there is a theorem as follows.

Theorem 1.6. *Suppose that a closed orientable 3-manifold M has a locally large distance two Heegaard splitting $V \cup_S W$. Let γ be an essential simple closed curve disjoint from two boundary curves of a pair of essential disks from different sides of S. Then M is hyperbolic if and only if either there is no torus domain containing γ in at least one of V and W or γ is a co-core for one side of S.*

We introduce some results of curve complex and Seifert fiber spaces in Section 2, construct a non hyperbolic 3-manifold in Section 3 and prove Theorem 1.3 and 1.6 in Sections 4 and 5.

Acknowledgement. The authors would like to thank Tao Li for careful checking our early version of this manuscript, pointing out the genus two case of Theorem 1.3 and many helpful conversations, thank referees for many suggestions. This work was partially supported by NSFC (No.11571110, 11601065, 11726609, 11726610) and Science and Technology Commission of Shanghai Municipality (STCSM), grant No. 18dz2271000.

2. Heegaard splittings of Seifert fiber spaces

Before studying Heegaard splittings of Seifert fiber spaces, we introduce some definitions and results of the curve complex.

Let S be a compact orientable surface of genus at least 1. A simple closed curve c in S is essential if c bounds no disk in S and is not parallel to ∂S. c also represents the isotopy class of an essential simple closed curve in S. For simplicity, without any further notation, we don't distinguish an essential simple closed curve c and its isotopy class.

Harvey [6] defined the curve complex $\mathcal{C}(S)$ as follows: the vertices of $\mathcal{C}(S)$ are the isotopy classes of essential simple closed curves on S, and $k+1$ distinct vertices $x_0, x_1, \ldots, x_k$ determine a k-simplex of $\mathcal{C}(S)$ if and only if they are represented by pairwise disjoint essential simple closed curves. For any two vertices x and y of $\mathcal{C}(S)$, the distance of x and y, denoted by $d_{\mathcal{C}(S)}(x, y)$, is defined to be the minimal number of 1-simplexes in a simplicial path joining x to y. In other words, $d_{\mathcal{C}(S)}(x, y)$ is the smallest integer $n \geq 0$ so that there is a sequence of vertices $x_0 = x, \ldots, x_n = y$ where x_{i-1} and x_i are represented by two disjoint essential simple closed curves on S for each $1 \leq i \leq n$. Therefore for any two sets of vertices in $\mathcal{C}(S)$, say X and Y, $d_{\mathcal{C}(S)}(X, Y)$ is defined to be $min\{d_{\mathcal{C}(S)}(x, y) \| \ x \in X, \ y \in Y\}$. When S is a torus or once punctured torus, there is a slight change on the definition of curve complex. Masur and Minsky [13] define $\mathcal{C}(S)$ as follows. The vertices of $\mathcal{C}(S)$ are the isotopy classes of essential simple closed curves on S, and $k+1$ distinct vertices $x_0, x_1, \ldots, x_k$ determine a k-simplex of $\mathcal{C}(S)$ if and only if x_i and x_j are represented by two simple closed curves c_i and c_j on S such that c_i intersects c_j in one point for each $0 \leq i \neq j \leq k$. The following lemma is well known, see [12–14].

Lemma 2.1. *$\mathcal{C}(S)$ is connected and the diameter of $\mathcal{C}(S)$ is infinite.*

A collection $\mathcal{G} = \{a_0, a_1, \ldots, a_n\}$ is a geodesic in $\mathcal{C}(S)$ if $a_i \in \mathcal{C}^0(S)$ and

$$d_{\mathcal{C}(S)}(a_i, a_j) = |i - j|,$$

for any $0 \leq i, j \leq n$. And the length of $\mathcal{G}$ denoted by $\mathcal{L}(\mathcal{G})$ is defined to be n. By Lemma 2.1, for any two vertices α and β, there is a shortest path in

$\mathcal{C}^1(S)$ connecting them. For any two simplicial sub-complex $X, Y \subset \mathcal{C}(S)$, we call a geodesic $\mathcal{G}$ realizing the distance between X and Y if $\mathcal{G}$ connects an element $\alpha \in X$ and an element $\beta \in Y$ so that $\mathcal{L}(\mathcal{G}) = d_{\mathcal{C}(S)}(X, Y)$.

If $\partial S \neq \emptyset$, then there are essential properly embedded simple arcs. Similar to the definition of the curve complex $\mathcal{C}(S)$, we define the arc and curve complex $\mathcal{AC}(S)$ as follows. Each vertex of $\mathcal{AC}(S)$ is the isotopy class of an essential simple closed curve or an essential properly embedded arc in S, and a set of vertices forms a simplex of $\mathcal{AC}(S)$ if these vertices are represented by pairwise disjoint arcs or curves in S. For any two disjoint vertices, we place an edge between them. All the vertices and edges form 1-skeleton of $\mathcal{AC}(S)$, denoted by $\mathcal{AC}^1(S)$. And for each edge, we assign it length 1. Thus for any two vertices α and β in $\mathcal{AC}^1(S)$, the distance $d_{\mathcal{AC}(S)}(\alpha, \beta)$ is defined to be the minimal length of paths in $\mathcal{AC}^1(S)$ connecting them.

Let F be a subsurface of S, not an annulus, a pair of pants or 4-holed sphere. F is essential in S if the induced map of the inclusion from $\pi_1(F)$ to $\pi_1(S)$ is injective. Moreover, F is a proper essential subsurface of S if F is essential in S and at least one boundary component of F is not ∂-parellel in S. It is known that each essential simple closed curve in F is essential in S. So there is some connection between the $\mathcal{AC}(F)$ and $\mathcal{C}(S)$. In other words, for any $\alpha \in \mathcal{C}^0(S)$, there is an essential simple closed curve α_{geo} that represents α so that the intersection number $i(\alpha_{geo}, \partial F)$ is minimal. Hence each component of $\alpha_{geo} \cap F$ (resp. $\alpha_{geo} \cap F$) is essential in F (resp. $S - F$). Let $\kappa_F(\alpha)$ be isotopy classes of the essential components of $\alpha_{geo} \cap F$. For any $\gamma \in \mathcal{AC}(F)$, $\sigma_F(\gamma)$ is the collection of all essential boundary curves of a closed regular neighborhood of $\gamma \cup \partial F$ in F. Then let $\pi_F = \sigma_F \circ \kappa_F$. So π_F is the subsurface projection defined in [14].

We say that $\alpha \in \mathcal{C}^0(S)$ cuts F if $\pi_F(\alpha) \neq \emptyset$. If α, $\beta \in \mathcal{C}^0(S)$ both cut F, then $d_{\mathcal{C}(F)}(\alpha, \beta)$ is defined to be $diam_{\mathcal{C}(F)}(\pi_F(\alpha), \pi_F(\beta))$. So if $d_{\mathcal{C}(S)}(\alpha, \beta) = 1$, then

$$d_{\mathcal{AC}(F)}(\alpha, \beta) \leq 1; \quad d_{\mathcal{C}(F)}(\alpha, \beta) \leq 2.$$

The following lemma immediately follows from the above observation.

Lemma 2.2. *Let F and S be as above. If $\mathcal{G} = \{\alpha_0, \ldots, \alpha_k\}$ is a geodesic of $\mathcal{C}(S)$ such that α_j cuts F for each $0 \leq i \leq k$, then $d_{\mathcal{C}(F)}(\alpha_0, \alpha_k) \leq 2k$.*

For essential curves α, β in S, let $|\alpha \cap \beta|$ be the minimal geometric intersection number. We call α and β intersect efficiently if the number of $\alpha \cap \beta$ is equal to $|\alpha \cap \beta|$.

One tool for studying the intersection between essential simple closed curves and arcs in S is the bigon criterion.

Lemma 2.3. *[3] Let surface S be as above. Then for any two essential curves α, β in S, α and β intersects efficiently if and only if $\alpha \cup \beta \cup \partial S$ bounds no bigon or half-bigon in S.*

Assume that V is a non-trivial compression body, i.e., not the product I-bundle of a closed surface. Then there is an essential simple closed curve in $\partial_+ V = S$ bounding an essential disk in V. Let F be an essential subsurface in S. We call F is a hole for V if for any essential disk $D \subset V$, $\pi_F(\partial D) \neq \emptyset$. Furthermore, we call an essential subsurface $F \subset S$ is an incompressible hole for V if F is not only a hole for V but also incompressible in V. Otherwise, F is a compressible hole for V. Masur and Schleimer[15] studied the subsurface projection of an essential disk and proved that:

Lemma 2.4. *Let V be a non-trivial compression body and F a compressible hole for V. Then for an essential disk D in V, there are two essential disks D_1 and D_2 satisfying:*

· ∂D, ∂D_1 and ∂F intersect efficiently;

· $\partial D_2 \subset F$;

· A component of $\partial D \cap F$ is disjoint from a component of $\partial D_1 \cap F$ and $\partial D_1 \cap \partial D_2 = \emptyset$. Furthermore, $d_{\mathcal{AC}(F)}(\pi_F(\partial D), \partial D_2) \leq 3$.

Proof. See the proof of Lemma 11.5 and Lemma 11.7 [15]. □

Let $\{x_1, x_2, \ldots, x_n\}$ be a collection of different points in S. In the 3-manifold $S \times S^1$, $\mathcal{SC} = \{x_i \times S^1, i = 1, \ldots, n\}$ is a collection of essential simple closed curves. A closed orientable 3-manifold is a Seifert fiber space if it is obtained by doing Dehn surgeries along $\mathcal{SC}$. More precisely, we remove a open regular neighborhood of $x_i \times S^1$ and attach a solid torus back where the meridian curve coincides with some β_i/α_i slope, where $\alpha_i \neq 0$. If $\beta_i/\alpha_i \neq 0$ and $|\alpha_i| > 1$, then this slope is called exceptional. It is known that if all of those exceptional slopes are removed, then $M - \cup N(\beta_i/\alpha_i)$ is $F \times S^1$. So there is another representation of $M = \{S, \beta_1/\alpha_1, \ldots, \beta_n/\alpha_n\}$, where S is called the base surface. Furthermore this representation is unique with some permutations in order, see [7].

Of all irreducible Heegaard splittings of a Seifert fiber space M, there are two standard ones named vertical and horizontal Heegaard splittings. To be clear, for $M = \{S, \beta_1/\alpha_1, \ldots, \beta_n/\alpha_n\}$ with projection $f : M \to S$, let $S = D \cup E \cup F$ be a cell decomposition where each component of D or F

contains at most one singular point in its interior and each component of E is a square with one pair of opposite edges in D and the other one in F, where both $D \cup E$ and $E \cup F$ are connected. Then the union of $H_1 = f^{-1}(D) \cup E \times [0, \frac{1}{2}]$ is a handlebody and H_2, the complement of H_1 in M, is also a handlebody which is homeomorphic to $f^{-1}(F) \cup E \times [\frac{1}{2}, 1]$, where $S^1 = [0,1]/\sim$. So $H_1 \cup_{\partial H_2} H_2$ is a Heegaard splitting of M, called a vertical Heegaard splitting. So by the construction of a vertical Heegaard splitting, every Seifert fiber space M admits a vertical Heegaard splitting. Knowing that fact, people wonder that whether there is any different type Heegaard splitting for a Seifert manifold in general or not. However, there is no other type Heegaard splitting in general because in [17], Moriah and Schultens proved that except some special cases, every irreducible Heegaard splitting of almost all of Seifert fiber spaces is vertical. For special cases, they [17] defined so called horizontal Heegaard splittings as follows. Taking a surface bundle $M_1 = F \times I/(x,0) \sim (\psi(x), 1)$, where $\chi(F) \leq 0$ with one boundary component and $\psi : F \to F$ is a periodic homeomorphism and fixes ∂F point by point. Let M be a Dehn filling of $M_1 \cup D \times S^1$, where the longitude goes to ∂F. Then $\partial F \times \{0, \frac{1}{2}\}$ bounds an annulus $\mathcal{A}$ in $D \times S^1$ which cuts out an I-bundle of $\mathcal{A}$. It is not hard to see that $F \times \{0, \frac{1}{2}\} \cup \mathcal{A}$ cuts M into two handlebodies, where both of these two handlebodies are compact surfaces product I-bundles. So it gives a Heegaard splitting of M, called a horizontal Heegaard splitting. A result in [17] says that M admits a horizontal Heegaard splitting if and only if its euler number is zero; each irreducible Heegaard splitting of M is either vertical or horizontal.

If the base surface of M has a genus at least one or is S^2 but with at least 4 exceptional slopes, then it contains an essential torus T^2. So for any strongly irreducible Heegaard splitting $H_1 \cup_{\partial H_1} H_2$, there are two essential annuli $\mathcal{A}_1 \subset H_1$ and $\mathcal{A}_2 \subset H_2$ with $\partial\mathcal{A}_1 \cap \partial\mathcal{A}_2 = \emptyset$. If M has S^2 as its base surface with at most three exceptional slopes, then

Lemma 2.5. *(1) for a vertical Heegaard splitting, there are essential disks D_1 and D_2 from two sides of Heegaard surface so that their boundaries intersects in at most two points;*

(2) for a horizontal Heegaard splitting, there is an essential simple closed curve C and two essential annuli $\mathcal{A}_1 = C \times [0, \frac{1}{2}]$ in H_1 and $\mathcal{A}_2 = C \times [\frac{1}{2}, 1]$ in H_2 so that $C \times \{0\} \cap C \times \{1\}$ contains at most one point.

Proof. The proof of second part is contained in the proof of Theorem 3.5 in [9]. So all we need to do is to prove the first part. Since a weakly reducible

Heegaard splitting satisfies the conclusion, we only consider all strongly irreducible vertical Heegaard splittings. If this vertical Heegaard splitting has genus at least 3, Corollary 3.3 in [9] says that it has distance at most 1. Hence there are two essential disks satisfying the conclusion of Lemma 2.5. If this vertical Heegaard splitting has genus 2, by the definition, one handlebody H_1 is the union of two closed neighborhood of exceptional fibers and a rectangle $\times[0, \frac{1}{2}]$ and the other one H_2 is homeomorphic to an I-bundle of one-holed torus with a non trivial Dehn surgery. It is not hard to see that removing these two exceptional fibers reduces M into a torus $\times$ I with a non trivial Dehn surgery, where we do the Dehn surgery along a simple closed curve C and the union of a longitude and C bounds an embedded annulus. The rectangle $\times[0, \frac{1}{2}]$ is isotopic to the closed neighborhood of an properly embedded unknotted arc which connects these two boundaries. After removing a rectangle $\times[0, \frac{1}{2}]$, it is changed into the handlebody H_2.

Let a be an properly embedded arc in this rectangle where it connects a pair of opposite edges. Then $a \times [0, \frac{1}{2}]$ bounds an essential disk D_1 in H_2. Let b be an properly embedded essential arc in the once punctured torus which intersects the longitude empty. Then $b \times [\frac{1}{2}, 1]$ bounds an essential disk D_2 in H_2. It is not hard to see that ∂D_1 intersects ∂D_2 in two points. □

By the proof of Lemma 2.5, if a vertical Heegaard splitting $V \cup_S W$ of a small Seifert fiber space is strongly irreducible, then it has genus 2. Conversely, for a genus 2 strongly irreducible and vertical Heegaard splitting $V \cup_S W$ of a Seifert fiber space, Hempel [9] proved that its genus is equal to the sum of the number of rectangles and 1. So it means that this Seifert fiber space is a small Seifert fiber space with S^2 as its base surface and three exceptional fibers. Then there is an interesting result as follows.

Corollary 2.6. *Let $V \cup_S W$ be a genus 2 strongly irreducible and vertical Heegaard splitting of a Seifert fiber space. Then there are two essential disks D_1 (resp. D_2) for V (resp. W) and two disjoint but non isotopy essential simple closed curve C_1 and C_2 in S so that both C_1 and C_2 are disjoint from $\partial D_1 \cup \partial D_2$.*

Proof. Let D_1 and D_2 be as in Lemma 2.5. It is known that H_2 is a once punctured torus I-bundle with a non trivial Dehn surgery, i.e., $H_2 = [\overline{T^2 - B} \times I](\frac{\beta}{\alpha})$ for some $\frac{\beta}{\alpha} \neq 0$. Let C_1 be a longitude of $\overline{T^2 - B} \times \{0\}$ while C_2 is a longitude of $\overline{T^2 - B} \times \{1\}$. Then both of these two curves are disjoint from $\partial D_1 \cup \partial D_2$. □

3. A non hyperbolic 3-manifold with a locally large distance 2 Heegaard splitting

In this section, we construct a non hyperbolic 3-manifold, which admits a locally large distance 2 Heegaard splitting.

Let M be a genus $2g-1$ compression body, where $g \geq 2$, with $\partial_- M$ a torus. Then there are two nonseparating spanning annuli A_1 and A_2, i.e., the boundary of an essential annulus lies in different components of ∂M, such that $\overline{M - A_1 \cup A_2}$ are two handlebodies V_1 and V_2 with same genera, see Figure 1.

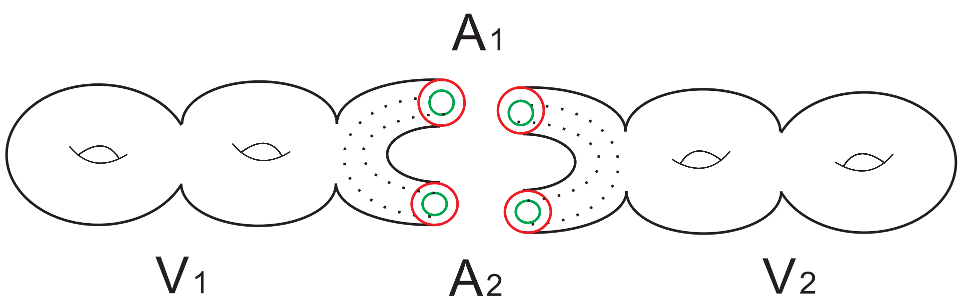

Figure 1: Annuli in M.

From Figure 1, ∂V_1 (resp. ∂V_2) consists of S_1 and an annulus A_1^1-the left two dotted line (resp. S_2 and an annulus A_2^1 -the right two dotted line). Since S_1 and S_2 are homeomorphic, there is a orientation reversing homeomorphism $f: S_1 \to S_2$ such that $f(\partial A_1^1 \cap S_1) = \partial A_2^1 \cap S_2$.

Since S_1 is a genus $g-1 \geq 1$ surface with two boundary curves, the Projective Measured Lamination space of S_1

$$\mathcal{PML}(S_1) \cong S^{6g-9}$$

is not an empty set. It is known that the isotopy class of the boundary of an essential disk in V_1 together with a counting measure is an element of $\mathcal{PML}(S_1)$. Then the collection of all essential simple closed curves bounding disks is a subset of $\mathcal{PML}(S_1)$. It is known that the intersection function on $\mathcal{ML}(S)$ defines a weak*-topology on $\mathcal{ML}(S)$, see [18]. Then there is a topology defined on $\mathcal{PML}(\mathcal{S})$ induced by the projection $P: \mathcal{ML}(S) \to \mathcal{PML}(S)$. Under this topology, let $\mathcal{DS}_1 \subset \mathcal{PML}(S_1)$ be the closure of the set consisting of all essential simple closed curves in S_1 which bound disks in V_1. So is $\mathcal{DS}_2$. By the symmetry of these two handlebodies V_1 and V_2, there is an automorphism of $h : S_1 \to S_1$ such that $h \circ f(\mathcal{DS}_2) \subset \mathcal{DS}_1$.

Fact 3.1. $\mathcal{DS}_1$ is nowhere dense.

Note 3.2. The proof is based on and contained in the proof of Theorem 1.2 [11]. For the integrity of this paper, we use the theory of Measured Lamination Space to rewrite it here.

Before proving Fact 3.1, let us introduce a definition here. For any essential simple closed curve $\alpha \subset S_1$ bounding an essential disk in V_1, there is a disk system Γ in S_1 so that

(1) one of its vertices is α;

(2) all of its vertices are the isotopy classes of the boundary curves of pairwise disjoint non-isotopic essential disks in V_1;

(3) it splits S_1 into a collection of pairs of pants.

Proof. All we need to prove is $\mathcal{DS}_1$ contains no open set in $\mathcal{PML}(S_1)$.

Choosing an element $\alpha \in \mathcal{DS}_1$ represented by an essential non-separating simple closed curve in S_1, by the above argument, there is a disk system Γ in S_1. For any element $\beta \in \mathcal{DS}_1$ represented by an essential simple closed curve in S_1, by Lemma 2.3, we can isotope β such that β intersects Γ efficiently. If $\beta \cap \Gamma \neq \emptyset$, then there is a wave w corresponding to the outermost disk component in the complement of Γ in S_1. Since ∂S_1 bounds no essential disk in V_1, the wave w is contained in a pair of pants bounded by the boundaries of essential disks. If β intersects Γ empty, then $\beta \in \Gamma$.

From Penner and Harer [18], there is a birecurrent maximal train track τ in S_1 such that it intersects all the wave like w for the disk system Γ. Then there is a minimal filling measured lamination $\mathcal{L}$ carried by τ intersecting all the wave like w so that its complement in S_1 is a disk or a one-holed disk with a finite points removed from its boundary, where the one-holed disk contains a boundary component of S_1. Moreover, $\mathcal{L}$ is not in $\mathcal{DS}_1$ because it intersects each element in $\mathcal{DS}_1$ nontrivially.

It is known that the collection of essential simple closed curves in S_1 is dense in $\mathcal{PML}(S_1)$. Then there is a sequence $\{c_1, \ldots, c_n, \ldots\}$ converging to $\mathcal{L}$ in $\mathcal{PML}(S_1)$, where c_i is represented by an essential simple closed curve. Hence there is a number N such that c_{N+1} intersects all the waves like w for the disk system Γ. So there is a neighborhood U of c_{N+1} in $\mathcal{PML}(S_1)$ disjoint from $\mathcal{DS}_1$ in $\mathcal{PML}(S_1)$.

Now suppose that there is an open subset $U' \subset \mathcal{DS}_1$. Then there is an automorphism $f : S_1 \to S_1$, where $f(\mathcal{DS}_1) = \mathcal{DS}_1$, and a nonseparating essential curve $\alpha_1 \in U'$ bounding a disk in V_1 such that $f(\alpha_1) = \alpha$ and $f(U') \subset \mathcal{DS}_1$ is an open neighborhood of α in $\mathcal{PML}(S_1)$.

For each essential simple closed curve $c \subset S_1$ which intersects α, let τ_α be the Dehn twist along α in S_1. It is known that $\tau_\alpha^n(c)$ is closed to α in $\mathcal{PML}(S_1)$ as n goes to the infinity. Then $\tau_\alpha^n(c_{N+1})) \subset f(U')$ for n large enough. Hence there is an open subset $U_1 \subset U$ such that $\tau_\alpha^n(U_1) \subset f(U')$. It means that $f^{-1} \circ \tau_\alpha^n(U_1) \subset U'$. Then $\tau_\alpha^{-n} \circ f(\mathcal{DS}_1)$ is not contained in $\mathcal{DS}_1$. But since α bounds an essential disk in V_1, both of these two maps τ_α and τ_α^{-1} map $\mathcal{DS}_1$ into $\mathcal{DS}_1$. Hence $\tau_\alpha^{-n} \circ f$ maps $\mathcal{DS}_1$ into $\mathcal{DS}_1$, a contradiction. □

By Fact 3.1, $\mathcal{DS}_1$ is not equal to $\mathcal{PML}(S_1)$. Since the collection of those stable and unstable laminations of all pseudo anosov automorphisms in S_1 is dense in $\mathcal{PML}(S_1)$, there is a pseudo anosov map g in S_1 such that the stable lamination are not in $\mathcal{DS}_1$. By the proof of Theorem 2.7 [9], if n is large enough, then $d_{\mathcal{C}(S_1)}(g^n(\alpha), h \circ f(\beta)) \geq 11$ for any α and β bounding two essential disks in V_1 and V_2 respectively.

For constructing a non hyperbolic 3-manifold, we set M_1 be $V_2 \cup_{g^{-n} \circ h \circ f} V_1$ along S_2 and $S = \partial V_1$ in M_1. After pushing S a little into the interior of M, S splits M_1 into a handlebody V and a compression body W, see Figure 2 (S is colored in green while S_1 is colored in red, where S is parallel to the union of S_1 and an annulus $A \subset \partial M_1$).

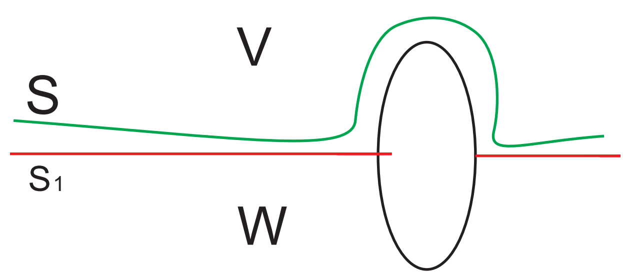

Figure 2: Heegaard surface S.

A Heegaard splitting is weakly reducible if there is a pair of essential disks from different sides of the Heegaard surface so that their boundary curves are disjoint. Otherwise, the Heegaard splitting is strongly irreducible.

Fact 3.3. The Heegaard splitting $V \cup_S W$ is strongly irreducible.

Proof. Suppose $V \cup_S W$ is weakly reducible. Then there is a pair of essential disks $D \subset V$ and $E \subset W$ so that $\partial D \cap \partial E = \emptyset$. From the construction of M_1, A is incompressible in M_1. Let $S_{1,1} \subset S_1$ be $S_1 - N(\partial S_1)$, where $N(\partial S_1)$ is a regular neighborhood of ∂S_1 in S_1. After pushing the closure of $A \cup N(\partial S_1)$ a little into M_1 so that it is disjoint from $S_{1,1}$, $\overline{A \cup N(\partial S_1)}$ is turned into an embedded annulus $A_{1,1}$. Then S is isotopic to $S_{1,1} \cup A_{1,1}$. It is not hard to see that every essential disk of V (resp. W) has the property that its boundary curve cuts $S_{1,1}$. It means that $S_{1,1}$ is a hole for both V and W. By the construction of $V \cup_S W$, there is an essential disk in V (resp. W) with its boundary in $S_{1,1}$. Then $S_{1,1}$ is a compressible hole for both V and W.

By Lemma 2.4, for the essential disk D, there is an essential disk $D_1 \subset V$ such that

(1) $\partial D_1 \subset S_{1,1}$;

(2) there is a component $a \subset \partial D \cap S_{1,1}$ such that $d_{\mathcal{C}(S_{1,1})}(\pi_{S_{1,1}}(a), \partial D_1) \leq 3$;

Similarly for the essential disk E, there is an essential disk $E_1 \subset W$ such that

(1) $\partial E_1 \subset S_{1,1}$;

(2) there is a component $b \subset \partial E \cap S_1$ such that $d_{\mathcal{C}(S_{1,1})}(\pi_{S_{1,1}}(b), \partial E_1) \leq 3$;

Since $\partial D \cap \partial E = \emptyset$, by Lemma 2.2, $d_{\mathcal{C}(S_{1,1})}(\pi_{S_{1,1}}(a), \pi_{S_{1,1}}(b)) \leq 2$. By triangle inequality,

$$d_{\mathcal{C}(S_{1,1})}(\partial D_1, \partial E_1) \leq 8.$$

Since $S_{1,1}$ is an essential subsurface of S_1, every essential simple closed curve in $S_{1,1}$ is also an essential simple closed curve in S_1. Then

$$d_{\mathcal{C}(S_1)}(\partial D_1, \partial E_1) \leq 8.$$

Since $S_1 \subset \partial V_1$, it is not hard to see that D_1 is also an essential disk in V_1. So is the disk E_1. Then the inequality above implies that

$$d_{\mathcal{C}(S_1)}(g^n(\alpha), h \circ f(\beta)) \leq 8,$$

for a pair of α and β bounding two essential disks in V_1 and V_2 respectively. It contradicts the assumption of M_1. □

It is known that every Heegaard splitting of a boundary reducible 3-manifold is weakly reducible. Then the torus boundary T_1^2 of $M_1 = V_1 \cup_{f \circ g^n} V_2$ is incompressible.

Let ST_1 and ST_2 be two solid tori. Let $A_1^2 \subset \partial(ST_1)$ be an incompressible annulus so that the core circle of A_1^2 intersects the meridian circle in at least two points up to isotopy. Similarly, choose an annulus A_2^2 in the boundary of ST_2. After gluing ST_1 and ST_2 together by a homeomorphism from A_1^2 to A_2^2, the resulted 3-manifold M_2 is a small Seifert fiber space with a torus boundary T_2^2, where T_2^2 is incompressible.

Let $h_1 : T_1^2 \to T_2^2$ be a homeomorphism such that $h_1(\partial S_1) = \partial A_1^2$. Then $M^* = M_1 \cup_{h_1} M_2$ is closed and T_2^2 is incompressible in M^*.

Let $S^* = S_1 \cup A_1^2$. Then S^* splits M^* into two 3-manifolds, denoted by V^* and W^* respectively. In this case, V^* is an amalgamation of V_1 and a solid torus ST_1 along the annulus $\partial V_1 - S_1$. Then there are finitely many disjoint essential disks cutting V^* into some 3-balls. So V^* is a genus g handlebody. Similarly, W^* is a genus g handlebody too. Hence $V^* \cup_{S^*} W^*$ is a Heegaard splitting of M^*.

Fact 3.4. $V^* \cup_{S^*} W^*$ is a distance 2 genus g Heegaard splitting.

Proof. Since ∂S_1 is essential in S^*, every compression disk of S_1 in M_1 is an essential disk of S^*. Since there are compression disks of S_1 in two sides, there are essential disks in V^* and W^* disjoint from ∂A_1^2 in S^*. Hence the Heegaard splitting $V^* \cup_{S^*} W^*$ has the distance at most two.

Suppose the Heegaard splitting $V^* \cup_{S^*} W^*$ has distance at most one. Then there are two essential disks $D \subset V^*$ and $E \subset W^*$ so that ∂D is disjoint from ∂E. It is not hard to see S_1 is a compressible hole for both V^* and W^*. By Lemma 2.4, for the essential disk D, there is an essential disk $D_1 \subset V^*$ such that

(1) $\partial D_1 \subset S_1$;

(2) there is a component $a \subset \partial D \cap S_1$ such that $d_{\mathcal{C}(S_1)}(\pi_{S_1}(a), \partial D_1) \leq 3$;

Since ∂S_1 bounds an essential annulus in V^*, after some isotopy, D_1 is a compressing disk for S_1 in M_1.

Similarly for the essential disk E, there is an essential disk $E_1 \subset W^*$ such that

(1) $\partial E_1 \subset S_1$;

(2) there is a component $b \subset \partial E \cap S_1$ such that $d_{\mathcal{C}(S_1)}(\pi_{S_1}(b), \partial E_1) \leq 3$;

(3) E_1 is a compression disk for S_1 in M_1.

Since $\partial D \cap \partial E = \emptyset$, by Lemma 2.2, $d_{\mathcal{C}(S_1)}(\pi_{S_1}(a), \pi_{S_1}(b)) \leq 2$. By triangle inequality,

$$d_{\mathcal{C}(S_1)}(\partial D_1, \partial E_1) \leq 8.$$

It contradicts the assumption of M_1. □

By Fact 3.4, M^* admits a distance 2, genus g Heegaard splitting. Furthermore, it contains an essential torus. Then there is a free abelian subgroup Z^2 in its fundamental group. So M^* is non hyperbolic.

4. Proof of Theorem 1.3

Before proving Theorem 1.3, we introduce some results, which are Facts 4.1, 4.2 and 4.3.

Let M be a closed orientable 3-manifold admitting a distance 2 Heegaard splitting $V \cup_S W$. By the definition of Heegaard distance, there are three essential simple closed curves $\{\alpha, \gamma, \beta\}$ so that $\alpha \cap \gamma = \emptyset, \gamma \cap \beta = \emptyset$ and α (resp. β) bounds an essential disk in V (resp. W). Set

$$\mathcal{G} = \{\alpha, \gamma, \beta\}.$$

Then it is a geodesic in $\mathcal{C}(S)$ realizing the distance of $V \cup_S W$.

By the Geometrization theorem, there are finitely many (possible no) essential pairwise disjoint and nonisotopic tori in M so that cutting M along these tori, each component is either hyperbolic, Seifert or solvable. Thus to understand the geometry of M, it is necessary to study the essential torus contained in M. Since M admits a distance 2 Heegaard splitting, by Schultens' Lemma [25], they can be isotoped into a general position so that $T^2 \cap S$ consists of essential simple closed curves in both S and T^2. After pushing the possible ∂-parallel annulus to the other side, we assume that each component of $T^2 \cap V$ (resp. $T^2 \cap W$) is an essential annulus in V (resp. W). As those two boundary curves of an essential annulus in V, for example, presents too many possibilities, we want to reduce it into finitely many cases so that we can exam all these intersection curves.

For doing this, let us recall the non hyperbolic example in Section 3. It says that there is an essential nonseparating simple closed curve γ so that for any pair of essential simple closed curves α and β disjoint from γ bounding essential disks in V and W respectively, $d_{\mathcal{C}(S_\gamma)}(\alpha, \beta) \geq 11$, or equivalently, this Heegaard splitting is locally large. Then it implies that

Fact 4.1. every geodesic realizing the distance of $V \cup_S W$ has γ as its vertex.

Proof. Suppose the conclusion is false. Then there is a geodesic

$$\mathcal{G}_1 = \{\alpha_1, \gamma_1, \beta_1\}$$

so that

(1) it realizes the Heegaard distance;

(2) γ_1 is not isotopic to γ.

Let $S_\gamma = S - N(\gamma)$. Since the nonseparating essential simple closed curve γ bounds no essential disk in V or W, S_γ is a compressible hole for both of these two disk complexes of V and W. By Lemma 2.4, for α_1 (resp. β_1), there is an essential disk D (E) so that

(1) ∂D (resp. ∂E) is disjoint from γ;

(2) there is an essential disk D_1 (resp. E_1) which is disjoint from D (resp. E);

(3) a component a of $\alpha_1 \cap S_\gamma$ (resp. b of $\beta_1 \cap S_\gamma$) is disjoint from a component of $\partial D_1 \cap S_\gamma$ (resp. $\partial E_1 \cap S_\gamma$).

Then by Lemma 2.2,

$$d_{\mathcal{C}(S_\gamma)}(\pi_{S_\gamma}(a), \partial D) \leq 3 \quad \text{and} \quad d_{\mathcal{C}(S_\gamma)}(\pi_{S_\gamma}(b), \partial E) \leq 3.$$

Since each component of $\gamma_1 \cap S_\gamma$ is disjoint from a and b and not isotopic to γ, by Lemma 2.2,

$$d_{\mathcal{C}(S_\gamma)}(\pi_{S_\gamma}(a), \pi_{S_\gamma}(b)) \leq 4.$$

Then by triangle inequality, $d_{\mathcal{C}(S_\gamma)}(\partial D, \partial E) \leq 10$. It contradicts the assumption of γ. □

The condition that the Heegaard splitting $V \cup_S W$ is locally large is natural because for a geodesic realizing Heegaard distance, we always want to study its local behavior. With the locally large property of the Heegaard splitting, each component of $T^2 \cap S$ is isotopic to γ. Because, on one hand, for each component C of $T^2 \cap S$, it is a boundary component of both an essential annulus A_1 of $T^2 \cap V$ and an essential annulus A_2 of $T^2 \cap W$. Doing a boundary compression on A_1 (resp. A_2) in V (resp. W) produces an essential disk D (resp. W). Then C is disjoint from both ∂D and ∂E. So the geodesic $\mathcal{G} = \{\partial D, C, \partial E\}$ realizes the Heegaard distance of $V \cup_S W$. On the other hand, by Fact 4.1, $\mathcal{G}$ has γ as a vertex. So C is isotopic to γ. Hence each component of $T^2 \cap S$ is isotopic to γ.

Doing a boundary compression on $T^2 \cap V$ results an essential separating disk D. Then D is disjoint from $T^2 \cap V$. Moreover,

(1) D cuts out a solid torus ST in V;

(2) each component of $T^2 \cap V$ lies in this solid torus ST.

Since $T^2 \cap V$ (resp. $T^2 \cap W$) are contained in ST, all components of $T^2 \cap V$ (resp. $T^2 \cap W$) are pairwise disjoint and parallel, i.e., any two components of $T^2 \cap V$ cuts out an I-bundle over an annulus. So are $T^2 \cap W$. Since the union of all those annuli is T^2, by combinatorial techniques, we have

Fact 4.2. T^2 intersects V in only one essential annulus.

Proof. Suppose there are at least two essential annuli in V. It is known that there is an essential disk $D \subset V$ such that D cuts out a solid torus containing $T^2 \cap V$. For $T^2 \cap W$, there is also an essential disk $E \subset W$ such that E cuts out a solid torus containing $T^2 \cap W$, see Figure 3.

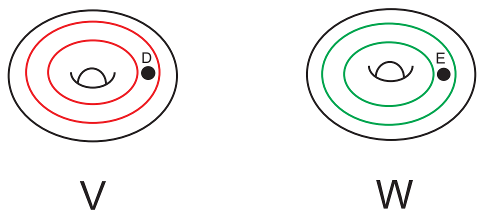

Figure 3: Parallel annuli.

Since the distance of Heegaard splitting $V \cup_S W$ is 2, $\partial D \cap \partial E \neq \emptyset$. It means that the red circles coincide with the blue circles in Figure 3. Then the essential annulus bounded by the red circles in V and the essential annulus bounded by the blue circles in W are patched together in T. And the resulted manifold is a torus or a Kleinian bottle. But T^2 contains no Kleinian bottle as its subset. So the resulted surface is a torus which is the T^2, where it intersects S in only two simple closed curves, a contradiction. $\square$

By the proof of Fact 4.2, we have

Fact 4.3. if M is toroidal, there is only one essential separating torus in M up to isotopy.

We begin to prove Theorem 1.3, which is rewritten as follows.

Theorem 4.4. *For a 3-manifold M admitting a distance two Heegaard splitting $V \cup_S W$ of genus at least two, if there is an essential nonseparating simple closed curve γ in S so that*

1) *γ bounds no essential disk in V or W;*
2) *there is a geodesic realizing Heegaard distance of $V \cup_S W$ with γ as its vertex;*

3) *for any two essential simple closed curves* α *and* β *bounding disks in* V *and* W *respectively, if they are both disjoint from* γ*, then* $d_{\mathcal{C}(S_\gamma)}(\alpha, \beta) \geq 11$*,*

then M *is either a hyperbolic 3-manifold or an amalgamation of a hyperbolic 3-manifold and a small Seifert fiber space along an incompressible torus.*

Proof. Since M admits a distance 2 Heegaard splitting, by Haken's Lemma, M is irreducible. It is known that every irreducible closed orientable 3-manifold M either contains an essential torus or not. In the later case, by Geometrization theorem, M is either a small Seifert fiber space or a hyperbolic 3-manifold.

Claim 4.5. M is not a small Seifert fiber space.

Proof. Suppose M is a small Seifert fiber space. Then it has S^2 as its base surface and at most three exceptional fibers. If M has only one or two exceptional fibers, then M is a Lens space. But all genus at least 2 Heegaard splitting of a Lens space is stabilized, reducible, i.e., they all have distance 0. So M has three exceptional fibers.

It is known that every irreducible Heegaard splitting of a Seifert fiber space is either vertical or horizontal, see [17]. If the Heegaard splitting $V \cup_S W$ is vertical, then it has genus 2. By Corollary 2.6, there are two essential disks D_1 and D_2 from two sides of S and two non isotopic disjoint essential simple closed curves C_1 and C_2 so that both C_1 and C_2 are disjoint from $\partial D_1 \cup \partial D_2$. But under the condition that $d_{\mathcal{C}(S_\gamma)}(\alpha, \beta) \geq 11$, by the proof of Fact 4.1, C_1 is isotopic to C_2. It is impossible. So it is a horizontal Heegaard splitting.

Recall that for a horizontal Heegaard splitting, $M_1 = F \times I/(x, 0) \sim (\psi(x), 1)$, where ∂F is connected and $\psi|_{\partial F} = Id$, and $M = M_1 \cup B^2 \times S^1$. And $V = F \times [0, \frac{1}{2}]$ (resp. W is homeomorphic to $F \times [\frac{1}{2}, 1]$). By Lemma 2.5, there is an essential simple closed curve $C \subset F$ so that $\mathcal{A}_1 = C \times [0, \frac{1}{2}]$ and $\mathcal{A}_2 = C \times [\frac{1}{2}, 1]$ so that $C \times \{0\}$ intersects $C \times \{1\}$ in at most one point.

It is not hard to see that there are a pair of essential disks of two sides of S so that their boundary disjoint from $C \times \{\frac{1}{2}\}$. By the proof of Fact 4.1, $C \times \{\frac{1}{2}\}$ is isotopic to γ. Let a be an arc in F disjoint from C. Then there is an essential disk $D_1 = a \times [0, \frac{1}{2}]$ (resp. $D_2 = a \times [\frac{1}{2}, 1]$) disjoint from $C \times$

$\{\frac{1}{2}\}$. Thus $D_1 \cap \mathcal{A}_1 = \emptyset$ (resp. $D_2 \cap \mathcal{A}_2 = \emptyset$). Hence

$$\begin{aligned} d_{\mathcal{C}(S_\gamma)}(\partial D_1, \partial D_2) &\leq diam_{\mathcal{C}(S_\gamma)}(\partial D_1, \partial \mathcal{A}_1)+ \\ &\quad + diam_{\mathcal{C}(S_\gamma)}(\partial \mathcal{A}_1, \partial \mathcal{A}_2)+ \\ &\quad + diam_{\mathcal{C}(S_\gamma)}(\partial D_2, \partial \mathcal{A}_2) \\ &\leq 1+2+1 \\ &= 4. \end{aligned}$$

It contradict the choice of γ. □

So M is not a small Seifert fiber space. Then it is either hyperbolic or toroidal.

If M is a hyperbolic 3-manifold, then the proof ends. So we assume that M contains an essential torus T^2. By Facts 4.1, 4.2 and 4.3,

(1) it contains only one essential separating torus T^2 up to isotopy;

(2) each component of $T^2 \cap S$ is isotopic to γ;

(3) $T^2 \cap V$ (resp. $T^2 \cap W$) splits V (resp.W) into a solid torus and a handlebody.

Let A be the annulus bounded by $T^2 \cap S$ in S. Then $S_A = \overline{S - A} = S_\gamma$. Let M_1 be the amalgamation of these two solid tori along A. It is not hard to see that M_1 is a small Seifert fiber space with a disk as its base surface.

Let $M_2 = \overline{M - M_1}$. In M_2, ∂S_A consists of two isotopic essential simple closed curves in $\partial M_2 = T^2$ and S_A cuts M_2 into two handlebodies. Let S_2 be the union of S_A and an annulus A^* bounded by ∂S_A in ∂M_2, see Figure 4.

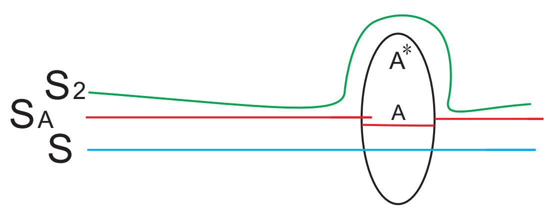

Figure 4: Heeegaard surface S_2.

After pushing S_2 a little into the interior of M_2, S_2 cuts M_2 into a handlebody and a compression body. So there is a Heegaard splitting $V_2 \cup_{S_2} W_2$ for M_2. Similar to the proof of Fact 3.3, S_2 is a strongly irreducible Heegaard surface.

Remember that S_2 is also contained in M. So

Fact 4.6. S_2 shares the essential subsurface S_A with S in common.

Proof. See Figure 4. □

From Figure 4, every essential disk in V_2 (resp. W_2) with its boundary disjoint from ∂S_A is a compression disk of S_A in V (resp. W) and a compression disk of S in V (resp. W).

Claim 4.7. M_2 is irreducible, boundary irreducible, atoroidal and anannular.

Proof. Since M is irreducible and T^2 is incompressible, M_2 is irreducible and boundary irreducible. By Fact 4.3, M contains only one essential torus T^2 up to isotopy. Then M_2 is atoroidal.

Now suppose M_2 contains an essential annulus A_1. By Schultens' Lemma [25], $A_1 \cap S_2$ are all essential simple closed curves in both A_1 and S_2. After pushing all these boundary parallel annuli to the different side of S_2, $A_1 \cap V_2$ (resp. $A_1 \cap W_2$) are essential annuli. We claim that at least one component $\gamma_1 \subset A_1 \cap S_2$ is not isotopic to γ. For if not, then there is an I-bundle of $\partial M_2 = T^2$ containing A_1 after some isotopy, which means that A_1 is inessential. Then there is an essential disk $D_1 \subset V_2$ (resp. $E_1 \subset W_2$)) disjoint from γ_1.

Since S_2 cuts M_2 into a handlebody and a compression body, let V_2 be the handlebody. From Figure 4, S_2 is the union of S_A and annulus A^*, where V_2 is a disk sum of a handlebody and I-bundle of the annulus A^*. Then the boundary of each essential disk in V_2 intersects S_A nontrivially. So S_A is a compressible hole. By the similar argument, S_A is also a compressible hole for W_2. Then by the proof of Fact 4.1, there is a pair of essential disks $D \subset V_2$ for D_1 and $E \subset W_2$ for E_1 so that ∂D and ∂E are both disjoint from ∂S_A and

$$d_{\mathcal{C}(S_A)}(\partial D, \partial E) \leq 10.$$

Remember that each essential disk in V_2 or W_2 disjoint from ∂S_A is still an essential disk in V or W respectively and $S_A = S_\gamma$. Then it contradicts the assumption of γ. □

By Thurston's hyperbolic theorem of Haken 3-manifolds, M_2 is hyperbolic. Then $M = M_1 \cup_{T^2} M_2$ is an amalgamation of a hyperbolic 3-manifold and a small Seifert fiber space. □

Remark 4.8. The main result (Theorem 1.1) of Johnson, Minsky and Moriah's paper [10] says that for a Heegaard splitting $V \cup_S W$, if there is an essential subsurface $F \subset S$ such that the distance of these two projections of disk complexes $\mathcal{D}(V)$ and $\mathcal{D}(W)$ into F, denoted by $d_F(S)$, satisfies that $d_F(S) > 2g(S) + 2$, then up to an ambient isotopy, any Heegaard splitting of M with genus less than or equal to $g(S)$ has the subsurface F in common. For the Heegaard splitting in Theorem 4.4, if condition (3) is updated into $d_{\mathcal{S}_1}(\alpha, \beta) \geq \max\{2g(S) + 3, 11\}$, then any Heegaard splitting $S^{'}$ of it with genus less than or equal to $g(S)$ has S_1 in common up to an ambient isotopy. Since ∂S_1 bounds no disk in M, $S_1 \subset S^{'}$ is essential. By the calculation of the Euler characteristic number, ∂S_1 bounds an annulus A in $S^{'}$. So the Heegaard surface S is a minimal Heegaard surface.

5. Proof of Theorem 1.6

Let M, $V \cup_S W$ and γ be the same as in Theorem 4.4. Recall that a once punctured torus is a torus domain in V (resp. W) if it is essential in S and its boundary curve bounds an essential disk in V (resp. W).

Throughout the proof of Theorem 4.4, the case that M contains an essential torus means that (1) two copies of γ bounds an essential annulus in V (resp. W), namely, there are two torus domains containing γ in both V and W; (2) γ is not a co-core in either of these two sides of S. To eliminate all possible essential tori in M, it is sufficient to add some conditions related to these two cases.

We assemble the above argument as the following proposition.

Proposition 5.1. *Let M, $V \cup_S W$ and γ be the same as in Theorem 4.4. If either there is no torus domain containing γ in at least one of V and W or γ is a co-core for one side of S, then M is hyperbolic.*

Proof. Suppose M is not hyperbolic. Since M admits a locally large distance two Heegaard splitting, by Theorem 4.4, M contains an essential annulus T^2.

The proof of Theorem 4.4 suggests that T^2 intersects S in two copies of γ. It means that these two copies of γ bounds an essential annulus A_1 (resp. A_2) in V (resp. W). Then doing a boundary compression on A_1 (resp. A_2) produces a separating essential disk in V (resp. W), which cuts out a solid torus in V (resp. W). So there are two torus domains containing γ in both V and W.

Claim 5.2. γ is not a co-core for either side of S.

Proof. Suppose the conclusion is not true. Without loss of generality, we assume γ is a co-core in V. Then there is an essential disk D so that $\partial D \cap \gamma$

in one point. So $\partial N(\partial D \cup \gamma)$ bounds an essential disk D_1, which cuts V into a solid torus ST and a smaller genus handlebody. Since the annulus A_1 is essential in V and V is irreducible, by the standard innermost disk argument, $A \cap D_1 = \emptyset$. Then $A_1 \subset ST$.

As γ is a co-core, the disk D intersects A_1 in one essential arc. Then there is a boundary compression disk $D_0 \subset D$ for A_1 in ST so that after doing a boundary compression along D_0, A_1 is changed into a trivial disk in V, a contradiction. □

Thus these two conclusions contradict the assumption of γ. □

Moreover, Proposition 5.1 can be updated into the following theorem, which is Theorem 1.6.

Theorem 5.3. *Let M, $V \cup_S W$ and γ be the same as in Proposition 5.1. Then M is hyperbolic if and only if either there is no torus domain containing γ in at least one of V and W or γ is a co-core for one side of S.*

Proof. For the forward direction. Suppose that (1) there are two torus domains F_1 and F_2 containing γ in V and W respectively; (2) γ is a co-core for neither of these two sides of S. Then ∂F_1 (resp. ∂F_2)) cuts out a solid torus in V (resp. W) containing γ. Let A be the closed regular neighborhood of γ. Since γ is not a co-core for either side of S, by standard combinatorial techniques, ∂A bounds two essential annuli A_1 and A_2 in both of V and W respectively, see Figure 5. In Figure 5, let a be a properly embedded essential arc in $F_1 - A$ with its two ends in ∂A. Then there is also a properly embedded essential arc a_1 in A_1 so that $a \cup a_1$ bounds a disk in this solid torus, which is a boundary compression disk of A_1 in V. It is not hard to see that doing a boundary compression along this disk on A_1 produces an disk D_1 in this solid torus, which is also an essential disk in V. Since D_1 is disjoint from A_1 and A_1 is incompressible in V, D_1 is parallel to the disk bounded by ∂F_1.

Then $A_1 \cup A_2$ is a torus T^2 or a Kleinian bottle K. Since $A_1 \cup A_2$ is separating in M, it is a torus T^2.

Claim 5.4. T^2 is essential in M.

Proof. By Figure 5, A_1 (resp. A_2) cuts out a solid torus ST_1 (resp. ST_2), where both of these two solid tori have the annulus A as their common boundary surface. Since γ, the core curve of A, is not a co-core of either of

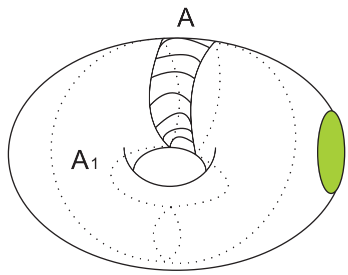

Figure 5: The essential annulus A_1.

these two handlebodies V and W, $M_2 = ST_1 \cup_A ST_2$ is a small Seifert fiber space. Since A is incompressible in both ST_1 and ST_2, A is incompressible in M_2. Then T^2 is incompressible in M_2. For if not, then $T^2 = \partial M_2$ is compressible in M_2. So the compression disk D intersects A nontrivially. Otherwise, either A_1 or A_2 is compressible in ST_1 or ST_2 respectively. Then there is an outermost disk $D_0 \subset D$ for A. Without loss of generality, we assume that $D_0 \subset ST_1$ in V. It is not hard to see that D_0 is a boundary compression disk of A_1 in V. After doing a boundary compression on A_1 along D_0, A_1 is changed into a trivial disk in V, which is impossible.

Let $M_1 = \overline{M - M_2}$. The proof of Fact 3.3 suggests that T^2 is incompressible in M_1. So T^2 is incompressible in M. □

So M contains an essential torus T^2. It contradicts the assumption that M is hyperbolic.

For the backward direction. The proof is already contained in the proof of Proposition 5.1. □

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Received August 23, 2016
Accepted June 3, 2017