Communications in
Analysis and Geometry
Volume 28, Number 3, 565–606, 2020

On Li-Yau gradient estimate for sum of squares of vector fields up to higher step

Der-Chen Chang, Shu-Cheng Chang, and Chien Lin

In this paper, we generalize the Cao-Yau's gradient estimate for the sum of squares of vector fields up to higher step under assumption of the generalized curvature-dimension inequality. With its applications, by deriving a curvature-dimension inequality, we are able to obtain the Li-Yau gradient estimate for the CR heat equation in a closed pseudohermitian manifold of nonvanishing torsion tensors. As consequences, we obtain the Harnack inequality and upper bound estimate for the CR heat kernel.

1. Introduction

One of the goals for differential geometry and geometric analysis is to understand and classify the singularity models of a nonlinear geometric evolution equation, and to connect it to the existence problem of geometric structures

The first author is partially supported by an NSF grant DMS-1408839 and a McDevitt Endowment Fund at Georgetown University. The second and the third author are partially supported by the MOST of Taiwan.

on manifolds. For instance in 1982, R. Hamilton ([H3]) introduced the Ricci flow. Then by studying the singularity models ([H2], [Pe1], [Pe2], [Pe3]) of Ricci flow, R. Hamilton and G. Perelman solved the Thurston geometrization conjecture and Poincare conjecture for a closed 3-manifold in 2002.

On the other hand, in the seminal paper of P. Li and S.-T. Yau ([LY]) established the parabolic Li-Yau gradient estimate and Harnack inequality for the positive solution of heat equation

$$\left(\Delta - \frac{\partial}{\partial t}\right) u\,(x,t) = 0$$

in a complete Riemannian manifold with nonnegative Ricci curvature. Here Δ is the time-independent Laplacian operator. Later, R. S. Hamilton ([H1]) obtained the so-called Li-Yau-Hamilton inequality for the Ricci flow in a complete Riemannian manifold with a bounded and nonnegative curvature operator. Recently, G. Perelman ([Pe1]) derived the remarkable entropy formula which is important in the study of the singularity models of Ricci flow. The derivation of the entropy formula resembles the Li-Yau gradient estimate for the heat equation. Since then, there were many additional works in this direction which cover various different geometric evolution equations such as the mean curvature flow ([H4]), the Kähler-Ricci flow ([Ca]), the Yamabe flow ([Ch]), etc.

In the paper of [CKW], following this direction, we propose to study the most important geometrization problem of closed CR 3-manifolds via the CR torsion flow (1.1). More precisely, let us recall that a strictly pseudoconvex CR structure on a pseudohermitian 3-manifold (M, J, θ) is given by a cooriented plane field $\ker\theta$, where θ is a contact form, together with a compatible complex structure J. Given this data, there is a natural connection, the so-called Tanaka-Webster connection or pseudohermitian connection. We denote the torsion of this connection by $A_{J,\theta}$, and the Webster curvature by W. We consider the torsion flow

$$\begin{cases} \frac{\partial J}{\partial t} = 2A_{J,\theta}, \\ \frac{\partial \theta}{\partial t} = -2W\theta, \end{cases} \tag{1.1}$$

on $(M, J, \theta) \times [0, T)$. It is the negative gradient flow of CR Einstein-Hilbert functional. Along this direction with the torsion flow (1.1), we have established the CR Li-Yau gradient estimate ([CKL]) and the Li-Yau-Hamilton

inequality ([CFTW], [CCF]) for the positive solution of CR heat equation

$$\left(\Delta_b - \frac{\partial}{\partial t}\right) u(x,t) = 0 \tag{1.2}$$

in a closed pseudohermitian $(2n+1)$-manifold with nonnegative pseudohermitian Ricci curvature and vanishing torsion tensors (see next section for definition). Here Δ_b is the time-independent sub-Laplacian operator. One of our goals in this paper is to find the CR Li-Yau gradient estimate in a closed pseudohermitian $(2n+1)$-manifold with nonvanishing torsion tensors.

Let us start with a more general setup for the Li-Yau gradient estimate in a closed manifold with a positive measure and an operator

$$L = \sum_{j=1}^{d} e_j^2 \tag{1.3}$$

with respect to the sum of squares of vector fields $e_1,\ e_2, \ldots, e_d$ which satisfies Hörmander's condition ([H]). More precisely, the vector fields $e_1,\ e_2, \ldots,$ e_d together with their commutators $Y_1, \ldots, Y_h$ up to finite order span the tangent bundle at every point of M with $d + h = \dim M$. It is to say that the commutators of $e_1,\ e_2, \ldots, e_d$ of order r (or called step r as well) can be expressed as linear combinations of $e_1,\ e_2, \ldots, e_d$ and their commutators up to the order $r-1$. The very first paper of H.-D. Cao and S.-T. Yau ([CY]) follows this line, and considers the heat equation

$$\left(L - \frac{\partial}{\partial t}\right) u(x,t) = 0. \tag{1.4}$$

They derived the gradient estimate of sum of squares of vector fields of step two ($r = 2$) in a closed manifold with a positive measure.

In this paper, with the help of a generalized curvature-dimension inequality explained below, we are able to obtain the Li-Yau gradient estimate for the CR heat equation in a closed pseudohermitian manifold of the nonvanishing torsion tensor. As consequences, we obtain the Harnack inequality and upper bound estimate for the heat kernel. With the same mentality, we generalize the Cao-Yau's gradient estimate for the sum of squares of vector fields up to order three and higher under assumption of a generalized curvature-dimension inequality.

One of the key steps in Li-Yau's method for the proof of gradient estimates is the Bochner formula involving the (Riemannian) Ricci curvature

tensor. Bakry and Emery ([BE], [BBBQ]) pioneered the approach to generalizing curvature in the context of gradient estimates by using curvature-dimension inequalities. In the CR analogue of the Li-Yau gradient estimate ([CKL]), the CR Bochner formula ([G]) is

$$\begin{aligned}(1.5)\qquad \frac{1}{2}\Delta_b\left|\nabla_b f\right|^2 &= \left|Hess(f)\right|^2 + \left\langle \nabla_b f, \nabla_b(\Delta_b f)\right\rangle + 2\left\langle J\nabla_b f, \nabla_b f_0\right\rangle \\ &\quad + (2Ric - (n-2)\,Tor)((\nabla_b f)_C\,, (\nabla_b f)_C),\end{aligned}$$

which involves a term $\langle J\nabla_b f,\ \nabla_b f_0\rangle$ that has no analogue in the Riemannian case. Here $f_0 := \mathbf{T}\varphi$ and T is the characteristic vector field. In order to deal with the extra term $\langle J\nabla_b f,\ \nabla_b f_0\rangle$ in case of vanishing torsion tensors, based on the CR Bochner formula (1.5), we can show the so-called curvature-dimension inequality (see Lemma 3.1):

$$(1.6)\qquad \Gamma_2(f,f) + \nu\Gamma_2^Z(f,f) \geq \frac{2}{n}\left|\Delta_b f\right|^2 + \left(-2k - \frac{8}{\nu}\right)\left|\nabla_b f\right|^2 + 2n\left|f_0\right|^2$$

for any smooth function $f \in C^\infty(M)$ and $\nu > 0$ and the pseudohermitian Ricci curvature bounded below by $-k$. Here

$$\Gamma_2^Z(f,\ f) := 2\left|\nabla_b f_0\right|^2$$

and

$$\Gamma_2(f,\ f) := 4\left|Hess(f)\right|^2 + 8Ric((\nabla_b f)_C\,, (\nabla_b f)_C) + 8\left\langle J\nabla_b f, \nabla_b f_0\right\rangle.$$

Before we introduce the generalized curvature-dimension inequality (1.7) which was first introduced by Baudoin and Garofalo ([BG]) in the content of sub-Riemannian geometry, it is useful to compare Cao-Yau's notations with pseudohermitian geometry.

Let J be a CR structure compatible with the contact bundle $\xi = \ker\theta$ and $\mathbf{T}$ be the characteristic vector field of the contact form θ in a closed pseudohermitian $(2n+1)$-manifold (M, J, θ) . The CR structure J decomposes $\mathbf{C}\otimes\xi$ into the direct sum of $T_{1,0}$ and $T_{0,1}$ which are eigenspaces of J with respect to i and $-i$, respectively. By choosing a frame $\{\mathbf{T}, Z_{ji}, Z_{\bar{j}}\}$ of $TM\otimes\mathbb{C}$ with respect to the Levi form such that

$$J(Z_j) = iZ_j \quad \text{and} \quad J(Z_{\bar{j}}) = -iZ_{\bar{j}},$$

then Y_1 will be the characteristic vector field $\mathbf{T}$ with $\alpha = 1$, $d = 2n$ and

$$Z_j = \frac{1}{2}(e_j - ie_{\tilde{j}}) \text{ and } Z_{\bar{j}} = \frac{1}{2}(e_j + ie_{\tilde{j}})$$

with $\widetilde{j} = n + j,\ j = 1, \ldots, n$. The operator that we are interested in this paper will be

$$L = \sum_{j=1}^{n} (e_j{}^2 + e_{\widetilde{j}}{}^2) = 2\ \Delta_b.$$

Definition 1.1. Let M be a smooth connected manifold with a positive measure and vector fields $\{e_i, Y_\alpha\}_{i \in I_d, \alpha \in \Lambda}$ spanning the tangent space TM. For $\rho_1 \in \mathbb{R},\ \rho_2 > 0,\ \kappa \geq 0,\ m > 0$, we say that M satisfies the generalized curvature-dimension inequality $CD(\rho_1, \rho_2, \kappa, m)$ if

$$\frac{1}{m}(Lf)^2 + (\rho_1 - \frac{\kappa}{\nu})\Gamma(f,\ f) + \rho_2 \Gamma^Z(f,\ f) \leq \Gamma_2(f,\ f) + \nu \Gamma_2^Z(f,\ f) \tag{1.7}$$

for any smooth function $f \in C^\infty(M)$ and $\nu > 0$. Here

$$\begin{aligned}
\Gamma(f,\ f) &:= \sum_{j \in I_d} |e_j f|^2, \\
\Gamma^Z(f,\ f) &:= \sum_{\alpha \in I_h} |Y_\alpha f|^2, \\
\Gamma_2(f,\ f) &:= \frac{1}{2}\left[L(\Gamma(f,\ f)) - 2 \sum_{j \in I_d} (e_j f)(e_j L f) \right], \\
\Gamma_2^Z(f,\ f) &:= \frac{1}{2}\left[L(\Gamma^Z(f,\ f)) - 2 \sum_{\alpha \in I_h} (Y_\alpha f)(Y_\alpha L f) \right].
\end{aligned}$$

Note that we also have

$$\Gamma_2(f,\ f) = \sum_{i,j \in I_d} |e_i e_j f|^2 + \sum_{j \in I_d} (e_j f)([L, e_j] f)$$

and

$$\Gamma_2^Z(f,\ f) = \sum_{i \in I_d, \alpha \in I_h} |e_i Y_\alpha f|^2 + \sum_{\alpha \in I_h} (Y_\alpha f)([L, Y_\alpha] f).$$

In Lemma 3.2, we will derive a curvature-dimension inequality (1.7) in a closed pseudohermitian manifold of the nonvanishing torsion tensor. As a result, we are able to obtain the following CR Li-Yau gradient estimate which is served as a generalization of the CR Li-Yau gradient estimate in a closed pseudohermitian $(2n+1)$-manifold with nonnegative pseudohermitian Ricci curvature and vanishing torsion as in [CKL], [CCKL] and [BG].

Theorem 1.1. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold with*

$$(2Ric - (n-2)\,Tor)(Z, Z) \geq -k\,\langle Z, Z\rangle$$

and

$$\max_{i,j\in I_n} |A_{ij}| \leq \overline{A}, \quad \max_{i,j\in I_n} \left|A_{ij,\bar{i}}\right| \leq \overline{B}$$

for $Z \in \Gamma(T_{1,0}M)$, $k \geq 0$ and $\overline{A}$, $\overline{B}$ as positive constants. Suppose that $u(x,t)$ is the positive solution of (1.2) on $M \times [0, \infty)$. Then there exist $\delta_0 = \delta_0(n, k, \overline{A}, \overline{B}) >> 1$ such that $f(x,t) = \ln u(x,t)$ satisfies the following gradient estimate

$$|\nabla_b f|^2 - \delta f_t < \frac{C_1}{t} + C_2 \tag{1.8}$$

for $\delta \geq \delta_0$ and

$$C_1 = \frac{1}{2}\max\left\{ n(n+1)\delta^2 + \frac{8\sqrt{3}(n+1)^2\delta^2}{(\delta-\delta_0)}, \right.$$
$$\left. \frac{3n(n+1)\delta^2}{4(\delta-\delta_0)^2}\left[\left(k + \frac{\overline{B}^2}{2(n+1)}\right)\frac{(\delta-\delta_0)}{2n(n+1)\overline{A}\delta} + \frac{16(n+1)}{n}\right]^2 \right\}.$$

$$C_2 = \frac{1}{2}\max\left\{ \left(k + \frac{\overline{B}^2}{2(n+1)}\right)\frac{\sqrt{3}n(n+1)\delta^2}{2(\delta-\delta_0)} + 16\sqrt{3}(n+1)^2\frac{\delta^3\overline{A}}{(\delta-\delta_0)^2}, \right.$$
$$\left. \frac{3(n+1)\delta}{8n\overline{A}(\delta-\delta_0)}\left(k + \frac{\overline{B}^2}{2(n+1)} + \frac{32n(n+1)\delta\overline{A}}{(\delta-\delta_0)}\right)^2 \right\}.$$

As a consequence, we have $C_2 = 0$ if $k = 0$ and $\overline{A} = 0$. Hence, we have

Corollary 1.1. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold with nonnegative pseudohermitian Ricci curvature and vanishing torsion. If $u(x,t)$ is the positive solution of (1.2) on $M \times [0, \infty)$. Then $f(x,t) = \ln u(x,t)$ satisfies the following gradient estimate*

$$|\nabla_b f|^2 - \delta f_t < \frac{C_1}{t}. \tag{1.9}$$

Remark 1.1. In fact, in [CCKL], we get the following CR Li-Yau gradient estimate in a closed pseudohermitian $(2n+1)$-manifold with nonnegative

pseudohermitian Ricci curvature and vanishing torsion. That is

$$|\nabla_b f|^2 - \left(1 + \frac{3}{n}\right) f_t + \frac{n}{3} t (f_0)^2 < \frac{(\frac{9}{n} + 6 + n)}{t},$$

but where we can not deal with the case of nonvanishing torsion tensors. The major different here is: we apply the generalized curvature-dimension inequality, which holds as in Lemma 3.2, and Cao-Yau's method ([CY]) to derive the gradient estimate in a closed pseudohermitian $(2n+1)$ -manifold with nonvanishing torsion tensors.

Next we have the CR version of Li-Yau Harnack inequality and upper bound estimate for the heat kernel as in [CFTW] and [CY].

Theorem 1.2. *Under the same hypothesis of Theorem 1.1, suppose that u is the positive solution of*

$$\left(\Delta_b - \frac{\partial}{\partial t}\right) u = 0$$

on $M \times [0, +\infty)$. Then for any $x_1, x_2 \in M$ and $0 < t_1 < t_2 < +\infty$, there exists a constant $\delta_0(n, k, \overline{A}, \overline{B}) > 1$ such that

$$\frac{u(x_1, t_1)}{u(x_2, t_2)} \leq \left(\frac{t_2}{t_1}\right)^{\frac{C_1'(n,\delta)}{\delta}} \exp\left(\frac{\delta}{4} \frac{d_{cc}(x_1, x_2)^2}{t_2 - t_1} + \frac{C_2'(n, k, \delta, \overline{A}, \overline{B})}{\delta}(t_2 - t_1)\right)$$

for $\delta \geq \delta_0(n, k, \overline{A}, \overline{B})$. Here we denote the Carnot-Carathéodory distance in (M, J, θ) by d_{cc}.

Theorem 1.3. *Under the same hypothesis of Theorem 1.1, suppose that $H(x, y, t)$ is the heat kernel of*

$$\left(\Delta_b - \frac{\partial}{\partial t}\right) u = 0$$

on $M \times [0, +\infty)$. Then there exists a constant $\delta_1 > 0$ such that

$$H(x, y, t) \leq C(\varepsilon)^{\delta_1} \frac{1}{\sqrt{vol\left(B_x(\sqrt{t})\right) vol\left(B_y(\sqrt{t})\right)}} \times \exp\left(\frac{C_2'(n, k, \delta, \overline{A}, \overline{B})}{\delta} \varepsilon t - \frac{d_* cc(x, y)^2}{(4 + \varepsilon) t}\right)$$

for $\varepsilon \in (0, 1)$ and $C(\varepsilon) \to +\infty$ as $\varepsilon \to 0^+$.

In the Cao-Yau gradient estimate for a positive solution of an operator with respect to the sum of squares of vector fields of step 2, the key estimates are (2.10), (2.12) and (2.14) of ([CY]). This in fact, resembles the generalized curvature-dimension inequality (1.7) with some certain ρ_1, ρ_2, κ and m. However this is not the case for step 3 and up. Then, as in Theorem 1.4, it was an important insight that one can use the generalized curvature-dimension inequality as a substitute for the lower Ricci curvature bound on spaces where a direct generalization of Ricci curvature is not available.

We start to setup the Li-Yau gradient estimate for a positive solution of an operator with respect to the sum of squares of vector fields of higher step. For simplicity, we assume that M is of step 3, i.e.

$$[e_i, [e_j, [e_k, e_l]]] = a^n_{ijkl} e_n + b^\eta_{ijkl} Y'_\eta + c^A_{ijkl} Y''_A \tag{1.10}$$

for $a^n_{ijkl}, b^\eta_{ijkl}, c^A_{ijkl} \in C^\infty(M)$ with

$$\{Y_\alpha\}_{\alpha\in\Lambda} := \left\{Y'_\eta = [e_i, e_j]\right\}_{i,j\in I_d} \cup \left\{Y''_A = [e_i, [e_j, e_k]]\right\}_{i,j,k\in I_d}.$$

We denote the supremum of coefficients as:

$$\begin{cases} a = \sup\left|a^n_{ijkl}\right|, \quad b = \sup\left|b^\eta_{ijkl}\right|, \quad c = \sup\left|c^A_{ijkl}\right|, \\ a' = \sup\left|e_h a^n_{ijkl}\right|, \quad b' = \sup\left|e_h b^\eta_{ijkl}\right|, \quad c' = \sup\left|e_h c^A_{ijkl}\right|. \end{cases}$$

Theorem 1.4. *Let M be a smooth connected manifold with a positive measure satisfying the generalized curvature-dimension inequality $CD(\rho_1, \rho_2, \kappa, m)$ and let L be an operator with respect to the sum of squares of vector fields $\{e_1, e_2, \ldots, e_d\}$ satisfying the condition (1.10). Suppose that u is the positive solution of*

$$\left(L - \frac{\partial}{\partial t}\right) u = 0 \tag{1.11}$$

on $M \times [0, +\infty)$. Then for all $\frac{1}{2} < \lambda < \frac{2}{3}$, there exists

$$\delta_0 = \delta_0(\lambda, \rho_1, \rho_2, \kappa, d, h) > 1$$

such that for any $\delta > \delta_0$

$$\sum_{j\in I_d} \frac{|e_j u|^2}{u^2} + \sum_{\alpha\in\Lambda} \left(1 + \frac{|Y_\alpha u|^2}{u^2}\right)^\lambda - \delta \frac{u_t}{u} \le \frac{C_1}{t} + C_2 + C_3 t^{\frac{\lambda}{\lambda-1}},$$

where C_1, C_2, C_3 *are all positive constants depending on* $d, \lambda, \delta, a, a', b, b', c, c', \rho_1, \rho_2, \kappa, m$.

Remark 1.2. 1. In the paper of [BG], they proved the L^p version of Li-Yau type gradient estimates for $2 \leq p \leq \infty$ under the assumption of the generalized curvature-dimension inequality via the semigroup method in the sub-Riemannian geometry setting.

2. We can obtain the Li-Yau Harnack inequality and upper bound estimate for the heat kernel of $L - \frac{\partial}{\partial t}$ with respect to the sum of squares of vector fields as in [CY]. We also refer to [JS], [KS1], [KS2] and [M] for some details along this direction.

We briefly describe the methods used in our proofs. In section 3, we derive a generalized curvature-dimension inequality in a closed pseudohermitian $(2n+1)$-manifold. In order to gain insight for the estimate, we first derive the CR Li-Yau gradient estimate and the Harnack inequality for the CR heat equation in a closed pseudohermitian manifold as in section 4. Then, for simplicity, we will derive the Li-Yau gradient estimate for the sum of squares of vector fields of step three as in section 5. Similar estimates will hold for the sum of squares of vector fields of higher step as well.

Acknowledgement. The authors would like to express their profound gratitude to Prof. S.-T. Yau for bringing this project to them and his inspirations of the Li-Yau gradient estimate for the sum of squares of vector fields.

2. Preliminary

We introduce some basic materials about a pseudohermitian manifold (see [DT] , [CKL], and [L] for more details). Let (M, ξ) be a $(2n+1)$-dimensional, orientable, contact manifold with contact structure ξ. A CR structure compatible with ξ is an endomorphism $J : \xi \to \xi$ such that $J^2 = -1$. We also assume that J satisfies the integrability condition: If X and Y are in ξ, then so are $[JX, Y] + [X, JY]$ and $J([JX, Y] + [X, JY]) = [JX, JY] - [X, Y]$.

Let $\{\mathbf{T}, Z_\alpha, Z_{\bar{\alpha}}\}$ be a frame of $TM \otimes \mathbb{C}$, where Z_α is any local frame of $T_{1,0}$, $Z_{\bar{\alpha}} = \overline{Z_\alpha} \in T_{0,1}$ and $\mathbf{T}$ is the characteristic vector field. Then $\{\theta, \theta^\alpha, \theta^{\bar{\alpha}}\}$, the coframe dual to $\{\mathbf{T}, Z_\alpha, Z_{\bar{\alpha}}\}$, satisfies

$$d\theta = ih_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\overline{\beta}} \tag{2.1}$$

for some positive definite hermitian matrix of functions $(h_{\alpha\bar{\beta}})$. If we have this contact structure, we also call such M a strictly pseudoconvex CR $(2n+1)$-manifold.

The Levi form $\langle\ ,\ \rangle_{L_\theta}$ is the Hermitian form on $T_{1,0}$ defined by

$$\langle Z, W\rangle_{L_\theta} = -i\left\langle d\theta, Z\wedge\overline{W}\right\rangle.$$

We can extend $\langle\ ,\ \rangle_{L_\theta}$ to $T_{0,1}$ by defining $\left\langle\overline{Z},\overline{W}\right\rangle_{L_\theta} = \overline{\langle Z, W\rangle}_{L_\theta}$ for all $Z, W \in T_{1,0}$. The Levi form induces naturally a Hermitian form on the dual bundle of $T_{1,0}$, denoted by $\langle\ ,\ \rangle_{L_\theta^*}$, and hence on all the induced tensor bundles. Integrating the Hermitian form (when acting on sections) over M with respect to the volume form $d\mu = \theta\wedge(d\theta)^n$, we get an inner product on the space of sections of each tensor bundle.

The pseudohermitian connection of (J,θ) is the connection ∇ on $TM\otimes\mathbb{C}$ (and extended to tensors) given in terms of a local frame $Z_\alpha \in T_{1,0}$ by

$$\nabla Z_\alpha = \omega_\alpha{}^\beta\otimes Z_\beta,\quad \nabla Z_{\bar{\alpha}} = \omega_{\bar{\alpha}}{}^{\bar{\beta}}\otimes Z_{\bar{\beta}},\quad \nabla T = 0,$$

where $\omega_\alpha{}^\beta$ are the 1-forms uniquely determined by the following equations:

$$\begin{aligned} d\theta^\beta &= \theta^\alpha\wedge\omega_\alpha{}^\beta + \theta\wedge\tau^\beta,\\ 0 &= \tau_\alpha\wedge\theta^\alpha,\\ 0 &= \omega_\alpha{}^\beta + \omega_{\bar{\beta}}{}^{\bar{\alpha}}, \end{aligned}$$

We can write (by Cartan lemma) $\tau_\alpha = A_{\alpha\gamma}\theta^\gamma$ with $A_{\alpha\gamma} = A_{\gamma\alpha}$. The curvature of Tanaka-Webster connection, expressed in terms of the coframe $\{\theta=\theta^0,\theta^\alpha,\theta^{\bar{\alpha}}\}$, is

$$\begin{aligned} \Pi_\beta{}^\alpha &= \overline{\Pi_{\bar{\beta}}{}^{\bar{\alpha}}} = d\omega_\beta{}^\alpha - \omega_\beta{}^\gamma\wedge\omega_\gamma{}^\alpha,\\ \Pi_0{}^\alpha &= \Pi_\alpha{}^0 = \Pi_0{}^{\bar{\beta}} = \Pi_{\bar{\beta}}{}^0 = \Pi_0{}^0 = 0. \end{aligned}$$

Webster showed that $\Pi_\beta{}^\alpha$ can be written

$$\Pi_\beta{}^\alpha = R_\beta{}^\alpha{}_{\rho\bar{\sigma}}\theta^\rho\wedge\theta^{\bar{\sigma}} + W_\beta{}^\alpha{}_\rho\theta^\rho\wedge\theta - W^\alpha{}_{\beta\bar{\rho}}\theta^{\bar{\rho}}\wedge\theta + i\theta_\beta\wedge\tau^\alpha - i\tau_\beta\wedge\theta^\alpha$$

where the coefficients satisfy

$$R_{\beta\bar{\alpha}\rho\bar{\sigma}} = \overline{R_{\alpha\bar{\beta}\sigma\bar{\rho}}} = R_{\bar{\alpha}\beta\bar{\sigma}\rho} = R_{\rho\bar{\alpha}\beta\bar{\sigma}},\quad W_{\beta\bar{\alpha}\gamma} = W_{\gamma\bar{\alpha}\beta}.$$

Here $R_\gamma{}^\delta{}_{\alpha\bar{\beta}}$ is the pseudohermitian curvature tensor, $R_{\alpha\bar{\beta}} = R_\gamma{}^\gamma{}_{\alpha\bar{\beta}}$ is the pseudohermitian Ricci curvature tensor and $A_{\alpha\beta}$ is the pseudohermitian

torsion. Furthermore, we define the bi-sectional curvature

$$R_{\alpha\bar{\alpha}\beta\bar{\beta}}(X,Y) = R_{\alpha\bar{\alpha}\beta\bar{\beta}}X_\alpha X_{\bar{\alpha}} Y_\beta Y_{\bar{\beta}}$$

and the bi-torsion tensor

$$T_{\alpha\bar{\beta}}(X,Y) := i(A_{\bar{\beta}\bar{\rho}}X^{\bar{\rho}}Y_\alpha - A_{\alpha\rho}X^\rho Y_{\bar{\beta}})$$

and the torsion tensor

$$Tor(X,Y) := h^{\alpha\bar{\beta}}T_{\alpha\bar{\beta}}(X,Y) = i(A_{\bar{\alpha}\bar{\rho}}X^{\bar{\rho}}Y^{\bar{\alpha}} - A_{\alpha\rho}X^\rho Y^\alpha)$$

for any $X = X^\alpha Z_\alpha,\ Y = Y^\alpha Z_\alpha$ in $T_{1,0}$.

We will denote the components of the covariant derivatives with indices preceded by a comma; thus write $A_{\alpha\beta,\gamma}$. The indices $\{0,\alpha,\bar{\alpha}\}$ indicate derivatives with respect to $\{T, Z_\alpha, Z_{\bar{\alpha}}\}$. For derivatives of a scalar function, we will often omit the comma, for instance, $u_\alpha = Z_\alpha u,\ u_{\alpha\bar{\beta}} = Z_{\bar{\beta}}Z_\alpha u - \omega_\alpha{}^\gamma(Z_{\bar{\beta}})Z_\gamma u$.In particular,

$$|\nabla_b u|^2 = 2\sum_\alpha u_\alpha u_{\bar{\alpha}}, \quad |\nabla_b^2 u|^2 = 2\sum_{\alpha,\beta}(u_{\alpha\beta}u_{\bar{\alpha}\bar{\beta}} + u_{\alpha\bar{\beta}}u_{\bar{\alpha}\beta}).$$

Also

$$\Delta_b u = Tr\left((\nabla^H)^2 u\right) = \sum_\alpha (u_{\alpha\bar{\alpha}} + u_{\bar{\alpha}\alpha}).$$

Next we recall the following commutation relations ([L]). Let φ be a scalar function and $\sigma = \sigma_\alpha\theta^\alpha$ be a $(1,0)$ form, $\varphi_0 = \mathbf{T}\varphi$, then we have

$$\begin{aligned}
\varphi_{\alpha\beta} &= \varphi_{\beta\alpha},\\
\varphi_{\alpha\bar{\beta}} - \varphi_{\bar{\beta}\alpha} &= ih_{\alpha\bar{\beta}}\varphi_0,\\
\varphi_{0\alpha} - \varphi_{\alpha 0} &= A_{\alpha\beta}\varphi^\beta,\\
\sigma_{\alpha,0\beta} - \sigma_{\alpha,\beta 0} &= \sigma_{\alpha,\bar{\gamma}}A^{\bar{\gamma}}{}_\beta - \sigma^{\bar{\gamma}}A_{\alpha\beta,\bar{\gamma}},\\
\sigma_{\alpha,0\bar{\beta}} - \sigma_{\alpha,\bar{\beta}0} &= \sigma_{\alpha,\gamma}A^\gamma{}_{\bar{\beta}} + \sigma^{\bar{\gamma}}A_{\bar{\gamma}\bar{\beta},\alpha},
\end{aligned}$$

and

$$\begin{aligned}
&(1)\ \varphi_{e_j e_{\tilde{k}}} - \varphi_{e_{\tilde{k}} e_j} = 2h_{j\bar{k}}\varphi_0,\\
&(2)\ \varphi_{e_j e_k} - \varphi_{e_k e_j} = 0,\\
&(3)\ \varphi_{0e_j} - \varphi_{e_j 0} = \varphi_{e_l}\,\mathrm{Re}\,A_j^{\bar{l}} - \varphi_{e_{\tilde{l}}}\,\mathrm{Im}\,A_j^{\bar{l}},\\
&(4)\ \varphi_{0e_{\tilde{j}}} - \varphi_{e_{\tilde{j}}0} = -\varphi_{e_l}\,\mathrm{Im}\,A_j^{\bar{l}} - \varphi_{e_{\tilde{l}}}\mathrm{Re}\,A_j^{\bar{l}}.
\end{aligned} \tag{2.2}$$

Finally we introduce the concept about the Carnot-Carathéodory distance in a closed pseudohermitian manifold.

Definition 2.1. A piecewise smooth curve $\gamma : [0,1] \to M$ is said to be horizontal if $\gamma\,'(t) \in \xi$ whenever $\gamma\,'(t)$ exists. The length of γ is then defined by

$$l(\gamma) = \int_0^1 \left\langle \gamma\,'(t), \gamma\,'(t) \right\rangle_{L_\theta}^{\frac{1}{2}} dt.$$

The Carnot-Carathéodory distance between two points $p,\ q \in M$ is

$$d_{cc}(p,q) = \inf \left\{ l(\gamma) |\ \gamma \in C_{p,q} \right\},$$

where $C_{p,q}$ is the set of all horizontal curves joining p and q. By Chow connectivity theorem [Cho], there always exists a horizontal curve joining p and q, so the distance is finite. The diameter d_c is defined by

$$d_c(M) = \sup \left\{ d_c(p,q) |\ p,q \in M \right\}.$$

Note that there is a minimizing geodesic joining p and q so that its length is equal to the distance $d_{cc}(p,q)$.

3. A generalized curvature-dimension inequality

Now we proceed to derive a curvature-dimension inequality in a closed pseudohermitian $(2n+1)$-manifold under the specific assumptions on the pseudohermitian Ricci curvature tensor and the torsion tensor. In particular, in the case of vanishing torsion tensors, we have the following lemma.

Lemma 3.1. *If (M, J, θ) is a pseudohermitian $(2n+1)$-manifold of vanishing torsion with*

$$2Ric\,(Z,Z) \geq -k \left\langle Z, Z \right\rangle \tag{3.1}$$

for $Z \in \Gamma\,(T_{1,0}M)$, $k \geq 0$, then M satisfies the curvature-dimension inequality $CD(-k, 2n, 4, 2n)$.

Proof. By the CR Bochner formulae (see [G])

$$\frac{1}{2}\Delta_b |\nabla_b f|^2 = |Hess(f)|^2 + \langle \nabla_b f, \nabla_b(\Delta_b f)\rangle + (2Ric - (n-2)\,Tor)((\nabla_b f)_c, (\nabla_b f)_c) + 2\,\langle J\nabla_b f, \nabla_b f_0\rangle\,,$$

where $(\nabla_b f)_c$ is the $T_{1,0}M$-component of $(\nabla_b f)$, we have

$$\frac{1}{4}\Gamma_2(f,f) = |Hess(f)|^2 + (2Ric - (n-2)\,Tor)((\nabla_b f)_c, (\nabla_b f)_c) + 2\,\langle J\nabla_b f, \nabla_b f_0\rangle\,.$$

With the equality

$$\frac{1}{2}\Gamma_2^Z(f,f) = |\nabla_b f_0|^2 + f_0[\Delta_b, T]f,$$

we have

$$\begin{aligned}(3.2)\qquad &\Gamma_2(f,f) + \nu\Gamma_2^Z(f,f)\\ &= 4[|Hess(f)|^2 + (2Ric - (n-2)\,Tor)((\nabla_b f)_c, (\nabla_b f)_c)\\ &\quad + 2\,\langle J\nabla_b f, \nabla_b f_0\rangle] + 2\nu\,|\nabla_b f_0|^2 + 2\nu f_0[\Delta_b, T]f.\end{aligned}$$

On the other hand, we have

$$(3.3)\qquad |Hess(f)|^2 = 2\left(\sum_{i,j\in I_n} |f_{ij}|^2 + \sum_{i,j\in I_n}\left|f_{i\bar{j}}\right|^2\right) \geq \frac{1}{2n}\,|\Delta_b f|^2 + \frac{n}{2}\,|f_0|^2$$

and

$$(3.4)\qquad \langle J\nabla_b f, \nabla_b f_0\rangle \geq -\frac{|\nabla_b f|^2}{\nu} - \frac{\nu}{4}\,|\nabla_b f_0|^2\,.$$

Now it follows from (3.2), (3.3), (3.4) and curvature assumptions

$$\begin{aligned}(3.5)\qquad &\Gamma_2(f,f) + \nu\Gamma_2^Z(f,f)\\ &\geq \left(\frac{2}{n}\,|\Delta_b f|^2 + 2n\,|f_0|^2\right) + 4(2Ric - (n-2)\,Tor)((\nabla_b f)_c, (\nabla_b f)_c)\\ &\quad - 8\frac{|\nabla_b f|^2}{\nu} + 2\nu f_0[\Delta_b, T]f\\ &\geq \frac{2}{n}\,|\Delta_b f|^2 + \left(-2k - \frac{8}{\nu}\right)|\nabla_b f|^2 + 2n\,|f_0|^2 + 2\nu f_0[\Delta_b, T]f.\end{aligned}$$

Finally, it follows from the commutation relation ([CKL]) that

$$\Delta_b f_0 = (\Delta_b f)_0 + 2[(A_{\alpha\beta} f^{\alpha})^{\beta} + (A_{\overline{\alpha}\overline{\beta}} f^{\overline{\alpha}})^{\overline{\beta}}]. \tag{3.6}$$

But $A_{\alpha\beta} = 0$, hence

$$[\Delta_b, T]\, f = 0.$$

All these imply

$$\Gamma_2(f,f) + \nu\Gamma_2^Z(f,f) \geq \frac{2}{n}\left|\Delta_b f\right|^2 + \left(-2k - \frac{8}{\nu}\right)\left|\nabla_b f\right|^2 + 2n\left|f_0\right|^2. \qquad \square$$

Remark 3.1. In a closed pseudohermitian $(2n+1)$-manifold of vanishing torsion tensors, the CR Bochner formulae (1.5) is equivalent to the curvature-dimension inequality (1.7) which also observed in the paper of [BG].

As for the curvature-dimension inequality in a closed pseudohermitian $(2n+1)$-manifold of nonvanishing torsion tensors, we have

Lemma 3.2. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold of*

$$(2Ric - (n-2)\,Tor)\,(Z,Z) \geq -k\,\langle Z, Z\rangle$$

for $Z \in \Gamma\,(T_{1,0}M)$, $k \geq 0$ and

$$\max_{i,j\in I_n} |A_{ij}| \leq \overline{A}, \quad \max_{i,j\in I_n} \left|A_{ij,\bar{i}}\right| \leq \overline{B}$$

for nonnegative constants $\overline{A}$, $\overline{B}$, Then M satisfies the curvature-dimension inequality $CD(-k - 2nN\varepsilon_1\overline{B}^2, \frac{2n}{m} - \frac{2n^2N}{\varepsilon_1} - \frac{2mn^2N^2\overline{A}^2}{m-1}, 4, 2mn)$ for $1 < m < +\infty$, $0 < \varepsilon_1 < +\infty$ and smaller $N > 0$ such that

$$\left(\frac{2n}{m} - \frac{2n^2N}{\varepsilon_1} - \frac{2mn^2N^2\overline{A}^2}{m-1}\right) > 0$$

and $0 < \nu \leq N$.

Proof. It follows from (3.2), (3.4) and (3.6) that

$$\Gamma_2(f,\ f)+\nu\Gamma_2^Z(f,\ f)\geq 8\left[\sum_{\alpha,\beta}\left(|f_{\alpha\beta}|^2+\left|f_{\alpha\overline{\beta}}\right|^2\right)\right]-\left(2k+\frac{8}{\nu}\right)|\nabla_b f|^2 \\ -8\nu\,|f_0|\sum_{\alpha,\beta}\left|(A_{\overline{\alpha}\overline{\beta},\alpha}f_\beta+A_{\overline{\alpha}\overline{\beta}}f_{\beta\alpha})\right|.$$

Note that by using the Young inequality

$$|f_0|\left(\left|A_{\overline{\alpha}\overline{\beta},\alpha}f_\beta\right|+\left|A_{\overline{\alpha}\overline{\beta}}f_{\beta\alpha}\right|\right) \\ \leq\frac{|f_0|^2}{4\varepsilon_1}+\varepsilon_1\left|A_{\overline{\alpha}\overline{\beta},\alpha}f_\beta\right|^2+\frac{|f_0|^2}{4\varepsilon_2}+\varepsilon_2\left|A_{\overline{\alpha}\overline{\beta}}f_{\beta\alpha}\right|^2, \tag{3.7}$$

for $\varepsilon_1,\varepsilon_2>0$. Choose

$$\varepsilon_2=\frac{m-1}{mN\overline{A}^2}$$

for $m>1$ and N with $\nu\leq N$. This implies that $\left(1-N\varepsilon_2\overline{A}^2\right)=\frac{1}{m}$.

It follows from (3.3) that

$$\begin{aligned}
&\Gamma_2(f,\ f)+\nu\Gamma_2^Z(f,\ f)\\
&\geq 8\sum_{\alpha,\beta}\left|f_{\alpha\overline{\beta}}\right|^2+8\sum_{\alpha,\beta}\left(1-\nu\varepsilon_2\left|A_{\overline{\alpha}\overline{\beta}}\right|^2\right)|f_{\beta\alpha}|^2-\left(2k+\frac{8}{\nu}\right)|\nabla_b f|^2\\
&\quad-2\nu\sum_{\alpha,\beta}\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\right)|f_0|^2-8\nu\varepsilon_1\sum_{\alpha,\beta}\left|A_{\overline{\alpha}\overline{\beta},\alpha}f_\beta\right|^2\\
&\geq\frac{8}{m}\sum_{\alpha,\beta}\left(\left|f_{\alpha\overline{\beta}}\right|^2+|f_{\beta\alpha}|^2\right)-\left(2k+\frac{8}{\nu}+4N\varepsilon_1 n\overline{B}^2\right)|\nabla_b f|^2\\
&\quad-2n^2N\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\right)|f_0|^2\\
&\geq\frac{4}{m}\left(\frac{1}{2n}|\Delta_b f|^2+\frac{n}{2}|f_0|^2\right)-\left(2k+\frac{8}{\nu}+4N\varepsilon_1 n\overline{B}^2\right)|\nabla_b f|^2\\
&\quad-2n^2N\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\right)|f_0|^2\\
&\geq\frac{1}{2mn}(Lf)^2+\left(-k-2nN\varepsilon_1\overline{B}^2-\frac{4}{\nu}\right)\Gamma(f,f)\\
&\quad+\left(\frac{2n}{m}-\frac{2n^2N}{\varepsilon_1}-\frac{2mn^2N^2\overline{A}^2}{m-1}\right)\Gamma^Z(f,f).
\end{aligned}$$

Now we make N smaller such that

$$\frac{2n}{m} - \frac{2n^2N}{\varepsilon_1} - \frac{2mn^2N^2\overline{A}^2}{m-1} > 0.$$

Then we are done. □

Remark 3.2. By choosing $\overline{A} = 0 = \overline{B}$, $m \to 1^+, \varepsilon_1 \to +\infty$ and noting the inequality (3.7) in Lemma 3.2, we are also able to have the same conclusion in Lemma 3.1.

4. The CR Li-Yau gradient estimate

In this section, based on methods of [CY] and [CKL], we first derive the CR Li-Yau gradient estimate and the Harnack inequality for the CR heat equation in a closed pseudohermitian manifold. Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold and $u(x,t)$ be a positive solution of the CR heat equation

$$\left(\Delta_b - \frac{\partial}{\partial t}\right) u\,(x,t) = 0$$

on $M \times [0,\ \infty)$. We denote that $f(x,t) = \ln u(x,t)$. Modified by [CKL], we define a real-valued function $F(x,t,\beta,\delta) : M \times 0, T) \times \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}$ by

$$F(x,t,\beta,\delta) = t\left[\sum_{j\in I_d} |e_j f|^2 + \beta t \sum_{\alpha\in I_h} |Y_\alpha f|^2 - \delta f_t\right] \tag{4.1}$$

for $x \in M,\ t \geq 0,\ \beta > 0,\ \delta > 0$. Note that $\beta \to 0^+$ if $T \to \infty$ as in the proof.

Lemma 4.1. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold and $u(x,t)$ be a positive solution of the CR heat equation*

$$\left(L - \frac{\partial}{\partial t}\right) u\,(x,t) = 0$$

on $M \times [0,\ \infty)$. We have the identity

$$\left(L - \frac{\partial}{\partial t}\right) F = -\frac{F}{t} + 2t[\Gamma_2(f,\ f) + \beta t \Gamma_2^Z(f,\ f)] + 4\beta t^2 \sum_{j \in I_d, \alpha \in I_h} (e_j f)(Y_\alpha f)([e_j, Y_\alpha] f) - 2\sum_{j \in I_d} (e_j f)(e_j F) - \beta t \sum_{\alpha \in I_h} |Y_\alpha f|^2 . \tag{4.2}$$

Proof. It follows from definitions of $\Gamma_2(f,\ f)$ and $\Gamma_2^Z(f,\ f)$ that

$$\begin{aligned} LF &= t\left[L(\Gamma(f,\ f)) + \beta t L(\Gamma^Z(f,\ f)) - \delta L f_t\right] \\ &= t\Bigg\{\left[2\Gamma_2(f,\ f) + 2\sum_{j \in I_d} (e_j f)\,(e_j L f)\right] \\ &\quad + \beta t\left[2\Gamma_2^Z(f,\ f) + 2\sum_{\alpha \in I_h} (Y_\alpha f)\,(Y_\alpha L f)\right] - \delta L f_t\Bigg\}. \end{aligned}$$

Then

$$\begin{aligned} \left(L - \frac{\partial}{\partial t}\right) F = -\frac{F}{t} + t\Bigg[&2\Gamma_2(f,\ f) \\ &+ 2\beta t \Gamma_2^Z(f,\ f) + 2\sum_j (e_j f)\, e_j \left(L - \frac{\partial}{\partial t}\right) f \\ &+ 2\beta t \sum_\alpha (Y_\alpha f)\, Y_\alpha \left(L - \frac{\partial}{\partial t}\right) f \\ &- \beta \sum_\alpha (Y_\alpha f)^2 - \delta \frac{\partial}{\partial t}\left(L - \frac{\partial}{\partial t}\right) f\Bigg]. \end{aligned} \tag{4.3}$$

Since

$$\left(L - \frac{\partial}{\partial t}\right) f = -\sum_j |e_j f|^2 = -\frac{F}{t} + \beta t \sum_\alpha |Y_\alpha f|^2 - \delta f_t,$$

we obtain

(4.4a)

$$
\begin{aligned}
&2\sum_j (e_j f)\, e_j \left(L - \frac{\partial}{\partial t}\right) f + 2\beta t \sum_\alpha (Y_\alpha f)\, Y_\alpha \left(L - \frac{\partial}{\partial t}\right) f \\
&- \beta \sum_\alpha (Y_\alpha f)^2 - \delta \frac{\partial}{\partial t}\left(L - \frac{\partial}{\partial t}\right) f \\
&= 2\sum_j (e_j f)\, e_j \left(-\frac{F}{t} + \beta t \sum_\alpha |Y_\alpha f|^2 - \delta f_t\right) \\
&\quad + 2\beta t \sum_\alpha (Y_\alpha f)\, Y_\alpha \left(-\sum_j |e_j f|^2\right) - \beta \sum_\alpha (Y_\alpha f)^2 - \delta \frac{\partial}{\partial t}\left(L - \frac{\partial}{\partial t}\right) f \\
&= 2\beta t \left[\sum_j (e_j f)\, e_j \left(\sum_\alpha |Y_\alpha f|^2\right) + \sum_\alpha (Y_\alpha f)\, Y_\alpha \left(-\sum_j |e_j f|^2\right)\right] \\
&\quad + 2\sum_j (e_j f)\, e_j \left(-\frac{F}{t} - \delta f_t\right) - \beta \sum_\alpha (Y_\alpha f)^2 - \delta \frac{\partial}{\partial t}\left(-\sum_j |e_j f|^2\right) \\
&= 4\beta t \sum_{j,\alpha} (e_j f)(Y_\alpha f)([e_j, Y_\alpha] f) - \frac{2}{t}\sum_j (e_j f)(e_j F) - \beta \sum_\alpha |Y_\alpha f|^2 .
\end{aligned}
$$

Substitute (4.4a) into (4.3), we have the identity (4.2). □

As a consequence of the identity (4.2), we have proposition 4.2.

Proposition 4.1. *If M satisfies the curvature-dimension inequality $CD(\rho_1, \rho_2, \kappa, m)$ for $\rho_1 \in \mathbb{R}, \rho_2 > 0, \kappa \geq 0, m > 0$, then*

$$
\begin{aligned}
\left(L - \frac{\partial}{\partial t}\right) F \geq -\frac{F}{t} + 2t \left[\frac{1}{m}(Lf)^2 + \left(\rho_1 - \frac{\kappa}{\beta t}\right)\Gamma(f,f) + \rho_2 \Gamma^Z(f,f)\right] \\
+ 4\beta t^2 \sum_{j \in I_d, \alpha \in I_l} (e_j f)(Y_\alpha f)([e_j, Y_\alpha] f) \\
- 2\sum_{j \in I_d} (e_j f)(e_j F) - \beta t \sum_{\alpha \in I_h} |Y_\alpha f|^2 .
\end{aligned} \tag{4.5}
$$

Now we proceed to prove **Theorem 1.1 :**

Proof. Note that M satisfies the curvature-dimension inequality $CD(\rho_1, \rho_2, \kappa, m)$ with $\rho_1 < 0, \rho_2 > 0, \kappa \geq 0, m > 0$ as in Lemma 3.2. Here we follow the method as in ([CY]). Set

$$\begin{cases} x = \left(\delta_0 \left|\nabla_b f\right|^2 - \delta f_t\right)(x_0, t_0) \quad \text{for } \delta > \delta_0 > 2 \\ \overline{x} = \left|\nabla_b f\right|^2 (x_0, t_0) \\ y = |f_0| (x_0, t_0) \end{cases}$$

where δ_0 and (x_0, t_0) will be chosen later and $f_0 = Tf$ with $T := Y_\alpha$. From now on, T denotes a positive real number instead of a vector field.

If F attains its maximum at $(x_0, t_0) \in M \times [0, T]$, then, by choosing a normal coordinate at (x_0, t_0) and (2.2), (4.5) becomes

$$\begin{aligned} 0 \geq -\frac{F}{t_0} + 2t_0 \left[\frac{1}{m}(2\Delta_b f)^2 + 2\left(\rho_1 - \frac{\kappa}{\beta t_0}\right)\overline{x} + \rho_2 y^2\right] \\ - 16n\beta t_0^2 \overline{A}\overline{x}y - \beta t_0 y^2 \end{aligned} \tag{4.6}$$

for $d = 2n$ and $h = 1$. More precisely from the commutation relations (2.2), we have at (x_0, t_0)

(4.7)

$$\begin{aligned} &\sum_{\alpha \in I_{2n}} (e_\alpha f)(Tf)([e_\alpha, T]f)(x_0, t_0) \\ &= f_0 \sum_{\alpha \in I_{2n}} f_{e_\alpha} (e_\alpha T f - T e_\alpha f) \\ &= f_0 \sum_{\alpha \in I_{2n}} f_{e_\alpha} \left[(f_{0e_\alpha} + (D_{e_\alpha} T) f) - (f_{e_\alpha 0} + (D_T e_\alpha) f)\right] \\ &= f_0 \left[\sum_{j \in I_n} f_{e_j} \left(f_{0e_j} - f_{e_j 0}\right) + \sum_{j \in I_n} f_{e_{\tilde{j}}} \left(f_{0e_{\tilde{j}}} - f_{e_{\tilde{j}} 0}\right) - \sum_{\alpha, \beta \in I_{2n}} \Gamma_{0e_\alpha}^{e_\beta} f_{e_\alpha} f_{e_\beta}\right] \\ &= f_0 \sum_{j,l \in I_n} f_{e_j} \left(f_{e_l} \operatorname{Re} A_{jl} - f_{e_{\tilde{l}}} \operatorname{Im} A_{jl}\right) \\ &\quad + f_0 \sum_{j,l \in I_n} f_{e_{\tilde{j}}} \left(-f_{e_l} \operatorname{Im} A_{jl} - f_{e_{\tilde{l}}} \operatorname{Re} A_{jl}\right) - f_0 \sum_{\alpha, \beta \in I_{2n}} \Gamma_{0e_\alpha}^{e_\beta} f_{e_\alpha} f_{e_\beta} \\ &\geq -|f_0| \overline{A} \sum_{j,l \in I_n} \left(\left|f_{e_j}\right| + \left|f_{e_{\tilde{j}}}\right|\right)\left(|f_{e_l}| + |f_{e_{\tilde{l}}}|\right) \\ &\geq -4n\overline{A}\overline{x}y. \end{aligned}$$

We divide the discussion into the following two cases:

(I) *Case* $I : x \geq 0$:

By

$$(\Delta_b f)^2 = \left(f_t - |\nabla_b f|^2\right)^2 = \left[\frac{x}{\delta} + \left(1 - \frac{\delta_0}{\delta}\right)|\nabla_b f|^2\right]^2 \geq \frac{x^2}{\delta^2} + \frac{(\delta - \delta_0)^2}{\delta^2}\left(|\nabla_b f|^2\right)^2,$$

we have

$$0 \geq -\frac{F}{t_0} + \frac{8t_0}{m\delta^2}x^2 + t_0\rho_2 y^2 + \frac{8t_0(\delta - \delta_0)^2}{m\delta^2}\overline{x}^2 + \left(4t_0\rho_1 - \frac{4\kappa}{\beta}\right)\overline{x} + t_0(\rho_2 - \beta)y^2 - 16n\beta t_0^2\overline{A}\overline{x}y. \tag{4.8}$$

Let

$$\mathcal{A} := t_0\left[\frac{8(\delta - \delta_0)^2}{m\delta^2}\overline{x}^2 + \left(4\rho_1 - \frac{4\kappa}{\beta t_0}\right)\overline{x} + (\rho_2 - \beta)y^2 - 16n\beta t_0\overline{A}\overline{x}y\right]$$

and

$$A := \frac{8(\delta - \delta_0)^2}{m\delta^2}.$$

We have

$$\begin{aligned}\mathcal{A} &= t_0\Bigg\{A\left(\overline{x} + \frac{2\rho_1 - \frac{2\kappa}{\beta t_0} - 8n\beta t_0\overline{A}y}{A}\right)^2 \\ &\quad - \frac{\left(2\rho_1 - \frac{2\kappa}{\beta t_0} - 8n\beta t_0\overline{A}y\right)^2}{A} + (\rho_2 - \beta)y^2\Bigg\} \\ &= t_0\Bigg\{A\left(\overline{x} + \frac{2\rho_1 - \frac{2\kappa}{\beta t_0} - 8n\beta t_0\overline{A}y}{A}\right)^2 + \left(\rho_2 - \beta - \frac{64n^2\beta^2 t_0^2\overline{A}^2}{A}\right)y^2 \\ &\quad + \frac{32n\beta t_0\overline{A}}{A}\left(\rho_1 - \frac{\kappa}{\beta t_0}\right)y - 4\frac{\left(\rho_1 - \frac{\kappa}{\beta t_0}\right)^2}{A}\Bigg\}.\end{aligned}$$

Choose

$$\beta = \beta_1 := \min\left\{\frac{\rho_2}{4}, \frac{\sqrt{A\rho_2}}{16nT\overline{A}}\right\}.$$

This implies that

$$B := \left(\rho_2 - \beta - \frac{64n^2\beta^2 t_0^2 \overline{A}^2}{A}\right) \geq \frac{\rho_2}{2}.$$

(i) Under the case

$$T \geq T_0 := \frac{\sqrt{A\rho_2}}{4n\rho_2\overline{A}},$$

we have

$$\begin{aligned}
\mathcal{A} &\geq t_0\left\{By^2 + \frac{32n\beta t_0\overline{A}}{A}\left(\rho_1 - \frac{\kappa}{\beta t_0}\right)y - 4\frac{\left(\rho_1 - \frac{\kappa}{\beta t_0}\right)^2}{A}\right\} \\
&= t_0\left\{B\left(y + \frac{16n\beta t_0\overline{A}}{BA}\left(\rho_1 - \frac{\kappa}{\beta t_0}\right)\right)^2\right. \\
&\qquad \left. - \frac{16^2n^2\beta^2t_0^2\overline{A}^2 + 4BA}{BA^2}\left(\rho_1 - \frac{\kappa}{\beta t_0}\right)^2\right\} \\
&\geq -\frac{1}{A}\left(\rho_1 - \frac{16n\kappa\overline{A}}{\sqrt{A\rho_2}}\frac{T}{t_0}\right)^2\left(1 + \frac{2t_0^2}{T^2}\right)t_0 \\
&\geq -\frac{3}{A}\left(\rho_1 - \frac{16n\kappa\overline{A}}{\sqrt{A\rho_2}}\frac{T}{t_0}\right)^2 t_0,
\end{aligned}$$

Set

$$z = t_0 x.$$

(a) If $x \geq \beta t_0 |f_0|^2$, it follows from

$$F = t_0\left[(2-\delta_0)|\nabla_b f|^2 + x + \beta t_0 |f_0|^2\right]$$

that (4.8) becomes

$$0 \geq -2t_0 x + \frac{8}{m\delta^2}(t_0 x)^2 - \frac{3}{A}\left(\rho_1 t_0 - \frac{16n\kappa\overline{A}}{\sqrt{A\rho_2}}T\right)^2$$

and then

$$\left(z - \frac{m\delta^2}{8}\right)^2 \leq \frac{m\delta^2}{8}\left(\frac{m\delta^2}{8} + \frac{3}{A}\left(\rho_1 t_0 - \frac{16n\kappa\overline{A}}{\sqrt{A\rho_2}}T\right)^2\right).$$

Thus

$$z \leq \frac{m\delta^2}{4} + \frac{\delta}{4}\sqrt{\frac{6m}{A}}\left|\left(\rho_1 t_0 - \frac{16n\kappa\overline{A}}{\sqrt{A\rho_2}}T\right)\right|.$$

This implies

$$F \leq 2t_0 x \leq \frac{m\delta^2}{2} + \frac{\delta}{2}\sqrt{\frac{6m}{A}}\left(-\rho_1 t_0 + \frac{16n\kappa\overline{A}}{\sqrt{A\rho_2}}T\right)$$

and then

$$\left[2\left|\nabla_b f\right|^2 + \beta_1 t\left|f_0\right|^2 - \delta f_t\right](x,T) \leq \frac{C_1^{'}}{T} + C_2^{'}$$

with $C_1^{'} := \frac{m\delta^2}{2}$ and

$$C_2^{'} := -\frac{\rho_1\delta}{2}\sqrt{\frac{6m}{A}} + \frac{8n\delta\kappa\overline{A}\sqrt{6m}}{A\sqrt{\rho_2}} > 0.$$

(b) If $x \leq \beta t_0\left|f_0\right|^2$, it follows that

$$0 \geq -2\beta t_0 y^2 + t_0\rho_2 y^2 - \frac{3}{A}\left(\rho_1 - \frac{16n\kappa\overline{A}}{\sqrt{A\rho_2}}\frac{T}{t_0}\right)^2 t_0$$

and then

$$y^2 \leq \frac{1}{(\rho_2 - 2\beta)}\frac{3}{A}\left(\rho_1 - \frac{16n\kappa\overline{A}}{\sqrt{A\rho_2}}\frac{T}{t_0}\right)^2.$$

Hence

$$\begin{aligned} F &\leq 2\beta t_0^2 y^2 \\ &\leq \frac{6\beta}{(\rho_2 - 2\beta)A}\left(\rho_1 t_0 - \frac{16n\kappa\overline{A}}{\sqrt{A\rho_2}}T\right)^2 \\ &\leq \frac{3}{4n\overline{A}\sqrt{A\rho_2}}\left(\rho_1\frac{t_0}{\sqrt{T}} - \frac{16n\kappa\overline{A}}{\sqrt{A\rho_2}}\sqrt{T}\right)^2. \end{aligned}$$

Finally we have

$$\left[2\left|\nabla_b f\right|^2 + \beta_1 t\left|f_0\right|^2 - \delta f_t\right](x,T) \leq C_2^{''}\left(\rho_1,\rho_2,\kappa,m,n,\delta,\overline{A}\right)$$

with

$$C_2^{''} := \frac{3}{4n\overline{A}\sqrt{A\rho_2}}\left(-\rho_1 + \frac{16n\kappa\overline{A}}{\sqrt{A\rho_2}}\right)^2.$$

(ii) Under the case

$$T \le T_0 := \frac{\sqrt{A\rho_2}}{4n\rho_2\overline{A}},$$

we have

$$\beta_1 = \frac{\rho_2}{4}$$

and then

$$\mathcal{A} \ge -\frac{3}{A}\left(\rho_1 - \frac{4\kappa}{\rho_2 t_0}\right)^2 t_0.$$

(a) If $x \ge \beta t_0 |f_0|^2$, then

$$\left[2|\nabla_b f|^2 + \beta_1 t |f_0|^2 - \delta f_t\right](x,T) \le \frac{C_1''}{T} + C_2'''$$

with $C_1'' := \frac{m\delta^2}{2} + \frac{2\delta\kappa}{\rho_2}\sqrt{\frac{6m}{A}}$ and

$$C_2''' := -\frac{\delta\rho_1}{2}\sqrt{\frac{6m}{A}} > 0.$$

(b) If $x \le \beta t_0 |f_0|^2$, then

$$\left[2|\nabla_b f|^2 + \beta_1 t |f_0|^2 - \delta f_t\right](x,T) \le \frac{C_1'''}{T}$$

with

$$C_1''' := \frac{3}{A}\left(\frac{\rho_1\sqrt{A\rho_2}}{4n\rho_2\overline{A}} - \frac{4\kappa}{\rho_2}\right)^2.$$

(II) *Case II* : $x \le 0$:
We may assume

$$(\delta_0 - 2)|\nabla_b f|^2 \le \beta t_0 |f_0|^2.$$

Otherwise,

$$F \le 0.$$

From (4.6)

$$0 \ge -\frac{F}{t_0} + 2t_0\left[2(\rho_1 - \frac{\kappa}{\beta t_0})\frac{\beta t_0 y^2}{(\delta_0 - 2)} + \rho_2 y^2\right] - 16n\beta t_0^2\overline{A}\overline{x}y - \beta t_0 y^2. \tag{4.9}$$

Set

$$\beta = \beta_2 := \min\left\{\frac{\rho_2}{2}, \frac{1}{n\overline{A}T}, \frac{1}{T\left\|f_0\right\|_{M\times[0,T]}}\right\}.$$

Hence

$$\begin{aligned}
0 &\geq -2\beta t_0 y^2 + 2\rho_2 t_0 y^2 + \rho_1 \frac{4\beta t_0^2}{(\delta_0 - 2)} y^2 - \frac{4\kappa t_0}{(\delta_0 - 2)} y^2 - 16n\beta\overline{A} t_0^2 y \frac{\beta t_0 y^2}{(\delta_0 - 2)} \\
&= 2\left(\rho_2 - \beta\right) t_0 y^2 + t_0 y^2 \left[\rho_1 \frac{4\beta t_0}{(\delta_0 - 2)} - \frac{4\kappa}{(\delta_0 - 2)} - \frac{16n\beta^2 \overline{A} t_0^2 y}{(\delta_0 - 2)}\right] \\
&\geq t_0 y^2 \left[2\left(\rho_2 - \beta\right) + \rho_1 \frac{4}{(\delta_0 - 2)\,\overline{A}} - \frac{4\kappa}{(\delta_0 - 2)} - \frac{16}{(\delta_0 - 2)}\right] \\
&\geq t_0 y^2 \left[\rho_2 + \rho_1 \frac{4}{(\delta_0 - 2)\,\overline{A}} - \frac{4\kappa}{(\delta_0 - 2)} - \frac{16}{(\delta_0 - 2)}\right].
\end{aligned}$$

Choose $\delta_0\left(\rho_1, \rho_2, \kappa, \overline{A}\right) > 2$ such that

$$\left(\rho_2 + \rho_1 \frac{4}{(\delta_0 - 2)\,\overline{A}} - \frac{4\kappa}{(\delta_0 - 2)} - \frac{16}{(\delta_0 - 2)}\right) > 0,$$

we obtain

$$y(x_0, t_0) = 0.$$

It follows that

$$F(x_0, t_0) \leq 0$$

and then

$$2\left|\nabla_b f\right|^2 + \beta_2 t\left|f_0\right|^2 - \delta f_t \leq 0$$

on $M \times [0, T]$. So if we choose

$$\beta \leq \min\left\{\beta_1, \beta_2, \frac{1}{4\left(n+1\right)nT}, \frac{1}{2\left(n+1\right)\overline{A}T}\right\}$$

and

$$m = n + 1, \varepsilon_1 = 1, N = \beta T$$

such that

$$\frac{2n}{m} - \frac{2n^2 N}{\varepsilon_1} - \frac{2n^2 N^2 \overline{A}^2 m}{m-1} > 0$$

with $0 < \nu \leq N$ as in *Lemma* 3.2, we obtain

$$\left[\left|\nabla_b f\right|^2 - \frac{\delta}{2} f_t\right] \leq \frac{C_1}{t} + C_2.$$

Here

$$C_1 = \frac{1}{2}\max\left\{ n(n+1)\delta^2 + \frac{8\sqrt{3}\left(n+1^2\right)\delta^2}{(\delta-\delta_0)}, \right.$$
$$\left. \frac{3n(n+1)\delta^2}{4(\delta-\delta_0)^2}\left[\left(k+\frac{\overline{B}^2}{2(n+1)}\right)\frac{(\delta-\delta_0)}{2n(n+1)\overline{A}\delta} + \frac{16(n+1)}{n}\right]^2 \right\},$$
$$C_2 = \frac{1}{2}\max\left\{ \left(k+\frac{\overline{B}^2}{2(n+1)}\right)\frac{\sqrt{3}n(n+1)\delta^2}{2(\delta-\delta_0)} + 16\sqrt{3}(n+1)^2\frac{\delta^3\overline{A}}{(\delta-\delta_0)^2}, \right.$$
$$\left. \frac{3(n+1)\delta}{8n\overline{A}(\delta-\delta_0)}\left(k+\frac{\overline{B}^2}{2(n+1)}+\frac{32n(n+1)\delta\overline{A}}{(\delta-\delta_0)}\right)^2 \right\}.$$

Note that $\beta \to 0^+$ if $T \to \infty$ and

$$\max\left\{C_1', C_1'', C_1'''\right\} \le C_1,$$
$$\max\left\{C_2', C_2'', C_2'''\right\} \le C_2.$$

These will complete the proof. □

The proof of **Theorem 1.2 :**

Proof. Define

$$\eta : [t_1, t_2] \longrightarrow M \times [t_1, t_2]$$
$$t \mapsto (\gamma(t), t)$$

where γ is a horizontal curve with $\gamma(t_1) = x_1,\ \gamma(t_2) = x_2$. Let $f = \ln u$, integrate $f'(t)$ along γ, so we get

$$f(x_1,t_1) - f(x_2,t_2) = -\int_{t_1}^{t_2} (f\circ\eta)'\,dt = -\int_{t_1}^{t_2} \left(\langle\gamma'(t), \nabla_b f\rangle + f_t\right) dt.$$

By applying Theorem 1.1**,** this yields

$$f(x_1,t_1) - f(x_2,t_2) < -\int_{t_1}^{t_2}\langle\gamma'(t), \nabla_b f\rangle\,dt + \int_{t_1}^{t_2}\frac{1}{\delta}\left(\frac{C_1}{t} + C_2 - |\nabla_b f|^2\right)dt$$
$$\le \int_{t_1}^{t_2}\left(\frac{\delta}{4}|\gamma'(t)|^2 + \frac{C_1}{\delta t} + \frac{C_2}{\delta}\right)dt.$$

We could choose

$$\left|\gamma'(t)\right| = \frac{d_{cc}(x_1, x_2)}{t_2 - t_1};$$

we reach

$$\frac{u(x_1,t_1)}{u(x_2,t_2)} < \left(\frac{t_2}{t_1}\right)^{\frac{C_1}{\delta}} \cdot \exp\left(\frac{\delta}{4}\frac{d_{cc}(x_1,x_2)^2}{t_2-t_1} + \frac{C_2}{\delta}(t_2-t_1)\right). \qquad \square$$

5. Li-Yau gradient estimates for sum of squares of vector fields

In the paper of H.-D. Cao and S.-T. Yau ([CY]), they derived the gradient estimate for step 2. Here we generalize the result to higher step under the assumption of the curvature-dimension inequality. Let M be a closed smooth manifold and L be an operator with respect to the sum of squares of vector fields $\{e_1,\ e_2, \ldots, e_d\}$

$$L = \sum_{j \in I_d} e_j^2.$$

Suppose that u is the positive solution of

$$\left(L - \frac{\partial}{\partial t}\right) u = 0$$

on $M \times [0, +\infty)$. Now we introduce another test function as in [CY] for $f(x,t) = \ln u(x,t)$

$$G(x,t) = t\left[\sum_{j \in I_d} |e_j f|^2 + \sum_{\alpha \in \Lambda} \left(1 + |Y_\alpha f|^2\right)^\lambda - \delta f_t\right] \tag{5.1}$$

for $\lambda \in \left(\frac{1}{2}, 1\right)$ to be determined later. Note that the power λ in this test function G is necessary due to (5.14).

By the same computation as in *Lemma 2.1* of [CY], we have Lemma 5.1.

Lemma 5.1. *Let M be a smooth connected manifold with a positive measure and L be an operator with respect to the sum of squares of vector fields*

$\{e_1,\ e_2,\ldots,e_d\}$. *Suppose that* u *is the positive solution of*

$$\left(L-\frac{\partial}{\partial t}\right)u=0$$

on $M\times[0,+\infty)$. *Then the following equality holds:*

$$\begin{aligned}\left(L-\frac{\partial}{\partial t}\right)G=&-\frac{G}{t}+2t\left(\sum_{i,j\in I_d}\left|e_ie_jf\right|^2+\sum_{j\in I_d}\left(e_jf\right)\left([L,e_j]\,f\right)\right)\\&+2\lambda t\sum_{i\in I_d,\alpha\in\Lambda}\left(1+\left|Y_\alpha f\right|^2\right)^{\lambda-2}\left|e_iY_\alpha f\right|^2\left[1+(2\lambda-1)\left|Y_\alpha f\right|^2\right]\\&+2\lambda t\sum_{\alpha\in\Lambda}\left(1+\left|Y_\alpha f\right|^2\right)^{\lambda-1}\left(Y_\alpha f\right)\left([L,Y_\alpha]\,f\right)\\&+4\lambda t\sum_{i\in I_d,\alpha\in\Lambda}\left(1+\left|Y_\alpha f\right|^2\right)^{\lambda-1}\left(e_if\right)\left(Y_\alpha f\right)\left([e_i,Y_\alpha]\,f\right)\\&-2\sum_{j\in I_d}\left(e_jf\right)\left(e_jG\right).\end{aligned}$$

Then, as a consequence of Lemma 5.1, we get Proposition 5.2.

Proposition 5.1. *If* M *satisfies the curvature-dimension inequality* $CD\left(\rho_1,\rho_2,\kappa,m\right)$ *for* $\rho_1\in\mathbb{R},\rho_2>0,\kappa\geq0,m>0$, *then*

$$\begin{aligned}\left(L-\frac{\partial}{\partial t}\right)G\geq&-\frac{G}{t}+\frac{t}{m}\left(Lf\right)^2+t\sum_{i,j}\left|e_ie_jf\right|^2+t\rho_2\Gamma^Z\left(f,\ f\right)\\&+2\lambda\left(2\lambda-1\right)t\sum_{i,\alpha}\left(1+\left|Y_\alpha f\right|^2\right)^{\lambda-1}\left|e_iY_\alpha f\right|^2\\&+t\sum_j\left(e_jf\right)\left([L,e_j]\,f\right)+t\left(\rho_1-\frac{\kappa}{\nu}\right)\Gamma\left(f,\ f\right)-\nu t\Gamma_2^Z\left(f,\ f\right)\\&+2\lambda t\sum_\alpha\left(1+\left|Y_\alpha f\right|^2\right)^{\lambda-1}\left(Y_\alpha f\right)\left([L,Y_\alpha]\,f\right)\\&+4\lambda t\sum_{i,\alpha}\left(1+\left|Y_\alpha f\right|^2\right)^{\lambda-1}\left(e_if\right)\left(Y_\alpha f\right)\left([e_i,Y_\alpha]\,f\right)\\&-2\sum_j\left(e_jf\right)\left(e_jG\right).\end{aligned}\tag{5.2}$$

Remark 5.1. With the help of the curvature-dimension inequality, we obtain the extra positive term $t\rho_2\Gamma^Z(f,f)$ in order to control some of the remaining negative terms in the upcoming estimates. Note that there are similar spirits as in [CY, (2.10)] and [CKL, (2.10)].

Now we are ready to prove the main theorem in this section.

The proof of **Theorem 1.4 :**

Proof. Here we follow the method as in ([CY, Proposition 2.1.]). We claim that there are positive constants C_1, C_2, C_3 such that

$$G \le C_1 + C_2 t + C_3 t^{\frac{2\lambda-1}{\lambda-1}}.$$

If not, then for arbitrary such C_1, C_2, C_3, we have

$$G > C_1 + C_2 t + C_3 t^{\frac{2\lambda-1}{\lambda-1}}$$

at its maximum (x_0, t_0) on $M \times [0, T]$ for some $T > 0$. Clearly,

$$\begin{cases} t_0 > 0, & (e_j G)(x_0, t_0) = 0, \\ \frac{\partial G}{\partial t}(x_0, t_0) \ge 0, & LG(x_0, t_0) \le 0, \end{cases}$$

for $j \in I_d$.

Choosing

$$\nu = \lambda(2\lambda - 1)\left(1 + \max_\alpha\left(|Y_\alpha f|^2(x_0, t_0)\right)\right)^{\lambda-1}$$

and evaluating (5.2) at (x_0, t_0), we obtain

$$\begin{aligned} 0 \ge & -\frac{G}{t_0} + \frac{t_0}{m}(Lf)^2 + t_0\sum_{i,j}|e_i e_j f|^2 + t_0\rho_2\sum_\alpha |Y_\alpha f|^2 \\ & + t_0\sum_j (e_j f)([L, e_j] f) \\ & + \lambda(2\lambda-1)\, t_0\sum_{j,\alpha}\left(1 + |Y_\alpha f|^2\right)^{\lambda-1}|e_j Y_\alpha f|^2 \\ & + \lambda(3-2\lambda)\, t_0\sum_\alpha\left(1 + |Y_\alpha f|^2\right)^{\lambda-1}(Y_\alpha f)([L, Y_\alpha] f) \\ & + 4\lambda t_0\sum_{j,\alpha}\left(1 + |Y_\alpha f|^2\right)^{\lambda-1}(e_j f)(Y_\alpha f)([e_j, Y_\alpha] f) \\ & - \frac{t_0\kappa}{\lambda(2\lambda-1)}\sum_\alpha\left(1 + |Y_\alpha f|^2\right)^{1-\lambda}\sum_j |e_j f|^2 + t_0\rho_1\sum_j |e_j f|^2. \end{aligned} \tag{5.3}$$

By straightforward computation, we have

$$|[L,e_j]\,f| = \left|2\sum_i e_i\,[e_i,e_j]\,f - \sum_i [e_i,[e_i,e_j]]\,f\right| \le 2\sum_i |e_i\,[e_i,e_j]\,f| + \sum_\alpha |Y_\alpha f| \tag{5.4}$$

and

$$\begin{aligned}\left|\left[L,Y'_\eta\right]f\right| &= \left|2\sum_i e_i\left[e_i,Y'_\eta\right]f - \sum_i \left[e_i,\left[e_i,Y'_\eta\right]\right]f\right| \\ &\le 2\sum_{i,A}\left|e_iY''_Af\right| + d\left(a\sum_j|e_jf| + b\sum_\eta\left|Y'_\eta f\right| + c\sum_A\left|Y''_Af\right|\right) \\ &\le 2\sum_{i,\alpha}|e_iY_\alpha f| + da\sum_j|e_jf| + d\,(b+c)\sum_\alpha|Y_\alpha f|\,.\end{aligned} \tag{5.5}$$

Similarly

$$\begin{aligned}\left|\left[L,Y''_A\right]f\right| &= \left|\sum_i e_i\left[e_i,Y''_A\right]f + \sum_i\left[e_i,Y''_A\right]e_if\right| \\ &\le \sum_i\left|e_i\left(a^n_{(i,A)}e_n + b^\eta_{(i,A)}Y'_\eta + c^B_{(i,A)}Y''_B\right)f\right| \\ &\quad + a\sum_{i,n}|e_ne_if| + b\sum_{i,\eta}\left|Y'_\eta e_if\right| + c\sum_{i,B}\left|Y''_Be_if\right| \\ &\le 2a\sum_{i,j}|e_ie_jf| + 2b\sum_{i,\eta}\left|e_iY'_\eta f\right| \\ &\quad + 2c\sum_{i,A}\left|e_iY''_Af\right| + b\sum_{i,\eta}\left|\left[e_i,Y'_\eta\right]f\right| \\ &\quad + c\sum_{i,A}\left|\left[e_i,Y''_A\right]f\right| + da'\sum_j|e_jf| \\ &\quad + db'\sum_\eta\left|Y'_\eta f\right| + dc'\sum_B\left|Y''_Bf\right|\end{aligned} \tag{5.6}$$

$$
\begin{aligned}
&\le 2a\sum_{i,j}\left|e_ie_jf\right|+2\left(b+c\right)\sum_{i,\alpha}\left|e_iY_\alpha f\right|+da'\sum_{j}\left|e_jf\right|+db'\sum_{\eta}\left|Y'_\eta f\right|\\
&\quad+\left(dc'+b\right)\sum_{B}\left|Y''_Bf\right|+c\sum_{i,B}\left|\left(a^n_{(i,B)}e_n+b^\eta_{(i,B)}Y'_\eta+c^A_{(i,B)}Y''_A\right)f\right|\\
&\le 2a\sum_{i,j}\left|e_ie_jf\right|+2\left(b+c\right)\sum_{i,\alpha}\left|e_iY_\alpha f\right|+d\left(a'+acd^4\right)\sum_{j}\left|e_jf\right|\\
&\quad+d\left(b'+bcd^4\right)\sum_{\eta}\left|Y'_\eta f\right|+\left(dc'+b+c^2d^4\right)\sum_{B}\left|Y''_Bf\right|\\
&\le 2a\sum_{i,j}\left|e_ie_jf\right|+2\left(b+c\right)\sum_{i,\alpha}\left|e_iY_\alpha f\right|+d\left(a'+acd^4\right)\sum_{j}\left|e_jf\right|\\
&\quad+\left(db'+bcd^5+dc'+b+c^2d^4\right)\sum_{\alpha}\left|Y_\alpha f\right|.
\end{aligned}
$$

Also

$$
\left|\left[e_i,Y'_\eta\right]f\right|\le\sum_{\alpha}\left|Y_\alpha f\right| \tag{5.7}
$$

and

$$
\left|\left[e_i,Y''_A\right]f\right|\le a\sum_{j}\left|e_jf\right|+\left(b+c\right)\sum_{\alpha}\left|Y_\alpha f\right|. \tag{5.8}
$$

Substituting (5.4)–(5.8) into (5.3) and noting that

$$
\left(L-\frac{\partial}{\partial t}\right)f=-\sum_{j}\left|e_jf\right|^2,
$$

we have

$$
\begin{aligned}
0\ge&-\frac{G}{t_0}+\frac{t_0}{m}\left(\sum_{j}\left|e_jf\right|^2-f_t\right)^2+t_0\sum_{i,j}\left|e_ie_jf\right|^2+t_0\rho_2\sum_{\alpha}\left|Y_\alpha f\right|^2\\
&+\lambda\left(2\lambda-1\right)t_0\sum_{j,\alpha}\left(1+\left|Y_\alpha f\right|^2\right)^{\lambda-1}\left|e_jY_\alpha f\right|^2\\
&-d\left(a+a'+acd^4\right)\lambda\left(3-2\lambda\right)t_0\sum_{j,\beta}\left(1+\left|Y_\beta f\right|^2\right)^{\lambda-1}\left|Y_\beta f\right|\left|e_jf\right|\\
&-\left(db+dc+db'+bcd^5+dc'+b+c^2d^4\right)\lambda\left(3-2\lambda\right)\\
&\quad\times t_0\sum_{\alpha,\beta}\left(1+\left|Y_\alpha f\right|^2\right)^{\lambda-1}\left|Y_\alpha f\right|\left|Y_\beta f\right|
\end{aligned} \tag{5.9}
$$

$$
\begin{aligned}
&\underbrace{-2t_0\sum_{i,j}|e_jf|\,|e_i\,[e_i,e_j]\,f|}_{(1)}\underbrace{-t_0\left(\sum_j(e_jf)\right)\left(\sum_\alpha|Y_\alpha f|\right)}_{(2)}\\
&\underbrace{-2a\lambda\,(3-2\lambda)\,t_0\sum_{i,j,\beta}\left(1+|Y_\beta f|^2\right)^{\lambda-1}|Y_\beta f|\,|e_ie_jf|}_{(3)}\\
&\underbrace{-2\,(1+b+c)\,\lambda\,(3-2\lambda)\,t_0\sum_{i,\alpha,\beta}\left(1+|Y_\beta f|^2\right)^{\lambda-1}|Y_\beta f|\,|e_iY_\alpha f|}_{(4)}\\
&\underbrace{-4a\lambda t_0\left(\sum_\beta\left(1+|Y_\beta f|^2\right)^{\lambda-1}|Y_\beta f|\right)\left(\sum_j|e_jf|\right)^2}_{(5)}\\
&\underbrace{-4\,(1+b+c)\,\lambda t_0\sum_{j,\alpha,\beta}\left(1+|Y_\beta f|^2\right)^{\lambda-1}|e_jf|\,|Y_\alpha f|\,|Y_\beta f|}_{(6)}\\
&\underbrace{-\frac{t_0\kappa}{\lambda\,(2\lambda-1)}\sum_\alpha\left(1+|Y_\alpha f|^2\right)^{1-\lambda}\cdot\sum_j|e_jf|^2}_{(7)}-t_0\rho_1\sum_j|e_jf|^2\,.
\end{aligned}
$$

Now we estimate each term (1)–(7) in the right hand of (5.9) as follows:

(1)
$$
\begin{aligned}
&\sum_{i,j}|e_jf|\,|e_i\,[e_i,e_j]\,f|\le\sum_{i,j,\alpha}|e_jf|\,|e_iY_\alpha f|\\
&\le\frac{d^2}{\lambda\,(2\lambda-1)}\left(\sum_j|e_jf|^2\right)\left(\sum_\alpha\left(1+|Y_\alpha f|^2\right)^{\frac{1-\lambda}{2}}\right)^2\\
&\quad+\frac{\lambda\,(2\lambda-1)}{4}\sum_{i,\alpha}\left(1+|Y_\alpha f|^2\right)^{\lambda-1}|e_iY_\alpha f|^2\\
&\le\varepsilon\left(\sum_j|e_jf|^2\right)^2+\frac{d^4}{\varepsilon\,(2\lambda-1)^2}\left(\sum_\alpha\left(1+|Y_\alpha f|^2\right)^{\frac{1-\lambda}{2}}\right)^4\\
&\quad+\frac{\lambda\,(2\lambda-1)}{4}\sum_{i,\alpha}\left(1+|Y_\alpha f|^2\right)^{\lambda-1}|e_iY_\alpha f|^2\,.
\end{aligned}
$$

$$(2)\qquad \left(\sum_j (e_j f)\right)\left(\sum_\alpha |Y_\alpha f|\right)$$
$$\le \frac{4d^2(1+d)}{\rho_2}\left(\sum_j |e_j f|\right)^2 + \frac{\rho_2}{16d^2(1+d)}\left(\sum_\alpha |Y_\alpha f|\right)^2$$
$$\le \frac{4d^2(1+d)}{\rho_2}\left(\sum_j |e_j f|\right)^2 + \frac{\rho_2}{16}\left(\sum_\alpha |Y_\alpha f|^2\right).$$

$$(3)\qquad t_0\left[2a\lambda(3-2\lambda)\sum_{i,j,\beta}\left(1+|Y_\beta f|^2\right)^{\lambda-1}|Y_\beta f|\,|e_i e_j f|\right]$$
$$= t_0\left[\left(2a\lambda(3-2\lambda)\sum_\beta\left(1+|Y_\beta f|^2\right)^{\lambda-1}|Y_\beta f|\right)\left(\sum_{i,j}|e_i e_j f|\right)\right]$$
$$\le t_0\left[\sum_{i,j}|e_i e_j f|^2 + d^2\lambda^2(3-2\lambda)^2 a^2\left(\sum_\beta\left(1+|Y_\beta f|^2\right)^{\lambda-1}|Y_\beta f|\right)^2\right].$$

(4)

$$2(1+b+c)\lambda(3-2\lambda)\,t_0\sum_{i,\alpha,\beta}\left(1+|Y_\beta f|^2\right)^{\lambda-1}|Y_\beta f|\,|e_i Y_\alpha f|$$
$$= t_0\gamma\sum_{i,\alpha,\beta}\left[\left(1+|Y_\beta f|^2\right)^{\lambda-1}|Y_\beta f|\left(1+|Y_\alpha f|^2\right)^{\frac{1-\lambda}{2}}\left(1+|Y_\alpha f|^2\right)^{\frac{\lambda-1}{2}}|e_i Y_\alpha f|\right]$$
$$\le t_0\gamma\sum_{i,\alpha,\beta}\left[\frac{d^2(1+d)\gamma}{2\lambda(2\lambda-1)}\left(\left(1+|Y_\beta f|^2\right)^{\lambda-1}|Y_\beta f|\right)^2\left(1+|Y_\alpha f|^2\right)^{1-\lambda}\right.$$
$$\left.+\frac{\lambda(2\lambda-1)}{2\gamma d^2(1+d)}\left(1+|Y_\alpha f|^2\right)^{\lambda-1}|e_i Y_\alpha f|^2\right]$$
$$\le \frac{\gamma^2 d^3(1+d)}{2\lambda(2\lambda-1)}t_0\left(\sum_\beta\left(1+|Y_\beta f|^2\right)^{\lambda-1}|Y_\beta f|\right)^2\left(\sum_\alpha\left(1+|Y_\alpha f|^2\right)^{\frac{1-\lambda}{2}}\right)^2$$
$$+\frac{\lambda(2\lambda-1)}{2}t_0\sum_{i,\alpha}\left(1+|Y_\alpha f|^2\right)^{\lambda-1}|e_i Y_\alpha f|^2$$

for $\gamma = 2(1+b+c)\lambda(3-2\lambda)$.

$$
\begin{aligned}
(5)\qquad & 4a\lambda t_0 \left(\sum_\beta \left(1+|Y_\beta f|^2\right)^{\lambda-1}|Y_\beta f|\right)\left(\sum_j |e_j f|\right)^2 \\
& \le \varepsilon\lambda t_0 \left(\sum_j |e_j f|^2\right)^2 + \frac{4\lambda t_0 d^2 a^2}{\varepsilon}\left(\sum_\beta \left(1+|Y_\beta f|^2\right)^{\lambda-1}|Y_\beta f|\right)^2.
\end{aligned}
$$

$$
\begin{aligned}
(6)\qquad & 4\left(1+b+c\right)\lambda t_0 \sum_{j,\alpha,\beta}\left(1+|Y_\beta f|^2\right)^{\lambda-1}|e_j f|\,|Y_\alpha f|\,|Y_\beta f| \\
& \le t_0 \frac{64d^2\left(1+d\right)\left(1+b+c\right)^2}{\rho_2} \\
& \quad \times \left(\sum_\beta \left(1+|Y_\beta f|^2\right)^{\lambda-1}|Y_\beta f|\right)^2\left(\sum_j |e_j f|\right)^2 \\
& \quad + t_0 \frac{\rho_2}{16d^2\left(1+d\right)}\left(\sum_\alpha |Y_\beta f|\right)^2 \\
& \le t_0 \frac{\varepsilon}{d^2}\left(\sum_j |e_j f|\right)^4 + t_0\frac{\rho_2}{16}\left(\sum_\alpha |Y_\alpha f|^2\right) \\
& \quad + t_0 \frac{1024d^6\left(1+d\right)^2\left(1+b+c\right)^4}{\varepsilon\rho_2^2}\left(\sum_\beta \left(1+|Y_\beta f|^2\right)^{\lambda-1}|Y_\beta f|\right)^4 \\
& \le t_0\varepsilon\left(\sum_j |e_j f|^2\right)^2 + t_0\frac{\rho_2}{16}\left(\sum_\alpha |Y_\beta f|^2\right) \\
& \quad + t_0 \frac{1024d^6\left(1+d\right)^2\left(1+b+c\right)^4}{\varepsilon\rho_2^2}\left(\sum_\beta \left(1+|Y_\beta f|^2\right)^{\lambda-1}|Y_\beta f|\right)^4.
\end{aligned}
$$

$$
\begin{aligned}
(7)\qquad & \frac{t_0\kappa}{\lambda\left(2\lambda-1\right)}\sum_\alpha \left(1+|Y_\alpha f|^2\right)^{1-\lambda}\cdot\sum_j |e_j f|^2 \\
& \le t_0\left[\varepsilon\left(\sum_j |e_j f|^2\right)^2 + \frac{1}{4\varepsilon}\frac{\kappa^2}{\lambda^2\left(2\lambda-1\right)^2}\left(\sum_\alpha \left(1+|Y_\alpha f|^2\right)^{\frac{1-\lambda}{2}}\right)^4\right].
\end{aligned}
$$

Let

$$\begin{aligned}
\overline{x} &= \sum_j |e_j f|^2 (x_0, t_0) , \\
x &= \left(\delta_0 \sum_j |e_j f|^2 - \delta f_t \right) (x_0, t_0) , \\
y &= \max_\alpha |Y_\alpha f| (x_0, t_0) .
\end{aligned}$$

We may assume that $y > 1$; otherwise, the similar method adopted as follows still holds for $y \leq 1$. Now we divide it into two cases:

(I) *Case* $I : x \geq 0$:

In this case, we have

$$\left(\sum_j |e_j f|^2 - f_t \right)^2 \geq \frac{x^2}{\delta^2} + \frac{(\delta - \delta_0)^2}{\delta^2} \left(\sum_j |e_j f|^2 \right)^2 . \tag{5.10}$$

Substituting (1)–(7) and (5.10) into (5.9), we obtain

(5.11)

$$\begin{aligned}
0 \geq & -\frac{G}{t_0} + \left\{ \frac{t_0}{m\delta^2} x^2 + \frac{(\delta - \delta_0)^2}{m\delta^2} t_0 \left(\sum_j |e_j f|^2 \right)^2 + \frac{7}{8} \rho_2 t_0 \left(\sum_\alpha |Y_\alpha f|^2 \right) \right\} \\
& - 2t_0 \varepsilon \left(\sum_j |e_j f|^2 \right)^2 - \frac{2d^4}{\varepsilon (2\lambda - 1)^2} t_0 \left(\sum_\alpha \left(1 + |Y_\alpha f|^2 \right)^{\frac{1-\lambda}{2}} \right)^4 \\
& - \frac{4d^2 (1 + d)}{\rho_2} t_0 \left(\sum_j |e_j f| \right)^2 \\
& - t_0 d^2 \lambda^2 (3 - 2\lambda)^2 a^2 \left(\sum_\alpha \left(1 + |Y_\alpha f|^2 \right)^{\lambda - 1} |Y_\alpha f| \right)^2 \\
& - \frac{\gamma^2 d^3 (1 + d)}{2\lambda (2\lambda - 1)} t_0 \left(\sum_\beta \left(1 + |Y_\beta f|^2 \right)^{\lambda - 1} |Y_\beta f| \right)^2 \left(\sum_\alpha \left(1 + |Y_\alpha f|^2 \right)^{\frac{1-\lambda}{2}} \right)^2
\end{aligned}$$

$$
\begin{aligned}
&- d\left(a + a' + acd^4\right)\lambda\left(3 - 2\lambda\right) t_0 \sum_{j,\beta} \left(1 + |Y_\beta f|^2\right)^{\lambda-1} |e_j f|\,|Y_\beta f| \\
&- \left(db + dc + db' + bcd^5 + dc' + b + c^2 d^4\right) \\
&\quad \times \lambda\left(3 - 2\lambda\right) t_0 \sum_{\alpha,\beta} \left(1 + |Y_\alpha f|^2\right)^{\lambda-1} |Y_\alpha f|\,|Y_\beta f| - t_0 \varepsilon \lambda \left(\sum_j |e_j f|^2\right)^2 \\
&- \frac{4\lambda t_0 d^2 a^2}{\varepsilon} \left(\sum_\beta \left(1 + |Y_\beta f|^2\right)^{\lambda-1} |Y_\beta f|\right)^2 - t_0 \varepsilon \left(\sum_j |e_j f|^2\right)^2 \\
&- t_0 \frac{1024 d^6 \left(1 + d\right)^2 \left(1 + b + c\right)^4}{\varepsilon \rho_2^2} \left(\sum_\beta \left(1 + |Y_\beta f|^2\right)^{\lambda-1} |Y_\beta f|\right)^4 \\
&- t_0 \varepsilon \left(\sum_j |e_j f|^2\right)^2 - t_0 \frac{1}{4\varepsilon} \frac{\kappa^2}{\lambda^2 \left(2\lambda - 1\right)^2} \left(\sum_\alpha \left(1 + |Y_\alpha f|^2\right)^{\frac{1-\lambda}{2}}\right)^4 \\
&- t_0 \rho_1 \left(\sum_j |e_j f|^2\right).
\end{aligned}
$$

Because

$$
\begin{aligned}
&\sum_{j,\beta} \left(1 + |Y_\beta f|^2\right)^{\lambda-1} |e_j f|\,|Y_\beta f| \\
&\leq \frac{1}{4} \left(\sum_j |e_j f|\right)^2 + \left(\sum_\alpha \left(1 + |Y_\alpha f|^2\right)^{\lambda-1} |Y_\alpha f|\right)^2,
\end{aligned}
$$

we could write the inequality (5.11) in $\overline{x}, y$:

$$
\begin{aligned}
0 \geq &- \frac{G}{t_0} + \frac{t_0}{m\delta^2} x^2 + \frac{3}{4} \rho_2 t_0 y^2 + t_0 [\frac{\left(\delta - \delta_0\right)^2}{m\delta^2} \overline{x}^2 - 2\varepsilon \overline{x}^2 - \frac{4d^3\left(1 + d\right)}{\rho_2} \overline{x} \\
&- \frac{d^2\left(a + a' + acd^4\right)}{4} \left(3 - 2\lambda\right) \lambda \overline{x} - \lambda \varepsilon \overline{x}^2 - \varepsilon \overline{x}^2 - \varepsilon \overline{x}^2 - \rho_1 \overline{x}] \\
&+ t_0 \left[\frac{1}{8} \rho_2 y^2 - \frac{2d^{12}(1 + d)^4}{\varepsilon \left(2\lambda - 1\right)^2} y^{4(1-\lambda)} - d^6 (1 + d)^2 \lambda^2 \left(3 - 2\lambda\right)^2 a^2 y^{2(2\lambda-1)}\right.
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\gamma^2 d^{11}\left(1+d\right)^5}{2\lambda\left(2\lambda-1\right)} y^{2\lambda} - d^5(1+d)^2\left(a+a'+acd^4\right)\lambda\left(3-2\lambda\right) y^{2(2\lambda-1)} \\
&- d^4(1+d)^2\left(db+dc+db'+bcd^4+dc'+b+c^2d^4\right)\lambda\left(3-2\lambda\right) y^{2\lambda} \\
&- \frac{4\lambda d^6(1+d)^2a^2}{\varepsilon} y^{2(2\lambda-1)} - \frac{1024 d^{14}\left(1+d\right)^6\left(1+b+c\right)^4}{\varepsilon\rho_2^2} y^{4(2\lambda-1)} \\
&- \frac{1}{4\varepsilon}\frac{\kappa^2 d^8(1+d)^4}{\lambda^2\left(2\lambda-1\right)^2} y^{4(1-\lambda)} - \text{lower order terms}\Bigg].
\end{aligned}
$$

Choose

$$\varepsilon = \frac{\left(\delta-\delta_0\right)^2}{10m\delta^2},$$

we obtain

$$0 \geq -\frac{G}{t_0} + \frac{t_0}{m\delta^2}x^2 + \frac{3}{4}\rho_2 t_0 y^2 - C_4 t_0. \tag{5.12}$$

(i) If $x \geq \sum_\alpha \left(1+|Y_\alpha f|^2\right)^\lambda$, then by the definition

$$G\left(x_0,t_0\right) = t_0\left[\left(1-\delta_0\right)\sum_j |e_j f|^2 + x + \sum_\alpha \left(1+|Y_\alpha f|^2\right)^\lambda\right]$$

we have

$$
\begin{aligned}
& 0 \geq -2t_0 x + \tfrac{(t_0 x)^2}{m\delta^2} - C_4 t_0^2 \\
\Longrightarrow\ & t_0 x \leq 2m\delta^2 + C_5 t_0 \\
\Longrightarrow\ & G \leq 2t_0 x \leq 4m\delta^2 + 2C_5 t_0 \\
\Longrightarrow\ & \left(\sum_{j\in I_d}|e_j f|^2 + \sum_{\alpha\in\Lambda}\left(1+|Y_\alpha f|^2\right)^\lambda - \delta f_t\right)(x_0,t_0) \leq \tfrac{4d\delta^2}{t_0} + C_6.
\end{aligned}
$$

(ii) If $x \leq \sum_\alpha \left(1+|Y_\alpha f|^2\right)^\lambda$, then

$$0 \geq -2C_7 y^{2\lambda} + \frac{3}{4}\rho_2 t_0 y^2 - C_4 t_0.$$

(a) If $t_0 < 1$, then

$$y^2\left(\frac{3}{4}\rho_2 t_0 - 2C_7 y^{2(\lambda-1)}\right) \leq C_4 t_0,$$

and

$$\frac{3}{4}\rho_2 t_0 \leq (C_4 + 2C_7)\, y^{2(\lambda-1)},$$

and

$$y \leq C_2 t_0^{\frac{1}{2(\lambda-1)}},$$

and

$$t_0 y^{2\lambda} \leq C_2 t_0^{\frac{2\lambda-1}{\lambda-1}}.$$

(b) If $t_0 \geq 1$, then

$$0 \geq -2C_7 y^{2\lambda} + \frac{3}{4}\rho_2 t_0 y^2 - C_4 t_0$$

and

$$0 \geq -2C_7 t_0 y^{2\lambda} + \frac{3}{4}\rho_2 t_0 y^2 - C_4 t_0$$

and

$$0 \geq -2C_7 y^{2\lambda} + \frac{3}{4}\rho_2 y^2 - C_4$$

and

$$y \leq C_8$$

and

$$t_0 y^{2\lambda} \leq C_9 t_0.$$

Combining (a) and (b), we have

$$G \leq 2t_0 \sum_{\alpha} \left(1 + |Y_\alpha f|^2\right)^{\lambda} \leq C_2' t_0 + C_3' t_0^{\frac{2\lambda-1}{\lambda-1}}$$

$$\Longrightarrow \quad \left(\sum_{j\in I_d} |e_j f|^2 + \sum_{\alpha\in\Lambda} \left(1 + |Y_\alpha f|^2\right)^{\lambda} - \delta f_t\right)(x_0, t_0) \leq C_2 + C_3 t_0^{\frac{\lambda}{\lambda-1}}.$$

(II) *Case II* : $x \leq 0$:

We may assume

$$(\delta_0 - 1)\sum_{j\in I_d}|e_j f|^2 \le \sum_{\alpha\in\Lambda}\left(1+|Y_\alpha f|^2\right)^{\lambda}; \tag{5.13}$$

otherwise,

$$G(x_0,t_0)\le 0.$$

By (1)–(4), (5.9) becomes

$$\begin{aligned}
0 \ge & -\frac{G}{t_0}+\frac{15}{16}\rho_2 t_0\left(\sum_\alpha |Y_\alpha f|^2\right)\\
& -\frac{2d^2}{\lambda(2\lambda-1)}\left(\sum_j |e_j f|^2\right)\left(\sum_\alpha\left(1+|Y_\alpha f|^2\right)^{\frac{1-\lambda}{2}}\right)^2\\
& -\frac{4d^2(1+d)}{\rho_2}t_0\left(\sum_j |e_j f|\right)^2\\
& -d^2\lambda^2(3-2\lambda)^2 a^2 t_0\left(\sum_\beta\left(1+|Y_\beta f|^2\right)^{\lambda-1}|Y_\beta f|\right)^2\\
& -\frac{\gamma^2 d^3(1+d)}{2\lambda(2\lambda-1)}t_0\left(\sum_\beta\left(1+|Y_\beta f|^2\right)^{\lambda-1}|Y_\beta f|\right)^2\\
& \quad\times\left(\sum_\alpha\left(1+|Y_\alpha f|^2\right)^{\frac{1-\lambda}{2}}\right)^2\\
& -d\left(a+a'+acd^4\right)\lambda(3-2\lambda)\,t_0\sum_{j,\beta}\left(1+|Y_\beta f|^2\right)^{\lambda-1}|Y_\beta f|\,|e_j f|\\
& -\left(db+dc+db'+bcd^5+dc'+b+c^2d^4\right)\\
& \quad\times\lambda(3-2\lambda)\,t_0\sum_{\alpha,\beta}\left(1+|Y_\alpha f|^2\right)^{\lambda-1}|Y_\alpha f|\,|Y_\beta f|\\
& -4a\lambda t_0\left(\sum_\beta\left(1+|Y_\beta f|^2\right)^{\lambda-1}|Y_\beta f|\right)\left(\sum_j |e_j f|\right)^2
\end{aligned} \tag{5.14}$$

$$
\begin{aligned}
&- 4\,(1+b+c)\,\lambda t_0 \sum_{j,\alpha,\beta} \left(1+|Y_\beta f|^2\right)^{\lambda-1} |e_j f|\,|Y_\alpha f|\,|Y_\beta f| \\
&- \frac{t_0\kappa}{\lambda\,(2\lambda-1)} \sum_{\alpha} \left(1+|Y_\alpha f|^2\right)^{1-\lambda} \cdot \sum_j |e_j f|^2 - t_0\rho_1 \sum_j |e_j f|^2 \\
\geq\ & -\frac{G}{t_0} + \frac{3}{4}\rho_2 t_0 y^2 + t_0 \Bigg[\frac{3}{16}\rho_2 y^2 - \frac{C_9}{\delta_0 - 1} y^2 - C_{10} y^{2\lambda} \\
&- C_{11} y^{2(2\lambda-1)} - C_{12} y^{2\lambda} - C_{13} y^{3\lambda-1} - C_{14} y^{2\lambda} - C_{15} y^{4\lambda-1} \\
&- C_{16} y^{3\lambda} - \frac{C_{17}}{\delta_0 - 1} y^2 - C_{18} y^{2\lambda} \Bigg].
\end{aligned}
$$

If we choose

$$
\delta_0 = 1 + \frac{8}{\rho_2}\,(C_9 + C_{17}) > 1 \quad \text{and} \quad \frac{1}{2} < \lambda < \frac{2}{3},
$$

then we derive the inequality

$$
0 \geq -\frac{G}{t_0} + \frac{3}{4}\rho_2 t_0 y^2 - C_{19} t_0 \tag{5.15}
$$

for some constant $C_{19} > 0$. Utilizing the same deductions as precedes and (5.15) instead of (5.12), the proof of this theorem is completed. □

References

[BBBQ] D. Bakry, F. Baudoin, M. Bonnefont, and B. Qian, *Subelliptic Li-Yau estimates on three dimensional model spaces*, Potential theory and stochastics in Albac, pp. 1–10, Theta Ser. Adv. Math. **11**, Theta, Bucharest, (2009).

[BE] D. Bakry and M. Emery, *Diffusions hypercontractives* (French) [Hypercontractive diffusions], Seminaire de Probabilite **XIX**, 1983/84, pp. 177–206; Lecture Notes in Math. **1123**, Springer, Berlin, (1985).

[BG] F. Baudoin and N. Garofalo, *Curvature-dimension inequalities and Ricci lower bounds for sub-Riemannian manifolds with transverse symmetries*, `arXiv:1101.3590v5`, (Oct. 4, 2014).

[Ca] H.-D. Cao, *On Harnack inequalities for the Kähler-Ricci flow*, Invent. Math. **109** (1992), 247–263.

[CCF] S.-C. Chang, T.-H. Chang, and Y.-W. Fan, *Linear trace Li-Yau-Hamilton inequality for the CR Lichnerowicz-Laplacian heat equation*, J. Geom. Anal. **25** (2015), no. 2, 783–819.

[CCKL] D.-C. Chang, S.-C. Chang, T.-L. Kuo, and S.-H. Lai, *CR Li-Yau Gradient Estimate and Entropy Formulae for Witten laplacian via Bakry-Emery pseudohermitian Ricci curvature*, to appear in AJM.

[CFTW] S.-C. Chang, Y.-W. Fan, J. Tie, and C.-T. Wu, *Matrix Li-Yau-Hamilton inequality for the CR heat equation in pseudohermitian* $(2n+1)$*-manifolds*, Math. Ann. **360** (2014), 267–306.

[Ch] B. Chow, *The Yamabe flow on locally conformally flat manifolds with positive Ricci curvature*, Commun. Pure and Appl. Math. **XLV** (1992), 1003–1014.

[Cho] W.-L. Chow, *Uber System Von Lineaaren Partiellen Differentialgleichungen erster Orduung*, Math. Ann. **117** (1939), 98–105.

[CKL] S.-C. Chang, T.-L. Kuo, and S.-H. Lai, *Li-Yau gradient estimate and entropy formulae for the CR heat equation in a closed pseudohermitian* 3*-manifold*, J. Diff. Geom. **89** (2011), 185–216.

[CKW] S.-C. Chang, Otto van Koert, and C.-T. Wu, *The torsion flow on a closed pseudohermitian* 3*-manifold*, `arXiv:1305.5391`.

[CY] H.-D. Cao and S.-T. Yau, *Gradient estimate, Harnack inequalities and estimates for heat kernel of the sum of squares of vector fields*, Mathematische Zeitschrift **221** (1992), 485–504.

[ChY] S.-Y. Cheng and S.-T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Commun. Pure Appl. Math. **28** (1975), 333–354.

[DT] S. Dragomir and G. Tomassini, Differential Geometry and Analysis on CR manifolds, Progress in Mathematics **246**, Birkhauser, (2006).

[G] A. Greenleaf, *The First eigenvalue of a Sublaplacian on a Pseudohermitian manifold*, Comm. Part. Diff. Equ. **10** (1985), no. 2, 191–217.

[H] L. Hörmander, *Hypoelliptic second order differential equations*, Acta Math. **119** (1967), 147–171.

[H1] R. S. Hamilton, *The Harnack estimate for the Ricci flow*, J. Differential Geom. **37** (1993), no. 1, 225–243.

[H2] R. S. Hamilton, *The formation of singularities in the Ricci flow*, Surveys in Differential Geometry, Vol. II, pp. 7–136, International Press, Cambridge, MA, (1995).

[H3] R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **41** (1982), 255–306.

[H4] R. S. Hamilton, *Harnack estimate for the mean curvature flow*, J. Differential Geom. **41** (1995), no. 1, 215–226.

[JS] D. Jerison and A. Sánchez-Calle, *Estimates for the heat kernel for the sum of squares of vector fields*, Indiana J. Math. **35** (1986), 835–854.

[KS1] S. Kusuoka and D. Stroock, *Application of Malliavin calculus. III*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **34** (1987), 391–442.

[KS2] S. Kusuoka and D. Stroock, *Long time estimates for the heat kernel associated with a uniformly sbellitic symmetric second order operator*, Ann. Math. **127** (1988), 165–189.

[L] J. M. Lee, *The Fefferman metric and pseudohermitian invariants*, Transaction A.M.S. **296** (1986), no. 1, 411–429.

[LY] P. Li and S.-T. Yau, *On the parabolic kernel of the Schrődinger operator*, Acta Math. **156** (1985), 153–201.

[M] R. Melrose, *Propagation for the wave group of a positive subelliptic second-order differential operator*, in: S. Mizohata (ed.) Hyperbolic Equations and Related Topics, pp. 181–192. Boston, MA: Academic Press (1986).

[Pe1] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, `arXiv:math.DG/0211159`.

[Pe2] G. Perelman, *The Ricci flow with surgery on three-manifolds*, `arXiv:math.DG/0303109`.

[Pe3] G. Perelman, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, `arXiv:math.DG/0307245`.

[T] N. Tanaka, A Differential Geometric Study on Stronly Pseudoconvex Manifolds, Kinokuniya Book Store Co. Ltd., Kyoto, (1975).

[W] S. M. Webster, *Pseudo-Hermitian structures on a real hypersurface*, J. Diff. Geom. **13** (1978), 25–41.

[Y] S.-T. Yau, *Harmonic functions on complete Riemannian manifolds*, Commun. Pure Appl. Math. **28** (1975), 201–228.

Department of Mathematics and Statistics, Georgetown University
Washington, DC 20057, USA
and Graduate Institute of Business Administration
College of Management, Fu Jen Catholic University
New Taipei City 242, Taiwan, ROC
E-mail address: chang@georgetown.edu

Department of Mathematics and Taida Institute for Mathematical Sciences (TIMS), National Taiwan University
Taipei 10617, Taiwan, ROC
E-mail address: scchang@math.ntu.edu.tw

Yau Mathematical Science Center, Tsinghua University
Haidian District, Beijing 100084, P. R. China
E-mail address: chienlin@mail.tsinghua.edu.cn

Received March 5, 2016
Accepted February 6, 2018