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A new invariant equation for umbilical points on real hypersurfaces in $\mathbb{C}^2$ and applications

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We introduce a new sequence of CR invariant determinants on a three dimensional CR manifold M embedded in $\mathbb{C}^2$ (which is a different and harder to handle case than $\mathbb{C}^n$ with $n \geq 3$). The lowest order invariant represents E. Cartan's 6th order invariant (the umbilical "tensor"), whose zero locus yields the set of umbilical points on M, whenever M is Levi-nondegenerate. Moreover, this invariant extends regularly to (and vanishes at) all Levi-degenerate points of M, implying e.g. real-analyticity (resp. real-algebraicity) of the umbilical set across such points whenever M is real-analytic (resp. real-algebraic). As a further application, we show that generic, almost circular perturbations of the sphere always contain curves of umbilical points.

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1. Introduction

The motivation behind this paper is to better understand E. Cartan's "6th order invariant" on a strictly pseudoconvex three dimensional CR manifold $M = M^3$. Points on M where this invariant vanishes are called *(CR) umbilical points*; they correspond to points where the CR structure can be locally osculated by the sphere to a higher-than-generic order, i.e., to order at least 7. Our aim is to understand the global behavior of this invariant and we will center our investigation on a basic question, due to S.-S. Chern and J. K. Moser [CM]: *Are there compact strictly pseudoconvex real hypersurfaces in $\mathbb{C}^2$ that do not possess umbilical points?* This question was very recently answered in the affirmative by the authors, jointly with with Son Duong, in the work [EDZ] where a family of such examples is constructed. These examples, however, all have the topology of a 3-torus $\mathbb{T}^3$, and one may instead ask if there are examples with "simple" topology. For instance, a version of the Chern–Moser question, which is still open, is the following:

Are there compact strictly pseudoconvex real hypersurfaces in $\mathbb{C}^2$, diffeomorphic to the sphere S^3, that do not possess umbilical points?

What makes this question even more intriguing is the existence of compact *non-embeddable* (in $\mathbb{C}^2$) strictly pseudoconvex CR manifolds of dimension 3 with simple topology and without umbilical points. Such examples arise as E. Cartan's 1-parameter family of real hypersurfaces $\mu_\alpha \subset \mathbb{P}^2$ that are compact, strictly pseudoconvex, homogeneous and non-spherical (hence obviously without umbilical points) [Ca]. A family of 2:1-covers of these CR manifolds are embeddable in $\mathbb{C}^3$ (but not in $\mathbb{C}^2$). The universal cover of μ_α is the sphere S^3, and by pulling back the CR structures of μ_α one obtains a family of CR structures on S^3 that do not possess any umbilical points. The latter structures, however, are well known to not be embeddable in $\mathbb{C}^n$ for any n (see e.g. [Is, R].) In particular, the above Chern-Moser question remains open.

We point out the analogy with the classical Caratheodory Conjecture (see, e.g., [Ha], [Iv]) regarding umbilical points on compact surfaces embedded in $\mathbb{R}^3$, and refer the interested reader to the paper [ED] for a closer discussion of the analogy between the notion of (CR) umbilical points in CR geometry and that of umbilical points in the classicial geometry of surfaces in $\mathbb{R}^3$.

Let $M = M^{2n+1}$ be a real hypersurface in $\mathbb{C}^{n+1}$ and $p \in M$. The well-known lowest order (local) invariant of M is its Levi form at p, which is a

Hermitian form

$$T_p^{1,0}M \times \overline{T_p^{1,0}M} \to (T_p\mathbb{C}^{n+1}/T_pM) \otimes \mathbb{C},$$

where $T_p^{1,0}M$ is the space of $(1,0)$-vectors tangent to M at p. A hypersurface is said to be *Levi-nondegenerate at* p if the Levi form is nondegenerate at p, and *strictly pseudoconvex* if the Levi form at p becomes definite after identifying the one-dimensional space $(T_p\mathbb{C}^{n+1}/T_pM) \otimes \mathbb{C}$ with $\mathbb{C}$.

Levi nondegeneracy of M at p can be detected by the condition $J_p \neq 0$, where $J = J(\rho)$ is Fefferman's Monge-Ampere operator [F]

$$J(\rho) := (-1)^{n+1} \det \begin{pmatrix} \rho & \rho_{\bar{Z}} \\ \rho_Z & \rho_{Z\bar{Z}} \end{pmatrix} \tag{1.1}$$

applied to a local defining function ρ of M near p; i.e., M is locally given by $\rho = 0$, and $d\rho \neq 0$ on M. In (1.1), the notation used is $\rho_{\bar{Z}} := (\rho_{\bar{Z}_1}, \ldots, \rho_{\bar{Z}_{n+1}})$, ρ_Z is its conjugate transpose $\rho_Z = (\rho_{\bar{Z}})^*$, and $\rho_{Z\bar{Z}}$ is the $(n+1) \times (n+1)$ matrix $(\rho_{Z_k\bar{Z}_j})$. One of the main contributions in this paper is the introduction of a *sequence of invariant determinants*, for hypersurfaces in $\mathbb{C}^2$, that can be viewed as higher order analogs of Fefferman's operator J in (1.1). The lowest order invariant determinant introduced below as $\det A_3(\rho)$, represents E. Cartan's 6th order invariant at Levi-nondegenerate points. However, in contrast to Cartan's definition, our tensors extend regularly to Levi-degenerate points. The higher order invariant determinants are of independent interest and appear to not have been explicitly considered in the literature before.

1.1. Umbilical points

Chern and Moser showed [CM] that if M is strictly pseudoconvex at p, then that there are formal holomorphic coordinates $Z = (z, w) = (z_1, \ldots, z_n, w)$ (convergent if M is real-analytic), vanishing at p, such that M can be expressed in Chern–Moser normal form. Rather than describing this normal form precisely here, we simply note that in these coordinates M is expressed as a graph

$$\mathsf{Im}\, w = \varphi(z, \bar{z}, \mathsf{Re}\, w), \tag{1.2}$$

where the graphing function has the form

$$\varphi(z,\bar z,\mathsf{Re}\,w)=\sum_{j=1}^{n}|z_j|^2+R_m(z,\bar z)+O(m+1). \tag{1.3}$$

Here, $R_m(z,\bar z)$ is a homogeneous Hermitian polynomial of degree m with $m=6$ for $n=1$ and $m=4$ for $n\ge 2$, $O(k)$ signify terms of weight $\ge k$ in $(z,\bar z,\mathsf{Re}\,w)$, where $(z,\bar z)$ are assigned weight one and $\mathsf{Re}\,w$ weight two. If $n\ge 2$, then $m=4$ and $R_4(z,\bar z)$ is of bidegree $(2,2)$; $R_4(z,\bar z)$ represents the CR curvature tensor $S_{\alpha\bar\beta\nu\bar\mu}$ at $p=(0,0)$ as its sectional curvature

$$R_4(z,\bar z)=\sum_{\alpha,\beta,\nu,\mu}S_{\alpha\bar\beta\nu\bar\mu}z_\alpha z_\nu\overline{z_\beta z_\mu}. \tag{1.4}$$

The CR curvature tensor $S_{\alpha\bar\beta\nu\bar\mu}$ has Hermitian curvature symmetries and zero trace. (The latter is equivalent to $R_4(z,\bar z)$ being a harmonic function of z.) The real dimension of the space of such curvature tensors (of Weyl–Bochner type) is $n^2(n-1)(n+3)/3$ (see [S]). From this it follows that the condition of being umbilical at a point p is equivalent to $n^2(n-1)(n+3)/3$ independent real equations on the 4-jet of the CR manifold M^{2n+1} at p. For $n=2$, this means 5 equations on a 5-dimensional manifold, but for $n\ge 3$, this is an "overdetermined" system. More precisely, an application of Thom's Transversality Theorem [GG] (see [BRZ] for similar arguments) shows that a *generic* strictly pseudoconvex hypersurface $M^{2n+1}\subset\mathbb{C}^{n+1}$ (i.e., in a dense open subset in the compact-open topology of C^∞-mappings $M^{2n+1}\hookrightarrow\mathbb{C}^{n+1}$) has no umbilical points when $n\ge 3$, and at most isolated umbilical points when $n=2$. Moreover, to further support the statement that umbilical points are rare when $n\ge 2$, we mention that Webster has shown [W] that every non-spherical real ellipsoid in $\mathbb{C}^{n+1}$ is free of umbilical points when $n\ge 2$.

In this paper, we shall consider the case $n=1$, i.e., strictly pseudoconvex hypersurfaces $M=M^3$ in $\mathbb{C}^2$. In this case, the CR curvature vanishes identically, and the lowest order nontrivial invariant in the Chern–Moser normal form occurs in weight $m=6$, and $R_6(z,\bar z)$ in (1.3) has the form

$$R_6(z,\bar z)=c_{24}z^2\bar z^4+c_{42}z^4\bar z^2,\quad c_{42}=\overline{c_{24}}\in\mathbb{C}, \tag{1.5}$$

where c_{24} represents E. Cartan's "6th order invariant/tensor" $Q=Q^1{}_{\bar 1}$ at $p=(0,0)$. The hypersurface M^3 is umbilical at $p=(0,0)$ if $c_{24}=0$. Thus, the condition of being umbilical on M^3 amounts to two independent real equations in the 6-jet space of CR manifolds in $\mathbb{C}^2$. Thus, we should expect umbilical points (if they exist!) to form real curves in M^3. If we consider the condition that M^3 is umbilical at p and the rank of the differential of the two real equations $c_{24} = 0$ is ≤ 1, which amounts to the vanishing of two real determinants, then we obtain an algebraic subvariety in the 7-jet space of codimension 4. An application of Thom's Transversality Theorem, as above, yields that a *generic* strictly pseudoconvex hypersurface $M^3 \subset \mathbb{C}^2$ either has no umbilical points at all or a set of umbilical points that consists of smooth real curves.

1.2. Summary of main results

The main goal of this paper is to construct a convenient global representation of Cartan's 6th order tensor Q and use this representation to address the Chern–Moser problem concerning existence of umbilical points on compact real hypersurfaces in $\mathbb{C}^2$. As is illustrated by the family of examples $\mu_\alpha \subset \mathbb{P}^2$ and their covers, there are compact three dimensional CR manifolds without umbilical points, that are however not embeddable in $\mathbb{C}^2$. We shall focus here on the Chern-Moser's question described above. Not much is known in general about this problem. It was shown by X. Huang and S. Ji [HJ] that every real ellipsoid in $\mathbb{C}^2$ must have umbilical points. More recently, it was proved by the first author and S. Duong [ED] that every *circular* $M^3 \subset \mathbb{C}^2$ has umbilical points; in fact, it was proved in [ED] that every compact three dimensional CR manifold with a transverse free CR $U(1)$ (circle) action must have umbilical points provided that the Riemann surface $M/U(1)$ has genus $g \neq 1$. As a first step towards answering the Chern-Moser's question more generally, we shall consider small perturbations $M_\varepsilon \subset \mathbb{C}^2$ of the unit sphere $M_0 = S^3 \subset \mathbb{C}^2$. We introduce certain *almost circular* perturbations and show that generic ones must possess umbilical points. This is the content of Theorem 6.11. Precise statements and definitions are given in Section 6. We also note in Remark 6.12 that real ellipsoids are not generic in the sense of Theorem 6.11, but we give separately a new proof of a special case of the Huang-Ji theorem that real ellipsoids possess umbilical points; namely, we prove the existence of real curves of umbilical points in the special case where the ellipsoids are sufficiently close to spherical. This is the content of Theorem 5.1.

One of the main obstacles in investigating the existence of umbilical points on a compact real hypersurface $M^3 \subset \mathbb{C}^2$ is the lack of a global convenient representation ("formula") of Cartan's tensor Q. In Chern–Moser normal form at $p = (0, 0)$, Q is represented at p by the coefficient c_{24}. In case M^3 has additional tranverse symmetry, Loboda [L] discovered a formula that represents Q for a graph $M^3 \subset \mathbb{C}^2$. One of the main results in this paper is a *global* representation of Q as a nonlinear Partial Differential Operator (PDO) acting on a defining function ρ for $M^3 \subset \mathbb{C}^2$. In fact, we construct a sequence of invariants $\det A_k(\rho)$, for $k \geq 3$, is introduced in Section 2 (see in particular Theorem 2.1). These invariant determinants are in some sense higher order versions of Fefferman's Monge-Ampere operator $J(\rho)$.

For every defining function ρ, we consider the $(1,0)$-vector field

$$L := -\frac{\partial \rho}{\partial w}\frac{\partial}{\partial z} + \frac{\partial \rho}{\partial z}\frac{\partial}{\partial w} = -\rho_w \partial_z + \rho_z \partial_w,$$

and then we have, as the reader can easily check,

$$J(\rho) = \det \begin{pmatrix} \rho_z & \bar{L}\rho_z \\ \rho_w & \bar{L}\rho_w \end{pmatrix} = \rho_{Z\bar{Z}}(L, L) \mod \rho, \tag{1.6}$$

where $J(\rho)$ is the Fefferman determinant (1.1). Our invariant determinants $\det A_k(\rho)$, $k \geq 3$, are higher order analogues of the determinant on the right in (1.6). The precise definition is given in (2.2). The relationship between $\det A_k(\rho)$ and the classical Chern–Moser invariants of M^3 is explored in Section 3. This relationship is expressed by (3.4); in particular, it is shown that $\det A_3(\rho)$ represents Cartan's tensor Q at every $p \in M^3$.

2. A family of invariant determinants

Consider a real hypersurface $M = M^3 \subset \mathbb{C}^2$ given by $\rho(z, w, \bar{z}, \bar{w}) = 0$ with $\partial\rho \neq 0$ on M. We use coordinates $Z = (z, w) \in \mathbb{C}^2$. Then the space of $(1,0)$-vectors on M at every point is spanned by the single vector field

$$L := -\rho_w \partial_z + \rho_z \partial_w. \tag{2.1}$$

We shall also use the complex Hessian ρ_{Z^2} evaluated at (L, L):

$$\rho_{Z^2}(L, L) = \rho_{z^2}\rho_w^2 - 2\rho_{zw}\rho_z\rho_w + \rho_{w^2}\rho_z^2.$$

For every $n \geq 3$, consider the $(2n-1)\times(2n-1)$ matrix PDO $A_n(\rho)$ acting on the smooth function ρ

$$A_n = A_n(\rho) := \begin{pmatrix} \bar{L}^j(\rho_z^k \rho_w^{n-k}) \\ \bar{L}^j(\rho_z^s \rho_w^{n-3-s}\rho_{Z^2}(L, L)) \end{pmatrix}_{0\leq j\leq 2n-2,\, 0\leq k\leq n,\, 0\leq s\leq n-3}, \tag{2.2}$$

where we regard j as column index and k and s as row indices (first followed by the second). In particular, for $n = 3, 4$, we obtain

$$A_3 = A_3(\rho) := \begin{pmatrix} \rho_w^3 & \bar{L}(\rho_w^3) & \cdots & \bar{L}^4(\rho_w^3) \\ \rho_z\rho_w^2 & \bar{L}(\rho_z\rho_w^2) & \cdots & \bar{L}^4(\rho_z\rho_w^2) \\ \rho_z^2\rho_w & \bar{L}(\rho_z^2\rho_w) & \cdots & \bar{L}^4(\rho_z^2\rho_w) \\ \rho_z^3 & \bar{L}(\rho_z^3) & \cdots & \bar{L}^4(\rho_z^3) \\ \rho_{Z^2}(L,L) & \bar{L}(\rho_{Z^2}(L,L)) & \cdots & \bar{L}^4(\rho_{Z^2}(L,L)) \end{pmatrix} \tag{2.3}$$

and

$$A_4 = A_4(\rho) := \begin{pmatrix} \rho_w^4 & \bar{L}(\rho_w^4) & \cdots & \bar{L}^6(\rho_w^4) \\ \rho_z\rho_w^3 & \bar{L}(\rho_z\rho_w^3) & \cdots & \bar{L}^6(\rho_z\rho_w^3) \\ \rho_z^2\rho_w^2 & \bar{L}(\rho_z^2\rho_w^2) & \cdots & \bar{L}^6(\rho_z^2\rho_w^2) \\ \rho_z^3\rho_w & \bar{L}(\rho_z^3\rho_w) & \cdots & \bar{L}^6(\rho_z^3\rho_w) \\ \rho_z^4 & \bar{L}(\rho_z^4) & \cdots & \bar{L}^6(\rho_z^4) \\ \rho_w\rho_{Z^2}(L,L) & \bar{L}(\rho_w\rho_{Z^2}(L,L)) & \cdots & \bar{L}^6(\rho_w\rho_{Z^2}(L,L)) \\ \rho_z\rho_{Z^2}(L,L) & \bar{L}(\rho_z\rho_{Z^2}(L,L)) & \cdots & \bar{L}^6(\rho_z\rho_{Z^2}(L,L)) \end{pmatrix}. \tag{2.4}$$

We also denote by $D_n = D_n(\rho)$ the upper left $(n+1)\times(n+1)$ sub-matrix of A_n, i.e.

$$D_n := \left(\bar{L}^j(\rho_z^k\rho_w^{n-k})\right)_{0\le j\le n,\, 0\le k\le n}. \tag{2.5}$$

The main interest in considering the matrices A_n and D_n is the following invariance property of their determinants:

Theorem 2.1. *For every M and $n \geq 3$, the properties $\det A_n = 0$ and $\det D_n = 0$ at points of M are independent of the choice of the defining function ρ as well as of the choice of the coordinates $Z = (z, w) \in \mathbb{C}^2$.*

More precisely, if L^*, A_n^* and D_n^* are given by (2.1), (2.2) and (2.5) respectively with ρ replaced by another defining function $\rho^* = a\rho$ (where a is any nonzero real smooth function), and $Z = (z, w)$ replaced by another (formal) holomorphic coordinate system $Z^* = (z^*, w^*)$, we have the transformation rule

$$\delta^{n^2-1}\bar{\delta}^{(n-1)(2n-1)} \det A_n^* = a^{(2n-1)^2} \det A_n, \tag{2.6}$$

$$|\delta|^{n(n+1)} \det D_n^* = a^{\frac{3n(n+1)}{2}} \det D_n, \tag{2.7}$$

where δ is the Jacobian determinant of the coordinate transformation $Z^* = H(Z)$.

Remark 2.2. It is important that L and ρ used in A_n are related via (2.1). The invariance of the property $\det A_n = 0$ does not hold for arbitrary choices of $(1,0)$ vector fields L. However, given that chosen L, the invariance of $\det A_n$ remains when replacing $\bar{L}$ by arbitrary $(0,1)$-vector fields.

To prove Theorem 2.1, we require two lemmas.

Lemma 2.3. *For any real smooth function $a = a(Z, \bar{Z})$, and L and L^* given respectively by* (2.1) *and by the same formula with ρ replaced with $\rho^* = a\rho$, we have on M the identities*

$$\rho^*_Z = a\rho_Z, \quad L^* = aL, \quad \rho^*_{Z^2}(L^*, L^*) = a^3 \rho_{Z^2}(L, L). \tag{2.8}$$

Proof. By Leibnitz' rule on M we have $\rho^*_Z = a\rho_Z$, which implies the first and second identities in (2.8). By Leibnitz' rule again, for any $(1,0)$ vectors ξ, η, we also have

$$(a\rho)_{Z^2}(\xi, \eta) = a\rho_{Z^2}(\xi, \eta) + a_Z(\xi)\rho_Z(\eta) + a_Z(\eta)\rho_Z(\xi) + a_{Z^2}(\xi, \eta)\rho.$$

Substituting $\xi = \eta = L$ and using the properties $\rho_Z(L) = 0$ and $\rho = 0$ on M, we obtain, on M,

$$\rho^*_{Z^2}(L, L) = a\rho_{Z^2}(L, L). \tag{2.9}$$

Together with the second identity in (2.8) this yields the third identity. □

Lemma 2.4. *For any (formal) biholomorphic transformations $Z^* = H(Z)$, $\zeta^* = K(\zeta)$ of $\mathbb{C}^2$, any complex formal power series $\rho^*(Z^*, \bar{Z}^*)$ and $\rho(Z, \bar{Z}) := \rho^*(H(Z), K(\bar{Z}))$, consider L, A_n, D_n and L^*, A^*_n, D^*_n given by* (2.1), (2.2), (2.5) *and respectively by the same identities with ρ replaced by ρ^*. Then the following hold:*

(i) *The identity*

$$\rho_{Z^2}(L, L) \cong (\det H_Z)^2 \rho^*_{Z^{*2}}(L^*, L^*) \tag{2.10}$$

holds, where $\cong$ here means equality modulo a cubic homogeneous polynomial in (ρ_z, ρ_w) with holomorphic coefficients in Z.

(ii) *The determinants of* A_n, D_n *and* A_n^*, D_n^* *are related by*

$$\det A_n = (\det H_Z)^{n^2-1}(\det K_{\bar{Z}})^{(n-1)(2n-1)} \det A_n^*, \tag{2.11}$$

$$\det D_n = (\det H_Z)^{\frac{n(n+1)}{2}}(\det K_{\bar{Z}})^{\frac{n(n+1)}{2}} \det D_n^*, \tag{2.12}$$

where the matrices A_n^* *and* D_n^* *are evaluated at*

$$(Z^*, \bar{Z}^*) = (H(Z), K(\bar{Z})).$$

Proof. We write

$$S_n := \begin{pmatrix} \rho_w^n \\ \rho_z \rho_w^{n-1} \\ \vdots \\ \rho_z^n \end{pmatrix}$$

for the standard basis of homogeneous monomials of order n in (ρ_z, ρ_w). In particular, S_n coincides with the $(n+1) \times 1$ matrix (column vector) consisting of the first $n+1$ entries of the first column of A_n. We also write S_n^* for corresponding column of monomials in $(\rho^*_{z^*}, \rho^*_{w^*})$. In particular,

$$S_1 = \begin{pmatrix} \rho_w \\ \rho_z \end{pmatrix}, \quad S_1^* = \begin{pmatrix} \rho^*_{w^*} \\ \rho^*_{z^*} \end{pmatrix}.$$

By the chain rule, we have

$$\rho_Z = \rho^*_{Z^*} \circ H_Z,$$

where $\rho^*_{Z^*}$ is evaluated at $(Z^*, \bar{Z}^*) = (H(Z), K(\bar{Z}))$. Writing as matrix identity we obtain

$$S_1 = H_Z S_1^*,$$

where by abuse of notation, for $H = (f, g)$, we identify H_Z with its induced matrix

$$\begin{pmatrix} g_w & f_w \\ g_z & f_w \end{pmatrix}. \tag{2.13}$$

Then viewing nth order homogenous monomials in ρ_z, ρ_w in the nth tensor power of the cotangent space, we have

$$S_n = (\otimes^n H_Z) S_n^*, \tag{2.14}$$

where, by another abuse of notation, we identify the tensor power transformation $\otimes^n H_Z$ with its induced matrix in the monomial basis. Further we have

$$\det(\otimes^n H_Z) = (\det H_Z)^{\frac{n(n+1)}{2}}, \tag{2.15}$$

which e.g. follows from Jordan normal form. Since $(\otimes^n H_Z)$ is holomorphic in Z and $\bar{L}$ is a $(0,1)$ vector field, we obtain for any j,

$$\bar{L}^j S_n = (\otimes^n H_Z)\bar{L}^j S_n^*. \tag{2.16}$$

In particular, we have

$$\det D_n = (\det H_Z)^{\frac{n(n+1)}{2}} \det E_n^*, \tag{2.17}$$

where E_n^* is the matrix obtained from D_n^* with $\bar{L}^*$ being replaced by $\bar{L}$.

We next turn to the relation between L and L^*. By definition

$$L = \begin{pmatrix}\rho_z & -\rho_w\end{pmatrix}\begin{pmatrix}\partial_w \\ \partial_z\end{pmatrix}, \quad L^* = \begin{pmatrix}\rho^*_{z^*} & -\rho^*_{w^*}\end{pmatrix}\begin{pmatrix}\partial_{w^*} \\ \partial_{z^*}\end{pmatrix}.$$

We further have

$$\begin{pmatrix}H_Z\partial_w \\ H_Z\partial_z\end{pmatrix} = \begin{pmatrix}g_w & f_w \\ g_z & f_z\end{pmatrix}\begin{pmatrix}\partial_{w^*} \\ \partial_{z^*}\end{pmatrix} = H_Z\begin{pmatrix}\partial_{w^*} \\ \partial_{z^*}\end{pmatrix},$$

where we continue our abuse of notation by writing H_Z also for its induced matrix. Similarly,

$$\begin{pmatrix}\rho_w & \rho_z\end{pmatrix} = \begin{pmatrix}\rho^*_{w^*} & \rho^*_{z^*}\end{pmatrix}\begin{pmatrix}g_w & g_z \\ f_w & f_z\end{pmatrix} = \begin{pmatrix}\rho^*_{w^*} & \rho^*_{z^*}\end{pmatrix} H_Z^t, \tag{2.18}$$

where H_Z^t is the transpose matrix. Furthermore,

$$\begin{pmatrix}\rho_z & -\rho_w\end{pmatrix} = \begin{pmatrix}\rho_w & \rho_z\end{pmatrix} J, \quad J := \begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix},$$

and hence

$$\begin{pmatrix}\rho_z & -\rho_w\end{pmatrix} = \begin{pmatrix}\rho^*_{z^*} & -\rho^*_{w^*}\end{pmatrix} J^{-1} H_Z^t J.$$

Putting everything together, we obtain

$$\begin{aligned} H_Z L = \begin{pmatrix}\rho_z & -\rho_w\end{pmatrix}\begin{pmatrix}H_Z\partial_w \\ H_Z\partial_z\end{pmatrix} &= \begin{pmatrix}\rho_z & -\rho_w\end{pmatrix} H_Z \begin{pmatrix}\partial_{w^*} \\ \partial_{z^*}\end{pmatrix} \\ &= \begin{pmatrix}\rho^*_{z^*} & -\rho^*_{w^*}\end{pmatrix} J^{-1} H_Z^t J H_Z \begin{pmatrix}\partial_{w^*} \\ \partial_{z^*}\end{pmatrix}. \end{aligned}$$

By direct calculation,

$$J^{-1}H_Z^t J H_Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} g_w & g_z \\ f_w & f_z \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} g_w & f_w \\ g_z & f_z \end{pmatrix}$$
$$= (\det H_Z) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and therefore

$$H_Z L = (\det H_Z) L^*. \tag{2.19}$$

Now, we introduce the holomorphic, or $(1,0)$, Hessian H_{Z^2} of the mapping H. We view this as a bilinear mapping on the space of $(1,0)$ vectors in the (z,w) coordinate system into that in the (z^*,w^*) system. Thus, by the chain rule, for any $(1,0)$ vectors ξ,η, we have

$$\rho_{Z^2}(\xi,\eta) = \rho^*_{Z^{*2}}(H_Z\xi, H_Z\eta) + \rho^*_{Z^*}(H_{Z^2}(\xi,\eta)).$$

Substituting $\xi=\eta=L$ and using (2.19), we obtain

$$\rho_{Z^2}(L,L) = (\det H_Z)^2 \rho^*_{Z^{*2}}(L^*,L^*) + \rho^*_{Z^*}(H_{Z^2}(L,L)).$$

Expanding the last term and using (2.18), we obtain the desired identity (2.10) modulo a homogeneous polynomial of degree 3 in (ρ_z,ρ_w) with coefficients that are polynomial in the derivatives of H, which proves the first statement (i) of the lemma.

We now turn to the $(n-2)\times 1$ matrix (column vector) consisting of the last $n-2$ entries of the first column in A_n, which in the notation introduced can be expressed as $\rho_{Z^2}(L,L)S_{n-3}$. Hence (2.14) implies

$$\rho_{Z^2}(L,L)S_{n-3} = \rho_{Z^2}(L,L)(\otimes^{n-3}H_Z)S^*_{n-3}.$$

By the first statement (i) of the lemma, already proved, we have

$$\rho_{Z^2}(L,L)S_{n-3} \cong (\det H_Z)^2(\otimes^{n-3}H_Z)\rho^*_{Z^{*2}}(L^*,L^*)S^*_{n-3},$$

where $\cong$ means equality modulo a homogeneous polynomial of order n in (ρ_z,ρ_w) with holomorphic coefficients in Z. Since $\bar{L}$ is $(0,1)$, it commutes with those holomorphic coefficients and, consequently, we can subtract from the last $n-2$ rows of A_n suitable linear combinations of the first $n+1$ rows to obtain a matrix with the same determinant of the form

$$\begin{pmatrix} \otimes^n H_Z & 0 \\ 0 & (\det H_Z)^2(\otimes^{n-3} H_Z) \end{pmatrix} B_n^*,$$

where B_n^* is the matrix obtained from A_n^* with $\bar{L}^*$ (but not L^*!) replaced by $\bar{L}$.

Finally, analogously to (2.19), we have the relation

$$K_{\bar{Z}}\bar{L} = (\det K_{\bar{Z}})\bar{L}^*.$$

Writing C_n^* for the first column of B_n^*, we obtain for any j,

$$(K_{\bar{Z}}\bar{L})^j C_n^* = (\det K_{\bar{Z}})^j (\bar{L}^*)^j C_n^*$$

modulo a linear combination of the columns $(\bar{L}^*)^s C_n^*$ with $s < j$. Hence, subtracting those linear combinations without changing the determinant, we obtain

$$\det B_n^* = (\det K_{\bar{Z}})^{(n-1)(2n-1)} \det A_n^*,$$

and similar

$$\det E_n^* = (\det K_{\bar{Z}})^{\frac{n(n+1)}{2}} \det D_n^*,$$

where E_n^* was defined after (2.17). Putting everything together and using (2.15) we obtain the second conclusion. $\square$

Proof of Theorem 2.1. Clearly it suffices to prove the proposition by separately considering changes of the defining function and the coordinates. The transformation formula under a change of coordinates follows from Lemma 2.4 when ρ is real-analytic. However, since the matrix A_n only depends on a finite order jet of ρ at a reference point, the corresponding transformation rule in (2.8) holds for any smooth ρ.

It remains to consider the change $\rho^* = a\rho$ of the defining function. By Lemma 2.3, for any k, s, n as in (2.2), we have

$$\begin{aligned} (\rho_z^*)^k(\rho_w^*)^{n-k} &= a^n \rho_z^k \rho_w^{n-k}, \\ (\rho_z^*)^s(\rho_w^*)^{n-3-k}\rho^*_{Z^2}(L^*, L^*) &= a^n \rho_z^k \rho_w^{n-k} \rho_{Z^2}(L, L). \end{aligned}$$

Then, writing

$$C_n := \begin{pmatrix} (\rho_z^k \rho_w^{n-k}) \\ (\rho_z^s \rho_w^{n-3-s} \rho_{Z^2}(L, L) \end{pmatrix}_{0\le k\le n,\, 0\le s\le n-3}$$

for the first column of the matrix A_n given by (2.2) and C_n^* for the corresponding first column of A_n^*, we obtain

$$C_n^* = a^n C_n.$$

Then using the relation $\bar{L}^* = a\bar{L}$, we conclude for every j,

$$(\bar{L}^*)^j C_n^* = a^{n+j} \bar{L}^j C_n$$

modulo linear combinations of the columns $\bar{L}^s C_n$ with $s < j$. Since the determinant does not change after subtracting a linear combination for columns from another column, we obtain the desired transformation rules

$$\det A_n^* = a^{(2n-1)^2} \det A_n, \quad \det D_n^* = a^{\frac{3n(n+1)}{2}} \det D_n. \qquad \square$$

3. Calculation in normalized coordinates

Note that the invariance property in Proposition 2.15 was obtained for any smooth real hypersurface given by $\rho(Z, \bar{Z}) = 0$. If the latter is Levi-nondegenerate, we can use special defining functions

$$\rho(z, w, \bar{z}, \bar{w}) = -\mathsf{Im}\, w + \varphi(z, \bar{z}, \mathsf{Re}\, w), \quad \varphi(z, \bar{z}, u) = \sum \varphi_{kl}(u) z^k \bar{z}^l, \tag{3.1}$$

in Chern-Moser normal form (in the formal sense if M is only smooth and not real-analytic) to compute the determinant of A_n at the origin. Recall [CM] that the normal form requires φ to satisfy

$$\varphi_{11} = 1, \quad \varphi_{0k} = \varphi_{1s} = \varphi_{22} = \varphi_{23} = \varphi_{33} = 0, \quad k \geq 0, s \geq 2. \tag{3.2}$$

In this normal form we have $(\rho_w, \rho_z)(0) = (i/2, 0)$ and furthermore

$$\bar{L}^j \rho_w(0) = \bar{L}^k \rho_z(0) = 0, \quad j \geq 1,\ k \neq 1,$$

and

$$\bar{L}\rho_z(0) = -i/2 \neq 0.$$

Furthermore,

$$\rho_{Z^2}(L, L) = \rho_w^2 \rho_{z^2} - 2\rho_z \rho_w \rho_{zw} + \rho_z^2 \rho_{w^2}$$

and

$$\rho_{\bar{z}^l w^s}(0) = \rho_{z\bar{z}^l w^s}(0) = 0, \quad l \geq 0, \quad s \geq 1,$$

imply

$$\begin{aligned}\bar{L}^k(\rho_{Z^2}(L,L))(0) &= (\rho_w(0))^2\bar{L}^k\rho_{z^2}(0)\\ &= (-i/2)^{k+2}\rho_{z^2\bar{z}^k}(0) = (-i/2)^{k+2}\varphi_{z^2\bar{z}^k}(0).\end{aligned}$$

We similarly observe that

$$\begin{aligned}\bar{L}^k(\rho_z^s\rho_w^{n-3-s}\rho_{Z^2}(L,L))(0) &= \binom{k}{s}(\rho_w(0))^{n-1-s}\bar{L}^s\rho_z^s(0)\bar{L}^{k-s}\rho_{z^2}(0)\\ &= (-1)^k(i/2)^{n+k-s-1}s!\binom{k}{s}\varphi_{z^2\bar{z}^{k-s}}(0).\end{aligned} \tag{3.3}$$

These calculations imply that that computation of $\det A_n|_{Z=0}$, modulo a non-zero constant, reduces to that of its lower right $(n-2)\times(n-2)$ minor, i.e.,

$$\det A_n|_{Z=0} = c_n \det\begin{pmatrix} \binom{n+1}{0}\varphi_{2,n+1} & \binom{n+2}{0}\varphi_{2,n+2} & \cdots & \binom{2n-2}{0}\varphi_{2,2n-2}\\ \binom{n+1}{1}\varphi_{2,n} & \binom{n+2}{1}\varphi_{2,n+1} & \cdots & \binom{2n-2}{1}\varphi_{2,2n-3}\\ \vdots & \vdots & \ddots & \vdots\\ \binom{n+1}{n-3}\varphi_{2,4} & \binom{n+2}{n-3}\varphi_{2,5} & \cdots & \binom{2n-2}{n-3}\varphi_{2,n+1}\end{pmatrix}, \tag{3.4}$$

where $c_n \neq 0$ is a universal constant (independent of φ). In particular,

$$\det A_3|_{Z=0} = c_3\varphi_{2,4}, \tag{3.5}$$

is Cartan's "6th order tensor",

$$\det A_4|_{Z=0} = c_4 \det\begin{pmatrix}\varphi_{2,5} & \varphi_{2,6}\\ 5\varphi_{2,4} & 6\varphi_{2,5}\end{pmatrix},$$

$$\det A_5|_{Z=0} = c_5 \det\begin{pmatrix}\varphi_{2,6} & \varphi_{2,7} & \varphi_{2,8}\\ 6\varphi_{2,5} & 7\varphi_{2,6} & 8\varphi_{2,7}\\ 15\varphi_{2,4} & 21\varphi_{2,5} & 28\varphi_{2,6}\end{pmatrix}.$$

The same calculations, for any hypersurface in the form (3.1) with $\varphi(z,0,s)=\varphi(0,\bar{z},s)\equiv 0$ as a formal power series (which can always be achieved; see [BER]), show also that each $\det D_n$ (given by (2.5)) equals a universal constant times $(\varphi_{11})^{\frac{n(n+1)}{2}}$. In a similar vein, when the Levi form of M vanishes at p, we obtain $\rho_z(0)=\bar{L}\rho_z(0)=0$ implying that the whole nth row in A_n must vanish at 0. Together with Theorem 2.1 this yields:

Proposition 3.1. *Let $M \subset \mathbb{C}^2$ be a real smooth hypersurface M given by $\rho(Z, \bar{Z}) = 0$. Then,* $\det D_n \neq 0$ *at $p \in M$ if and only if M is Levi-nondegenerate at p. In addition,* $\det A_n = 0$ *at every point where M is Levi-degenerate.*

4. Umbilical points on real hypersurfaces in $\mathbb{C}^2$

Let $M \subset \mathbb{C}^2$ be a smooth Levi-nondegenerate real hypersurface and $p \in M$. Recall that p is said to be an *umbilical point* if in Chern–Moser normal coordinates (z, w) vanishing at p, the coefficient $\varphi_{2,4}$ in the Chern–Moser normal form ((3.1) and (3.2)) vanishes, i.e., $\varphi_{2,4} = 0$. While Chern-Moser normal coordinates and normal form are not unique, it is well known [CM] that the vanishing of $\varphi_{2,4}$ is an invariant. By Theorem 2.1, Proposition 3.1, and Equation (3.5) above, we have:

Proposition 4.1. *Let $M \subset \mathbb{C}^2$ be defined by $\rho = 0$. Then the set of points $p \in M$ defined by* $\det A_3(\rho) = 0$ *consists of those at which either*

(i) *M is Levi-degenerate at p, or*

(ii) *M is Levi-nondegenerate but umbilical at p.*

In particular if M is Levi-nondegenerate at all points, then the set $\mathcal{U}$ of umbilical points on M is given by the equation $\det A_3(\rho) = 0$.

4.1. Umbilical indices

For a fixed global defining equation $\rho = 0$ for M, where ρ is defined in a neighborhood of M, denote by $Q := \det A_3(\rho)$, so that the set $\mathcal{U} \subset M$ of umbilical points is given by $Q = 0$. We assume M to be oriented and choose ρ compatible with that orientation, i.e. such that the gradient of ρ completes positively oriented frames in M to those in $\mathbb{C}^2$. Note that such ρ always exists e.g. the oriented distance function. Further, ρ is unique up to multiplication with a positive real function in a neighborhood of M in $\mathbb{C}^2$.

We shall say that $p \in \mathcal{U} \subset M$ is a 1-*regular* umbilical point of M if $\mathcal{U}$ is a smooth real curve (1-manifold) at p. By Thom's transversality, every hypersurface M can be approximated by one having only 1-regular umbilical points.

Definition 4.2. For every oriented closed curve C in M avoiding the umbilic set $\mathcal{U}$, define its *umbilical index* to be $-1/2$ times the winding number of Q along C. For every 1-regular umbilical point p with chosen orientation of $\mathcal{U}$, define its *local umbilical index* (or umbilical index of M at p) to be the umbilical index of the positively oriented boundary of any sufficiently small disk transversal to $\mathcal{U}$.

Since ρ is unique up to multiplication with positive real function, it follows from Theorem 2.1 that the index as defined is independent of the choice of ρ. It further follows from the same theorem that the umbilic index of C is also independent of the choice of the ambient coordinates in $\mathbb{C}^2$ as long as C is null-homotopic.

If $\Sigma \subset M$ is an oriented surface (2-manifold) that meets $\mathcal{U}$ transversely at a 1-regular point p, the index of M at p is given by

$$\iota_\Sigma(p) = -\frac{1}{2}\deg\left(\frac{Q}{|Q|} : \partial\Sigma_p \to S^1\right), \tag{4.1}$$

where Σ_p is the boundary of a sufficiently small topological disk containing p (topologically a circle S^1) oriented positively with respect to Σ, and where deg denotes the topological degree (which is the same as winding number in this case). We note that we can also express the index using an integral,

$$\iota_\Sigma(p) = \frac{i}{4\pi}\int_{\partial\Sigma_p}\frac{dQ}{Q} = \frac{i}{4\pi}\int_{\partial\Sigma_p} d\log Q. \tag{4.2}$$

Since the 1-form dQ/Q is closed away from the zeros of Q, we observe from (4.2) that in fact the index at p only depends on the orientation of Σ and not on the choice of transversal Σ itself, and that the index is constant along every component of the 1-manifold of nondegenerate umbilical points $\mathcal{U}_1$. If $p \in M$ is a 1-regular umbilical point and the index of M at p (with respect to any transversal Σ) is non-zero, then we shall say that p is a *stable* umbilical point. In view of Thom's transversality, any sufficiently small perturbation of the CR structure of M near a stable umbilical point p will have stable umbilical points near p, which motivates this terminology.

We shall use the notation $W_\gamma(R)$ for the winding number of a function R on M along an oriented closed curve γ (defined only when R does not vanish on γ);

$$W_\gamma(R) = \frac{1}{2\pi i}\int_\gamma \frac{dR}{R} = \frac{1}{2\pi i}\int_\gamma d\log R. \tag{4.3}$$

Thus, in particular, if Σ is a surface, transversal to $\mathcal{U}$ at p and Σ_p is its intersection with a small tubular neighborhood of μ near p, then by definition of the index:

$$\iota_\Sigma(p) = -\frac{1}{2} W_{\partial \Sigma_p}(Q).$$

If M is real-analytic, then $\mathcal{U}$ is a real-analytic subvariety of M, and the set $\mathcal{U}_1$ of 1-regular umbilical points consist of the subset of $\mathcal{U}$ of regular points of dimension one. For simplicity, we shall proceed under the assumption that M is real-analytic. In this case, $\mathcal{U}$ is either all of M (we assume that M is connected), in which case M is locally spherical, or $\mathcal{U}$ is a proper subvariety. In the latter case, points of $\mathcal{U}$ are either 0-, 1-, or, 2-dimensional, and we decompose $\mathcal{U}$ accordingly, $\mathcal{U} = \mathcal{U}^0 \cup \mathcal{U}^1 \cup \mathcal{U}^2$; recall that the (topological) dimension of a real-analytic subvariety $\mathcal{V}$ at a point p is the largest dimension of a nonsingular component of $\mathcal{V}$ with p in its closure.

We note that the set of 1-regular umbilical points $\mathcal{U}_1$ equals $\mathcal{U}^0 \cup \mathcal{U}^1$ minus a discrete set of points. Thus, if there are no points of dimension 2 on $\mathcal{U}$, then any surface Σ that intersects $\mathcal{U}$ can be locally perturbed to only intersect $\mathcal{U}$ along the set of 1-regular points $\mathcal{U}_1$. We have the following simple consequence of Stokes Theorem:

Proposition 4.3. *Let $M \subset \mathbb{C}^2$ be a real-analytic hypersurface, and assume that $\mathcal{U}$ has no points of dimension 2 or 3, i.e., $\mathcal{U} = \mathcal{U}^0 \cup \mathcal{U}^1$ (possibly empty). Let $\gamma \subset M$ be a oriented closed curve, homologous to 0 in M and not intersecting $\mathcal{U}$, and $\Sigma \subset M$ an oriented surface, intersecting $\mathcal{U}$ transversally along $\mathcal{U}_1$ and with $\partial\Sigma = \gamma$. Then,*

$$-\frac{1}{2} W_\gamma(Q) = \sum_{p \in \mathcal{U}_1 \cap \Sigma} \iota_\Sigma(p). \tag{4.4}$$

Proof. Let $p_1, \ldots, p_k$ be the finite set of points in $\mathcal{U}_1 \cap \Sigma$ and Σ_{p_j} the intersection of Σ with a sufficiently small tubular neighborhood of $\mathcal{U}_1$ near p_j (so small that the closures of the Σ_{p_j} are disjoint). Since dQ/Q is closed in the punctured surface

$$\Sigma' := \Sigma \setminus \bigcup_{j=1}^{k} \Sigma_{p_j},$$

being locally $d \log Q$ on Σ', we conclude by Stokes Theorem that:

$$W_\gamma(Q) = \frac{1}{2\pi i} \int_{\partial\Sigma} \frac{dQ}{Q} = \sum_{j=1}^{k} \frac{1}{2\pi i} \int_{\partial \Sigma_{p_j}} \frac{dQ}{Q} = -2 \sum_{p \in \mathcal{U}_1 \cap \Sigma} \iota_\Sigma(p). \tag{4.5}$$

□

5. Umbilical points on Real Ellipsoids

We shall consider real ellipsoids $E \subset \mathbb{C}^2$. A general real ellipsoid can be defined by an equation of the form

$$A(z^2+\bar z^2)+2(2+A)|z|^2+B(w^2+\bar w^2)+2(2+B)|w|^2=4, \quad A,B\geq 0. \tag{5.1}$$

We shall fix $A, B \geq 0$, and assume that at least one of these, say A, is nonzero (so that the ellipsoid does not degenerate to a sphere). We consider a 1-parameter family of ellipsoids E_ϵ, defined by $\rho_\epsilon = 0$, where

$$\begin{aligned}\rho_\epsilon &:= \epsilon A(z^2+\bar z^2)+2(2+\epsilon A)|z|^2+\epsilon B(w^2+\bar w^2)+2(2+\epsilon B)|w|^2-4\\ &= -4+4(|z|^2+|w|^2)\\ &\quad +\epsilon\big(A(z^2+\bar z^2+2|z|^2)+B(w^2+\bar w^2+2|w|^2)\big), \quad \epsilon>0.\end{aligned} \tag{5.2}$$

Note that E_0 is the unit sphere. We shall mainly be concerned with small perturbations of the sphere, and shall thus consider small $\epsilon > 0$. Our aim is to prove the following result:

Theorem 5.1. *For $\epsilon > 0$ sufficiently small, the subset of umbilical points on E_ϵ contains a curve of umbilical points.*

To this end, we shall compute the matrix $A_3 = A_3(\rho_\epsilon)$ on $\rho = \rho_\epsilon = 0$. Since we will be concerned with small perturbations it suffices, as we shall see, to compute A_3 mod $O(\epsilon^3)$. We note first that

$$\rho_z = 4\bar z + 2\epsilon A(\bar z + z), \quad \rho_w = 4\bar w + 2\epsilon B(\bar w + w) \tag{5.3}$$

and

$$\rho_{z^2} = 2\epsilon A, \quad \rho_{zw} = 0, \quad \rho_{w^2} = 2\epsilon B. \tag{5.4}$$

Therefore, we have

$$\begin{aligned}\rho_{Z^2}(L,L) &= (\rho_w)^2\rho_{z^2} - 2\rho_z\rho_w\rho_{zw} + (\rho_z)^2\rho_{w^2}\\ &= 8\epsilon\left(A(2\bar w+\epsilon B(\bar w+w))^2 + B(2\bar z+\epsilon A(\bar z+z))^2\right)\\ &= 32\epsilon(A\bar w^2+B\bar z^2)+32\epsilon^2AB(|z|^2+|w|^2+\bar w^2+\bar z^2)\\ &\quad + 8\epsilon^3AB\left(A(z+\bar z)^2+B(w+\bar w)^2\right).\end{aligned} \tag{5.5}$$

To calculate $A_3(\rho)$, we shall repeatedly apply $\bar{L}$, where

$$
\begin{aligned}
\bar{L} &= -\rho_{\bar{w}}\frac{\partial}{\partial \bar{z}} + \rho_{\bar{z}}\frac{\partial}{\partial \bar{w}} \\
&= -2(2w+\epsilon B(\bar{w}+w))\frac{\partial}{\partial \bar{z}} + 2(2z+\epsilon A(\bar{z}+z)\frac{\partial}{\partial \bar{w}} \\
&= 4\left(-w\frac{\partial}{\partial \bar{z}} + z\frac{\partial}{\partial \bar{w}}\right) + 2\epsilon\left(-B(w+\bar{w})\frac{\partial}{\partial \bar{z}} + A(z+\bar{z})\frac{\partial}{\partial \bar{w}}\right) \\
&= \bar{L}_0 + \epsilon \bar{L}_1,
\end{aligned}
\tag{5.6}
$$

to $\rho_{Z^2}(L, L)$, and subsequently evaluate at $w = 0$. Since $\bar{L}$ only involves differentiation in $\bar{z}$ and $\bar{w}$, the result for $w = 0$ will not change when replacing with 0 all occurences of w (but not $\bar{w}$). Thus for $w = 0$, we obtain

$$
\begin{aligned}
\frac{1}{2^k\cdot 32}\bar{L}^k\rho_{Z^2}(L,L) &= \left(2z\frac{\partial}{\partial \bar{w}} + \epsilon\left(-B\bar{w}\frac{\partial}{\partial \bar{z}} + A(z+\bar{z})\frac{\partial}{\partial \bar{w}}\right)\right)^k \\
&\qquad \times \left(\epsilon(A\bar{w}^2 + B\bar{z}^2) + \epsilon^2 AB(|z|^2 + \bar{z}^2 + \bar{w}^2).\right)
\end{aligned}
\tag{5.7}
$$

Then we obtain for $w = 0$,

$$
\begin{aligned}
&\bar{L}\rho_{Z^2}(L,L) = \bar{L}^3\rho_{Z^2}(L,L) = O(\epsilon^2), \quad \bar{L}^4\rho_{Z^2}(L,L) = O(\epsilon^3), \\
&\frac{1}{2^2\cdot 32}\bar{L}^2\rho_{Z^2}(L,L) = 2^3 z^2 A\epsilon + O(\epsilon^2).
\end{aligned}
\tag{5.8}
$$

5.1. Terms of the form $\bar{L}^k\rho_z^3$; first row

We note that

$$
\rho_z^3 = (4\bar{z} + 2\epsilon A(\bar{z}+z))^3 = 8(2\bar{z} + \varepsilon A(\bar{z}+z))^3. \tag{5.9}
$$

We shall be interested in A_3 mod $O(\epsilon^3)$, and since all terms in the last row (computed above) are already $O(\varepsilon)$, we shall compute $\bar{L}^k(\rho_z)^3$ mod $O(\varepsilon^2)$. Thus, we have

$$
\begin{aligned}
\rho_z^3 &= 8(8\bar{z}^3 + 12\varepsilon A\bar{z}^2(\bar{z}+z)) + O(\varepsilon^2) \\
&= 2^5(2\bar{z}^3 + 3\varepsilon A(\bar{z}^3 + z\bar{z}^2)) + O(\varepsilon^2).
\end{aligned}
\tag{5.10}
$$

We obtain for $w = 0$,

$$
\frac{1}{2^k}\bar{L}^k\rho_z^3 = \left(2z\frac{\partial}{\partial \bar{w}} + \epsilon\left(-B\bar{w}\frac{\partial}{\partial \bar{z}} + A(z+\bar{z})\frac{\partial}{\partial \bar{w}}\right)\right)^k \rho_z^3 \tag{5.11}
$$

and since ρ_z is independent of w,

$$\bar{L}\rho_z^3 = \bar{L}^3\rho_z^4 = \bar{L}^4\rho_z^3 = O(\varepsilon^2), \tag{5.12}$$

and

$$\frac{1}{4}\bar{L}^2\rho_z^3 = -2z\epsilon B\frac{\partial}{\partial\bar{z}}\rho_z^3 = -2^7\cdot 3\epsilon Bz\bar{z}^2 + O(\varepsilon^2) \tag{5.13}$$

5.2. Terms of the form $\bar{L}^k\rho_z^2\rho_w$; second row

As before, replacing occurences of w (but not $\bar{w}$) with 0 (written $\cong$), we obtain

$$\begin{aligned}\rho_z^2\rho_w &\cong (4\bar{z} + 2\varepsilon A(\bar{z}+z))^2(4+2\varepsilon B)\bar{w} \\ &= 2^5(2\bar{z}^2\bar{w} + \varepsilon((2A+B)\bar{z}^2\bar{w} + 2Az\bar{z}\bar{w}) + O(\varepsilon^2),\end{aligned} \tag{5.14}$$

We compute, mod $O(\varepsilon^2)$

$$\begin{aligned}\frac{1}{2^{k+5}}\bar{L}^k(\rho_z^2\rho_w) \cong &\left(2z\frac{\partial}{\partial\bar{w}} + \epsilon\left(-B\bar{w}\frac{\partial}{\partial\bar{z}} + A(z+\bar{z})\frac{\partial}{\partial\bar{w}}\right)\right)^k \\ &\times\left((2\bar{z}^2\bar{w} + \varepsilon((2A+B)\bar{z}^2\bar{w} + 2Az\bar{z}\bar{w})\right),\end{aligned} \tag{5.15}$$

and hence for $w = 0$,

$$\bar{L}^k(\rho_z^2\rho_w) = O(\epsilon),\ k \neq 1; \quad \bar{L}^4(\rho_z^2\rho_w) = O(\epsilon^2), \tag{5.16}$$

and

$$\bar{L}(\rho_z^2\rho_w) = 2^8 z\bar{z}^2 + O(\epsilon). \tag{5.17}$$

5.3. Terms of the form $\bar{L}^k\rho_z\rho_w^2$; third row

We can obtain the formulas in this case by considering the previous subsection and interchanging the roles of z and w. We obtain

$$\begin{aligned}\rho_z\rho_w^2 &\cong (4\bar{z} + 2\varepsilon A(\bar{z}+z))(4+2\varepsilon B)^2\bar{w}^2 \\ &= 2^5\left(2\bar{z} + \varepsilon((2B+A)\bar{z} + Az)\right)\bar{w}^2 + O(\varepsilon^2),\end{aligned} \tag{5.18}$$

As before we obtain mod $O(\varepsilon^2)$,

$$\frac{1}{2^{k+5}}\bar{L}^k(\rho_z\rho_w^2) \cong \left(2z\frac{\partial}{\partial\bar{w}} + \epsilon\left(-B\bar{w}\frac{\partial}{\partial\bar{z}} + A(z+\bar{z})\frac{\partial}{\partial\bar{w}}\right)\right)^k \times (2\bar{z} + \varepsilon((2B+A)\bar{z} + Az))\,\bar{w}^2, \tag{5.19}$$

from which as before, for $w=0$, we obtain

$$\bar{L}^k(\rho_z\rho_w^2) = O(\epsilon), \quad k \neq 2, \tag{5.20}$$

$$\bar{L}^2(\rho_z\rho_w^2) = 2^{11}z^2\bar{z} + O(\epsilon), \tag{5.21}$$

and

$$\begin{aligned}\bar{L}^4(\rho_z\rho_w^2) &= 2^7\binom{4}{2}\left(2z\frac{\partial}{\partial\bar{w}} - \epsilon B\bar{w}\frac{\partial}{\partial\bar{z}}\right)^2(4\bar{z}z^2) + O(\epsilon^2)\\ &= -2^{10}\binom{4}{2}\epsilon Bz^3 + O(\epsilon^2).\end{aligned} \tag{5.22}$$

5.4. Terms of the form $\bar{L}^k\rho_w^3$; fourth row

Following the same strategy as above, we obtain

$$\rho_w^3 \cong (4+2\varepsilon B)^3\bar{w}^3 = 2^3(8+12\varepsilon B)\bar{w}^3 + O(\varepsilon^2), \tag{5.23}$$

and for $w=0$,

$$\bar{L}^k(\rho_w^3) = O(\epsilon^2), \quad k \neq 3, \tag{5.24}$$

$$\bar{L}^3(\rho_w^3) = 2^9\cdot 6z^3 + O(\epsilon). \tag{5.25}$$

5.5. Calculation of ε^2-term of $A_3(\rho_\varepsilon)$ along $w=0$

From our calculations in the subsections above we obtain for $w=0$:

(5.26)

$$A_3(z,\bar{z}) = \begin{pmatrix} 2^6\bar{z}^3 + O(\epsilon) & 0 & O(\epsilon) & 0 & 0\\ O(\epsilon) & 8z\bar{z}^2 + O(\epsilon) & O(\epsilon) & O(\epsilon) & 0\\ O(\epsilon) & O(\epsilon) & 2^{11}z^2\bar{z} & O(\epsilon) & -2^{10}\binom{4}{2}\epsilon Bz^3\\ 0 & 0 & O(\epsilon) & 2^{10}\cdot 3z^3 & 0\\ 2^5\epsilon B\bar{z}^2 & 0 & 2^{10}\epsilon Az^2 & 0 & 0\end{pmatrix} + O(\epsilon^2).$$

Since the last row as well as the last column each has a factor ε, and we recall from (5.8) that the lower right hand corner element in A_3 is in fact $O(\varepsilon^3)$, we conclude

$$\det A_3(\rho_\varepsilon)|_{w=0} = \varepsilon^2 \Delta_2 + O(\varepsilon^3), \quad \Delta_2 = NABz^9\bar{z}^5, \tag{5.27}$$

where N is a non-zero integer.

5.6. Umbilical points on the ellipsoids E_ε

Let S_ε be the ellipse (in the z-plane) obtained by intersecting E_ε with the complex line $w = 0$. If both $A, B > 0$, we easily conclude that the winding number $W_{S_0}(\Delta_2)$ of $\Delta_2(z, \bar{z})$ around the circle S_0, traversed in the positive direction, equals 4; recall that the winding number is defined by (4.3) from which $W_{S_0}(\Delta_2) = 4$ follows immediately.

Proof of Theorem 5.1. Let us first assume that both $A, B > 0$. Since, in this case, Δ_2 does not vanish on S_0, it follows that $Q_\varepsilon|_{w=0}$ does not vanish on S_ε for $\varepsilon > 0$ sufficiently small, where $Q_\varepsilon = \det A_3(\rho_\varepsilon)$. It is also clear by continuity that, for sufficiently small $\varepsilon > 0$, the winding number of Q_ε around S_0 coincides with that of Δ_2 around S_0, and also around S_ε. We conclude that

$$W_{S_\varepsilon}(Q_\varepsilon) = 4, \tag{5.28}$$

for sufficiently small $\varepsilon > 0$. Now, either the set of umbilical points $\mathcal{U} \subset E_\varepsilon$ contains points of dimension at least 2 (in which case there are plenty of curves of umbilical points), or there is a surface Σ^ε in E_ε that is bounded by S_ε and meets the subset of 1-regular umbilical points $\mathcal{U}_1$ transversally; indeed, we can always find even a simply connected Σ^ε in E_ε with $\partial\Sigma^\varepsilon = S_\varepsilon$, and if $\mathcal{U}$ has only components of dimension 0 and 1, then small local deformations of Σ^ε along the intersection will result in only transversal intersections along $\mathcal{U}_1$. It now follows from (5.28) and Proposition 4.3 that

$$\sum_{p \in \Sigma^\varepsilon \cap \mathcal{U}_1} \iota_{\Sigma^\varepsilon_p}(p) = -2. \tag{5.29}$$

In particular, either $\mathcal{U}$ has points of dimension at least 2 or contains at least one curve of stable umbilical points when both $A, B > 0$.

In the remaining case where, say, $B = 0$, the ellipsoid is invariant under the circle action $(z, w) \mapsto (z, e^{it}w)$ and therefore has umbilical points along the curve of fixed points $(z, 0)$, in view of the special Chern–Moser normalization at non-umbilical points [CM], pp. 246–247. $\square$

6. Umbilical points on perturbations of the sphere

We shall consider perturbations $M_\varepsilon \subset \mathbb{C}^2$ of the unit sphere given by $\rho = \rho^\epsilon = 0$, where

$$\rho^\varepsilon := \rho^0 + \varepsilon\rho', \quad \rho^0 := -1 + z\bar{z} + w\bar{w}, \tag{6.1}$$

ρ' is a smooth real-valued function, and ε is a small real parameter. For $\varepsilon = 0$ we recover the unit sphere $S^3 = M_0$ and hence $\det A_3 = 0$. For $\varepsilon \neq 0$, we shall consider the power series expansion of $\det A_3$ in ε. In that expansion, we shall compute the linear term in ε. Since the expansion of ρ_{Z^2} begins with a linear term in ε, the only nonzero contribution to the linear term in ε in the determinant (2.3) will come from 0th order terms (in ε) in the first 4 rows and 1st order terms in the last row. Furthermore, only 0th order terms in the expansion of L will contribute. Thus for our computation, we only need use the terms with

$$(\rho^0_w, \rho^0_z) = (\bar{w}, \bar{z}), \quad L_0 = -\bar{w}\partial_z + \bar{z}\partial_w,$$

and hence the desired coefficient of ε is

$$\det \begin{pmatrix} D^0_3 & 0 \\ * & \bar{L}_0^4(\rho'_{Z^2}(L_0, L_0)) \end{pmatrix} = (\det D^0_3)\bar{L}_0^4(\rho'_{Z^2}(L_0, L_0)), \tag{6.2}$$

where D^0_3 is calculated using ρ^0. By Proposition 3.1, we conclude:

Proposition 6.1. *For a perturbation of the form* (6.1)*,*

$$\det A_3(\rho^\varepsilon) = c_0\bar{L}_0^4(\rho'_{Z^2}(L_0, L_0))\varepsilon + O(\varepsilon^2), \tag{6.3}$$

where c_0 is a universal polynomial that does not vanish on the unit sphere $\rho^0 = 0$.

We note that

$$\rho'_{Z^2}(L_0, L_0) = (-\bar{w})^2\rho'_{z^2} - 2\bar{z}\bar{w}\rho'_{zw} + \bar{z}^2\rho'_{w^2}, \tag{6.4}$$

and observe that the coefficients in $\bar{L}_0$ are holomorphic, and hence repeated applications of $\bar{L}_0$ will not result in any differentiations of the coefficients, and we obtain

$$\begin{aligned}(6.5)\qquad \bar L_0^4 &= (-w\partial_{\bar z} + z\partial_{\bar w})^4 \\ &= w^4\partial_{\bar z}^4 - 4zw^3\partial_{\bar z}^3\partial_{\bar w} + 6z^2w^2\partial_{\bar z}^2\partial_{\bar w}^2 - 4z^3w\partial_{\bar z}\partial_{\bar w}^3 + z^4\partial_{\bar w}^4.\end{aligned}$$

We shall consider polynomial perturbations of the form $\rho' = \sum_{k=2}^m \rho'_k$, where ρ'_k are homogeneous polynomials of degree k in $Z = (z, w)$ and $\bar Z$. We may decompose ρ'_k further into bidegree, $\rho'_k = \sum_{p+q=k} \rho'_{p,q}$, where each $\rho_{p,q}$ is of bidegree (p, q). Since our perturbations ρ' are real-valued, we must have $\rho'_{q,p} = \overline{\rho'_{p,q}}$.

We shall use the notation $\mathcal{H}_k$ for the space of homogeneous polynomials of degree k, and $\mathcal{H}_{p,q}$ for those of bidegree (p, q). We note that if $R \in \mathcal{H}_{p,q}$, then $\bar L_0^4(R_{Z^2}(L_0, L_0)) \in \mathcal{H}_{p+2,q-2}$. We also note that in this case $R_{Z^2}(L_0, L_0) \in \mathcal{H}_{p-2,q+2}$, and we conclude that $\bar L_0^4(R_{Z^2}(L_0, L_0)) = 0$ unless both p and q satisfy $p, q \geq 2$. Let us for brevity use the notation

$$(6.6)\qquad Q^0(R) := \bar L_0^4(R_{Z^2}(L_0, L_0)),$$

so that

$$(6.7)\qquad Q = Q(\rho^\varepsilon) := \det A_3(\rho^\varepsilon) = c_0 Q^0(\rho')\varepsilon + O(\varepsilon^2).$$

We may then summarize the discussion above as follows.

Proposition 6.2. *For a real-valued polynomial ρ' of degree m, decomposed into homogeneous components ρ'_k and further decomposed into bidegree $\rho'_{p,q}$, of the form*

$$(6.8)\qquad \rho' = \sum_{k=2}^m \rho'_k = \sum_{k=2}^m \sum_{p+q=k} \rho'_{p,q}, \quad \rho'_{p,q} = \overline{\rho'_{q,p}},$$

it holds that

$$(6.9)\qquad Q^0(\rho') = \sum_{k=4}^m \sum_{l=4}^k Q^0_{l,k-l}, \quad Q^0_{l,k-l} = Q^0(\rho'_{l-2,k-l+2}).$$

We have the following technical result.

Proposition 6.3. *Let ρ' be a real-valued polynomial of degree m, and decompose $Q^0(\rho')$ as in (6.9). Assume that:*

(i) *The real-algebraic variety* $\mathcal{V} := \{Q^0(\rho') = 0\} \cap S^3$ *in* S^3 *has no points of dimension* ≥ 2.

(ii) $Q^0_{l,k-l} = 0$ *for* $4 \leq l \leq k/2$.

Then, for sufficiently small $\varepsilon > 0$*, the set of umbilical points* $\mathcal{U}$ *on the perturbation* M_ε *contains either points of dimension* ≥ 2 *or a curve of stable umbilical points.*

Remark 6.4. We make a few observations:

- If the degree $m \leq 7$, then condition (ii) is vacuous, and hence only (i) is required in this case.
- If the degree $m \leq 3$, then condition (i) is never satisfied, since $Q^0(\rho')$ vanishes completely. In particular, as noted in the previous section, for real ellipsoid perturbations E_ε we have $Q^0(\rho') = 0$. Nevertheless, as is proved in Theorem 5.1, real ellipsoids do have umbilical points.

To prove Proposition 6.3, we need the following lemma:

Lemma 6.5. *Let* ρ' *be a real-valued polynomial such that condition* (i) *in Proposition* 6.3 *holds. Then, there is point* $Z_0 = (z_0, w_0) \in S^3$ *such that* $Q^0(\rho')$ *does not vanish on the circle* $S_0 \colon t \mapsto e^{it} Z_0$ *in* S^3.

Proof. For $Z_1 := (z_1, w_1) \in S^3$, consider the circle S_1 in S^3 parametrized by $t \mapsto e^{it}(z_1, w_1)$. Let $\Sigma \subset S^3$ be a germ at Z_1 of an open, real-analytic surface, transverse to S_1 at this point. Consider the real-analytic map $\Gamma \colon \Sigma \times S^1 \to S^3$, given by $\Gamma(z, w, t) = e^{it}(z, w)$ in local coordinates $t \to e^{it}$ on S^1. This map realizes an open subset Ω of S^3 as an S^1-fibration over Σ. If we let $\pi \colon \Omega \to \Sigma$ be the projection, then we can consider $\pi(\mathcal{V}) \subset \Sigma$, where $\mathcal{V}$ is the zero locus of $Q^0(\rho')$ as in Proposition 6.3. Since $\mathcal{V}$, by condition (i), has no points of dimension 2 or 3, $\pi(\mathcal{V})$ is a proper sub-analytic subset of the open surface Σ. Thus, by choosing $Z_0 = (z_0, w_0)$ in Σ outside this projection, we find the desired oriented circle S_0, parametrized by $t \to \Gamma(z_0, w_0, t)$. $\square$

Remark 6.6. We may parametrize all great circles on S^3 by blowing up the origin in $\mathbb{C}^2$. In this way, S^3 becomes the unit circle in the universal line bundle $O(-1)$ over $\mathbb{P}^1$. The corresponding projection $\pi \colon O(-1) \to \mathbb{P}^1$ is algebraic and then the set $\pi(\mathcal{V})$ of unit circles to avoid is a closed semialgebraic set in $\mathbb{P}^1$. This is explained in greater detail in Subsection 6.1 below.

We now proceed with the proof of Proposition 6.3. Let $S_0 \colon t \mapsto e^{it} Z_0$, with $Z_0 = (z_0, w_0) \in S^3$, be the circle provided by Lemma 6.5, and define a polynomial $P(\zeta, \bar\zeta)$ of degree m in the variable $\zeta \in \mathbb{C}$ by

$$P(\zeta, \bar\zeta) := Q^0(\rho')(\zeta Z_0, \overline{\zeta Z_0}). \tag{6.10}$$

By construction of S_0, P does not vanish on the unit circle. The decomposition of $Q^0(\rho')$ into bidegree, given by (6.9), yields a decomposition of P into bidegree:

$$P(\zeta, \bar\zeta) = \sum_{k=4}^{m} \sum_{l=4}^{k} p_{l,k-l} \zeta^l \bar\zeta^{k-l}, \quad p_{l,k-l} := Q_{l,k-l}(\zeta Z_0, \overline{\zeta Z_0}) / \zeta^l \bar\zeta^{k-l}. \tag{6.11}$$

On the unit circle $\zeta = e^{it}$, $P(\zeta, \bar\zeta)$ coincides with a rational function $R(\zeta)$,

$$R(\zeta) = P(\zeta, 1/\zeta) = \frac{\sum_{k=4}^{m} \sum_{l=4}^{k} p_{l,k-l} \zeta^{2l+m-k}}{\zeta^m}. \tag{6.12}$$

If we define

$$p(\zeta) := \sum_{k=4}^{m} \sum_{l=4}^{k} p_{l,k-l} \zeta^{2l+m-k}, \tag{6.13}$$

then by the construction of $p(\zeta)$ and the argument principle we conclude:

Lemma 6.7. *Let n denote the number of zeros (counted with multiplicities) of $p(\zeta)$ in the unit disk $|\zeta| < 1$. Then*

$$W_{S_0}(Q^0) = n - m, \tag{6.14}$$

where $Q^0 = Q^0(\rho')$ and S_0 is the circle in the construction of $p(\zeta)$ above.

We may now complete the proof of Proposition 6.3.

Proof of Proposition 6.3. Recall that the set of umbilical points $\mathcal{U}$ on M_ε is given by $Q = 0$, where Q is as in (6.7). Let S_0 the circle on S^3 as above, and let S_ε be the perturbed oriented curve on M_ε obtained as the intersection between the complex subspace through $Z_0 = (z_0, w_0) \in S^3$ and M_ε. As in the proof of Theorem 5.1, for sufficiently small $\varepsilon > 0$, we have

$$W_{S_\varepsilon}(Q) = W_{S_0}(Q^0) = n - m, \tag{6.15}$$

where n and m are as in Lemma 6.7. We claim that $n - m \neq 0$. Indeed, by condition (ii), the coefficients $p_{l,k-l} = 0$ for $l \leq k/2$. If we let r denote the minimum integer $r = 2l + m - k$ for which $p_{l,k-l} \neq 0$, then $p(\zeta)$ is divisible by ζ^r and since $r > m$, we conclude that $n > m$. The proof of Proposition 6.3 is now completed in the same way as the proof of Theorem 5.1. □

6.1. The sphere S^3 as a circle bundle over $\mathbb{P}^1$

We recall here the idea of realizing the sphere as the unit circle in the universal bundle $\pi\colon J := O(-1) \to \mathbb{P}^1$; this idea has been extended to more general three-dimensional CR manifolds with a CR circle action by Bland–Duchamp [BD] and Epstein [Ep]. Recall that J naturally embeds into $\mathbb{P}^1 \times \mathbb{C}^2$ in such a way that the fiber J_Z over a point Z in homogeneous coordinates, $Z = [z\colon w] \in \mathbb{P}^1$, is the complex line through $(z, w) \in \mathbb{C}^2$; J_Z is parametrized by $\zeta \mapsto \zeta(z, w)$. The standard metric $|\cdot|$ on J is the one induced by the Euclidian metric on $\mathbb{C}^2$; if $s(Z)$ is a non-vanishing local section in J and we write $s(Z) = (u, v) \in \mathbb{C}^2$, then $|s|^2 := |u|^2 + |v|^2$. The unit circle bundle $\tilde{S}^3 := \{\lambda \in J\colon |\lambda|^2 = 1\}$ is CR isomorphic to the unit sphere S^3 in $\mathbb{C}^2$. Indeed, if we view the total space J as the blow-up of the origin in $\mathbb{C}^2$, then the CR isomorphism $\tilde{\pi}|_{\tilde{S}^3}\colon \tilde{S}^3 \to S^3$ is the blow-down map $\tilde{\pi}\colon J \to \mathbb{C}^2$ restricted to $\tilde{S}^3$. For convenience, we shall simply identify S^3 with $\tilde{S}^2$ via this isomorphism; in this identification, the fibers $\pi^{-1}(Z)$ in $\tilde{S}^3$ correspond to the great circles $t \mapsto e^{it}Z$.

We note that if ρ' is a real-valued polynomial, then the projection $\pi(\mathcal{V}) \subset \mathbb{P}^1$ of the real-algebraic subvariety $\mathcal{V} \subset S^3 \cong \tilde{S}^3$, defined to be the zero locus of $Q^0 = Q^0(\rho')$ as in Proposition 6.3, is a closed semialgebraic subset. An inspection of the proof of Proposition 6.3 reveals immediately that condition (i) in the assumptions of this proposition can be replaced by the assumption that $\pi(\mathcal{V}) \neq \mathbb{P}^1$. As is shown in Lemma 6.5, condition (i) implies $\pi(\mathcal{V}) \neq \mathbb{P}^1$, and the latter property is the only one used in the proof of Proposition 6.3. For convenience, we state the result here.

Proposition 6.8. *Let ρ' be a real-valued polynomial of degree m, and decompose $Q^0(\rho')$ as in* (6.9). *Assume that:*

(i′) $\mathbb{P}^1 \setminus \pi(\mathcal{V}) \neq \emptyset$, *where $\mathcal{V} \subset S^3 \cong \tilde{S}^3$ and $\pi\colon \tilde{S}^3 \to \mathbb{P}^1$ are as above.*

(ii) $Q^0_{l,k-l} = 0$ *for* $4 \le l \le k/2$.

Then, for sufficiently small $\varepsilon > 0$, *the set of umbilical points* $\mathcal{U}$ *on the perturbation* M_ε *contains either points of dimension* ≥ 2 *or a curve of stable umbilical points.*

6.2. Generic perturbations of the sphere

We shall denote by $\mathcal{P}_m$ the space of all polynomials in $Z = (z, w)$ and $\bar{Z}$ of degree at most m, and by $\mathcal{P}^{\mathbb{R}}_m$ the real subspace of those that are real-valued. Thus, we have

$$\mathcal{P}_m = \bigoplus_{p+q \le m} \mathcal{H}_{p,q}$$

and $\rho' \in \mathcal{P}_m$ belongs to $\mathcal{P}^{\mathbb{R}}_m$ when $\rho'_{p,q} = \overline{\rho_{q,p}}$ for all p, q. We shall show that condition (i′) in Proposition 6.8 is generic. More precisely, we shall prove the following:

Proposition 6.9. *The set* Π_m *of polynomials* ρ' *in* $\mathcal{P}^{\mathbb{R}}_m$ *such that* $\pi(\mathcal{V}) = \mathbb{P}^1$, *where* $\mathcal{V} \subset S^3 \cong \tilde{S}^3$ *and* $\pi \colon \tilde{S}^3 \to \mathbb{P}^1$ *are as in Proposition* 6.8, *is a real-analytic subvariety in* $\mathcal{P}^{\mathbb{R}}_m$. *Moreover, if* $\mathcal{A} \subset \mathcal{P}^{\mathbb{R}}_m$ *is any real subspace containing* $\mathcal{A}_p := \{e z^p \bar{w}^p + \bar{e} w^p \bar{z}^p \colon e \in \mathbb{C}\}$ *for some* $2 \le p \le m/2$, *then* $\Pi_m \cap \mathcal{A}$ *has strictly smaller dimension than* $\mathcal{A}$.

Proof. Let $\tilde{z} = z/w$ be a local coordinate in the chart $U_0 = \{[z \colon w] \in \mathbb{P}^1 \colon w \ne 0\}$ in $\mathbb{P}^1$ and $\tilde{Q}^0 = \tilde{Q}^0(\tilde{z}, \bar{\tilde{z}}; \zeta, \bar{\zeta})$ the polynomial $Q^0 = Q^0(\rho')$ for some $\rho' \in \mathcal{P}^{\mathbb{R}}_m$ in the local trivialization

$$U_0 \times \mathbb{C} = \mathbb{C} \times \mathbb{C} \cong J|_{U_0} \subset U_0 \times \mathbb{C}^2,$$

given by

$$(\tilde{z}, \zeta) \mapsto (\tilde{z}; \tilde{\pi}(\tilde{z}, \zeta)), \;\; \tilde{\pi}(\tilde{z}, \zeta) := \zeta(\tilde{z}, 1).$$

In other words, $\tilde{Q}^0 = Q^0 \circ \tilde{\pi}$; we shall denote by $\tilde{Q}^0_{p,q} = Q^0_{p,q} \circ \tilde{\pi}$, so that we have the decomposition (see Proposition 6.2)

$$\tilde{Q}^0 = \sum_{k=4}^{m} \sum_{l=4}^{k} \tilde{Q}^0_{l,k-l}, \tag{6.16}$$

where each component $\tilde{Q}^0_{p,q}$ takes the form

$$\tilde{Q}^0_{p,q}(\tilde{z}, \bar{\tilde{z}}; \zeta, \bar{\zeta}) = q_{p,q}(\tilde{z}, \bar{\tilde{z}}) \zeta^p \bar{\zeta}^q = q_{p,q}(\tilde{z}, \bar{\tilde{z}}) \zeta^{p-q} |\zeta|^{2q}, \tag{6.17}$$

with

$$q_{p,q}(\tilde{z},\bar{\tilde{z}})=\tilde{Q}^0_{p,q}((\tilde{z},1),(\bar{\tilde{z}},1))=\left(\sum_{\alpha\leq p,\,\gamma\leq q}c_{pq;\alpha\bar{\gamma}}\tilde{z}^{\alpha}\bar{\tilde{z}}^{\gamma}\right). \tag{6.18}$$

for suitable coefficients $c_{pq;\alpha\bar{\gamma}}$. Recall that $\tilde{S}^3\subset J$ is given in these coordinates by

$$|\zeta|^2(1+|\tilde{z}|^2)=1. \tag{6.19}$$

Consequently, each $\tilde{Q}^0_{p,q}$ coincides on $\tilde{S}^3$ with the function

$$R_{p,q}(\tilde{z},\bar{\tilde{z}};\zeta,\bar{\zeta})=\frac{q_{p,q}(\tilde{z},\bar{\tilde{z}})}{(1+|\tilde{z}|^2)^q}\,\zeta^{p-q}, \tag{6.20}$$

and $\tilde{Q}^0$ coincides with R, where

$$R=\sum_{k=4}^{m}\sum_{l=4}^{k}R_{l,k-l}=\sum_{k=4}^{m}\sum_{l=4}^{k}\frac{q_{l,k-l}(\tilde{z},\bar{\tilde{z}})}{(1+|\tilde{z}|^2)^{k-l}}\,\zeta^{2l-k}, \tag{6.21}$$

a rational function in ζ with coefficients that are rational functions in $\tilde{z}$ and $\bar{\tilde{z}}$. Note that the powers of ζ range from $8-m$ to m. Let us collect terms of equal powers in ζ and rewrite R in (6.21) in the form

$$R=\sum_{r=8-m}^{m}\frac{b_r(\tilde{z},\bar{\tilde{z}})}{(1+|\tilde{z}|^2)^{s_r}}\,\zeta^r=\frac{1}{\zeta^{m-8}}\sum_{r=0}^{2m-8}\frac{b_{r+8-m}(\tilde{z},\bar{\tilde{z}})}{(1+|\tilde{z}|^2)^{s_{r+8-m}}}\,\zeta^r, \tag{6.22}$$

where the s_r are (easily computable but not important) positive integers, and the b_r are polynomials in $(\tilde{z},\bar{\tilde{z}})$.

Now, by definition of the set Π_m, we have $\rho'\in\Pi_m$ precisely when R as a rational function in ζ has at least one root on the circle (6.19) for every $\tilde{z}\in U_0\subset\mathbb{P}^1$. Observe that that set B_k of coefficients $a=(a_0,\dots,a_k)\in\mathbb{C}^{k+1}$ such that the polynomial $a_0+a_1\zeta+\cdots+a_k\zeta^k$ has a root on the unit circle forms a real-algebraic, Levi flat (singular) hypersurface. Thus, $\rho'\in\Pi_m$ translates into the condition that

$$\left(\frac{b_{8-m}(\tilde{z},\bar{\tilde{z}})}{(1+|\tilde{z}|^2)^{s_{m-8}}},\frac{b_{9-m}(\tilde{z},\bar{\tilde{z}})}{(1+|\tilde{z}|^2)^{s_m+1/2}},\dots,\frac{b_m(\tilde{z},\bar{\tilde{z}})}{(1+|\tilde{z}|^2)^{s_m+(2m-8)/2}}\right)\in B_{2m-8}, \tag{6.23}$$
$$\forall\tilde{z}\in U_0.$$

By unraveling the construction of R, we note that if we expand ρ' in the monomial basis $Z^I = z^\alpha w^\beta$ of $\mathcal{P}_m$, i.e.,

$$\rho' = \sum_{|I|+|J|\le m} e_{I\bar J} Z^I \bar Z^J, \quad e_{I\bar J} = \overline{e_{J\bar I}}, \tag{6.24}$$

then the components in (6.23)

$$\frac{b_r(\tilde z, \bar{\tilde z})}{(1+|\tilde z|^2)^{s_r+r/2}}$$

are linear in $e_{I\bar J}$ and $\overline{e_{I\bar J}}$. Consequently, we deduce from the above discussion and (6.23) that Π_m is a real-algebraic subvariety in $\mathcal{P}_m^{\mathbb{R}}$.

To complete the proof of Proposition 6.9, we must show that if $\mathcal{A}$ is as in the statement of the proposition, then the dimension of $\Pi_m \cap \mathcal{A}$ is strictly less than that of $\mathcal{A}$. For this, it suffices to show that $\Pi_m \cap \mathcal{A} \neq \mathcal{A}$. To this end, we compute $Q^0(z^p \bar w^p)$, for $p \ge 2$,

$$\begin{aligned} Q^0(z^p \bar w^p) &= \bar L_0^4(p(p-1) z^{p-2} \bar w^{p+2}) \\ &= (p+2)(p+1)p^2(p-1)^2 z^{p+2} \bar w^{p-2}, \end{aligned} \tag{6.25}$$

and similarly,

$$\begin{aligned} Q^0(w^p \bar z^p) &= \bar L_0^4(p(p-1) w^{p-2} \bar z^{p+2}) \\ &= (p+2)(p+1)p^2(p-1)^2 w^{p+2} \bar z^{p-2}. \end{aligned} \tag{6.26}$$

Thus, if ρ' is any polynomial in $\mathcal{A}$, resulting in the polynomial R as in (6.22), then $\rho' + e z^p \bar w^p + \bar e w^p \bar z^p$, which is also in $\mathcal{A}$ for all $e \in \mathbb{C}$, results in

$$R' = R + \frac{(e \tilde z^{p+2} + \bar e \bar{\tilde z}^{p-2})}{(1+|\tilde z|^2)^{p-2}} \zeta^4. \tag{6.27}$$

From this we easily deduce that if $\rho' \in \Pi_m$, then $\rho' + e z^p \bar w^p + \bar e w^p \bar z^p$ will not be in Π_m for $e \neq 0$; indeed, since

$$\tilde z \mapsto \frac{(e \tilde z^{p+2} + \bar e \bar{\tilde z}^{p-2})}{(1+|\tilde z|^2)^{p-2}}, \quad e \neq 0, \tag{6.28}$$

maps onto an open neighborhood of 0 in $\mathbb{C}$, this statement follows from the following simple observation:

Lemma 6.10. *If $p(\zeta) = \zeta^n + a_{n-1}\zeta^{n-1} + \cdots + a_0$ has a root on the unit circle, then the set of $b \in \mathbb{C}$ such that $p(\zeta) + b\zeta^k$ has a root on the unit* *circle is a real-algebraic, possibly singular curve (real-algebraic variety of dimension one).*

Proof. Consider the (symmetric) finite polynomial mapping $\Phi\colon \mathbb{C}^n \to \mathbb{C}^n$ sending a collection of roots $\tau = (\tau_1, \ldots, \tau_n)$ to the collection of coefficients $a = (a_0, \ldots, a_{n-1})$ of the polynomial

$$p(\zeta) = \zeta^n + a_{n-1}\zeta^{n-1} + \cdots + a_0 := (\zeta - \tau_1)\cdots(\zeta - \tau_n). \tag{6.29}$$

Pick $p_0(\zeta)$ such that one its roots is on the unit circle, i.e., $a^0 = \Phi(\tau^0)$ with τ^0 in the Levi flat (singular) hypersurface $H = \cup_{j=1}^n H_j$, with $H_j := \{\tau\colon |\tau_j| = 1\}$. The polynomials $p_e(\zeta) := p_0(\zeta) + b\zeta^k$ correspond to points $a^b = a^0 + (0, \ldots, b, \ldots, 0)$ (with b in the $(k+1)$th component) and hence their roots τ^b belong to the complex 1-dimensional subvariety $\Phi^{-1}(X_k)$, where X_k denotes the complex curve $b \mapsto a^0 + (0, \ldots, b, \ldots, 0)$. We claim that $\Phi^{-1}(X_k)$ is not contained in H, which will prove the conclusion of the lemma. Indeed, $\Phi^{-1}(X_k)$ could only be contained in the Levi flat H if it were contained in one of its leaves $\tau_j = c$, with c constant, which is clearly impossible. $\square$

As mentioned above, we have now shown that the real-algebraic subvariety Π_m satisfies $\Pi_m \cap \mathcal{A} \neq \mathcal{A}$, which completes the proof of Proposition 6.9. $\square$

6.3. Generic perturbations of almost circular type

Recall that a real hypersurface $M \subset \mathbb{C}^2$ is called *circular* if $Z \in M$ implies $e^{it}Z \in M$ for all $e^{it} \in S^1$. For perturbations M_ε of the sphere, as in (6.1), it is straightforward to verify that the M_ε are circular for all sufficiently small $\varepsilon > 0$ if and only if in the decomposition (6.8) we have $\rho'_{p,q} = 0$ for $|p - q| \neq 0$. It was shown in [ED] that compact, circular real hypersurfaces in $\mathbb{C}^2$ always have umbilical points. Here we shall consider perturbations M_ε that are *almost circular*, which we define to be those for which, in the decomposition (6.8) of ρ', we have $\rho'_{p,q} = 0$ when $|p - q| \geq 4$; we also say that such ρ' are almost circular. We easily observe that a polynomial $P = P(Z, \bar{Z})$ is almost circular if and only if its Fourier coefficients $\hat{P}_k$ vanish for $|k| \geq 4$:

$$\hat{P}_k(Z, \bar{Z}) := \frac{1}{2\pi}\int_0^{2\pi} P(e^{it}Z, e^{-it}\bar{Z})e^{-ikt}dt = 0, \quad |k| \geq 4.$$

Recall that $\mathcal{P}_m$ denotes the space of all polynomials in $Z = (z, w)$ and $\bar{Z}$ of degree at most m. We shall denote by $\mathcal{AC}_m$ the real subspace of those that are real-valued and almost circular. Thus, $\rho' \in \mathcal{P}_m$ belongs to $\mathcal{AC}_m$ when ρ' is real-valued (i.e., $\rho' \in \mathcal{P}_m^{\mathbb{R}}$) and $\rho'_{p,q} = 0$ for $|p-q| \geq 4$. We note that $\mathcal{A} = \mathcal{AC}_m$ satisfies the hypothesis in Proposition 6.9 for all $m \geq 2$ and with any $2 \leq p \leq m/2$.

Theorem 6.11. *For $m \geq 4$, there is a real-algebraic subvariety $\Xi_m \subset \mathcal{AC}_m$ of dimension strictly less than that of $\mathcal{AC}_m$ such that if $\rho' \in \mathcal{AC}_m \setminus \Xi_m$, then, for sufficiently small $\varepsilon > 0$, the set of umbilical points $\mathcal{U}$ on the perturbation M_ε, given by* (6.1)*, contains either points of dimension ≥ 2 or a curve of stable umbilical points.*

Proof. We shall let Ξ_m be $\Xi_m := \Pi_m \cap \mathcal{AC}_m$, where Π_m is as defined in Proposition 6.9. As noted above, $\mathcal{A} = \mathcal{AC}_m$ satisfies the hypotheses in Proposition 6.9 and, hence, we conclude that Ξ_m is a real-algebraic subvariety of strictly lower dimension that $\mathcal{AC}_m$. The conclusion of Theorem 6.11 now follows from Proposition 6.8, since $\rho' \in \mathcal{AC}_m$ clearly guarantees that condition (ii) in that proposition holds; indeed, for $\rho'_{p,q}$, we have $Q^0(\rho'_{p,q}) = Q^0_{p+2,q-2}$ and if $|p-q| \leq 3$, then $l = p+2 \geq (p+q+1)/2 > k/2$, which is the requirement in condition (ii). $\square$

Remark 6.12.

- Recall that if, for example, $m = 2p$ and

$$\rho' = \rho'_{p-1,p+1} + \rho_{p,p} + \rho'_{p+1,p-1}, \quad \rho'_{p+1,p-1} = \overline{\rho'_{p-1,p+1}}$$

($\implies \rho' \in \mathcal{AC}_m$), then

$$Q^0 = Q^0(\rho') = Q^0_{p+1,p-1} + Q^0_{p+2,p-2} + Q^0_{p+3,p-3}.$$

We note that there are plenty of polynomials of this form,

$$Q = Q_{p+1,p-1} + Q_{p+2,p-2} + Q_{p+3,p-3},$$

such that $\pi(\mathcal{V}) = \mathbb{P}^1$, where $\mathcal{V}$ denotes the zero locus of Q in $\tilde{S}^3$. For example, any Q of the form

$$Q = (z+\bar{z})(Q'_{p-1,p} + Q'_{p,p-1})$$

will satisfy this, as the reader can easily verify. However, we do not know any non-trivial examples of such Q that are also in the image of the linear map Q^0, i.e., of the form $Q = Q^0(\rho')$ with $\rho' \in \mathcal{AC}_m$.

- It is clear from the calculations in Section 5 that the real ellipsoids E_ε are not generic in the sense of Theorem 6.11, i.e., these belong to Π_m.

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