Communications in
Analysis and Geometry
Volume 28, Number 2, 263–319, 2020

On classification of toric surface codes of dimension seven

Naveed Hussain, Xue Luo, Stephen S.-T. Yau, Mingyi Zhang, and Huaiqing Zuo

In this paper, we give an almost complete classification of toric surface codes of dimension less than or equal to 7, according to monomially equivalence. This is a natural extension of our previous work [YZ], [LYZZ]. More pairs of monomially equivalent toric codes constructed from non-equivalent lattice polytopes are discovered. A new phenomenon appears, that is, the monomially non-equivalence of two toric codes $C_{P_7^{(10)}}$ and $C_{P_7^{(19)}}$ can be discerned on $\mathbb{F}_q$, for all $q \geq 8$, except $q = 29$. This sudden break seems to be strange and interesting. Moreover, the parameters, such as the numbers of codewords with different weights, depends on q heavily. More meticulous analyses have been made to have the possible distinct families of reducible polynomials.

This work is financially supported by National Natural Science Foundation of China (NSFC) (11961141005, 11531007). Zuo thanks the support of NSFC (11771231); Luo acknowledges the support of NSFC (11871003) and the Fundamental Research Funds for the Central Universities (YWF-20-BJ-J-644); and Yau is supported by Tsinghua University start-up fund and Tsinghua University education foundation fund (042202008).

1. Introduction

Tsfasman and Vlăduţ [TV] proposed a general framework of constructing error-correction codes by using algebraic varieties in their seminal work "Algebraic-Geometry Codes". Taking the connection of toric varieties to geometry of lattice polytopes into consideration, the explicit calculations can be made in toric varieties, on which the toric codes construct first by J. Hansen [H1, H2]. The toric codes may be defined as evaluation codes obtained from monomials corresponding to integer lattice points in an integral convex polytope $P \subset \mathbb{R}^n$, see Definition 1, [LS2]. Briefly, we recall the toric surface codes as below:

Given a finite field $\mathbb{F}_q$ where q is a power of prime number. Let P be any convex lattice polygon contained in $\square_{q-1} = [0, q-2]^2$. We associate P with a $\mathbb{F}_q$-vector space of polynomials spanned by the bivariate power monomials:

$$\mathcal{L}(P) = \mathrm{Span}_{\mathbb{F}_q} \left\{ x^{m_1} y^{m_2} \mid (m_1, m_2) \in P \right\}.$$

For any $f \in \mathcal{L}(P)$, regard $(f(t), t \in (\mathbb{F}_q^*)^2)$ as a vector with $(q-1)^2$ coordinates, i.e. an ordered $(q-1)^2$-tuple. We define a toric surface code C_P (cf. [SS1]), the set of the codewords associated to the polytope P, by

$$C_P := \{(f(t), t \in (\mathbb{F}_q^*)^2) \mid f \in \mathcal{L}(P)\}.$$

Back to the general framework of [TV], given a finite field $\mathbb{F}_q$, the lattice polytope $P \subset \mathbb{R}^n$ defines a projective toric variety X_P over $\mathbb{F}_q$ and an ample line bundle L_P. If the finite set $S = \{p_1, \ldots, p_s\}$ of $\mathbb{F}_q$-rational points on X_P takes the algebraic torus $\mathbb{T} = \left(\mathbb{F}_q^*\right)^n$, one defines an evaluation code $\mathcal{C}_P := \mathcal{C}(X_P, L_P, \mathbb{T})$ by evaluating global sections $\Gamma(X_P, L_P)$ of L_P at the points of $\mathbb{T}$. Thus, it is clear that the toric code is completely determined by P, since the global section can be identified with polynomials whose monomials are the lattice points in P, see section 3.4, [F]. Therefore, it is not surprising that the parameters of a toric code are expressible in terms of the polytope P, such as the dimension and the minimum distance.

Before J. Little and R. Schwarz [LS2], Hansen and other researchers proposed more advanced techniques from algebraic geometry to study the toric codes, see [H2, Jo, LS1] and [Z]. It is [LS2] that for the first time used a more elementary approach based on a sort of multivariate generalization of Vandermonde determinants to determine the minimum distance of toric codes from simplices and rectangular polytopes. They also proved an interesting and intriguing result that if there is a unimodular integer affine transformation taking one polytopes P_1 to another P_2, i.e. P_1 and P_2 are lattice equivalent[1] , then the corresponding toric codes are monomially equivalent (hence have the same parameters), see section 4, [LS2]. Two codes of block length n and dimension k over $\mathbb{F}_q$, denoted as C_1 and C_2, are said to be monomially equivalent if there is an invertible $n \times n$ diagonal matrix Δ and an $n \times n$ permutation matrix Π such that $G_2 = G_1 \Delta \Pi$, where G_i is the generator matrix for C_i, $i = 1, 2$. Unfortunately, the statement given in [LS2] is not necessary to examine the monomially equivalence of two toric codes. The corresponding toric codes of two lattice non-equivalent polytopes could be monomially equivalent. Explicit examples are discovered first in [LYZZ], and more pairs in this paper.

In our classification of the toric codes, the parameters such as the minimum distance and the numbers of codewords with different weights[2] are important indices to discern the monomially non-equivalence. Thanks to close tie to the geometry of the toric surface X_P associated with the normal fan $\triangle_P$ of the polygon P, Ruano [Ru] estimated the minimum distance using intersection theory and mixed volumes, extending the methods of Hansen's for plane polygons. Little and Schenck [LS1] obtained upper and lower bounds on the minimum distance of a toric code constructed from a polygons $P \subset \mathbb{R}^2$ by examining Minkowski sum decompositions of subpolygons of P. Soprunov and Soprunova [SS1] proved new lower bounds for the minimum distance of the toric code $\mathcal{C}_P$ defined by a convex lattice polygon $P \subset \mathbb{R}^2$, where the bounds involve a geometric invariant $L(P)$, the full Minkowski length of P. Later, they [SS2] investigated the computation of the minimum distance of higher dimensional toric codes defined by lattice polytopes in $\mathbb{R}^n$. Unfortunately, as far as we know, there is no convenient formula to get (the bounds of) the numbers of codewords with different weights, which are obtained by meticulous analyses to enumerate all the possible distinct families of reducible polynomials in this paper.

In this paper, we give a (almost) complete classification of toric surface codes of dimension equal to 7, according to monomially equivalence. This is a natural extension of the previous work [LS2, YZ, LYZZ]. [LS2, YZ] gave the complete classification of toric codes with dimension less than or equal

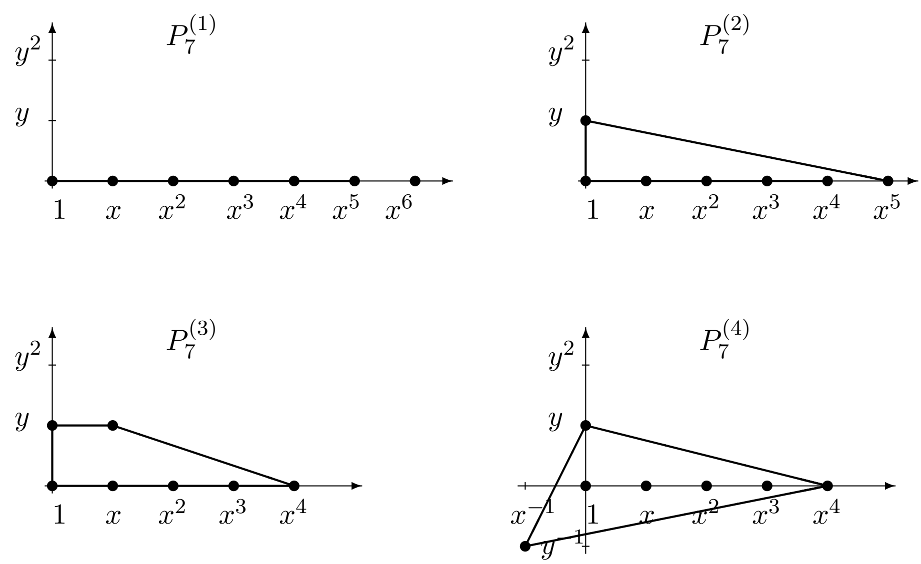
[1] Two integral convex polytopes P and $\tilde{P}$ in $\mathbb{Z}^m$ are said to be lattice equivalent if there exists an invertible integer affine transformation T as above such that $T(P) = \tilde{P}$. An affine transformation of $\mathbb{R}^m$ is a mapping of the form $T(x) = Mx + \lambda$, where λ is a fixed vector and M is an $m \times m$ matrix. The affine mappings T, where $M \in GL(m, \mathbb{Z})$ (so $Det(M) = \pm 1$) and λ have integer entries, are precisely the bijective affine mappings from the integer lattice $\mathbb{Z}^m$ to itself.

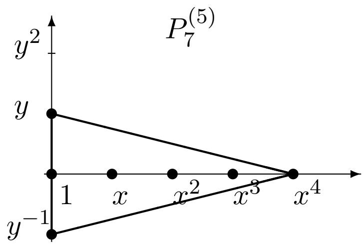
[2] The weight of each nonzero codeword in $\mathcal{C}_P$ is the number of points $(x, y) \in \left(\mathbb{F}_q^*\right)^2$ where the corresponding polynomial does not vanish.

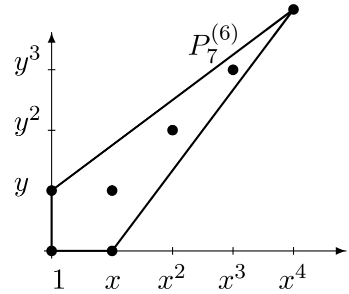
to 5. The 6-dimensional toric surface codes are classified in [LYZZ]. Though the strategy (of grouping the codes sharing the same minimum distance, and then comparing the numbers of codewords with different weights in each group) is similar as that in [LYZZ], the novelty of this paper are two folds: On the one hand, we obtain more explicit examples of two monomially equivalent toric codes constructed from two lattice non-equivalent polygons, see Proposition 3.4. Also the monomially equivalence of more pairs of toric codes remains to be undetermined. It is interesting to notice that the open case can happen solely on $\mathbb{F}_q$, q slightly large, for instance the pair $C_{P_7^{(10)}}$ and $C_{P_7^{(19)}}$ on $\mathbb{F}_{29}$. This phenomenon never happened before. On the other hand, the number of the codewords in C_P over $\mathbb{F}_q$ with some particular weight varies with respect to q. The dependence on q is more complicated than before. For example, the argument of finding the number of codewords with weight $(q-1)^2 - 2(q-1)$ for $C_{P_7^{(9)}}$ in the proof Proposition 3.6 and Table 3.7.

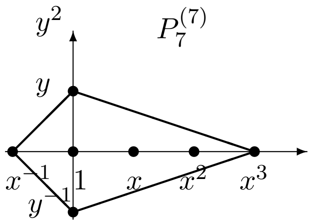
The main results in this paper are stated below.

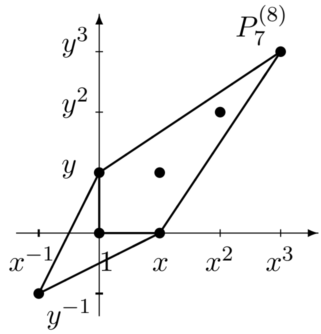
Theorem 1.1. *Every toric surface code with $k = 7$, where k is the dimension of the code, is monomially equivalent to one constructed from the one of the polygons in Fig.1 as following.*

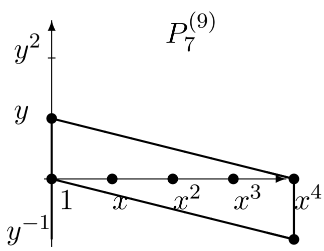


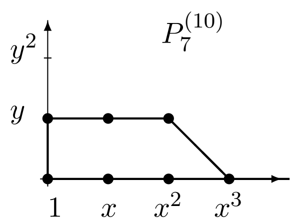

$P_7^{(5)}$
y^2
y
y^{-1}
1 x x^2 x^3 x^4

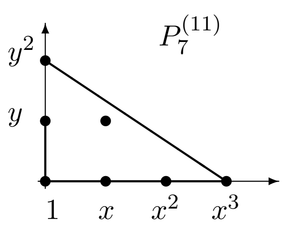

$P_7^{(6)}$
y^3
y^2
y
1 x x^2 x^3 x^4

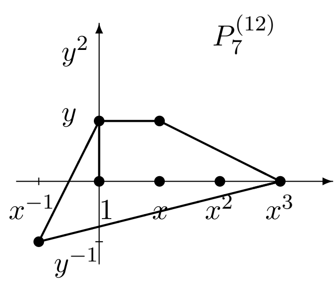

$P_7^{(7)}$
y^2
y
y^{-1}
x^{-1} 1 x x^2 x^3

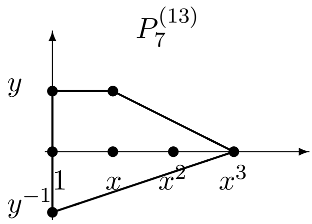

$P_7^{(8)}$
y^3
y^2
y
y^{-1}
x^{-1} 1 x x^2 x^3

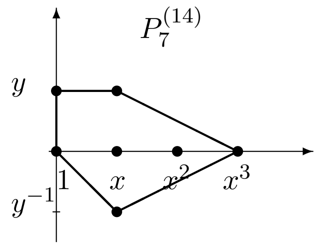

$P_7^{(9)}$
y^2
y
y^{-1}
1 x x^2 x^3 x^4

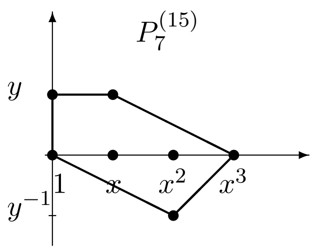

$P_7^{(10)}$
y^2
y
1 x x^2 x^3

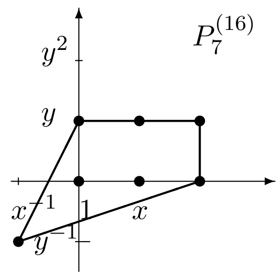

$P_7^{(11)}$
y^2
y
1 x x^2 x^3

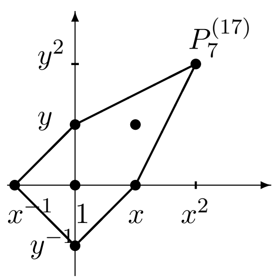

$P_7^{(12)}$
y^2
y
y^{-1}
x^{-1} 1 x x^2 x^3

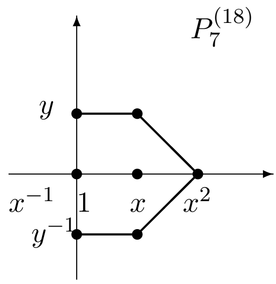

$P_7^{(13)}$
y
y^{-1}
1 x x^2 x^3

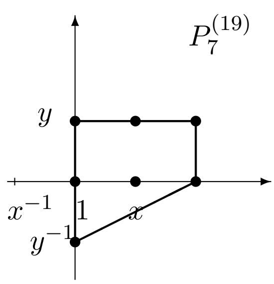

$P_7^{(14)}$
y
y^{-1}
1 x x^2 x^3

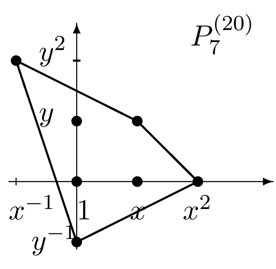

$P_7^{(15)}$
y
y^{-1}
1 x x^2 x^3

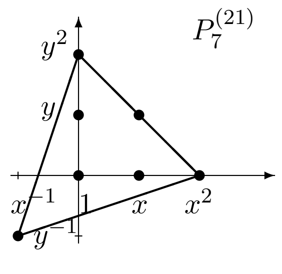

$P_7^{(16)}$
y^2
y
x^{-1} 1 x
y^{-1}


$P_7^{(17)}$
y^2
y
x^{-1} 1 x x^2
y^{-1}


$P_7^{(18)}$
y
x^{-1} 1 x x^2
y^{-1}


$P_7^{(19)}$
y
x^{-1} 1 x
y^{-1}


$P_7^{(20)}$
y^2
y
x^{-1} 1 x x^2
y^{-1}



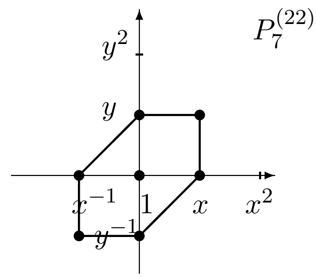


Figure 1. Polygons yielding toric codes with $k = 7$.

Theorem 1.2. *$C_{P_7^{(i)}}$ and $C_{P_7^{(j)}}$ are not monomially equivalent over $\mathbb{F}_q$ for all $q \geq 7$, except that the following three pairs are monomially equivalent:*

(1) *$C_{P_7^{(4)}}$ and $C_{P_7^{(8)}}$ over $\mathbb{F}_7$*

(2) *$C_{P_7^{(5)}}$ and $C_{P_7^{(7)}}$ over $\mathbb{F}_7$*

(3) *$C_{P_7^{(12)}}$ and $C_{P_7^{(13)}}$ over $\mathbb{F}_7$*

and the monomially equivalence of the following three pairs remain open:

(4) *$C_{P_7^{(10)}}$ and $C_{P_7^{(19)}}$ over $\mathbb{F}_{29}$*

(5) *$C_{P_7^{(4)}}$ and $C_{P_7^{(7)}}$ over $\mathbb{F}_8$*

(6) *$C_{P_7^{(5)}}$ and $C_{P_7^{(6)}}$ over $\mathbb{F}_8$*

.

Remark 1.1. More precisely, $C_{P_7^{(1)}}$ is defined only on $\mathbb{F}_q$ with $q \geq 8$, others are defined on $\mathbb{F}_q$ with $q \geq 7$.

Theorem 1.2 and the main results in [LS2, YZ, LYZZ] yield a (almost) complete classification of the toric codes of dimension ≤ 7 up to monomial equivalence. Based on the observation that the enumerator polynomial of the three pairs (4)-(6) in Theorem 1.2 are exactly the same, see Table A.1. We propose the following conjecture:

Conjecture 1.1. The following pairs of toric codes are monomially equivalent:

(1) $C_{P_7^{(4)}}$ and $C_{P_7^{(7)}}$ over $\mathbb{F}_8$

(2) $C_{P_7^{(5)}}$ and $C_{P_7^{(6)}}$ over $\mathbb{F}_8$

(3) $C_{P_7^{(10)}}$ and $C_{P_7^{(19)}}$ over $\mathbb{F}_{29}$

The preliminaries have been introduced in section 2; Section 3 is devoted to the proof of the theorems; All the datum computed by Magma code (or program) from [Jo] to support the proof in Section 3 has been listed in Appendix A, and the Magma program is appended in Appendix B.

2. Preliminaries

In this section, we shall recall some basic definitions and results which are needed in this paper. We will follow the terminology and notation for toric codes from [LS2].

2.1. Minkowski sum and minimum distance of toric codes

For some special polygons P, we can compute the minimum distance of the toric surface code C_P, say the rectangles and triangles.

Let $P_{k,l}^{\square} = \text{conv}\{(0,0),(k,0),(0,l),(k,l)\}$ be the convex hull of the vectors $(0,0)$, $(k,0)$, $(0,l)$, (k,l). Let $\square_{q-1} = [0,q-2]^2 \subset \mathbb{Z}^2$. The minimum distance of $C_{P_{k,l}^{\square}}$ is given in the following theorem.

Theorem 2.3. ([LS2]) *Let $k,l < q-1$, so that $P_{k,l}^{\square} \subset \square_{q-1} \subset \mathbb{R}^2$. Then the minimum distance of the toric surface code $C_{P_{k,l}^{\square}}$ is*

$$d(C_{P_{k,l}^{\square}}) = (q-1)^2 - (k+l)(q-1) + kl = ((q-1)-k)((q-1)-l).$$

Let $P_{k,l}^{\triangle} = \text{conv}\{(0,0),(k,0),(0,l)\}$ be the convex hull of the vectors $(0,0),(k,0),(0,l)$. Similarly, the minimum distance of $C_{P_{k,l}^{\triangle}}$ is given below:

Theorem 2.4. ([LS2]) *If $P_{k,l}^{\triangle} \subset \square_{q-1} \subset \mathbb{R}^2$, and $m = \max\{k,l\}$, then*

$$d(C_{P_{k,l}^{\triangle}}) = (q-1)^2 - m(q-1).$$

Remark 2.2. Theorems 2.3 and 2.4 can be generalized to high dimensional case, see [LS2].

In [SS1], the authors give a good bound for the minimum distance of C_P in terms of certain geometric invariant $L(P)$, the so-called full Minkowski length of P.

Definition 2.1. Let P and Q be two subsets of $\mathbb{R}^n$. The Minkowski Sum is obtained by taking the pointwise sum of P and Q:

$$P + Q = \{x + y \mid x \in P, y \in Q\}.$$

Let P be a lattice polytope in $\mathbb{R}^n$. Consider a Minkowski decomposition

$$P = P_1 + \cdots + P_l$$

into lattice polytopes P_i of positive dimension. Let $l(P)$ be the largest number of summands in such decompositions of P, and call the Minkowski length of P.

Definition 2.2. ([SS1]) The full Minkowski length of P is the maximum of the Minkowski lengths of all subpolytopes Q in P,

$$L(P) := max\{l(Q) \mid Q \subset P\}.$$

We shall use the results in [SS1] to give a bound of the minimum distance of C_P:

Theorem 2.5. ([SS1]) *Let $P \subset \square_{q-1}$ be a lattice polygon with area A and full Minkowski length L. For $q \geq max(23, (c + \sqrt{c^2 + 5/2})^2)$, where $c = A/2 - L + 9/4$, the minimum distance of the toric surface code C_P satisfies*

$$d(C_P) \geq (q-1)^2 - L(q-1) - 2\sqrt{q} + 1.$$

With the condition that no factorization $f = f_1 \cdots f_{L(P)}$ for all $f \in \mathcal{L}(P)$ contains an exceptional triangle(a triangle with exactly 1 interior and 3 boundary lattice points), we have a better bound for the minimum distance of C_P:

Proposition 2.1. ([SS1]) Let $P \subset \square_{q-1}$ be a lattice polygon with area A and full Minkowski length L. Under the above condition on P, for $q \geq \max(37, (c + \sqrt{c^2 + 2})^2)$, where $c = A/2 - L + 11/4$, the minimum distance of the toric surface code $\mathcal{C}_P$ satisfies

$$d(\mathcal{C}_P) \geq (q-1)^2 - L(q-1).$$

2.2. Some theorems about classification of toric codes

The monomial equivalence is actually an equivalent relation on codes since a product $\Pi\Delta$ equals $\Delta'\Pi$ for another invertible diagonal matrix Δ'. It is also a direct consequence of the definition that monomially equivalent codes C_1 and C_2 have the same dimension and the same minimum distance (indeed, the same full weight enumerator).

Generally speaking, it is impractical to determine two given toric codes to be monomially equivalent directly from the definition. A more practical criteria comes from the nice connection between the monomially equivalence class of the toric codes C_P and the lattice equivalence class of the polygon P in [LS2].

Theorem 2.6. *If two polytopes P and $\tilde{P}$ are lattice equivalent, then the toric codes C_P and $C_{\tilde{P}}$ are monomially equivalent.*

For the sake of completeness, we list simples facts about lattice equivalence of two polytopes P and $\tilde{P}$ in $\mathbb{Z}^2$.

Proposition 2.2. (i) If P can be transformed to $\tilde{P}$ by translation, rotation and reflection with respect to x-axis or y-axis, then P and $\tilde{P}$ are lattice equivalent;

(ii) If P and $\tilde{P}$ are lattice equivalent, then they have the same number of sets of n collinear points and the same number of sets of n concurrent segments;

(iii) If P and $\tilde{P}$ are lattice equivalent, then they are both n-side polygons;

(iv) If P and $\tilde{P}$ are lattice equivalent, then they have the same number of interior integer lattices.

These properties are directly derived from the definition. Besides the properties of lattice equivalence, Pick's Formula is also a useful tool in the proof of Theorem 1.1.

Theorem 2.7. (Pick's Formula) *Assume P is a convex rational polytope in the plane, then*

$$\sharp(P) = A(P) + \frac{1}{2} \cdot \partial(P) + 1,$$

where $\sharp(P)$ represents the number of lattice points in P, $A(P)$ is the area of P and $\partial(P)$ is the perimeter of P, with the length of an edge between two lattice points defined as one more than the number of lattice points lying strictly between them.

Remark 2.3. Generally speaking, $\partial(P)$ is the number of lattice points on the boundary of P. The only exception in plane is line segment, which should follow the precise definition of length of the edge above.

2.3. Some theorems to lower the upper bound of q

Let us introduce the so-called Hasse-Weil bounds, which will be used in the proof of Theorem 1.2 frequently to help specifying the exact number of the codewords with some particular weight, for q large. Recall that a polynomial f defined over a field K is absolutely irreducible if it is irreducible over the complex field $\mathbb{C}$.

Theorem 2.8. ([AP]) *If Y is an absolutely irreducible but possibly singular curve, g is the arithmetic genus of Y, $Y(\mathbb{F}_q)$ is the set of $\mathbb{F}_q$-rational points of curve, then*

$$1+q-2g\sqrt{q} \leq |Y(\mathbb{F}_q)| \leq 1+q+2g\sqrt{q}.$$

These two bounds are called the Hasse-Weil bounds.

Let $f \in \mathcal{L}(P)$ and P_f denotes its Newton polygon, which is the convex hull of the lattice points in $(\mathbb{F}_q^*)^2$. Denote

$$f = \sum_{m=(m_1,m_2)\in P_f} \lambda_m x^{m_1} y^{m_2}, \quad \lambda_m \in \mathbb{F}_q^*.$$

Let X be a smooth toric surface over $\overline{\mathbb{F}}_q$ defined by a fan $\Sigma_X \subset \mathbb{R}^2$ which is a refinement of the normal fan of P_f. Let C_f be the closure in X of the affine curve given by $f=0$. If f is absolutely irreducible, then C_f is irreducible. By Theorem 2.8,

$$|C_f(\mathbb{F}_q)| \leq q+1+2g\sqrt{q},$$

where g is the arithmetic genus of C_f.

Let $Z(f)$ be the number of zeros of f in the torus $(\mathbb{F}_q^*)^2$. It is well known that the arithmetic genus g of C_f equals to the number of interior lattice points in P_f(see [LS1] for the curves).

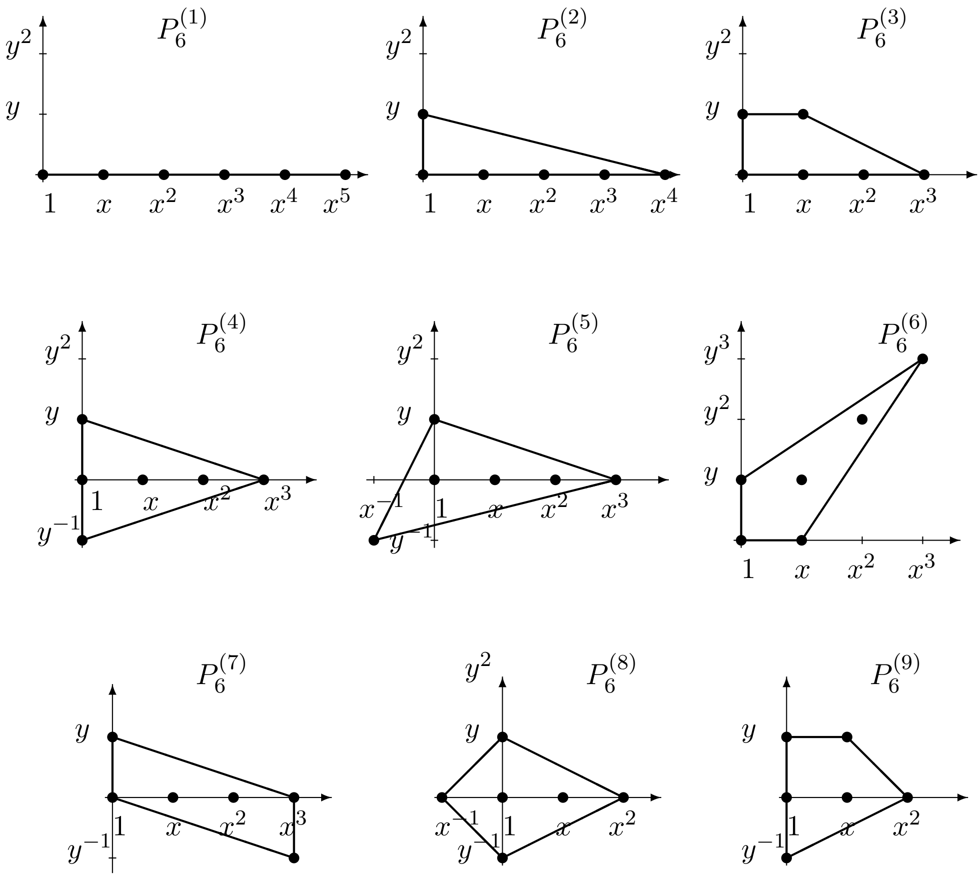
Proposition 2.3. Let f be absolutely irreducible with Newton polygon P_f. Then

$$Z(f) \leq q+1+2I(P_f)\sqrt{q},$$

where $I(P_f)$ is the number of interior lattice points.

3. Proof of the theorems

Let P_i denote an integral convex polygon in $\mathbb{Z}^2$ with i lattice points, $P_i^{(j)}$ is the jth lattice equivalence class of P_i, V is the additional lattice point added to $P_i^{(j)}$ and $P_{i,V}^{(j)} := \mathrm{conv}\{P_i^{(j)}, V\}$ denote a new integral convex polygon with lattices points in $P_i^{(j)}$ and V. Starting from $P_6^{(j)}$ in Fig. 2 (Fig. 2, [LYZZ]), we obtain all lattice equivalence classes of P_7 with the help of Pick's formula, by adding all possible one more additional lattice point V.



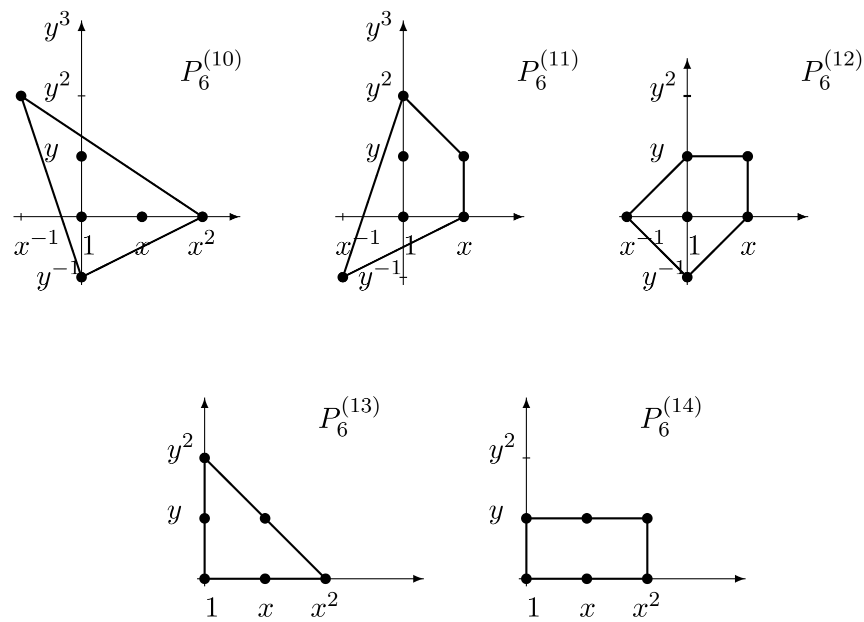


Figure 2. Polygons yielding toric codes with $k = 6$.

Sketchy proof of Theorem 1.1. There are total 22 lattice equivalence classes $P_7^{(i)}$, $i = 1, \dots, 22$ as shown in Fig. 1. We list all the equivalence classes and their corresponding $P_6^{(i)}$, for $i = 1, \dots, 14$ and the additional lattice point V in Table 3.1. The arguments of finding possible V with the help of Pick's formula is exactly the same as those in the proof of Theorem 1.1, [LYZZ]. The verification is left to the interested readers. □

Table 3.1.

$P_6^{(i)}$	possible additional lattice point V	lattice equivalent class $(P_7^{(i)})$
$P_6^{(1)}$	$(6,0)$, $(-1,0)$	$P_7^{(1)}$
	$(x_0, \pm 1)$, x_0 is integer	$P_7^{(2)}$
	$(5,0)$, $(-1,0)$	$P_7^{(2)}$
	$(1,1)$, $(-1,1)$	$P_7^{(3)}$
	$(-1,-1)$, $(9,-1)$	$P_7^{(4)}$
$P_6^{(2)}$	$(0,-1)$, $(8,-1)$	$P_7^{(5)}$

	$(1,-1)$, $(7,-1)$	$P_7^{(6)}$
	$(2,-1)$, $(6,-1)$	$P_7^{(7)}$
	$(3,-1)$, $(5,-1)$	$P_7^{(8)}$
	$(4,-1)$	$P_7^{(9)}$
$P_6^{(3)}$	$(-1,0)$, $(4,0)$	$P_7^{(3)}$
	$(-1,1)$, $(2,1)$	$P_7^{(10)}$
	$(0,2)$, $(-1,2)$	$P_7^{(11)}$
	$(-1,-1)$, $(6,-1)$	$P_7^{(12)}$
	$(0,-1)$, $(5,-1)$	$P_7^{(13)}$
	$(1,-1)$, $(4,-1)$	$P_7^{(14)}$
	$(2,-1)$, $(3,-1)$	$P_7^{(15)}$
$P_6^{(4)}$	$(4,0)$	$P_7^{(5)}$
	$(-1,0)$	$P_7^{(7)}$
	$(-1,1)$, $(-1,-1)$	$P_7^{(12)}$
	$(1,1)$, $(1,-1)$	$P_7^{(13)}$
$P_6^{(5)}$	$(4,0)$	$P_7^{(4)}$
	$(-1,0)$	$P_7^{(6)}$
	$(1,1)$, $(0,-1)$	$P_7^{(12)}$
$P_6^{(6)}$	$(4,4)$	$P_7^{(6)}$
	$(-1,-1)$	$P_7^{(8)}$
	$(-1,0)$, $(0,-1)$	$P_7^{(13)}$
	$(2,1)$, $(1,2)$	$P_7^{(14)}$
$P_6^{(7)}$	$(-1,0)$, $(4,0)$	$P_7^{(8)}$
	$(2,-1)$, $(4,-1)$, $(1,1)$, $(-1,1)$	$P_7^{(15)}$
$P_6^{(8)}$	$(3,0)$	$P_7^{(7)}$
	$(-2,0)$	$P_7^{(9)}$
	$(-1,1)$, $(-1,-1)$	$P_7^{(14)}$
	$(1,1)$, $(1,-1)$	$P_7^{(15)}$
$P_6^{(9)}$	$(0,2)$	$P_7^{(11)}$
	$(3,0)$	$P_7^{(13)}$
	$(-1,0)$	$P_7^{(15)}$
	$(-1,1)$, $(1,2)$	$P_7^{(16)}$
	$(-1,-1)$	$P_7^{(17)}$

	$(1,-1)$	$P_7^{(18)}$
	$(2,1)$	$P_7^{(19)}$
	$(-1,2)$, $(-1,-2)$	$P_7^{(20)}$
$P_6^{(10)}$	$(1,1)$, $(-1,1)$, $(1,-1)$	$P_7^{(20)}$
$P_6^{(11)}$	$(0,3)$	$P_7^{(12)}$
	$(0,-1)$	$P_7^{(14)}$
	$(1,2)$	$P_7^{(16)}$
	$(-1,0)$	$P_7^{(17)}$
	$(2,1)$	$P_7^{(20)}$
	$(2,0)$	$P_7^{(21)}$
$P_6^{(12)}$	$(-2,0)$, $(0,-2)$	$P_7^{(14)}$
	$(2,0)$, $(0,2)$	$P_7^{(15)}$
	$(2,2)$	$P_7^{(17)}$
	$(1,-1)$, $(-1,1)$	$P_7^{(18)}$
	$(2,1)$, $(1,2)$	$P_7^{(19)}$
	$(-1,-1)$	$P_7^{(22)}$
$P_6^{(13)}$	$(-1,0)$, $(0,-1)$,$(3,0)$, $(0,3)$, $(3,-1)$, $(-1,3)$,$(2,1)$, $(1,2)$	$P_7^{(11)}$
	$(1,-1)$, $(2,-1)$, $(-1,1)$, $(-1,2)$	$P_7^{(19)}$
	$(-1,-1)$, $(4,-1)$, $(-1,4)$	$P_7^{(21)}$
$P_6^{(14)}$	$(-1,0)$, $(-1,1)$, $(3,0)$, $(3,1)$	$P_7^{(10)}$
	$(-1,-1)$, $(3,-1)$, $(3,2)$, $(-1,2)$	$P_7^{(16)}$
	$(1,-1)$, $(1,2)$	$P_7^{(18)}$
	$(0,-1)$, $(0,2)$, $(2,0)$, $(2,2)$	$P_7^{(19)}$

It will be soon clear to see that the monomial equivalence of two toric codes not only depends on which polygons $P_7^{(i)}$, $i=1,\ldots,22$ they constructed from, but also relies heavily on the field $\mathbb{F}_q$. In principle, for q small, the enumerator polynomials of $C_{P_7^{(i)}}$, $1\leq i\leq 22$, can be computed directly by Magma program. The monomial non-equivalence of any two toric codes is discerned by different enumerator polynomials. However, in case that the enumerator polynomial of two toric codes is the same, further investigations are required. For q large, we shall compare several invariants of the codes, say minimum distance, the number of the codewords with some particular weight, etc. Once we could identify certain invariant of one code to be different from that of another one, then we conclude that they are monomially non-equivalent. It is the key component and heavy task to enumerate all distinct families of reducible polynomials that yields the codewords with some particular weight and to count the total number of the codewords.

3.1. For q small

Table A.1 list all the enumerator polynomials of $C_{P_7^{(i)}}, 1 \leq i \leq 22$ and these enumerator polynomials are distinct except that of

$$C_{P_7^{(4)}} \text{ and } C_{P_7^{(8)}} \text{ over } \mathbb{F}_7, \quad C_{P_7^{(5)}} \text{ and } C_{P_7^{(7)}} \text{ over } \mathbb{F}_7,$$
$$C_{P_7^{(12)}} \text{ and } C_{P_7^{(13)}} \text{ over } \mathbb{F}_7, \quad C_{P_7^{(4)}} \text{ and } C_{P_7^{(7)}} \text{ over } \mathbb{F}_8,$$
$$C_{P_7^{(5)}} \text{ and } C_{P_7^{(6)}} \text{ over } \mathbb{F}_8, \quad C_{P_7^{(10)}} \text{ and } C_{P_7^{(19)}} \text{ over } \mathbb{F}_{29}.$$

Using Magma to give the exact monomially mapping between the generate matrices, we can identify two toric codes constructed from two lattice non-equivalent polygons to be monomially equivalent. The Magma code for three cases in the following proposition is provided in Appendix B.

Proposition 3.4. The following three pairs of toric codes are monomially equivalent:

(1) $C_{P_7^{(4)}}$ and $C_{P_7^{(8)}}$ over $\mathbb{F}_7$,

(2) $C_{P_7^{(5)}}$ and $C_{P_7^{(7)}}$ over $\mathbb{F}_7$,

(3) $C_{P_7^{(12)}}$ and $C_{P_7^{(13)}}$ over $\mathbb{F}_7$.

Unfortunately, the other three pairs $C_{P_7^{(4)}}$ and $C_{P_7^{(7)}}$ over $\mathbb{F}_8$, $C_{P_7^{(5)}}$ and $C_{P_7^{(6)}}$ over $\mathbb{F}_8$ and $C_{P_7^{(10)}}$ and $C_{P_7^{(19)}}$ over $\mathbb{F}_{29}$ cannot be determined by the same way as in Proposition 3.4, since the command "IsEquivalent" in Magma only works for toric codes over $\mathbb{F}_q$ with $q = 4$ or small prime numbers. Moreover, it is also infeasible to show the monomial equivalence by definition directly. We leave the problem open. Based on the same enumerator polynomials of these three pairs and the results in Proposition 3.4, we give the conjecture of monomially equivalence of these three pairs.

3.2. For q large

Next, we shall classify $C_{P_7^{(i)}}$, $1 \leq i \leq 22$, for $q > 9$. The first invariant to be examined is the minimum distance (or the minimum weight), denoted as $d(C_{P_7^{(i)}})$.

Proposition 3.5. For $q \geq 37$, no two codes in the following different groups are monomially equivalent to each other:

(i) $C_{P_7^{(1)}}$;

(ii) $C_{P_7^{(2)}}$;

(iii) $C_{P_7^{(i)}}$, for $3 \leq i \leq 9$;

(iv) $C_{P_7^{(i)}}$, for $10 \leq i \leq 16$ or $i = 18, 19, 22$;

(v) $C_{P_7^{(i)}}$, for $i = 17$ or $20 \leq i \leq 22$.

Proof. The minimal distances of the codes $C_{P_7^{(i)}}$, $1 \leq i \leq 22$, have been summarized in Table 3.3. The estimates of the distances follow from Theorem 2.3 and Theorem 2.4. It is clear to see that $d(C_{P_7^{(1)}}) = (q-1)^2 - 6(q-1)$, $d(C_{P_7^{(2)}}) = (q-1)^2 - 5(q-1)$, and $d(C_{P_7^{(11)}}) = (q-1)^2 - 3(q-1)$.

For $C_{P_7^{(3)}}$, it is a subcode of $C_{P_{4,2}^{\triangle}}$ with $d(C_{P_{4,2}^{\triangle}}) = (q-1)^2 - 4(q-1)$, by Theorem 2.4; while it is also a supercode of $C_{P_{4,1}^{\triangle}}$ with the same minimum distance as $C_{P_{4,2}^{\triangle}}$, again by Theorem 2.4. Therefore, $d(C_{P_7^{(3)}}) = (q-1)^2 - 4(q-1)$.

For $C_{P_7^{(4)}}$, it is a supercode of $C_{P_{4,1}^{\triangle}}$ with $d(C_{P_{4,1}^{\triangle}}) = (q-1)^2 - 4(q-1)$, so $d(C_{P_7^{(4)}}) \leq (q-1)^2 - 4(q-1)$. By Proposition 2.1, when $q \geq 37$, $d(C_{P_7^{(4)}}) \geq (q-1)^2 - 4(q-1)$. Thus, $d(C_{P_7^{(4)}}) = (q-1)^2 - 4(q-1)$.

Similarly, $d(C_{P_7^{(i)}}) = (q-1)^2 - 4(q-1)$ for $5 \leq i \leq 9$; $d(C_{P_7^{(i)}}) = (q-1)^2 - 3(q-1)$ for $10 \leq i \leq 16$ and $i = 18, 19$; $d(C_{P_7^{(i)}}) = (q-1)^2 - 2(q-1)$ for $20 \leq i \leq 21$.

For $C_{P_7^{(17)}}$, it is a supercode of $C_{P_{1,1}^{\square}}$ with $d(C_{P_{1,1}^{\square}}) = (q-1)^2 - (2q-3)$. By Theorem 2.5, when $q \geq 23$, $(q-1)^2 - 2(q-1) - 2\sqrt{q} + 1 \leq d(C_{P_7^{(17)}}) \leq (q-1)^2 - (2q-3)$.

For $C_{P_7^{(22)}}$, it follows from Proposition 2.1, when $q \geq 37$, $d(C_{P_7^{(22)}}) \geq (q-1)^2 - 3(q-1)$. $\square$

Table 3.3: The (bounds of) minimal distances of the codes $C_{P_7^{(i)}}$, $1 \leq i \leq 22$, when $q \geq 37$.

$C_{P_7^{(i)}}$	minimal distance $d(C_{P_7^{(i)}})$
$C_{P_7^{(1)}}$	$= (q-1)^2 - 6(q-1)$
$C_{P_7^{(2)}}$	$= (q-1)^2 - 5(q-1)$
$C_{P_7^{(i)}}$, $3 \leq i \leq 9$	$= (q-1)^2 - 4(q-1)$

$C_{P_7^{(i)}}$, $10 \le i \le 16$ or $i = 18, 19$	$= (q-1)^2 - 3(q-1)$
$C_{P_7^{(17)}}$	$(q-1)^2 - 2(q-1) - 2\sqrt{q} + 1 \le d(C_{P_7^{(17)}}) \le (q-1)^2 - (2q-3)$
$C_{P_7^{(i)}}$, $i = 20, 21$	$= (q-1)^2 - 2(q-1)$
$C_{P_7^{(22)}}$	$\ge (q-1)^2 - 3(q-1)$

Just according to the minimum distance, the monomially equivalent (resp. non-equivalent) of any two codes within (iii), (iv) and (v) in Proposition 3.5 are still unknown. The next two invariants we shall rely on are the numbers of the codewords with weight $(q-1)^2 - 2(q-1)$ and $(q-1)^2 - (2q-3)$, denoted by $n_1(C_{P_7^{(i)}})$ and $n_2(C_{P_7^{(i)}})$, respectively. We start by dealing with $C_{P_7^{(i)}}$, $3 \le i \le 9$. With the similar argument, we can have $C_{P_7^{(i)}}$, $10 \le i \le 22$, done later.

The basic idea to figure out the monomially non-equivalent of any two codes of $C_{P_7^{(i)}}$, $3 \le i \le 9$, is: first, we find out $n_1(C_{P_7^{(i)}})$, $3 \le i \le 9$, compare them and sort the codes with the same $n_1(C_{P_7^{(i)}})$ into subgroups to be determined later; next, we give the range of $n_2(C_{P_7^{(i)}})$ among the codes with the same $n_1(C_{P_7^{(i)}})$ and compare them to give the final classification. Fortunately, in our situation, these two invariants are enough to give a complete monomially equivalent classification to $C_{P_7^{(i)}}$, $3 \le i \le 9$.

To be more detailed, the way to compute $n_1(C_{P_7^{(i)}})$ is based on enumerating the families of evaluations that contribute to weight $(q-1)^2 - 2(q-1)$. The completeness of the enumeration above is followed by Theorem 2.8, which requires the largeness of q, say $q \ge 23$ in most of the cases. Thus, as a supplement, we adopt the Magma program again to make up the gap $9 \le q \le 19$, see Table A.1. For $q \ge 23$, with the help of $n_1(C_{P_7^{(i)}})$, we already could exclude some codes and sort the left ones into several subgroups with the same $n_1(C_{P_7^{(i)}})$. Moreover, by enumerating the families of evaluations that contribute to weight $(q-1)^2 - (2q-3)$, the range of $n_2(C_{P_7^{(i)}})$ can be obtained to classify the subgroups.

Proposition 3.6. For $q > 9$, no two codes of $C_{P_7^{(i)}}$, $3 \le i \le 9$, are monomially equivalent to each other.

Proof. We shall identify $n_1(C_{P_7^{(i)}})$, for $3 \le i \le 9$, one by one.

For $C_{P_7^{(3)}}$, $n_1(C_{P_7^{(3)}}) \ge [10 + \binom{q}{2}]\binom{q-1}{2}(q-1)$, since there are eight distinct families of reducible polynomials. The polynomials and their numbers of codewords are list in Table 3.7. We claim that for $q \ge 11$, there are exactly

$[10 + \binom{q}{2}]\binom{q-1}{2}(q-1)$ such codewords in $C_{P_7^{(3)}}$. Any other codewords could only come from evaluating a linear combination of $\{1, x, x^2, x^3, x^4, y, xy\}$ in which at least one of y and xy with nonzero coefficients.

1) If either the coefficient of y or that of xy is zero, such polynomial will be absolutely irreducible. In fact, if y has nonzero coefficient but xy doesn't, then by Proposition 2.3, the number of zeros of such polynomial f in the torus $(\mathbb{F}_q^*)^2$ has a bound:

$$Z(f) \leq q + 1 + 2I(P_f)\sqrt{q} = q + 1 + 0 = q + 1 < 2q - 2,$$

for all $q > 3$; while if xy has nonzero coefficient but y doesn't, similarly,

$$Z(f) \leq q + 1 + 2I(P_f)\sqrt{q} = q + 1 + 0 = q + 1 < 2q - 2,$$

for all $q > 3$. In either case, such polynomial can never have $2q - 2$ zeros.

2) If both of them have nonzero coefficients, the only possible cases for the polynomials to be reducible are list in Table **??**.

Table 3.5: Possible cases of polynomials to be reducible for $C_{P_7^{(3)}}$.

Possible polynomials to be reducible	♯ of zeros
$a(y + bx^3 + cx^2 + dx + e)(x + k)$, where $a, b, c, d, e, k \in \mathbb{F}_q^*$ and $bc \neq 0$, $a, k \neq 0$ and $bx^3 + cx^2 + dx + e$ is irreducible	$2q - 3$
$a[y + b(x + c)(x + d)(x + e)](x + k)$, where $a, b, c, d, e, k \in \mathbb{F}_q^*$ and $c \neq d \neq e \neq k$	$2q - 6$
$a[y + b(x + c)(x + d)(x + e)](x + e)$, where $a, b, c, d, e \in \mathbb{F}_q^*$ and $c \neq d \neq e$	$2q - 5$
$a[y + bx(x + c)(x + d)](x + e)$, where $a, b, c, d, e \in \mathbb{F}_q^*$ and $c \neq d \neq e$	$2q - 5$
$a[y + bx(x + c)(x + d)](x + d)$, where $a, b, c, d \in \mathbb{F}_q^*$ and $c \neq d$	$2q - 4$
$a[y + b(x + c)(x + d)^2](x + e)$, where $a, b, c, d, e \in \mathbb{F}_q^*$ and $c \neq d \neq e$	$2q - 5$
$a[y + b(x + c)(x + d)^2](x + c)$, where $a, b, c, d \in \mathbb{F}_q^*$ and $c \neq d$	$2q - 4$
$a[y + b(x + c)(x + d)^2](x + d)$, where $a, b, c, d \in \mathbb{F}_q^*$ and $c \neq d$	$2q - 4$
$a[y + b(x + c)^3](x + d)$, where $a, b, c, d \in \mathbb{F}_q^*$ and $c \neq d$	$2q - 4$
$a[y + b(x + c)^3](x + c)$, where $a, b, c \in \mathbb{F}_q^*$	$2q - 3$

$a[y+b(x^2+cx+d)(x+e)](x+k)$, where $a,b,c,d,e,k\in\mathbb{F}_q^*$ and $e\neq k$ and x^2+cx+d is irreducible	$2q-4$
$a[y+b(x^2+cx+d)(x+e)](x+e)$, where $a,b,d,e,\in\mathbb{F}_q^*$ and x^2+cx+d is irreducible	$2q-4$
$a[y+bx(x+c)^2](x+d)$, where $a,b,c,d\in\mathbb{F}_q^*$ and $c\neq d$	$2q-4$
$a[y+bx(x+c)^2](x+c)$, where $a,b,c\in\mathbb{F}_q^*$	$2q-3$
$a[y+bx^2(x+c)](x+d)$, where $a,b,c,d\in\mathbb{F}_q^*$ and $c\neq d$	$2q-4$
$a[y+bx^2(x+c)](x+c)$, where $a,b,c\in\mathbb{F}_q^*$	$2q-3$
$a(y+bx^3)(x+c)$, where $a,b,c\in\mathbb{F}_q^*$	$2q-3$
$a[y+b(x+c)(x+d)](x+e)$, where $a,b,c,d,e\in\mathbb{F}_q^*$ and $c\neq d\neq e$	$2q-5$
$a[y+b(x+c)(x+d)](x+c)$, where $a,b,c,d\in\mathbb{F}_q^*$ and $c\neq d$	$2q-4$
$a[y+b(x+c)^2](x+d)$, where $a,b,c,d\in\mathbb{F}_q^*$ and $c\neq d$	$2q-4$
$a[y+b(x+c)^2](x+c)$, where $a,b,c\in\mathbb{F}_q^*$	$2q-3$
$a[y+bx(x+c)](x+d)$, where $a,b,c,d\in\mathbb{F}_q^*$ and $c\neq d$	$2q-4$
$a[y+bx(x+c)](x+c)$, where $a,b,c\in\mathbb{F}_q^*$	$2q-3$
$a(y+bx^2)(x+c)$, where $a,b,c\in\mathbb{F}_q^*$	$2q-3$
$a[y+b(x+c)](x+d)$, where $a,b,c,d\in\mathbb{F}_q^*$ and $c\neq d$	$2q-4$
$a[y+b(x+c)](x+c)$, where $a,b,c\in\mathbb{F}_q^*$	$2q-3$
$a(y+bx)(x+c)$, where $a,b,c\in\mathbb{F}_q^*$	$2q-3$
$a(y+b)(x+c)$, where $a,b,c\in\mathbb{F}_q^*$	$2q-3$

So over $\mathbb{F}_q^*$, such curves cannot give the codewords with weight $(q-1)^2-2(q-1)$. The claim has been proven.

For $C_{P_7^{(4)}}$, $n_1(C_{P_7^{(4)}})\geq[10+\binom{q}{2}]\binom{q-1}{2}(q-1)$, since there are eight distinct families of reducible polynomials. The polynomials and their numbers of codewords are list in Table 3.7. We claim that for $q\geq 23$ there are exactly $[10+\binom{q}{2}]\binom{q-1}{2}(q-1)$ such codewords in $C_{P_7^{(4)}}$. In fact , any other such codewords could only come from evaluating a linear combination of $\{1,x,x^2,x^3,x^4,y,x^{-1}y^{-1}\}$ in which at least one of y and $x^{-1}y^{-1}$ with nonzero coefficients.

1) If either the coefficient of y or that of $x^{-1}y^{-1}$ is zero, such polynomial is absolutely irreducible. Let f be such polynomial. By Proposition 2.3, the number of zeros of f in the torus $(\mathbb{F}_q^*)^2$ has a bound:

$$Z(f)\leq q+1+2I(P_f)\sqrt{q}=q+1+0=q+1< 2q-2,$$

so such polynomial can never have $2q-2$ zeros.

2) If both of them have nonzero coefficients, the polynomial is also absolutely irreducible. Let f be such a polynomial. By Proposition 2.3, the number of zeros of f in the torus $(\mathbb{F}_q^*)^2$ has a bound:

$$Z(f) \leq q + 1 + 2I(P_f)\sqrt{q} = q + 1 + 8\sqrt{q} < 2q - 2,$$

if $q \geq 71$. So this kind of polynomials cannot have $2(q-1)$ zero points when $q \geq 71$.

For $C_{P_7^{(5)}}$, $n_1(C_{P_7^{(5)}}) \geq [11 + \binom{q}{2}]\binom{q-1}{2}(q-1)$, since there are nine distinct families of reducible polynomials. The polynomials and their numbers of codewords are list in Table 3.7. We claim that $q \geq 43$, there are exactly $[11 + \binom{q}{2}]\binom{q-1}{2}(q-1)$. Any other such codewords could only come from evaluating a linear combination of $\{1, x, x^2, x^3, x^4, y, y^{-1}\}$ in which at least one of $\{y, y^{-1}\}$ and $\{x, x^2, x^3, x^4\}$ with nonzero coefficients. All these polynomial are all irreducible.

1) If the coefficient of y is nonzero but not that of y^{-1}, and there is one of $\{x, x^2, x^3, x^4\}$ has nonzero coefficient, such polynomial is absolutely irreducible. In fact, let f be such a polynomial. By Proposition 2.3, the number of zeros of f in the torus $(\mathbb{F}_q^*)^2$ has the bound:

$$Z(f) \leq q + 1 + 2I(P_f)\sqrt{q} = q + 1 + 0 = q + 1 < 2q - 2.$$

So such polynomial can never have $2q - 2$ zeros.

2) If the coefficient of y is zero but not that of y^{-1}, and there is one of $\{x, x^2, x^3, x^4\}$ has nonzero coefficient, such polynomial is absolutely irreducible. In fact, let f be such a polynomial. By Proposition 2.3, the number of zeros of f in the torus $(\mathbb{F}_q^*)^2$ has the bound:

$$Z(f) \leq q + 1 + 2I(P_f)\sqrt{q} = q + 1 + 0 = q + 1 < 2q - 2.$$

So such polynomial can never have $2q - 2$ zeros.

3) If both of y and y^{-1} have nonzero coefficients, and there is one of $\{x, x^2, x^3, x^4\}$ has nonzero coefficient, the polynomial is also absolutely irreducible. In fact, let f be such a polynomial. By Proposition 2.3, the number of zeros of f in the torus $(\mathbb{F}_q^*)^2$ has the bound:

$$Z(f) \leq q + 1 + 2I(P_f)\sqrt{q} = q + 1 + 6\sqrt{q} < 2q - 2,$$

if $q \geq 43$. So this kind of polynomials cannot have $2(q-1)$ zero points when $q \geq 43$. Table A.1 illustrates that $n_1(C_{P_7^{(5)}})$ is still $[11+\binom{q}{2}]\binom{q-1}{2}(q-1)$, for $q < 43$.

For $C_{P_7^{(6)}}$, $n_1(C_{P_7^{(6)}}) \geq [10+\binom{q}{2}]\binom{q-1}{2}(q-1)$, since there are eight distinct families of reducible polynomials. We claim that when $q \geq 43$, there are exactly $[10+\binom{q}{2}]\binom{q-1}{2}(q-1)$ such codewords. Any other such codewords could only come from evaluating a linear combination of $\{1, xy, x^2y^2, x^3y^3, x^4y^4, x, y\}$ with at least one of x and y with nonzero coefficients.

1) If x has nonzero coefficient and y doesn't, the only reducible polynomials are $ax(y+b)$, $ax(xy^2+b)$, $ax(x^2y^3+b)$, $ax(y+bxy^2+c)$, $ax(y+bx^2y^3+c)$, $ax(y+bxy^2+cx^2y^3+d)$, $ax(xy^2+bx^2y^3+c)$, and $ax(bc+cxy^2+dx^2y^3+ex^3y^4)$. They all have at most $q-1$ zeros. Otherwise such polynomial would be absolutely irreducible, then by Proposition 2.3, the number of zeros of such polynomial in the torus $(\mathbb{F}_q^*)^2$ has the bound:

$$Z(f) \leq q+1+2I(P_f)\sqrt{q} = q+1+0 = q+1 < 2q-2.$$

So such polynomial can never have $2q-2$ zeros.

2) If y has nonzero coefficient and x doesn't, similarly,

$$Z(f) \leq q+1+2I(P_f)\sqrt{q} = q+1+0 = q+1 < 2q-2.$$

So such polynomial can never have $2q-2$ zeros.

3) If both of them have nonzero coefficients, the only possible case for the polynomial to be reducible is that : $a(x+b)(y+c)$ where $a, b, c \in \mathbb{F}_q^*$. It has zeros of two types:
   - $(-b, j)$ for $j \in \mathbb{F}_q^*$;
   - $(i, -c)$ for $i \in \mathbb{F}_q^*$.

   There is one common zero $(-b,-c)$ of both types. Thus the polynomial has exactly $2q-3$ zeros. Otherwise the polynomial would be absolutely irreducible. Then by Proposition 2.3,

$$Z(f) \leq q+1+2I(P_f)\sqrt{q} = q+1+6\sqrt{q} < 2q-2,$$

if $q \geq 43$. So over $\mathbb{F}_q$, such curve cannot give the codewords with weight $(q-1)^2 - 2(q-1)$ when $q \geq 43$. Table A.1 illustrates that $n_1(C_{P_7^{(6)}})$ is still $[10+\binom{q}{2}]\binom{q-1}{2}(q-1)$, for $q < 43$.

For $C_{P_7^{(7)}}$, $n_1(C_{P_7^{(7)}}) \geq [11 + \binom{q}{2}]\binom{q-1}{2}(q-1)$, since there are ten distinct families of reducible polynomials. The polynomials and their numbers of codewords are list in Table 3.7. Any other such codewords could only come from evaluating a linear combination of $\{1, x^{-1}, x, x^2, x^3, y^{-1}, y\}$ and belong to one of the following two possible cases:

1) $\{y, x^{-1}, x\}$, $\{y, x^{-1}, x^2\}$, $\{y^{-1}, x^{-1}, x\}$, $\{y^{-1}, x^{-1}, x^2\}$, $\{y, x^{-1}, x, x^2\}$, $\{y^{-1}, x^{-1}, x, x^2\}$, $\{y^{-1}, x^{-1}, y, 1\}$, $\{y^{-1}, x^{-1}, y, 1, x\}$, $\{y^{-1}, x^{-1}, y, 1, x^2\}$, $\{y^{-1}, x^{-1}, y, x, x^2\}$, $\{y^{-1}, x^{-1}, y, x^2\}$, $\{y^{-1}, y, 1, x\}$, $\{y^{-1}, y, 1, x^2\}$, $\{y^{-1}, y, x\}$ or $\{y^{-1}, y, x, x^2\}$ with nonzero coefficients;
   In this case, such polynomial is absolutely irreducible. In fact, let f be such polynomial. Then
   $$Z(f) \leq q + 1 + 2I(P_f)\sqrt{q} = q + 1 + 4\sqrt{q} < 2q - 2,$$
   if $q \geq 23$. The last equality follows from the fact that the maximal number of $I(P_f) = 2$ and P_f has 4 primitive edges. So such polynomial can never have $2q - 2$ zeros when $q \geq 23$.

2) $\{y^{-1}, y, x^{-1}, x\}$ with nonzero coefficients.
   In this case, the only possible case for the polynomial to be reducible is that : $cx^{-1}y^{-1}(y - ax)(b - xy)$ where $a, b, c \in \mathbb{F}_q^*$. It has zeros of two types:
   - (i, ai) for $i \in \mathbb{F}_q^*$;
   - (i, j) for $i, j \in \mathbb{F}_q^*$ such that $ij = b$.

   In both types, there are $q - 1$ zeros. If $q = 2^n$, then there exists an $i \in \mathbb{F}_q^*$ such that $a^{-1}b \neq i^2$, therefore there is a common zero. Thus the curve has at most $2q - 3$ zeros. If $q \neq 2^n$ for all $n \in \mathbb{Z}_+$, let $\mathbb{F}_q^* = \{1, \alpha, \ldots, \alpha^{q-2}\}$, then the curve have $2q - 2$ zeros only if $a^{-1}b = \alpha^{2k-1}$, $k = 1, \ldots, \frac{q-1}{2}$. Therefore there are $\frac{(q-1)^3}{2}$ such curves. Otherwise the polynomial would be absolutely irreducible. Then by Proposition 2.3,
   $$Z(f) \leq q + 1 + 2I(P_f)\sqrt{q} = q + 1 + 6\sqrt{q} < 2q - 2,$$
   if $q \geq 43$. So over $\mathbb{F}_q$, such curves cannot give the codewords with weight $(q-1)^2 - 2(q-1)$, when $q \geq 43$.

Therefore, when $q \geq 43$, if $q = 2^n$, then $n_1(C_{P_7^{(7)}}) = [11 + \binom{q}{2}]\binom{q-1}{2}(q-1)$; if $q \neq 2^n$, then $n_1(C_{P_7^{(7)}}) = [11 + \binom{q}{2}]\binom{q-1}{2}(q-1) + \frac{(q-1)^3}{2}$. Table A.1 verifies $n_1(C_{P_7^{(7)}})$ for $q < 43$.

For $C_{P_7^{(8)}}$, $n_1(C_{P_7^{(8)}}) \geq [10+\binom{q}{2}]\binom{q-1}{2}(q-1)$, since there are nine distinct families of reducible polynomials. The polynomials and their numbers of codewords are list in Table 3.7. We claim that there are exactly $[10+\binom{q}{2}]\binom{q-1}{2}(q-1)$ such codewords, if $q \geq 43$ and $3 \nmid (q-1)$; and $[10+\binom{q}{2}]\binom{q-1}{2}(q-1)+\frac{2}{3}(q-1)^3$ such codewords, if $q \geq 43$ and $3 \mid (q-1)$. In fact, any other such codewords could only come from evaluating a linear combination of $\{x^{-1}y^{-1}, 1, xy, x^2y^2, x^3y^3, x, y\}$ with at least one of x and y with nonzero coefficients.

1) If x has nonzero coefficient and y doesn't, the only reducible polynomials are $ax(y+b)$, $ax(xy^2+b)$, $ax(x^2y^3+b)$, $ax(y+bxy^2+c)$, $ax(y+bx^2y^3+c)$, $ax(y+bxy^2+cx^2y^3+d)$, $ax(xy^2+bx^2y^3+c)$, and $ax(bc+cxy^2+dx^2y^3+ex^{-2}y^{-1})$. They all have at most $q-1$ zeros. Otherwise such polynomial would be absolutely irreducible, then by Proposition 2.3, the number of zeros of such polynomial f in the torus $(\mathbb{F}_q^*)^2$ has the bound:

$$Z(f) \leq q+1+2I(P_f)\sqrt{q} = q+1+0 = q+1 < 2q-2.$$

So such polynomial can never have $2q-2$ zeros.

2) If y has nonzero coefficient and y doesn't, similarly,

$$Z(f) \leq q+1+2I(P_f)\sqrt{q} = q+1+0 = q+1 < 2q-2.$$

So such polynomial can never have $2q-2$ zeros.

3) If both of them have nonzero coefficients, the possible cases for the polynomial to be reducible are:

(3.1) $a(x+b)(y+c)$ where $a, b, c \in \mathbb{F}_q^*$, whose zeros are of two types:
- $(-b, j)$ for $j \in \mathbb{F}_q^*$;
- $(i, -c)$ for $i \in \mathbb{F}_q^*$.

There is one common zero $(-b, -c)$ of both types. Thus the polynomial has exactly $2q-3$ zeros.

(3.2) $cx^{-1}y^{-1}(x^2y+a)(xy^2+b), a, b, c \in \mathbb{F}_q^*$. If $3 \nmid (q-1)$, then it is easy to see that $cx^{-1}y^{-1}(x^2y+a)(xy^2+b)$ has at most $2q-3$ zeros. If $3 \mid (q-1)$, $cx^{-1}y^{-1}(x^2y+a)(xy^2+b)$ may has $2q-2$ zeros. In fact, it is easy to conclude there are $\frac{2(q-1)^3}{3}$ such curves with $2q-2$ zeros.

Otherwise the polynomial should be absolutely irreducible. Then by Proposition 2.3,

$$Z(f) \leq q+1+2I(P_f)\sqrt{q} = q+1+6\sqrt{q} < 2q-2,$$

if $q \geq 43$. So over $\mathbb{F}_q$, such curves cannot give the codewords with weight $(q-1)^2 - 2(q-1)$ when $q \geq 43$. Table A.1 verifies $n_1(C_{P_7^{(8)}})$ for $q < 43$.

For $C_{P_7^{(9)}}$, $n_1(C_{P_7^{(9)}}) \geq [11 + \binom{q}{2}]\binom{q-1}{2}(q-1)$, since there are twelve distinct families of reducible polynomials. The polynomials and their numbers of codewords are list in Table 3.7. With the similar arguments, there are two different cases:

1) If either the coefficient of y or that of x^4y^{-1} is zero, such polynomial would be absolutely irreducible. In fact, no matter which coefficient is nonzero, by Proposition 2.3, the number of zeros of such polynomial in the torus $(\mathbb{F}_q^*)^2$ has the bound:

$$Z(f) \leq q+1+2I(P_f)\sqrt{q} = q+1+0 = q+1 < 2q-2.$$

So such polynomial can never have $2q-2$ zeros.

2) If both of them have nonzero coefficients, the only possible cases for the polynomial to be reducible are:

(2.1) $cy^{-1}(y-a)(y-bx^4)$ where $a, b, c \in \mathbb{F}_q^*$, which has zeros of two types:
- (i, a) for $i \in \mathbb{F}_q^*$, with $q-1$ zeros;
- (i, j) for $i, j \in \mathbb{F}_q^*$ such that $j = bi^4$, with $q-1$ zeros as (i, bi^4) for all $i \in \mathbb{F}_q^*$.

If $q = 2^n, n \geq 2$, then there must be some $i \in \mathbb{F}_q^*$ such that $i^4 = ab^{-1}$. So there is a common zero of both types. Thus the polynomial has at most $2q-3$ zeros. If $q \neq 2^n$, then $q-1$ is even. We have two possibilities: $q-1 \equiv 2 \bmod 4$ and $4 \mid (q-1)$. If $q-1 \equiv 2 \bmod 4$, then it is easy to conclude that there are $\frac{(q-1)^3}{2}$ such curves having $2q-2$ zeros. If $4 \mid (q-1)$, then there are $\frac{3(q-1)^3}{4}$ such curves having $2q-2$ zeros.

(2.2) $cy^{-1}(y-ax)(y-bx^3)$ where $a, b, c \in \mathbb{F}_q^*$ (note that the family $cy^{-1}(y-ax^2)(y-bx^2)$, $a, b, c \in \mathbb{F}_q^*$, $a \neq b$ has already been considered before). In this case, it has zeros of two types: (1) (i, ai) for $i \in \mathbb{F}_q^*$; (2) (i, bi^3) for $i \in \mathbb{F}_q^*$. Both types have $q-1$ zeros. If $q = 2^n, n \geq 2$, then there must be some $i \in \mathbb{F}_q^*$ such that $i^2 = ab^{-1}$.

So there is a common zero of both types. Thus the polynomial has at most $2q-3$ zeros. If $q \neq 2^n$, then it is easy to conclude that there are $\frac{(q-1)^3}{2}$ such curves having $2q-2$ zeros.

Otherwise the polynomial would be absolutely irreducible. Then by Proposition 2.3,

$$Z(f) \leq q+1+2I(P_f)\sqrt{q} = q+1+6\sqrt{q} < 2q-2,$$

if $q \geq 43$. Over $\mathbb{F}_q$, such curves cannot give the codewords with weight $(q-1)^2 - 2(q-1)$ when $q \geq 43$.

So, for $q \geq 43$, if $q = 2^n$, then $n_1(C_{P_7^{(9)}}) = [11 + \binom{q}{2}]\binom{q-1}{2}(q-1)$. If $q \neq 2^n$ and $q-1 \equiv 2 \bmod 4$, then $n_1(C_{P_7^{(9)}}) = [11+\binom{q}{2}]\binom{q-1}{2}(q-1)) + (q-1)^3$. If $q \neq 2^n$ and $4 \mid q-1$, then $n_1(C_{P_7^{(9)}}) = [11+\binom{q}{2}]\binom{q-1}{2}(q-1) + \frac{5(q-1)^3}{4}$. Table A.1 verifies $n_1(C_{P_7^{(9)}})$ for $q < 43$.

Table 3.7: The distinct families of reducible polynomials which evaluate to give the codewords with weight $(q-1)^2 - 2(q-1)$ for $C_{P_7^{(i)}}$, $3 \leq i \leq 9$.

$C_{P_7^{(i)}}$	Distinct families of reducible polynomials	♯ of codewords	$n_1(C_{P_7^{(i)}})$
$C_{P_7^{(3)}}$, for $q \geq 11$	$c(x-a)(x-b)$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$	$\binom{q-1}{2}(q-1)$	$= [10 + \binom{q}{2}]\binom{q-1}{2}(q-1)$
	$cx(x-a)(x-b)$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$	$\binom{q-1}{2}(q-1)$	
	$c(x-a)^2(x-b)$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$	$2\binom{q-1}{2}(q-1)$	
	$c(x-a)^3(x-b)$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$	$2\binom{q-1}{2}(q-1)$	
	$c(x-a)^2(x-b)^2$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$	$\binom{q-1}{2}(q-1)$	
	$cx^2(x-a)(x-b)$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$	$\binom{q-1}{2}(q-1)$	
	$cx(x-a)^2(x-b)$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$	$2\binom{q-1}{2}(q-1)$	
	$e(x^2+ax+b)(x-c)(x-d)$, $b,c,d,e \in \mathbb{F}_q^*$, $c \neq d$, x^2+ax+b is irreducible, $a \in \mathbb{F}_q$	$\binom{q}{2}\binom{q-1}{2}(q-1)$	
$C_{P_7^{(4)}}$, for $q \geq 71$	$c(x-a)(x-b)$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$	$\binom{q-1}{2}(q-1)$	$= [10 + \binom{q}{2}]\binom{q-1}{2}(q-1)$
	$cx(x-a)(x-b)$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$	$\binom{q-1}{2}(q-1)$	
	$c(x-a)^2(x-b)$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$	$2\binom{q-1}{2}(q-1)$	
	$c(x-a)^3(x-b)$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$	$2\binom{q-1}{2}(q-1)$	
	$c(x-a)^2(x-b)^2$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$	$\binom{q-1}{2}(q-1)$	
	$cx^2(x-a)(x-b)$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$	$\binom{q-1}{2}(q-1)$	
	$cx(x-a)^2(x-b)$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$	$2\binom{q-1}{2}(q-1)$	
	$e(x^2+ax+b)(x-c)(x-d)$, $b,c,d,e \in \mathbb{F}_q^*$, $c \neq d$, x^2+ax+b is irreducible, $a \in \mathbb{F}_q$	$\binom{q}{2}\binom{q-1}{2}(q-1)$	

<table>
<tr><td rowspan="9">$C_{P_7^{(5)}}$, for $q \geq 43$</td><td>$c(x-a)(x-b)$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$</td><td>$\binom{q-1}{2}(q-1)$</td><td rowspan="9">$= [11 + \binom{q}{2}]\binom{q-1}{2}(q-1)$</td></tr>
<tr><td>$cx(x-a)(x-b)$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$</td><td>$\binom{q-1}{2}(q-1)$</td></tr>
<tr><td>$c(x-a)^2(x-b)$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$</td><td>$2\binom{q-1}{2}(q-1)$</td></tr>
<tr><td>$c(x-a)^3(x-b)$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$</td><td>$2\binom{q-1}{2}(q-1)$</td></tr>
<tr><td>$c(x-a)^2(x-b)^2$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$</td><td>$\binom{q-1}{2}(q-1)$</td></tr>
<tr><td>$cx^2(x-a)(x-b)$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$</td><td>$\binom{q-1}{2}(q-1)$</td></tr>
<tr><td>$cx(x-a)^2(x-b)$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$</td><td>$2\binom{q-1}{2}(q-1)$</td></tr>
<tr><td>$cy^{-1}(y-a)(y-b), a,b,c \in \mathbb{F}_q^*, a \neq b$</td><td>$\binom{q-1}{2}(q-1)$</td></tr>
<tr><td>$e(x^2+ax+b)(x-c)(x-d)$, $b,c,d,e \in \mathbb{F}_q^*$, $c \neq d$, x^2+ax+b is irreducible, $a \in \mathbb{F}_q$</td><td>$\binom{q}{2}\binom{q-1}{2}(q-1)$</td></tr>
<tr><td rowspan="8">$C_{P_7^{(6)}}$, for $q \geq 43$</td><td>$c(xy-a)(xy-b)$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$</td><td>$\binom{q-1}{2}(q-1)$</td><td rowspan="8">$= [10 + \binom{q}{2}]\binom{q-1}{2}(q-1)$</td></tr>
<tr><td>$cxy(xy-a)(xy-b)$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$</td><td>$\binom{q-1}{2}(q-1)$</td></tr>
<tr><td>$c(xy-a)^2(xy-b)$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$</td><td>$2\binom{q-1}{2}(q-1)$</td></tr>
<tr><td>$c(xy-a)^3(xy-b)$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$</td><td>$2\binom{q-1}{2}(q-1)$</td></tr>
<tr><td>$c(xy-a)^2(xy-b)^2$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$</td><td>$\binom{q-1}{2}(q-1)$</td></tr>
<tr><td>$cx^2y^2(xy-a)(xy-b)$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$</td><td>$\binom{q-1}{2}(q-1)$</td></tr>
<tr><td>$cxy(xy-a)^2(xy-b)$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$</td><td>$2\binom{q-1}{2}(q-1)$</td></tr>
<tr><td>$e(x^2y^2+axy+b)(xy-c)(xy-d)$, $b,c,d,e \in \mathbb{F}_q^*$, $c \neq d$, $x^2y^2+axy+b$ is irreducible, $a \in \mathbb{F}_q$</td><td>$\binom{q}{2}\binom{q-1}{2}(q-1)$</td></tr>
<tr><td rowspan="8">$C_{P_7^{(7)}}$, for $q \geq 43$</td><td>$c(x-a)(x-b), a,b,c \in \mathbb{F}_q^*, a \neq b$</td><td>$\binom{q-1}{2}(q-1)$</td><td rowspan="8">$= [11 + \binom{q}{2}]\binom{q-1}{2}(q-1)$, if $q = 2^m$;

$= [11 + \binom{q}{2}]\binom{q-1}{2}(q-1) + \frac{(q-1)^3}{2}$, if $q \neq 2^m$</td></tr>
<tr><td>$cx^{-1}(x-a)(x-b), a,b,c \in \mathbb{F}_q^*, a \neq b$</td><td>$\binom{q-1}{2}(q-1)$</td></tr>
<tr><td>$cx^{-1}(x-a)^2(x-b), a,b,c \in \mathbb{F}_q^*, a \neq b$</td><td>$2\binom{q-1}{2}(q-1)$</td></tr>
<tr><td>$cy^{-1}(y-a)(y-b), a,b,c \in \mathbb{F}_q^*, a \neq b$</td><td>$\binom{q-1}{2}(q-1)$</td></tr>
<tr><td>$cx^{-1}(x-a)^3(x-b), a,b,c \in \mathbb{F}_q^*, a \neq b$</td><td>$2\binom{q-1}{2}(q-1)$</td></tr>
<tr><td>$cx^{-1}(x-a)^2(x-b)^2, a,b,c \in \mathbb{F}_q^*, a \neq b$</td><td>$\binom{q-1}{2}(q-1)$</td></tr>
<tr><td>$cx(x-a)(x-b), a,b,c \in \mathbb{F}_q^*, a \neq b$</td><td>$\binom{q-1}{2}(q-1)$</td></tr>
<tr><td>$c(x-a)^2(x-b), a,b,c \in \mathbb{F}_q^*, a \neq b$</td><td>$2\binom{q-1}{2}(q-1)$</td></tr>
</table>

	$e(x^2+ax+b)(x-c)(x-d)$, $b,c,d,e \in \mathbb{F}_q^*$, $c \neq d$, x^2+ax+b is irreducible, $a \in \mathbb{F}_q$	$\binom{q}{2}\binom{q-1}{2}(q-1)$	
	$cx^{-1}y^{-1}(y-ax)(b-xy)$, $m \in \mathbb{Z}_+$, $a,b,c \in \mathbb{F}_q^*$, if $q \neq 2^m$	$\frac{(q-1)^3}{2}$	
$C_{P_7^{(8)}}$, for $q \geq 43$	$c(xy-a)(xy-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	$= [10 + \binom{q}{2}]\binom{q-1}{2}(q-1)$, for $3 \nmid q-1$; $= [10 + \binom{q}{2}]\binom{q-1}{2}(q-1) + \frac{2}{3}(q-1)^3$, for $3 \mid q-1$.
	$cx^{-1}y^{-1}(xy-a)(xy-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	
	$cx^{-1}y^{-1}(xy-a)^2(xy-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$2\binom{q-1}{2}(q-1)$	
	$cx^{-1}y^{-1}(xy-a)^3(xy-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$2\binom{q-1}{2}(q-1)$	
	$cx^{-1}y^{-1}(xy-a)^2(xy-b)^2, a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	
	$cxy(xy-a)(xy-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	
	$c(xy-a)^2(xy-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$2\binom{q-1}{2}(q-1)$	
	$ex^{-1}y^{-1}(x^2y^2+axy+b)(xy-c)(xy-d)$, $b,c,d,e \in \mathbb{F}_q^*$, $c \neq d$, x^2+ax+b is irreducible, $a \in \mathbb{F}_q$	$\binom{q}{2}\binom{q-1}{2}(q-1)$	
	$cx^{-1}y^{-1}(x^2y+a)(xy^2+b), a,b,c \in \mathbb{F}_q^*$, $3 \mid q-1$	$\frac{2}{3}(q-1)^3$	
$C_{P_7^{(9)}}$, for $q \geq 43$	$c(x-a)(x-b)$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$	$\binom{q-1}{2}(q-1)$	$= [11 + \binom{q}{2}]\binom{q-1}{2}(q-1)$, if $q = 2^n$; $= [11 + \binom{q}{2}]\binom{q-1}{2}(q-1) + \frac{5(q-1)^3}{4}$, if $q \neq 2^n, 4 \mid (q-1)$; $= [11 + \binom{q}{2}]\binom{q-1}{2}(q-1)) + (q-1)^3$, if $q \neq 2^n, (q-1) \equiv 2 \bmod 4$.
	$cx(x-a)(x-b)$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$	$\binom{q-1}{2}(q-1)$	
	$c(x-a)^2(x-b)$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$	$2\binom{q-1}{2}(q-1)$	
	$c(x-a)^3(x-b)$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$	$2\binom{q-1}{2}(q-1)$	
	$c(x-a)^2(x-b)^2$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$	$\binom{q-1}{2}(q-1)$	
	$cx^2(x-a)(x-b)$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$	$\binom{q-1}{2}(q-1)$	
	$cx(x-a)^2(x-b)$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$	$2\binom{q-1}{2}(q-1)$	
	$e(x^2+ax+b)(x-c)(x-d)$, $b,c,d,e \in \mathbb{F}_q^*$, $c \neq d$, x^2+ax+b is irreducible, $a \in \mathbb{F}_q$	$\binom{q}{2}\binom{q-1}{2}(q-1)$	
	$cy^{-1}(y-ax^2)(y-bx^2)$, $a,b,c \in \mathbb{F}_q^*$, $a \neq b$	$\binom{q-1}{2}(q-1)$	
	$cy^{-1}(y-a)(y-bx^4)$, $a,b,c \in \mathbb{F}_q^*$, $q \neq 2^n$, $4 \mid (q-1)$	$\frac{3}{4}(q-1)^3$	
	$cy^{-1}(y-a)(y-bx^4)$, $a,b,c \in \mathbb{F}_q^*$, $q \neq 2^n$, $(q-1) \equiv 2 \bmod 4$	$\frac{(q-1)^3}{2}$	
	$cy^{-1}(y-ax)(y-bx^3)$, $a,b,c \in \mathbb{F}_q^*$, $q \neq 2^n$	$\frac{(q-1)^3}{2}$	

According to Table 3.7, further investigations are need to discern the monomially non-equivalence of two codes in the following cases:

1) Any two codes of $C_{P_7^{(3)}}$, $C_{P_7^{(4)}}$ and $C_{P_7^{(6)}}$ over $\mathbb{F}_q$, $q \geq 43$;
2) Any two codes of $C_{P_7^{(3)}}$, $C_{P_7^{(4)}}$, $C_{P_7^{(6)}}$ and $C_{P_7^{(8)}}$ over $\mathbb{F}_q$, $q \geq 43$, $3 \nmid (q-1)$;
3) Any two codes of $C_{P_7^{(5)}}$, $C_{P_7^{(7)}}$ and $C_{P_7^{(9)}}$ over $\mathbb{F}_q$, $q \geq 43$, $q = 2^m$, $m \in \mathbb{Z}_+$;
4) Any two codes of $C_{P_7^{(1)}}$, $C_{P_7^{(4)}}$ and $C_{P_7^{(i)}}$, $i = 2, 3, 5, \ldots, 9$ over $\mathbb{F}_q$, $9 < q \leq 11$ or $9 < q \leq 71$ or $9 < q \leq 43$.

With the similar procedure, the distinct families of reducible polynomials which evaluate to give the codewords with weight $(q-1)^2 - (2q-3)$ of each codes $C_{P_7^{(i)}}$, $3 \leq i \leq 9$, for $q \geq 25$, are list in Table 3.9. Due to lengthy argument, the verifications are left to interested readers.

Table 3.9: The distinct families of reducible polynomials which evaluate to give the codewords with weight $(q-1)^2 - (2q-3)$ for some $C_{P_7^{(i)}}$, $3 \leq i \leq 9$.

$C_{P_7^{(i)}}$	Distinct families of reducible polynomials	$\sharp$ of codewords	$n_2(C_{P_7^{(i)}})$
$C_{P_7^{(5)}}, q \geq 25$	None	0	$= 0$
$C_{P_7^{(7)}}, q \geq 25$, $q = 2^m$, $m \in \mathbb{Z}_+$	$cx^{-1}y^{-1}(y-ax)(b-xy)$ and $a^{-1}b = \alpha^2$ for one unique $\alpha \in \mathbb{F}_q^*$	$(q-1)^3$	$= (q-1)^3$
$C_{P_7^{(3)}}$, $q \geq 25$	$d(x-a)(y-bx-c), a, d \in \mathbb{F}_q^*$, if $b = 0$, $c \neq 0$ or $c = 0$, $b \neq 0$ or $b, c \neq 0$, $a = -\frac{c}{b}$	 $(q-1)^3$ $(q-1)^3$ $(q-1)^3$	$> 3(q-1)^3$
	$a(y + bx^3 + cx^2 + dx + e)(x + f)$	> 0	
	$a(y + bx^2 + cx + d)(x + e)$	> 0	
$C_{P_7^{(4)}}, q \geq 27$	None	0	$= 0$ †
$C_{P_7^{(6)}}, q \geq 25$	$c(x-a)(y-b), a, b, c \in \mathbb{F}_q^*$	$(q-1)^3$	$= (q-1)^3$
$C_{P_7^{(8)}}, q \geq 25$, $3 \nmid (q-1)$	$c(x-a)(y-b), a, b, c \in \mathbb{F}_q^*$	$(q-1)^3$	$= 2(q-1)^3$
	$cx^{-1}y^{-1}(x^2y + a)(xy^2 + b)$, $a, b, c \in \mathbb{F}_q^*$	$(q-1)^3$	
$C_{P_7^{(9)}}, q \geq 25$, $4 \nmid (q-1)$	$cy^{-1}(y - ax^4)(y-b), a, b, c \in \mathbb{F}_q^*$	$(q-1)^3$	$= 2(q-1)^3$
	$cy^{-1}(y - ax^3)(y - bx)$, $a, b, c \in \mathbb{F}_q^*$	$(q-1)^3$	

† Only if $q \geq 71$, $n_2(C_{P_7^{(4)}})$ could be shown to be 0. Table A.1 illustrates that $n_2(C_{P_7^{(4)}}) = 0$, for $27 \leq q \leq 71$.

It follows from Table 3.9 that

1) $C_{P_7^{(i)}}, i = 3, 4, 6$, have different n_2s, for $q \geq 27$;
2) $n_2(C_{P_7^{(8)}})$ when $q \geq 27$ and $3 \nmid (q-1)$ is different from those of $C_{P_7^{(i)}}$, $i = 3, 4, 6$;
3) $C_{P_7^{(i)}}, i = 5, 7, 9$ have different n_2s, for $q \geq 27$ and $q = 2^m$, $m \in \mathbb{Z}_+$,

which verifies the monomially non-equivalent of the codes in these groups. Table A.1 make up the gap $9 < q \leq 11$ for $C_{P_7^{(1)}}$, $9 < q \leq 71$ for $C_{P_7^{(4)}}$ and $9 < q \leq 43$ for $C_{P_7^{(i)}}$, $i = 2, 3, 5, \ldots, 9$. They confirms that any two codes from these small q's groups are monomially non-equivalent. □

The monomially non-equivalent of any two codes from the groups (iv) and (v) in Proposition 3.5, except $C_{P_7^{(22)}}$, are discussed in the following proposition.

Proposition 3.7. For $q > 9$, no two codes of $C_{P_7^{(i)}}$, $10 \leq i \leq 16$, or $i = 18, 19, 22$ are monomially equivalent, except the pair $C_{P_7^{(10)}}$ and $C_{P_7^{(19)}}$ over $\mathbb{F}_{29}$ to be undetermined.

Proof. First, let us compute $n_1(C_{P_7^{(i)}})$, $10 \leq i \leq 16$, $i = 18, 19, 22$.

Table 3.11: The distinct families of reducible polynomials which evaluate to give the codewords with weight $(q-1)^2 - 2(q-1)$ for $C_{P_7^{(i)}}$, $10 \leq i \leq 16$, $i = 18, 19, 22$.

$C_{P_7^{(i)}}$	Distinct families of reducible polynomials	♯ of codewords	$n_1(C_{P_7^{(i)}})$
$C_{P_7^{(10)}}$, $q \geq 11$	$c(x-a)(x-b)$, $a, b, c \in \mathbb{F}_q^*$, $a \neq b$	$\binom{q-1}{2}(q-1)$	$= 5\binom{q-1}{2}(q-1)$
	$cx(x-a)(x-b)$, $a, b, c \in \mathbb{F}_q^*$, $a \neq b$	$\binom{q-1}{2}(q-1)$	
	$c(x-a)^2(x-b)$, $a, b, c \in \mathbb{F}_q^*$, $a \neq b$	$2\binom{q-1}{2}(q-1)$	
	$cy(x-a)(x-b)$, $a, b, c \in \mathbb{F}_q^*$, $a \neq b$	$\binom{q-1}{2}(q-1)$	
$C_{P_7^{(11)}}$, $q \geq 11$	$c(y-a)(y-b), a, b, c \in \mathbb{F}_q^*, a \neq b,$	$\binom{q-1}{2}(q-1)$	$= 6\binom{q-1}{2}(q-1)$
	$c(x-a)(x-b), a, b, c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	
	$cx(x-a)(x-b), a, b, c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	
	$c(x-a)^2(x-b), a, b, c \in \mathbb{F}_q^*, a \neq b$	$2\binom{q-1}{2}(q-1)$	
	$c(x-ay)(x-by), a, b, c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	

$C_{P_7^{(12)}}$, $q \geq 43$	$c(x-a)(x-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	$= 5\binom{q-1}{2}(q-1)$
	$c(x-a)^2(x-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$2\binom{q-1}{2}(q-1)$	
	$cx(x-a)(x-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	
	$cx^{-1}y^{-1}(xy-a)(xy-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	
$C_{P_7^{(13)}}$, $q \geq 23$	$c(x-a)(x-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	$= 5\binom{q-1}{2}(q-1)$
	$cy^{-1}(y-a)(y-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	
	$cx(x-a)(x-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	
	$c(x-a)^2(x-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$2\binom{q-1}{2}(q-1)$	
$C_{P_7^{(14)}}$, $q \geq 23$	$c(x-a)(x-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	$= 5\binom{q-1}{2}(q-1)$, if $q = 2^m$; $= 5\binom{q-1}{2}(q-1) + \frac{1}{2}(q-1)^3$, if $q \neq 2^m$
	$cx(x-a)(x-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	
	$c(x-a)^2(x-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$2\binom{q-1}{2}(q-1)$	
	$cxy^{-1}(y-a)(y-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	
	$c(x-ay)(x-by^{-1})$, $m \in \mathbb{Z}_+$, $a,b,c,\alpha \in \mathbb{F}_q^*$, if $q \neq 2^m$, $\frac{a}{b} \neq \alpha^{2i}$ for $0 \leq i \leq \frac{q-3}{2}$	$\frac{1}{2}(q-1)^3$	
$C_{P_7^{(15)}}$, $q \geq 23$	$c(x-a)(x-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	$= 5\binom{q-1}{2}(q-1) + \frac{2}{3}(q-1)^3$, if $q = 2^m$ and $3 \mid q-1$; $= 5\binom{q-1}{2}(q-1) + \frac{7}{6}(q-1)^3 = 7\binom{q-1}{2}(q-1) + \frac{q+5}{6}(q-1)^2$, if $q \neq 2^m$ and $3 \mid q-1$; the same as $n_1(C_{P_7^{(14)}})$, if $3 \nmid q-1$.
	$cx(x-a)(x-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	
	$c(x-a)^2(x-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$2\binom{q-1}{2}(q-1)$	
	$cy(xy^{-1}-a)(xy^{-1}-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	
	$cy^{-1}(y-a)(x^2-by)$, $m \in \mathbb{Z}_+$, $a,b,c,\alpha \in \mathbb{F}_q^*$, if $q \neq 2^m$, $ab \neq \alpha^{2i}$ for $0 \leq i \leq \frac{q-3}{2}$	$\frac{1}{2}(q-1)^3$	
	$cy^{-1}(xy+a)(y+bx^2)$,when $3 \mid q-1$	$\frac{2}{3}(q-1)^3$	
$C_{P_7^{(16)}}$, $q \geq 23$	$c(x-a)(x-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	$= 3\binom{q-1}{2}(q-1)$
	$cy(x-a)(x-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	
	$cx^{-1}y^{-1}(xy-a)(xy-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	
$C_{P_7^{(18)}}$, $q \geq 11$	$c(x-a)(x-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	$> 3\binom{q-1}{2}(q-1) + \frac{q(q-1)^3}{2}$, if $q = 2^m$; $> 3\binom{q-1}{2}(q-1) + \frac{(q+2)}{2}(q+1)^3$, if $q \neq 2^m$
	$cx(y-a)(y-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	
	$cy^{-1}(y-a)(y-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	
	$cy^{-1}(y-ax)(b-xy)$, $m \in \mathbb{Z}_+$, $a,b,c,\alpha \in \mathbb{F}_q^*$, if $q \neq 2^m$, $\frac{a}{b} \neq \alpha^{2i}$ for $0 \leq i \leq \frac{q-3}{2}$	$\frac{1}{2}(q-1)^3$	
	$cy^{-1}(x+a)(y^2+b)$, $m \in \mathbb{Z}_+$, $a,b,c,\alpha \in \mathbb{F}_q^*$, if $q \neq 2^m$, $b \neq \alpha^{2i}$ for $0 \leq i \leq \frac{q-3}{2}$	$\frac{1}{2}(q-1)^3$	

	$dy^{-1}(xy+ay^2+b)(x+c)$, $a,b,c,d\in\mathbb{F}_q^*$	$\frac{q(q-1)^3}{2}$	
	$ey^{-1}(xy+ay^2+by+c)(x+d)$, $a,b,c,d,e\in\mathbb{F}_q^*$, if ay^2+by+c and $ay^2+(b-d)y+c$ are absolutely irreducible	$\le (q-1)^5$	
$C_{P_7^{(19)}}$, $q\ge 11$	$c(x-a)(x-b), a,b,c\in\mathbb{F}_q^*, a\ne b$	$\binom{q-1}{2}(q-1)$	$>4\binom{q-1}{2}(q-1)+\frac{q(q-1)^3}{2}$, if $q=2^m$;
	$cy^{-1}(y-a)(y-b), a,b,c\in\mathbb{F}_q^*, a\ne b$	$\binom{q-1}{2}(q-1)$	
	$cy(x-a)(x-b), a,b,c\in\mathbb{F}_q^*, a\ne b$	$\binom{q-1}{2}(q-1)$	
	$cy^{-1}(xy-a)(xy-b), a,b,c\in\mathbb{F}_q^*, a\ne b$	$\binom{q-1}{2}(q-1)$	
	$cy^{-1}(y+a)(x^2y+b)$, $a,b,c\in\mathbb{F}_q^*$, if $q\ne 2^m$	$\frac{1}{2}(q-1)^3$	$>4\binom{q-1}{2}(q-1)+\frac{(q-1)^3}{2}(q+1)$, if $q\ne 2^m$
	$dy^{-1}(x^2y+ay+b)(y+c)$, $a,b,c,d\in\mathbb{F}_q^*$	$\frac{q(q-1)^3}{2}$	
	$ey^{-1}(x^2y+axy+by+c)(y+d)$, $a,b,c,d,e\in\mathbb{F}_q^*$, if x^2+ax+b and $x^2+ax+b-cd^{-1}$ are absolutely irreducible	$\le (q-1)^5$	
$C_{P_7^{(22)}}$, $q\ge 11$	$cx^{-1}(x-a)(x-b), a,b,c\in\mathbb{F}_q^*, a\ne b$	$\binom{q-1}{2}(q-1)$	$=3\binom{q-1}{2}(q-1)$, if $q=2^m$; $=3\binom{q-1}{2}(q-1)+\frac{3}{2}(q-1)^3$, if $q\ne 2^m$
	$cy^{-1}(y-a)(y-b), a,b,c\in\mathbb{F}_q^*, a\ne b$	$\binom{q-1}{2}(q-1)$	
	$cx^{-1}y^{-1}(xy-a)(xy-b), a,b,c\in\mathbb{F}_q^*, a\ne b$	$\binom{q-1}{2}(q-1)$	
	$cx^{-1}y^{-1}(y+a)(x^2y+b)$, if $q\ne 2^m$	$\frac{1}{2}(q-1)^3$	
	$cy^{-1}(y-ax)(b-xy)$, $m\in\mathbb{Z}_+$, $a,b,c,\alpha\in\mathbb{F}_q^*$, if $q\ne 2^m$, $\frac{a}{b}\ne\alpha^{2i}$ for $0\le i\le\frac{q-3}{2}$	$\frac{1}{2}(q-1)^3$	
	$cx^{-1}y^{-1}(x-ay)(x-by^{-1})$, $m\in\mathbb{Z}_+$, $a,b,c,\alpha\in\mathbb{F}_q^*$, if $q\ne 2^m$, $\frac{a}{b}\ne\alpha^{2i}$ for $0\le i\le\frac{q-3}{2}$	$\frac{1}{2}(q-1)^3$	

According to Table 3.11, the following codes share the same n_1:

1) For $q\ge 43$, $C_{P_7^{(10)}}$, $C_{P_7^{(12)}}$, $C_{P_7^{(13)}}$, $C_{P_7^{(14)}}$ with $q=2^m$, $m\in\mathbb{Z}_+$, and $C_{P_7^{(15)}}$ with $q=2^m$, $m\in\mathbb{Z}_+$, $3\nmid(q-1)$;
2) For $q\ge 23$, $C_{P_7^{(16)}}$ and $C_{P_7^{(22)}}$ with $q=2^m$, $m\in\mathbb{Z}_+$;
3) For $q\ge 11$, $C_{P_7^{(18)}}$ and $C_{P_7^{(19)}}$.

The above three statements can be easily distinguished by n_2 list in Table 3.13. In Table 3.13, the analyses of the number of codewords of some families are novel compared with those in [LYZZ]. For example, in $C_{P_7^{(10)}}$, the reducible family $e(x+a)(x^2+bx+cy+d)$, $a,b,c,d,e\in\mathbb{F}_q^*$. In the sequel, we shall detail how to count the number of codewords with weight $(q-1)^2-(2q-3)$.

The zeros of this type polynomials are given by $x+a=0$ or $x^2+bx+cy+d=0$. That is, $(-a,y)$, $y\in\mathbb{F}_q^*$, contributes $q-1$ zeros. Or $(x,-\frac{ax^2+bx+d}{c})$, $x\in\mathbb{F}_q^*$, could be the zeros. Thus, we shall discuss in what situation $(x,-\frac{x^2+bx+d}{c})$ are zeros and how many such codewords.

1) x^2+bx+d is absolutely irreducible

In this case, for any $x\in\mathbb{F}_q^*$, $x^2+bx+d\neq 0$, so $(x,-\frac{x^2+bx+d}{c})$, $x\in\mathbb{F}_q^*$, are zeros of $x^2+bx+cy+d=0$, including $(-a,-\frac{a^2-ab+d}{c})$. Thus, the codewords have exactly $2q-3$ zeros.

Next, we shall count the number of irreducible polynomials of the form x^2+bx+d. The total possible number of such polynomials is $(q-1)^2$, due to $b,d\in\mathbb{F}_q^*$ and the reducible ones will be excluded. It is clear that the reducible polynomials are of the form $(x-m_1)(x-m_2)$ with $m_1,m_2\in\mathbb{F}_q^*$.

(1.1) $q=2^n$

We just need to count the number of polynomials with $m_1\neq m_2$, since $(x-m)^2=x^2-2mx+m^2=x^2+m^2$ yields $b=0\notin\mathbb{F}_q^*$. There are $\frac{(q-1)(q-2)}{2}$ possible choices. So the number of irreducible polynomials is $(q-1)^2-\frac{(q-1)(q-2)}{2}=\binom{q}{2}$.

(1.2) $q\neq 2^n$

The polynomial $(x+m)(x-m)=x^2-m^2$ yields $b=0\notin\mathbb{F}_q^*$. Similar as in (1.1) before, we just need to count the number of $(x-m_1)(x-m_2)$ with $m_1\neq -m_2$. There are $\frac{(q-1)^2}{2}$ possible choices. So the number of irreducible polynomials is $(q-1)^2-\frac{(q-1)^2}{2}=\frac{(q-1)^2}{2}$.

2) x^2+bx+d is reducible

Only $x^2+bx+d=(x+a)^2$ makes $e(x+a)(x^2+bx+cy+d)$ having exactly $2q-3$ zeros, due to the fact that for any $m\neq -a$, $m^2+bm+d\neq 0$, so $(m,-\frac{m^2+bm+d}{c})$ is a zero. It contributes $q-2$ solutions for $m\neq -a$. Adding the $q-1$ zeros of the form $(-a,r)$, with $r\in\mathbb{F}_q^*$, there are exactly $2q-3$ zeros.

Next, we count the number of the polynomials of the form $(x+a)^2$, $a\in\mathbb{F}_q^*$.

(2.1) $q=2^n$

It is easy to see that $b=2a=0\notin\mathbb{F}_q^*$. It contributes no codewords.

(2.2) $q\neq 2^n$

This type of codewords are $e(x+a)((x+a)^2+cy)$, the number of all possible such codewords is $(q-1)^3$, since $a,c,e\in\mathbb{F}_q^*$.

We summarize the result:

$$\sharp \text{ of codewords} = \begin{cases} \frac{1}{2}(q-1)^5, & q \neq 2^n, x^2+bx+d \text{ irreducible} \\ \binom{q}{2}(q-1)^3, & q = 2^n, x^2+bx+d \text{ irreducible} \\ (q-1)^3, & q \neq 2^n, x^2+bx+d = (x+a)^2 \\ 0, & q = 2^n, x^2+bx+d=(x+a)^2 \end{cases}.$$

Table 3.13: The distinct families of reducible polynomials which evaluate to give the codewords with weight $(q-1)^2-(2q-3)$ for $C_{P_7^{(i)}}$, $10 \le i \le 16$, $i = 18, 19, 22$.

$C_{P_7^{(i)}}$	Distinct families of reducible polynomials	$\sharp$ of codewords	$n_2(C_{P_7^{(i)}})$
$C_{P_7^{(10)}}$, $q \geq 11$	$e(x+a)(x^2+bx+cy+d), a,b,c,d,e \in \mathbb{F}_q^*$, if x^2+bx+d is absolutely irreducible and $q \neq 2^n$ or x^2+bx+d is absolutely irreducible and $q = 2^n$ or $x^2+bx+d=(x+a)^2$ and $q \neq 2^n$	$\frac{1}{2}(q-1)^5$ $\binom{q}{2}(q-1)^3$ $(q-1)^3$	$> (13+\frac{q}{2}+\frac{q^2}{2})(q-1)^3$, if $q=2^n$; $> (11+\frac{3q}{2}+\frac{q^2}{2})(q-1)^3$, if $q \neq 2^n$
	$a(y+bx+c)(x^2+dx+e)$, $a,b,c,d,e \in \mathbb{F}_q^*$, if $q \neq 2^n$	$(q-2)(q-1)^3$	
	$d(x+a)(x^2+by+c), a,b,c,d \in \mathbb{F}_q^*$, if $q \neq 2^n$ and $c \neq -\alpha^2$ for all $\alpha \in \mathbb{F}_q^*$ or if $q=2^n$ and $c=-a^2$	$\frac{1}{2}(q-1)^4$ $(q-1)^3$	
	$c(x+a)[x(x+a)+by)], a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$c(x+a)(x^2+by), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$c(x+a)(y+b), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$c(x+a)(x+by), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$c(x+a)(x+by+c), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$cx(x+a)(y+b), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$cx(x+a)(x+by), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$cx(x+a)(x+by+c), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$c(x+a)(xy+b), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$c(xy+ay+b)(x+a), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$c(xy+ax+by)(x+b), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$c(xy+ax+ab)(x+b), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$c(xy+ax+by+ab)(x+b), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	

	$a(y+bx+c)(x^2+d), a,b,c,d \in \mathbb{F}_q^*$, if $q=2^n$	$(q-2)(q-1)^3$	
	$a(y+bx)(x^2+c), a,b,c \in \mathbb{F}_q^*$, if $q=2^n$	$(q-1)^3$	
	$a(y+bx)(x^2+cx+d), a,b,c,d \in \mathbb{F}_q^*$, if $q \neq 2^n$	$(q-1)^4$	
$C_{P_7^{(12)}}$, $q \geq 47$	$c(x+a)(y+b), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	$= 6(q-1)^3 + \binom{q}{2}(q-1)^3$, if $q=2^n$;
	$c(x+a)(x+by), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$c(x+a)(x+by+a), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$e(x+a)(x^2+bx+cy+d), a,b,c,d,e \in \mathbb{F}_q^*$, if x^2+bx+d is absolutely irreducible and $q \neq 2^n$ if x^2+bx+d is absolutely irreducible and $q=2^n$ or $x^2+bx+d=(x+a)^2$ and $q \neq 2^n$	 $\frac{1}{2}(q-1)^5$ $\binom{q}{2}(q-1)^3$ $(q-1)^3$	
	$c(x+a)[x(x+a)+by], a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$d(x+a)(x^2+by+c), a,b,c,d \in \mathbb{F}_q^*$, if $q \neq 2^n$ and $c \neq -\alpha^2$ for all $\alpha \in \mathbb{F}_q^*$ or if $q=2^n$ and $c=-a^2$	 $\frac{1}{2}(q-1)^4$ $(q-1)^3$	$= (q-1)^3(6 - \frac{q}{2} + \frac{q^2}{2})$, if $q \neq 2^n$
	$c(x+a)(x^2+by), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
$C_{P_7^{(13)}}$, $q \geq 25$	$c(x+a)(y+b), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	$= 8(q-1)^3 + \binom{q}{2}(q-1)^3$, if $q=2^n$; $= (8 + \frac{q^2}{2} - \frac{q}{2})(q-1)^3$, if $q \neq 2^n$
	$c(x+a)(x+by), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$c(x+a)(x+by+a), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$e(x+a)(x^2+bx+cy+d), a,b,c,d,e \in \mathbb{F}_q^*$, if x^2+bx+d is absolutely irreducible and $q \neq 2^n$ if x^2+bx+d is absolutely irreducible and $q=2^n$ or $x^2+bx+d=(x+a)^2$ and $q \neq 2^n$	 $\frac{1}{2}(q-1)^5$ $\binom{q}{2}(q-1)^3$ $(q-1)^3$	
	$c(x+a)[x(x+a)+by], a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$d(x+a)(x^2+by+c), a,b,c,d \in \mathbb{F}_q^*$, if $q \neq 2^n$ and $c \neq -\alpha^2$ for all $\alpha \in \mathbb{F}_q^*$ or if $q=2^n$ and $c=-a^2$	 $\frac{1}{2}(q-1)^4$ $(q-1)^3$	
	$c(x+a)(x^2+by), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$cy^{-1}[xy+a(y+b)](y+b), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$cy^{-1}(xy+a)(y+b),\ a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$c(x+a)(y+b), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$c(x+a)(x+by), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$c(x+a)(x+by+a), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	

	$e(x+a)(x^2+bx+cy+d), a,b,c,d,e \in \mathbb{F}_q^*$, if x^2+bx+d is absolutely irreducible and $q \neq 2^n$ if x^2+bx+d is absolutely irreducible and $q = 2^n$ or $x^2+bx+d=(x+a)^2$ and $q \neq 2^n$;	 $\frac{1}{2}(q-1)^5$ $\binom{q}{2}(q-1)^3$ $(q-1)^3$	
	$c(x+a)[x(x+a)+by], a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$d(x+a)(x^2+by+c), a,b,c,d \in \mathbb{F}_q^*$, if $q \neq 2^n$ and $c \neq -\alpha^2$ for all $\alpha \in \mathbb{F}_q^*$ or if $q = 2^n$ and $c = -a^2$	 $\frac{1}{2}(q-1)^4$ $(q-1)^3$	
$C_{P_7^{(14)}}$, $q \geq 25$	$c(x+a)(x^2+by), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	$= 9(q-1)^3 + \frac{q^2}{2}(q-1)^3$, if $q = 2^n$; $= (\frac{15}{2} + \frac{q^2}{2})(q-1)^3$, if $q \neq 2^n$
	$cy^{-1}(xy-a)(y-bx), a,b,c \in \mathbb{F}_q^*$, if $q = 2^n$ if $q \neq 2^n$	 $(q-1)^3$ 0	
	$cy^{-1}(y+a)[x(y+a)+by], a,b,c \in \mathbb{F}_q^*$,	$(q-1)^3$	
	$cy^{-1}(x+ay)(y+b), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$dy^{-1}(xy+ay+b)(x+cy), a,b,c,d \in \mathbb{F}_q^*$, if cy^2-ay-b is absolutely irreducible and $q \neq 2^n$ if cy^2-ay-b is absolutely irreducible and $q = 2^n$	 $\frac{1}{2}(q-1)^4$ $\binom{q}{2}(q-1)^2$	
	$c(x+a)(y+b), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$c(x+a)(x+by), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$c(x+a)(x+by+a), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
$C_{P_7^{(15)}}$, $q \geq 25$	$e(x+a)(x^2+bx+cy+d), a,b,c,d,e \in \mathbb{F}_q^*$, if x^2+bx+d is absolutely irreducible and $q \neq 2^n$ if x^2+bx+d is absolutely irreducible and $q = 2^n$ or $x^2+bx+d=(x+a)^2$ and $q \neq 2^n$	 $\frac{1}{2}(q-1)^5$ $\binom{q}{2}(q-1)^3$ $(q-1)^3$	$(10 + \frac{q^2}{2})(q-1)^3 < n_2 < (\frac{19}{2} + \frac{q}{2} + \frac{q^2}{2})(q-1)^3$, if $q = 2^n$ and $3 \nmid q-1$; $(\frac{28}{3} + \frac{q^2}{2})(q-1)^3 < n_2 < (\frac{53}{6} + \frac{q}{2} + \frac{q^2}{2})(q-1)^3$, if $q = 2^n$ and $3 \mid q-1$;
	$c(x+a)[x(x+a)+by], a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$d(x+a)(x^2+by+c), a,b,c,d \in \mathbb{F}_q^*$, if $q \neq 2^n$ and $c \neq -\alpha^2$ for all $\alpha \in \mathbb{F}_q^*$ or if $q = 2^n$ and $c = -a^2$	 $\frac{1}{2}(q-1)^4$ $(q-1)^3$	
	$c(x+a)(x^2+by), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$a(y+b)(x^2y^{-1}+c), a,b,c \in \mathbb{F}_q^*$, if $q = 2^n$ if $q \neq 2^n$	 $(q-1)^3$ 0	

	$cy^{-1}(xy+ax+aby)(x+by), a,b,c \in \mathbb{F}_q^*$, if $q \neq 2^n$	$(q-1)^3$	$(\frac{17}{2}+\frac{q^2}{2})(q-1)^3 < n_2 < (8+\frac{q}{2}+\frac{q^2}{2})(q-1)^3$, if $q \neq 2^n$ and $3 \nmid q-1$; $(\frac{47}{6}+\frac{q^2}{2})(q-1)^3 < n_2 < (\frac{22}{3}+\frac{q}{2}+\frac{q^2}{2})(q-1)^3$, if $q \neq 2^n$ and $3 \mid q-1$
	$cy^{-1}(x^2+ay)(y+b)$, $a,b,c \in \mathbb{F}_q^*$, if $q = 2^n$	$(q-1)^3$	
	$cxy^{-1}(x+ay)(y+b)$, $a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$dy^{-1}(xy+ay+bx^2)(y+c)$, $a,b,c,d \in \mathbb{F}_q^*$, if $bx^2-cx-ac$ is absolutely irreducible and $q \neq 2^n$ if $bx^2-cx-ac$ is absolutely irreducible and $q = 2^n$	$\frac{1}{2}(q-1)^4$ $\binom{q}{2}(q-1)^2$	
	$cy^{-1}(xy+a)(y+bx^2)$, $a,b,c \in \mathbb{F}_q^*$, if $3 \nmid q-1$ if $3 \mid q-1$	$(q-1)^3$ $\frac{(q-1)^3}{3}$	
	$dy^{-1}(xy+ay+b)(y+cx^2)$, $a,b,c,d \in \mathbb{F}_q^*$, if cx^3+acx^2-b is absolutely irreducible	$0 < \sharp < \frac{(q-1)^4}{2}$	
$C_{P_7^{(16)}}$, $q \geq 25$	$c(x+a)(y+b), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	$= 11(q-1)^3$
	$c(x+a)(xy+b), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$c(x+a)(x+by), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$cx(x+a)(y+b), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$c(xy+ay+b)(x+a), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$c(xy+ax+by)(x+b)$, $a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$c(x+ay+b)(x+b)$, $a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$c(xy+ax+ab)(x+b)$, $a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$c(xy+ax+by+ab)(x+a), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$cx^{-1}y^{-1}(xy+a)(x^2y+b)$, $a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$cx^{-1}y^{-1}(x^2y+axy+ab)(xy+b)$, $a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$c(x+a)(y+b), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
$C_{P_7^{(22)}}$, $q \geq 11$	$c(x+a)(y+b)$, $a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	$= (15+3q)(q-1)^3 + \frac{3q}{2}(q-1)^4$, if $q = 2^n$; $= (9+3q)(q-1)^3$, if $q \neq 2^n$
	$cx^{-1}(x+a)(xy+b)$, $a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$cx^{-1}y^{-1}(x+a)(xy+b), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$cx^{-1}y^{-1}(x+a)(y+b)$, $a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$cx^{-1}y^{-1}(x+a)(xy^2+b)$, $a,b,c \in \mathbb{F}_q^*$, if $q = 2^n$ if $q \neq 2^n$	$(q-1)^3$ 0	
	$cy^{-1}(y+a)(xy+b)$, $a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$cx^{-1}y^{-1}(y+a)(xy+b), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$cx^{-1}y^{-1}(y+a)(x^2y+b)$, $a,b,c \in \mathbb{F}_q^*$, if $q = 2^n$ if $q \neq 2^n$	$(q-1)^3$ 0	
	$cx^{-1}y^{-1}(xy+a)(x+by)$, $a,b,c \in \mathbb{F}_q^*$,		

if $q = 2^n$ if $q \neq 2^n$	$(q-1)^3$ 0
$dx^{-1}y^{-1}(x+ay+b)(xy+c)$, $a,b,c,d \in \mathbb{F}_q^*$, if $x^2+bx-ac$ is absolutely irreducible if $q \neq 2^n$ if $q = 2^n$	 $\frac{1}{2}(q-1)^4$ $\frac{q}{2}(q-1)^3$
$dx^{-1}y^{-1}(xy+ax+by)(xy+c)$, $a,b,c,d \in \mathbb{F}_q^*$, if $ax^2-cx-bc$ is absolutely irreducible if $q \neq 2^n$ if $q = 2^n$	 $\frac{1}{2}(q-1)^4$ $\frac{q}{2}(q-1)^3$
$cy^{-1}(xy+ay+ab)(y+b)$, $a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$
$cx^{-1}y^{-1}(xy+ay+b)(x+a)$, $a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$
$cx^{-1}y^{-1}(xy+ay+b)(xy+b)$, $a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$
$cx^{-1}(xy+ax+ab)(x+b)$, $a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$
$cx^{-1}y^{-1}(xy+ax+b)(y+a)$, $a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$
$cx^{-1}y^{-1}(xy+ax+b)(xy+b)$, $a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$
$d(x^{-1}y^{-1}+a+bx)(y+c)$, $a,b,c,d \in \mathbb{F}_q^*$, if $q \neq 2^n$ if $q = 2^n$	 $\frac{1}{2}(q-1)^4$ $\frac{q}{2}(q-1)^3$
$dy^{-1}(x^{-1}+a+bxy)(y+c)$, $a,b,c,d \in \mathbb{F}_q^*$, if $q \neq 2^n$ if $q = 2^n$	 $\frac{1}{2}(q-1)^4$ $\frac{q}{2}(q-1)^3$
$dx^{-1}(y^{-1}+a+bxy)(x+c)$, $a,b,c,d \in \mathbb{F}_q^*$, if $q \neq 2^n$ if $q = 2^n$	 $\frac{1}{2}(q-1)^4$ $\frac{q}{2}(q-1)^3$
$d(x^{-1}y^{-1}+a+by)(x+c)$, $a,b,c,d \in \mathbb{F}_q^*$, if $q \neq 2^n$ if $q = 2^n$	 $\frac{1}{2}(q-1)^4$ $\frac{q}{2}(q-1)^3$
$ex^{-1}y^{-1}(xy+ax+by+c)(xy+d)$, $a,b,c,d,e \in \mathbb{F}_q^*$, if $q = 2^n$ if $q \neq 2^n$	 $\frac{q}{2}(q-1)^4$ 0
$e(x^{-1}y^{-1}+ax^{-1}+b+cy)(x+d)$, $a,b,c,d,e \in \mathbb{F}_q^*$, if $q = 2^n$ if $q \neq 2^n$	 $\frac{q}{2}(q-1)^4$ 0

	$ey^{-1}(x^{-1}+a+by+cxy)(y+d)$, $a,b,c,d,e \in \mathbb{F}_q^*$, if $q=2^n$ if $q \neq 2^n$	 $\frac{q}{2}(q-1)^4$ 0	
$C_{P_7^{(18)}}$, $q \geq 11$	$cy^{-1}(x+a)(y+b), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	$=(q-1)^3(14+2q)$, if $q=2^n$; $=(11+2q)(q-1)^3$, if $q \neq 2^n$
	$c(x+a)(x+by), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$cy^{-1}(x+a)(xy+b), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$cy^{-1}(y+a)(x+by), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$cy^{-1}(y+a)(xy+b), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$cy^{-1}(xy+a)(x+by), a,b,c \in \mathbb{F}_q^*$, if $q=2^n$ if $q \neq 2^n$	 $(q-1)^3$ 0	
	$cy^{-1}(xy+ay+ab)(y+b), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$dy^{-1}(xy+ay+b)(x+cy), a,b,c,d \in \mathbb{F}_q^*$ if cy^2-ay-b is absolutely irreducible and $q \neq 2^n$ if cy^2-ay-b is absolutely irreducible and $q=2^n$	 $\frac{1}{2}(q-1)^4$ $\binom{q}{2}(q-1)^2$	
	$ay^{-1}(xy+cy+b)(x+c), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$c(x+ay+b)(x+b),\ a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$cy^{-1}(x+ay+ab)(y+b)$, $a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$dy^{-1}(x+ay+b)(xy+c), a,b,c,d \in \mathbb{F}_q^*$, if $q \neq 2^n$ if $q=2^n$	 $\frac{1}{2}(q-1)^4$ $\binom{q}{2}(q-1)^2$	
	$cy^{-1}(xy+ax+by)(y+a),\ a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$cy^{-1}(xy+ax+b)(y+a),\ a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$dy^{-1}(xy+ay^2+b)(x+c)$, $a,b,c,d \in \mathbb{F}_q^*$, if ay^2-cy+b is absolutely irreducible and $q=2^n$ or if $-a^{-1}b$ is not a square, $ay^2-cy+b=a(y-e)^2$ for some e and $q \neq 2^n$	 $\frac{q}{2}(q-1)^3$ $(q-1)^3$	
	$cy^{-1}(xy+ax+by+ab)(y+a), a,b,c,d \in \mathbb{F}_q^*$	$(q-1)^3$	

	$ey^{-1}(xy+ay^2+by+c)(x+d)$, $a,b,c,d \in \mathbb{F}_q^*$, if ay^2+by+c is absolutely irreducible and $ay^2+(b-d)y+c=0$ has only one root; or if $ay^2+(b-d)y+c$ is absolutely irreducible and $ay^2+by+c=0$ has only one root if $q=2^n$ if $q \neq 2^n$	 $\frac{q}{2}(q-1)^3$ $(q-2)(q-1)^3$	
$C_{P_7^{(19)}}$, $q \geq 11$	$c(x+a)(y+b), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	$=(14+q)(q-1)^3$, if $q=2^n$; $=(10+2q)(q-1)^3$, if $q \neq 2^n$
	$c(x+a)(xy+b), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$c(x+a)(x+by), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$cx(x+a)(y+b), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$cy^{-1}(xy+a)(y+b), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$cy^{-1}(y+a)(x^2y+b), a,b,c \in \mathbb{F}_q^*$, if $q=2^n$ if $q \neq 2^n$	 $(q-1)^3$ 0	
	$c(xy+ay+b)(x+a), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$dy^{-1}(x^2y+axy+b)(y+c), a,b,c,d \in \mathbb{F}_q^*$, if $q \neq 2^n$ if $q=2^n$	 $\frac{1}{2}(q-1)^4$ $2(q-1)^3$	
	$cy^{-1}(xy+ay+ab)(y+b), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$cy^{-1}(xy+ay+b)(xy+b),\ a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$c(xy+ax+by)(x+b),\ a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$dy^{-1}(x^2y+ay+b)(y+c), a,b,c,d \in \mathbb{F}_q^*$, if $q=2^n$ if $q \neq 2^n$	 $(q-1)^3$ 0	
	$c(x+ay+b)(x+b),\ a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$c(xy+ax+ab)(x+b),\ a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$d(xy+ax+by+ac)(x+c)$, $a,b,c,d \in \mathbb{F}_q^*$	$(q-1)^4$	
	$ey^{-1}(x^2y+axy+by+c)(y+d)$, $a,b,c,d,e \in \mathbb{F}_q^*$, if x^2+ax+b is absolutely irreducible and $x^2+ax+b-cd^{-1}=0$ has only one root; or if $x^2+ax+b-cd^{-1}$ is absolutely irreducible and $x^2+ax+b=0$ has only one root if $q=2^n$ if $q \neq 2^n$	 0 $\frac{q+1}{2}(q-1)^3$	

For q small, say $q \leq 42$ for $C_{P_7^{(12)}}$, the codes are discerned by the generating polynomials list in Table A.1 and Table A.1. According to these tables, the only pair that shares the same generating polynomial is $C_{P_7^{(10)}}$ and $C_{P_7^{(19)}}$ over $\mathbb{F}_{29}$, which is left to be undetermined the monomially equivalence. $\square$

Proposition 3.8. For $q > 9$, no two codes of $C_{P_7^{(i)}}$, $i = 17, 20, 21$ or 22 are monomially equivalent.

Proof. Let us compute $n_1(C_{P_7^{(i)}})$, $i = 17, 20, 21$ and 22:

Table 3.15: The distinct families of reducible polynomials which evaluate to give the codewords with weight $(q-1)^2 - 2(q-1)$ for $C_{P_7^{(i)}}$, $i = 17$ and $20 \leq i \leq 22$.

$C_{P_7^{(i)}}$	Distinct families of reducible polynomials	$\sharp$ of codewords	$n_1(C_{P_7^{(i)}})$
$C_{P_7^{(17)}}$, $q \geq 23$	$c(xy-a)(xy-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	$\geq 3\binom{q-1}{2}(q-1) + 2(q-1)^3$, if $q = 2^m$; $\geq 3\binom{q-1}{2}(q-1) + \frac{5}{2}(q-1)^3$, if $q \neq 2^m$
	$cx^{-1}(x-a)(x-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	
	$cy^{-1}(y-a)(y-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	
	$cx^{-1}y^{-1}(y-ax)(b-xy)$, if $q \neq 2^m$, $m \in \mathbb{Z}_+$, $a,b,c,\alpha \in \mathbb{F}_q^*$, $\frac{a}{b} \neq \alpha^{2i}$ for $1 \leq i \leq q-1$	$\frac{1}{2}(q-1)^3$	
	$ay^{-1}(xy+b)(xy^2+c), a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$dx^{-1}y^{-1}(xy+a)(x+bxy+cy)$, $a,b,c,d \in \mathbb{F}_q^*$, if $x^2 - abx + ac$ is absolutely irreducible and has one solution	$(q-1)^3$	
	$dx^{-1}y^{-1}(xy+a)(x+bx^2y^2+cy)$, $a,b,c,d \in \mathbb{F}_q^*$	≥ 0	
	$ex^{-1}y^{-1}(xy+a)(x+bx^2y^2+cxy+dy)$, $a,b,c,d,e \in \mathbb{F}_q^*$	≥ 0	
$C_{P_7^{(20)}}$, $q \geq 43$	$c(x-a)(x-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	$3\binom{q-1}{2}(q-1)$
	$cy^{-1}(y-a)(y-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	
	$cx^{-1}(x-ay)(x-by), a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	
$C_{P_7^{(21)}}$, $q \geq 43$	$c(x-a)(x-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	$4\binom{q-1}{2}(q-1)$
	$c(y-a)(y-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	
	$cx^{-1}y^{-1}(xy-a)(xy-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	
	$c(x-ay)(x-by), a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	
	$cx^{-1}(x-a)(x-b), a,b,c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	

	$cy^{-1}(y-a)(y-b), a, b, c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	$= 3\binom{q-1}{2}(q-1)$, if $q = 2^m$; $= 3\binom{q-1}{2}(q-1) + \frac{3}{2}(q-1)^3$, if $q \neq 2^m$
$C_{P_7^{(22)}}$, $q \geq 11$	$cx^{-1}y^{-1}(xy-a)(xy-b), a, b, c \in \mathbb{F}_q^*, a \neq b$	$\binom{q-1}{2}(q-1)$	
	$cx^{-1}y^{-1}(y+a)(x^2y+b)$, $a, b, c \in \mathbb{F}_q^*$, if $q \neq 2^m$	$\frac{1}{2}(q-1)^3$	
	$cy^{-1}(y-ax)(b-xy)$, $m \in \mathbb{Z}_+$, $a, b, c, \alpha \in \mathbb{F}_q^*$, if $q \neq 2^m$, $\frac{a}{b} \neq \alpha^{2i}$ for $0 \leq i \leq \frac{q-3}{2}$, $\alpha \in \mathbb{F}_q^*$	$\frac{1}{2}(q-1)^3$	
	$cx^{-1}y^{-1}(x-ay)(x-by^{-1})$, $m \in \mathbb{Z}_+$, $a, b, c \in \mathbb{F}_q^*$, if $q \neq 2^m$, $\frac{a}{b} \neq \alpha^{2i}$ for $0 \leq i \leq \frac{q-3}{2}$, $\alpha \in \mathbb{F}_q^*$	$\frac{1}{2}(q-1)^3$	

It is easy to see from Table 3.15 that only $C_{P_7^{(20)}}$ and $C_{P_7^{(22)}}$ with $q = 2^m$, $m \in \mathbb{Z}_+$ share the same n_1. We shall investigate n_2 in Table 3.17.

Table 3.17: The distinct families of reducible polynomials which evaluate to give the codewords with weight $(q-1)^2 - (2q-3)$ for $C_{P_7^{(i)}}$, $i = 20, 22$.

$C_{P_7^{(i)}}$	Distinct families of reducible polynomials	♯ of codewords	$n_2(C_{P_7^{(i)}})$
	$c(x+a)(y+b), a, b, c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$c(x+ay)(x+b),\ a, b, c \in \mathbb{F}_q^*$	$(q-1)^3$	
$C_{P_7^{(20)}}$, $q \geq 47$	$cx^{-1}(x+ay)(x^2+by),\ a, b, c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$cy^{-1}(xy+a)(y+b),\ a, b, c \in \mathbb{F}_q^*$	$(q-1)^3$	$7(q-1^3)$
	$c(x+ay+b)(x+b),\ a, b, c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$cy^{-1}(xy+ay+ab)(y+b),\ a, b, c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$cx^{-1}(x^2+abx+ay)(y+bx)$, $a, b, c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$c(x+a)(y+b),\ a, b, c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$cx^{-1}(x+a)(xy+b),\ a, b, c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$cx^{-1}y^{-1}(x+a)(xy+b), a, b, c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$cx^{-1}y^{-1}(x+a)(y+b),\ a, b, c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$cx^{-1}y^{-1}(x+a)(xy^2+b)$, $a, b, c \in \mathbb{F}_q^*$, if $q = 2^n$ if $q \neq 2^n$	 $(q-1)^3$ 0	
	$cy^{-1}(y+a)(xy+b),\ a, b, c \in \mathbb{F}_q^*$	$(q-1)^3$	
	$cx^{-1}y^{-1}(y+a)(xy+b), a, b, c \in \mathbb{F}_q^*$	$(q-1)^3$	
$C_{P_7^{(22)}}$, $q \geq 11$	$cx^{-1}y^{-1}(y+a)(x^2y+b)$, $a, b, c \in \mathbb{F}_q^*$, if $q = 2^n$ if $q \neq 2^n$	 $(q-1)^3$ 0	$(15+3q)(q-1)^3 + \frac{3q}{2}(q-1)^4$, if $q = 2^n$; $(9+3q)(q-1)^3$, if $q \neq 2^n$
	$cx^{-1}y^{-1}(xy+a)(x+by),\ \ a, b, c \in \mathbb{F}_q^*$, if $q = 2^n$	 $(q-1)^3$	

if $q \neq 2^n$	0
$dx^{-1}y^{-1}(x+ay+b)(xy+c)$, $a,b,c,d \in \mathbb{F}_q^*$, if $x^2+bx-ac$ is absolutely irreducible if $q \neq 2^n$ if $q = 2^n$	 $\frac{1}{2}(q-1)^4$ $\frac{q}{2}(q-1)^3$
$dx^{-1}y^{-1}(xy+ax+by)(xy+c)$, $a,b,c,d \in \mathbb{F}_q^*$, if $ax^2-cx-bc$ is absolutely irreducible if $q \neq 2^n$ if $q = 2^n$	 $\frac{1}{2}(q-1)^4$ $\frac{q}{2}(q-1)^3$
$cy^{-1}(xy+ay+ab)(y+b)$, $a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$
$cx^{-1}y^{-1}(xy+ay+b)(x+a)$, $a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$
$cx^{-1}y^{-1}(xy+ay+b)(xy+b)$, $a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$
$cx^{-1}(xy+ax+ab)(x+b)$, $a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$
$cx^{-1}y^{-1}(xy+ax+b)(y+a)$, $a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$
$cx^{-1}y^{-1}(xy+ax+b)(xy+b)$, $a,b,c \in \mathbb{F}_q^*$	$(q-1)^3$
$d(x^{-1}y^{-1}+a+bx)(y+c)$, $a,b,c,d \in \mathbb{F}_q^*$, if $q \neq 2^n$ if $q = 2^n$	 $\frac{1}{2}(q-1)^4$ $\frac{q}{2}(q-1)^3$
$dy^{-1}(x^{-1}+a+bxy)(y+c)$, $a,b,c \in \mathbb{F}_q^*$, if $q \neq 2^n$ if $q = 2^n$	 $\frac{1}{2}(q-1)^4$ $\frac{q}{2}(q-1)^3$
$dx^{-1}(y^{-1}+a+bxy)(x+c)$, $a,b,c,d \in \mathbb{F}_q^*$, if $q \neq 2^n$ if $q = 2^n$	 $\frac{1}{2}(q-1)^4$ $\frac{q}{2}(q-1)^3$
$d(x^{-1}y^{-1}+a+by)(x+c)$, $a,b,c,d \in \mathbb{F}_q^*$, if $q \neq 2^n$ if $q = 2^n$	 $\frac{1}{2}(q-1)^4$ $\frac{q}{2}(q-1)^3$
$ex^{-1}y^{-1}(xy+ax+by+c)(xy+d)$, $a,b,c,d,e \in \mathbb{F}_q^*$, if $q = 2^n$ if $q \neq 2^n$	 $\frac{q}{2}(q-1)^4$ 0
$e(x^{-1}y^{-1}+ax^{-1}+b+cy)(x+d)$, $a,b,c,d,e \in \mathbb{F}_q^*$, if $q = 2^n$ if $q \neq 2^n$	 $\frac{q}{2}(q-1)^4$ 0
$ey^{-1}(x^{-1}+a+by+cxy)(y+d)$, $a,b,c,d,e \in \mathbb{F}_q^*$,	

	if $q = 2^n$ if $q \neq 2^n$	$\frac{q}{2}(q-1)^4$ 0	

For q small, say $q \leq 42$ for $C_{P_7^{(20)}}$, the codes are discerned by the generating polynomials list in Table A.1. □

Appendix A. Tables of the enumerator polynomials for toric codes

In this appendix, we list all the tables mentioned in our proof of Theorem 1.2, where the weight enumerator polynomials are defined as follows:

$$W_C(x) = \sum_{i=0}^{(q-1)^2} A_i x^i$$

where $A_i = |\{w \in C : wt(w) = i\}|$, for the $k = 7$ toric codes. All the polynomials are computed by using Magma code from [Joy].

Table A.1: Generator polynomials for toric codes.

Over $\mathbb{F}_7$	${P_7}^{(3)}$:	$1 + 90x^{12} + 600x^{18} + 2790x^{24} + \cdots$
	${P_7}^{(4)}$:	$1 + 90x^{12} + 600x^{18} + 3870x^{24} + \cdots$
	${P_7}^{(5)}$:	$1 + 90x^{12} + 600x^{18} + 4860x^{24} + \cdots$
	${P_7}^{(6)}$:	$1 + 90x^{12} + 600x^{18} + 3222x^{24} + \cdots$
	${P_7}^{(7)}$:	$1 + 90x^{12} + 600x^{18} + 4860x^{24} + \cdots$
	${P_7}^{(8)}$:	$1 + 90x^{12} + 600x^{18} + 3870x^{24} + \cdots$
	${P_7}^{(9)}$:	$1 + 90x^{12} + 600x^{18} + 5688x^{24} + \cdots$
	${P_7}^{(10)}$:	$1 + 120x^{18} + 2160x^{20} + 2160x^{21} + \cdots$
	${P_7}^{(11)}$:	$1 + 120x^{18} + 864x^{24} + 7776x^{25} + \cdots$
	${P_7}^{(12)}$:	$1 + 120x^{18} + 810x^{24} + 8640x^{25} + \cdots$
	${P_7}^{(13)}$:	$1 + 120x^{18} + 810x^{24} + 8640x^{25} + \cdots$
	${P_7}^{(14)}$:	$1 + 120x^{18} + 774x^{24} + 9072x^{25} + \cdots$
	${P_7}^{(15)}$:	$1 + 120x^{18} + 990x^{24} + 8856x^{25} + \cdots$
	${P_7}^{(16)}$:	$1 + 540x^{20} + 918x^{24} + 4104x^{25} + \cdots$
	${P_7}^{(17)}$:	$1 + 216x^{21} + 432x^{22} + 1080x^{23} + \cdots$
	${P_7}^{(18)}$:	$1 + 540x^{20} + 2322x^{24} + 5400x^{25} + \cdots$
	${P_7}^{(19)}$:	$1 + 540x^{20} + 2088x^{24} + 5616x^{25} + \cdots$
	${P_7}^{(20)}$:	$1 + 216x^{22} + 216x^{23} + 1350x^{24} + \cdots$
	${P_7}^{(21)}$:	$1 + 108x^{22} + 864x^{23} + 1656x^{24} + \cdots$
	${P_7}^{(22)}$:	$1 + 216x^{20} + 1080x^{21} + 594x^{24} + \cdots$
	${P_7}^{(1)}$:	$1 + 49x^{7} + 1029x^{14} + 12005x^{21} + \cdots$
	${P_7}^{(2)}$:	$1 + 147x^{14} + 1470x^{21} + 10535x^{28} + \cdots$
	${P_7}^{(3)}$:	$1 + 245x^{21} + 1225x^{28} + 5586x^{35} + \cdots$

Over $\mathbb{F}_8$	${P_7}^{(4)}$:	$1 + 245x^{21} + 1225x^{28} + 6762x^{35} + \cdots$
	${P_7}^{(5)}$:	$1 + 245x^{21} + 1225x^{28} + 7791x^{35} + 3773x^{36} + 16464x^{37} + 34986x^{38} + \cdots$
	${P_7}^{(6)}$:	$1 + 245x^{21} + 1225x^{28} + 7791x^{35} + 3773x^{36} + 16464x^{37} + 34986x^{38} + \cdots$
	${P_7}^{(7)}$:	$1 + 245x^{21} + 1225x^{28} + 6762x^{35} + \cdots$
	${P_7}^{(8)}$:	$1 + 245x^{21} + 1225x^{28} + 5586x^{35} + 7546x^{36} + \cdots$
	${P_7}^{(9)}$:	$1 + 245x^{21} + 1225x^{28} + 7791x^{35} + 3773x^{36} + 16464x^{37} + 36015x^{38} + \cdots$
	${P_7}^{(10)}$:	$1 + 245x^{28} + 4116x^{30} + 5145x^{31} + \cdots$
	${P_7}^{(11)}$:	$1 + 245x^{28} + 882x^{35} + 14749x^{36} + \cdots$
	${P_7}^{(12)}$:	$1 + 245x^{28} + 882x^{24} + 15778x^{25} + \cdots$
	${P_7}^{(13)}$:	$1 + 245x^{28} + 735x^{35} + 16121x^{36} + \cdots$
	${P_7}^{(14)}$:	$1 + 245x^{28} + 735x^{35} + 17493x^{36} + \cdots$
	${P_7}^{(15)}$:	$1 + 245x^{28} + 735x^{35} + 18522x^{36} + \cdots$
	${P_7}^{(16)}$:	$1 + 1029x^{30} + 441x^{35} + 6174x^{36} + \cdots$
	${P_7}^{(17)}$:	$1 + 2058x^{33} + 1029x^{34} + 5586x^{35} + \cdots$
	${P_7}^{(18)}$:	$1 + 1029x^{30} + 4457x^{35} + 10290x^{36} + \cdots$
	${P_7}^{(19)}$:	$1 + 1029x^{30} + 4704x^{35} + 7546x^{36} + \cdots$
	${P_7}^{(20)}$:	$1 + 49x^{28} + 1372x^{34} + 441x^{35} + \cdots$
	${P_7}^{(21)}$:	$1 + 1029x^{33} + 3675x^{35} + 7889x^{36} + \cdots$
	${P_7}^{(22)}$:	$1 + 343x^{30} + 2058x^{31} + 441x^{35} + \cdots$
Over $\mathbb{F}_9$	${P_7}^{(1)}$:	$1 + 224x^{16} + 3136x^{24} + 31920x^{32} + \cdots$
	${P_7}^{(2)}$:	$1 + 448x^{24} + 3360x^{32} + 22848x^{40} + \cdots$
	${P_7}^{(3)}$:	$1 + 560x^{32} + 2240x^{40} + 10304x^{48} + \cdots$
	${P_7}^{(4)}$:	$1 + 560x^{32} + 2240x^{40} + 12352x^{48} + 5120x^{49} + \cdots$
	${P_7}^{(5)}$:	$1 + 560x^{32} + 2240x^{40} + 14944x^{48} + \cdots$
	${P_7}^{(6)}$:	$1 + 560x^{32} + 2240x^{40} + 12352x^{48} + 5120x^{49} + \cdots$
	${P_7}^{(7)}$:	$1 + 560x^{32} + 2240x^{40} + 14368x^{48} + \cdots$
	${P_7}^{(8)}$:	$1 + 560x^{32} + 2240x^{40} + 12352x^{48} + 8192x^{49} + \cdots$
	${P_7}^{(9)}$:	$1 + 560x^{32} + 2240x^{40} + 14624x^{48} + \cdots$
	${P_7}^{(10)}$:	$1 + 448x^{40} + 7168x^{42} + 10752x^{43} + \cdots$
	${P_7}^{(11)}$:	$1 + 448x^{40} + 1344x^{48} + 23040x^{49} + \cdots$
	${P_7}^{(12)}$:	$1 + 448x^{40} + 2400x^{48} + 27648x^{49} + \cdots$
	${P_7}^{(13)}$:	$1 + 448x^{40} + 1888x^{48} + 25600x^{49} + \cdots$
	${P_7}^{(14)}$:	$1 + 448x^{40} + 1376x^{48} + 28160x^{49} + \cdots$
	${P_7}^{(15)}$:	$1 + 448x^{40} + 1888x^{48} + 27648x^{49} + \cdots$
	${P_7}^{(16)}$:	$1 + 1792x^{42} + 672x^{48} + 8192x^{49} + \cdots$
	${P_7}^{(17)}$:	$1 + 1024x^{44} + 3072x^{46} + 4096x^{47} + \cdots$
	${P_7}^{(18)}$:	$1 + 1792x^{42} + 9120x^{48} + 14848x^{49} + \cdots$
	${P_7}^{(19)}$:	$1 + 1792x^{42} + 8320x^{48} + 14336x^{49} + \cdots$
	${P_7}^{(20)}$:	$1 + 512x^{46} + 1024x^{47} + 4256x^{48} + \cdots$
	${P_7}^{(21)}$:	$1 + 64x^{40} + 1024x^{47} + 6272x^{48} + \cdots$
	${P_7}^{(22)}$:	$1 + 512x^{42} + 3584x^{43} + 1440x^{48} + \cdots$
	${P_7}^{(1)}$:	$1 + 2100x^{40} + 17640x^{50} + 159600x^{60} + \cdots$
	${P_7}^{(2)}$:	$1 + 2520x^{50} + 12600x^{60} + 84000x^{70} + \cdots$

Over $\mathbb{F}_{11}$	${P_7}^{(3)}$: $1 + 2100x^{60} + 6000x^{70} + 29250x^{80} + 615000x^{81} + \cdots$
	${P_7}^{(4)}$: $1 + 2100x^{60} + 6000x^{70} + 29450x^{80} + 4000x^{81} + \cdots$
	${P_7}^{(5)}$: $1 + 2100x^{60} + 6000x^{70} + 35700x^{80} + 2000x^{81} + \cdots$
	${P_7}^{(6)}$: $1 + 2100x^{60} + 6000x^{70} + 31450x^{80} + 5000x^{81} + \cdots$
	${P_7}^{(7)}$: $1 + 2100x^{60} + 6000x^{70} + 33200x^{80} + 1000x^{81} + \cdots$
	${P_7}^{(8)}$: $1 + 2100x^{60} + 6000x^{70} + 30450x^{80} + 7000x^{81} + \cdots$
	${P_7}^{(9)}$: $1 + 2100x^{60} + 6000x^{70} + 39200x^{80} + 29000x^{82} + \cdots$
	${P_7}^{(10)}$: $1 + 1200x^{70} + 18000x^{72} + 36000x^{73} + 2250x^{80} + 183000x^{81} + \cdots$
	${P_7}^{(11)}$: $1 + 1200x^{70} + 2700x^{80} + 64000x^{81} + \cdots$
	${P_7}^{(12)}$: $1 + 1200x^{70} + 2450x^{80} + 64000x^{81} + 50000x^{82} + \cdots$
	${P_7}^{(13)}$: $1 + 1200x^{70} + 2450x^{80} + 65000x^{81} + 58000x^{82} + \cdots$
	${P_7}^{(14)}$: $1 + 1200x^{70} + 2950x^{80} + 69000x^{81} + \cdots$
	${P_7}^{(15)}$: $1 + 1200x^{70} + 2750x^{80} + 73000x^{81} + \cdots$
	${P_7}^{(16)}$: $1 + 4500x^{72} + 2550x^{80} + 13000x^{81} + \cdots$
	${P_7}^{(17)}$: $1 + 2000x^{75} + 1000x^{76} + 7000x^{77} + 8000x^{78} + 18050x^{80} + 35000x^{81} + \cdots$
	${P_7}^{(18)}$: $1 + 4500x^{72} + 26850x^{80} + 33000x^{81} + \cdots$
	${P_7}^{(19)}$: $1 + 4500x^{72} + 24800x^{80} + 32000x^{81} + \cdots$
	${P_7}^{(20)}$: $1 + 1000x^{79} + 5550x^{80} + 21000x^{81} + \cdots$
	${P_7}^{(21)}$: $1 + 500x^{76} + 1500x^{78} + 1000x^{79} + 9200x^{80} + 26000x^{81} + \cdots$
	${P_7}^{(22)}$: $1 + 1000x^{72} + 9000x^{73} + 2850x^{80} + 42000x^{81} + \cdots$
Over $\mathbb{F}_{13}$	${P_7}^{(3)}$: $1 + 5940x^{96} + 13200x^{108} + 69696x^{120} + 1679616x^{121} + \cdots$
	${P_7}^{(4)}$: $1 + 5940x^{96} + 13200x^{108} + 70272x^{120} + 5184x^{121} + \cdots$
	${P_7}^{(5)}$: $1 + 5940x^{96} + 13200x^{108} + 78120x^{120} + 50112x^{122} + \cdots$
	${P_7}^{(6)}$: $1 + 5940x^{96} + 13200x^{108} + 71424x^{120} + 6912x^{121} + \cdots$
	${P_7}^{(7)}$: $1 + 5940x^{96} + 13200x^{108} + 71640x^{120} + 1728x^{121} + \cdots$
	${P_7}^{(8)}$: $1 + 5940x^{96} + 13200x^{108} + 70484x^{120} + 3456x^{121} + \cdots$
	${P_7}^{(9)}$: $1 + 5940x^{96} + 13200x^{108} + 82152x^{120} + 28512x^{122} + \cdots$
	${P_7}^{(10)}$: $1 + 2640x^{108} + 38016x^{110} + 95040x^{111} + 3960x^{120} + 435456x^{121} + \cdots$
	${P_7}^{(11)}$: $1 + 2640x^{108} + 4752x^{120} + 150336x^{121} + 142560x^{122} + \cdots$
	${P_7}^{(12)}$: $1 + 2640x^{108} + 4248x^{120} + 145152x^{121} + 95040x^{122} + \cdots$
	${P_7}^{(13)}$: $1 + 2640x^{108} + 4248x^{120} + 148608x^{121} + 100224x^{122} + \cdots$
	${P_7}^{(14)}$: $1 + 2640x^{108} + 4824x^{120} + 158976x^{121} + 97632x^{122} + \cdots$
	${P_7}^{(15)}$: $1 + 2640x^{108} + 5976x^{120} + 165888x^{121} + 109728x^{122} + \cdots$
	${P_7}^{(16)}$: $1 + 9504x^{110} + 2376x^{120} + 19008x^{121} + 143424x^{122} + \cdots$
	${P_7}^{(17)}$: $1 + 1728x^{114} + 5184x^{116} + 27648x^{117} + 6912x^{118} + 48384x^{119} + 23976x^{120} + 25920x^{121} + \cdots$
	${P_7}^{(18)}$: $1 + 9504x^{110} + 65448x^{120} + 63936x^{121} + 408672x^{122} + \cdots$
	${P_7}^{(19)}$: $1 + 9504x^{110} + 61056x^{120} + 62208x^{121} + 299808x^{122} + \cdots$
	${P_7}^{(20)}$: $1 + 1728x^{119} + 9288x^{120} + 25920x^{121} + 93312x^{122} + \cdots$
	${P_7}^{(21)}$: $1 + 432x^{116} + 3456x^{118} + 3456x^{119} + 9216x^{120} + 27648x^{121} + \cdots$
	${P_7}^{(22)}$: $1 + 1728x^{110} + 19008x^{111} + 4968x^{120} + 82944x^{121} + \cdots$
	${P_7}^{(3)}$: $1 + 20475x^{165} + 34125x^{180} + 204750x^{195} + 5838750x^{196} + \cdots$

	${P_7}^{(4)}$:	$1+20475x^{165}+34125x^{180}+217800x^{195}+27000x^{197}+\cdots$
Over $\mathbb{F}_{16}$	${P_7}^{(5)}$:	$1+20475x^{165}+34125x^{180}+233325x^{195}+16875x^{197}+\cdots$
	${P_7}^{(6)}$:	$1+20475x^{165}+34125x^{180}+205425x^{195}+3375x^{196}+\cdots$
	${P_7}^{(7)}$:	$1+20475x^{165}+34125x^{180}+206325x^{195}+20250x^{196}+\cdots$
	${P_7}^{(8)}$:	$1+20475x^{165}+34125x^{180}+207675x^{195}+3375x^{196}+\cdots$
	${P_7}^{(9)}$:	$1+20475x^{165}+34125x^{180}+213075x^{195}+6750x^{196}+\cdots$
	${P_7}^{(10)}$:	$1+6825x^{180}+94500x^{182}+307125x^{183}+7875x^{195}+1275750x^{196}+\cdots$
	${P_7}^{(11)}$:	$1+6825x^{180}+9450x^{195}+435375x^{196}+\cdots$
	${P_7}^{(12)}$:	$1+6825x^{180}+8550x^{195}+432000x^{196}+219375x^{197}+\cdots$
	${P_7}^{(13)}$:	$1+6825x^{180}+8550x^{195}+432000x^{196}+249750x^{197}+\cdots$
	${P_7}^{(14)}$:	$1+6825x^{180}+8550x^{195}+462375x^{196}+\cdots$
	${P_7}^{(15)}$:	$1+6825x^{180}+10125x^{195}+479250x^{196}+\cdots$
	${P_7}^{(16)}$:	$1+23625x^{182}+5400x^{195}+37125x^{196}+\cdots$
	${P_7}^{(17)}$:	$1+27000x^{189}+6750x^{190}+67500x^{191}+13500x^{192}+20250x^{194}+194400x^{195}+87750x^{196}+\cdots$
	${P_7}^{(18)}$:	$1+23625x^{182}+193725x^{195}+155250x^{196}+\cdots$
	${P_7}^{(19)}$:	$1+23625x^{182}+195300x^{195}+101250x^{196}+\cdots$
	${P_7}^{(20)}$:	$1+5400x^{195}+23625x^{196}+\cdots$
	${P_7}^{(21)}$:	$1+6750x^{194}+7650x^{195}+20250x^{196}+\cdots$
	${P_7}^{(22)}$:	$1+3375x^{182}+47250x^{183}+4725x^{195}+212625x^{196}+\cdots$
	${P_7}^{(3)}$:	$1+29120x^{192}+44800x^{208}+280320x^{224}+8396800x^{225}+\cdots$
	${P_7}^{(4)}$:	$1+29120x^{192}+44800x^{208}+280320x^{224}+20480x^{225}+\cdots$
	${P_7}^{(5)}$:	$1+29120x^{192}+44800x^{208}+300160x^{224}+16384x^{226}+\cdots$
	${P_7}^{(6)}$:	$1+29120x^{192}+44800x^{208}+280320x^{224}+4096x^{225}+\cdots$
	${P_7}^{(7)}$:	$1+29120x^{192}+44800x^{208}+284288x^{224}+20480x^{226}+\cdots$
	${P_7}^{(8)}$:	$1+29120x^{192}+44800x^{208}+280320x^{224}+8192x^{225}+\cdots$
	${P_7}^{(9)}$:	$1+29120x^{192}+44800x^{208}+295552x^{224}+43008x^{226}+\cdots$
	${P_7}^{(10)}$:	$1+8960x^{208}+122880x^{210}+430080x^{211}+9600x^{224}+1744896x^{225}+\cdots$
Over $\mathbb{F}_{17}$	${P_7}^{(11)}$:	$1+8960x^{208}+11520x^{224}+593920x^{225}+\cdots$
	${P_7}^{(12)}$:	$1+8960x^{208}+9600x^{224}+589824x^{225}+313344x^{226}+\cdots$
	${P_7}^{(13)}$:	$1+8960x^{208}+9600x^{224}+589824x^{225}+307200x^{226}+\cdots$
	${P_7}^{(14)}$:	$1+8960x^{208}+11648x^{224}+622592x^{225}+\cdots$
	${P_7}^{(15)}$:	$1+8960x^{208}+11648x^{224}+651264x^{225}+\cdots$
	${P_7}^{(16)}$:	$1+30720x^{210}+5760x^{224}+45056x^{225}+\cdots$
	${P_7}^{(17)}$:	$1+57344x^{219}+20480x^{220}+118784x^{221}+86016x^{222}+147072x^{224}+208896x^{225}+\cdots$
	${P_7}^{(18)}$:	$1+30720x^{210}+269952x^{224}+184320x^{225}+\cdots$
	${P_7}^{(19)}$:	$1+30720x^{210}+255488x^{224}+180224x^{225}+\cdots$
	${P_7}^{(20)}$:	$1+4096x^{223}+5760x^{224}+32768x^{225}+\cdots$
	${P_7}^{(21)}$:	$1+2048x^{222}+20480x^{224}+61440x^{225}+\cdots$
	${P_7}^{(22)}$:	$1+4096x^{210}+61440x^{211}+11904x^{224}+245760x^{225}+\cdots$
	${P_7}^{(3)}$:	$1+55080x^{252}+73440x^{270}+498474x^{288}+16347096x^{289}+\cdots$
	${P_7}^{(4)}$:	$1+55080x^{252}+73440x^{270}+501714x^{288}+11664x^{290}+\cdots$
	${P_7}^{(5)}$:	$1+55080x^{252}+73440x^{270}+501228x^{288}+17496x^{290}+\cdots$

Over $\mathbb{F}_{19}$	${P_7}^{(6)}$:	$1 + 55080x^{252} + 73440x^{270} + 498474x^{288} + 5832x^{289} + \cdots$
	${P_7}^{(7)}$:	$1 + 55080x^{252} + 73440x^{270} + 504144x^{288} + 8748x^{290} + \cdots$
	${P_7}^{(8)}$:	$1 + 55080x^{252} + 73440x^{270} + 502362x^{288} + 5832x^{289} + \cdots$
	${P_7}^{(9)}$:	$1 + 55080x^{252} + 73440x^{270} + 512892x^{288} + 23328x^{290} + \cdots$
	${P_7}^{(10)}$:	$1 + 14688x^{270} + 198288x^{272} + 793152x^{273} + 13770x^{288} + 3096792x^{289} + \cdots$
	${P_7}^{(11)}$:	$1 + 14688x^{270} + 16524x^{288} + 1049760x^{289} + \cdots$
	${P_7}^{(12)}$:	$1 + 14688x^{270} + 13770x^{288} + 1032264x^{289} + \cdots$
	${P_7}^{(13)}$:	$1 + 14688x^{270} + 13770x^{288} + 1043928x^{289} + \cdots$
	${P_7}^{(14)}$:	$1 + 14688x^{270} + 16686x^{288} + 1096416x^{289} + \cdots$
	${P_7}^{(15)}$:	$1 + 14688x^{270} + 20574x^{288} + 1131408x^{289} + \cdots$
	${P_7}^{(16)}$:	$1 + 49572x^{272} + 8262x^{288} + 64152x^{289} + \cdots$
	${P_7}^{(17)}$:	$1 + 81648x^{282} + 163296x^{284} + 151632x^{285} + 46656x^{286} + 104976x^{289} + \cdots$
	${P_7}^{(18)}$:	$1 + 49572x^{272} + 483570x^{288} + 285768x^{289} + \cdots$
	${P_7}^{(19)}$:	$1 + 49572x^{272} + 460080x^{288} + 279936x^{289} + 1851660x^{290} + \cdots$
	${P_7}^{(20)}$:	$1 + 8262x^{288} + 40824x^{289} + \cdots$
	${P_7}^{(21)}$:	$1 + 8748x^{286} + 16848x^{288} + 64152x^{289} + \cdots$
	${P_7}^{(22)}$:	$1 + 5832x^{272} + 99144x^{273} + 17010x^{288} + 384912x^{289} + \cdots$
Over $\mathbb{F}_{23}$	${P_7}^{(3)}$:	$1 + 160930x^{396} + 169400x^{418} + 1336566x^{440} + 51291416x^{441} + \cdots$
	${P_7}^{(4)}$:	$1 + 160930x^{396} + 169400x^{418} + 1336566x^{440} + 10648x^{443} + \cdots$
	${P_7}^{(5)}$:	$1 + 160930x^{396} + 169400x^{418} + 1341648x^{440} + 31944x^{444} + \cdots$
	${P_7}^{(6)}$:	$1 + 160930x^{396} + 169400x^{418} + 1336566x^{440} + 10648x^{441} + \cdots$
	${P_7}^{(7)}$:	$1 + 160930x^{396} + 169400x^{418} + 1346972x^{440} + 5324x^{442} + \cdots$
	${P_7}^{(8)}$:	$1 + 160930x^{396} + 169400x^{418} + 1336566x^{440} + 21296x^{441} + \cdots$
	${P_7}^{(9)}$:	$1 + 160930x^{396} + 169400x^{418} + 1336566x^{440} + 21296x^{442} + \cdots$
	$P_7^{(10)}$:	$1 + 33880x^{418} + 447216x^{420} + 2236080x^{421} + 25410x^{440} + 8273496x^{441} + \cdots$
	$P_7^{(11)}$:	$1 + 33880x^{418} + 30492x^{440} + 2789776x^{441} + \cdots$
	$P_7^{(12)}$:	$1 + 33880x^{418} + 25410x^{440} + 2757832x^{441} + \cdots$
	$P_7^{(13)}$:	$1 + 33880x^{418} + 25410x^{440} + 2779128x^{441} + \cdots$
	$P_7^{(14)}$:	$1 + 33880x^{418} + 30734x^{440} + 2896256x^{441} + \cdots$
	$P_7^{(15)}$:	$1 + 33880x^{418} + 30734x^{440} + 2992088x^{441} + \cdots$
	$P_7^{(16)}$:	$1 + 111804x^{420} + 15246x^{440} + 117128x^{441} + \cdots$
	$P_7^{(17)}$:	$1 + 74536x^{432} + 159720x^{434} + 510378x^{440} + 905080x^{441} + \cdots$
	$P_7^{(18)}$:	$1 + 111804x^{420} + 1308978x^{440} + 606936x^{441} + 6990412x^{442} + \cdots$
	$P_7^{(19)}$:	$1 + 111804x^{420} + 1255496x^{440} + 596288x^{441} + \cdots$
	$P_7^{(20)}$:	$1 + 15246x^{440} + 74536x^{441} + 681472x^{442} + \cdots$
	$P_7^{(21)}$:	$1 + 25652x^{440} + 63888x^{441} + 1032856x^{442} + \cdots$
	$P_7^{(22)}$:	$1 + 10648x^{420} + 223608x^{421} + 31218x^{440} + 830544x^{441} + \cdots$
	${P_7}^{(3)}$:	$1 + 255024x^{480} + 242880x^{504} + 2053440x^{528} + 84464640x^{529} + \cdots$
	${P_7}^{(4)}$:	$1 + 225024x^{480} + 242880x^{504} + 2053440x^{528} + 13824x^{529} + \cdots$

Over $\mathbb{F}_{25}$	$P_7^{(5)}$:	$1+255024x^{480}+242880x^{504}+2103264x^{528}+13824x^{529}+\cdots$
	$P_7^{(6)}$:	$1+255024x^{480}+242880x^{504}+2053440x^{528}+27648529x^{529}+\cdots$
	$P_7^{(7)}$:	$1+255024x^{480}+242880x^{504}+2066976x^{528}+20736x^{530}+\cdots$
	$P_7^{(8)}$:	$1+255024x^{480}+242880x^{504}+2062656x^{528}+13824x^{529}+\cdots$
	$P_7^{(9)}$:	$1+255024x^{480}+242880x^{504}+2087712x^{528}+6912x^{530}+\cdots$
	$P_7^{(10)}$:	$1+48576x^{504}+635904x^{506}+33120x^{528}+12690432x^{529}+\cdots$
	$P_7^{(11)}$:	$1+48576x^{504}+39744x^{528}+4271616x^{529}+\cdots$
	$P_7^{(12)}$:	$1+48576x^{504}+33120x^{528}+4243968x^{529}+\cdots$
	$P_7^{(13)}$:	$1+48576x^{504}+33120x^{528}+4257792x^{529}+\cdots$
	$P_7^{(14)}$:	$1+48576x^{504}+40032x^{528}+4423680x^{529}+\cdots$
	$P_7^{(15)}$:	$1+48576x^{504}+49248x^{528}+4534272x^{529}+\cdots$
	$P_7^{(16)}$:	$1+158976x^{506}+19872x^{528}+152064x^{529}+\cdots$
	$P_7^{(17)}$:	$1+110592x^{519}+55296x^{520}+1022112x^{528}+290304x^{529}+\cdots$
	$P_7^{(18)}$:	$1+158976x^{506}+2017440x^{528}+843264x^{529}+\cdots$
	$P_7^{(19)}$:	$1+158976x^{506}+1941120x^{528}+829440x^{529}+\cdots$
	$P_7^{(20)}$:	$1+19872x^{528}+110592x^{529}+953856x^{530}+\cdots$
	$P_7^{(21)}$:	$1+36864x^{528}+96768x^{529}+1430784x^{530}+\cdots$
	$P_7^{(22)}$:	$1+13824x^{506}+317952x^{507}+40608x^{528}+1161216x^{529}+\cdots$
Over $\mathbb{F}_{27}$	$P_7^{(3)}$:	$1+388700x^{572}+338000x^{598}+3050450x^{624}+133841240x^{625}+\cdots$
	$P_7^{(4)}$:	$1+388700x^{572}+338000x^{598}+3050450x^{624}+52728x^{630}+\cdots$
	$P_7^{(5)}$:	$1+388700x^{572}+338000x^{598}+3058900x^{624}+52728x^{628}+\cdots$
	$P_7^{(6)}$:	$1+388700x^{572}+338000x^{598}+3050450x^{624}+17576x^{625}+105456x^{630}+\cdots$
	$P_7^{(7)}$:	$1+388700x^{572}+338000x^{598}+3067688x^{624}+8788x^{626}+\cdots$
	$P_7^{(8)}$:	$1+388700x^{572}+338000x^{598}+3050450x^{624}+35152x^{625}+105456x^{630}+\cdots$
	$P_7^{(9)}$:	$1+388700x^{572}+338000x^{598}+3076476x^{624}+17576x^{626}+\cdots$
	$P_7^{(10)}$:	$1+67600x^{598}+878800x^{600}+42250x^{624}+18823896x^{625}+\cdots$
	$P_7^{(11)}$:	$1+67600x^{598}+50700x^{624}+6327360x^{625}+\cdots$
	$P_7^{(12)}$:	$1+67600x^{598}+42250x^{624}+6276632x^{625}+\cdots$
	$P_7^{(13)}$:	$1+67600x^{598}+42250x^{624}+6309748x^{625}+\cdots$
	$P_7^{(14)}$:	$1+67600x^{598}+51038x^{624}+6538272x^{625}+\cdots$
	$P_7^{(15)}$:	$1+67600x^{598}+51038x^{624}+6714032x^{625}+\cdots$
	$P_7^{(16)}$:	$1+219700x^{600}+25350x^{624}+193336x^{625}+\cdots$
	$P_7^{(17)}$:	$1+615160x^{617}+105456x^{618}+139594x^{624}+1986088x^{625}+\cdots$
	$P_7^{(18)}$:	$1+219700x^{600}+3004482x^{624}+1142440x^{625}+\cdots$
	$P_7^{(19)}$:	$1+219700x^{600}+2898688x^{624}+1124864x^{625}+\cdots$
	$P_7^{(20)}$:	$1+25350x^{624}+123032x^{625}+\cdots$
	$P_7^{(21)}$:	$1+33800x^{624}+123032x^{625}+\cdots$
	$P_7^{(22)}$:	$1+17576x^{600}+439400x^{601}+51714x^{624}+1581840x^{625}+\cdots$

Over $\mathbb{F}_{29}$	$P_7^{(3)}$:	$1 + 573300x^{672} + 458640x^{700} + 4402944x^{728} + 205207296x^{729} + \cdots$
	$P_7^{(4)}$:	$1 + 573300x^{672} + 458640x^{700} + 4402944x^{728} + 21952x^{733} + \cdots$
	$P_7^{(5)}$:	$1 + 573300x^{672} + 458640x^{700} + 4413528x^{728} + 43904x^{734} + \cdots$
	$P_7^{(6)}$:	$1 + 573300x^{672} + 458640x^{700} + 4402944x^{728} + 21952x^{729} + \cdots$
	$P_7^{(7)}$:	$1 + 573300x^{672} + 458640x^{700} + 4424504x^{728} + 10976x^{730} + \cdots$
	$P_7^{(8)}$:	$1 + 573300x^{672} + 458640x^{700} + 4402944x^{728} + 43904x^{729} + \cdots$
	$P_7^{(9)}$:	$1 + 573300x^{672} + 458640x^{700} + 440968x^{728} + 10976x^{730} + \cdots$
	$P_7^{(10)}$:	$1 + 91728x^{700} + 1185408x^{702} + 52920x^{728} + 27132672x^{729} + \cdots$
	$P_7^{(11)}$:	$1 + 91728x^{700} + 63504x^{728} + 9110080x^{729} + \cdots$
	$P_7^{(12)}$:	$1 + 91728x^{700} + 52920x^{728} + 9044224x^{729} + \cdots$
	$P_7^{(13)}$:	$1 + 91728x^{700} + 52920x^{728} + 9088128x^{729} + \cdots$
	$P_7^{(14)}$:	$1 + 91728x^{700} + 63896x^{728} + 9395456x^{729} + \cdots$
	$P_7^{(15)}$:	$1 + 91728x^{700} + 63896x^{728} + 9636928x^{729} + \cdots$
	$P_7^{(16)}$:	$1 + 296352x^{702} + 31752x^{728} + 241472x^{729} + \cdots$
	$P_7^{(17)}$:	$1 + 439040x^{720} + 790272x^{722} + 1315944x^{728} + 2392768x^{729} + \cdots$
	$P_7^{(18)}$:	$1 + 296352x^{702} + 4345320x^{728} + 1514688x^{729} + \cdots$
	$P_7^{(19)}$:	$1 + 91728x^{700} + 1185408x^{702} + 52920x^{728} + 27132672x^{729} + \cdots$
	$P_7^{(20)}$:	$1 + 31752x^{728} + 153664x^{729} + \cdots$
	$P_7^{(21)}$:	$1 + 42336x^{728} + 131712x^{729} + \cdots$
	$P_7^{(22)}$:	$1 + 21952x^{702} + 592704x^{703} + 64680x^{728} + 2107392x^{729} + \cdots$
Over $\mathbb{F}_{31}$	$P_7^{(3)}$:	$1 + 822150x^{780} + 609000x^{810} + 6198750x^{840} + 305775000x^{841} + \cdots$
	$P_7^{(4)}$:	$1 + 822150x^{780} + 609000x^{810} + 6198750x^{840} + 135000x^{848} + \cdots$
	$P_7^{(5)}$:	$1 + 822150x^{780} + 609000x^{810} + 6211800x^{840} + 243000x^{848} + \cdots$
	$P_7^{(6)}$:	$1 + 822150x^{780} + 609000x^{810} + 6198750x^{840} + 27000x^{841} + \cdots$
	$P_7^{(7)}$:	$1 + 822150x^{780} + 609000x^{810} + 6225300x^{840} + 135000x^{842} + \cdots$
	$P_7^{(8)}$:	$1 + 822150x^{780} + 609000x^{810} + 6216750x^{840} + 27000x^{841} + \cdots$
	$P_7^{(9)}$:	$1 + 822150x^{780} + 609000x^{810} + 6238800x^{840} + 27000x^{842} + \cdots$
	$P_7^{(10)}$:	$1 + 121800x^{810} + 1566000x^{812} + 65250x^{840} + 38151000x^{841} + \cdots$
	$P_7^{(11)}$:	$1 + 121800x^{810} + 78300x^{840} + 12798000x^{841} + 5872500x^{842} + \cdots$
	$P_7^{(12)}$:	$1 + 121800x^{810} + 65250x^{840} + 12717000x^{841} + 3132000x^{842} + \cdots$
	$P_7^{(13)}$:	$1 + 121800x^{810} + 65250x^{840} + 12771000x^{841} + 3915000x^{842} + \cdots$

	$P_7^{(14)}$: $1+121800x^{810}+78750x^{840}+13176000x^{841}+3955500x^{842}+\cdots$
	$P_7^{(15)}$: $1+121800x^{810}+96750x^{840}+13446000x^{841}+\cdots$
	$P_7^{(16)}$: $1+391500x^{812}+39150x^{840}+297000x^{841}+5481000x^{842}+\cdots$
	$P_7^{(17)}$: $1+540000x^{831}+54000x^{832}+862650x^{840}+648000x^{841}+\cdots$
	$P_7^{(18)}$: $1+391500x^{812}+6127650x^{840}+1971000x^{841}+30685500x^{842}+\cdots$
	$P_7^{(19)}$: $1+391500x^{812}+59338200x^{840}+1944000x^{841}+19264500x^{842}+\cdots$
	$P_7^{(20)}$: $1+39150x^{840}+189000x^{841}+2349000x^{842}+\cdots$
	$P_7^{(21)}$: $1+52200x^{840}+162000x^{841}+3523500x^{842}+\cdots$
	$P_7^{(22)}$: $1+27000x^{812}+783000x^{813}+79650x^{840}+2754000x^{841}+\cdots$
Over $\mathbb{F}_{32}$	$P_7^{(3)}$: $1+975415x^{837}+696725x^{868}+7293990x^{899}+369706310x^{900}+\cdots$
	$P_7^{(4)}$: $1+975415x^{837}+696725x^{868}+7293990x^{899}+1042685x^{909}+\cdots$
	$P_7^{(5)}$: $1+975415x^{837}+696725x^{868}+7308405x^{899}+148955x^{905}+\cdots$
	$P_7^{(6)}$: $1+975415x^{837}+696725x^{868}+7293990x^{899}+29791x^{900}+\cdots$
	$P_7^{(7)}$: $1+975415x^{837}+696725x^{868}+7308405x^{899}+29791x^{900}+\cdots$
	$P_7^{(8)}$: $1+975415x^{837}+696725x^{868}+7293990x^{899}+59582x^{900}+\cdots$
	$P_7^{(9)}$: $1+975415x^{837}+696725x^{868}+7308405x^{899}+59582x^{900}+\cdots$
	$P_7^{(10)}$: $1+139345x^{868}+72075x^{899}+44865246x^{900}+\cdots$
	$P_7^{(11)}$: $1+139345x^{868}+86490x^{899}+15044455x^{900}+\cdots$
	$P_7^{(12)}$: $1+139345x^{868}+72075x^{899}+14955082x^{900}+\cdots$
	$P_7^{(13)}$: $1+139345x^{868}+72075x^{899}+15014664x^{900}+\cdots$
	$P_7^{(14)}$: $1+139345x^{868}+72075x^{899}+15521111x^{900}+\cdots$
	$P_7^{(15)}$: $1+139345x^{868}+72075x^{899}+15878603x^{900}+\cdots$
	$P_7^{(16)}$: $1+446865x^{870}+43245x^{899}+327701x^{900}+\cdots$
	$P_7^{(17)}$: $1+1638505x^{891}+148955x^{892}+5256670x^{899}+1281013x^{900}+\cdots$
	$P_7^{(18)}$: $1+446865x^{870}+7193085x^{899}+2323698x^{900}+\cdots$
	$P_7^{(19)}$: $1+446865x^{870}+7207500x^{899}+1370386x^{900}+\cdots$
	$P_7^{(20)}$: $1+43245x^{899}+208537x^{900}+\cdots$
	$P_7^{(21)}$: $1+57660x^{899}+208537x^{900}+\cdots$
	$P_7^{(22)}$: $1+29791x^{870}+893730x^{871}+43245x^{899}+3306801x^{900}+\cdots$
Over $\mathbb{F}_{37}$	$P_7^{(3)}$: $1+2120580x^{1152}+1285200x^{1188}+15331680x^{1224}+880865280x^{1225}+\cdots$
	$P_7^{(4)}$: $1+2120580x^{1152}+1285200x^{1188}+15336864x^{1224}+46656x^{1230}+\cdots$
	$P_7^{(5)}$: $1+2120580x^{1152}+1285200x^{1188}+15354360x^{1224}+279936x^{1234}+\cdots$
	$P_7^{(6)}$: $1+2120580x^{1152}+1285200x^{1188}+15331680x^{1224}+46656x^{1225}+\cdots$

	$P_7^{(7)}$:	$1 + 2120580x^{1152} + 1285200x^{1188} + 15377668x^{1244} + 23328x^{1226} + \cdots$
	$P_7^{(8)}$:	$1 + 2120580x^{1152} + 1285200x^{1188} + 15362784x^{1224} + 46656x^{1225} + \cdots$
	$P_7^{(9)}$:	$1 + 2120580x^{1152} + 1285200x^{1188} + 15412680x^{1224} + 23328x^{1226} + \cdots$
	$P_7^{(10)}$:	$1 + 257040x^{1188} + 3265920x^{1190} + 113400x^{1224} + 94058496x^{1225} + \cdots$
	$P_7^{(11)}$:	$1 + 257040x^{1188} + 136080x^{1224} + 31492800x^{1225} + \cdots$
	$P_7^{(12)}$:	$1 + 257040x^{1188} + 113400x^{1224} + 31352832x^{1225} + \cdots$
	$P_7^{(13)}$:	$1 + 257040x^{1188} + 113400x^{1224} + 31446144x^{1225} + \cdots$
	$P_7^{(14)}$:	$1 + 257040x^{1188} + 136728x^{1224} + 32285952x^{1225} + \cdots$
	$P_7^{(15)}$:	$1 + 257040x^{1188} + 167832x^{1224} + 32845824x^{1225} + \cdots$
	$P_7^{(16)}$:	$1 + 816480x^{1190} + 68040x^{1224} + 513216x^{1225} + \cdots$
	$P_7^{(17)}$:	$1 + 1959552x^{1215} + 279936x^{1216} + 2050920x^{1224} + 1259712x^{1225} + \cdots$
	$P_7^{(18)}$:	$1 + 816480x^{1190} + 15207912x^{1224} + 3965760x^{1225} + \cdots$
	$P_7^{(19)}$:	$1 + 816480x^{1190} + 14810688x^{1224} + 3919104x^{1225} + \cdots$
	$P_7^{(20)}$:	$1 + 68040x^{1224} + 326592x^{1225} + \cdots$
	$P_7^{(21)}$:	$1 + 90720x^{1224} + 279936x^{1225} + \cdots$
	$P_7^{(22)}$:	$1 + 46656x^{1190} + 138024x^{1224} + 5598720x^{1225} + \cdots$
Over $\mathbb{F}_{41}$	$P_7^{(3)}$:	$1 + 3655600x^{1440} + 1976000x^{1480} + 25896000x^{1520} + 1627520000x^{1521} + \cdots$
	$P_7^{(4)}$:	$1 + 3655600x^{1440} + 1976000x^{1480} + 25896000x^{1520} + 64000x^{1532} + \cdots$
	$P_7^{(5)}$:	$1 + 3655600x^{1440} + 1976000x^{1480} + 25927200x^{1520} + 448000x^{1532} + \cdots$
	$P_7^{(6)}$:	$1 + 3655600x^{1440} + 1976000x^{1480} + 25896000x^{1520} + 64000x^{1521} + \cdots$
	$P_7^{(7)}$:	$1 + 3655600x^{1440} + 1976000x^{1480} + 25959200x^{1520} + 32000x^{1522} + \cdots$
	$P_7^{(8)}$:	$1 + 3655600x^{1440} + 1976000x^{1480} + 25896000x^{1520} + 128000x^{1521} + \cdots$
	$P_7^{(9)}$:	$1 + 3655600x^{1440} + 1976000x^{1480} + 26007200x^{1520} + 32000x^{1522} + \cdots$
	$P_7^{(10)}$:	$1 + 395200x^{1480} + 4992000x^{1482} + 156000x^{1520} + 158592000x^{1521} + \cdots$
	$P_7^{(11)}$:	$1 + 395200x^{1480} + 187200x^{1520} + 53056000x^{1521} + \cdots$
	$P_7^{(12)}$:	$1 + 395200x^{1480} + 156000x^{1520} + 52864000x^{1521} + \cdots$
	$P_7^{(13)}$:	$1 + 395200x^{1480} + 156000x^{1520} + 52992000x^{1521} + \cdots$
	$P_7^{(14)}$:	$1 + 395200x^{1480} + 188000x^{1520} + 54272000x^{1521} + \cdots$
	$P_7^{(15)}$:	$1 + 395200x^{1480} + 188000x^{1520} + 55232000x^{1521} + \cdots$
	$P_7^{(16)}$:	$1 + 1248000x^{1482} + 93600x^{1520} + 704000x^{1521} + \cdots$
	$P_7^{(17)}$:	$1 + 1792000x^{1509} + 128000x^{1510} + 6717600x^{1520} + 12224000x^{1521} + \cdots$

	$P_7^{(18)}$:	$1 + 1248000x^{1482} + 25725600x^{1520} + 5952000x^{1521} + \cdots$
	$P_7^{(19)}$:	$1 + 1248000x^{1482} + 25116800x^{1520} + 5888000x^{1521} + \cdots$
	$P_7^{(20)}$:	$1 + 93600x^{1520} + 448000x^{1521} + \cdots$
	$P_7^{(21)}$:	$1 + 124800x^{1520} + 384000x^{1521} + \cdots$
	$P_7^{(22)}$:	$1 + 64000x^{1482} + 2496000x^{1483} + 189600x^{1520} + 8448000x^{1521} + \cdots$
Over $\mathbb{F}_{43}$	$P_7^{(4)}$:	$1 + 4701060x^{1596} + 2410800x^{1638} + 33015906x^{1680} + 148176x^{1692} + \cdots$
	$P_7^{(5)}$:	$1 + 4701060x^{1596} + 2410800x^{1638} + 33052068^{1680} + 889056x^{1694} + \cdots$
	$P_7^{(6)}$:	$1 + 4701060x^{1596} + 2410800x^{1638} + 33015906x^{1680} + 74088x^{1681} + \cdots$
	$P_7^{(7)}$:	$1 + 4701060x^{1596} + 2410800x^{1638} + 33089112x^{1680} + 37044x^{1682} + \cdots$
	$P_7^{(8)}$:	$1 + 4701060x^{1596} + 2410800x^{1638} + 33065298x^{1680} + 74088x^{1681} + \cdots$
	$P_7^{(9)}$:	$1 + 4701060x^{1596} + 2410800x^{1638} + 33126156x^{1680} + 74088x^{1682} + \cdots$
	$P_7^{(12)}$:	$1 + 482160x^{1638} + 180810x^{1680} + 67345992x^{1681} + \cdots$
	$P_7^{(20)}$:	$1 + 108486x^{1680} + 518616x^{1681} + 9112824x^{1682} + \cdots$
	$P_7^{(21)}$:	$1 + 144648x^{1680} + 444528x^{1681} + \cdots$
Over $\mathbb{F}_{47}$	$P_7^{(4)}$:	$1 + 7506510x^{1932} + 3491400x^{1978} + 51942510x^{2024} + 97336x^{2038} + \cdots$
	$P_7^{(12)}$:	$1 + 698280x^{1978} + 238050x^{2024} + 105804232x^{2025} + 17520480x^{2026} + \cdots$
	$P_7^{(20)}$:	$1 + 142830x^{2024} + 681352x^{2025} + 13140360x^{2026} + 97336x^{2038} + \cdots$
	$P_7^{(21)}$:	$1 + 190440x^{2024} + 584016x^{2025} + 19710540x^{2026} + \cdots$
Over $\mathbb{F}_{49}$	$P_7^{(4)}$:	$1 + 830208x^{2160} + 270720x^{2208} + 130719744x^{2209} + \cdots$
Over $\mathbb{F}_{53}$	$P_7^{(4)}$:	$1 + 14077700x^{2496} + 5746000x^{2548} + 95705376x^{2600} + 562432x^{2618} + \cdots$
Over $\mathbb{F}_{59}$	$P_7^{(4)}$:	$1 + 24607660x^{3132} + 8948240x^{3190} + 164999154x^{3248} + 585336x^{3270} + \cdots$
Over $\mathbb{F}_{61}$	$P_7^{(4)}$:	$1 + 29258100x^{3360} + 10266000x^{3420} + 195408000x^{3480} + 648000x^{3503} + \cdots$
Over $\mathbb{F}_{64}$	$P_7^{(4)}$:	$1 + 37526895x^{3717} + 12508965x^{3780} + 249304797x^{3843} + 250047x^{3868} + \cdots$
Over $\mathbb{F}_{67}$	$P_7^{(4)}$:	$1 + 47567520x^{4092} + 15100800x^{4158} + 314426970x^{4224} + 287496x^{4249} + \cdots$
Over $\mathbb{F}_{71}$	$P_7^{(4)}$:	$1 + 64182650x^{4620} + 19159000x^{4690} + 421779750x^{4760} + 34300x^{4788} + \cdots$

Appendix B. Magma program of Proposition 3.4

```
> cart_prod_lists:=function(R)
> N:=#R;
> if N eq 1 then return R[1]; end if;
> if N gt 1 then
> L:=[];
> R0:=[R[k]:k in [1..(N-1)]];
> R1:=$$(R0);
> for i in R1 do
> for j in R[N] do
> if N eq 2 then L:=Append(L,Append([i],j)); end if;
> if N gt 2 then L:=Append(L,Append(i,j)); end if;
> end for;
> end for;
> return L;
> end if;
> end function;

> toric_points:=function(n,F)
> T:=[x : x in F | x ne 0];
> L:=cart_prod_lists([T:i in [1..n]]);
> return L;
> end function;

> toric_code:=function(L,F)
> u:=L;
> gens:=[];
> d:=#L[1];
> n:=#toric_points(d,F);
> V:=VectorSpace(F,n);
> Z:=Integers();
> for v in u do
> Append(~gens,[Z!v[i]: i in [1..d]]);
> end for;
> B0:=;
> for e in gens do
> Include(~B0, V![&*[t[i]^e[i]:i in [1..d]]:t
        in toric_points(d,F)]);
> end for;
```

```
> B:=SetToSequence(B0);
> C1:=VectorSpaceWithBasis(B);
> C:=LinearCode(C1);
> return C;
> end function;

> C1:=toric_code([[-1,-1],[0,1],[0,0],[1,0],
                  [2,0],[3,0],[4,0]],GF(7));
> C2:=toric_code([[0,0],[1,0],[0,1],[1,1],
                  [2,2],[3,3],[-1,-1]],GF(7));

> IsEquivalent(C1,C2);
```

References

[AP] Y. Aubry and M. Perret, *A Weil theorem for singular curves*, in: R. Pellikaan, M. Perret, S. G. Vladut (eds.) Arithmetic, and Coding Theory. pp. 1–7, de Gruyter, Berlin (1996).

[BK] G. Brown and A. M. Kasprzyk, *Seven new champion linear codes*, LMS J. Computation and Mathematics. **16** (2013), 109–117.

[BP] P. Beelen and R. Pellikaan, *The Newton polygon of plane curves with many rational points*, Des. Codes Cryptogr. **21** (2000), 41–67.

[DGV] P. Diaz, C. Guevara, and M. Vath, *Codes from n-dimensional polyhedra and n-dimensional cyclic codes*, Proceedings of SIMU Summer Institute (2001).

[F] W. Fulton, Introduction to Toric Varieties, Princeton Univ. Press, Princeton (1993).

[H1] J. P. Hansen, *Toric surfaces and error-correcting codes*, in: Coding Theory, Cryptography and Related Areas (Guanajuato, 1998). pp. 132–142, Berlin: Springer (2000).

[H2] J. P. Hansen, *Toric varities Hirzebruch surfaces and error-correcting codes*, Applicable Algebra in Engineering, Communication and Computing **13** (2002), 289–300.

[Jo] D. Joyner, *Toric codes over finite fields*, Applicable Algebra in Engineering, Communication and Computing **15** (2004), 63–79.

[LYZZ] X. Luo, S. S.-T. Yau, M.-Y. Zhang, and H.-Q. Zuo, *On classification of toric surface codes of low dimension*, Finite Fields and Their Applications **33** (2015), 90–102.

[LS1] J. Little and H. Schenck, *Toric surface codes and minkowski sums*, SIAM J. Discrete Math. **20** (2006), 999–1014.

[LS2] J. Little and R. Schwarz, *On toric codes and multivariate Vandermonde matrices*, Applicable Algebra in Engineering, Communication and Computing **18** (2007), 349–367.

[Ru] D. Ruano, *On the parameters of r-dimensional toric codes*, Finite Fields and Their Applications. **13** (2007), no. 4, 962–976.

[SS1] I. Soprunov and J. Soprunova, *Toric surface codes and Minkowski length of polygons*, SIAM J. Discrete Math. **23** (2009), 384–400.

[SS2] I. Soprunov and J. Soprunova, *Bring toric codes to the next dimension*, SIAM J. Discrete Math. **24** (2010), no. 2, 655–665.

[TV] M. A. Tsfasman and S. G. Vlăduţ, Algebraic-Geometry Codes. Kluwer, Dordrecht, The Netherlands (1991).

[YZ] S. S.-T. Yau and H.-Q. Zuo, *Notes on classification of toric surface codes of dimension* 5, Applicable Algebra in Engineering, Communication and Computing **20** (2009), 175–185.

[Z] G. Ziegler, Lectures on Polytopes, Springer, Berlin (1995).

[ZZL] H.-Q. Zuo, P. Zhang, and X. Luo, *Parameters of two special toric surface codes (Chinese)*, Chinese Ann. Math. Ser. A **31** (2010), no. 5, 517–524. Translation in Chinese J. Contemp. Math. **31** (2010), no. 4, 319–328.

Department of Mathematical Sciences
Tsinghua University, Beijing 100084, P. R. China
Current address:
School of Data Sciences, Huashang College Guangdong University of Finance and Economics
Guangzhou Guangdong, 511300, P. R. China
E-mail address: naveed1573@gmail.com

School of Mathematical Sciences
Beihang University (Shahe campus), Beijing 102206, P. R. China
E-mail address: xluo@buaa.edu.cn

Department of Mathematical Sciences
Tsinghua University, Beijing 100084, P. R. China
E-mail address: yau@uic.edu

Department of Mathematics, Northwestern University
Evanston, IL 60208, USA
E-mail address: mingyi@math.northwestern.edu

Department of Mathematical Sciences
Tsinghua University, Beijing 100084, P. R. China
E-mail address: hqzuo@mail.tsinghua.edu.cn

Received May 14, 2017
Accepted November 2, 2017