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# On LMO invariant constraints for cosmetic surgery and other surgery problems for knots in $S^3$

Tetsuya Ito

We use the LMO invariant to find constraints for a knot to admit a purely or reflectively cosmetic surgery. We also get a constraint for knots to admit a lens space surgery, and some information about characterizing slopes.

## 1. Introduction

For a knot $K$ in $S^3$ and $r = p/q \in \mathbb{Q} \cup \{\infty\}$, let $S^3(K, r)$ be the oriented closed 3-manifold obtained by Dehn surgery on $K$ along the slope $r$. We denote the 3-manifold $M$ with opposite orientation by $-M$, and we write $M \cong M'$ if two 3-manifolds are homeomorphic by an orientation preserving homeomorphism.

Two Dehn surgeries along a knot $K$ with different slopes $r$ and $r'$ are *purely cosmetic* if $S^3(K, r) \cong S^3(K, r')$ and *reflectively cosmetic* (or, *chirally cosmetic*) if $S^3(K, r) \cong -S^3(K, r')$.

Two slopes are called *equivalent* if there exists a homeomorphism (which may reverse the orientation) of $S^3 \setminus K$ that sends one slope to the other. A famous cosmetic surgery conjecture (for a knot in $S^3$) [Kir, Problem 1.81] states that the Dehn surgeries along inequivalent slopes are never purely cosmetic. By the knot complement theorem [GoLu], for a non-trivial knot $K$ two different slopes $r$ and $r'$ are always inequivalent unless $K$ is amphicheiral and $r = -r'$. For an amphicheiral knot $K$, $S^3(K, r) \cong -S^3(K, -r) \not\cong S^3(K, -r)$. Therefore for a knot in $S^3$ the cosmetic surgery conjecture can be rephrased that a non-trivial knot does not admit a purely cosmetic surgery.

There are various constraints for two Dehn surgeries to be purely cosmetic. Among them, using Heegaard Floer homology theory in [NiWu, Theorem 1.2] the following strong restrictions are shown.

$$\text{(1.1)} \qquad \begin{array}{l} \text{If } S^3(K,p/q) \cong S^3(K,p'/q'), \\ \text{then } p'/q' = \pm p/q \text{ and } q^2 \equiv -1 \pmod{p}. \end{array}$$

The cosmetic surgery conjecture can be regarded as a statement saying that when we fix a knot $K$ then Dehn surgery gives an injective map $S^3(K,*) : \{\text{Slopes}\} = \mathbb{Q} \cup \{\infty\} \to \{\text{(oriented) 3-manifolds}\}$ except in the case where $K$ is the unknot. From this point of view, it is natural to ask about the injectivity of the Dehn surgery map when we fix a slope; Is the Dehn surgery map $S^3(*,r) : \{\text{Knots}\} \to \{\text{(oriented) 3-manifolds}\}$ injective ? A slope $r$ is a called a *characterizing slope* of $K$ if near $K$ the answer is affirmative, meaning that $S^3(K,r) \cong S^3(K',r)$ implies $K = K'$.

Compared with purely cosmetic surgeries, the situation for characterizing slopes is more complicated. There are various examples of non-characterizing slopes. Among them, in [BaMo] hyperbolic knots with infinitely many (integral) non-characterizing slopes are given. On the other hand, if $K$ is the unknot [KMOS, OzSa2], the trefoil, or the figure-eight knot [OzSa1] then all the slopes are characterizing. Moreover, if $K$ is a torus knot, a slope $r$ is characterizing provided $r$ is sufficiently large [NiZh, Theorem 1.3], whereas some small slopes are not characterizing.

In this paper we use the LMO invariant to study Dehn surgery along knots. We obtain various constraints for a knot to admit a purely or reflectively cosmetic surgery, or, a slope $r$ to be characterizing.

The LMO invariant is an invariant of closed oriented 3-manifolds which takes values in a certain graded algebra $\mathcal{A}(\emptyset)$. In this paper, we restrict our attention to the case where $M$ is a rational homology sphere, and we use a normalization as in [BGRT3] for the LMO invariants of rational homology spheres – see section 2). For rational homology spheres, the degree one part $\lambda_1$ of the LMO invariant, under the normalization as in [BGRT3], is $\frac{1}{4}\lambda_{CW}$ [LMMO], where $\lambda_{CW}$ denotes the Casson-Walker invariant with Walker's normalization [Wal]. By the surgery formula of the Casson-Walker invariant [BoLi, Wal] we have

$$\text{(1.2)} \qquad \lambda_1(S^3(K,p/q)) = a_2(K)\frac{q}{2p} + \lambda_1(L(p,q)).$$

Here $a_2(K)$ denotes the coefficient of $z^2$ in the Conway polynomial, and $L(p,q) = S^3(\mathsf{Unknot}, p/q)$ denotes the $(p,q)$-lens space.

Using (1.1) and the surgery formula (1.2) we immediately get the following constraint for cosmetic surgery and characterizing slopes. (In [BoLi], this is proved without using (1.1) – instead they used the Casson-Gordon invariant to get an additional constraint.)

**Theorem 1.1.** *[BoLi, Proposition 5.1] Let $K$ and $K'$ be knots in $S^3$ and $r, r' \in \mathbb{Q} \setminus \{0\}$ with $r \neq r'$.*

(i) *If $S^3(K,r) \cong S^3(K',r)$ then $a_2(K) = a_2(K')$.*

(ii) *If $S^3(K,r) \cong S^3(K,r')$ then $a_2(K) = 0$ $(= a_2(\mathsf{Unknot}))$.*

Our purpose is to get further constraints that generalize Theorem 1.1 by looking at higher order parts of the LMO invariants.

Two knots $K$ and $K'$ are called $C_{n+1}$*-equivalent* if $v(K) = v(K')$ for all finite type invariant $v$ whose degree is less than or equal to $n$. A knot which is $C_{n+1}$ equivalent to the unknot is called a $C_{n+1}$*-trivial* knot. In [Gou, Hab] it is shown that two knots are $C_{n+1}$-equivalent if and only if they are related by a sequence of certain local moves, called $C_{n+1}$*-moves*.

In this terminology and knowing that $a_2$ is the only finite type invariant of degree 2 up to constant multiple, Theorem 1.1 can be understood to say that Dehn surgery characterizes a knot or a slope *up to* $C_3$*-equivalence*: (i) says that if Dehn surgeries on two knots $K$ and $K'$ along the same slope are homeomorphic then $K$ and $K'$ are $C_3$-equivalent, and (ii) says that the cosmetic surgery conjecture is true unless $K$ is $C_3$-trivial.

In [BaL] Bar-Natan and Lawrence gave a rational surgery formula for the LMO invariant. First we write down a rational surgery formula for the degree two and three parts of the (primitive) LMO invariants of $S^3(K,r)$.

**Theorem 1.2 (Surgery formula for $\lambda_2$ and $\lambda_3$).** *Let $K$ be a knot in $S^3$. We have*

$$\begin{aligned}
\lambda_2(S^3(K,p/q)) &= \left(v_2(K)^2 + \frac{1}{24}v_2(K) + \frac{5}{2}v_4(K)\right)\frac{q^2}{p^2} - v_3(K)\frac{q}{p} \\
&\quad + \frac{v_2(K)}{24}\left(\frac{1}{p^2} - 1\right) + \lambda_2(L(p,q)) \\
&= \left(\frac{7a_2(K)^2 - a_2(K) - 10a_4(K)}{8}\right)\frac{q^2}{p^2} - v_3(K)\frac{q}{p} \\
&\quad + \frac{a_2(K)}{48}\left(1 - \frac{1}{p^2}\right) + \lambda_2(L(p,q))
\end{aligned}$$

$$\begin{aligned}
&\lambda_3(S^3(K,p/q))\\
&\quad = -\left(\frac{35}{4}v_6(K)+\frac{5}{24}v_4(K)+10v_2(K)v_4(K)+\frac{4}{3}v_2(K)^3+\frac{1}{12}v_2(K)^2\right)\frac{q^3}{p^3}\\
&\qquad -\left(\frac{5}{24}v_4(K)+\frac{1}{288}v_2(K)+\frac{1}{12}v_2(K)^2\right)\frac{q}{p^3}\\
&\qquad +\left(\frac{5}{2}v_5(K)+2v_3(K)v_2(K)+\frac{1}{24}v_3(K)\right)\frac{q^2}{p^2}+\frac{v_3(K)}{24}\left(\frac{1}{p^2}-1\right)\\
&\qquad -\left(w_4(K)-\frac{1}{12}v_2(K)^2-\frac{1}{288}v_2(K)-\frac{5}{24}v_4(K)\right)\frac{q}{p}+\lambda_3(L(p,q))
\end{aligned}$$

The formula for $\lambda_2(L(p,q))$ and $\lambda_3(L(p,q))$ will be given (2.3). In particular $\lambda_3(L(p,q))=0$.

Here $v_2(K), v_3(K), v_4(K), w_4(K), v_5(K)$ and $v_6(K)$ are certain finite type invariants of the knot $K$ (see Section 2 for details – as we will see in Lemma 2.1, except $v_5$ they are determined by the Alexander and the Jones polynomials). Also, $a_{2n}(K)$ is the coefficient of $z^{2n}$ in $\nabla_K(z)$, the Conway polynomial of $K$.

The degree two part of the LMO invariant (combined with (1.1)) gives rise to the following.

**Corollary 1.3.** *Let $K$ and $K'$ be knots in $S^3$, and $r, r' \in \mathbb{Q}\setminus\{0\}$ with $r \neq r'$.*

(i) *If $S^3(K,r)\cong S^3(K,r')$ then $v_3(K)=0$.*

(ii) *If $S^3(K,r)\cong -S^3(K,-r)$ then $v_3(K)=0$.*

(iii) *If $S^3(K,r)\cong -S^3(K,r')$ for $r'\neq \pm r$ then either*
 (iii-a) *$v_3(K)=0$, or,*
 (iii-b) *$v_3(K)\neq 0$ and $\dfrac{rr'}{r+r'}=\dfrac{7a_2(K)^2-a_2(K)-10a_4(K)}{8v_3(K)}$.*

(iv) *If $S^3(K,r)\cong S^3(K',r)$ then either*
 (iv-a) *$a_4(K)=a_4(K')$, $v_3(K)=v_3(K')$, or,*
 (iv-b) *$a_4(K)\neq a_4(K')$, $v_3(K)\neq v_3(K')$, and $r=\dfrac{5(a_4(K)-a_4(K'))}{4(v_3(K)-v_3(K'))}$.*

(i) was proven in [IcWu] by a similar argument using Lescop's surgery formula for the Kontsevich-Kuperberg-Thurston invariant [Ko2, KuTh] (see Remark 3.2).

We note that the degree two part gives the following constraint for a knot to admit a lens space surgery.

**Corollary 1.4.** *If $S^3(K,p/q)$ is a lens space, then*

$$\left(\frac{7a_2(K)^2 - a_2(K) - 10a_4(K)}{8}\right)\frac{q^2}{p^2} - v_3(K)\frac{q}{p} + \frac{a_2(K)}{48}\left(1 - \frac{1}{p^2}\right) = 0.$$

By the cyclic surgery theorem [CGLS], if $K$ is not a torus knot, then $q = 1$ hence we get

$$a_2(K)p^2 - 48v_3(K)p + \big(42a_2(K)^2 - 7a_2(K) - 60a_4(K)\big) = 0. \tag{1.3}$$

Combined with the fact that $a_2(K), 4v_3(K)$ and $a_4(K)$ are integers, (1.3) brings some interesting information. For example, the integer $p$ must be a solution of the quadratic equation (1.3) so $576v_3(K)^2 - a_2(K)(42a_2(K)^2 - 7a_2(K) - 60a_4(K))$ is a perfect square. If a non-torus knot $K$ admits more than one lens space surgeries, the surgery slopes are successive integers [CGLS, Corollary 1] so such a knot has $a_2(K) \neq \pm 1$.

The formula for the degree three part is more complicated. Fortunately, as for cosmetic surgery, using (1.1) we get the following simple constraints.

**Corollary 1.5.** *Let $K$ and $K'$ be knots in $S^3$ and $r = p/q \in \mathbb{Q} \setminus \{0\}$.*

(i) *If $S^3(K,r) \cong S^3(K,r')$ for $r' \neq r$, then*

$$p^2(24w_4(K) - 5v_4(K)) + 5v_4(K) + q^2(210v_6(K) + 5v_4(K)) = 0.$$

(ii) *If $S^3(K,r) \cong -S^3(K,-r)$, then $v_5(K) = 0$.*

In [IcWu], the cosmetic surgery conjecture is confirmed for all prime knots with less than or equal to 11 crossings, with eight exceptions. Corollary 1.5 (i) shows that seven of them do not admit purely cosmetic surgery so we conclude the following.

**Corollary 1.6.** *The cosmetic surgery conjecture is true for all prime knots with less than or equal to 11 crossings, possibly except $10_{118}$.*

Using the higher degree parts of the LMO invariant, adding suitable $C_n$-equivalence assumptions we prove the following more direct generalizations of Theorem 1.1.

**Theorem 1.7.** *Let $K$ and $K'$ be knots in $S^3$ and $r, r' \in \mathbb{Q} \setminus \{0\}$ with $r \neq r'$.*

(i) *Assume that $K$ and $K'$ are $C_{2m+2}$-equivalent. If $S^3(K,r) \cong S^3(K',r)$ then $a_{2m+2}(K) = a_{2m+2}(K')$.*

(ii) *Assume that $K$ is $C_{4m+2}$-trivial. If $S^3(K,r) \cong S^3(K,r')$ then $a_{4m+2}(K) = 0$.*

We say that a (complex-valued) finite type invariant $v$ is *of odd type* if $v(K) = -v(\text{mirror of } K)$ for every knot $K$. For example, $v_3$ and $v_5$ in Theorem 1.2 are finite type invariants of odd type. We say that $K$ and $K'$ are *odd $C_{n+1}$*-equivalent if $v(K) = v(K')$ for all finite type invariant $v$ of odd type with degree $\leq n$, and that $K$ is *odd $C_{n+1}$*-trivial if it is odd $C_{n+1}$-equivalent to the unknot.

In a similar spirit, as a corollary to the general argument based on the higher degree parts of the LMO invariant (Theorem 3.4), we also prove a vanishing of certain finite type invariants that come from the colored Jones polynomials (Quantum $\mathfrak{sl}_2$ invariant). Let $V_n(K;t)$ be the $n$-colored Jones polynomial, normalized so that $V_n(\mathsf{Unknot};t) = 1$. The colored Jones polynomials have the following expansion called the *loop expansion*, or *Melvin-Morton expansion* [MeMo].

$$V_n(K;e^{-h}) = \sum_{e\geq 0} \left( \sum_{k\geq 0} j_{e,k}(K)(nh)^k \right) h^e.$$

It is known that the coefficient $j_{e,k}(K) \in \mathbb{Q}$ is a *canonical* finite type invariant of degree $e+k$, a finite type invariant that has some nice theoretical properties (see Section 2.2 for definition). In particular, $j_{e,k}$ is a finite type invariant of odd type if $e+k$ is odd.

**Corollary 1.8.** *Let $K$ and $K'$ be knots in $S^3$ and $r \in \mathbb{Q} \setminus \{0\}$.*

(i) *Assume that $K$ and $K'$ are $C_{2m+1}$-equivalent. If $S^3(K,r) \cong S^3(K',r)$ and $a_{2m+2}(K) = a_{2m+2}(K')$, then $j_{1,2m}(K) = j_{1,2m}(K')$.*

(ii) *Assume that $K$ is odd $C_{4m+1}$-trivial. If $S^3(K,r) \cong -S^3(K,-r)$ then $j_{1,4m}(K) = 0$.*

(iii) *Assume that $K$ is odd $C_{4m+3}$-trivial. If $S^3(K,r) \cong \pm S^3(K,-r)$ then $j_{1,4m+2}(K) = 0$.*

By definition, if $K$ is amphicheiral then $v(K) = 0$ for all finite type invariants of odd type. It is conjectured that the converse is true (this is related to a more familiar conjecture that finite type invariants do not detect the orientation of knots [Kir, Problem 1.89]). Corollary 1.3 (i), Corollary 1.5 (ii), and Corollary 1.8 (ii),(iii) say that if $S^3(K,r) \cong -S^3(K,-r)$ then various finite type invariants of odd type vanish for $K$. Thus, they bring supporting evidence for an affirmative answer to the following question.

*If* $S^3(K,r) \cong -S^3(K,-r)$ *for some* $r \neq 0,\infty$, *then is* $K$ *amphicheiral?*

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## 2. LMO invariant and rational surgery formula

In this section we briefly review the basics of the Kontsevich and the LMO invariants. We use the Århus integral construction of the LMO invariant developed in [BGRT1, BGRT2, BGRT3] and a rational surgery formula for the LMO invariant due to Bar-Natan and Lawrence [BaL]. For basics of the Kontsevich and the LMO invariants we refer to [Oht].

### 2.1. Open Jacobi diagrams

An *(open) Jacobi diagram* or *(vertex-oriented) uni-trivalent graph* is a graph $D$ whose vertex is either univalent or trivalent, such that at each trivalent vertex $v$ a cyclic order on the three edges around $v$ is given. The *degree* of $D$ is half the number of vertices. We will often call a univalent vertex a *leg*, and denote the number of legs of a Jacobi diagram $D$ by $k(D)$. For a Jacobi diagram $D$, let $e(D) = -\chi(D)$ be minus the euler characteristic of $D$. We call $e(D)$ the *euler degree* of $D$. Then $\deg(D) = e(D) + k(D)$.

Let $\mathcal{B}$ (resp. $\mathcal{A}(\emptyset)$) be the vector space over $\mathbb{C}$ spanned by Jacobi diagrams (resp. Jacobi diagrams without a univalent vertex), modulo the AS and IHX relations given in Figure 1.

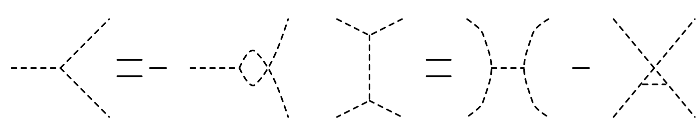


Figure 1: The AS and IHX relations: we understand that at each trivalent vertex, the cyclic order is defined by the counter-clockwise direction.

By taking disjoint union $\sqcup$ as the product, both $\mathcal{B}$ and $\mathcal{A}(\emptyset)$ have the structure of graded algebras. Since the IHX and the AS relations and the disjoint union product respect both $k(D)$ and $e(D)$, we view $\mathcal{B}$ as a bigraded algebra. For $X \in \mathcal{B}$ we denote by $X_{e,k}$ the part of $X$ whose bigrading is $(e,k)$. Strictly speaking, we will use the completion of $\mathcal{B}$ and $\mathcal{A}(\emptyset)$ with respect to degrees which we denote by the same symbol $\mathcal{B}$ and $\mathcal{A}(\emptyset)$ by abuse of notation.

Let $\exp_{\sqcup} : \mathcal{B} \to \mathcal{B}$ (or, $\mathcal{A}(\emptyset) \to \mathcal{A}(\emptyset)$) be the exponential map with respect to the $\sqcup$ product operation, defined by

$$\exp_{\sqcup}(D) = 1 + D + \frac{1}{2} D \sqcup D + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{(D \sqcup \cdots \sqcup D)}_{n}.$$

We will simply denote $\underbrace{(D \sqcup \cdots \sqcup D)}_{n}$ by $D^n$.

For a Jacobi diagram $C$, let $\partial_C : \mathcal{B} \to \mathcal{B}$ be the differential operator defined by

$$\partial_C(D) = \begin{cases} 0 & \text{if } k(C) > k(D) \\ \sum(\text{glue } \textit{all} \text{ the legs of } C \text{ to some legs of } D) & \text{if } k(C) \leq k(D) \end{cases}$$

In a similar manner, we define the pairing $\langle C, D \rangle \in \mathcal{A}(\emptyset)$ of $C, D \in \mathcal{B}$ by

$$\langle C, D \rangle = \begin{cases} 0 & \text{if } k(C) \neq k(D) \\ \sum(\text{glue the legs of } C \text{ to the legs of } D) & \text{if } k(C) = k(D) \end{cases}$$

Thus $\partial_C(D) = \langle C, D \rangle$ if $k(C) = k(D)$. In both cases, the summation runs over all possible ways of gluing *all* the legs of $C$ to *some* legs of $D$. We denote this summation by a box, as in Figure 2. It is known that $\partial_{C \sqcup C'} = \partial_{C'} \circ \partial_C$. Thus if $C \in \mathcal{B}$ is invertible (with respect to the $\sqcup$ product) then $\partial_C$ is invertible with $\partial_C^{-1} = \partial_{C^{-1}}$ (see [BGRT2, BLT, BaL] for details).

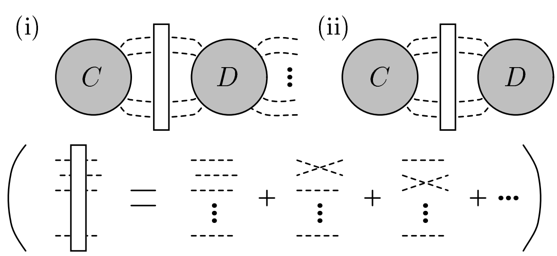
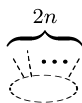


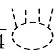

Figure 2: (i) Differential operator $\partial_C(D)$ (ii) Pairing $\langle C, D \rangle$.

Let $b_{2i}$ be the modified Bernoulli numbers, defined by

$$\frac{1}{2}\log\frac{\sinh(\frac{x}{2})}{\frac{x}{2}} = \sum_{i=0}^{\infty} b_{2i}x^{2i} = 1 + \frac{1}{48}x^2\frac{1}{5760}x^4 + \frac{1}{362880}x^6 + \cdots . \tag{2.1}$$

For $q \in \mathbb{Z}\setminus\{0\}$, let

$$\Omega_q = \exp_{\sqcup}\left(\sum_{n=1}^{\infty}\frac{b_{2n}}{q^{2n}}\right.$$  $$\left.\right)$$

$$= 1 + \frac{1}{48q^2}$$  $$- \frac{1}{5760q^4}$$  $$+ \frac{1}{4608q^4}$$  $$+ \cdots .$$

The element $\Omega = \Omega_1$ is called the *wheel element.*

## 2.2. Wheeled Kontsevich invariant

The *Kontsevich invariant* $Z(K)$ is an invariant of a framed knot, which takes values in $\mathcal{A}(S^1)$, the space of Jacobi diagrams over $S^1$ [Ko1, Bar]. Throughout the paper, we will always assume that the knot $K$ is zero-framed. The target space $\mathcal{A}(S^1)$ is isomorphic to $\mathcal{B}$ as a graded vector space, by the Poincaré-Birkhoff-Witt isomorphism $\chi : \mathcal{B} \to \mathcal{A}(S^1)$. Let $\sigma : \mathcal{A}(S^1) \to \mathcal{B}$ be the inverse of $\chi$. In the rest of the paper, we will always view the Kontsevich invariant as taking values in $\mathcal{B}$, by defining

$$Z^{\sigma}(K) = \sigma(Z(K)) \in \mathcal{B}.$$

We will denote by $Z^{\sigma}(K)_{e,k}$ the bigrading $(e,k)$ part of the Kontsevich invariant. See [GR] for a topological meaning of this bigrading.

Let $V_n$ be the vector space spanned by $\mathbb{C}$-valued finite type invariants of degree $\leq n$. The Kontsevich invariant gives a map $\mathcal{Z} : (\mathcal{B}_{\mathrm{deg}=n})^* \to V_n$, by $\mathcal{Z}(w)(K) = w(Z^{\sigma}(K)_{\mathrm{deg}=n})$. Here $w : \mathcal{B}_{\mathrm{deg}=n} \to \mathbb{C}$ is an element of $(\mathcal{B}_{\mathrm{deg}=n})^*$, the dual space of the degree $n$ part of $\mathcal{B}$ and $Z^{\sigma}(K)_{\mathrm{deg}=n} \in \mathcal{B}_{\mathrm{deg}=n}$ denotes the degree $n$ part of $Z^{\sigma}(K)$. On the other hand, there is a map called *symbol* $\mathsf{Symb} : V_n \to (\mathcal{B}_{\mathrm{deg}=n})^*$ (see [Bar, BaGa] for definition). A finite type invariant $v \in V_n$ is called

- *canonical*, if $v = \mathcal{Z}(\mathsf{Symb}(v))$.
- *primitive*, if $v(K\#K') = v(K) + v(K')$.

For a canonical finite type invariant $v$ of degree $d$, $v(K) = (-1)^d v(\text{mirror of } K)$ for all knots $K$. Thus, a canonical finite type invariant is of odd type if and only if its degree is odd.

The *wheeled Kontsevich invariant* $Z^{\mathsf{Wheel}}(K) \in \mathcal{B}$ is a version of the Kontsevich invariant defined as follows.

Let $\partial_\Omega = 1 + \frac{1}{48}\partial$ $+ \cdots$ be the differential operator defined by the wheel element $\Omega$. The *wheeling map* $\Upsilon = \chi \circ \partial_\Omega : \mathcal{B} \to \mathcal{A}(S^1)$ is the composite of $\partial_\Omega$ and the Poincaré-Birkhoff-Witt isomorphism $\chi$. The wheeling map $\Upsilon$ gives an isomorphism of *algebras* [BLT, Wheeling theorem], whereas the Poincaré-Birkhoff-Witt isomorphism $\chi$ only gives an isomorphism of vector spaces.

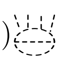
The wheeled Kontsevich invariant is the image of the Kontsevich invariant under the inverse of the wheeling map $\Upsilon$:

$$\begin{aligned} Z^{\mathsf{Wheel}}(K) &= \Upsilon^{-1}(Z(K)) \\ &= (\partial_\Omega)^{-1} \circ \sigma(Z(K)) \\ &= \partial_{\Omega^{-1}} Z^\sigma(K). \end{aligned}$$

The wheel element $\Omega$ is equal to the Kontsevich invariant of the unknot [BLT, Wheel theorem]: $Z^\sigma(\mathsf{Unknot}) = \Omega$. Therefore instead of $Z^\sigma$ or $Z^{\mathsf{Wheel}}$, it is often useful to use $Z^\sigma(K) \sqcup \Omega^{-1}$ since $Z^\sigma(\mathsf{Unknot}) \sqcup \Omega^{-1} = 1$.

The Kontsevich invariant is group-like, so $Z^\sigma(K) = \exp_\sqcup(z^\sigma(K))$, where $z^\sigma(K)$ denotes the primitive part of $Z^\sigma(K)$. By taking a basis of the primitive subspace of $\mathcal{B}_{e,k}$ for $e + \frac{k}{2} \leq 3$ (see [Kni] for the dimension of these spaces) we express the low degree part of the primitive Kontsevich invariant as

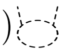
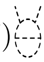
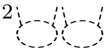
$Z^\sigma(K) \sqcup \Omega^{-1}$

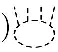
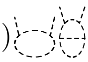
$= \exp_\sqcup \Big( v_2(K)$  $+ v_3(K)$  $+ v_4(K)$  $+ w_4(K)$  $+ v_5(K)$ 

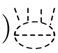
$+ v_6(K)$  $+ \text{(bigrading } (e,k) \text{ parts with } e + \frac{k}{2} > 3) \Big).$

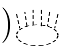
Here $v_2(K), v_3(K), v_4(K), w_4(K), v_5(K), v_6(K)$ are canonical, primitive finite type invariants of degree $2, 3, 4, 4, 5, 6$, respectively.

Thus the bigrading $(e,k)$ parts of $Z^\sigma(K)$ with $e + \frac{k}{2} \leq 3$ are explicitly written as

$Z^\sigma(K) = 1 + \big(v_2(K) + b_2\big)$  $+ v_3(K)$  $+ \frac{1}{2}\big(v_2(K) + b_2\big)^2$ 

$+ (v_4(K) + b_4)$  $+ w_4(K)$  $+ v_3(K)(v_2(K) + b_2)$ 

$+ v_5(K)$  $+ \frac{1}{6}\big(v_2(K) + b_2\big)^3$ 

$+ \big(v_2(K) + b_2\big)\big(v_4(K) + b_4\big)$  $+ \big(v_6(K) + b_6\big)$ 

$+ \left(\text{bigrading } (e,k)\text{parts with } e + \frac{k}{2} > 3\right)$

Here $b_{2i}$ denotes the modified Bernoulli numbers given by (2.1).

Except $v_5(K)$, these finite type invariants can be expressed in terms of the Conway polynomial and the Jones polynomial. Let $a_{2i}(K)$ be the coefficient of $z^{2i}$ in the Conway polynomial $\nabla_K(z)$ of $K$, and let $j_n(K)$ be the coefficient of $h^n$ in the Jones polynomial $V_K(e^h)$ of $K$, putting the variable as $t = e^h$. Then we have the following (see Section 4 for proof).

**Lemma 2.1.** (i) $v_2(K) = -\frac{1}{2}a_2(K)$.

(ii) $v_3(K) = -\frac{1}{24}j_3(K)$.

(iii) $v_4(K) = -\frac{1}{2}a_4(K) - \frac{1}{24}a_2(K) + \frac{1}{4}a_2(K)^2$.

(iv) $w_4(K) = \frac{1}{96}j_4(K) + \frac{3}{32}a_4(K) - \frac{9}{2}a_2(K)^2$.

(v) $v_6(K) = -\frac{1}{2}a_6(K) - \frac{1}{12}a_4(K) - \frac{1}{720}a_2(K) + \frac{1}{24}a_2(K)^2 + \frac{1}{2}a_2(K)a_4(K) - \frac{1}{6}a_2(K)^3$.

Since the Jones polynomial is an integer coefficient polynomial, $j_3(K) \in 6\mathbb{Z}$ so $4v_3(K) \in \mathbb{Z}$. The degree three finite type invariant $v_3$ takes the value $\frac{1}{4}$ for a right-handed trefoil.

In general, the euler degree zero part of the Kontsevich invariant can be expressed in terms of the Alexander polynomial, using the following formula (this is a consequence of the Melvin-Morton-Rozansky conjecture [BaGa] as discussed in [Kri]).

**Proposition 2.2.** *Let* $-\frac{1}{2}\log\Delta_K(e^x) = \sum_{n=0}^{\infty} d_{2n}(K)x^{2n}$ *where* $\Delta_K(t)$ *is the Alexander polynomial of* $K$*, normalized so that* $\Delta_K(t) = \Delta_K(t^{-1})$*,*

$\Delta_K(1) = 1$. *Then the euler degree zero part of the Kontsevich invariant is*

$$\exp_{\sqcup}\left(\sum_{k=0}^{\infty} d_{2k}(K) \overbrace{\quad}^{2n}\right).$$ 

*In particular, since* $\nabla_K(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) = \Delta_K(t)$, *if* $a_2(K) = a_4(K) = \cdots = a_{2m}(K) = 0$ *for some* $m \geq 0$ *then* $d_2(K) = d_4(K) = \cdots = d_{2m}(K) = 0$ *and* $d_{2m+2}(K) = -\frac{1}{2}a_{2m+2}(K)$.

### 2.3. The LMO invariant and the rational surgery formula

The LMO invariant $\widehat{Z}^{LMO}(M)$ is an invariant of an oriented closed 3-manifold $M$ that takes values in $\mathcal{A}(\emptyset)$. In the rest of the paper we restrict our attention to the case where $M$ is of the form $M = S^3(K, r)$ for some knot $K$ in $S^3$ and $r \neq 0, \infty$. In particular, we will always assume that $M$ is a rational homology sphere. We use the Århus integral normalization of the LMO invariant of rational homology spheres [BGRT3]: $\widehat{Z}^{LMO}(M)$ is $|H_1(M;\mathbb{Z})|^{-\deg}\Omega(M)$, where $\Omega(M)$ is the LMO invariant as defined in [LMO] and $|H_1(M;\mathbb{Z})|^{-\deg}$ is the operation that multiplies any degree $m$ elements in $\mathcal{A}(\emptyset)$ by $|H_1(M;\mathbb{Z})|^{-m}$.

To make computation simpler, we will use the following simplification. Let  be the theta-shaped Jacobi diagram which generates the degree one part of $\mathcal{A}(\emptyset)$.

Let $\mathcal{A}^{red}$ be the quotient of $\mathcal{A}(\emptyset)$ by the ideal generated by , and let $\pi : \mathcal{A}(\emptyset) \to \mathcal{A}^{red}$ be the quotient map. We call $\pi(\widehat{Z}^{LMO}(M)) \in \mathcal{A}^{red}$ the *reduced LMO invariant* and denote it by $Z^{LMO}(M)$. By abuse of notation, we will simply refer to the reduced LMO invariant as the LMO invariant. When we change the orientation, the (reduced) LMO invariant changes as

$$Z_n^{LMO}(-M) = (-1)^n Z_n^{LMO}(M)$$

where $Z_n^{LMO}(M)$ denotes the degree $n$ part of the LMO invariant.

The low degree part of the (reduced) LMO invariant is written as

$Z^{LMO}(M) = 1 + \lambda_2(M)$  $+ \lambda_3(M)$  $+ (\text{degree} > 4 \text{ parts}).$

where $\lambda_2(M), \lambda_3(M) \in \mathbb{C}$ are finite type invariant of rational homology spheres. In the rest of the arguments, unless otherwise specified, we will always work in $\mathcal{A}^{red}$. For example, we will always view the pairing $\langle D, D' \rangle$ so that it takes values in $\mathcal{A}^{red}$, by composing with the quotient map $\pi$.

Using the simplification $\ominus = 0$, the (reduced) LMO invariant of the 3-manifold obtained by rational Dehn surgery along a knot $K$ is given by the following simpler formula. Let $\frown$ be the strut, the Jacobi diagram homeomorphic to the interval.

**Theorem 2.3 (Rational surgery formula [BaL]).** *Let $K$ be a knot in $S^3$. Then the (reduced) LMO invariant of the $(p/q)$-surgery along $K$ is given by*

$$Z^{LMO}(S^3(K, p/q)) = \left\langle Z^{\mathsf{Wheel}}(K) \sqcup \Omega_q\, , \, \exp_{\sqcup}\left(-\frac{q}{2p} \frown \right)\right\rangle .$$

As an application of the rational surgery formula above, in [BaL, Proposition 5.1] it is shown that the (reduced) LMO invariant of the lens space $L(p, q)$ is given by

$$Z^{LMO}(L(p,q)) = \langle \Omega, \Omega^{-1} \sqcup \Omega_p \rangle . \tag{2.2}$$

Thus the (reduced) LMO invariant of the lens space only depends on $p$. In particular,

$$\lambda_2(L(p,q)) = \frac{1}{24}\left(\frac{1}{48p^2} - \frac{1}{48}\right), \quad Z^{LMO}_{2m+1}(L(p,q)) = 0 \ (m \in \mathbb{Z}). \tag{2.3}$$

## 3. Proofs of Theorems

First of all we determine which part of the Kontsevich invariant contributes to the degree $n$ part of the (reduced) LMO invariant.

**Proposition 3.1.** *The degree $n$ part of the LMO invariant for $S^3(K, p/q)$ is determined by the slope $p/q$ and the bigrading $(e, k)$ part $Z^\sigma(K)_{e,k}$ of $Z^\sigma(K)$ with $e + \frac{k}{2} \leq n$.*

*Proof.* By definition of the pairing, for $D \in \mathcal{B}_{e,k}$ we have $\langle D, \exp_{\sqcup}(-\frac{q}{2p} \frown )\rangle \in \mathcal{A}(\emptyset)_{e+\frac{k}{2}}$. Thus the degree $n$ part of the LMO invariant of $S^3(K, p/q)$ is determined by the bigrading $(e, k)$ part of $Z^{\mathsf{Wheel}}(K) \sqcup \Omega_q$, with $e + \frac{k}{2} = n$.

The bigrading $(e, k)$ part of $Z^{\mathsf{Wheel}}(K) \sqcup \Omega_q$ is determined by the bigrading $(e', k')$ part of $Z^{\mathsf{Wheel}}(K)$ with $(e', k') \in \{(e, k), (e, k-2), (e, k-4), \ldots\}$. Also, by definition of $\partial_D$, if $D \in \mathcal{B}_{e,k}$ and $D' \in \mathcal{B}_{e',k'}$, $\partial_D(D') \in \mathcal{B}_{e'+e+k, k'-k}$. This shows that the bigrading $(e, k)$ part of $Z^{\mathsf{Wheel}}(K)$ is determined by the bigrading $(e', k')$ part of $Z^\sigma(K)$ with $(e', k') \in \{(e, k), (e-2, k+2), (e-4, k+4), \ldots\}$.

These observations show that the degree $n$ part of the LMO invariant of $S^3(K, p/q)$ is determined by the bigrading $(e,k)$ part of $Z^\sigma(K)_{e,k}$ with $e + \frac{k}{2} \leq n$ (and the surgery slope $p/q$). $\square$

*Proof of Theorem 1.2.* By Proposition 3.1, to compute the degree 2 and 3 part of the LMO invariant for $S^3(K, p/q)$, it is sufficient to consider the bigrading $(e,k)$ part of $Z^\sigma(K)$ for $e + \frac{k}{2} \leq 3$. As we have already seen, this is given by

$$\begin{aligned} Z^\sigma(K) = 1 &+ \big(v_2(K)+b_2\big) + v_3(K) + \frac{1}{2}\big(v_2(K)+b_2\big)^2 \\ &+ (v_4(K)+b_4) + w_4(K) + v_3(K)(v_2(K)+b_2) \\ &+ v_5(K) + \frac{1}{6}\big(v_2(K)+b_2\big)^3 \\ &+ \big(v_2(K)+b_2\big)\big(v_4(K)+b_4\big) + \big(v_6(K)+b_6\big) \\ &+ \left(\text{bigrading } (e,k) \text{ parts with } e+\frac{k}{2} > 3\right). \end{aligned}$$

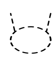
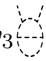
Here $b_2 = \frac{1}{48}, b_4 = -\frac{1}{5760}, b_6 = \frac{1}{362880}$ are modified Bernoulli numbers. Since

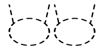
$$\begin{cases} \partial_{\Omega^{-1}}\left(\ \right) = \ - 2b_2\ , \quad \partial_{\Omega^{-1}}\left(\ \right) = \ - 2b_2\ , \\ \partial_{\Omega^{-1}}\left(\ \right) = \ - 8b_2\ + \big(\text{bigrading } (e,k) \text{ parts with } e+\frac{k}{2}>3\big) \\ \partial_{\Omega^{-1}}\left(\ \right) = \ - 10b_2\ + \big(\text{bigrading } (e,k) \text{ parts with } e+\frac{k}{2}>3\big), \end{cases}$$

the wheeled Kontsevich invariant is given by

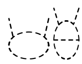
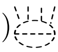
$$\begin{aligned} Z^{\mathsf{Wheel}}(K) = 1 &+ \big(v_2(K)+b_2\big) + v_3(K) + \frac{1}{2}\big(v_2(K)+b_2\big)^2 \\ &+ (v_4(K)+b_4) + w_4(K) + v_3(K)(v_2(K)+b_2) + v_5(K) \\ &+ \frac{1}{6}\big(v_2(K)+b_2\big)^3 + \big(v_2(K)+b_2\big)\big(v_4(K)+b_4\big) \\ &+ \big(v_6(K)+b_6\big) + \big(-2b_2\big)\big(v_2(K)+b_2\big) + (-2b_2v_3(K)) \\ &+ \left(-8b_2\frac{1}{2}\big(v_2(K)+b_2\big)^2 - 10b_2v_4(K)\right) \\ &+ \left(\text{bigrading } (e,k) \text{ parts with } e+\frac{k}{2} > 3\right). \end{aligned}$$

Thus $Z^{\mathsf{Wheel}}(K)\sqcup\Omega_q$ is equal to

$Z^{\mathsf{Wheel}}(K)\sqcup\Omega_q = 1 + \left(v_2(K)+b_2+\dfrac{b_2}{q^2}\right)$  $+\, v_3$ 

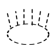
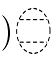
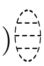
$+ \left\{\dfrac{1}{2}\big(v_2(K)+b_2\big)^2 + (v_2(K)+b_2)\dfrac{b_2}{q^2} + \dfrac{b_2^2}{2q^4}\right\}$ 

$+ \left(v_4(K)+b_4+\dfrac{b_4}{q^4}\right)$  $+ \big(w_4(K) - 4b_2\big(v_2(K)+b_2\big)^2 - 10b_2v_4(K)\big)$ 

$+ \left(v_3(K)(v_2(K)+b_2) + v_3(K)\dfrac{b_2}{q^2}\right)$  $+\, v_5(K)$ 

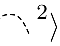
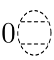
$+ \left(\dfrac{1}{6}\big(v_2(K)+b_2\big)^3 + (v_2(K)+b_2)^2\dfrac{b_2}{2q^2} + (v_2(K)+b_2)\dfrac{b_2^2}{2q^4} + \dfrac{b_2^3}{6q^6}\right)$

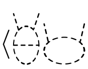
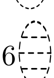
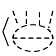
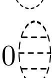
$+ \left(\big(v_2(K)+b_2\big)\big(v_4(K)+b_4\big) + (v_4(K)+b_4)\dfrac{b_2}{q^2} + (v_2(K)+b_2)\dfrac{b_4}{q^4} + \dfrac{b_2b_4}{q^6}\right)$

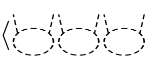
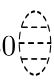
$+ \left(v_6(K)+b_6+\dfrac{b_6}{q^6}\right)$  $+\big(-2b_2\big)\big(v_2(K)+b_2\big)$  $+ (-2b_2v_3(K))$ 

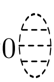
$+ \left(\text{bigrading } (e,k) \text{ parts with } e+\dfrac{k}{2} > 3\right).$

By direct computations (or, by using the $\mathfrak{sl}_2$ weight system evaluations as we will do in Section 4), the pairing with struts (in $\mathcal{A}^{red}$) are given by

$\langle$ ,  $\rangle = 2$ , $\langle$ ,  $^2\rangle = 16$ , $\langle$ ,  $^2\rangle = 20$ ,

$\langle$ ,  $^2\rangle = 2$ , $\langle$ ,  $^2\rangle = 16$ , $\langle$ ,  $^2\rangle = 20$ ,

$\langle$ ,  $^3\rangle = 384$ , $\langle$ ,  $^3\rangle = 480$ ,

$\langle$ ,  $^3\rangle = 420$ .

Consequently, we get

$$\begin{aligned}
\lambda_2(S^3_K(p/q)) &= v_3(K)\left(-\frac{q}{2p}\right)\cdot 2 \\
&+ \left(\frac{(v_2(K)+b_2)^2}{2} + \frac{(v_2(K)+b_2)b_2}{q^2} + \frac{b_2^2}{2q^4}\right)\frac{1}{2}\left(-\frac{q}{2p}\right)^2\cdot 16 \\
&+ \left(v_4(K)+b_4+\frac{b_4}{q^4}\right)\frac{1}{2}\left(-\frac{q}{2p}\right)^2\cdot 20 + \big(-2b_2\big)\big(v_2(K)+b_2\big)\big),
\end{aligned}$$

and

$$\lambda_3(S^3(K,p/q)) = \big(w_4(K) - 4b_2\big(v_2(K)+b_2\big)^2 - 10b_2v_4(K)\big)\left(-\frac{q}{2p}\right)\cdot 2$$
$$+\left((v_3(K)(v_2(K)+b_2)+\frac{v_3(K)b_2}{q^2}\right)\frac{1}{2}\left(-\frac{q}{2p}\right)^2\cdot 16 + v_5(K)\frac{1}{2}\left(-\frac{q}{2p}\right)^2\cdot 20$$
$$+\left(\frac{\big(v_2(K)+b_2\big)^3}{6}+\frac{(v_2(K)+b_2)^2b_2}{2q^2}+\frac{(v_2(K)+b_2)b_2^2}{2q^4}+\frac{b_2^3}{6q^6}\right)$$
$$\cdot\frac{1}{6}\left(-\frac{q}{2p}\right)^3\cdot 384$$
$$+\left(\big(v_2+b_2\big)\big(v_4+b_4\big)+\frac{(v_4(K)+b_4)b_2}{q^2}+\frac{(v_2(K)+b_2)b_4}{q^4}+\frac{b_2b_4}{q^6}\right)$$
$$\cdot\frac{1}{6}\left(-\frac{q}{2p}\right)^3\cdot 480$$
$$+\left(v_6(K)+b_6+\frac{b_6}{q^6}\right)\frac{1}{6}\left(-\frac{q}{2p}\right)^3\cdot 420 + (-2b_2v_3(K)).$$

Thus we conclude

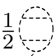
$$\begin{aligned}
&\lambda_2(S^3(K,p/q)) - \lambda_2(L(p,q))\\
&\quad= \left(v_2(K)^2+\frac{1}{24}v_2(K)+\frac{5}{2}v_4(K)\right)\frac{q^2}{p^2} - v_3(K)\frac{q}{p}+\frac{v_2(K)}{24}\left(\frac{1}{p^2}-1\right)\\
&\quad= \left(\frac{7a_2(K)^2-a_2(K)-10a_4(K)}{8}\right)\frac{q^2}{p^2} - v_3(K)\frac{q}{p}+\frac{a_2(K)}{48}\left(1-\frac{1}{p^2}\right),
\end{aligned}$$

$$\begin{aligned}
&\lambda_3(S^3(K,p/q)) - \lambda_3(L(p,q))\\
&= -\left(\frac{35}{4}v_6(K)+\frac{5}{24}v_4(K)+10v_2(K)v_4(K)+\frac{4}{3}v_2(K)^3+\frac{1}{12}v_2(K)^2\right)\frac{q^3}{p^3}\\
&\quad -\left(\frac{5}{24}v_4(K)+\frac{1}{288}v_2(K)+\frac{1}{12}v_2(K)^2\right)\frac{q}{p^3}\\
&\quad +\left(\frac{5}{2}v_5(K)+2v_3(K)v_2(K)+\frac{1}{24}v_3(K)\right)\frac{q^2}{p^2}+\frac{v_3(K)}{24}\left(\frac{1}{p^2}-1\right)\\
&\quad -\left(w_4(K)-\frac{1}{12}v_2(K)^2-\frac{1}{288}v_2(K)-\frac{5}{24}v_4(K)\right)\frac{q}{p}
\end{aligned}$$

□

**Remark 3.2.** In [Les, Theorem7.1] Lescop proved a similar formula

$$\lambda_2^{KKT}(S^3(K,p/q)) = \lambda_2^{\prime\prime KKT}(K)\frac{q^2}{p^2} + w_3(K)\frac{q}{p} + c(p/q)a_2(K) + \lambda_2^{KKT}(L(p,q)) \tag{3.1}$$

for the degree two part $\lambda_2^{KKT}$ of the Kontsevich-Kuperberg-Thurston universal finite type invariant $Z^{KKT}$, which is defined by configuration space integrals [Ko2, KuTh]. Assuming the conjectural equality $Z^{KKT} = Z^{LMO}$, we have $\lambda_2^{KKT} = 2\lambda_2$ (note that in [Les] Lescop used the coefficient of the Jacobi diagram  $= \frac{1}{2}$ ) and Theorem 1.2 gives formulae for the invariants in Lescop's formula (3.1), namely,

$$\lambda_2^{\prime\prime KKT}(K) = \frac{7a_2(K)^2 - a_2(K) - 10a_4(K)}{4},$$
$$w_3(K) = -2v_3(K), \quad c(p/q) = \frac{1}{24} - \frac{1}{24p^2}.$$

Indeed, the equality $w_3 = -2v_3(K)$ is confirmed in [IcWu], without assuming $\lambda_2^{KKT} = 2\lambda_2$.

*Proof of Corollary 1.3.*

(i, ii): By (1.1), it is sufficient to consider the case $r' = -r$. Assume that $S^3(K,p/q) \cong \pm S^3(K,-p/q)$. Since $\lambda_2(M) = \lambda_2(-M)$ for any rational homology sphere $M$, by Theorem 1.2

$$\lambda_2(S^3(K,p/q)) - \lambda_2(\pm S^3(K,-p/q)) = -2v_3(K)q/p = 0$$

Therefore $v_3(K) = 0$.

(iii): Assume that $S^3(K,p/q) \cong \pm S^3(K,-p'/q')$. Since $H_1(S^3(K,p/q)) \cong \mathbb{Z}/p\mathbb{Z}$, we have $p = p'$. By Theorem 1.2

$$\begin{aligned} 0 &= \lambda_2(S^3(K,p/q)) - \lambda_2(-S^3(K,p/q')) \\ &= \left(\frac{7a_2(K)^2 - a_2(K) - 10a_4(K)}{8}\right)\frac{q^2 - q'^2}{p^2} - v_3(K)\frac{q-q'}{p}. \end{aligned}$$

Since $q' \neq \pm q$, either $v_3(K) \neq 0$ (and $7a_2(K)^2 - a_2(K) - 10a_4(K) = 0$), or,

$$\frac{p}{q+q'} = \frac{7a_2(K)^2 - a_2(K) - 10a_4(K)}{8v_3(K)}.$$

(iv): Assume that $S^3(K,p/q) \cong \pm S^3(K',p/q)$. By Theorem 1.1 (i), we have $a_2(K) = a_2(K')$. By Theorem 1.2

$$\begin{aligned} 0 &= \lambda_2(S^3(K,p/q)) - \lambda_2(S^3(K',p/q)) \\ &= \frac{5}{4}(a_4(K) - a_4(K'))\frac{q^2}{p^2} - (v_3(K) - v_3(K'))\frac{q}{p}. \end{aligned}$$

□

*Proof of Corollary 1.4.* Assume that $S^3(K,p/q) \cong L(p',q')$. Then

$$|H_1(S^3(K,p/q);\mathbb{Z})| = p = p' = H_1(L(p',q));\mathbb{Z})$$

so $p = p'$. By (2.3) $\lambda_2(L(p,q)) = \lambda_2(L(p,q'))$ so Theorem 1.2 gives the desired equality. □

*Proof of Corollary 1.5.* (i): By (1.1), it is sufficient to consider the case $r' = -r$. By Theorem 1.1 (ii) and Corollary 1.3 (i), $a_2(K) = v_2(K) = v_3(K) = 0$. Thus by Theorem 1.2

$$\begin{aligned} 0 &= \lambda_3(S^3(K,p/q)) - \lambda_3(S^3(K,-p/q)) \\ &= -\left(\frac{35}{2}v_6(K) + \frac{5}{12}v_4(K)\right)\frac{q^3}{p^3} - \frac{5}{12}v_4(K)\frac{q}{p^3} - \left(2w_4(K) - \frac{5}{12}v_4(K)\right)\frac{q}{p}. \end{aligned}$$

(ii): If $S^3_K(p/q)) \cong -S^3_K(-p/q)$ then by Corollary 1.3 (ii) $v_3(K) = 0$. Therefore by Theorem $\lambda_3(S^3(K,p/q)) - \lambda_3(-S^3(K,-p/q)) = 5v_5(K) = 0$. □

*Proof of Corollary 1.6.* By Lemma 2.1 and Corollary 1.5 (i), if $S^3(K,p/q) \cong S^3(K,-p/q)$ then we get

$$(3.2) \qquad (19a_4(K) + j_4(K))p^2 - 10a_4(K) - (420a_6(K) + 80a_4(K))q^2 = 0.$$

According to [IcWu], the cosmetic surgery conjecture was confirmed for prime knots with less than or equal to 11 crossings, with 8 exceptions

$$10_{33}, 10_{118}, 10_{146}, 11a_{91}, 11a_{138}, 11a_{285}, 11n_{86}, 11n_{157}$$

in the table KnotInfo [ChLi].

For these knots, the values of $a_4, j_4$ and $a_6$ are given as follows.

| | $10_{33}$ | $10_{118}$ | $10_{146}$ | $11a_{91}$ | $11a_{138}$ | $11a_{285}$ | $11n_{86}$ | $11n_{157}$ |
|---|---|---|---|---|---|---|---|---|
| $a_4$ | 4 | 2 | 2 | 0 | 2 | 2 | -2 | 0 |
| $j_4$ | -12 | -6 | -6 | 0 | -6 | -6 | 6 | 0 |
| $a_6$ | 0 | 3 | 0 | -2 | -2 | -2 | -1 | -1 |

Note that (3.2) gives a diophantine equation of the form $ap^2 - bq^2 = c$, whose solvability can be checked algorithmically [AnAn]. For these knots the equation (3.2) has no integer solutions, except in the case $K = 10_{118}$ (The author used the computer program at [Sol]. In the case $K = 10_{118}$ we get the equation $32p^2 - 20 - 1420q^2 = 0$ which has the solutions $p = 20u + 1065v$ and $q = 3u - 160v$, where $(u, v)$ are the solutions of Pell's equation $u^2 - 2840v^2 = 1$.) □

Next we proceed to see the higher degree part. To use the $C_n$-equivalence assumption, we observe the following.

**Lemma 3.3.** *If $K$ and $K'$ are $C_{2m+1}$-equivalent, then for $e + \frac{k}{2} \leq m + 1$,*

$$\begin{aligned}(Z^{\mathsf{Wheel}}(K) \sqcup \Omega_q - Z^{\mathsf{Wheel}}(K') \sqcup \Omega_q)_{e,k} \\ &= (Z^\sigma(K) \sqcup \Omega^{-1})_{e,k} - (Z^\sigma(K') \sqcup \Omega^{-1})_{e,k} \\ &= \begin{cases} (Z^\sigma(K) - Z^\sigma(K'))_{e,k} & \text{if } (e,k) = (0, 2m+2), (1, 2m) \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

*Similarly, if $K$ and $K'$ are odd $C_{2m+1}$-equivalent, then for $e + \frac{k}{2} \leq m + 1$ with odd $e + k$,*

$$\begin{aligned}(Z^{\mathsf{Wheel}}(K) \sqcup \Omega_q - Z^{\mathsf{Wheel}}(K') \sqcup \Omega_q)_{e,k} \\ &= (Z^\sigma(K) \sqcup \Omega^{-1})_{e,k} - (Z^\sigma(K') \sqcup \Omega^{-1})_{e,k} \\ &= \begin{cases} (Z^\sigma(K) - Z^\sigma(K'))_{e,k} & \text{if } (e,k) = (1, 2m) \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

*Proof.* By definition of $\partial_{\Omega^{-1}}$ and the wheel element $\Omega$, for $D \in \mathcal{B}_{e,k}$, $\partial_{\Omega^{-1}}(D)$ is of the form

$$(3.3)\quad \partial_{\Omega^{-1}}(D) = D + \sum \left( \text{Jacobi diagram } D' \text{ with } e(D') + \frac{k(D')}{2} > e(D) + \frac{k(D)}{2} \right)$$

Since $K$ and $K'$ are $C_{2m+1}$-equivalent, if $e + k \leq 2m$ then $Z^\sigma(K)_{e,k} - Z^\sigma(K')_{e,k} = 0$. Therefore, for $(e, k)$ with $e + \frac{k}{2} \leq m + 1$ we have $Z^\sigma(K)_{e,k} - Z^\sigma(K')_{e,k} = 0$, unless $(e, k) = (0, 2m + 2), (1, 2m)$.

By (3.3), this shows that if $e+\frac{k}{2} \leq m+1$ then we have

$$\begin{aligned} Z^{\mathsf{Wheel}}(K) - Z^{\mathsf{Wheel}}(K') &= \partial_{\Omega^{-1}}(Z^{\sigma}(K) - Z^{\sigma}(K'))_{e,k} \\ &= (Z^{\sigma}(K) - Z^{\sigma}(K'))_{e,k}. \end{aligned}$$

For $(e,k)$ with $e+\frac{k}{2} \leq m+1$, and $(Z^{\sigma}(K) - Z^{\sigma}(K'))_{e,k} = 0$ unless $(e,k) = (0,2m+2),(1+2m)$. On the other hand,

$$D \sqcup \Omega_q = D + \sum \big(\text{Jacobi diagram } D' \text{ with } k(D') \geq k(D)+2\big).$$

Hence we have

$$\begin{aligned} \big((Z^{\sigma}(K) - Z^{\sigma}(K')) \sqcup \Omega_q\big)_{e,k} &= (Z^{\sigma}(K) - Z^{\sigma}(K'))_{e,k} \\ &= (Z^{\sigma}(K) \sqcup \Omega^{-1})_{e,k} - (Z^{\sigma}(K') \sqcup \Omega^{-1})_{e,k}. \end{aligned}$$

Therefore for $(e,k)$ with $e+\frac{k}{2} \leq m+1$ we conclude

$$\begin{aligned} &(Z^{\mathsf{Wheel}}(K) \sqcup \Omega_q - Z^{\mathsf{Wheel}}(K') \sqcup \Omega_q)_{e,k} \\ &\quad = \Big(\Big(Z^{\mathsf{Wheel}}(K) - Z^{\mathsf{Wheel}}(K')\Big) \sqcup \Omega_q\Big)_{e,k} \\ &\quad = \big((Z^{\sigma}(K) - Z^{\sigma}(K')) \sqcup \Omega_q\big)_{e,k} \\ &\quad = (Z^{\sigma}(K) \sqcup \Omega^{-1})_{e,k} - (Z^{\sigma}(K') \sqcup \Omega^{-1})_{e,k} \\ &\quad = \begin{cases} (Z^{\sigma}(K) - Z^{\sigma}(K'))_{e,k} & \text{if } (e,k) = (0,2m+2),(1,2m) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

To see the latter assertion, we note that both $\partial_{\Omega^{-1}}$ and $\sqcup\Omega_q$ preserve the parity of the degree of $D$. Namely, if $D \in \mathcal{B}_{odd}$, where we denote by $\mathcal{B}_{odd}$ the odd degree part of $\mathcal{B}$, then $\partial_{\Omega^{-1}}(D), D \sqcup \Omega_q \in \mathcal{B}_{odd}$. Therefore the same argument, restricted to bigrading $(e,k)$ with odd $e+k$, proves the desired result. □

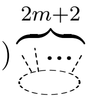
*Proof of Theorem 1.7.*

(i): By the proof of Proposition 3.1, the degree $m+1$ part of the LMO invariant of $S^3(K,p/q)$ is determined by $(Z^{\mathsf{Wheel}}(K) \sqcup \Omega_q)_{e,k}$ with $e+\frac{k}{2} = m+1$. Since $K$ and $K'$ are $C_{2m+2}$-equivalent, by Lemma 3.3, for $(e,k)$ with $e+\frac{k}{2} = m+1$,

$$(Z^{\mathsf{Wheel}}(K) \sqcup \Omega_q - Z^{\mathsf{Wheel}}(K') \sqcup \Omega_q)_{e,k} = 0$$

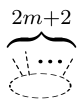
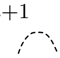
unless $(e,k) = (0,2m+2),(1,2m)$.

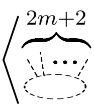
In the current situation, we are assuming the slightly stronger condition that $K$ and $K'$ are $C_{2m+2}$-equivalent. Thus $(Z^{\mathsf{Wheel}}(K) \sqcup \Omega_q - Z^{\mathsf{Wheel}}(K') \sqcup \Omega_q)_{1,2m} = 0$ as well. By Proposition 2.2 we have

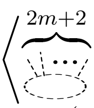
$$(Z^\sigma(K) \sqcup \Omega^{-1})_{0,2m+2} - (Z^\sigma(K') \sqcup \Omega^{-1})_{0,2m+2} = -\frac{1}{2}(a_{2m+2}(K) - a_{2m+2}(K'))$$ 

Therefore

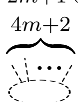
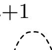
(3.4)

$$Z^{LMO}_{m+1}(S^3(K,p/q)) - Z^{LMO}_{m+1}(S^3(K',p/q)) = \left\langle -\frac{1}{2}(a_{2m+2}(K) - a_{2m+2}(K'))\right.$$  $$, \frac{1}{(m+1)!}\left(-\frac{q}{2p}\right)^{2m+1}$$  $$\left.{}^{m+1}\right\rangle$$

$$= -\frac{a_{2m+2}(K) - a_{2m+2}(K')}{2(m+1)!}\left(-\frac{q}{2p}\right)^{m+1}\Bigg\langle$$  ,  $$^{m+1}\Bigg\rangle .$$

As we will see in Lemma 4.1, $\Big\langle$  ,  $^{m+1}\Big\rangle \neq 0$. This shows that $S^3(K,p/q) \cong S^3(K',p/q)$ implies $a_{2m+2}(K) = a_{2m+2}(K')$.

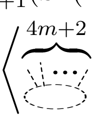
(ii) By (3.4), when $K$ is $C_{4m+2}$-trivial, then

$$Z^{LMO}_{2m+1}(S^3(K,p/q)) - Z^{LMO}_{2m+1}(L(p,q)) = \left\langle -\frac{1}{2}a_{4m+2}(K)\right.$$  $$, \frac{1}{(2m+1)!}\left(-\frac{q}{2p}\right)^{2m+1}$$  $$\left.{}^{2m+1}\right\rangle .$$

By (1.1), if $S^3(K,p/q) \cong S^3(K,p'/q')$ then

$$p/q = -p'/q' \quad \text{and} \quad Z^{LMO}_{2m+1}(L(p,q)) - Z^{LMO}_{2m+1}(L(p',q')) = 0$$

hence

$$0 = Z^{LMO}_{2m+1}(S^3(K,p/q)) - Z^{LMO}_{2m+1}(S^3(K,-p/q)) = -\frac{a_{4m+2}(K)}{(2m+1)!}\left(-\frac{q}{2p}\right)^{2m+1}\Bigg\langle$$  ,  $$^{2m+1}\Bigg\rangle .$$

Therefore $a_{4m+2}(K)=0$. □

A similar argument shows the following. Let $K_{e,k}\subset \mathcal{B}_{e,k}$ denotes the kernel of the Århus integration (diagram pairing) $\langle *, \frown^{\frac{k}{2}}\rangle : \mathcal{B}_{e,k}\to \mathcal{A}(\emptyset)_{e+\frac{k}{2}}$ (See Section 2). Then we have similar vanishing results.

**Theorem 3.4.** *Let $K$ and $K'$ be a knot in $S^3$ and $r\in\mathbb{Q}\setminus\{0\}$.*

- (i) *Assume that $K$ and $K'$ are $C_{2m+1}$-equivalent. If $S^3(K,r)\cong S^3(K',r)$ and $a_{2m+2}(K)=a_{2m+2}(K')$, then $(Z^\sigma(K)\sqcup\Omega^{-1})_{1,2m}-(Z^\sigma(K)\sqcup\Omega^{-1})_{1,2m}\in K_{1,2m}$.*
- (ii) *Assume that $K$ is odd $C_{4m+1}$-trivial. If $S^3(K,r)\cong -S^3(K,-r)$ then $(Z^\sigma(K)\sqcup\Omega^{-1})_{1,4m}\in K_{1,4m}$.*
- (iii) *Assume that $K$ is odd $C_{4m+3}$-trivial. If $S^3(K,r)\cong \pm S^3(K,-r)$ then $(Z^\sigma(K)\sqcup\Omega^{-1})_{1,4m+2}\in K_{1,4m+2}$.*

*Proof.* (i): By the same argument as in Theorem 1.7 (i), Lemma 3.3 and Proposition 2.2 show that

$$\begin{aligned}
&Z^{LMO}_{m+1}(S^3(K,p/q))-Z^{LMO}_{m+1}(S^3(K',p/q))\\
&=\left\langle -\frac{1}{2}(a_{2m+2}(K)-a_{2m+2}(K'))\overbrace{\bigcirc}^{2m+2}, \frac{1}{(m+1)!}\left(-\frac{q}{2p}\right)^{m+1}\frown^{m+1}\right\rangle\\
&\quad+\left\langle (Z^\sigma(K)\sqcup\Omega^{-1})_{1,2m}-(Z^\sigma(K')\sqcup\Omega^{-1})_{1,2m}, \frac{1}{m!}\left(-\frac{q}{2p}\right)^{m}\frown^{m}\right\rangle .
\end{aligned}$$

Thus, if $S^3(K,p/q)\cong S^3(K',p/q)$ and $a_{2m+2}(K)=a_{2m+2}(K')$ then

$$\left\langle\Big((Z^\sigma(K)\sqcup\Omega^{-1})_{1,4m+2}-(Z^\sigma(K')\sqcup\Omega^{-1})_{1,4m+2}\Big), \frown^{2m+1}\right\rangle=0.$$

(ii) Assume that $K$ is odd $C_{4m+1}$-trivial. Since

$$\begin{aligned}
&Z^{LMO}_{2m+1}(S^3(K,p/q))\\
&=\sum_{e=0}^{m}\left\langle (Z^{\mathsf{Wheel}}(K)\sqcup\Omega_q)_{2e,4m+2-4e}, \frac{1}{(2m+1-2e)!}\left(-\frac{q}{2p}\frown\right)^{2m+1-2e}\right\rangle\\
&\quad+\sum_{e=0}^{m}\left\langle (Z^{\mathsf{Wheel}}(K)\sqcup\Omega_q)_{2e+1,4m-4e}, \frac{1}{(2m-2e)!}\left(-\frac{q}{2p}\frown\right)^{2m-2e}\right\rangle
\end{aligned}$$

we get

$$\begin{aligned}0 &= Z^{LMO}_{2m+1}(S^3(K,p/q)) - Z^{LMO}_{2m+1}(-S^3(K,-p/q))\\ &= Z^{LMO}_{2m+1}(S^3(K,p/q)) + Z^{LMO}_{2m+1}(S^3(K,-p/q))\\ &= 2\sum_{e=0}^{m}\left\langle (Z^{\mathsf{Wheel}}(K)\sqcup\Omega_q)_{2e+1,4m-4e}\,,\, \frac{1}{(2m-2e)!}\left(-\frac{q}{2p}\frown\right)^{2m-2e}\right\rangle.\end{aligned}$$

By Lemma 3.3, $(Z^{\mathsf{Wheel}}(K)\sqcup\Omega_q)_{2e+1,4m-4e} = 0$ unless $e=0$, and we have

$$\begin{aligned}(Z^{\mathsf{Wheel}}(K)\sqcup\Omega_q)_{1,4m} = {}& (Z^{\mathsf{Wheel}}(\mathsf{Unknot})\sqcup\Omega_q)_{1,4m} + (Z^{\sigma}(K)\sqcup\Omega^{-1})_{1,4m}\\ &- (Z^{\sigma}(\mathsf{Unknot})\sqcup\Omega^{-1})_{1,4m}.\end{aligned}$$

Since for any $X\in\mathcal{B}$ and $q\in\mathbb{Z}\setminus\{0\}$, $(X\sqcup\Omega_q^{\pm1})_{1,k} = X_{1,k}$ we have

$$(Z^{\mathsf{Wheel}}(\mathsf{Unknot})\sqcup\Omega_q)_{1,4m} = (Z^{\sigma}(\mathsf{Unknot})\sqcup\Omega^{-1})_{1,4m}.$$

Thus we get

$$(Z^{\mathsf{Wheel}}(K)\sqcup\Omega_q)_{1,4m} = (Z^{\sigma}(K)\sqcup\Omega^{-1})_{1,4m}.$$

Hence

$$0 = \frac{2}{(2m)!}\left(-\frac{q}{2p}\right)^{2m}\left\langle (Z^{\sigma}(K)\sqcup\Omega^{-1})_{1,4m}\,,\, \frown^{2m}\right\rangle.$$

(iii) is proved similarly.

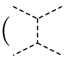
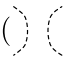
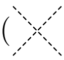
□

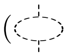
*Proof of Corollary 1.8.* By Theorem 3.4, under the assumptions (i),(ii), and (iii) of Corollary 1.8 we have $(Z^{\sigma}(K)\sqcup\Omega^{-1})_{1,2m} - (Z^{\sigma}(K)\sqcup\Omega^{-1})_{1,2m}\in K_{1,2m}$, $(Z^{\sigma}(K)\sqcup\Omega^{-1})_{1,4m} = 0$, and $(Z^{\sigma}(K)\sqcup\Omega^{-1})_{1,4m+2}$, respectively. By Lemma 4.2 in the next section, if $(Z^{\sigma}(K)\sqcup\Omega^{-1})_{1,2m}\in K_{1,2m}$ then $j_{1,2m}(K)=0$. Thus under the assumptions (i),(ii), and (iii) of Corollary 1.8, we have $j_{1,2m}(K) = j_{1,2m}(K')$, $j_{1,4m}(K)=0$, and $j_{1,4m+2}(K)=0$ respectively. □

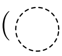
## 4. Some $\mathfrak{sl}_2$ weight system computations

In this section we use the $(\mathfrak{sl}_2, V_n)$ weight system, which is a linear map $W_{(\mathfrak{sl}_2,V_n)}:\mathcal{B}$ (or, $\mathcal{A}(\emptyset)$) $\to\mathbb{C}[[h]]$ that comes from the Lie algebra $\mathfrak{sl}_2$ and its $n$-dimensional irreducible representation, to confirm some assertions used in previous sections.

The image of $W_{(\mathfrak{sl}_2,V_n)}$ can be calculated recursively by the following relations [ChVa]:

(i) $W_{(\mathfrak{sl}_2,V_n)}($  $) = 2h\Big(W_{(\mathfrak{sl}_2,V_n)}($  $) - W_{(\mathfrak{sl}_2,V_n)}($  $)\Big)$

(ii) $W_{(\mathfrak{sl}_2,V_n)}($  $) = 4hW_{(\mathfrak{sl}_2,V_n)}($  $)$.

(iii) $W_{(\mathfrak{sl}_2,V_n)}($  $) = 3$.

(iv) $W_{(\mathfrak{sl}_2,V_n)}(D \sqcup$  $) = h^{\frac{n^2-1}{2}} W_{(\mathfrak{sl}_2,V_n)}(D)$.

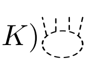
Note that $\deg_h W_{(\mathfrak{sl}_2,V_n)}(D) = \deg(D)$ and the relations (i)–(iii) do not depend on $n$. Thus for $D \in \mathcal{A}(\emptyset)$, $W_{(\mathfrak{sl}_2,V_n)}(D)$ does not depend on $n$ so we will simply write by $W_{\mathfrak{sl}_2}(D)$.

The colored Jones polynomial is a knot invariant from quantum $\mathfrak{sl}_2$ with its $n$-dimensional irreducible representation. By the Drinfel'd-Kohno theorem [Dri, Koh], which shows the equivalence of quantum braid group representation from quantum groups and the monodromy representation from the KZ equation, we have

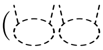
$$\begin{aligned} W_{(\mathfrak{sl}_2,V_n)}(Z^\sigma(K) \sqcup \Omega^{-1}) &= V_n(K; e^{-h}) \\ &= \sum_{e\geq 0}\left(\sum_{k\geq 0} j_{e,k}(K)(nh)^k\right) h^e. \end{aligned} \tag{4.1}$$

We remark that we put the variable $t$ in the colored Jones polynomial equal not to $e^h$ but to $e^{-h}$, due to the difference of normalization of the colored Jones polynomial and quantum $\mathfrak{sl}_2$ invariants.

We use this to check that some finite type invariants which we used can be written in terms of the Jones and the Conway polynomials.

*Proof of Lemma 2.1.* (i),(iii) and (v) follow from Proposition 2.2 so we prove (ii) and (iv). The degree three and four parts of $(Z^\sigma(K) \sqcup \Omega^{-1})$ are given $v_3(K)$  and $\frac{1}{8}a_2(K)^2$  $- \frac{1}{2}a_4(K)$  $+ w_4(K)$ , respectively. Thus by (4.1) applying $W_{(\mathfrak{sl}_2,V_2)}$ we get

$j_3(K)(-h)^3 = v_3(K)W_{(\mathfrak{sl}_2,V_2)}($  $) = 24v_3(K)h^3$

$$j_4(K)(-h)^4 = \frac{1}{8}a_2(K)^2 W_{(\mathfrak{sl}_2,V_2)}($$  $$) - \frac{1}{2}a_4(K)W_{(\mathfrak{sl}_2,V_2)}($$  $$)$$

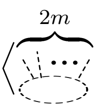
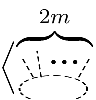
$$+ w_4(K)W_{(\mathfrak{sl}_2,V_2)}($$  $$)$$

$$= \frac{1}{8}a_2(K)^2 36h^4 - \frac{1}{2}a_4(K)18h^4 + w_4(K)96h^4$$

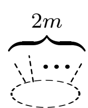
$$= \left(\frac{9}{2}a_2(K)^2 - 9a_4(K) + 96w_4(K)\right)h^4.$$

□

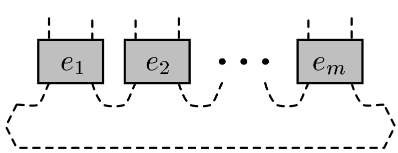
For two Jacobi diagrams $D$ and $D'$ we write $D \equiv D'$ if $D$ is equal to $D'$ by using the $\mathfrak{sl}_2$ weight system relations (i)–(iii) which are independent of $V_n$. By the $\mathfrak{sl}_2$ weight system relations (i)–(iii) we can remove all the trivalent vertices of a Jacobi diagram when the number of univalent vertices is even.

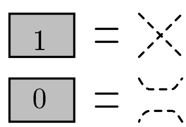
**Lemma 4.1.** $W_{\mathfrak{sl}_2}\Big(\Big\langle$  $,$  $^m\Big\rangle\Big) = 2(2h)^m(2m+1)!$. *In particular,* $\Big\langle$  $,$  $^m\Big\rangle \neq 0$.

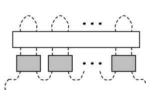
*Proof.* First we observe that

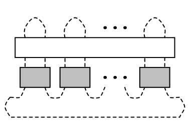
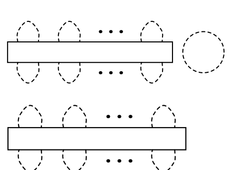
 $$\equiv (2h)^m \sum_{\mathbf{e}=(e_1,\dots,e_m)\in\{0,1\}^m} (-1)^{e_1+\cdots+e_m} D_{\mathbf{e}}$$

Here for $\mathbf{e} = (e_1, \dots, e_m) \in \{0,1\}^m$, $D_{\mathbf{e}}$ denotes the Jacobi diagram



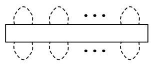



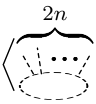

Then the pairing $\langle D_{\mathbf{e}},$  $^m\rangle =$  is given by

(4.2)  $=$ 

if $e_1 = e_2 = \cdots = e_m = 0$

otherwise

By definition, $W_{\mathfrak{sl}_2}\Big($  $\Big) = W_{\mathfrak{sl}_2}\left(\left\langle \frown^{m}, \frown^{m}\right\rangle\right) = (2m+1)!$ (see [BLT, Lemma 6.1]). Therefore by (4.2)

$$W_{\mathfrak{sl}_2}\Big(\Big\langle$$  $$, \frown^{m}\Big\rangle\Big) = 2(2h)^m(2m+1)!$$

□

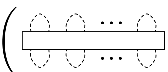
**Lemma 4.2.**

$$W_{\mathfrak{sl}_2}\left(\left\langle (Z^{\sigma}(K)\sqcup\Omega^{-1})_{1,2m}, \frown^{m}\right\rangle\right) = 2^m h^{m+1}(2m+1)! j_{1,2m}(K)$$

*Proof.* Let us put $(Z^{\sigma}(K)\sqcup\Omega^{-1})_{e,2k} \equiv c_{e,2k}(K)h^{e+k}\frown^{k}$. Then

$$\begin{aligned} W_{(\mathfrak{sl}_2,V_n)}((Z^{\sigma}(K)\sqcup\Omega^{-1})_{e,2k}) &= c_{e,2k}(K)h^{e+2k}\left(\frac{n^2-1}{2}\right)^k \\ &= \frac{c_{e,2k}(K)}{2^k}h^e(nh)^{2k} - \frac{c_{e,2k}(K)}{2^k}kh^{e+2}(nh)^{2k-2} \\ &\quad + \frac{c_{e,2k}(K)}{2^k}\binom{k}{2}h^{e+4}(nh)^{2k-4} - \cdots . \end{aligned}$$

By (4.1) we conclude that $j_{1,2m}(K) = \frac{c_{1,2m}(K)}{2^m}$. Then the $\mathfrak{sl}_2$ weight system evaluation of the desired pairing is

$$\begin{aligned} &W_{\mathfrak{sl}_2}\left(\left\langle (Z^{\sigma}(K)\sqcup\Omega^{-1})_{1,2m}, \frown^{m}\right\rangle\right) \\ &= 2^m j_{1,2m}(K)h^{m+1}W_{\mathfrak{sl}_2}\Big(\end{aligned}$$  $$\Big)$$

$$= 2^m h^{m+1}(2m+1)! j_{1,2m}(K)$$

□

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Department of Mathematics, Kyoto University
Kyoto 606-8502, Japan
*E-mail address*: `tetitoh@math.kyoto-u.ac.jp`

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